The Spinor bundle on loop space

Stephan Stolz and Peter Teichner *

June 15, 2005

Abstract

We give a definition of 6-connected covering groups String(n) $\to$ Spin(n) in terms of "local fermions" on the circle. These are certain very explicit von Neumann algebras, the easiest examples of hyperfinite type III_1 factors. Given a Riemaniann *string* manifold M^n, i.e. an n-dimensional manifold with prescribed lifts of the (derivatives of the) coordinate changes to String(n), we define a classical 2-dimensional conformal field theory. The fields on space-time, a conformal surface, are maps to M. Another way to view the construction is to say that it gives the spinor bundle over the loop space LM, together with a *conformal connection*, see Theorem 4. In physics lingo we have constructed a 2-dimensional classical conformal field theory of central charge dim(M), with the extra structure of fusion and gluing along intervals (or open strings).

On the way, we give precise definitions and proofs for the fact that orientations of LM are in canonical 1-1 correspondence to spin structures on M (Theorem 9). Moreover, we show that the conformal anomaly in the spinor bundle over LM can be resolved in the presence of a string structure on M (Theorems 1 and 2). Our new idea is to make systematic use of the *fusion* operation on LM, which already has been successfully applied to define a level preserving product of finite energy representations of loop groups in [Wa].

*We thank the Max-Planck Institute for it's hospitality and support, and for providing an incredible research setting. Both of us are also supported by NSF grants.

1 Introduction

For large n, the first few homotopy groups of the orthogonal groups $O(n)$ are given in the following table:

k	0	1	2	3	4	5	6	7
$\pi_k O(n)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

It is well known that there are topological groups and homomorphisms

$$\mathrm{String}(n) \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to O(n)$$

which kill exactly the first few homotopy groups. More precisely, $\mathrm{SO}(n)$ is connected, $\mathrm{Spin}(n)$ is simply-connected and $\mathrm{String}(n)$ is 6-connected, and the above maps induce isomorphisms on all higher homotopy groups. This homotopy theoretical description of k-connected covers actually works for any topological group in place of $O(n)$ but it only determines the groups up to homotopy equivalence. For the 0-th and 1-st homotopy groups, it is also well known how to construct the groups explicitely, giving the smallest possible models. In our case, $\mathrm{SO}(n)$ is an (index 2) subgroup of $O(n)$, namely the identity component, and $\mathrm{Spin}(n)$ is the universal (double) covering of $\mathrm{SO}(n)$. In particular, both of these groups are Lie groups. However, a group $\mathrm{String}(n)$ cannot have the homotopy type of a Lie group since π_3 vanishes, and there has yet not been found a canonical construction which gives a smallest possible model for it.

In this note we construct such a concrete model for $\mathrm{String}(n)$ in terms of "local fermions" on the circle. These are certain very explicit von Neumann algebras, the easiest examples of hyperfinite type III_1 factors. We then use this model to give a geometric explanation of the meaning of lifting the (derivatives of the) coordinate changes of a manifold to $\mathrm{String}(n)$. Given a smooth manifold M of dimension n, recall that a lift to $O(n)$ is nothing but the choice of a Riemannian metric. Lifting further to $\mathrm{SO}(n)$ gives an orientation on M, and a lift to $\mathrm{Spin}(n)$ is a spin structure. Continuing in this spirit, we call a lift to $\mathrm{String}(n)$ a *string structure* on M. From the above homotopy theoretical description it follows that

1. M is orientable if and only if the Stiefel-Whitney class $w_1 M$ vanishes. Orientations of M are in 1-1 correspondence with $H^0(M;\mathbb{Z}/2)$.

2. M is spin if and only if the Stiefel-Whitney classes w_1M and w_2M vanish. Spin structures on M are in 1-1 correspondence with $H^1(M;\mathbb{Z}/2)$.

3. M is string if and only if the Stiefel-Whitney classes w_1M, w_2M and the characteristic class $p_1/2(M) \in H^4(M;\mathbb{Z})$ vanish. String structures on M are in 1-1 correspondence with $H^3(M;\mathbb{Z})$.

Recall that for any vector bundle E over M with spin structure, there is a *canonical* cohomology class in $H^4(M;\mathbb{Z})$, twice of which is the first Pontrjagin class $p_1(E)$. Lacking a better name, one simply denotes it by $p_1/2(E)$.

The geometric relevance of a spin structure is that it enables one to

- define the *spinor bundle* S_M on M,
- define the *Dirac operator*, acting on the sections of S_M,
- compute the *index* of the Dirac operator which is the $\widehat{A}$-*genus* of M,
- show that this index is an obstruction to the existence of a metric with positive scalar curvature on M.

The ultimate goal of our project is to generalize all of the above to loop spaces. More precisely, let LM be the space of all piecewise smooth loops in M. Then we would like to prove that for a string manifold M

- there is a *spinor bundle* S_{LM} on LM,
- there is a *Dirac operator*, acting on the sections of S_{LM},
- it has an *index* which is the *Witten genus* of M.
- show that this index is an obstruction to the existence of a metric with positive Ricci curvature on M.

In this note we shall only explain the first point in this list, so far the other points seem out of reach for mathematicians. To motivate the first point, we will show in Theorem 9 that orientations of LM are in canonical 1-1 correspondence to spin structures on M. This slogan is well known and was made precise in [McL] in the simply-connected case by a purely topological argument. Our definition and proof works for every manifold and we use the Riemannian metric, hence our argument is better suited for our ultimate purposes.

Using our new model for $\mathrm{String}(n)$, we then explain how a string structure on M gives a spin structure on LM and hence allows the definition of the spinor bundle S_{LM} on LM. To be more specific, we have to recall the most important properties of spinor bundles. Just like for M, the fibres of S_{LM} are irreducible modules over the Clifford algebra of the tangent spaces of LM. This implies that these fibres are infinite dimensional. However, there is a very important new structure showing up in this infinite dimensional case which is not usually discussed in the finite dimensional setting. It has to do with the fact that two loops which agree on a common interval can be *fused* to a third loop. Then one wants the relevant fibres of S_{LM} to glue up accordingly. More precisely, it is convenient to think of a piecewise smooth loop $\gamma\colon S^1 \to X$ as given by a pair of piecewise smooth paths $\gamma_1, \gamma_2\colon I \to X$ with common endpoints; we write $\gamma = \gamma_1 \cup \bar{\gamma}_2$. Now given two loops $\gamma_1 \cup \bar{\gamma}_2$ and $\gamma_2 \cup \bar{\gamma}_3$ with common segment γ_2, we can form a new loop $\gamma_1 \cup \bar{\gamma}_3$; we say that $\gamma_1 \cup \bar{\gamma}_3$ is obtained by *fusing* the loops $\gamma_1 \cup \bar{\gamma}_2$ and $\gamma_2 \cup \bar{\gamma}_3$, see figure .

Theorem 1. *For a string manifold M there is an (infinite dimensional) vector bundle $\mathcal{F} = S_{LM} \to LM$ with the following properties:*

1. **(bimodule)** *The fiber $\mathcal{F}(\gamma)$ over a loop $\gamma = \gamma_1 \cup \bar{\gamma}_2$ is an irreducible bimodule over $A(\gamma_1) - A(\gamma_2)$, where $A(\gamma_i)$ is the Clifford von Neumann algebra generated by the real Hilbert space $L^2(\gamma_i^* E)$ of L^2-sections of the pull-back bundle $\gamma_i^* E$.*

2. **(fusion)** *For three paths $\gamma_1, \gamma_2, \gamma_3$ with common endpoints, there is an isomorphism of $A(\gamma_1) - A(\gamma_3)$-bimodules*

$$G(\gamma_1, \gamma_2, \gamma_3) : \mathcal{F}(\gamma_1 \cup \bar{\gamma}_2) \boxtimes_{A(\gamma_2)} \mathcal{F}(\gamma_2 \cup \bar{\gamma}_3) \xrightarrow{\cong} \mathcal{F}(\gamma_1 \cup \bar{\gamma}_3),$$

   where the left hand side is the fusion product a la Connes [Co] of the bimodules $\mathcal{F}(\gamma_1 \cup \bar{\gamma}_2)$ and $\mathcal{F}(\gamma_2 \cup \bar{\gamma}_3)$. Moreover, these fusion isomorphisms satisfy associativity constraints (for four paths with common endpoints).

It turns out that the fusion condition makes this bundle over LM behave locally in M, and hence we call the structure constructed in Theorem 1 the *stringor bundle on M*. We can then summarize the analogies with the spinor bundle on M which is a $Cl(M) - C_n$ -bimodule bundle, where $Cl(M)$ is the Clifford bundle on M and C_n is the (constant) Clifford algebra. $Cl(M)$ exists canonically for every Riemannian manifold M. A spinor bundle on M is the choice of a graded, irreducible $Cl(M) - C_n$ -bimodule bundle on M. Note that over a point $m \in M$, there are exactly two *isomorphism classes* of such modules, and each of them corresponds to a local orientation at m. Given an orientation on M, there is still no reason, why these unique isomorphism classes should fit together to a bimodule bundle over M.

Theorem 2. *Let M be a Riemannian manifold.*

1. *If M is oriented then it is spin if and only if a spinor bundle on M exists. Moreover, isomorphism classes of spinor bundles are in 1-1 correspondence with spin structures on M, and hence with $H^1(M;\mathbb{Z}/2)$.*

2. *If M is spin then it is string if and only if a stringor bundle on M exists. Moreover, isomorphism classes of stringor bundles are in 1-1 correspondence with string structures on M, and hence with $H^3(M;\mathbb{Z})$.*

This theorem is really new in the sense that it shows in particular that the fusion property of S_{LM} implies the vanishing of $p_1/2(M)$, not just of its transgression in $H^3(LM)$. The latter class actually vanishes if one has a bundle $\mathcal{F}$ which only satisfies the first property in Theorem 1.

It turns out that we can make one tiny step towards the definition of the Dirac operator on LM. To explain this step, recall the Feynman-Kac formula for the Dirac operator D on a spin manifold M. It says that the operator e^{-tD^2} is an integral operator, i.e.,

$$(e^{-tD^2}\psi)(x) = \int_M K_t(x,y)\psi(y)dy,$$

whose operator kernel $K_t(x,y) \in \mathrm{Hom}(S_y, S_x)$ is given by an integral

$$K_t(x,y) = \int_\gamma ||^S(\gamma)\mathfrak{D}_t\gamma.$$

Here the integral is taken over the space of continuous paths $\gamma\colon [0,t] \to M$ with starting point y and end point x, and $||^S(\gamma)\colon S_y \to S_x$ is (stochastic) parallel translation in the spinor bundle along γ (this is induced by the Levi-Civita connection on TM). The measure $\mathfrak{D}_t$ is the *Wiener measure* on the space of continuous paths connecting y and x. From the McKean-Singer Formula

$$\mathrm{index}(D) = \mathrm{str}\, e^{-tD^2},$$

it follows that the above path integral is all one needs for index calculations.

Moving further to LM, one can formally write down quite analoguous expressions as above. The relevant operator kernel would then be a functional integral over the space of maps of a surface into M. Even though a reasonable measure on this space has not been found, it is still interesting to ask what the integrand in that functional integral could be. It should be the analogue of parallel translation in the spinor bundle of M. We shall explain in Section 2 why the following structure is indeed exactly such an analogue.

Definition 3. *Assume $\mathcal{F}$ is a stringor bundle as in Theorem 1. A* conformal connection *on $\mathcal{F}$ is given by a family of triangles, one for each Riemaniann spin manifold Σ with boundary S^1, as follows:*

$$\begin{tikzcd} & \mathcal{F} \arrow[d] \\ P_\Sigma M \arrow[ur, "\mathcal{V}"] \arrow[r, "\mathrm{res}"] & LM \end{tikzcd}$$

Here res *denotes the restriction to the boundary. There are two axioms for the section $\mathcal{V}$, namely*

- *$\mathcal{V}$ is conformal, i.e. it depends only on conformal structure on Σ, and an element ϵ in the Pfaffian line of D_Σ. This dependence satisfies for all $\lambda \in \mathbb{C}$:*
$$\mathcal{V}(\lambda\epsilon) = \lambda^{\dim(M)} \cdot \mathcal{V}(\epsilon).$$
- *If $\partial\Sigma_1 = \gamma_1 \cup \bar{\gamma}_2$ and $\partial\Sigma_2 = \gamma_2 \cup \bar{\gamma}_3$ there is the* gluing law
$$\mathcal{V}(\Sigma_1 \cup_{\gamma_2} \Sigma_2) = G(\gamma_1, \gamma_2, \gamma_3)(\mathcal{V}(\Sigma_1) \otimes \mathcal{V}(\Sigma_2)),$$

As we shall see, the isomorphisms $G(\gamma_1, \gamma_2, \gamma_3)$ are actually *determined* be the gluing laws above.

Theorem 4. *The stringor bundle S_{LM} of a string manifold M comes equipped with a conformal connection as defined above.*

It is worth pointing out that in physics lingo we have constructed a 2-dimensional classical conformal field theory of central charge $\dim(M)$, with the extra structure of fusion and gluing along intervals (or open strings).

2 Spinor bundle and heat kernel in finite dimensions

Let M^n be a closed Riemaniann spin manifold. The spin structure is a $\mathrm{Spin}(n)$-principal bundle P whose underlying $\mathrm{SO}(n)$-bundle is the oriented frame bundle of TM. then

$$S_M = P \times_{\mathrm{Spin}(n)} C_n$$

is the *spinor bundle* on M. It is $\mathbb{Z}/2$-graded and carries a right action of the Clifford algebra C_n. Moreover, the fibres S_x are irreducible $C(T_xM) - C_n$-bimodules.

We start with a description of the heat kernel $e^{-tD^2}(x, y)$, for $x, y \in M$ and $t > 0$. This is a section of the endomorphism bundle $\mathrm{End}_{C_n}(S)$ of the spinor bundle which is a bundle over over $M \times M$. It describes the distribution (as a function of $y \in M$) of "Dirac heat" at time t, assuming that at time 0 a unit of heat was concentrated at $x \in M$. It is an intuitive but difficult fact that this heat distribution can be obtained as a functional integral over the space of all path $\gamma \in P_t(x, y)$ (along which heat moves). Here the path space $P_t(x, y)$ consists of continuous maps

$$\gamma : [0, t] \longrightarrow M, \text{ with } \gamma(0) = x, \gamma(t) = y$$

and

$$e^{-tD^2}(x, y) = \int_\gamma v(\gamma)\mathfrak{D}_t\gamma.$$

Such a functional integral expression is the content of the *Feynman-Kac formula* for the heat kernel. Ignoring the details about the Wiener measure $\mathfrak{D}_t$ on $P_t(x, y)$ (which is defined using the exponential of the energy of γ and the scalar curvature of M), we just need to define the integrand $v(\gamma)$. It must take values in the fibre $\mathrm{Hom}_{C_n}(S_x, S_y)$ of the endomorphism bundle of the spinor bundle and it is the beast that's averaged over the paths γ connecting x and y to give $e^{-tD^2}(x, y)$. This integrand $v(\gamma)$ is usually described as the parallel translation in the spinor bundle. However, in the generalization to loop spaces we are trying to *define* this spinor bundle and hence we shall avoid using it in finite dimensions.

It is important to remark that parallel translation along γ does *not* depend on the parametrization or metric of the interval. One way to express this is to say that it is a *conformal* invariant, just like in the case of loop spaces. The serious dependence on the length t of the interval only enters through the Wiener measure. In the following, we carry the length along just to make the gluing of intervals into an associative operation.

In order to be able to extract the Dirac operator from the heat kernel $e^{-tD^2}(x,y)$, we need to make sure that it has the semi-group property. This means that the vectors $v(\gamma)$ have to be defined compatibly with the gluing of intervals. More precisely, there are obvious gluing maps

$$P_{t_1}(x,y) \times P_{t_2}(y,z) \longrightarrow P_{t_1+t_2}(x,z)$$

covered by compositions

$$\mathrm{Hom}(S_x, S_y) \times \mathrm{Hom}(S_y, S_z) \xrightarrow{\circ} \mathrm{Hom}(S_x, S_z)$$

and we ask that $v(\gamma_2\gamma_1) = v(\gamma_1) \circ v(\gamma_2)$. This condition is obvious in the case of parallel transport and it shows that the above definitions determine a 1-dimensional classical conformal field theory. The fields in the theory are maps of 0- and 1-manifolds into M, the action at a point $x \in M$ is just the spinor fibre S_x, and the action of a path γ is the parallel transport along γ.

We shall next explain the definition of the "vacuum" vectors $v(\gamma)$, up to sign, without even having to define the spinor bundle. Instead, we'll only define a *Fock bundle* W over $M \times M$ such that the fibre over (x,y) is a real irreducible graded representation $W(x,y)$ of the real graded Clifford algebra

$$C(x,y) := C(-T_xM \perp T_yM) = C(-T_xM) \otimes C(T_yM) = C^{\mathrm{op}}(x) \otimes C(y)$$

Using these *canonical* isomorphisms of graded algebras, $W(x,y)$ can be thought of as either a left $C(x,y)$-module or a $C(y)-C(x)$-bimodule. In the presence of a spinor bundle, these representations satisfy $W(x,y) \cong \mathrm{Hom}_{C_n}(S_x, S_y)$ as $C(x,y)$-modules.

Definition 5. *A* Fock bundle *is a vector bundle W over $M \times M$ whose fibre over (x,y) is a real irreducible graded representations of $C(x,y)$. Moreover, W comes equipped with* fusion isomorphisms *of $C(x,z)$-modules*

$$G(x,y,z) : W(x,y) \otimes_{C(y)} W(y,z) \xrightarrow{\cong} W(x,z)$$

which lie over the relevant maps of base spaces. A conformal connection *on a Fock bundle W is given by a triangle:*

$$\begin{tikzcd} & W \arrow[d] \\ \mathrm{Maps}(I,M) \arrow[ur, "v"] \arrow[r, "\mathrm{res}"] & M \times M \end{tikzcd}$$

Here res *denotes the restriction of a path to its boundary. There are two axioms for the section v, namely*

- *v is conformal, i.e. it does not depend on a Riemaniann metric on the interval I.*
- *If γ_1 goes from x to y and γ_2 goes from y to z then there is the* gluing law
$$v(\gamma_2 \cup_y \gamma_1) = G(x,y,z)(v(\gamma_1) \otimes v(\gamma_2))$$

We shall see that the maps $G(x,y,z)$ are actually *determined* be the gluing equation for the vacuum vectors. The following result motivates our Definition 3 of a Fock bundle with conformal connection.

Theorem 6. *For a spin manifold M, let $W(x,y) := \mathrm{Hom}_{C_n}(S_x, S_y)$. Then these fibres fit together to give a Fock bundle W as in Definition 5, with composition as gluing maps. Moreover, parallel transport in S_M defines a conformal connection on W.*

The next purpose is to define the Fock bundle W with a conformal connection, without using the spinor bundle S_M. This new construction then generalizes in a straight forward manner to loop spaces. First recall that there are graded algebra isomorphism

$$C(x,y) \cong C_{n,n} \cong \mathbb{R}(2^n)$$

and thus there is, up to isomorphism, a unique (ungraded) real irreducible representation Δ (of dimension 2^n) of $C(x,y)$. The grading is picked out by orientations of T_xM and T_yM which are given by assumption. Note that since our algebra is a real matrix ring, it follows that

$$\mathrm{End}_{C(x,y)}(\Delta) \cong \mathbb{R}$$

Moreover, there is an inner product on Δ such that Δ is a $*$-representation. Then the group of Clifford linear self isometries of Δ only consists of $\pm\,\mathrm{id}$. This is the undetermined sign in our discussion. We shall explain below why the sign indeterminancy can be resolved, consistently with gluing, if and only if M has a spin structure.

In our next linear algebra section we explain how to *canonically* define a graded irreducible $C(x,y)$-module $W(\phi)$, the *Fock space*, for any orientation preserving isometry $\phi : T_xM \to T_yM$. It will come equipped with an inner product as well as a canonical vacuum vector $v(\phi)$. It satisfies the gluing law in the sense that there are isomorphisms of $C(x,z)$-modules

$$G(x,y,z) : W(\phi_1) \otimes_{C(y)} W(\phi_2) \xrightarrow{\cong} W(\phi_2 \circ \phi_1)$$

such that the vacuum vectors glue by $v(\phi_2 \circ \phi_1) = G(v(\phi_1) \otimes v(\phi_2))$. These isomorphisms satisfy the obvious associativity condition. Assuming this linear algebra construction (and that M is connected), we *define*

$$W(x,y) \stackrel{\text{def}}{=} W(\gamma) \stackrel{\text{def}}{=} W(p(\gamma)),$$

where γ is a path from x to y and $p(\gamma)$ is the isometry between the tangent spaces given by parallel translation *in the tangent bundle*. Then the vacuum vector $v(\gamma) \in W(x,y)$ is just given by $v(p(\gamma))$ and the required gluing laws follow from the fact that parallel translation composes nicely.

Note that for a different choice of γ there is a unique isomorphism (up to sign) between the corresponding $C(x,y)$-modules, and hence we have defined a projective version of the bundle W over $M \times M$. Note the we have *not* used the spin structure on M, only the orientation (to get a grading on W). The spin structure will come in when we resolve the sign indeterminancy.

Theorem 7. *Let M be a closed Riemannian spin manifold, and let S be the graded spinor bundle on M. Up to sign, there are canonical isomorphism of $C(x,y)$-modules*

$$\Phi : W(x,y) \cong \mathrm{Hom}_{C_n}(S_x, S_y)$$

which carry the vacuum vectors $v(\gamma)$ to the parallel transport in the spinor bundle. Moreover, the sign indeterminancy can be resolved using the spin structure as follows:

There is a canonical 1-1 correspondence between spin structures on M and unprojective versions of the projective bimodule bundle W (with vacuum vectors) constructed above. Using this correspondence, the above isomorphism

Φ *is determined canonically and the vacuum vectors* $v(\gamma)$ *map to the parallel transport in the spinor bundle.*

The proof of this theorem will be given after explaining the precise role of the spin structure in resolving the sign indeterminancy.

Consider the following double covering $\hat{L}M$ of the loop space LM: Choose, once and for all, the two points $\pm i$ on S^1, subdividing the circle into a left and a right interval. Then $\hat{L}M$ consists of pairs (α, φ) where α is a loop in M and φ is an $C(x, y)$-module isometry between the representations $W(\gamma_1)$ and $W(\gamma_2)$. Here α decomposes into two paths $\alpha = \gamma_1 \cup r(\gamma_2)$ (right and left), where the paths γ_i are parametrized by the left semicircle, they both connect x and y and r reflects γ_2 so that it becomes a map on the right part of the circle (with orientation reversed) and hence $\gamma_1 \cup r(\gamma_2)$ is a loop in M.

This double covering has the following operation of *fusion*: Assume that γ_k, for $k = 1, 2, 3$, are three path connecting x and y, parametrized by the left semicircle. Then there are composition maps

$$\mathrm{Hom}(W(\gamma_1), W(\gamma_2)) \times \mathrm{Hom}(W(\gamma_2), W(\gamma_3)) \xrightarrow{\circ} \mathrm{Hom}(W(\gamma_1), W(\gamma_3))$$

which lie over the following composition of maps on LM coming from the theta graph (formed from the three path γ_k meeting at x and y):

$$\Theta(\gamma_1 \cup r(\gamma_2), \gamma_2 \cup r(\gamma_3)) \stackrel{\mathrm{def}}{=} \gamma_1 \cup r(\gamma_3).$$

We call a section s of $\hat{L}M$ *fusion preserving* if $s(\Theta(\alpha_1, \alpha_2)) = s(\alpha_1) \circ s(\alpha_2)$ for loops α_i which happen to share the correct segments.

Definition 8. *An* orientation *of LM is the choice of a fusion preserving section of $\hat{L}M$.*

Note the analogy of this definition to an orientation of M, as the choice of a section of the orientation double cover of M. One only needs to build in the additional structure of fusion on LM.

Theorem 9. *There is a canonical 1-1 correspondence between spin structures on M and orientations of LM.*

In particular, a spin structure on M exists if and only if an orientation on LM exists. The same results hold for any oriented vector bundle (with connection) over M in place of the tangent bundle.

Proof of Theorems 7 and 9. We shall describe the claimed correspondences as a circle in the following order:

- a spin structure on M induces
- an orientation of LM induces
- a bimodule bundle W on $M \times M$ induces ...

Start with a spin structure on M and recall that this is a bundle (with induced connection) over M of bimodules $S(x)$ over $C(x) - C_n$. For a path γ connecting points x and y, there are canonical isomorphisms of $C(x, y)$-modules

$$W(\gamma) = \mathrm{Hom}_{C_n}(S_x, S_y)$$

uniquely defined by sending the vacuum vector $v(\gamma)$ to the parallel translation $p_S(\gamma)$ from $S(x)$ to $S(y)$. The existence of such isomorphisms follows from the defining property of $W(\gamma)$, namely that $v(\gamma)$ is annihilated by the Lagrangian graph$(p(\gamma))$. So one just has to check that the $C(x, y)$-module on the right hand side has this property with respect to $p_S(\gamma)$.

It is then clear how to fit these isomorphisms together to give a fusion preserving section of $\hat{L}M$ as required for an orientation of LM. All one has to use is that parallel translation in the bundle V composes under gluing of path.

Now assume given a fusion preserving section s of $\hat{L}M$. Consider the space of pairs (γ, w) where $w \in W(\gamma)$ and γ is a path from the left semicircle to M. Then the required bundle W_s over $M \times M$ is obtained by putting the following relation on these pairs:

$$(\gamma_1, w_1) \sim (\gamma_2, w_2) :\Longleftrightarrow \gamma_1(\pm i) = \gamma_2(\pm i), w_2 = s(\gamma_1 \cup r(\gamma_2)) \cdot w_1$$

This relation is an equivalence relation if and only if the section s is fusion preserving: Transitivity follows directly from the fact that s preserves fusion (and this actually motivates the definition of fusion!). Fusion also implies reflexivity, i.e. that $s(\gamma \cup r(\gamma)) = \mathrm{id}$ because one may cancel a factor in the fusion equation

$$s(\gamma \cup r(\gamma)) = s(\gamma \cup r(\gamma)) \circ s(\gamma \cup r(\gamma)).$$

Finally, the relation is symmetric by another application of the fusion rule:

$$\mathrm{id} = s(\gamma_1 \cup r(\gamma_1)) = s(\gamma_1 \cup r(\gamma_2)) \circ s(\gamma_2 \cup r(\gamma_1)).$$

Finally, assume that we have a module bundle $W \to M \times M$ over the Clifford algebras $C(x, y)$. Choose a base point $x_0 \in M$ and identify, once and for all, $T_{x_0}M = \mathbb{R}^n$. Then the restriction map

$$M \longrightarrow M \times M, \quad x \mapsto (x_0, x)$$

can be used to pull back W to a bimodule bundle on M over the algbras $C(x) - C_n$. But this is exactly a spin structure on M.

The last thing to check is that these 3 constructions, when run in a circle starting from a spin structure on M, produce an equivalent spin structure. This is left to the reader. □

Remark 10. It is easy to check that the characteristic class of the double covering $\hat{L}M$ comes from $w_2(M)$ under the transgression

$$H^2(M; \mathbb{Z}/2) \longrightarrow H^1(LM; \mathbb{Z}/2).$$

If this transgression is zero, then a section of $\hat{L}M$ exists but by the above theorem it can be chosen to be fusion preserving if and only if $w_2(M) = 0$.

It is a fun exercise for the reader to compute the affine space of fusion preserving sections. This is the best place to see what can go wrong if M is not simply-connected, and why the fusion property resolves the problems. Assume that s is a fusion preserving section. To get a second section, we can add any map to $\mathbb{Z}/2$ from the set $\pi_0(LM)$, which is isomorphic to $\pi_1(M)$ modulo conjugation. However, we only have an inclusion

$$H^1(M; \mathbb{Z}/2) \cong \mathrm{Hom}(\pi_1(M), \mathbb{Z}/2) \hookrightarrow \mathrm{Maps}(\pi_1(M)/\mathrm{conjugation}, \mathbb{Z}/2)$$

and the exercise is to check that the resulting section is again fusion preserving if and only if the map we add comes from $H^1(M; \mathbb{Z}/2)$, i.e. is a homomorphism. Moreover, under our isomorphism from the above theorem, this operation behaves well with changing a spin structure by an element of $H^1(M; \mathbb{Z}/2)$ in the usual way.

Remark 11. It is possible to formulate (and prove) Theorem 9 in the setting where we, instead of an oriented vector bundle, start with a gerbe with band $\mathbb{Z}/2$ over M. Isomorphism classes of such gerbes are given by $H^2(M; \mathbb{Z}/2)$ and the theorem (and proof) generalizes to give an equivalence of categories between

- gerbes with band $\mathbb{Z}/2$ on M.

- double coverings $\hat{L}M$ of LM with fusion.

The theorem above treats in this language "trivializations" of objects in the above categories. So by definition, a trivialization of a gerbe is the analogue of a spin structure, and a trivialization of $\hat{L}M$ is a fusion preserving section. If the gerbe comes from a real vector bundle E then one can construct the projective bimodule bundle $P(W_E)$ as above and a trivialization of $P(W_E)$ is an unprojective version W_E of this bimodule bundle.

In [Bry] it is explained how to construct $\hat{L}M$ out of a gerbe. To go backwards, consider the following bundle gerbe over M: The total space are path starting at a fixed base point x_0, with the end-point projection to M. The necessary $\mathbb{Z}/2$-torsor associated to a pair of path with the same end-point is then given by the fibre of $\hat{L}M$ over the loop constructed by gluing together the two path. Note that the fusion product on $\hat{L}M$ is the same information as the composition of morphisms in this bundle gerbe.

The same exact results is true if one replaces the band $\mathbb{Z}/2$ by S^1. This then gives an interpretation of the sheaf cohomology group $H^2(M;\mathbb{T}) \cong H^3(M;\mathbb{Z})$.

Remark 12. After talking to Dan Freed, we realized that the above discussion fits beautifully with the supersymmetric version of the path integral expression for the heat kernel of the Dirac operator. Namely, supersymmetry predicts that for a given path γ from x to y, the integrand (or vacuum vector) $v(\gamma)$ in the path integral should really be a fermionic integral over the odd fields lying over γ. In the usual model, these odd fields ψ are just vector fields along γ and the odd part of the action functional is given by

$$e^{\int_I \langle \psi, \nabla_{\dot{\gamma}}(\psi)\rangle dt}$$

with covariant differentiation $\nabla_{\dot{\gamma}}$ acting on the sections of γ^*TM. This is a real skew adjoint operator and it has a real Pfaffian line after fixing the boundary conditions. The Pfaffian $\mathrm{pf}(\nabla_{\dot{\gamma}})$ of the operator is then an element in this line, or better, it is a section of the Pfaffian line bundle $\mathrm{Pf}(\nabla_{\dot{\gamma}})$ over the space of boundary conditions. Recall that an elliptic boundary condition for $\nabla_{\dot{\gamma}}$ is given by a Lagrangian subspace of $-T_xM \perp T_yM$. We denote the manifold of all Lagrangian subspaces by $Gr(x,y)$.

Hence one expects the result of the fermionic path integral (over all odd ψ but with fixed γ) to be the section $\mathrm{pf}(\nabla_{\dot{\gamma}})$ of $\mathrm{Pf}(\nabla_{\dot{\gamma}})$.

We shall next discuss why this is explained by our vacuum vector in the Fock bundle W. For this purpose, it is useful to study the reason why the fermionic path integral should be a section of some line bundle. This involves the Hamiltonian formalism: The classical solutions (γ, ψ) of the supersymmetric Lagrangian

$$\langle \dot\gamma, \dot\gamma \rangle + \langle \psi, \nabla_{\dot\gamma}(\psi) \rangle$$

are solutions of the ODE's

$$\nabla_{\dot\gamma}\dot\gamma = \frac{1}{2} R(\psi, \psi)\dot\gamma \text{ and } \nabla_{\dot\gamma}\psi = 0.$$

Note that this implies that spinning particles ($\psi \neq 0$) do not move along geodesics. It is still true, however, that one may parametrize the solutions for $\gamma : \mathbb{R} \to M$ by TM and that for such a fixed even part of a classical solution the odd parts ψ form the space of vectorfields parallel along γ and hence are determined by the initial value. This is a discussion where space is a point and time is the real line.

Now fix two points x and y in M of opposite orientation. Then the odd parts of the solution set form the odd symplectic manifold $-T_xM \perp T_yM$, which happens to be a vector space (with the symmetric inner product given by the metric on M). In the Hamiltonian formalism, we need to quantize this odd symplectic manifold to figure out the possible values of the path integral. In the linear case, this quantization is determined by a Lagrangian subspace of $-T_xM \perp T_yM$, and these subspaces form the Grassmannian $Gr(x, y)$ of boundary conditions. As explained in Section 3, a base point $L \in Gr(x, y)$ determines a real line bundle, the Pfaffian bundle $\mathrm{Pf}(L)$ over $Gr(x, y)$, and a "holomorphic" section $\mathrm{pf}(L)$ of this Pfaffian. More precisely, L determines the Fock space $W(L)$ together with a vacuum vector $v(L) \in W(L)$. Given any other $L' \in Gr(x, y)$ we can then consider the corresponding annihilator line $\mathrm{Pf}(L : L') \subset W(L)$ and we may project $v(L)$ onto $\mathrm{Pf}(L : L')$ (using the inner product on $W(L)$) to obtain a section $\mathrm{pf}(L)$. By definition, the *holomorphic* sections of $\mathrm{Pf}(L)$ are comprised by this construction from vectors in $W(L)$ hence a quantization of $-T_xM \perp T_yM$ is given by $W(L) = \Gamma^{hol}(\mathrm{Pf}(L))$ itself.

Given a path γ from x to y, the graph of parallel translation along γ *in the tangent bundle* provides a base point in $Gr(x, y)$, and then the quantization is just the Fock space $W(\gamma)$ used in Definition 5. Moreover, the vacuum vector $v(\gamma) \in W(\gamma)$ is a "holomorphic" section of the Pfaffian line bundle as explained above. This motivates the name "Pfaffian" because of the identifications

$$v(\gamma) = \mathrm{pf}(\nabla_{\dot\gamma}) \in W(\gamma) = \Gamma^{hol}(\mathrm{Pf}(\nabla_{\dot\gamma})).$$

They also explain why the fermionic integration (for fixed γ) is expected to lead to the vacuum vector $v(\gamma)$.

For the path integral for the heat kernel to make sense, we need to be able to add the vectors $v(\gamma)$ for all paths γ from x to y. This is in general impossible since $v(\gamma) \in W(\gamma)$ are varying vector spaces. But as explained in Theorem 9 that's exactly where the spin structure on M is used: In the guise of a fusion preserving section of the Pfaffian line bundle over LM, we showed that it enables one to identify any two vector spaces $W(\gamma)$ and $W(\gamma')$ in a consistent way. Thus there really is a well defined quantization $W(x, y)$ of $-T_xM \perp T_yM$, and these fit together to give the Fock bundle on $M \times M$.

3 Linear algebra of Fock spaces

Let V be a real or complex vector space equipped with a bilinear form b. Assume that there is an isometric involution $v \mapsto \bar{v}$ on V ($\mathbb{C}$-*anti*-linear in the complex case) such that

$$\langle v, w\rangle \stackrel{\text{def}}{=} b(\bar{v}, w)$$

is an inner product (positive definite, and hermitian in the complex case). We will then construct canonical $*$-representations (real respectively complex) of the Clifford algebra $C(V, b)$. They will be graded irreducible and come equipped with an inner product and a vacuum vector. The input datum is a Lagrangian L of (V, b), which means that b vanishes identically on L and that $V = L \oplus \bar{L}$. Then define

$$W(L) \stackrel{\text{def}}{=} \Lambda^*(L),$$

equipped with the usual inner product induced from $\langle\ \rangle$. If V is infinite dimensional, we actually complete $\Lambda^*(L)$ with respect to this inner product to get $W(L)$. The vacuum vector $v(L)$ is given by the zero-form 1. Finally, $C(V, b)$ acts as creation operator for L and annihilation operators for $\bar{L}$. More precisely, it is clear that $W(L)$ is an irreducible module over the Clifford algebra $C(L \oplus L^*)$ (formed with respect to the hyperbolic form). But our assumptions give a canonical isomorphism $L^* \cong \bar{L}$ which lead to an isometry of the hyperbolice form on $L \oplus L^*$ with (V, b). Hence $C(V, b)$ acts irreducibly on $W(L)$. The grading is also obvious.

For $V = \mathbb{C}^{2n}$, $W(L)$ is *the* irreducible $C(V)$-module, and similarly for $(V, b) = \mathbb{R}^{n|n}$. This is the case needed in the application to heat kernels. We can actually start more generally with two real inner product spaces V_1 and V_2 of the same dimension and then define

$$(V, b) \stackrel{\text{def}}{=} (-V_1 \perp V_2)$$

with the involution $\overline{(v_1, v_2)} := (-v_1, v_2)$. It is then clear that our induced form $\langle\ \rangle$ is just $V_1 \perp V_2$ and hence an inner product. Finally, one easily checks that Lagrangians for b are exactly graphs of isometries $\phi : V_1 \to V_2$.

If V is complex and infinite dimensional, we'll need the following classification theorem.

Theorem 13 (Segal's equivalence criterion)**.** *Two complex representation $W(L)$ and $W(L')$ of $C(V, b)$ are isomorphic if and only if the composition of inclusion and projection maps, orthogonal with respect to $\langle\ \rangle$,*

$$L' \hookrightarrow V \twoheadrightarrow \bar{L}$$

is a Hilbert-Schmidt operator. Moreover, this isomorphism preserves the grading if and only if $\dim(\bar{L} \cap L')$ is even.

Note that the above relative dimension makes sense because the composition $L' \hookrightarrow V \twoheadrightarrow L$ is a Fredholm operator if $L' \hookrightarrow V \twoheadrightarrow \bar{L}$ is Hilbert-Schmidt.

Definition 14. *Let (V, b) be infinite dimensional. A* sub-Lagrangian *is an isotropic subspace L of (V, b) such that $L \oplus \bar{L}$ has finite codimension. Furthermore, a* polarization *of (V, b) is an equivalence class of sub-Lagrangians, where L and L' are identified if they satisfy the above Segal criterion that $L' \hookrightarrow V \twoheadrightarrow \bar{L}$ is a Hilbert-Schmidt operator.*
If the grading is relevant, we also ask that $\dim(\bar{L} \cap L')$ is even.

Example 15. If $S^1 \xrightarrow{\gamma} M$ is a loop in a Riemannian manifold M, then we form the Hilbert space

$$V(\gamma) \stackrel{\text{def}}{=} L^2(S_{S^1} \otimes \gamma^* TM)$$

Here S_{S^1} is the spinor bundle on the circle with respect to some chosen spin structure. Since $C_1 \cong \mathbb{C}$ we have a complex structure on $V(\gamma)$ and the grading involution is $\mathbb{C}$-antilinear. Thus we have all the structures needed for our linear algebra above, in particular we have a $\mathbb{C}$-biliear form b and we can try to construct representations of the Clifford algebra $C(\gamma) := C(V(\gamma), b)$.

For this we need a sub-Lagrangian in $V(\gamma)$ constructed as follows: The Levi-Civita connection on TM induces a connection on f^*TM and hence we can form the twisted Dirac operator D_γ acting on $V(\gamma)$. This is a self-adjoint elliptic operator of order one, and we can consider the Hilbert sum of Eigenspaces corresponding to Eigenvalues $\lambda > 0$. This is a sub-Lagrangian $L_>$ for the form b and hence defines the isomorphism class of the representation $W(L_>)$. There is a missing detail here since one cannot always enlarge $L_>$ to a Lagrangian. However, one can instead pick a representation of the Clifford algebra on the Kernel of D_γ.

Another way to obtain a Lagrangian in $V(\gamma)$ is as follows. Let Σ^2 be a Riemannian spin manifold with boundary S^1, and let $\Gamma\colon \Sigma \to M$ be a smooth map extending γ. Let S_Σ be the Clifford linear spinor bundle on Σ, a $\mathbb{Z}/2$-graded vector bundle over Σ whose fibers are graded rank 1 modules over $C_2 \cong \mathbb{H}$. We note that the restriction of S_Σ^+ to the boundary can be identified with with the full spinor bundle bundle S_{S^1}. Let

$$D_\Gamma \colon C^\infty(S_\Sigma^+ \otimes \Gamma^*TM) \longrightarrow C^\infty(S_\Sigma^+ \otimes \Gamma^*TM)$$

be the Dirac operator on S_Σ^+ twisted by Γ^*TM (equipped with the pull-back of the Levi-Civita connection on TM) and composed with right multiplication $e_2 : S_\Sigma^- \longrightarrow S_\Sigma^+$. The elements in the kernel of D_Γ are referred to as (twisted) *harmonic spinors*. The boundary values of harmonic spinors form a Lagrangian subspace $L_\Gamma \subset V(\gamma)$ by the usual arguments for elliptic operators.

Theorem 16. *The Lagrangian $L_\Gamma \subset V(\gamma)$ represents the same polarisation as the sub-Lagrangian $L_>$. Moreover,* $\dim(\bar{L}_\Gamma \cap L_>) = \mathrm{index}(D_\Gamma) \in KO_2 = \mathbb{Z}/2$*. Here we use the elliptic operator D_Γ obtained by requiring the skew-adjoint boundary conditions given by $L_>$.*

Corollary 17. *Let $\Sigma_1 \xrightarrow{\Gamma_1} M$ and $\Sigma_2 \xrightarrow{\Gamma_2} M$ be smooth maps with $\partial\Gamma_1 = \partial\Gamma_2 = \gamma$. Then the associated Fock type modules $W(L_{\Gamma_1})$ and $W(L_{\Gamma_2})$ are isomorphic as modules over the Clifford algebra $C(\gamma)$.*

4 Extensions of compact Lie groups

In this section we construct extensions of topological groups

$$PU(A_\rho) \longrightarrow G_\rho \longrightarrow G, \tag{18}$$

one for each projective unitary representation ρ of the loop group LG of a Lie group G. Here A_ρ is a certain von Neumann algebra, the "local loop algebra", and the projective unitary group $PU(A_\rho) \stackrel{\text{def}}{=} U(A_\rho)/\mathbb{T}$ has the homotopy type of a $K(\mathbb{Z}, 2)$. Thus our extension has an obstruction class in $H^3(G)$, which we call the *level* of ρ.

In the special case where $G = \mathrm{Spin}(n)$ and ρ is the level 1 positive energy representation which is the trivial 1-dimensional representation of G at energy 0, this is the extension $\mathrm{String}(n) \to \mathrm{Spin}(n)$ announced in the introduction. Then A_ρ is a hyperfinite type III_1-factor, the "local fermions" on the circle.

Remark 19. The loop group considered in the literature consists of all *smooth* loops $\gamma\colon S^1 \to G$. For technical reasons that will become clear below, we prefer to work with the larger loop group consisting of all piecewise smooth (and continuous) loops. This will be what we mean by the 'free loop group' LG. The important fact is that the theory of positive energy representations of loop groups still works for these larger groups (cf. [PS]).

4.1 Construction of the local loop algebra.

Let ρ be a projective unitary representation of LG, i.e., a homomorphism $\rho\colon LG \to PU(H)$ from LG to the projective unitary group $PU(H) \stackrel{\text{def}}{=} U(H)/\mathbb{T}$ of some complex Hilbert space H. Note that by definition, we are assuming that ρ is defined for all piecewise smooth loops in G. Pulling back the canonical circle group extension

$$\mathbb{T} \longrightarrow U(H) \longrightarrow PU(H)$$

via ρ, we obtain an extension

$$\mathbb{T} \longrightarrow \tilde{L}G \longrightarrow LG,$$

and a unitary representation $\tilde{\rho}\colon \tilde{L}G \to U(H)$.

Let $I \subset S^1$ be the upper semi-circle consisting of all $z \in S^1$ with non-negative imaginary part. Let $L^I G \subset LG$ be the subgroup consisting of those loops $\gamma\colon S^1 \to G$ with support in I (i.e., $\gamma(z)$ is the identity element of G for $z \notin I$). Let $\tilde{L}^I G < \tilde{L}G$ be the preimage of $L^I G$. We define

$$A_\rho \stackrel{\text{def}}{=} \tilde{\rho}(\tilde{L}^I G)'' \subset B(H).$$

to be the von Neumann algebra generated by the operators $\tilde{\rho}(\gamma)$ with $\gamma \in \tilde{L}^I G$. Recall that von Neumann's double commutant theorem implies that this is precisely the weak (and strong) closure of $\tilde{L}^I G$ in the algebra $B(H)$ of all bounded operators on H.

To construct the group extension (18) we start with the group extension

$$L^I G \longrightarrow P^I_{1\!\!1} G \longrightarrow G,$$

where $P^I_{1\!\!1} G = \{\gamma\colon I \to G \mid \gamma(1) = 1\!\!1\}$, the left map is given by restriction to $I \subset S^1$ (alternatively we can think of $L^I G$ as maps $\gamma\colon I \to G$ with $\gamma(1) = \gamma(-1) = 1\!\!1$), and the right map is given by evaluation at $z = -1$. The idea is to modify this extension by replacing the normal subgroup $L^I G$ by the projective unitary group $PU(A_\rho)$ of the von Neumann algebra A_ρ (the unitary group $U(A_\rho) \subset A_\rho$ consists of all $a \in A_\rho$ with $aa^* = a^*a = 1$), using the homomorphism

$$\rho\colon L^I G \longrightarrow PU(A_\rho), \tag{20}$$

given by restricting the representation ρ to $L^I G \subset LG$. We note that by definition of $A_\rho \subset B(H)$, we have $\rho(L^I G) \subset PU(A_\rho) \subset PU(H)$.

4.2 The group extension of G

In order to construct the desired extension (18), we observe that $P^I_{1\!\!1} G$ acts on $L^I G$ by conjugation and that this action extends to a left action on $PU(A_\rho)$. In fact, this action exists for the group $P^I G$ of all piecewise smooth path $I \to G$ (of which $P^I_{1\!\!1} G$ is a subgroup): To describe how $\delta \in P^I G$ acts on $PU(A_\rho)$, extend $\delta\colon I \to G$ to a piecewise smooth loop $\gamma\colon S^1 \to G$ and pick a lift $\tilde{\gamma} \in \tilde{L}G$ of $\gamma \in LG$. We decree that $\delta \in P^I G$ acts on $PU(A_\rho)$ via

$$[a] \mapsto [\tilde{\rho}(\tilde{\gamma}) a \tilde{\rho}(\tilde{\gamma}^{-1})].$$

Here $a \in U(A_\rho) \subset B(H)$ is a representative for $[a] \in PU(A_\rho)$. It is clear that $\tilde{\rho}(\tilde{\gamma}) a \tilde{\rho}(\tilde{\gamma}^{-1})$ is a unitary element in $B(H)$; to see that it is in fact in A_ρ, we may assume that a is of the form $a = \tilde{\rho}(\tilde{\gamma}')$ for some $\tilde{\gamma}_0 \in \tilde{L}^I G$ (these elements generate A_ρ as von Neumann algebra). Then $\tilde{\rho}(\tilde{\gamma}) a \tilde{\rho}(\tilde{\gamma}^{-1}) = \tilde{\rho}(\tilde{\gamma}\tilde{\gamma}_0\tilde{\gamma}^{-1})$, which shows that this element is in fact in A_ρ and that it is independent of how we extend the path $\delta\colon I \to G$ to a loop $\gamma\colon S^1 \to G$, since $\gamma_0(z) = 1$ for $z \notin I$.

Lemma 21. *With the above left action of P^IG on $PU(A_\rho)$, the representation $\rho : L^IG \to PU(A_\rho)$ is P^IG-equivariant. Therefore, there is a well defined monomorphism*

$$r : L^IG \longrightarrow PU(A_\rho) \rtimes P^IG, \quad r(\gamma) \stackrel{def}{=} (\rho(\gamma^{-1}), \gamma)$$

into the semidirect product, whose image is a normal subgroup.

Before giving the proof of this Lemma, we note that writing the semidirect product in the order given, one indeed needs a *left* action of the right hand group on the left hand group. This follows from the equality

$$(u_1g_1)(u_2g_2) = u_1(g_1u_2g_1^{-1})g_1g_2$$

because $u \mapsto gug^{-1} \stackrel{\mathrm{def}}{=} u^g$ is a left action on $u \in U$. The fact that we put the g on the upper right is an annoying TeX problem.

Proof. The first statement is obvious from our definition of the action on $PU(A_\rho)$. To check that r is a homomorphism, we compute

$$\begin{aligned} r(\gamma_1)r(\gamma_2) &= (\rho(\gamma_1^{-1}), \gamma_1)(\rho(\gamma_2^{-1}), \gamma_2) \\ &= (\rho(\gamma_1^{-1})[\rho(\gamma_2^{-1})^{\gamma_1}], \gamma_1\gamma_2) \\ &= (\rho(\gamma_1^{-1})[\rho(\gamma_1)\rho(\gamma_2^{-1})\rho(\gamma_1^{-1})], \gamma_1\gamma_2) \\ &= (\rho(\gamma_2^{-1})\rho(\gamma_1^{-1}), \gamma_1\gamma_2) = (\rho(\gamma_1\gamma_2)^{-1}, \gamma_1\gamma_2) \\ &= r(\gamma_1\gamma_2) \end{aligned}$$

To check that the image of r is normal, it suffices to check invariance under the two subgroups $PU(A_\rho)$ and P^IG. For the latter, invariance follows directly from the P^IG-equivariance of ρ. For the former, we check

$$\begin{aligned} (u^{-1}, 1)(\rho(\gamma^{-1}), \gamma)(u, 1) &= (u^{-1}\rho(\gamma^{-1}), \gamma)(u, 1) \\ &= (u^{-1}\rho(\gamma^{-1})u^{\gamma}, \gamma) \\ &= (u^{-1}\rho(\gamma^{-1})\rho(\gamma)u\rho(\gamma)^{-1}, \gamma) \\ &= (r(\gamma^{-1}), \gamma) \end{aligned}$$

This actually shows that the two subgroups $r(L^IG)$ and $PU(A_\rho)$ commute in the semidirect product group. Finally, projecting to the second factor P^IG one sees that r is injective. □

Definition 22. We define the group G_ρ to be the quotient of $PU(A_\rho) \rtimes P^I_{\mathbb{1}} G$ by the normal subgroup $r(L^I G)$, in short

$$G_\rho \stackrel{\text{def}}{=} PU(A_\rho) \rtimes_{L^I G} P^I_{\mathbb{1}} G$$

Then there is a projection onto G by sending $[u, \gamma]$ to $\gamma(-1)$ which has kernel $PU(A_\rho)$.

We observe that there is a group extension

$$G_\rho \longrightarrow PU(A_\rho) \rtimes_{L^I G} P^I G \longrightarrow G$$

where the right hand map sends $[u, \gamma]$ to $\gamma(1)$. This extension splits because we can map g to $[\mathbb{1}, \gamma(g)]$, where $\gamma(g)$ is the constant path with value g. This implies the isomorphism

$$G_\rho \rtimes G \cong PU(A_\rho) \rtimes_{L^I G} P^I G \tag{23}$$

with the action of G on G_ρ defined by the previous split extension. Note that after projecting G_ρ to G this action becomes the conjugation action of G on G because the splitting used constant paths.

Lemma 24. *There is a homomorphism*

$$\Phi : PU(A_\rho) \rtimes_{L^I G} P^I G \longrightarrow \operatorname{Aut}(A_\rho) \quad \Phi([u], \gamma) \stackrel{def}{=} c_u \circ \phi(\gamma)$$

where c_u is conjugation by $u \in U(A_\rho)$ and $\phi(\gamma)$ is the previously defined action of $P^I G$ on A_ρ (which was so far only used for its induced action on $PU(A_\rho)$).

Proof. The statement follows (by calculations very similar to the ones given above) from the fact that

$$\phi(\gamma) \circ c_u = c_{u^\gamma} \circ \phi(\gamma)$$

□

We summarize the above results as follows.

Proposition 25. *There is a homomorphism $G_\rho \rtimes G \longrightarrow \operatorname{Aut}(A_\rho)$ which reduces to the conjugation action $PU(A_\rho) \twoheadrightarrow \operatorname{Inn}(A_\rho) \subset \operatorname{Aut}(A_\rho)$ on*

$$PU(A_\rho) = \ker(G_\rho \longrightarrow G) = \ker(G_\rho \rtimes G \longrightarrow G \rtimes G)$$

The action of G on G in the right hand semidirect product is given by conjugation which implies the isomorphism $G \rtimes G \cong G \times G$ sending $(g, 1)$ to $(g, 1)$ and $(1, g)$ to (g, g).

5 Bimodule bundles associated to principal G_ρ-bundles

5.1 Positive energy representations as bimodules.

Let $\rho : LG \to PU(H)$ be a positive energy representation of the loop group LG on a complex Hilbert space H and assume that G is semisimple and simply connected. Let $\tilde{L}G$ be the extension of LG obtained by pull-back via the extension $U(H) \to PU(H)$, and recall that the subgroup $G < LG$ of constant loops admits a canonical splitting $G \to U(H)$. Moreover, we shall need that this splitting is equivariant with respect to the action of G on A_ρ constructed in Proposition 25. Here $A_\rho \subset B(H)$ acts canonically on H.

Recall also that the Möbius group acts on H intertwining with ρ. Here the orientation preserving (resp. reversing) Möbius transformations act by complex linear (resp. anti-linear) operators on H. In particular the 'flip' $S^1 \to S^1$ given by $z \mapsto \bar{z}$ is implemented by a complex anti-linear operator $J\colon H \to H$ with $J^2 = 1$.

As in the previous section, we may define the *local loop algebra* A_ρ. This algebra can be defined for an arbitrary interval $I \subset S^1$ mand we want to ephasize this dependence by the notation $A^I = A_\rho$ (since ρ is now fixed).

An important fact is that if $I^c \subset S^1$ is the complementry segment $S^1 \setminus \bar{I}$, then the elements of $A^I \subset B(H)$ commute with the elements of A^{I^c}. This follows from the fact that $L^I G$ and $L^{I^c} G$ are commuting subgroups of LG and hence the centrally extended groups must commute up to elements in the center $\mathbb{T}$. One gets homomorphisms (for fixed $\gamma^c \in \tilde{L}^{I^c} G$)

$$L^I G \longrightarrow \mathbb{T}, \quad \gamma \mapsto \tilde{\gamma}\gamma^c(\tilde{\gamma})^{-1}(\gamma^c)^{-1}$$

which by the semisimplicity of G must factor through the connected component group $\pi_1 G$. By assumption, this group also vanishes and hence the two groups $\tilde{L}^I G$ and $\tilde{L}^{I^c} G$ commute (and so do the von Neumann algebras generated by them). This important *locality* property allows us to regard our Hilbert space H as a module over $A^I \otimes A^{I^c}$.

If I is upper semi-circle, then the flip interchanges I and I^c. It follows the anti-unitary $J\colon H \to H$ which implements the flip on H gives a isomorphism of von Neumann algebras:

$$(A^I)^{op} \xrightarrow{\cong} A^{I^c} \qquad a \mapsto Ja^*J,$$

where $(A^I)^{op}$ is the opposite von Neumann algebra. Note that the map $a \mapsto JaJ$ is *not* a $*$-homomorphism of von Neumann algebras since it is not complex linear due to the fact that F is anti-linear. Moreover, the element $a^* \in B(H)$ is $\mathbb{C} - linear$ with the property $(\lambda a)^* = \bar{\lambda} a^*$ for all $\lambda \in \mathbb{C}$.

This allows us to interpret H as an $(A^I - A^I)$-bimodule. It is an important fact that no information about the representation ρ is lost when passing to the corresponding bimodule in the sense that the map from isomorphism classes of positive energy representations to isomorphism classes of bimodules injective (this is the irreducibility statement in [Wa, p. 472]).

5.2 The bimodule bundle over based loop space.

Let G be a semisimple and simply connected Lie group, and let $\rho\colon LG \to PU(H)$ be the vacuum representation of the loop group LG (at some level $k \in H^3(G)$ defined by our previously constructed extension $\widehat{G} \stackrel{\text{def}}{=} G_\rho$ of G. Then the $A - A$-bimodule H constructed in the previous section is actually isomorphic to the canonical bimodule $L^2 A$, a fact which we shall use instantly.

Given a principal G-bundle $E \to X$, there is an obstruction $k(E) \in H^4(X)$ which measures whether the structure group of E lifts to $\widehat{G}$. This obstruction is just the pull back of the level $k \in H^3(G) \cong H^4(BG)$ under a classifying map $X \to BG$ of the bundle E. We would like to define the following structure on X, provided $k(E) = 0$.

- A bundle $\mathcal{A} \to P^I X$ of von Neumann algebras over path space, whose fibers are isomorphic to A.
- A bundle $\mathcal{B} \to LX$ over loop space, whose fiber over $\gamma \cup -\gamma^c$ is a bimodule over $\mathcal{A}_\gamma - \mathcal{A}_{\gamma^c}$, where $S^1 = I \cup I^c$. This bimodule is supposed to satisfy the fusion property from Theorem 1.

We first explain this structure over the based loop space ΩX, where we fix base points $x_0 \in X$ and $e_0 \in E$. Then the algebra bundle is defined to be the trivial A-bundle and the $A - A$-bimodule structure over a based loop γ is given as follows. Pick a $\widehat{G}$-structure on E which is a $\widehat{G}$-principle bundle over X, and equip it with a connection. Then we get a fusion preserving holonomy map

$$\Omega X \longrightarrow \widehat{G} \xrightarrow{\Phi} \mathrm{Aut}(A)$$

which we have composed with the homomorphism Φ from Proposition 25. But given the automorphism $F = \Phi(\mathrm{hol}(\gamma))$ of the algebra A, we may use it to change the original $A-A$-bimodule structure on H by twisting the (say) left multiplication with F, and leaving the right multiplication unchanged. This has the advantage that by a Lemma of Connes [Co], the composition of based loops (which is sent to the composition of automorphisms) will then correspond to Connes fusion of $A-A$-bimodules, as desired. This follows from the fact that $H \cong L^2A$. Note that this property of fusion was one of the motivations to introduce bimodules and their fusion as a generalization of homomorphisms (of von Neumann algebras) and their composition.

It is easy to check that this construction does not depend on the choice of e_0, up to a canonical isomorphism. We have thus finished the proof of Theorem 1 for based loops (but for all groups G, not just Spin(n). Note that the choice of a $\widehat{G}$-structure (or string structure for $G = \text{Spin}(n)$) was essential in the construction, and we shall indeed show that these are in 1-1 correspondence with isomorphism classes of bimodule bundles as claimed in Theorem 2.

References

[Bry] J.-L. Brylinski, *Loop spaces, Characteristic classes and Geometric quantization.* Progress in Math., Birkhäuser 1993.

[Co] A. Connes, *Noncommutative Geometry*

[Fr2] D. Freed *Classical Chern-Simons theory, Part 1*

[LM] Lawson and Michelsohn, *Spin Geometry*

[McL] MacLaughlin

[PR] R. J. Plymen and P. L. Robinson, *Spinors in Hilbert space*, Cambridge University Press 1994, Cambridge Tracts in Mathematics 114

[PS] Pressley and Segal, *Loop groups*, Princeton University Press

[Se 2] G. Segal, *The definition of conformal field theory*, unpublished

[Wa] A. Wassermann, *Operator algebras and conformal field theory*, Invent. math. 133, 467–538 (1998)