Braidless Topological Quantum Teleportation

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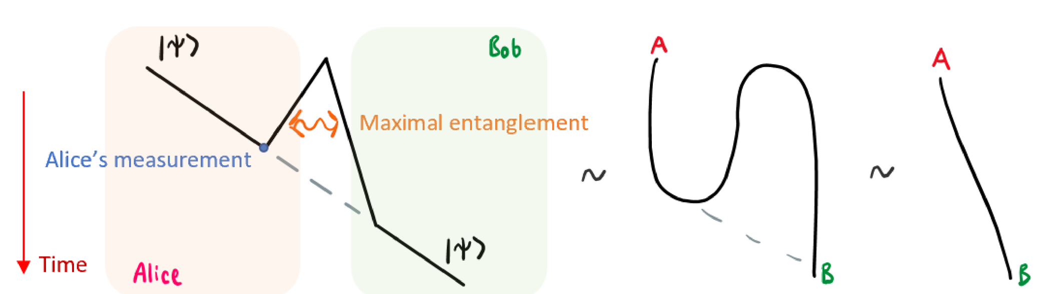
Quantum Teleportation

Scenario: Alice has one copy of an unknown qubit $|\psi\rangle$. She wants to send the state to Bob at but can only transmit *cbits*. They share two maximally entangled qubits i.e. 1 *ebit*.

On the face of it, Alice is in a predicament. With infinitely many copies of $|\psi\rangle$ (no-cloning aside!), she could make infinitely many measurements and send the resulting bits to Bob. He could then reconstruct $|\psi\rangle$. In stark contrast, the teleportation protocol allows Alice to complete her task by exploiting the shared ebit and sending Bob just 2 cbits. Very loosely, teleportation can be *thought of* as using work done in the past (by the preshared ebit) as a tunnel for transmitting quantum information. This is somewhat captured by the homotopy equivalence of a ziz-zag and straight line (a notion made precise in [1]) as processes in spacetime. Below, this is seen as deforming the spike to the dashed line. We take time to run downwards in all diagrams.



These characteristic zig-zags will become hardcore in our anyonic approach to teleportation.

Anyons and String Diagrams

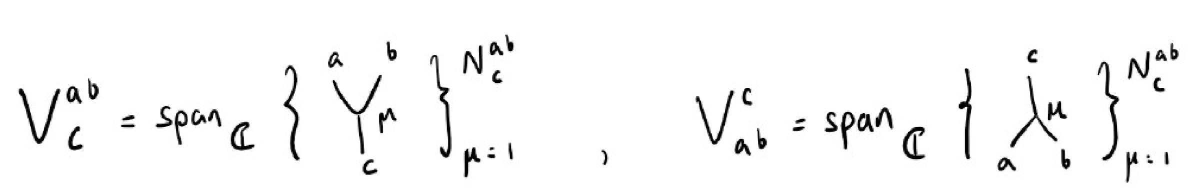
Anyons are localised emergent particles in two spatial dimensions. Any theory of anyons has an underlying finite set of labels $\mathfrak{L} = \{0, a, b, \cdots\}$ that represent their distinct possible types or *charges*. The trivial label 0 represents the vacuum. Anyons can

(1) *Braid* i.e. a sequence of particle exchanges represented as worldlines in (2+1)D.

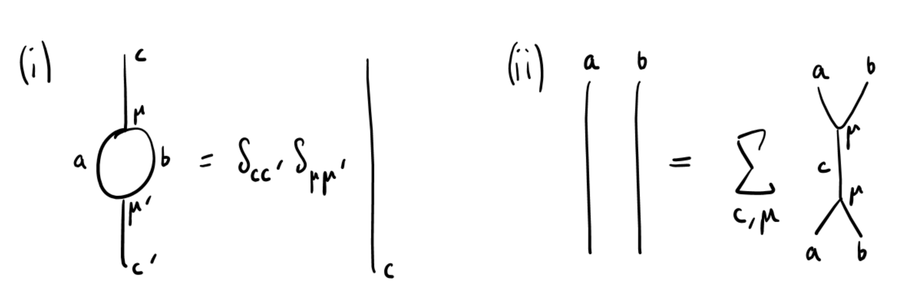
(2) *Twist* i.e. a 2π self-rotation of specified orientation.

(3) *Fuse*. Two anyons of charges a and b can generally have total charge $a \times b = \sum_c N^{ab}_c c$ where fusion coefficients $N^{ab}_c \in \mathbb{Z}_{\geq 0}$ for all $a, b, c \in \mathfrak{L}$. Fusion is *commutative* and *associative*. The summation indicates that the total charge may be a *superposition* of charges. Note that $N^{a0}_b = N^{0a}_b = \delta_{ab}$. Also, each $a \in \mathfrak{L}$ has a unique *dual* charge $\bar{a}$ such that $N^{a\bar{a}}_0 = N^{\bar{a}a}_0 = 1$.

The (fusion) state space of a and b is $V^{ab} \cong \bigoplus_c V^{ab}_c$ where $\dim(V^{ab}_c) = N^{ab}_c$. Measuring the charge of such a pair results in their state collapsing to a subspace V^{ab}_c. The dual space of V^{ab}_c is denoted V^c_{ab}. Its orthogonal basis elements can be thought of as the distinguishable ways in which a pair a and b may be initialised from c. With an appropriate normalisation, we use trivalent vertices to represent orthonormal basis (ONB) elements of these spaces.



We can diagrammatically express (i) orthogonal pairings and (ii) the completeness relation on V^{ab}.

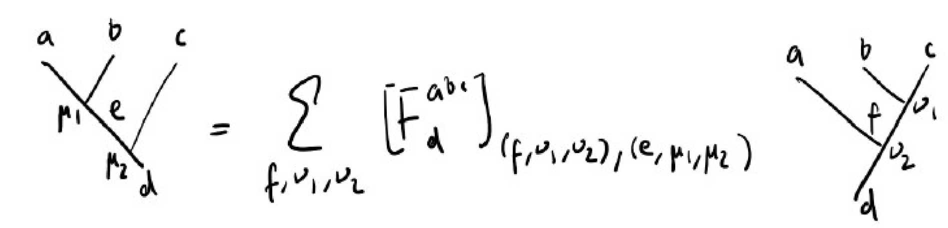


A collection of n anyons can be fused in C_{n-1} different ways (i.e. fixing a sequence of pairwise fusions), where C_n is the n^{th} Catalan number. Each distinct sequence defines a *fusion basis*. Such a basis determines a decomposition of the n-anyon space.

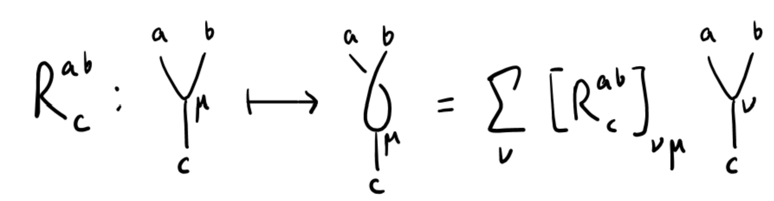
By associativity of fusion, all such decompositions are isomorphic e.g.

$$V^{abc}_d \cong \bigoplus_e V^{ab}_e \otimes V^{ec}_d \cong \bigoplus_f V^{af}_d \otimes V^{bc}_f \tag{1}$$

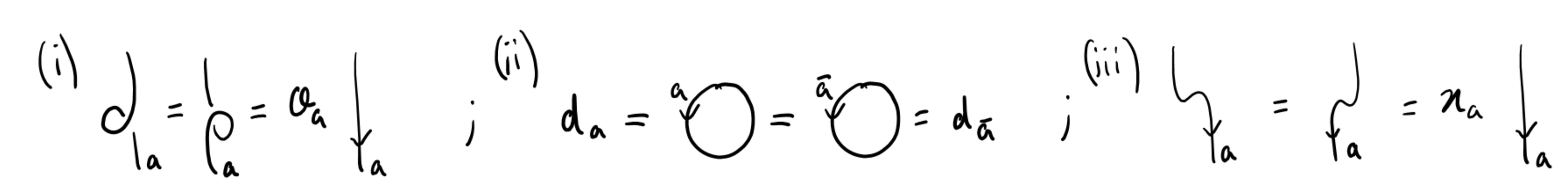
Fixing a fusion basis is akin to a choice of measurement ONB. Change of ONB is realised by (some sequence of) unitary F-matrices. Diagrammatically, these F-moves recouple trivalent vertices. E.g. we illustrate the F-move for F-matrix $F^{abc}_d : \bigoplus_e V^{ab}_e \otimes V^{ec}_d \to \bigoplus_f V^{af}_d \otimes V^{bc}_f$.



A clockwise exchange of two anyons a and b is described by the unitary R-matrix $R^{ab} : V^{ab} \to V^{ba}$ where $R^{ab} = \bigoplus_c R^{ab}_c$. Diagrammatically, $R^{ab}_c : V^{ab}_c \to V^{ba}_c$ is given by the R-move below.



(i) Clockwise twisting an anyon a induces an evolution $\vartheta_a \in U(1)$; the topological spins $\{\vartheta_a\}_a$ can be expressed in terms of the F and R-symbols of a theory.
(ii) The value $d_a \geq 1$ given by an unnormalised loop with label a is called its *quantum dimension*. When $d_a = 1$, a is called an *abelian* anyon; else, $d_a \geq \sqrt{2}$ and a is called a *non-abelian* anyon.
(iii) Zig-zags labelled by a self-dual charge a may be straightened at the cost of a gauge-invariant sign $\varkappa_a = d_a[F^{aaa}_a]_{00} \in \{\pm 1\}$ called its *Frobenius-Schur indicator*.



Altogether, a theory of anyons is specified by its label set, fusion rules, F-symbols and R-symbols i.e. a skeleton of a *unitary braided fusion category* (UBFC). All F and R-matrices satisfy consistency equations known as *pentagon* and *hexagon* equations. There is some gauge-freedom for the symbols (arising from choice of orthonormal basis for each V^{ab}_c); physically meaningful quantities are necessarily gauge-invariant.

Topological Qudits and Braidless Teleportation

The careful control of quantum states is made difficult by the interaction between a system and its environment. One idea is to sidestep this noise by encoding information in anyonic fusion states (whereby the quantum information is nonlocal, and thus hidden from the environment). Quantum gates are typically thought of as being realised via braiding in this approach. In recent work [2, 3] the qubit teleportation protocol was investigated in the context of braiding *Ising* anyons. An Ising theory has $\mathfrak{L} = \{0, \psi, \sigma\}$, where σ is the Ising anyon and ψ is an abelian anyon (often called a 'fermion'). The nontrivial fusion rules of the theory are

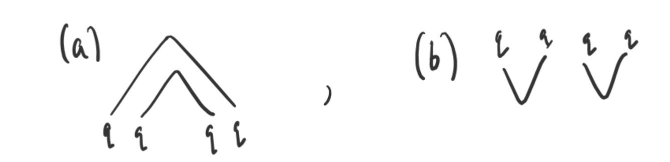
$$\sigma \times \sigma = 0 + \psi\ ,\ \sigma \times \psi = \sigma\ ,\ \psi \times \psi = 0 \tag{2}$$

The controlled transport of anyons presents a formidable exercise in engineering. Moreover, in some candidate systems for Ising anyons, it is not clear how braiding should be carried out. With this in mind, we introduce a braid-free version of the protocol for a family of theories. We will call an anyon q *Tambara-Yamagami* (TY) when it is realised by a UBFC with fusion rule

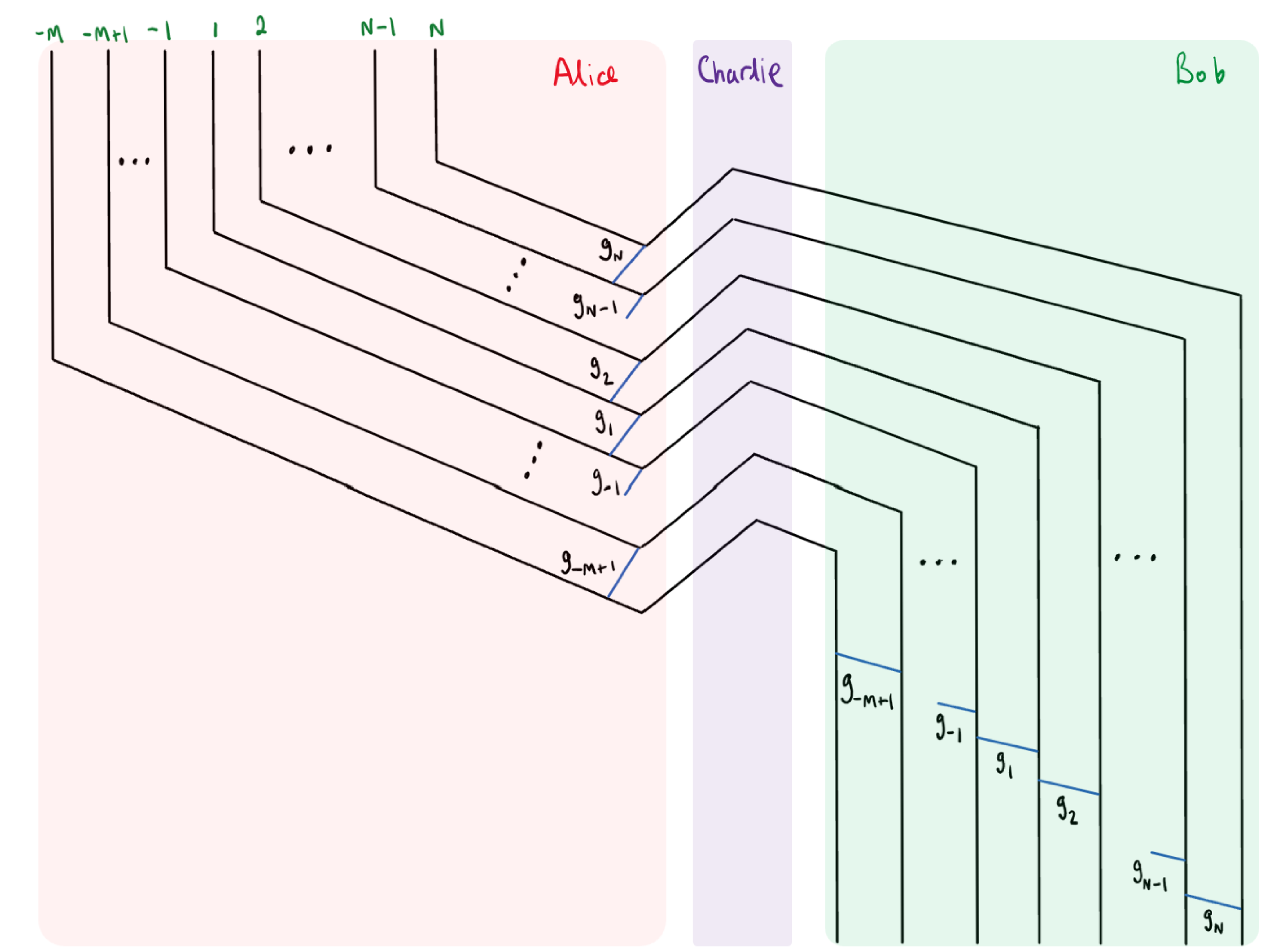
$$q \times q = \sum_g g\ ,\ g \times g' = gg' \tag{3}$$

for $g, g' \in G$ a finite abelian group or order ≥ 2. We note that $q \times g = g \times q = q$ and that $g \in G$ are abelian anyons.

For G of order 2, note that q is an Ising anyon. For $\mathrm{Ord}(G) = d$, we can initialise a maximally entangled qudit $d^{-1/2}\sum_g |g\bar{g}\rangle$ by pair-creating as in (a), and then recoupling as in (b).

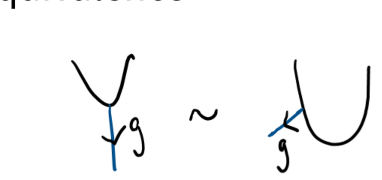


Suppose Alice wishes to teleport a state of $N \geq 2$ TY-anyons (i.e. the state of at most $\lfloor \frac{N}{2} \rfloor$ topological qudits). This can be achieved (without braiding) as follows.

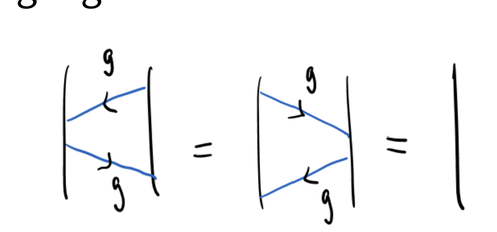


where $g_j \in G$. We use M ancillae where M is the first $j < 0$ such that $g_j = 0$. The probability that Alice measures $g_j = g \in G$ is $\frac{1}{d}$. The distributor of the $k \geq N + M$ pairs of TY-anyons is labelled Charlie (this could also just be either Alice or Bob). The proof that the above process realises the desired teleportation follows from these key facts.

(i) Up to a global phase, we have the equivalence



(ii) It holds for any $g \in G$ and in any gauge that



References

[1] Coecke, B. (2006). "Kindergarten Quantum Mechanics: Lecture Notes". *AIP Conference Proceedings*, 810(1).

[2] Huang, H.L. et al. (2021). "Emulating Quantum Teleportation of a MZM Qubit". *Physical Review Letters*, 126(9).

[3] Xu, C.Q. and Zhou, D.L. (2022). "Quantum Teleportation using Ising Anyons". *Physical Review A*, 106(1).