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COHOMOLOGY OPERATIONS AND OBSTRUCTIONS TO EXTENDING CONTINUOUS FUNCTIONS

by

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Colloquium Lectures

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§1. Introduction.

The class of problems known as extension problems is central to nearly all of topology. Many of the basic theorems of topology, and some of its most successful applications in other areas of mathematics are solutions of particular extension problems. The deepest results of this kind have been obtained by the method of algebraic topology. The essence of the method is a conversion of the geometric problem into an algebraic problem which is sufficiently complex to embody the essential features of the geometric problem, yet sufficiently simple to be solvable by standard algebraic methods. Many extension problems remain unsolved, and much of the current development of algebraic topology is inspired by the hope of finding a truly general solution.

To place my contribution to these developments in its proper setting, I will begin with a discussion of the extension problem, and the methods of finding solutions in special cases.

§2. The extension problem.

Let X and Y be topological spaces. Let A be a closed subset of X, and let $h: A \longrightarrow Y$ be a mapping, i.e. a continuous function from A to Y. A mapping $f: X \longrightarrow Y$ is called an extension of h if $f(x) = h(x)$ for each $x \in A$. The inclusion mapping $g: A \longrightarrow X$ is defined by $g(x) = x$ for $x \in A$.

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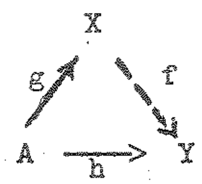
Then the condition that f be an extension can be restated: h is the composition fg of f and g.

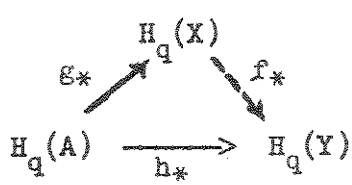


$fg = h.$

When X,Y,A and h are given, we have an extension problem: Does an extension f of h exist?

§3. Transforming geometric into algebraic problems.

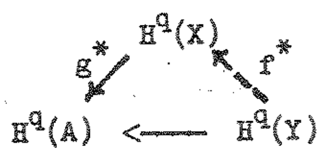
The general method of attack on an extension problem is to apply homology theory to transform the problem into an algebraic problem. To the diagram of spaces and mappings we assign a diagram of groups and homomorphisms. Each space has a homology group H_q for each dimension q, and each mapping induces homomorphisms of the corresponding groups. Thus, for each q, we have an algebraic diagram



$f_* g_* = h_*.$

Given the three groups and the homomorphisms g_*, h_*, we can now ask the question: Does there exist a homomorphism ϕ such that $\phi g_* = h_*$? (It should be noted that g_* is not usually an inclusion, because a non-bounding cycle of A may bound in X). If an extension f exists, setting $\phi = f_*$ solves the algebraic problem because of the property $(fg)_* = f_* g_*$ of induced homomorphisms. Thus, the existence of a solution of the algebraic problem is a necessary condition for the existence of an extension. But it is not usually a sufficient condition. The reason for this is that much of the geometry has been lost in the transition to algebra.

It is a prime objective of research in algebraic topology to improve the algebraic machinery so as to give a sharper algebraic picture of the geometric problem. For example, in place of homology we may use cohomology. We obtain an analogous diagram



$$g^*f^* = h^*.$$

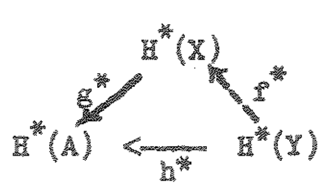
The chief difference is the reversal of the directions of the induced homomorphisms. If we consider cohomology solely as additive groups, they have no real advantage over homology groups. However, unlike homology, the cohomology groups of a space admit a ring structure: if $u \varepsilon H^p(Y)$ and $v \varepsilon H^q(Y)$, then they have a product, called the cup-product,

$$u \smile v \;\varepsilon\; H^{p+q}(Y).$$

This product is bilinear, and satisfies the commutative law $u \smile v = (-1)^{pq} v \smile u$. Furthermore a mapping $f\colon X \longrightarrow Y$ induces a ring homomorphism

$$f^*(u \smile v) = f^*u \smile f^*v.$$

Letting $H^*(Y) = \{H^q(Y),\ q = 0,1,\ldots\}$ denote the resulting graded ring, the algebraic diagram becomes



$$g^*f^* = h^*,$$

and the algebraic problem is sharpened by the requirement that the solution ϕ of $g^*\phi = h^*$ must be a ring homomorphism.

This provides a considerable improvement in the algebraic picture of the geometric problem. However it is not the best that can be done. The cohomology groups possess not only a ring structure but also a more involved structure referred to as the <u>system of cohomology operations</u>. A <u>cohomology operation</u> T, relative to dimensions q and r, is a collection of functions $\{T_X\}$, one for each space X, such that

$$T_X: H^q(X) \longrightarrow H^r(X),$$

and, for each mapping $f: X \longrightarrow Y$,

$$f^*T_Y u = T_X f^* u \quad \text{for all} \quad u \in H^q(Y).$$

The simplest non-trivial operations are the squaring operations. For each dimension q and each integer $i \geqq 0$, there is a cohomology operation, called square-i,

$$Sq^i: H^q(X;Z_2) \longrightarrow H^{q+i}(X;Z_2).$$

Here the coefficient group Z_2 consists of the integers Z reduced modulo 2. Also for each prime $p > 2$, there are cohomology operations generalizing the squares called cyclic reduced p th powers. These are functions

$$\mathcal{P}^i: H^q(X;Z_p) \longrightarrow H^{q+2i(p-1)}(X;Z_p).$$

I will discuss these operations in detail later on. At the present time I wish only to emphasize the importance of cohomology operations to the study of the extension problem: In the derived algebraic problem using cohomology, the solution $\phi: H^*(Y) \longrightarrow H^*(X)$ of the algebraic problem $g^*\phi = h^*$ must be a ring homomorphism, and also must satisfy $\phi T_Y = T_X \phi$ for every cohomology operation T. Thus, by cramming as much structure as possible into cohomology theory, we endeavor to obtain the strongest possible necessary conditions for a solution of the extension problem.

The ultimate objective is to so refine the algebraic machinery that the derived algebraic problem is a faithful picture of the geometric problem. This has not yet been accomplished; but it appears to be within reach.

We turn now to a more detailed discussion of the ideas presented so far.

§4. Examples of extension problems.

Examples of solutions of extension problems are plentiful even in the most elementary aspects of topology. The Urysohn lemma is an example. In this case X is a normal space, $A = A_0 \cup A_1$ is the union of two disjoint closed subsets, Y is the interval $[0,1]$ of real numbers, and $h(A_0) = 0$, $h(A_1) = 1$. The conclusion of the lemma asserts that an extension always exists.

The Tietze extension theorem is another example. In this case X is normal, $Y = [0,1]$, and h is arbitrary. Again an extension always exists.

The study of the arcwise connectivity of a space Y is another example. In this case $X = [0,1]$, A consists of the two points 0 and 1, and $h(0) = y_0$, $h(1) = y_1$. An extension f of h is a path in Y from y_0 to y_1.

There is a special class of extension problems called <u>retraction</u> problems. If $A \subset X$, then a mapping $f: X \longrightarrow A$ is called a retraction if $f(x) = x$ for each $x \in A$. Given a space X and a closed subspace A, there is the problem of deciding whether or not such a retraction exists. By setting $Y = A$, and taking $h: A \longrightarrow Y$ to be the identity, it is seen that each retraction problem is an extension problem.

An important example from elementary algebraic topology is the following. Let E be the closed n-cell, i.e. the set $\Sigma_{i=1}^{n} x_i^2 \leqq 1$ in cartesian n-space, and let S be its boundary, i.e. the (n-1)-sphere $\Sigma_{i=1}^{n} x_i^2 = 1$. Then

The boundary S of the n-cell E is not a retract of E.

The proof of this for $n = 1$ is readily deduced from the fact that E is connected and S is not. For $n > 1$, the proof is not trivial, although the conclusion for $n = 2$ is intuitively appealing to anyone who has tightened a drum head, or stretched canvas tautly over a frame. The proof utilizes the general method of converting the problem into an algebraic one. We take homology groups in the dimension n-1, and obtain the diagram

$$\begin{array}{ccccc} & & H_{n-1}(E) & & \\ & {\scriptstyle g_*}\nearrow & & \searrow{\scriptstyle f_*} & \\ H_{n-1}(S) & & \xrightarrow[h_*]{} & & H_{n-1}(S) \end{array}$$

The dimension n-1 is used since this gives the only non-trivial homology group of S. Using integer coefficients Z, we have $H_{n-1}(S) \approx Z$, and $H_{n-1}(E) = 0$. Now h = identity implies h_* = identity. This gives an impossibility: the identity homomorphism of Z cannot be factored into homomorphisms $Z \xrightarrow{g_*} 0 \xrightarrow{f_*} Z$. Therefore the retraction f of E into S does not exist.

It may be felt that a non-existence theorem is of little use. This is not the case. By a mild twist, a negative result can be given a positive form. In the case at hand, we obtain as a corollary the well-known Brouwer fixed-point theorem: *Each mapping* $g: E \longrightarrow E$ *has at least one fixed point.* For suppose to the contrary that there is a g with no fixed-point. As x and $g(x)$ are distinct points, they lie on a unique straight line, and x divides this line into two half lines. The half line not containing $g(x)$ meets S in a single point denoted by $f(x)$. The continuity of g implies that of f. In case

$x \in S$, it is clear that $f(x) = x$. So f is a retraction $E \longrightarrow S$. As this is impossible, a fixed point free g cannot exist.

§5. The use of the cohomology ring.

The next example is one in which the cohomology ring must be used to arrive at a decision. Let X denote the complex projective plane, i.e. the space of 3 homogeneous complex variables $[z_0, z_1, z_2]$ not all zero. It is a compact manifold of dimension 4. Let A be the complex projective line in X defined by the equation $z_2 = 0$. Topologically, A is a 2-sphere. In this case the conclusion is that A is not a retract of X.

Suppose that $f\colon X \longrightarrow A$ is a retraction so that fg = identity where $g\colon A \longrightarrow X$ is the inclusion. Passing to cohomology, we have the diagram

$$H^*(X) \underset{f^*}{\overset{g^*}{\rightleftarrows}} H^*(A), \qquad g^* f^* = \text{identity}.$$

When two groups are so related by homomorphisms, the left hand group splits into a direct sum:

$$H^*(X) = \text{Image of } f^* + \text{Kernel of } g^*$$

The abbreviated notation is

$$H^*(X) = \operatorname{Im} f^* + \operatorname{Ker} g^*. \tag{5.1}$$

Furthermore g^* gives an isomorphism

$$g^*\colon \operatorname{Im} f^* \approx H^*(A). \tag{5.2}$$

If we include the ring structure, and use the fact that f^*, g^* are ring homomorphisms, then

(5.3) Im f^* is a subring, and Ker g^* is an ideal.

Turning to the example under consideration, we are given X, A and the inclusion g, and we can ask if Ker g^* is a direct summand. The cohomology of X is zero in dimensions > 4. In dimensions $\leqq 4$, the cohomology of X and A with integer coefficients Z is given by the table

	0	1	2	3	4
A	Z	0	Z	0	0
X	Z	0	Z	0	Z

Furthermore g^* is an isomorphism in the dimensions 0 and 2. It is seen then that the direct sum decomposition required by 5.1 does exist and, in fact, is unique. Namely, in the dimensions 0 and 2, Ker g^* is zero so that Im f^* is the whole group, and in the dimension 4, Ker g^* is the whole group and Im $f^* = 0$.

However, on examining the ring structure, we find that the uniquely determined candidate for Im f^* is not a subring. For let u be a generator of $H^2(X)$ so that $u \in$ Im f^*. Since X is a manifold, the Poincaré duality theorem asserts that H^2 is self-dual under the cup product pairing to H^4. It follows that $u \smile u$ must generate $H^4(X)$. Therefore $u \smile u$ is not in Im f^*; and therefore A is not a retract.

This example is intimately related to the mapping $h: S^3 \longrightarrow S^2$ of the 3-sphere into the 2-sphere studied first by H. Hopf [14]. In the space of two complex variables, let S^3 be the unit sphere $z_0\bar{z}_0 + z_1\bar{z}_1 = 1$, and E^4 the unit 4-cell $z_0\bar{z}_0 + z_1\bar{z}_1 \leqq 1$. Let S^2 be the space of two homogeneous complex variables $[z_0, z_1]$. Then h sends the point (z_0, z_1) of S^3 into $[z_0, z_1]$ in S^2. This is a very smooth mapping. The inverse images of points of S^2 give a fibration of S^3 into great circles. Hopf proved that h is not extendable to a mapping $E^4 \longrightarrow S^2$. (Notice that $(z_0,z_1) \longrightarrow [z_0,z_1]$ has a singularity at $(0,0)$.) If we form a new space out of E^4 by collapsing its boundary S^3 into S^2 according to h, the resulting space is homeomorphic to the complex projective plane X, and S^2 corresponds to the complex projective line A. Since A is not a retract of X, it follows that h cannot be extended over E^4.

§6. The use of the squaring operations.

The next example is a retraction problem for which the cohomology ring does not provide an answer; but the squaring operations do give an answer. Let P^5 denote the real projective space of dimension 5 (6 homogeneous real variables). Let $P^4 \supset P^3 \supset P^2$ be projective subspaces of the indicated dimensions. Let X be the space obtained from P^5 by collapsing P^2 to a point, and let $A \subset X$ be the image of P^4 under the collapsing map : $P^5 \longrightarrow X$. Again the assertion is that <u>A is not a retract of X.</u>

We tackle this problem in the same manner as the preceding one, and begin by asking whether $\operatorname{Ker} g^*$ is a direct summand of $H^*(X)$. Knowing the cohomology of P^5, one readily deduces that of X and A. With Z_2 as coefficients, the cohomology is given by the following table

	0	1	2	3	4	5
A	Z_2	0	0	Z_2	Z_2	0
X	Z_2	0	0	Z_2	Z_2	Z_2

Furthermore, g^* is an isomorphism in dimensions < 5. Therefore there is a direct sum splitting as in 5.1 and it is unique: $\operatorname{Im} f^*$ must be the whole group in dimensions < 5, and it is zero in the dimension 5.

In this case the candidate for $\operatorname{Im} f^*$ is obviously a subring. The reason is that the cup product of elements of dim ≥ 3 has dim ≥ 6, and is therefore zero. Thus, insofar as the cohomology ring is concerned, A could be a retract of X. To show that it is not a retract, we must use the cohomology operation

$$Sq^2 \colon H^3(X;Z_2) \longrightarrow H^5(X;Z_2).$$

If u is the non-zero element of H^3, a suitable calculation shows that $Sq^2 u$ is the non-zero element of H^5. Now the unique candidate for $\operatorname{Im} f^*$ contains u and is zero in dimension 5; hence it is not closed under Sq^2. But it would have to be closed if a retraction f existed because $f^* Sq^2 = Sq^2 f^*$. Therefore a retraction does not exist.

This result has a good application in differential geometry. It is well known that a differentiable manifold has a continuous field F of non-zero tangent vectors if and only if its Euler number is zero. This implies that the n-sphere S^n has a tangent field F if and only if n is odd. S^3 in fact has 3 fields which are independent at each point because it is a group manifold (unit quaternions). The question arises as to the maximum number of fields tangent to S^5 which are independent at each point. The answer is 1. For, by a direct construction, two independent fields can be made to yield a retraction of X into A (see [28]).

The same method can be used to prove a more general result [28]. If n is a positive integer, and 2^k is the largest power of 2 dividing $n+1$, then any set of 2^k vector fields tangent to S^n are dependent at some point. This result is the best possible for $n < 15$.

§7. Homotopies.

Having demonstrated the need of finer and finer algebraic tools, it is natural to ask if there is an end to the process. The answer is that there is real hope of achieving a complete solution. To exhibit the basis for my hope, I must delve more deeply into the geometric aspects of the extension problem. For this, the concept of homotopy is vital. Let h be a mapping $A \longrightarrow Y$, and let $I = [0,1]$ be the unit interval, then a mapping $H: A \times I \longrightarrow Y$ is called a homotopy of h if $H(x,0) = h(x)$ for $x \varepsilon A$. Setting $h'(x) = H(x,1)$, H is called a homotopy of h into h' and we write $h \simeq h'$ (h is homotopic to h'). This is an equivalence relation, and the set of maps homotopic to h is called the homotopy class of h. The set of homotopy classes of mappings $A \longrightarrow Y$ is denoted by $\mathrm{Map}(X, Y)$.

A basic result, due to Borsuk, is the

Homotopy Extension Theorem. If $f: X \longrightarrow Y$, A is closed in X, and $h = f|A$. Then any homotopy H of h may be extended to a homotopy of f. Precisely, the mapping G of the subset $X \times 0 \cup A \times I$ of $X \times I$ into Y, given by $G(x,0) = f(x)$ for $x \varepsilon X$ and $G(x,t) = H(x,t)$ for $x \varepsilon A$, $t \varepsilon I$, may be extended to a mapping $F: X \times I \longrightarrow Y$.

The intuitive idea of the theorem is that if we grab hold of the image of A and pull it along, then the image of X will come sliding after.

The theorem is not true in the generality stated; some restriction on X, A or Y is necessary. It suffices for example if Y is triangulable or if X and A are triangulable. It also suffices to impose the condition of being an absolute neighborhood retract on Y or on X and A. In the future we assume some such restriction without further mention.

Notice that the theorem asserts the extendability of certain kinds of mappings. This solution of a special extension problem is of the utmost importance for the general problem because of the following

<u>Corollary</u>. The extendability of $h: A \longrightarrow Y$ to a mapping $f: X \longrightarrow Y$ depends only on the homotopy class of h: If h is extendable and $h \simeq h'$, then h' is extendable.

It is only necessary to extend the homotopy to $F: X \times I \longrightarrow Y$ and set $f'(x) = F(x,1)$.

One advantage this gives us is that, in any particular extension problem, we may vary h by a homotopy and obtain a simpler but equivalent problem. For example, suppose it were known that h is homotopic to a constant mapping h' (i.e. $h'(A)$ is a single point). Since such an h' is obviously extendable, so is h.

The result also enables us to rephrase the extension problem in an apparently weaker form: <u>Does there exist an f such that</u> $fg \simeq h$? Given such an f, we have that $f|A$ is obviously extendable, and $f|A \simeq h$, and so h is extendable.

Having freed one aspect of the extension problem (replacing $fg = h$ by $fg \simeq h$), it is natural to consider freeing other parts of unnecessary retrictions. The condition that g be the inclusion mapping $A \subset X$ is no longer an essential feature. Let X, A, Y be any three spaces and let $h: A \longrightarrow Y$ and $g: A \longrightarrow X$ be mappings. Does there exist a mapping $f: X \longrightarrow Y$ such that $fg \simeq h$? This problem is called the "left-factorization" problem. The class of these problems includes the extension problems and many more. Broadening thus the class of problems does not increase the difficulties because of the following result.

Each left-factorization problem is equivalent to some retraction problem.

To see this, we start with a left-factorization problem as above, and construct a space Z as follows. In the union of X, $A \times I$ and Y, identify each point $(a,0)$ with $g(a)$ in x, and identify each point $(a,1)$ with $h(a)$ in Y. The resulting space Z contains X and Y and a homotopy of g into h. It follows quickly that Y is a retract of Z if and only if there exists a mapping $f: X \longrightarrow Y$ such that $fg \simeq h$.

Thus the broadest type of problem is equivalent to the narrowest type.

It is easily shown that a left-factorization problem depends only on the homotopy classes of g and h. Even more it depends only on the homotopy types of the three spaces involved. Two spaces X, X' have the same homotopy type (are homotopically equivalent) if there exist mappings $\phi: X \longrightarrow X'$ and $\phi': X' \longrightarrow X$ such that $\phi\phi' \simeq$ identity of X' and $\phi'\phi \simeq$ identity of X. We may substitute X' for X in any problem if we set $g' = \phi g$. Analogous substitutions can be made for A and Y.

An advantage of this flexibility is that any particular problem can often be greatly simplified by homotopic alterations of the spaces and mappings involved.

More important however is the light which it casts on the class of all problems. If we consider only those spaces admitting finite triangulations, then there are only a countable number of homotopy types of spaces, and for any two spaces there are only a countable number of homotopy classes of mappings. This statement can be proved by the use of the well-known simplicial approximation theorem. It is a consequence that there are only a countable number of extension problems. This in itself makes it reasonable to hope for effective methods of solving any extension problem.

To substantiate this hope, consider the notion of the induced homomorphism

f^* of cohomology assigned to a mapping $f: X \longrightarrow Y$. A well-known property is that homotopic maps induce the same homomorphism. Hence we have a function

$$R_{XY}: \mathrm{Map}(X,Y) \longrightarrow \mathrm{Hom}\,(H^*(Y),H^*(X))$$

defined by $R_{XY}(f) = f^*$. By Hom we mean all functions preserving whatever algebraic structure we are able to put into the cohomology theory of spaces. Suppose we have an extension problem with spaces X, A, Y such that R_{XY} is onto, and R_{AY} is 1-1 into. Suppose moreover that the algebraic problem $g^*\phi = h^*$ has a solution ϕ. Since R_{XY} is onto, there exists an $f: X \longrightarrow Y$ such that $f^* = \phi$. Then $(fg)^* = h^*$. Since R_{AY} is 1-1 into, this implies $fg \simeq h$. Hence the solvability of the algebriac problem is both necessary and sufficient for solving the geometric problem.

Thus we would have a complete hold on the extension problem if we knew that R_{XY} is 1-1 onto for all triangulable spaces X, Y. This is true for some spaces and false for others. For example, let $X = S^3$ and $Y = S^2$; then $\mathrm{Hom}\,(H^*(S^2),H^*(S^3)) = 0$, and $\mathrm{Map}(S^3,S^2) = \pi_3(S^2)$ is infinite. However our point of view above has been too narrow in specifying the range of R_{XY}. Some more intricate algebraic gadget should do the trick. The possibilities are many. For example $R_{XY}(f)$ could be taken to be the cohomology sequence associated with the mapping cylinder of f.

The finding of a suitable 1-1 mapping R_{XY} of $\mathrm{Map}(X,Y)$ into a computable algebraic object is called the homotopy classification problem. Solving it completely will solve the extension problem completely. Why should we be hopeful of solving this? First, $\mathrm{Map}(X,Y)$ is a countable set, and is therefore suitable for algebraization. Secondly, in many special cases (as will be shown) we have obtained solutions. Thirdly, we have available now a variety of functions R_{XY} which taken together may provide the complete solution.

§8. Lifting problems.

There is a class of problems called lifting problems which are dual in a certain sense to extension problems. In a lifting problem, we are given a fibre bundle X over a base space Y with projection $f\colon X \longrightarrow Y$. This means that each $y \in Y$ has a neighborhood V such that $f^{-1}V$ is representable as a product space $V \times F$ for some fixed space F called the fibre. Furthermore, f restricted to $f^{-1}V$ is the projection $V \times F \longrightarrow V$. In the lifting problem, we are also given a space A and a mapping $h\colon A \longrightarrow Y$; and the problem is to decide whether there exists a mapping $g\colon A \longrightarrow X$ such that $fg = h$.

$$\begin{array}{ccc} & X & \\ {\scriptstyle g}\nearrow & & \searrow{\scriptstyle f} \\ A & \longrightarrow & Y \end{array} \qquad\qquad fg = h.$$

The condition that $X \xrightarrow{f} Y$ is a bundle is dual to the condition of an extension problem that $A \xrightarrow{g} X$ is an inclusion mapping.

An elementary example of a lifting problem and its solution is the

Monodromy theorem. If X is a covering space of Y with projection f, then a mapping $h\colon A \longrightarrow Y$ can be lifted to $g\colon A \longrightarrow X$ if and only if the algebraic problem posed by the fundamental groups has a solution:

$$\begin{array}{ccc} & \pi_1(X) & \\ {\scriptstyle g_*}\nearrow & & \searrow{\scriptstyle f_*} \\ \pi_1(A) & \xrightarrow[h_*]{} & \pi_1(Y) \end{array} \qquad\qquad f_* g_* = h_* .$$

It should be recalled that, since X covers Y, f_* imbeds $\pi_1(X)$ isomorphically into $\pi_1(Y)$. Also, since base points are not specified, the images of f_* and h_* are only defined up to inner automorphisms of $\pi_1(Y)$. Thus to decide whether the algebraic problem has a solution it suffices to determine whether some conjugate of $f_*\pi_1(X)$ contains $h_*\pi_1(A)$.

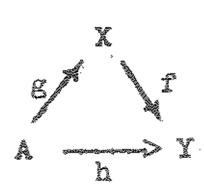
The monodromy theorem is used in complex variable theory in order to find a single-valued branch of the composition of a single-valued and a multiple-valued function.

If X is a bundle over Y with projection f, we obtain a special lifting problem by taking $A = Y$ and h = identity. A solution $g: Y \longrightarrow X$ of fg = identity is called a cross-section of the bundle. Cross-sectioning problems are the duals of retraction problems.

A great variety of these problems arise in differential geometry (see [24]). Let Y be a differentiable manifold. For any tensor of a specified algebraic type, the set of all such tensors at all points of Y forms a fibre bundle X over Y. A cross-section of this bundle is a tensor field defined on Y of the specified type. For example, let X be the manifold of non-zero tangent vectors of Y. A cross-section is a continuous field of non-zero vectors. For a compact Y, such a field exists if and only if the Euler number of Y is zero. This is proved by using cohomology groups of the dimension of Y. Many applications of algebraic topology to problems of this type have been made. But many more remain out of reach.

We propose to show now that the duality between extension and lifting persists in considerable detail. The dual of the homotopy extension theorem is the

Covering homotopy theorem. In the situation



$fg = h$

where X is a bundle over Y, let H be any homotopy of h. Then there exists a homotopy G of g such that $fG = H$, i.e. any motion in the base space Y can be covered by a motion in the bundle space X.

The proposition asserts that a certain kind of lifting problem always has a solution. In analogy with the case of the extension problem, we have the

Corollary. In any lifting problem, the liftability of a mapping $h: A \longrightarrow Y$ depends only on the homotopy class of h.

In any lifting problem, a solution g' of the weaker problem $fg' \simeq h$ leads to a solution of the problem $fg = h$. It is only necessary to cover the homotopy of fg' into h by a homotopy of g'.

As before we can abandon now the restriction that X is a fibre bundle over Y. We define a right-factorization problem to consist of three spaces A, X, Y and mappings $h: A \longrightarrow Y$ and $f: X \longrightarrow Y$. A solution is a mapping $g: A \longrightarrow X$ such that $fg \simeq h$. The existence of a solution depends only on the homotopy classes of the mappings and the homotopy types of the spaces.

The general method of handling a lifting problem or a right-factorization problem is the same as that used for extension and left-factorization problems. We transform the problem to an algebriac one by applying a functor from topology to algebra. All of the discussion of the derived algebraic problems applies equally well to the new situation. When we are able to cram into the algebraic functor enough structure to be able to solve the homotopy classification problem, then we will be able to solve any lifting problem.

§9. The classification theorems of Hopf and Hurewicz

There are certain restricted situations where homology and cohomology, considered as having additive structure only, are adequate to solve the homotopy classification problem. Two theorems proved about 1935 mark high spots in this direction. These are the theorems of Hopf and Hurewicz.

Hopf's classification theorem. If K is a finite complex, and $n > 0$ is an integer such that $H^q(K) = 0$ for all $q > n$, then the natural function

$$\mathrm{Map}(K, S^n) \longrightarrow \mathrm{Hom}(H^n(S^n), H^n(K))$$

is one-to-one and onto.

let T_n = full subset of finite complexes with $H^q K = 0$ $q > n$
Then
H^n is corepresentable
$H^n K = [K, S^n]$

Since $H^n(S^n)$ is infinite cyclic, we have

$$\mathrm{Hom}(H^n(S^n), H^n(K)) \approx H^n(K);$$

therefore $\mathrm{Map}(K, S^n)$ is in 1-1 correspondence with $H^n(K)$.

Hurewicz's classification theorem. If Y is a connected and simply-connected space, and n is an integer such that $H_i(Y) = 0$ for $0 < i < n$, then the natural function

$$\mathrm{Map}(S^n, Y) \longrightarrow \mathrm{Hom}(H_n(S^n), H_n(Y))$$

is one-to-one and onto.

Again $H_n(S^n)$ is infinite cyclic, and therefore $\mathrm{Map}(S^n, Y)$ is in 1-1 correspondence with $H_n(Y)$.

As is well known, Hurewicz defined a group structure in $\mathrm{Map}(S^n, Y)$ giving an abelian group denoted by $\pi_n(Y)$ and called the n^{th} homotopy group.

The conclusion of the theorem can be restated: <u>Then</u> $\pi_i(Y) = 0$ <u>for</u> $o < i < n$, <u>and</u> $\pi_n(Y) \approx H_n(Y)$.

The homotopy groups, like the homology groups, form a functor from topology to algebra, and convert geometric problems into algebraic ones. They can be and are used to solve extension problems. However, unlike homology groups, there is a severe restriction on their use. Homotopy groups are very difficult to calculate effectively. Computing a homotopy group requires us to solve a homotopy classification problem; and this may be a problem of the same order of difficulty as the extension problem under consideration. A chief virtue of Hurewicz's theorem is that it reduces the calculation of a particular homotopy group to that of a homology group.

The Hopf and Hurewicz theorems have an intersection: the homotopy classes of mappings $S^n \longrightarrow S^n$ are in 1-1 correspondence with the homomorphisms $H_n(S^n) \longrightarrow H_n(S^n)$. Since $H_n(S^n)$ is infinite cyclic, any such homomorphism f_* is characterized by an integer d called the <u>degree</u> of the mapping f, and it satisfies $f_*(z) = dz$ for $z \in H_n(S^n)$.

There is a union of the two theorems which is due to Eilenberg [11]:

<u>Homotopy classification theorem.</u> Let K be a finite complex and n a positive integer such that $H^q(K) = 0$ for $q > n$. Let Y be a connected and simply-connected space such that $H_i(Y) = 0$ for $0 < i < n$. Then $Map(K,Y)$ is in 1-1 correspondence with $H^n(K;H_n(Y))$, i.e. the n^{th} cohomology group of K using $H_n(Y)$ as coefficients.

Notice that the hypotheses allow only a single dimension n in which the cohomology of X is non-zero and the homology of Y is non-zero. As soon as we allow an overlapping of non-triviality in more than one dimension, the additive structure of homology and cohomology becomes inadequate.

§10. Obstructions.

The method introduced by Eilenberg to prove the above result has very general applicability, and is called obstruction theory (see [24, Part III]). Let K be a complex, L a subcomplex and $f: L \longrightarrow Y$. For the sake of simplicity assume that Y is arcwise connected and simply-connected. Let K^q denote the q-dimensional skeleton of K. The subcomplexes $L \cup K^q$ for $q = 0, 1, \ldots$ form an expanding sequence. Let us attempt to extend f over each in turn. An extension f_0 over $L \cup K^0$ is obtained by defining f_0 to be f on L, and to have arbitrary values on the vertices of K not in L. For any 1-cell σ of $K - L$, f_0 is defined on its vertices and gives two points in Y. As Y is arcwise connected, we may map σ into a path joining the two points. Doing this for each such σ gives an extension f_1 of f_0 over $L \cup K^1$. For each 2-cell σ of $K - L$, f_1 is defined on its boundary $\dot{\sigma}$ giving a loop in Y. Since $\pi_1(Y) = 0$, the mapping f_1 on $\dot{\sigma}$ may be extended over σ. Doing this for each σ gives an extension f_2 of f_1 over $L \cup K^2$. Now if each $\pi_i(Y) = 0$ for $i < \dim(K - L)$, there is nothing to stop us from continuing this process and obtaining an extension over all of K. But this is too severe a requirement, and we must ask what happens in the general case.

Assume now that somehow an extension f_q of f over $L \cup K^q$ has been achieved for some q, and consider the extension problem posed by each $(q+1)$-cell σ of $K - L$. We have that $f_q|\dot{\sigma}$ is defined, and is a mapping of a q-sphere into Y. This determines an element of the homotopy group $\pi_q(Y)$ provided we give $\dot{\sigma}$ an orientation. This is done by first orienting σ, and then giving $\dot{\sigma}$ the orientation of the algebraic boundary $\partial\sigma$. Then, for each oriented cell σ, $f_q|\partial\sigma$ defines an element of $\pi_q(Y)$ denoted by $c(f_q, \sigma)$. This function of $(q+1)$-cells may be regarded as a $(q+1)$-dim.cochain of K with coefficients in $\pi_q(Y)$, and is denoted by $c(f_q)$. Since f_q can be extended over σ if and only if $c(f_q,\sigma) = 0$, we call $c(f_q)$ the obstruction to extending f_q. Since f_q is defined on each cell of L, $c(f_q)$ is zero on L. It is therefore a cochain of K modulo L.

Most important is the fact that $c(f_q)$ is a cocycle, i.e. it vanishes on boundaries. This follows because it was defined using the boundary, and $\partial\partial = 0$. It determines therefore a cohomology class

$$\bar{c}(f_q) \in H^{q+1}(K,L;\pi_q(Y)).$$

Consider now what happens if we retreat one stage to f_{q-1} and extend it over $L \cup K^q$ in some other fashion obtaining f'_q. On any q-cell τ of $K-L$, the two mappings f_q, f'_q agree on the boundary, and give two cells in Y with a common boundary. These determine a map of a q-sphere in Y, and hence an element of $\pi_q(Y)$ denoted by $d(f_q,f'_q,\tau)$. The resulting q-cochain is called the difference cochain. Its main property is that its coboundary is the difference of the two obstruction cocycles.

$$\delta d(f_q,f'_q) = c(f_q) - c(f'_q)$$

This gives $\bar{c}(f_q) = \bar{c}(f'_q)$. Therefore $\bar{c}(f_q)$ depends only on f_{q-1} and can be written $\bar{c}^{q+1}(f_{q-1})$. It is the obstruction to extending f_{q-1} over $L \cup K^{q+1}$ knowing that it can be extended over $L \cup K^q$.

Now suppose we retreat two stages to f_{q-2} and extend over $L \cup K^{q-1}$ in some other fashion obtaining f'_{q-1}. This gives a (q-1)-cochain $d(f_{q-1},f'_{q-1})$. Its coboundary is $-c^q(f'_{q-1})$. So if the alteration f'_{q-1} is chosen so that $d(f_{q-1},f'_{q-1})$ is a cocycle, it may be extended to a map f'_q of $L \cup K^q$. In this case $\bar{c}(f_q)$ and $\bar{c}(f'_q)$ can be different cohomology classes. Their difference is some function of the cocycle $d(f_{q-1},f'_{q-1})$. In fact they are related by the squaring operation

$$Sq^2\,\bar{d}(f_{q-1}, f'_{q-1}) = \bar{c}(f_q) - \bar{c}(f'_q).$$

It follows that the obstruction to extending f_{q-2} over $L \quad K^{q+1}$, assuming it can be extended over $L \cup K^q$, is an element of the quotient group $H^{q+1}/\,Sq^2\,H^{q-1}$.

If we now retreat three stages to f_{q-3} and extend over $L \cup K^q$ in some other fashion obtaining f'_q, then $d(f_{q-2}, f'_{q-2})$ is a (q-2)-cycle, and Sq^2 of its cohomology class is zero. The difference $\bar{c}(f_q) - \bar{c}(f'_q)$ is some function of $\bar{d}(f_{q-2}, f'_{q-2})$. The relationship in this case has been studied by Adem [1]. He has defined quite generally a cohomology operation, denoted by Φ^3, which increases dimension by 3, is defined on the kernel of Sq^2 and has values in the cokernel of Sq^2. The operation provides the desired connection.

The three stage retreat is as far as this game has been analysed in a detailed and effective manner. The general pattern is clear. If f_q and f'_q are two extensions of f over $L \cup K^q$ which agree on $L \cup K^r$ $(0 \leqq r < q-3)$, then $d(f_{r+1}, f'_{r+1})$ is an (r+1)-cocycle. Furthermore it lies in the kernel of Sq^2; hence Φ^3 is defined on it, and it lies in the kernel of Φ^3; hence some unknown operation Φ^4 is defined on it. If $r < q-4$, it lies in the kernel of Φ^4; and some operation Φ^5 is defined on it. This continues up to Φ^{q-4}, and this operation applied to $d(f_{r+1}, f'_{r+1})$ gives the difference $\bar{c}(f_q) - \bar{c}(f'_q)$ modulo images of $Sq^2, \Phi^3, \ldots, \Phi^{q-r-1}$.

The method of successive obstructions has two main phases. First one must compute effectively those homotopy groups $\pi_i(Y)$ which appear as coefficient groups. This in itself is a difficult problem. It is worth noting in this connection that E.H. Brown [6] has shown that the homotopy groups of a simply-connected finite complex are effectively computable. The second phase is to give effective methods of computing the operations Φ^i for $i > 3$. Much work remains to be done. But enough has been accomplished to make one hopeful of ultimate success.

§11. The cohomology ring.

We shall turn now to the methods of constructing cohomology operations. Perhaps the simplest operation is the cup product which gives the ring structure to the cohomology groups. When first discovered about 1936 by Alexander, Čech and Whitney, the cup product appeared to be very mysterious. It was not known for example why cohomology admits a ring structure but homology does not. The formulas defining the cup product gave little insight into the structure of the cohomology ring.

Lefschetz in his Colloquium book of 1942 presented a new approach to products which dispelled much of the mystery. It was based on products of complexes. If K and L are cell complexes, then their topological product $K \times L$ may be regarded as a cell complex in which the cells are the products $\sigma \times \tau$ of cells $\sigma \in K$ and $\tau \in L$. It follows that the chain groups of $K \times L$ are sums of tensor products of the chain groups of K and L

$$C_r(K \times L) \approx \Sigma_{p+q=r} C_p(K) \otimes C_q(L)$$

Introducing orientations suitably (i.e. defining incidence numbers in $K \times L$ in terms of those in K and L), one arrives at the boundary formula

$$\partial(a \otimes b) = \partial a \otimes b + (-1)^p a \otimes \partial b, \quad \dim a = p.$$

From this it follows that the product of two cycles is a cycle, and if either is a boundary so is their product. Thus we have an induced homomorphism

$$\alpha: H_p(K) \otimes H_q(L) \longrightarrow H_{p+q}(K \times L).$$

In fact, with integer coefficients, α is an isomorphism of $\Sigma_{p+q=r} H_p \otimes H_q$ with a direct summand of $H_r(K \times L)$. Abbreviating $\alpha(x \otimes y)$ by $x \times y$, we obtain a bilinear product which is associative and commutative: if T interchanges K and L, then

$$T_*(x \times y) = (-1)^{pq} y \times x.$$

An entirely analogous game can be played with cochains and cohomology. If u and v are cochains of K and L respectively, define $u \times v$ by specifying its values on product cells as follows

$$(u \times v) \cdot (\sigma \times \tau) = (u \cdot \sigma)(v \cdot \tau)$$

(It is understood here that $u \cdot \sigma$ is zero if u and σ have different dimensions). This gives an isomorphism (K or L finite)

$$C^r(K \times L) \approx \Sigma_{p+q=r} C^p(K) \otimes C^q(L),$$

satisfying the coboundary relation

$$\delta(u \times v) = \delta u \times v + (-1)^p u \times \delta v,$$

and inducing

$$\alpha\colon H^p(K) \otimes H^q(L) \longrightarrow H^{p+q}(K \times L).$$

This yields a bilinear product which is associative, and commutative. It is also highly non-trivial in that α maps $\Sigma_{p+q=r} H^p \otimes H^q$ isomorphically onto a direct summand of $H^r(K \times L)$.

Up to this point the results for homology and cohomology are on a par. Now take $K = L$, and let

$$d\colon K \longrightarrow K \times K$$

be the diagonal mapping $d(x) = (x,x)$. Passing to homology and cohomology gives two diagrams of homomorphisms

$$H_p(K) \otimes H_q(K) \xrightarrow{\alpha} H_{p+q}(K \times K) \xleftarrow{d_*} H_{p+q}(K) ,$$

$$H^p(K) \otimes H^q(K) \xrightarrow{\alpha} H^{p+q}(K \times K) \xrightarrow{d^*} H^{p+q}(K).$$

Clearly d_* and α cannot be composed, but d^* and α can be because cohomology is contravariant. The cup-product of $u \varepsilon H^p(K)$ and $v \varepsilon H^q(K)$ is defined by

$$uv = d^* \alpha(u \otimes v) = d^*(u \times v).$$

This gives a product in the cohomology of K which is associative and commutative: $uv = (-1)^{pq} vu$.

This method of Lefschetz makes it completely clear why cohomology has a ring structure but homology does not. It also shows that the study of the cohomology ring reduces to the study of the homomorphism d^*, i.e. to an investigation of the way in which the diagonal is imbedded in the product.

A very beautiful application of the ring structure was made by Hopf [15] in determining the cohomology of Lie groups as follows

Hopf's theorem on group manifolds. If G is the space of a Lie group, then the cohomology ring of G over a field of coefficients of characteristic 0 is the same as the cohomology ring of the product space of a collection of spheres of odd dimensions. Equivalently, $H^*(G)$ is an exterior algebra with odd dimensional generators.

There is an extension theorem hidden in this proposition. To see this, let K be a finite complex, and let 1 denote a selected vertex of K. In the product $K \times K$, let $K \vee K$ denote the union of the subsets $K \times 1$ and

$1 \times K$; it is the union of two copies of K with a point in common.
Define

$$h: K \vee K \longrightarrow K \quad \text{by} \quad h(x,1) = x = h(1,x).$$

Then for each K, we have an extension problem: Can h be extended to $f: K \times K \longrightarrow K$? A very strong necessary condition for this is that $H^*(K)$ must be an exterior algebra with odd dimensional generators. For the existence of f defines a continuous multiplication in K by $xy = f(x,y)$ having 1 as a two-sided unit. And such a multiplication was all that Hopf assumed in proving his theorem.

An extensive generalization of Hopf's theorem has been given by A. Borel [4]. He relaxes the hypotheses by allowing the space G to be infinite dimensional, and the coefficient field to have a prime characteristic (providing the field is perfect). He concludes that $H^*(G)$ is a tensor product of exterior algebras and polynomial rings (which may be truncated).

Another application of the cohomology ring was made by Pontrjagin to the computation of an obstruction [19]. A simplified form of the result goes as follows. Let $h: K^3 \longrightarrow S^2$ map the 3-skeleton of a complex K into the 2-sphere; and let u be a generator of the infinite cyclic group $H^2(S^2)$ using integer coefficients. Since K^3 is the 3-skeleton, the inclusion $K^3 \subset K$ induces an isomorphism $\phi: H^2(K) \approx H^2(K^3)$. Then the cohomology class of the obstruction to extending h over K^4 is the square of $\phi^{-1}f^*u$. Therefore $(\phi^{-1}f^*u)^2 = 0$ is a necessary and sufficient condition that $h|K^2$ be extendable over K^4.

§12. Motivation for Sq^2.

Because the method of constructing the squaring operations appears to be somewhat arbitrary, it is worthwhile to give the motivation which led their discovery. Briefly, obstruction theory gave a non-constructive proof of the existence of Sq^2.

To see this clearly, let K be a complex, and let v be an n-cocycle of K representing $u \in H^n(K;Z)$. Construct a mapping f of the (n+1)-skeleton K^{n+1} into the n-sphere S^n as follows. First, shrink K^{n-1} to a point to be mapped by f into a point $y_0 \in S^n$. Each oriented n-cell σ of K becomes an n-sphere and may be mapped onto S^n with the degree $v \cdot \sigma$ (= the value of v on σ). For each (n+1)-cell τ, the boundary of τ is mapped on S^n with total degree $= v \cdot \partial \sigma$. By definition of coboundary, we have $v \cdot \partial \sigma = \delta v \cdot \sigma = 0$ because v is a cocycle. So the mapping of the boundary of τ extends over τ. Doing this for each τ defines $f: K^{n+1} \longrightarrow S^n$.

The obstruction to extending f over K^{n+2} is an (n+2)-cocyle $c(f)$ with coefficients in $\pi_{n+1}(S^n)$. Its cohomology class $\bar{c}(f)$ depends only on the class u of v, and may be written $\bar{c}(f) = Sq^2 u$. When $n = 2$, we have $\pi_3(S^2) = Z$, and Pontrjagin's extension theorem (§11) gives $\bar{c}(f) = u \smile u$. When $n > 2$, Freudenthal proved that $\pi_{n+1}(S^n) = Z_2$, and therefore Sq^2 is a mapping $H^n(K;Z) \longrightarrow H^{n+2}(K;Z_2)$.

§13. The homology of groups.

The first effective definition of the squares used explicit formulas in simplicial complexes. These were generalizations of the Alexander formula for the cup product, and they gave no intuitive insight. To obtain such insight it was important to find a conceptual definition analogous to Lefschetz's construction of cup products using $K \times K$ and the diagonal mapping $d: K \longrightarrow K \times K$. This was found; and, surprisingly, it revealed a connection with another development of algebraic topology, namely, the homology groups of a group. We turn to this now.

Let π be a group (possibly non-abelian). In the applications we have in mind π is a finite group. A complex W is called a <u>π-complex</u> if π is represented as a group of automorphisms of W. A π-complex W is said to be <u>π-free</u> if, for each cell σ of W, the transforms of σ under the various elements of π are all distinct. Let W/π denote the complex obtained by identifying points of W equivalent under π. Then π-freeness implies that the collapsing map $W \longrightarrow W/\pi$ is a covering with π as the group of covering transformations. Let $A(\pi)$ denote the family of π-free complexes which are also acyclic (i.e. all homology groups are zero). There are two important facts about the family $A(\pi)$. First, it is non-empty. Secondly, if W and W' are in $A(\pi)$, then there are chain mappings $W/\pi \longrightarrow W'/\pi \longrightarrow W/\pi$ giving a homotopy equivalence. It follows that the homology of W/π depends on π alone, and we define the homology of π by

$$H_q(\pi) = H_q(W/\pi) \qquad \text{for } W \in A(\pi).$$

This concept was developed first by Eilenberg and MacLane, and independently by Hopf.

As an example, let π be the cyclic group of order 2 with generator T. Let W be the union of a sequence of spheres

$$S^0 \subset S^1 \subset S^2 \subset \cdots \subset S^n \subset \cdots$$

where the n-sphere S^n is an equator of S^{n+1}. Let T be the antipodal transformation in each S^n. The two hemispheres of S^n determined by S^{n-1}

are n-cells denoted by d_n and Td_n. The collection of these cells for $n = 0, 1, 2, \cdots$ gives a cellular structure on W. Obviously W is π-free. We orient the cells so that the following are the boundary relations:

$$\partial d_1 = Td_0 - d_0, \quad \partial Td_1 = d_0 - Td_0,$$

$$\partial d_2 = d_1 + Td_1, \quad \partial Td_2 = Td_1 + d_1,$$

$$\partial d_3 = Td_2 - d_2, \quad \partial Td_3 = d_2 - Td_2,$$

$$\cdots$$

$$\partial d_{2n} = d_{2n-1} + Td_{2n-1}, \quad \partial Td_{2n} = Td_{2n-1} + d_{2n-1},$$

$$\partial d_{2n+1} = Td_{2n} - d_{2n}, \quad \partial Td_{2n+1} = d_{2n} - Td_{2n}.$$

In an even (odd) dimension, every cycle is a multiple of $Td_{2n} - d_{2n}$ ($d_{2n+1} + Td_{2n+1}$) and this cycle bounds. Therefore W is acyclic. Collapsing $W \longrightarrow W/\pi$ gives a sequence of real projective spaces

$$P^0 \subset P^1 \subset \cdots \subset P^n \subset \cdots .$$

The cells d_n, Td_n come together to form a single cell d'_n; and the boundary relations become

$$\partial d'_{2n} = 2d'_{2n-1}, \quad \partial d'_{2n+1} = 0 .$$

Using Z_2 (= the integers mod 2) as coefficients, we obtain $H_q(\pi; Z_2) \approx Z_2$ for all q.

§14. Construction of the squares.

We are prepared now to define the squaring operations in a complex K. Recall that the diagonal mapping $d: K \longrightarrow K \times K$ is used to construct cup products by the rule $u \smile v = d^*(u \times v)$. To compute d^*, one must obtain from d a chain mapping

$$d_0: C_q(K) \longrightarrow C_q(K \times K), \quad q \geqq 0 .$$

Since the cells of $K \times K$ are the products of cells of K, the diagonal is not a subcomplex of $K \times K$. Hence there is no uniquely determined d_0, but one must choose d_0 from a collection of algebraic approximations to d. We proceed to describe these. For each cell σ of K, define its carrier $C(\sigma) = |\sigma \times \sigma|$ to be the subcomplex of $K \times K$ consisting of $\sigma \times \sigma$ and all of its faces. We refer to $C(\sigma)$ as the diagonal carrier. Because σ and its faces form an acyclic complex, $C(\sigma)$ is likewise acyclic. It is the minimal carrier of d because $C(\sigma)$ is the least subcomplex containing $d(\sigma)$. Any chain mapping d_0 such that $d_0\sigma$ is a chain on $C(\sigma)$ is called an approximation to d. The two principal facts about such approximations are that they exist, and any two are chain homotopic. These facts are proved by constructing the chain map, or the chain homotopy, inductively with respect to the dimension starting in the dimension zero. The acyclicity of the carrier is all that is needed for the general step. Any approximation d_0 induces a homomorphism d^*, and the homotopy equivalence of any two insures that they give the same d^*.

The important point to be observed about the construction of d_0 is this: although the mapping d is symmetric, there is no symmetric approximation d_0. Precisely, if T is the automorphism of $K \times K$ which interchanges the two factors, then $Td = d$ but there is no d_0 such that $Td_0 = d_0$. This is easily seen by taking K to be a 1-simplex σ so that $K \times K$ is a square. The 1-chain $d_0\sigma$ must connect the two end points of the diagonal and lie on the periphery of the square, so it must go around one way or the other.

This difficulty can be restated in a more illuminating fashion. Let T act also on K as the identity map of K. Then d is equivariant, i.e. $Td = dT$. But there is no chain approximation d_0 which is equivariant. The reason is that π acts freely on the set of possible choices for $d_0\sigma$ but leaves σ fixed.

Given a d_0, we can measure its deviation from symmetry. Since d_0 and Td_0 are carried by C, there is a chain homotopy d_1 of d_0 into Td_0. Precisely, for each q-cell σ, there is a (q+1)-chain $d_1\sigma$ on $C(\sigma)$ such that

$$\partial d_1\sigma = Td_0\sigma - d_0\sigma - d_1\partial\sigma .$$

Then

$$\partial Td_1\sigma = d_0\sigma - Td_0\sigma - Td_1\partial\sigma .$$

It follows that $d_1 + Td_1$ is a homotopy of d_0 around a circuit back into itself. For each q-cell σ this homotopy lies on $C(\sigma)$; it is therefore homotopic to the constant homotopy of d_0. Precisely, there is a (q+2)-chain $d_2\sigma$ on $C(\sigma)$ such that

$$\partial d_2\sigma = d_1\sigma + Td_1\sigma + d_2\partial\sigma$$

At this stage, the construction should remind one of the construction, given in the preceding section, of the π-free complex W. The analogy is made precise as follows. Form the product complex $W \times K$. Define the action of π in $W \times K$ by $T(w \times \sigma) = (Tw) \times \sigma$. The composition of the projection $W \times K \longrightarrow K$ and $d: K \longrightarrow K \times K$ has the minimal carrier $C(w \times \sigma) = |\sigma \times \sigma|$; it is acyclic, and satisfies $TC = CT$. Since W is π-free, so also is $W \times K$. It follows that there is a chain mapping

$$\phi: W \otimes K \longrightarrow K \otimes K$$

carried by C which is equivariant: $\phi T = T\phi$. (The tensor product $\otimes$ is used instead of $\times$ because W and K are now regarded as chain complexes). Recalling that W consists of cells d_i, Td_i, we now identity $\phi(d_0 \otimes \sigma)$ with the diagonal approximation $d_0\sigma$, and $\phi(d_1 \otimes \sigma)$ with the chain homotopy $d_1\sigma$, etc. Then the ∂-relations given above for $d_0\sigma$, $d_1\sigma$, $d_2\sigma$ correspond exactly to the fact that ϕ is a chain mapping: $\partial\phi = \phi\partial$.

For each integer $i \geqq 0$, we define a product called cup-i, as follows. If $u \in C^p(K)$, and $v \in C^q(K)$, then $u \smile_i v \in C^{p+q-i}(K)$ is defined by

$$(u \smile_i v) \cdot c = u \otimes v \cdot \phi(d_i \otimes c), \quad c \in C_{p+q-i}(K).$$

Using the fact that ϕ is equivariant we obtain the coboundary relations modulo 2

$$\delta(u \smile_i v) = u \smile_{i-1} v + v \smile_{i-1} u + \delta u \smile_i v + u \smile_i \delta v .$$

(By convention, $u \smile_{-1} v = 0$). If we set $u = v$ and assume $\delta u = 0 \bmod 2$, it follows that $u \smile_i u$ is a cocycle mod 2. Passing to cohomology classes gives a function denoted by

$$Sq_i: H^p(K;Z_2) \longrightarrow H^{2p-i}(K;Z_2)$$

which assigns to the class of u the class of $u \smile_i u$. It is notationally more convenient to define

$$Sq^j \colon H^p(K;Z_2) \longrightarrow H^{p+j}(K;Z_2)$$

by setting $Sq^j u = Sq_{p-j} u$.

The cup-i products depend on the choice of ϕ. However any two ϕ's are connected by a chain homotopy which is equivariant. It follows that Sq^j is independent of the choice of ϕ.

§15. Properties of the Squares.

The elementary properties of the Sq^i are as follows.

1. If f is a mapping, then $f^* Sq^i = Sq^i f^*$. This expresses the toplogical invariance of Sq^i.

2. Sq^i is a homomorphism.

3. Sq^0 = identity.

4. $Sq^p u = u \smile u$ if $p = \dim u$.

5. $Sq^i u = 0$ if $i > \dim u$.

6. If $L \subset K$, and $\delta \colon H^p(L) \longrightarrow H^{p+1}(K,L)$ is the usual coboundary, then $\delta Sq^i = Sq^i \delta$.

7. If $\delta^* \colon H^p(K;Z_2) \longrightarrow H^{p+1}(K;Z_2)$ is the Bockstein coboundary for the coefficient sequence $0 \longrightarrow Z_2 \longrightarrow Z_4 \longrightarrow Z_2 \longrightarrow 0$, then $Sq^1 = \delta^*$ and

$$Sq^{2j+1} = \delta^* Sq^{2j} \qquad \text{for } j \geqq 0 .$$

These can be proved readily by using the machinery already set up. Less elementary is the Cartan formula:

8. $$Sq^j(u \smile v) = \sum_{i=0}^{j} Sq^i u \smile Sq^{j-i} v .$$

This can be proved by an explicit computation of a π-mapping $W \longrightarrow W \otimes W$.

Using these properties one can compute the squares in many special cases. If $\dim u = 1$, its only non-zero squares are $Sq^0 u = u$ and $Sq^1 u = \delta^* u = u \smile u$. If $\dim u = 2$, its only non-zero squares are $Sq^0 u = u$, $Sq^1 u = \delta^* u$, and $Sq^2 u = u \smile u$. These facts combined with formula 8 enable us to compute squares in the subring of $H^*(K;Z_2)$ generated by 1 and 2-dimensional classes. For example,

$$9. \quad Sq^i(u^k) = \binom{k}{i} u^{k+i} \qquad \text{if } \dim u = 1.$$

In this formula, $\binom{k}{i}$ is the binomial coefficient mod 2, and is zero if $i > k$.

In the real projective n-space P^n, the cohomology ring is the polynomial ring generated by the non-zero element $u \in H^1(P^n;Z_2)$, truncated by the relation $u^{n+1} = 0$. Clearly formula 9 gives all squares in P^n. Let P^r be a projective subspace of P^n $(0 < r < n)$, and form a space P^n/P^r by collapsing P^r to a point. The collapsing map $f: P^n \longrightarrow P^n/P^r$ induces isomorphisms $f^*: H^k(P^n/P^r) \approx H^k(P^n)$ for all $k > r$ because P^r is an r-dimensional skeleton of P^n. Let $w_k \in H^k(P^n/P^r)$ be such that $f^* w_k = u^k$. Using 9 and 1, we have $Sq^i w_k = \binom{k}{i} w_{k+i}$ for $k > r$ and all i. In particular, when $n=5$ and $r=2$, we have $Sq^2 w_3 = w_5$. I used this example in §6 to show that P^4/P^2 is not a retract of P^5/P^2. This is the simplest case known to me where a Sq^i gives a relation between cocycles which are not already related by a cup product or a Bockstein coboundary operator.

§16. Reduced power operations.

The squaring operations are associated with the symmetric group of degree 2. It is to be expected that more cohomology operations are to be obtained by studying the n-fold power $K^n = K \times \cdots \times K$, and the action of the symmetric group $S(n)$ as permutations of the factors of K^n. This is the case. The general definition goes as follows.

Let π be any subgroup of $S(n)$; and let W be a π-free acyclic complex. Let $C(d \times \sigma) = |\sigma|^n$ be the diagonal carrier from $W \times K$ to K^n. As it is equivariant and acyclic, there is an equivariant chain mapping

$$\phi: W \otimes K \longrightarrow K^n .$$

Let $K^* = \mathrm{Hom}(K,Z)$ be the cochain complex of K. Define a cochain complex $W \otimes K^{*n}$ by

$$C^r(W \otimes K^{*n}) = \sum_{i=0}^{\infty} C_i(W) \otimes C^{r+i}(K^{*n}) .$$

The terms of the sum are zero for $i > n \dim K - r$. If $w \in C_i(W)$ and $v \in C^{r+i}(K^{*n})$, set

$$\delta(w \otimes v) = \partial w \otimes v + (-1)^i w \otimes \delta v .$$

This defines δ in $W \otimes K^{*n}$ and makes of it a cochain complex. Define a cochain mapping

$$\phi': W \otimes K^{*n} \longrightarrow K^*$$

dual to ϕ as follows

$$\phi'(w \otimes v) \cdot \sigma = (-1)^{i(i-1)/2}\, v \cdot \phi(w \otimes \sigma)$$

where $i = \dim w$, v is a cochain of $K^{*n} \approx K^{n*}$, and σ is a chain of K with $\dim \sigma = \dim v - i$.

The action of π in $W \otimes K^{*n}$ is defined by

$$t(w \otimes v) = Tw \otimes Tv, \quad T \in \pi.$$

And π acts as the identity in K^*. Then the equivariance of ϕ implies that of ϕ'. It follows that ϕ' transforms cochains equivalent under π into the same cochain. If we identify equivalent cochains of $W \otimes K^{*n}$, we obtain the quotient complex denoted by $W \otimes_\pi K^{*n}$. Then ϕ' induces a cochain mapping

$$\phi'': W \otimes_\pi K^{*n} \longrightarrow K^*.$$

Passing to cohomology with coefficient group G, we obtain an induced homomorphism

$$\phi^*: H^r(W \otimes_\pi K^{*n} \otimes G) \longrightarrow H^r(K^* \otimes G) = H^r(K;G)$$

Now let u be a q-cocycle mod Θ of K^*. Treating u as an integer cochain, we have $\delta u = \Theta v$ for some v. Then the multiples of u and v form a cochain subcomplex M of K^*. Let ψ denote the inclusion mapping $M \longrightarrow K^*$. The product mapping $\psi^n: M^n \longrightarrow K^{*n}$ is equivariant, hence ψ^n and the identity map of W induce a cochain mapping

$$\psi': W \otimes_\pi M^n \longrightarrow W \otimes_\pi K^{*n}.$$

Tensoring with G and passing to cohomology gives an induced mapping

$$\psi^*: H^r(W \otimes_\pi M^n \otimes G) \longrightarrow H^r(W \otimes_\pi K^{*n} \otimes G).$$

Composing ψ^* and ϕ^* gives a mapping

$$\Phi: H^r(W \otimes_{\pi} M^n \otimes G) \longrightarrow H^r(K;G).$$

It depends apparently on the choice of ϕ and the cocycle u mod θ. In fact it is independent of ϕ (any two ϕ's are equivariantly homotopic); and it depends only on the cohomology class $\bar{u}$ of u. The image of Φ, for all r, is called the set of π-reduced powers of $\bar{u} \in H^q(K;Z_\theta)$.

The groups $H^*(W \otimes_{\pi} M^n \otimes G)$ depend only on the groups π, G and the integers θ, q, n. They are generalizations of the ordinary homology groups of π. In the special case that u is an integral cocycle $(v = 0)$, and q is even, we have

$$H^r(W \otimes_{\pi} M^n \otimes G) \approx H_{nq-r}(\pi;G).$$

For, in this case, $M \approx Z$ is generated by u, and $M^n \approx Z$ is generated by u^n with π acting as the identity. Therefore $W \otimes_{\pi} M^n \approx W \otimes_{\pi} Z \approx W/\pi$. If we take account of the dimensional indexing,the assertion follows. Then, to put it roughly, <u>each homology class of a permutation group of degree n gives a cohomology operation.</u>

If we recall that the squares Sq_i are the mod 2 homology classes of $S(2)$, it is clear that we have available a great wealth of cohomology operations, and that these demand analysis.

§17. <u>A basis for reduced power operations.</u>

A rather elaborate analysis [26,27] shows that a relatively small collection of reduced power operations generate all others by forming compositions. The analysis has two main steps. The first shows that we do not need to consider <u>all</u> permutation groups; it suffices to consider, for each prime p, the cyclic group ρ_p of order p and degree p. The second step analyzes the homology (in the generalized sense) of ρ_p.

Just as $H_j(\rho_2;Z_2) \approx Z_2$, we have $H_j(\rho_p;Z_p) \approx Z_p$. A generator ξ_j for this group gives a cohomology operation analogous to Sq_j. When $p > 2$, this operation is identically zero for most of the values of j. The reason for this is that the homomorphism of homology induced by the inclusion mapping $\rho_p \longrightarrow S_p$ has a large kernel for $p > 2$. If we discard the operations which are zero, we obtain an infinite sequence of operations called the <u>cyclic</u> reduced powers

$$\mathcal{P}^i_p : H^q(K;Z_p) \longrightarrow H^{q+2i(p-1)}(K;Z_p), \qquad i = 0,1,\ldots$$

The operation $\mathcal{P}^i_p$ reduces to Sq^{2i} when $p = 2$, and the main properties of these operations are mild modifications of the properties of Sq^{2i} listed in §15.

To complete our list of basic cohomology operations, we need to adjoin for each prime p the Pontrjagin p^{th} power. For each integer $k > 0$, it is a function

$$\mathfrak{P}_p : H^{2q}(K;Z_{p^k}) \longrightarrow H^{2pq}(K;Z_{p^{k+1}}).$$

At first glance, the operation may seem mysterious; however it is only a mild modification of the p^{th} power in the sense of cup products. For, if $\mathfrak{P}_p u$ is reduced mod p^k, it becomes u^p. Pontrjagin [20] discovered the operation for $p = 2$. He observed that, if u is a cocycle mod 2^k, then

$$u \smile_0 u + u \smile_1 \delta u$$

is a cocycle mod 2^{k+1}. The operations for $p > 2$ were found and studied by P. E. Thomas [29,30].

There are certain elementary cohomology operations which are taken for granted but must be mentioned in order to state the main result. These are: addition, cup products, homomorphisms induced by homomorphisms of coefficient groups, and Bockstein coboundary operators associated with exact coefficient sequences $0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$. Then the main result becomes:

The elementary operations and the operations $Sq^i, \mathfrak{P}_2, \mathcal{P}^i_p, \mathfrak{P}_p$ generate all reduced power operations by forming compositions.

§18. Relations on the basic operations.

The generators listed above satisfy numerous relations. Some of the relations satisfied by the Sq^i are given in §15. They satisfy also a more complicated set of relations which were found by J. Adem [1]: If $a < 2b$, then

$$Sq^a Sq^b = \sum_{i=0}^{[a/2]} \binom{b-i-1}{a-2i} Sq^{a+b-i} Sq^i .$$

This holds for the indicated operations applied to a cocycle of any dimension. To clarify the rough implication, let us call an iterated square $Sq^i Sq^j$ reducible if $i < 2j$. Then the formula expresses each reducible iterated square as a sum of irreducible ones. Iterated squares, as reduced power operations, appear as homology classes of the 2-Sylow subgroup of $S(4)$. These relations were found by computing the kernel of the homomorphism induced by the inclusion of the subgroup in the whole group. They have two important consequences.

1. Each Sq^i can be expressed as a sum of iterates of Sq^{2^j}, $j = 0, 1, 2, \ldots$.

2. Let us call the iterated square $Sq^{i_1} Sq^{i_2} \cdots Sq^{i_r}$ <u>admissible</u> if

$$i_1 \geqq 2i_2,\ i_2 \geqq 2i_3, \ldots,\ i_{r-1} \geqq 2i_r.$$

Then every iterated square is uniquely expressible as a sum of admissible iterated squares.

The first result shows that the system of generators given in §17 is too large, we can throw out each Sq^i for which i is not a power of 2. (It is to be noted that if we do this, then the relations satisfied by the remaining squares are not readily written).

The second result was proved first by J.-P. Serre [22] using an entirely different method involving the Eilenberg-MacLane complexes. The result can be expressed in a more illuminating fashion. Let A be the associative (non-commutative) algebra over Z_2 generated by the Sq^i subject to the relations of Adem with $Sq^0 = 1$. Then the admissible elements form an additive basis for A.

J. Milnor has shown [17] that the mapping $\phi: A \longrightarrow A \otimes A$ given by

$$\phi(Sq^i) = \sum_{j=0}^{i} Sq^j \otimes Sq^{i-j}$$

(compare with formula 8 of §15) defines a homomorphism of algebras, and converts A into a Hopf algebra. He shows that the dual Hopf algebra A^* (which is commutative) is a polynomial ring in an easily specified set of generators. Dualizing gives an additive base for A quite different from that of Serre. An important consequence of Milnor's work is that the algebra A is nilpotent.

Analogous results have been obtained for the $\mathcal{P}^i_p$ for primes $p > 2$. Adem [2,3] and Cartan [7] found independently the iteration relations, and proved the analogs of proposition 1 and 2 above. Milnor handles also the case $p > 2$.

To state the situation roughly, we have a very good hold on the relations satisfied by the cyclic reduced powers in spite of the fact that these relations are complicated.

As for the Pontrjagin p^{th} powers, the situation is not as satisfactory; however it is exceedingly interesting. Thomas has given a set of relations which the p^{th} powers satisfy, but in a most indirect fashion. He takes as coefficient domain a graded ring A with divided powers. The divided powers are functions $\gamma_n : A_r \longrightarrow A_{nr}$ having the formal properties of the function $x^n/n!$. The cohomology $H^*(K;A)$ becomes a bigraded ring. He then extends the definition of $\mathfrak{P}_p$ to operations $\mathfrak{P}_n$ for all integers $n \geqq 0$. The collection $(\mathfrak{P}_n)$ are then shown to form a set of divided powers in the subring of $H^*(K;A)$ of elements with even bidegrees. In this way he obtains relations such as

$$\mathfrak{P}_r(u) \smile \mathfrak{P}_s(u) = \binom{r+s}{r} \mathfrak{P}_{r+s}(u),$$

$$\mathfrak{P}_r(u+v) = \sum_{s=0}^{r} \mathfrak{P}_s(u) \smile \mathfrak{P}_{r-s}(v),$$

$$\mathfrak{P}_s \mathfrak{P}_r(u) = \binom{2r-1}{r-1}\binom{3r-1}{r-1} \cdots \binom{sr-1}{r-1} \mathfrak{P}_{sr}(u).$$

Although each $\mathfrak{P}_n$ is expressible in terms of the $\mathfrak{P}_p$ for primes p dividing n, it would be exceedingly clumsy to write the above relations using only the powers with prime indices.

It is not yet known whether we have a complete set of relations on the basic generators. One can ask, for example, whether expressions of the form $\mathfrak{P}_p \mathcal{P}^i_p$ are reducible?

§19. The Eilenberg-MacLane complexes.

There is another approach to the subject of cohomology operations which makes use of the special complexes, called (π,n)-spaces, due to Eilenberg and MacLane [12,13]. These spaces appear to be fundamental to any study of homotopy; and it seems likely that the complete solution of the extension problem will make vital use of them.

If π is an abelian group and $n > 0$ is an integer, then a space Y is said to be a <u>(π,n)-space</u> if it is arcwise connected and all of its homotopy groups are zero except $\pi_n(Y)$ which is isomorphic to π.

There are a few relatively simple examples. The circle S^1 is a $(Z,1)$-space (all its higher homotopy groups are zero because its universal covering space, the straight line, is contractible). The infinite dimensional real projective space (§13) is a $(Z_2,1)$-space (it is covered twice by S^∞ whose homotopy groups are zero). Another example is the complex projective space of infinite dimension. It is a $(Z,2)$-space because it is the base space of a fibration of S^∞ by circles, i.e. by fibres which are $(Z,1)$-spaces.

There are (π,n)-spaces for any prescribed π and n. This fact is not evident, and will be discussed in some detail in later sections. For the present, it is helpful to anticipate two broad conclusions of this discussion. First, a (π,n)-space is usually infinite dimensional. Secondly, although the homotopy structure of a (π,n)-space is simple, its homology structure is usually most intricate. This is in sharp contrast with a space such as S^n whose homology is simple, and whose homotopy is intricate.

Let Y be a (π,n)-space. Attached to Y is its <u>fundamental</u> class u_0. This is an element of $H^n(Y;\pi)$ obtained as follows. Since $\pi_1(Y) = 0$

for $i < n$, Hurewicz's theorem asserts that the natural map ϕ of $\pi_n(Y)$ into $H_n(Y)$ is an isomorphism. Since also $H_{n-1}(Y) = 0$, it follows that the natural mapping

$$H^n(Y;\pi_n(Y)) \longrightarrow \mathrm{Hom}\,(H_n(Y),\pi_n(Y))$$

is an isomorphism. Then u_0 is the element on the left whose image on the right is ϕ^{-1}. We may also describe u_0 is the primary obstruction to contracting Y to a point [24; p. 187]. The first important result about (π,n)-spaces, is the

Homotopy classification theorem: If Y is a (π,n)-space, and X is a complex, then the assignment to each $f: X \longrightarrow Y$ of f^*u_0 sets up a 1-1 correspondence between $\mathrm{Map}(X,Y)$ and $H^n(X;\pi)$.

A proof of this proposition, in the geometric case, can be found in [11; p.243, Th.II]; and, for the purely algebraic case of semi-simplicial complexes, see [13; paper III, pp.520-521]. In essence, the argument is the one used in proving Hopf's theorem (§9). If X, in the theorem, is also a (π,n)-space, the conclusion asserts that there is a map $f: X \longrightarrow Y$ such that f^*u_0 is the fundamental class of X, and this mapping is a homotopy equivalence:

Corollary. Within the realm of complexes, any two (π,n)-spaces have the same homotopy type. Thus their homology and cohomology depend only on π and n; hence $H^*(Y;G)$ may be written $H^*(\pi,n;G)$.

The importance of (π,n)-spaces to the study of cohomology operations is seen as follows. Recall that a cohomology operation T, relative to dimensions q, r and coefficient groups G, G' is a set of functions

$$T_X: H^q(X;G) \longrightarrow H^r(X;G')$$

for each space X such that $f^* T_Y = T_X f^*$ for each mapping $f: X \longrightarrow Y$. Let $\mathcal{O}(q,G;r,G')$ denote the set of all such operations. If we add operations in the usual way $(T+T')_X = T_X + T'_X$, then $\mathcal{O}(q,G;r,G')$ is an abelian group.

Now let Y be a (G,q)-space, and let u_0 be its fundamental class. If $T \in \mathcal{O}(q,G;r,G')$, then

$$Tu_0 \in H^r(Y;G') = H^r(G,q;G').$$

Theorem. The assignment $T \longrightarrow Tu_0$ defines an isomorphism

$$\mathcal{O}(q,G;r,G') \approx H^r(G,q;G').$$

This result is due to Serre [22;p.220], and independently to Eilenberg-MacLane [13]. The proof runs as follows. Suppose T, T' are operations such that $Tu_0 = T'u_0$. Let X be a complex and $u \in H^q(X;G)$. By the classification theorem, there is a mapping $f: X \longrightarrow Y$ such that $f^* u_0 = u$. Therefore

$$Tu = Tf^* u_0 = f^* Tu_0 = f^* T' u_0 = T' f^* u_0 = T' u.$$

Thus $T = T'$ in $\mathcal{O}(q,G;r,G')$. For the other part, let $w \in H^r(G,q;G')$. Construct a $T \in \mathcal{O}(q,G;r,G')$ as follows. If X is any complex, and $u \in H^q(X;G)$, choose a mapping $f: X \longrightarrow Y$ such that $f^* u_0 = u$ and define $Tu = f^* w$. One verifies that $Tu_0 = w$ by taking $X = Y$, $u = u_0$ and $f = \text{identity}$.

§20. Semi-simplicial complexes.

The rough conclusion of the preceding section is that the determination of all cohomology operations is equivalent to the problem of computing the cohomology of the (π, n)-spaces. The latter problem has been the subject of extensive research by Eilenberg-MacLane [13], H. Cartan [7,8,9], and others. A brief review of their work is in order.

The basic construction of (π, n)-spaces is given in the language of semi-simplicial complexes. This appears to be a most convenient concept for nearly all question concerned with homotopy. The following definition of an abstract semi-simplicial complex K is obtained by writing down fairly obvious properties of the singular complex of a space.

First, for each dimension $q \geqq 0$, there is a set K_q whose elements are called q-simplexes (to be thought of as ordered simplexes). For each q and each $i = 0, 1, \ldots, q$, there is a function $\partial_i : K_q \longrightarrow K_{q-1}$ called the i^{th} face operator, and if $x \in K_q$, then $\partial_i x$ is the i^{th} face of x. Again, for each q and each $i = 0, 1, \ldots, q$, there is a function $s_i : K_q \longrightarrow K_{q+1}$ called the i^{th} degeneracy operator. (Picture the collapsing of a (q+1)-simplex into a q-simplex obtained by bringing the i^{th} and $(i+1)^{st}$ vertices together; then s_i is the inverse operation). The definition is completed by imposing the identities:

$$\partial_i \partial_j = \partial_{j-1} \partial_i, \qquad i < j,$$

$$s_i s_j = s_j s_{i-1}, \qquad i > j,$$

$$\partial_i s_j = s_{j-1} \partial_i, \qquad i < j,$$

$$\partial_i s_i = \partial_{i+1} s_i = \text{identity}$$

$$\partial_i s_j = s_j \partial_{i-1}, \qquad i > j+1.$$

A mapping $f: K \longrightarrow L$ of one semi-simplicial complex into another consists of a function $f_q: K_q \longrightarrow L_q$ for each q such that $\partial_i f_q = f_{q-1}\partial_i$ and $s_i f_q = f_{q+1} s_i$.

An ordinary simplicial complex K can be converted in various ways into a semi-simplicial complex K'. For example, if an ordering of the vertices of K is given, one defines K'_q to be the set of order preserving (monotonic) simplicial mappings of the standard ordered q-simplex Δ_q into K.

As already remarked, the concept of the singular complex of a space is a functor S from the category $\mathcal{A}$ of spaces and mappings to the category $\mathcal{B}$ of semi-simplicial complexes and mappings. There is a functor $R: \mathcal{B} \longrightarrow \mathcal{A}$ called the geometric realization. In fact if $K \varepsilon \mathcal{B}$, then $R(K)$ is a CW-complex. The particular realization given by Milnor [18] has very useful properties. Each non-degenerate simplex of K determines a cell of $R(K)$. Also R behaves well with respect to standard operations such as suspensions and products. Now there are natural mappings

$[RK, X] = [K, SX]$

$$RS(X) \longrightarrow X \qquad \text{for} \quad X \varepsilon \mathcal{A},$$
$$K \longrightarrow SR(K) \qquad \text{for} \quad K \varepsilon \mathcal{B}.$$

The second of these is always a homotopy equivalence. If X is a reasonable space (e.g. triangulable), the first mapping is also a homotopy equivalence. The conclusion is that all questions in $\mathcal{A}$, depending on homotopy type only, are equivalent to the corresponding questions in $\mathcal{B}$. This is true in particular of extension problems and homotopy classification problems. Since this is our main concern we will limit all subsequent discussion to the category $\mathcal{B}$.

Each $K \varepsilon \mathcal{B}$ determines a simplicial chain complex $C(K)$ as follows. The free abelian group generated by the set K_q is denoted by $C_q(K)$ and is called the group of q-chains. The functions ∂_i, s_i extend uniquely to homomorphisms of the chain groups denoted by the same symbols. The identities listed above remain valid. Now define $\partial: C_q(K) \longrightarrow C_{q-1}(K)$ by $\partial = \Sigma_{i=0}^{q}(-1)^i\partial_i$. Then $\partial\partial = 0$, and one defines homology and cohomology in the usual way.

§21. Constructions of (π,n)-spaces.

Eilenberg and MacLane assign to (π,n) a semi-simplicial complex $K(\pi,n)$ in the following rather simple way. Let Δ_q denote the complex of the standard q-simplex with ordered vertices. Let $Z^n(\Delta_q,\pi)$ be the group of n-cocycles of Δ_q with coefficients in π. These are normalized cocycles in the sense that they have the value zero on degenerate n-simplexes. Then a q-simplex of $K(\pi,n)$ is defined to be such a cocycle: $K_q = Z^n(\Delta_q,\pi)$. The standard map $\Delta_{q-1} \longrightarrow \Delta_q$, gotten by skipping the i^{th} vertex, induces a homomorphism $Z^n(\Delta_q,\pi) \longrightarrow Z^n(\Delta_{q-1},\pi)$ which is denoted by $\partial_i: K_q \longrightarrow K_{q-1}$. The degeneracy s_i is likewise induced by the i^{th} degeneracy $\Delta_{q+1} \longrightarrow \Delta_q$.

Much work must be done to show that the homotopy groups of $K(\pi,n)$ are zero save $\pi_n = \pi$, i.e. it is a (π,n)-space. Granting this, one can ask what hinders a successful computation of its homology or cohomology. If π is infinite, e.g. $\pi = Z$, then each K_q is infinite. This means that $C_q(K)$ is not finitely generated; and therefore the standard methods of computation can not be applied. If π is a finite group, each K_q is finite, and we are in the realm of effective computability. But due to the large number of n-dimensional faces of Δ_q, the standard methods are not practical. So, in either case, some large scale reduction of the problem must be achieved.

The first observation is that $K(\pi,n)$ and $K(\pi,n+1)$ are related. Define a complex $W(\pi,n)$ in the same manner as $K(\pi,n)$ except for setting $W_q = C^n(\Delta_q,\pi)$. Since Δ_q is acyclic, $Z^n(\Delta_q;\pi)$ is the kernel of $\delta\colon C^n(\Delta_q;\pi) \longrightarrow Z^{n+1}(\Delta_q;\pi)$, and δ is an epimorphism. From this it follows that we have semi-simplicial mappings

$$K(\pi,n) \xrightarrow{\ i\ } W(\pi,n) \xrightarrow{\ p\ } K(\pi,n+1)$$

where p is a fibre mapping with the fibre $K(\pi,n)$. The argument which shows that $K(\pi,n)$ is a (π,n)-space, shows also that $W(\pi,n)$ is homotopically equivalent to a point. The second observation is that $K(\pi,n)$ is an abelian group complex. This means that each K_q is an abelian group, i.e. $Z^n(\Delta_q;\pi)$, and each ∂_i, s_i is a homomorphism. The group structure of K_q induces a ring structure in $C_q(K)$.

These observations motivate the construction of a new sequence of complexes $A(\pi,n)$ given by Eilenberg and MacLane. They start with $A(\pi,0) = K(\pi,0)$. Then, for any abelian group complex Γ, they construct a homotopically trivial complex $B(\Gamma)$, and a fibre mapping

$$\Gamma \xrightarrow{\ i\ } B(\Gamma) \xrightarrow{\ p\ } \overline{B}(\Gamma)$$

with fibre Γ. Finally, $A(\pi,n)$ is defined inductively by $A(\pi,n) = \overline{B}(A(\pi,n-1))$. This construction is referred to as the <u>bar</u> construction. An inductive argument based on the two fibrations leads to the conclusion that $A(\pi,n)$ is homotopically equivalent to $K(\pi,n)$.

In case π is finitely generated, the complexes $A(\pi,n)$ are finite in each dimension, and hence their homologies are effectively computable. This is a large reduction of the problem. Using the $A(\pi,n)$, Eilenberg and MacLane successfully computed the first few non-trivial homology groups, and obtained important applications. However the computation problem was still far from solved.

The next large reduction of the problem was made by H. Cartan. He formulated a general concept of fibre space construction of which the two constructions given above are examples. He showed that any two acyclic constructions applied to homotopically equivalent group complexes gave homotopically equivalent base spaces. He was then able to give relatively simple constructions for cyclic groups π. Using these, the computation of $H^*(\pi,n)$ for finitely generated π's is almost practical.

To illustrate the complexity of the situation, we will state Cartan's result on the structure of the ring $H^*(\pi,n;Z_p)$ when π is infinite cyclic and p is an odd prime. First, there is a sequence $x_1, x_2, \ldots$ of elements of H^* such that H^* is isomorphic to the tensor product $\bigotimes_{i=1}^{\infty} P(x_i)$ where $P(x_i)$ is the polynomial ring over Z_p generated by x_i if $\dim x_i$ is even, and it is the exterior algebra generated by x_i if $\dim x_i$ is odd. For any dimension q, only a finite number of x_i's have dimensions $< q$. It remains to specify the x_i's. This is done most efficiently by using the cyclic reduced p^{th} powers $\mathcal{P}^i$. A finite sequence of positive integers $(a_1, \ldots, a_k)$ is called admissible if

(i) each a_i has the form $2\lambda_i(p-1)+\epsilon_i$ where λ_i is a positive integer, and ϵ_i is 0 or 1,

(ii) $a_{i+1} \geqq p a_i$, $\qquad 1 \leqq i < k$.

(iii) $p a_k < (p-1)(n+a_1+\cdots+a_k)$

Define $St^{a_i} = \mathcal{P}^{\lambda_i}$ if $\epsilon_i = 0$ and $St^{a_i} = \delta^*\mathcal{P}^{\lambda_i}$ if $\epsilon_i = 1$ where δ^* is the Bockstein operator for $0 \longrightarrow Z_p \longrightarrow Z_{p^2} \longrightarrow Z_p \longrightarrow 0$. Let u_0 be the fundamental class of $K(\pi,n)$. Then the set $\{x_i\}$ consists of the element u_0 mod p and the elements

$$St^{a_k} \cdots St^{a_1} u_0$$

as $(a_1,\ldots,a_k)$ ranges over all admissible sequences.

A corollary of this result is that all cohomology operations with Z as initial and Z_p as terminal coefficient group are generated by the operations: addition, cup-product, δ^* and the $\mathcal{P}^i_p$.

Using the full strength of Cartan's results, Moore [18] has shown that all cohomology operations, whose initial coefficient groups are finitely generated, are generated by the cohomology operations listed at the end of §17.

§22. Symmetric products.

We have described two methods of obtaining cohomology operations. The first involved n^{th} powers of complexes and the action of the symmetric group on the factors. The second made use of the Eilenberg-MacLane complexes. Each method has its advantages. The first gives specific operations with convenient properties. The second gives all operations. Since they lead to the same results, it should be possible to bring the two methods together as a single method. The basis for accomplishing this is provided by a theorem of Dold and Thom [10] as follows.

Let SP^nX denote the symmetric n^{th} power of a space (or complex) X, i.e. collapse X^n by identifying points equivalent under $S(n)$. Choose a base point $x_0 \in X$, and use it to give an imbedding

(22.1) $$SP^n X \subset SP^{n+1} X$$

by identifying $(x_1,\ldots,x_n) \in X^n$ with $(x_0,x_1,\ldots,x_n) \in X^{n+1}$. The union over n of $SP^n X$ gives the <u>infinite</u> symmetric product $SP^\infty X$. The rough assertion of the Dold-Thom result is that there are isomorphisms

(22.2) $$\pi_i(SP^\infty X) \approx H_i(X), \qquad i \geqq 1.$$

This is a most surprising result. It offers an entirely new method of constructing (π,n)-spaces. For example, if X is the n-sphere S^n, it follows that $SP^\infty(S^n)$ is a (Z,n)-space. The case $n=2$ of this was already known for elementary reasons: <u>$SP^\infty(S^2)$ is the infinite dimensional complex projective space</u>. To see this, regard S^2 as the space of 2 homogeneous complex variables $[a_0,a_1]$, and also as the space of linear functions $a_0 + a_1 z$. Consider the complex projective n-space CP^n as the space of $n+1$ homogeneous variables $[a_0,a_1,\ldots,a_n]$, and also as the space of polynomials $\Sigma_{i=0}^n a_i z^i$. Each polynomial factors into the product of n linear functions, and determines thereby an unordered set of n elements of S^2. This gives a 1-1 correspondence between $SP^n(S^2)$ and CP^n. Letting $n \longrightarrow \infty$ yields the above assertion.

It is easy to construct a space X whose homology is zero save $H_n(X)$ which is prescribed. If $H_n(X)$ is a cyclic group of order θ, let X be S^n with an $(n+1)$-cell attached by a map of degree θ. If $H_n(X)$ is a direct sum of cyclic groups, let X be a cluster of n-spheres having a common point together with $(n+1)$-cells attached to the spheres with suitable degrees.

All of this can be done quite effectively. The question of the moment is the effectiveness of the construction of $SP^\infty X$. The latter, as a complex, appears to have infinitely many cells in each dimension. The fact which renders the construction effective is a natural direct sum decomposition of the chain complex $C(SP^{\infty}X)$. The basic step is a splitting into chain subcomplexes

$$C(SP^{k}X) \approx C(SP^{k-1}X) + U_k \tag{22.3}$$

The existence of such a subcomplex U_k is easily established in the language of semi-simplicial complexes as follows. Let X be semi-simplicial, let 1 denote the 0-simplex which acts as the base point x_0, and let 1_q be the q-simplex $(s_0)^q\, 1$. The k^{th} power X^k is taken in the sense of cartesian products, and SP^kX has as q-simplexes unordered sequences $x_1 \ldots x_k$ of q-simplexes of X. Such a simplex is in $SP^{k-1}X$ if some $x_i = 1_q$. The q-dimensional part of U_k is defined to be those q-chains generated by chains of the form

$$(x_1 - 1_q) \ldots (x_k - 1_q). \tag{22.4}$$

(It is clear that expanding this product into a sum gives a chain of SP^kX). Under a face or degeneracy operator, this expression retains the same form or becomes zero. Thus U_k is a chain subcomplex (FD-complex in the language of Eilenberg-MacLane).

If we iterate the decomposition 22.3, we obtain

$$C(SP^nX) \approx \Sigma_{k=0}^{n} U_k,$$

$$C(SP^{\infty}X) \approx \Sigma_{k=0}^{\infty} U_k.$$

Passing to cohomology gives

(22.5) $$H^*(SP^\infty X) \approx \Sigma_{k=0}^{\infty} H^*(U_k).$$

By 22.3,

(22.6) $$H^*(U_k) \approx H^*(SP^k X, SP^{k-1}X)$$

Since the finiteness of X (in each dimension) implies the same for $SP^k X$, there is no question about the effective computability of U_k and $H^*(U_k)$. If X is connected, there is an additional fact: $H^i(U_k) = 0$ for $i < k$. Thus for any dimension q, the sum in 22.5 is finite. To state it otherwise:

(22.7) $$H^q(SP^\infty X) \approx H^q(SP^k X) \quad \text{for} \quad k \geqq q.$$

Elements of $H^*(U_k)$ are said to be of <u>rank</u> k. We obtain then a natural bigrading of $H^*(SP^\infty X)$ by dimension and rank. Dold [18] has shown that the decomposition 22.5 depends only on the homology groups of X. It follows that $H^*(\pi,n)$ admits a natural bigrading by dimension and rank. In $H^*(\pi,n)$ the rank of a product is the sum of the ranks (for homogeneous elements). In fact this holds in $H^*(SP^\infty X)$ whenever X is a suspension.

When the decomposition by rank was discovered through the symmetric products, it was then seen how to define it directly through the constructions of Cartan. It follows that Cartan's methods of computation may be applied to compute effectively the homology of $SP^n X$. This is an old problem of algebraic topology, and many papers have treated special cases. Now, for the first time, we have a generally valid method.

The welding together of the two methods of constructing cohomology operations is not yet complete. By the methods described in §16, one can define a homomorphism

$$H^r(W \otimes_\pi M^n) \longrightarrow H^r(SP^n M)$$

which, for $\pi = S(n)$, is an isomorphism for large r but not for all. Much work remains to be done to complete the picture.

§23. Spaces with two non-zero homotopy groups.

A good start has been made on the analysis of spaces with just two non-zero homotopy groups. The rough overall picture is known but most of the details are missing.

First, we know how to construct such spaces. Suppose the prescribed non-zero groups are $\pi_n(Y) = \pi$, and $\pi_q(Y) = \pi'$ with $q > n$. The product space $K(\pi,n) \times K(\pi',q)$ has the required homotopy groups; but there are many others which are homotopically distinct. To obtain these, we must consider fibre spaces having $K(\pi,n)$ as base and $K(\pi',q)$ for fibre. Recall (§21) that $W(\pi',q)$ is an acyclic fibre space over the base space $K(\pi',q+1)$ with fibre $K(\pi',q)$. Any mapping $f: K(\pi,n) \longrightarrow K(\pi',q+1)$ induces a fibre space Y_f over $K(\pi,n)$ with the same fibre (see [24, §10]). Using a semi-simplicial version of the classification theorem [24, §19], it follows that the assignment of Y_f to f sets up a 1-1 correspondence between equivalence classes of such fibre spaces and homotopy classes of mappings. That such a fibre space has the prescribed homotopy groups follows from the exactness of the homotopy sequence of the fibre space [24, §17].

The homotopy classification theorem of §19 implies that the homotopy classes of mappings $K(\pi,n) \longrightarrow K(\pi',q+1)$ are in 1-1 correspondence with the elements of $H^{q+1}(\pi,n;\pi')$. Thus to any element $k \in H^{q+1}(\pi,n;\pi')$ corresponds a homotopy class of spaces with the prescribed homotopy groups. In fact this gives all such in a 1-1 manner. If Y has the prescribed homotopy groups, there is a unique k such that Y belongs to the class corresponding to k. This is seen by mapping $Y \xrightarrow{g} K(\pi,n)$ so as to carry the fundamental class of $K(\pi,n)$ into that of Y, and defining $k(Y) \in H^{q+1}(\pi,n;\pi')$ to be the primary obstruction to retracting the mapping cylinder of g into Y. The class $k(Y)$ is called the Eilenberg-MacLane <u>k-invariant</u> of Y (see [12]).

Automorphisms of π and π' induce automorphisms of $H^{q+1}(\pi,n;\pi')$. If k_1 and $k_2 \in H^{q+1}(\pi,n;\pi')$ are equivalent under such an automorphism then the corresponding spaces have the same homotopy type. Thus the homotopy type problem for such spaces reduces to determining equivalence classes of elements of $H^{q+1}(\pi,n;\pi')$ under such automorphisms. This problem is not yet solved. In essence we know how to compute the group $H^{q+1}(\pi,n;\pi')$; but, if two elements of the group are given, we do not know how to tell in a finite number of steps, whether or not they are equivalent under automorphisms of π,π'.

Recall (§10) that, in the theory of obstructions, we have need of <u>secondary</u> cohomology operations (such as Adem's Φ^3) which are defined only on the kernel of an ordinary (primary) cohomology operation. In §19 we have seen that any $k \in H^{q+1}(\pi,n;\pi')$ determines a primary operation $T(k)$: for any space X,

$$T(k): H^n(X;\pi) \longrightarrow H^{q+1}(X;\pi').$$

Furthermore k determines, as above, a fibre space Y over $K(\pi,n)$ with fibre $K(\pi';q)$.

<u>Each cohomology class $y \in H^r(Y;G)$ determines a secondary operation defined on the kernel of $T(k)$.</u>

To see this, suppose $u \in H^n(X;\pi)$ lies in the kernel of $T(k)$. There is a mapping $h: X \longrightarrow K(\pi,n)$ which carries the fundamental class of $K(\pi,n)$ into u, and its homotopy class is unique. Since $T(k)u = 0$, we must have $h^*k = 0$ (see §19). Since k is the characteristic class of Y (i.e. the obstruction to lifting $K(\pi,n)$ into Y), there is a mapping $g: X \longrightarrow Y$ which composes with the projection $Y \longrightarrow K(\pi,n)$ to give h. Define the secondary operation $T(k,y)$, when applied to u, to be the set of images $g^* y$ for all liftings g of h. In the stable case $r < n+q$, one can describe precisely the nature of the set $T(k,y)u$ as follows. The restriction of y to the fibre $K(\pi',q)$ determines a primary cohomology operation $T(y)\colon H^q(X;\pi') \longrightarrow H^r(X;G)$. Then the set of possible images $g^* y$ is obtained by adding one of them to the image of $T(y)$.

The result just proved emphasizes the importance of computing the cohomology of Y. This problem has barely been touched. As a fibre space, we know the cohomology of its base $K(\pi,n)$ and its fibre $K(\pi',q)$, and we know also its characteristic class k. This gives us a hold on its cohomology structure via the spectral sequence. But we are far from having it in our grasp.

§24. Postnikov systems

Spaces with three or more non-zero homotopy groups can be built by continuing the pattern of the preceding section. Suppose we wish to build spaces having homotopy groups π, π', π'' in the dimensions $n < q < r$ respectively. First we build a space Y having two non-zero homotopy groups π, π' in the dimensions n, q. Let $k \in H^{q+1}(\pi,n;\pi')$ be its k-invariant. Now choose an element $k' \in H^{r+1}(Y;\pi'')$. The homotopy classification theorem (§19) assigns to k' a mapping $f\colon Y \longrightarrow K(\pi'',r+1)$. Let Y' be the fibre space over Y induced by f and the acyclic fibre space $W(\pi'',r) \longrightarrow K(\pi'',r+1)$. Then $Y' \longrightarrow Y$ has $K(\pi'',r)$ as its fibre; and therefore Y' has the required three non-zero homotopy groups.

Given a fourth homotopy group, say σ, to be inserted in the dimension $s > r$, we start with the Y' above, choose a cohomology class $k'' \in H^{s+1}(Y';\sigma)$, select a corresponding map $Y' \longrightarrow K(\sigma,s+1)$, and form the fibre space Y'' over Y' induced by $W(\sigma,s) \longrightarrow K(\sigma,s+1)$.

It is clear that we have described a semi-effective method of building a great variety of spaces using the Eilenberg-MacLane complexes as building blocks. The fact of the matter is that <u>any</u> space can be built, in the sense of homotopy type, by a sequence of such constructions. This idea is due to Postnikov [21]. Precisely, with any connected space X, we can associate a sequence of spaces X_n, $n = 0, 1, 2, \ldots$, a sequence of projections $p_n\colon X_n \longrightarrow X_{n-1}$, and a sequence of mappings $f_n\colon X \longrightarrow X_n$ such that X_0 is a single point, and for each $n > 0$

(i) $\pi_i(X_n) = 0$ for $i > n$,

(ii) $f_{n*}\colon \pi_i(X) \approx \pi_i(X_n)$ for $i \leqq n$,

(iii) $p_n f_n \simeq f_{n-1}$,

(iv) X_n is a fibre space over X_{n-1} with respect to p_n, the fibre is a $(\pi_n(X),n)$-space and can be taken to be $K(\pi_n(X),n)$.

Such a system is called a Postnikov system for X. It is not unique but any two $\{X_n\}$, $\{X'_n\}$ are equivalent in the sense that there are mappings $X_n \longrightarrow X'_n \longrightarrow X_n$ which give a homotopy equivalence, and, in fact, a fibre homotopy equivalence of the fibre spaces $X_n \longrightarrow X_{n-1}$ and $X'_n \longrightarrow X'_{n-1}$.

This is indeed a most interesting way of dissecting a space. It provides a fresh point of view, and raises many questions whose answers may cast light on our basic problems. Some useful answers have already been obtained. E.H. Brown [6] has proved the following theorem:

<u>If X is a finite complex which is connected and simply-connected, then a Postnikov system for X is effectively constructible.</u>

An immediate corollary is that the homotopy groups of X are effectively computable. At one time this problem was thought to be of the same order of magnitude as the extension problem itself. It was regarded as a basic weakness of obstruction theory that it used homotopy groups as coefficients when these groups were not known to be computable.

It may be useful to conclude with some questions suggested by these results. Can Brown's result be improved? If X is a finite connected complex, and the word problem for $\pi_1(x)$ is effectively solvable, does it follow that a Postnikov system for X is effectively constructible? A useful special case is that in which $\pi_1(X)$ is abelian. It will be important to find efficient methods of computing the Postnikov systems of special kinds of spaces such as spheres and spaces with one or two non-zero homology groups.

Perhaps it is more important to analyse the basic extension problem in terms of the Postnikov systems of the spaces involved in the problem. Brown has given a partial result in this direction.

Let X, Y be finite simplicial complexes, let A be a subcomplex of X, and let $h: A \longrightarrow Y$ be simplicial. Also let Y be simply-connected and such that $H_q(Y;Z)$ is a finite group for all $q > 0$. Then there is a finite procedure for deciding whether h is extendable to a mapping $X \longrightarrow Y$.

This result is obtained by studying a Postnikov system for Y. The restriction that each $H_q(Y)$ be finite is most severe, and should ultimately be unnecessary.

It may be that what is needed is a method of dissecting a mapping (or its homotopy class) similar to the dissection of spaces. One can always treat a mapping as an inclusion mapping (into the mapping cylinder). This suggests trying to construct simultaneous Postnikov systems for a pair consisting of a space and a subspace. Again, a mapping is always homotopically equivalent to the projection of some fibre space onto its base. Starting with such a projection one can represent it as the composition of a sequence of fibre space projections for which the successive fibres are Eilenberg-MacLane complexes. This is done by dissecting the original fibre a homotopy group at a time. How effective is this procedure? How does it behave under compositions of mappings? It is easy to ask questions, it is hard to find good ones.

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