Tetracategories

The terminology for tricategories is taken from

R. Gordon, A. J. Power, and Ross Street, Coherence for Tricategories. Memoirs of the AMS, vol. 117, no. 558, September 1995.

A *tetracategory* is given by the following data:

(1) A collection of *0-cells* $a, b, c, \ldots$

(2) For each pair of 0-cells (a, b), a tricategory $C(a, b)$. Objects of these tricategories, called *1-cells* of the tetracategory C, are denoted by letters x, y, z, u, v, w. 1-morphisms, 2-morphisms, and 3-morphisms of the local tricategories $C(a,b)$ are called 2-cells, 3-cells, and 4-cells of the tetracategory C.

(3) For each triple of 0-cells (a, b, c), a trifunctor

$$\otimes_{a,b,c} : C(a,b) \times C(b,c) \longrightarrow C(a,c)$$

called *composition*, and for each 0-cell a, a 1-cell $I_a \in C(a,a)$, called a *unit*.

In what follows, the notation $\otimes$ is usually suppressed to save space. Thus, instead of writing $\otimes_{a,b,c}(x, y)$ where $x \in C(a,b)$, $y \in C(b,c)$ are 1-cells, we write xy. Similarly, we typically suppress the index of unit objects I_a, and write simply I.

(4) For each 4-tuple of 0-cells (a, b, c, d), a triequivalence

$$\alpha_{a,b,c,d} : \otimes_{a,c,d}(\otimes_{a,b,c} \times 1) \longrightarrow \otimes_{a,b,d}(1 \times \otimes_{b,c,d})$$

called the *associativity*. This is comprised of 2-cells of the form $(xy)z \to x(yz)$ and attendant structural data ("product cells") for a tritransformation, all of which are equivalences at the appropriate level.

Also, for each pair of 0-cells (a,b), there are triequivalences

$$\lambda_{a,b} : \otimes_{a,a,b} (I_a \times 1) \longrightarrow 1$$

$$\rho_{a,b} : \otimes_{a,b,b} (1 \times I_b) \longrightarrow 1$$

comprised of 2-cells of the form $\lambda : Ix \longrightarrow x$, $\rho : xI \longrightarrow I$, called <u>left</u> and <u>right unit actions</u>.

(5) For each 5-tuple of 0-cells, a trimodification π called the <u>pentagonator</u>, which is a local equivalence comprised of 3-cells

$$\begin{tikzcd}
 & (xy)(zw) \arrow[dr, "\alpha"] & \\
((xy)z)w \arrow[ur, "\alpha"] \arrow[d, "\alpha w"'] & \Downarrow \pi & x(y(zw)) \\
(x(yz))w \arrow[r, "\alpha"'] & x((yz)w) \arrow[ur, "x\alpha"'] &
\end{tikzcd}$$

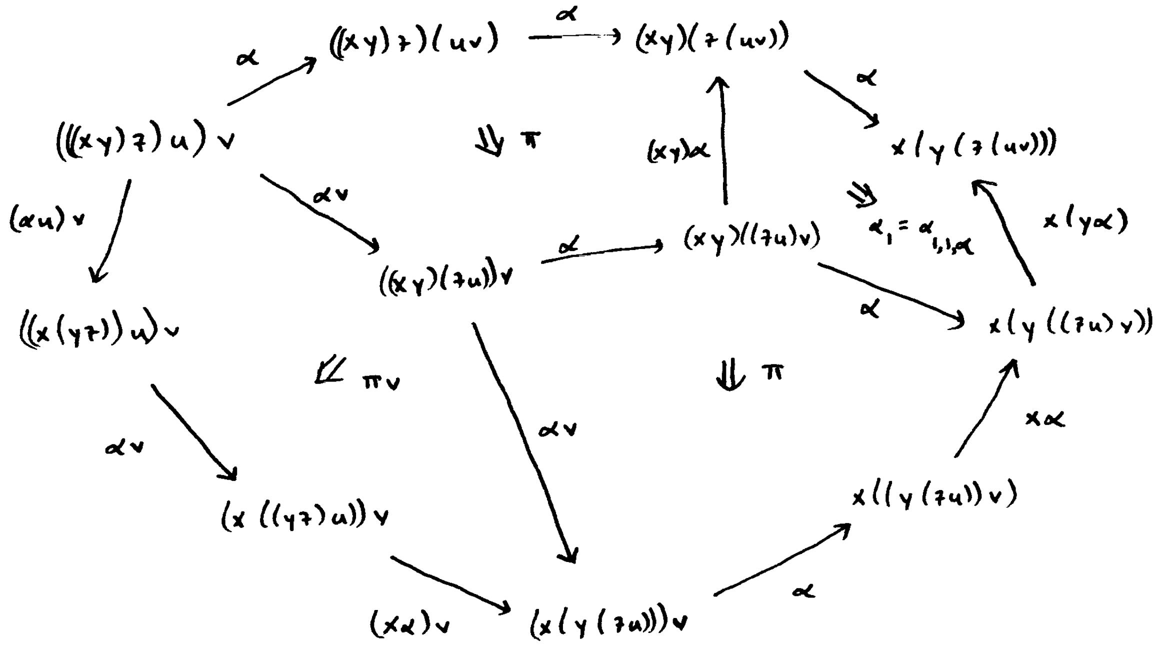
and attendant structural data ("product cells") for a trimodification that are invertible.

Also, for each triple of 0-cells, trimodifications ℓ, m, r called <u>unit mediators</u>, that are local equivalences comprised of 3-cells

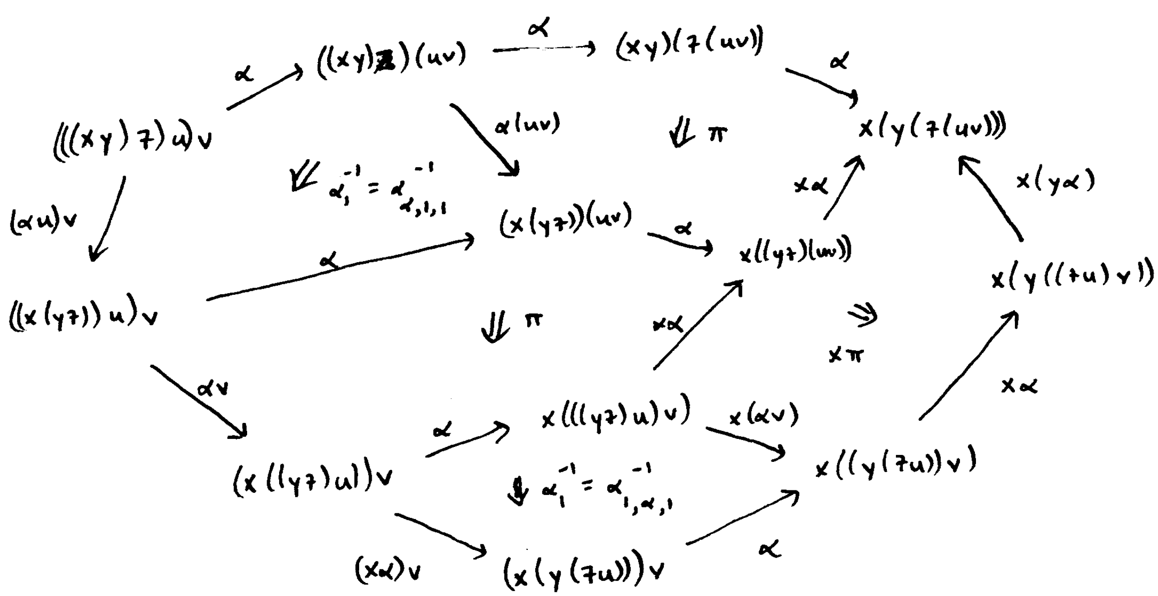
$$\begin{tikzcd}
(Ix)y \arrow[r, "\alpha"] \arrow[dr, "\lambda y"'] & I(xy) \arrow[d, "\lambda"] \\
 & xy
\end{tikzcd} \; \ell \Rrightarrow
\qquad
\begin{tikzcd}
(xI)y \arrow[r, "\alpha"] \arrow[dr, "\rho y"'] & x(Iy) \arrow[d, "x\lambda"] \\
 & xy
\end{tikzcd} \; m \Rrightarrow
\qquad
\begin{tikzcd}
(xy)I \arrow[r, "\alpha"] \arrow[dr, "\rho"'] & x(yI) \arrow[d, "x\rho"] \\
 & xy
\end{tikzcd} \; r \Rrightarrow$$

and attendant structural data for a trimodification that are invertible.

(6) For each 6-tuple of 0-cells, a pertubation K_5 called a (non-abelian) <u>4-cocycle</u>, comprised of invertible 4-cells of the form



$$\Rrightarrow$$



Also, for each 4-tuple of 0-cells, four perturbations $U_{4,1}$, $U_{4,2}$, $U_{4,3}$, $U_{4,4}$ called <u>unit</u> <u>cocycle</u> perturbations, comprised of invertible 4-cells of the form

$$\begin{tikzcd}
& ((Ix)y)z \arrow[r, "(\lambda y)z"] \arrow[dl, "\alpha z"'] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\swarrow \alpha_{\lambda,1,1}"] & (xy)z \arrow[d, "\alpha"] \\
(I(xy))z \arrow[d, "\alpha"'] \arrow[r, phantom, "\Leftarrow \pi"] & (Ix)(yz) \arrow[r, "\lambda(yz)"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\Downarrow l"] & x(yz) \\
I((xy)z) \arrow[r, "I\alpha"'] & I(x(yz)) \arrow[ur, "\lambda"'] &
\end{tikzcd}
\quad \overset{U_{4,1}}{\Rrightarrow} \quad
\begin{tikzcd}
& ((Ix)y)z \arrow[rr, "(\lambda y)z"] \arrow[dl, "\alpha z"'] \arrow[d, phantom, "\swarrow lz"] & & (xy)z \arrow[d, "\alpha"] \\
(I(xy))z \arrow[urrr, "\lambda z"'] \arrow[dr, "\alpha"'] & \Downarrow l & \Downarrow \lambda_\alpha^{-1} & x(yz) \\
& I((xy)z) \arrow[rr, "I\alpha"'] \arrow[uurr, "\lambda"] & & I(x(yz)) \arrow[u, "\lambda"']
\end{tikzcd}$$

$$\begin{tikzcd}
& ((xI)y)z \arrow[r, "(\rho y)z"] \arrow[dl, "\alpha z"'] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\swarrow \alpha_{\rho,1,1}"] & (xy)z \arrow[d, "\alpha"] \\
(x(Iy))z \arrow[d, "\alpha"'] \arrow[r, phantom, "\Leftarrow \pi"] & (xI)(yz) \arrow[r, "\rho(yz)"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\Downarrow m"] & x(yz) \\
x((Iy)z) \arrow[r, "x\alpha"'] & x(I(yz)) \arrow[ur, "x\lambda"'] &
\end{tikzcd}
\quad \overset{U_{4,2}}{\Rrightarrow} \quad
\begin{tikzcd}
& ((xI)y)z \arrow[rr, "(\rho y)z"] \arrow[dl, "\alpha z"'] \arrow[d, phantom, "\swarrow mz"] & & (xy)z \arrow[d, "\alpha"] \\
(x(Iy))z \arrow[urrr, "(x\lambda)z"'] \arrow[dr, "\alpha"'] & & \swarrow \alpha_{1,\lambda,1} & x(yz) \\
& x((Iy)z) \arrow[urr, "x(\lambda z)"] \arrow[r, "x\alpha"'] & x(I(yz)) \arrow[ur, "x\lambda"'] \arrow[u, phantom, "\Downarrow xl"] &
\end{tikzcd}$$

$$\begin{tikzcd}
& ((xy)I)z \arrow[rr, "\rho z"] \arrow[dl, "\alpha z"'] \arrow[dr, "\alpha"] & \Downarrow m & (xy)z \arrow[dr, "\alpha"] & \\
(x(yI))z \arrow[d, "\alpha"'] & \Leftarrow \pi & (xy)(Iz) \arrow[ur, "(xy)\lambda"'] \arrow[d, "\alpha"] & \swarrow \alpha_{1,1,\lambda} & x(yz) \\
x((yI)z) \arrow[rr, "x\alpha"'] & & x(y(Iz)) \arrow[urr, "x(y\lambda)"'] & &
\end{tikzcd}
\quad \overset{U_{4,3}}{\Rrightarrow} \quad
\begin{tikzcd}
& ((xy)I)z \arrow[rr, "\rho z"] \arrow[dl, "\alpha z"'] \arrow[d, phantom, "\swarrow rz"] & & (xy)z \arrow[d, "\alpha"] \\
(x(yI))z \arrow[urrr, "(x\rho)z"'] \arrow[dr, "\alpha"'] & & \swarrow \alpha_{1,\rho,1} & x(yz) \\
& x((yI)z) \arrow[urr, "x(\rho z)"] \arrow[r, "x\alpha"'] & x(y(Iz)) \arrow[ur, "x(y\lambda)"'] \arrow[u, phantom, "\Downarrow xm"] &
\end{tikzcd}$$

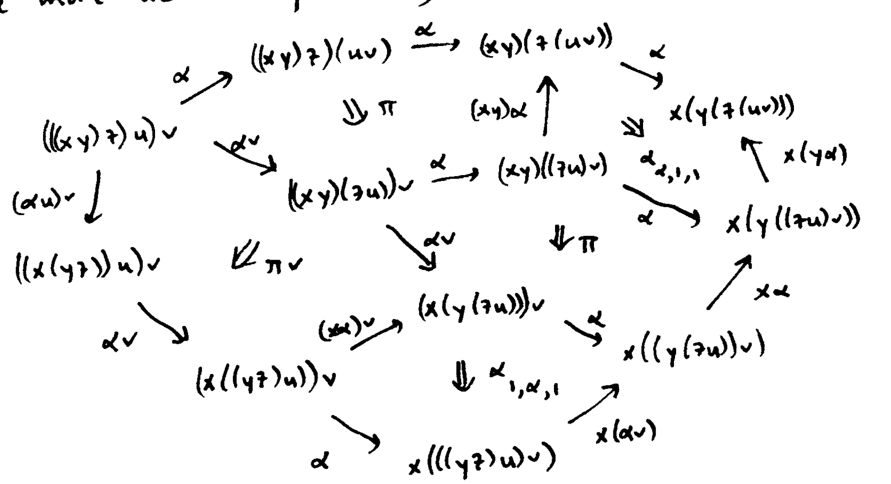
$$\begin{tikzcd}
& ((xy)z)I \arrow[rr, "\rho"] \arrow[dl, "\alpha I"'] \arrow[dr, "\alpha"] & & (xy)z \arrow[dr, "\alpha"] & \\
(x(yz))I \arrow[d, "\alpha"'] & \Leftarrow \pi & (xy)(zI) \arrow[ur, "(xy)\rho"'] \arrow[d, "\alpha"] & \swarrow \alpha_{1,1,\rho} & x(yz) \\
x((yz)I) \arrow[rr, "x\alpha"'] & & x(y(zI)) \arrow[urr, "x(y\rho)"'] & &
\end{tikzcd}
\quad \overset{U_{4,4}}{\Rrightarrow} \quad
\begin{tikzcd}
& ((xy)z)I \arrow[rr, "\rho"] \arrow[dl, "\alpha I"'] \arrow[d, phantom, "\swarrow \rho_\alpha^{-1}"] & & (xy)z \arrow[d, "\alpha"] \\
(x(yz))I \arrow[rrr, "\rho"] \arrow[dr, "\alpha"'] & \Downarrow r & & x(yz) \\
& x((yz)I) \arrow[urr, "x\rho"] \arrow[r, "x\alpha"'] & x(y(zI)) \arrow[ur, "x(y\rho)"'] \arrow[u, phantom, "\Downarrow xr"] &
\end{tikzcd}$$

These data are subject to four equations:

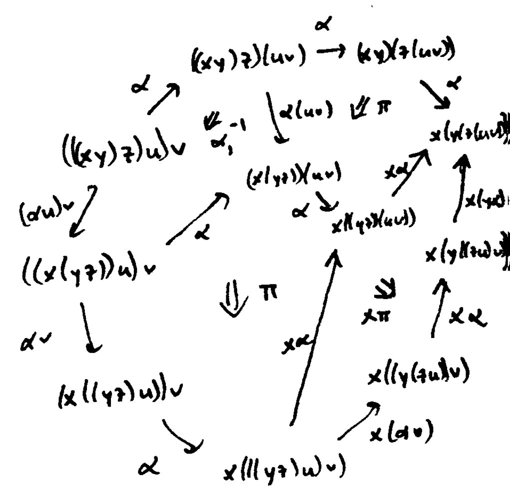
- K_6 Associativity Condition
- $U_{5,2}$ Unit Condition
- $U_{5,3}$ Unit Condition
- $U_{5,4}$ Unit Condition

which are described by the 4-cell pasting diagrams which follow. The interpretations of the pasting composites as 4-cells in local hom-tricategories is (essentially) unambiguous due to the coherence theorem for tricategories stated by Gordon-Power-Street.

<u>Note</u> The original diagrams, drawn up in 1995 at the request of Ross Street, involved the K_5 perturbation as presented on page 3. Only the K_6 associativity survived the span of years 1995 - 2006; the unit diagrams were redrawn in 2006, but using a slightly "twisted" version of K_5 — in the opinion of the author, this leads to unit diagrams which are more aesthetically pleasing.



Twisted K_5 $\Rrightarrow$



Various twists occur frequently throughout the pasting diagrams, and seem to be an unavoidable fact of life, due to an apparent inexistence of a lift of the Stasheff operad to the level of parity complexes. (Possibly related to similar difficulties in early attacks on the Deligne conjecture.)