Contemporary Mathematics
Volume 19, 1983

THE HOMOTOPY LIMIT PROBLEM

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ABSTRACT. I describe a problem which encompasses Segal's Burnside ring conjecture, Atiyah's theorem on the K-theory of classifying spaces, my descent theorem for algebraic K-theory, Quillen's conjecture on the algebraic K-groups of an algebraically closed field in characteristic p, and results of Giffen, Karoubi, and Guin on the relation between K- and L-theory.

The general problem is that of the relation of the lax limit of a group action on a symmetric monoidal category and the homotopy limit of the group action on the associated spectrum.

I would like to thank C. Giffen, who first formulated the general homotopy limit problem and who incited my interest in it. The formulation I use here is somewhat different from Giffen's, which is better adapted to L-theory.

1. I begin by recalling some basic facts about homotopy limits of group actions. Let G be a group acting on a space X. Let EG be the free acyclic G space which is the classifying space of the category $\underline{\underline{EG}}$ below. The homotopy limit of G acting on X is the space of equivariant maps from EG to X (1.1).

$$\text{(1.1)} \qquad \operatorname*{holim}_G X = \mathrm{Map}_G(EG, X)$$

This construction commutes with the loop space functor, so if G acts on a spectrum X, the homotopy limit is a spectrum.

Filtering EG by skeleta induces a tower of fibrations on the homotopy limit. The resulting long exact sequences of homotopy groups assemble into an exact couple, yielding a spectral sequence

$$\text{(1.2)} \qquad E_2^{p,q} = H^p(G;\ \pi_q X) \Rightarrow \pi_{q-p}\mathrm{Map}_G(EG,\ X)$$

The indexing is funny, so the differential d_r has bidegree $(r, r-1)$. For X a spectrum, this is a half-plane spectral sequence. If X is a space,

AMS (MOS) classification 18G99

* partially supported by NSF

there is some trouble in that $\pi_q X$ is not abelian if $q = 1$, not a group if $q = 0$, and not defined for $q < 0$. In both cases, convergence of the spectral sequence is problematic.

If X is a K(M, 0) spectrum for M a G-module, the homotopy limit is a generalized Eilenberg-MacLane spectrum whose homotopy groups are the cohomology groups of G with coefficients in M. If X is a generalized Eilenberg-MacLane spectrum, it corresponds to a chain complex under the Dold-Kan equivalence of categories. The homotopy groups of the homotopy limit are then the hypercohomology groups of G with coefficients in the chain complex corresponding to X. In general, I interpret the homotopy limit (1.1) as the hypercohomology spectrum of G with coefficients in X. This point of view is inspired by Quillen's "homotopical algebra" which generalizes homological algebra by replacing the category of chain complexes in an abelian category with more general categories like the category of spectra. This generalization is necessary for the development of higher algebraic K-theory, and accounts for the role of topology in that subject.

Note that if G acts trivially on X, there is an isomorphism

$$\mathrm{Map}_G(EG, X) = \mathrm{Map}(BG, X) \tag{1.3}$$

The notion of homotopy limit may be defined more generally for any diagram of spaces or spectra parameterized by a small category $\underline{\underline{K}}$. The basic reference, written in terms of simplicial sets, is chapter XI of [BK]. The case of diagrams of spectra is explicitly developed in [T] §5. Justification for the statements above is found in these references.

2. Let the group G act on a small category $\underline{\underline{C}}$. Let $\underline{\underline{EG}}$ be the category whose objects are the elements of G, and with a unique morphism between any two objects. The lax limit of G acting on $\underline{\underline{C}}$ is the category of equivariant functors and natural transformations from $\underline{\underline{EG}}$ to $\underline{\underline{C}}$

$$\underset{G}{\mathrm{laxlim}}\ \underline{\underline{C}} = \mathrm{Cat}_G(\underline{\underline{EG}}, \underline{\underline{C}}) \tag{2.1}$$

An explicit description is this: an object of the lax limit is a pair (C, ψ) where C is an object of $\underline{\underline{C}}$ and ψ is a function assigning to each $g \in G$ a morphism $\psi(g)$ in $\underline{\underline{C}}$

$$\psi(g):\ C \longrightarrow gC \tag{2.2}$$

The function ψ must satisfy normalization and cocycle identities

$\psi(1) = 1$

(2.3)

$$\begin{array}{ccc} C & \xrightarrow{\psi(g)} & gC \\ & {\scriptstyle \psi(gh)} \searrow & \swarrow {\scriptstyle g\psi(h)} \\ & ghC & \end{array}$$

$$\psi(gh) = g\psi(h) \cdot \psi(g)$$

Note $1 = \psi(1) = \psi(gg^{-1}) = g\psi(g^{-1}) \cdot \psi(g)$, and $1 = g1 = g(g^{-1}\psi(g)\ \psi(g^{-1}))$ $= \psi(g) \cdot g\psi(g^{-1})$, so each $\psi(g)$ is an isomorphism

A morphism $(C,\psi) \longrightarrow (C', \psi')$ in the lax limit is a morphism $c\colon\ C \longrightarrow C'$ in $\underline{\underline{C}}$ such that (2.4) commutes

(2.4)

$$\begin{array}{ccc} C & \xrightarrow{\psi(g)} & gC \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle gc} \\ C' & \xrightarrow[\psi'(g)]{} & gC' \end{array}$$

This construction produces many interesting categories as we'll see below. For now, note that if G acts trivially on $\underline{\underline{C}}$, the lax limit is the category of representations of G in $\underline{\underline{C}}$.

If $\underline{\underline{C}}$ is a symmetric monoidal category and the G action respects this structure, the lax limit inherits a symmetric monoidal structure with

(2.5) $$(C,\psi) \oplus (C',\psi') = (C \oplus C', \psi \oplus \psi')$$

The concept of lax limit is defined for any diagram of small categories parameterized by a small category $\underline{\underline{K}}$, and even for pseudo- and lax-diagrams. The construction in its explicit form is due to Street [S].

3. Let N: $\underline{\text{Cat}} \longrightarrow \underline{\Delta^{op}\text{-sets}}$ be the functor sending a category to its nerve. The geometric realization of the nerve is the usual classifying space of a category. Let the category $\underline{\underline{n}}$ have objects the integers 0, 1, 2, ..., n, with a morphism $i \to j$ if i is less than j. Then $N\underline{\underline{n}}$ is the standard n-simplex $\Delta[n]$.

An n-simplex of $N\ \text{Cat}_G(\underline{\underline{EG}}, \underline{\underline{C}})$ is a functor $\underline{\underline{n}} \longrightarrow \text{Cat}_G(\underline{\underline{EG}}, \underline{\underline{C}})$, which corresponds to an equivariant functor $\underline{\underline{EG}} \times \underline{\underline{n}} \longrightarrow \underline{\underline{C}}$. As N is full and faithful, this equivariant functor corresponds to an equivariant simplicial map

$$(3.1) \qquad EG \times \Delta[n] = N(\underline{\underline{EG}} \times \underline{\underline{n}}) \longrightarrow N\underline{\underline{C}}$$

This in turn corresponds to an n-simplex of the simplicial mapping space $Map_G(EG, N\underline{\underline{C}})$. Thus there is an isomorphism of simplicial sets

$$(3.2) \qquad N\ Cat_G(\underline{\underline{EG}}, \underline{\underline{C}}) \cong Map_G(EG, N\underline{\underline{C}})$$

Applying geometric realization, which is a closed functor and so is compatible with mapping space constructions, one gets a canonical map of spaces

$$(3.3) \qquad B\ Cat_G(\underline{\underline{EG}}, \underline{\underline{C}}) \cong |Map_G(EG, N\underline{\underline{C}})| \longrightarrow Map_G(EG, B\underline{\underline{C}})$$

If $\underline{\underline{C}}$ is symmetric monoidal or permutative, one gets a similar map of the associated spectra built by infinite loop space machines

$$(3.4) \qquad Spt(Cat_G(\underline{\underline{EG}}, \underline{\underline{C}})) \longrightarrow Map_G(EG, Spt\ \underline{\underline{C}})$$

These are maps from the lax limit to the homotopy limit. The homotopy limit problem asks whether the maps (3.3), (3.4) are homotopy equivalences, or rather how close they come to being homotopy equivalences.

If every morphism in $\underline{\underline{C}}$ is an isomorphism, then $N\underline{\underline{C}}$ is a Kan complex and (3.3) is a homotopy equivalence. If $\underline{\underline{C}}$ is also symmetric monoidal, (3.4) needn't be a homotopy equivalence. The map (3.3) is not a homotopy equivalence for general $\underline{\underline{C}}$. However, many examples below show that (3.4) becomes a homotopy equivalence if some appropriate modification is made. I would like a general principle to explain this phenomenon.

There are general principles that relate lax colimits with homotopy colimits, both in the case of spaces and of spectra [TH], [TF1], [TF2]. This dual problem appears much easier.

I should point out that the isomorphism (3.2) was first published by John Gray in [G]. I now turn form the general problem to a series of interesting examples.

4. Suppose that $\underline{\underline{C}}$ is a symmetric monoidal category on which G acts trivially. Then laxlim $\underline{\underline{C}}$ is the category $Rep(G, \underline{\underline{C}})$ of representations of G in $\underline{\underline{C}}$. The homotopy limit is the spectrum of maps from BG to Spt $\underline{\underline{C}}$. The homotopy limit problem asks how close the canonical map (4.1) is to being a homotopy equivalence.

(4.1) $$\mathrm{Spt}(\mathrm{Rep}(G, \underline{\underline{C}})) \longrightarrow \mathrm{Map}(BG, \mathrm{Spt}\ \underline{\underline{C}})$$

If $\underline{\underline{C}}$ is the category of finite sets and isomorphisms, with symmetric monoidal structure given by disjoint union, then Spt $\underline{\underline{C}}$ is the sphere spectrum by the Barratt-Priddy-Quillen theorem. Rep(G, $\underline{\underline{C}}$) is the category of finite G-sets and isomorphisms. For finite G, π_0 Spt(Rep(G, $\underline{\underline{C}}$)) is thus the Burnside ring A(G). One form of Segal's conjecture is that (4.1) induces an isomorphism on homotopy groups π_* after completing the homotopy groups on the left side with respect to the augmentation ideal of A(G). Carlsson says this is true at least in non-positive degrees. Note that the necessity of completing with respect to the augmentation ideal shows that (4.1) is not strictly a homotopy equivalence.

Now let $\underline{\underline{C}}$ be the category of finite dimensional vector spaces over a field k. The morphisms are linear isomorphisms, and the symmetric monoidal structure is given by direct sum. Spt $\underline{\underline{C}}$ is the spectrum K(k) whose homotopy groups are the algebraic K-groups of k. If G is finite with order invertible in k, Rep(G, $\underline{\underline{C}}$) is the category of finitely generated projective k[G] modules and isomorphisms. Thus Spt(Rep(G, $\underline{\underline{C}}$)) is the algebraic K-theory spectrum K(k[G]).

One can extend the formalism to the case where $\underline{\underline{C}}$ is a topological category. If $\underline{\underline{C}}$ is the category of finite dimensional complex vector spaces and isomorphims with the usual topology on $GL_n(\mathbf{C})$, Spt($\underline{\underline{C}}$) is the connective spectrum for topological K-theory, bu. For G finite, π_0 Spt(Rep(G, $\underline{\underline{C}}$)) is the complex representation ring R(G). Spt(Rep(G, $\underline{\underline{C}}$)) is a product of bu's indexed by a basis of R(G). The homotopy limit problem asks how close this product is to Map(BG, bu). Atiyah's theorem [A] says that if bu is replaced by the periodic spectrum BU obtained by inverting the Bott element and if R(G) is completed with respect to the augmentation ideal, then (4.1) becomes a homotopy equivalence

(4.2) $$R(G)^{\wedge} \otimes_{\mathbb{Z}} BU \xrightarrow{\ \sim\ } \mathrm{Map}(BG, BU)$$

There is an analogue for Atiyah's theorem in algebraic K-theory. For simplicity, I'll restrict to the case where G is a finite ℓ-group for a prime ℓ. Instead of completing, I'll reduce all spectra mod a power of ℓ by smashing with a $\mathbb{Z}/\ell^{\nu}$ Moore spectrum. I get periodic spectra by inverting the Bott element in algebraic K-theory [T].

Theorem 4.1: Let G be a finite ℓ-group, ℓ a prime. Let k be a field of characteristic not ℓ (and which contains primitive 16th or 9th roots of unity if ℓ = 2 or 3 respectively). Then there is a homotopy equivalence of

K-theory spectra, induced by (4.1)

$$(4.3)\qquad K/\ell^{\nu}(k[G])[\beta^{-1}] \simeq \mathrm{Map}(BG, K/\ell^{\nu}(k)[\beta^{-1}])$$

If in addition $k = \mathbb{F}_q$ is a finite field, there is a homotopy equivalence of non-periodic spaces (not spectra)

$$(4.4)\qquad K/\ell^{\nu}(\mathbb{F}_q[G]) \simeq \mathrm{Map}(BG, K/\ell^{\nu}(\mathbb{F}_q))$$

Pf: If k contains all $|G|$th roots of unity, k[G] is Morita equivalent to a product of copies of k indexed by a basis of R(G). Thus Atiyah's proof in [A] generalizes to this case. If k doesn't contain enough roots of unity, the result follows by etale cohomological descent [T] from an extension of k. The result for $\mathbb{F}_q$ follows as inverting β affects only the negative K-groups of $\mathbb{F}_q$ and $\mathbb{F}_q[G]$, which is Morita equivalent to a products of various finite fields.

5. Let L'/L be a Galois extension of fields with Galois group G. Let $\underline{\underline{C}}$ be the category of finite dimensional vector spaces over L' and isomorphisms. G acts on $\underline{\underline{C}}$ via its action on L'. If V' is in $\underline{\underline{C}}$, gV' is the abelian group V' with new L' action given by pulling back the old action along $g^{-1}: L' \to L'$. The category laxlim $\underline{\underline{C}}$ is the category of semilinear representations of G; its objects are vector spaces V' together with a compatible family of isomorphisms $\psi(g)\colon V' \to gV'$. If V is a vector space over L, let (V', ψ) be defined by (5.1)

$$(5.1)\qquad V' = L' \otimes_L V \quad , \qquad \psi(g) = g \otimes 1 : L' \otimes_L V \xrightarrow{\ \simeq\ } gL' \otimes_L V$$

This extends to a functor from the category of finite dimensional vector spaces over L to laxlim $\underline{\underline{C}}$. The theory of faithfully flat descent says that this functor is an equivalence of categories. See [SGA $4\frac{1}{2}$], Arcata I.4 for more details, in particular for an explanation of how this equivalence implies Hilbert's Theorem 90.

In this case, the homotopy limit problem asks whether there is an equivalence of K-theory spectra

$$(5.2)\qquad K(L) \simeq \mathrm{Map}_G(EG, K(L'))$$

If this were an equivalence, the spectral sequence (1.2) would be the spectral sequence relating the K-groups of a field extension as conjectured by Quillen and Lichtenbaum. This conjecture turns out to be false. However my cohomological descent theorem [T] §2 shows this conjecture is true after reducing mod a prime power and inverting the Bott element.

Theorem 5.1: Let L'/L be a finite Galois extension with Galois group G. Let ℓ be a prime invertible in L, and suppose L contains primitive 16th or 9th roots of unity if $\ell = 2$ or 3 respectively. Then the map (3.4) induces a homotopy equivalence

$$(5.3) \qquad K/\ell^{\nu}(L)[\beta^{-1}] \longrightarrow \mathrm{Map}_G(EG,\ K/\ell^{\nu}(L')[\beta^{-1}])$$

Pf: This is proved in [T]. Specifically it results from [T] 2.20, 2.21, 2.30, (3.29), and 3.23. The cases $\ell = 2,3$ will be handled in the second edition of [T], and in [TE].

This theorem is an important step in understanding the relation between algebraic and topological K-theory. One of the other important steps also fits in the framework of the homotopy limit problem. Let k be an algebraically closed field of characteristic p. Let ϕ^q be the qth power Frobenius map on k. The infinite cyclic group $\mathbb{Z}$ acts on the category of finite vector spaces over k with a generator acting via pullback along ϕ^q. Lang's theorem [L] identifies the lax limit to the category of finite vector spaces over $\mathbb{F}_q$. If the map (3.4) were an equivalence, at least on connected components of zeroth spaces, the spectral sequence (1.2) would yield a fibration sequence of K-theory spaces

$$(5.4) \qquad BGL(\mathbb{F}_q)^+ \longrightarrow BGL(k)^+ \xrightarrow{1-\phi^q} BGL(k)^+$$

This sequence was conjectured by Quillen. It's importance is that it is equivalent to the sequence (5.5) as shown by Hiller [H]

$$(5.5) \qquad BGL(\overline{\mathbb{F}}_p)^+ \longrightarrow BGL(k)^+ \longrightarrow BGL(k)^+ \otimes \mathbb{Q}$$

This sequence and Quillen's computation of the K-groups of the algebraically closed field $\overline{\mathbb{F}}_q$ would yield a calculation of the mod ℓ^{ν} K-groups of the general algebraically closed field of characteristic p, k. If follows from [TQ] that these fibration sequences do exist for mod ℓ^{ν} K-theory after inverting the Bott element. This suffices for most applications as these also need the descent theorem 5.1 which itself requires inverting the Bott element. However, it would be nice to know if (5.4) and (5.5) are true as stated.

6. Let R be a ring with involution, and $\underline{\underline{C}}$ the category of finitely generated free R modules and isomorphisms, $\coprod GL_n(R)$. The group $G = \mathbb{Z}/2$ acts on $\underline{\underline{C}}$ by sending an isomorphism represented by a matrix M to $(\overline{M}^t)^{-1}$, the conjugate transpose inverse. The laxlimit is the category of free R-modules together with an invertible matrix M satisfying the cocycle

condition, $M \cdot (\overline{M}^t)^{-1} = 1$. Thus laxlim $\underline{\underline{C}}$ is the category of free R modules with non-singular hermitian symmetric form. The higher homotopy groups of Spt(laxlim $\underline{\underline{C}}$) are the Karoubi L-groups of R. If the map (3.4) were an equivalence, the spectral sequence (1.2) would relate $\mathbb{Z}/2$ cohomology with coefficients in $K_*(R)$ to the Karoubi L-groups. Guin [Gu] has results similar to this, and his paper led me to try Karoubi periodicity as a general method of attack on the homotopy limit problem. While this method fails to prove the Segal conjecture, it did lead to the proof of cohomological descent for algebraic K-theory, and it can be used to prove Atiyah's theorem.

Giffen has shown how to extend the general formalism to include skew-hermitian and other types of forms, and has proved general Karoubi periodicity theorems in L-theory. None of the L-theory Karoubi periodicities are as simple as the form discussed below because of the alternation between hermitian and skew-hermitian in L-theory.

7. In this section I indicate how a Karoubi periodicity theorem can solve a homotopy limit problem. Let G be a finite group acting symmetrically monoidally on $\underline{\underline{C}}$. There is the usual forgetful functor sending (C, ψ) to C

$$(7.1) \qquad \lambda^* : \operatorname{laxlim}_G \underline{\underline{C}} \longrightarrow \underline{\underline{C}}$$

There is also a transfer functor λ_* , with $\lambda_* C = (\oplus\, gC, \psi)$ where the "sum" is taken over all $g \in G$ and $\psi(h)\colon \oplus\, gC \cong \oplus\, hgC$ is the obvious permutation isomorphism.

$$(7.2) \qquad \lambda_* : \underline{\underline{C}} \longrightarrow \operatorname{laxlim}_G \underline{\underline{C}}$$

If G is the Galois group of L' over L and $\underline{\underline{C}}$ is the category of finite dimensional vector spaces over L', $\mathrm{Spt}(\lambda^*)$ and $\mathrm{Spt}(\lambda_*)$ are the usual and transfer maps respectively on the algebraic K-theory spectra.

If one works with spectra reduced mod ℓ^ν and has a sufficiently general class of homotopy limit problems to be solved, one can reduce to the case $G = \mathbb{Z}/\ell$. See [A] and [T] §2 for examples of each reduction. Henceforth, I'll assume $G = \mathbb{Z}/\ell$ for ℓ prime, and let τ be a generator of G. Define U and V by the extended homotopy fibre sequences (7.3)

$$(7.3) \qquad \begin{array}{l} \Omega\mathrm{Spt}(\underline{\underline{C}}) \xrightarrow{\partial} U(\underline{\underline{C}}) \xrightarrow{u} \mathrm{Spt}(\operatorname{laxlim} \underline{\underline{C}}) \xrightarrow{\lambda^*} \mathrm{Spt}(\underline{\underline{C}}) \\ \Omega\mathrm{Spt}(\operatorname{laxlim} \underline{\underline{C}}) \xrightarrow{\partial} V(\underline{\underline{C}}) \xrightarrow{v} \mathrm{Spt}(\underline{\underline{C}}) \xrightarrow{\lambda_*} \mathrm{Spt}(\operatorname{laxlim} \underline{\underline{C}}) \end{array}$$

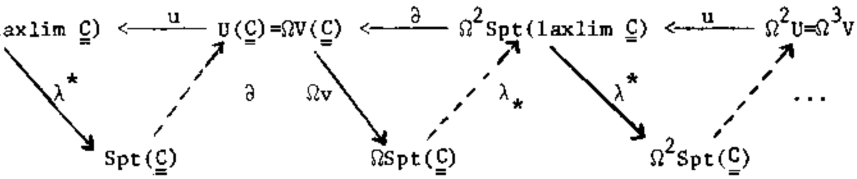
My slogan for remembering which is which is "U for usual, V for Verlagerung". However, when this is specialized to the L-theory examples of §6, my U becomes Karoubi's V and vice-versa.

Suppose there is a Karoubi periodicity homotopy equivalence

$$(7.4) \qquad \theta : U(\underline{\underline{C}}) \xrightarrow{\sim} \Omega V(\underline{\underline{C}})$$

such that $\Omega v \cdot \theta \cdot \partial = 1-\tau$ as an endomorphism of $\Omega \mathrm{Spt}(\underline{\underline{C}})$. Then there is a horizontal tower of fibrations with fibre sequence triangles

(7.5)



The tower extends to the right periodically. Suppose that the homotopy inverse limit of the tower is contractible, as happens if $u \cdot \theta^{-1} \cdot \partial$ is null homotopic. Then just as an Adams resolution yields the Adams spectral sequence, this tower yields a spectral sequence converging to $\pi_* \mathrm{Spt}(\mathrm{laxlim}\ \underline{\underline{C}})$. The E_1 term is a sum of copies of $\pi_* \mathrm{Spt}(\underline{\underline{C}})$, with differential d_1 induced by $\Omega v \cdot \theta \cdot \partial = 1-\tau$, and $\lambda^* \lambda_* = 1+\tau+\tau^2+\ldots+\tau^{\ell-1}$. This E_1 term is thus the canonical periodic resolution for computing the cohomology of the cyclic group $\mathbb{Z}/\ell$ with coefficients in $\pi_* \mathrm{Spt}(\underline{\underline{C}})$. Thus from the E_2 term on, the spectral sequence is

$$(7.6) \qquad E_2^{p,q} = H^p(\mathbb{Z}/\ell\,;\ \pi_q \mathrm{Spt}(\underline{\underline{C}})) \Longrightarrow \pi_{q-p} \mathrm{Spt}(\mathrm{laxlim}\ \underline{\underline{C}})$$

This looks suspiciously like the spectral sequence (1.2). In fact, if Karoubi periodicity also holds for various $\mathrm{Cat}(\tilde{\underline{E}}G, \underline{\underline{C}})$ with twisted G-action, one can show that the map (3.4) is a homotopy equivalence. For details in the Galois case, see [T] 2.25–2.30. Similar results hold after reducing all spectra mod ℓ^ν or after inverting the Bott element β, provided Karoubi periodicity holds after doing this.

8. I'll now describe the ideas behind the proof of Karoubi periodicity for Atiyah's theorem and for my descent theorem. The technical details of the latter are quite elaborate, and the discussion below contains several omissions, oversimplifications, and outright lies. For an accurate and honest account, see [T] and [TE]. Karoubi's proof of Karoubi periodicity in [K] was the paradigm for my proof, but is technically simpler.

For Atiyah's theorem, the sequences (7.3) become

$$(8.1) \qquad \begin{array}{l} U \longrightarrow \prod_1^\ell bu \longrightarrow bu \\ V \longrightarrow bu \longrightarrow \prod_1^\ell bu \end{array}$$

Here the products are indexed by the characters of $\mathbb{Z}/\ell$, a natural basis of the representation ring $R(\mathbb{Z}/\ell)$. Thus V is a product of $\ell-1$ copies of Ωbu. Inverting the Bott element replaces bu by the periodic spectrum BU. The equivalence $BU \simeq \Omega^2 BU$ yields a Karoubi periodicity equivalence $U \simeq \Omega V$ after inverting the Bott element. It's necessary to choose the correct equivalence and to complete with respect to the augmentation ideal to get convergence in the tower (7.5).

Now let L'/L be a Galois extension of fields with Galois group $\mathbb{Z}/\ell$. For any algebra A over L, one has fibration sequences (8.2) of algebraic K-theory spectra

(8.2)
$$U(A) \longrightarrow K(A) \xrightarrow{\lambda^*} K(A \otimes_L L')$$
$$V(A) \longrightarrow K(A \otimes_L L') \xrightarrow{\lambda_*} K(A)$$

Consider (8.3)

$$\begin{array}{ccccc} \Omega K(L) & \xrightarrow{\lambda^*} & \Omega K(L') & \xrightarrow{\partial} & U(L) \\ & \swarrow & \downarrow{\scriptstyle 1-\tau} & \swarrow & \\ \Omega V(L) & \xrightarrow{v} & \Omega K(L') & \xrightarrow{\lambda_*} & \Omega K(L) \end{array}$$

The composition of $1-\tau$ with either λ^* or λ_* is null homotopic. Choice of null homotopies provide lifts $U(L) \longrightarrow \Omega K(L')$ and $\Omega K(L') \longrightarrow \Omega V(L)$. There is a choice of compatible null homotopies, so the Toda bracket $\langle \lambda_*, 1-\tau, \lambda^* \rangle$ vanishes and there is a θ : $U(L) \longrightarrow \Omega V(L)$ compatible with the lifts. The compatibilities are non-trivial, so the map θ is non-trivial. To convert this mush into mathematics, one deploys a maze of fibre sequences to express the Toda bracket on the primary homotopy level. In fact, one uses diagram (8.4), in which all columns are fibre sequences

(8.4)
$$\begin{array}{ccccc} U(A) & & \xrightarrow{\theta(A)} & & \Omega V(A) \\ \downarrow & & & & \downarrow \\ K(A) & \xrightarrow{\cup x} & D(A) & \longrightarrow & D(A) \\ \downarrow{\scriptstyle \lambda^*} & & \downarrow{\scriptstyle \lambda^*} & & \downarrow{\scriptstyle d} \\ K(A \otimes L') & \xrightarrow{\cup x} & D(A \otimes L') & \longrightarrow & V(A \otimes L') \\ & & & & \downarrow{\scriptstyle \lambda_*} \\ & & & & V(A) \end{array}$$

Here the cup product map is induced by an $x \in \pi_0 D(L)$. $\pi_0 V(L')$ is the augmentation ideal of the group ring $\mathbb{Z}[\mathbb{Z}/\ell]$, and x is defined so dx is $1-\tau$. The existence of the lift x expresses the compatibilities mentioned above, as is evident by the construction in [TE].

This $\theta(L)$ satisfies the condition $\Omega v \cdot \theta \cdot \partial = 1-\tau^{-1}$. If it were a homotopy equivalence, all would be well. One calculates low dimensional homotopy groups of $U(L)$ and $\Omega V(L)$ from (8.2).

(8.5)
$$\pi_{-2}\, U(L) = 0 \qquad \pi_{-2}\, \Omega V(L) = \mathbb{Z}/\ell$$
$$\pi_{-1}\, U(L) = 0 \qquad \pi_{-1}\, \Omega V(L) = \mathrm{coker}(\lambda_*:\ L'^* \to L^*)$$
$$\pi_0\ U(L) = L'^*/L^*$$
$$0 \longrightarrow \mathrm{coker}\ \lambda_* K_2 \longrightarrow \pi_0 \Omega V(L) \longrightarrow \ker(\lambda_*:\ L'^* \to L^*) \longrightarrow 0$$

Note $\pi_{-2}\ \theta$ is not an isomorphism, so something must be done. On the other hand, Hilbert's Theorem 90 says that $1-\tau^{-1}$ or θ induces an isomorphism of L'^*/L^* on the kernal of the norm map, ker λ_* . Hilbert's Theorem 90 is part of the faithfully flat descent theory discussed in §5. Hilbert's Theorem 90 is also the basis of Kummer theory, which says that if L is of characteristic not ℓ and contains a primitive ℓth root of unity ζ , then $L' = L(\alpha)$ for α an ℓth root of $a \in L$. One can chose α so that $\tau\alpha/\alpha = \zeta$. I use this information to show $\theta[\beta^{-1}]$ is an equivalence.

Let $y \in \pi_0\ U(L)$ be the image of $\alpha \in K_1(L') = L'^*$ under the boundary map. Then $\Omega v\theta(y) = \Omega v\theta\partial(\alpha) = \alpha/\tau^{-1}\alpha = \zeta$. Suppose L' contains a primitive ℓ^2 root of unity γ , then $\zeta = \gamma^\ell$, so ζ is 0 mod ℓ in $K_1(L')$. Consider the fibre sequence of spectra reduced mod ℓ

(8.6)
$$F/\ell(L) \longrightarrow U/\ell(L) \xrightarrow{\ \Omega v\theta\ } \Omega K/\ell(L')$$

I've shown that the reduction of y in $\pi_0 U/\ell(L)$ dies under $\pi_0 \Omega v\theta$, so it lifts to a t in $\pi_0 F/\ell(L)$. If θ were an equivalence, $F/\ell(L)$ would be $\Omega^2 K/\ell(L)$, and (8.6) would be a shift of a sequence in (8.2). The element t would be a basis of $\Omega^2 K/\ell(L)$ as a module over $K/\ell(L)$. One does have (8.7), inducing the map ϕ on fibres.

(8.7)
$$\begin{array}{ccc} F/\ell(L) & \overset{\phi}{\dashrightarrow} & \Omega^2 K/\ell(L) \\ \downarrow & & \downarrow \\ U/\ell(L) & \xrightarrow{\ \theta\ } & \Omega V/\ell(L) \\ \Omega v\theta \downarrow & & \downarrow \\ \Omega K/\ell(L') & \xrightarrow{\ 1\ } & \Omega K/\ell(L') \end{array}$$

Everything in sight is a module spectrum over the ring spectrum $K/\ell(L)$, and all maps are module maps. Thus the element t determines a cup product map $\cup t:\ K/\ell(L) \longrightarrow F/\ell(L)$. The Bockstein lemma reveals that $\phi \cdot \cup t:\ K/\ell(L) \longrightarrow \Omega^2 K/\ell(L)$ is cup product with the Bott element β . This is because β Bocksteins to $\zeta = \Omega v\theta(y)$ in $K_1(L')$, and it's the divisibility of this element by ℓ that allows t to exist.

Consider the map Ξ induced by the shifted vertical fibre sequences on the left half of (8.8). Here Ω^{-1} is a delooping functor on spectra.

$$
(8.8)\qquad
\begin{array}{ccccc}
K/\ell(L) & \xrightarrow{\cup t} & F/\ell(L) & \xrightarrow{\phi} & \Omega^2 K/\ell(L) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{-1}V/\ell(L) & \xrightarrow{\Xi} & U/\ell(L) & \xrightarrow{\theta} & \Omega V/\ell(L) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{-1}K/\ell(L') & \xrightarrow{\cup\beta} & \Omega K/\ell(L') & \xrightarrow{1} & \Omega K/\ell(L')
\end{array}
\qquad (\cup\beta : K/\ell(L) \to \Omega^2 K/\ell(L))
$$

If one inverts β so that $\cup\beta$ is a homotopy equivalence, the 5-lemma shows that $\theta\ \Xi\ [\beta^{-1}]$ is a homotopy equivalence.

There is also a commutative diagram (8.9), which shows $\Xi\ \theta\ [\beta^{-1}]$ is a homotopy equivalence.

$$
(8.9)\qquad
\begin{array}{ccc}
\Omega K/\ell(L)[\beta^{-1}] & \xrightarrow{\cup\beta} & \Omega^3 K/\ell(L)[\beta^{-1}] \\
\downarrow & & \downarrow \\
\Omega K/\ell(L')[\beta^{-1}] & \xrightarrow{\cup\beta} & \Omega^3 K/\ell(L')[\beta^{-1}] \\
\partial\downarrow & & \downarrow \\
U/\ell(L)[\beta^{-1}] & \xrightarrow{\Xi\theta} & \Omega^2 U/\ell(L)[\beta^{-1}]
\end{array}
$$

Thus $\theta:\ U/\ell(L)[\beta^{-1}] \longrightarrow \Omega V/\ell(L)[\beta^{-1}]$ is the required Karoubi periodicity equivalence.

The diagram (8.9) results from the commutative diagram (8.10), which requires some work to verify.

$$
(8.10)\qquad
\begin{array}{ccccccc}
 & & \Omega K/\ell(L' \otimes_L L') & \xrightarrow{\cup\beta} & & & \Omega^3 K/\ell(L' \otimes_L L') \\
 & \nearrow m & & & & \swarrow \partial & \downarrow (\lambda \otimes 1)_* \\
\Omega K/\ell(L') & \xrightarrow{\cup dx} & \Omega V/\ell(L') & \xrightarrow{\Xi(L')} & \Omega^2 U/\ell(L') & & \Omega^3 K/\ell(L') \\
\partial\downarrow & & \downarrow\lambda_* & & \downarrow\lambda_* & \swarrow \partial & \\
U/\ell(L) & \xrightarrow{\theta} & \Omega V/\ell(L) & \xrightarrow{\Xi} & \Omega^2 U/\ell(L) & & \\
\downarrow & & & & & & \\
K/\ell(L) & & & & & &
\end{array}
\qquad (\cup\beta : \Omega K/\ell(L') \to \Omega^3 K/\ell(L'))
$$

I hope this makes the idea of the proof clear. To get a real proof one just patches all the holes.

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