May 23, 2009

COHOMOLOGICAL PHYSICS IN THE XXTH CENTURY: A SURVEY

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Abstract. Cohomological physics is a phrase I introduced sometime ago in the context of anomalies in gauge theory, but it all began with Gauss in 1833. The cohomology referred to in Gauss was that of differential forms, div, grad, curl and especially Stokes Theorem (the de Rham complex). This survey is limited to the years before 2001 since there has been an explosion of cohomological applications in theoretical physics (even of K-theory) in the new century.

MORE include disclaimer and appeal

version of 3-03-03

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1. Introduction

This survey is limited to the years before 2001 since there has been an explosion of cohomological applications in theoretical physics (even of K-theory) in the new century. Since 1931 but especially toward the end of the XXth century, there has been increased use of cohomological and more recently homotopy theoretical techniques in mathematical physics. The chart on the next page will give you some indication, though I'm sure it is not complete and would appreciate any additions.

In this survey intended for a mixed sudience of mathematically inclined physicists and physically inclined mathematicians, I'll paint with a very broad brush, hoping to provide an overview and guide to the literature. I will emphasize the comparatively recent development of two aspects, the use of configuration and moduli spaces (cf. operads) and the use of homological algebra, where I've been actively involved, at least in spreading the gospel. Each section begins with a brief synopsis.

Research supported in part by NSF grant DMS-9504871, DMS-9803435 and in part by W.R. Kenan and Departmental Research and Study Leaves from the University of North Carolina - Chapel Hill and the hospitality of the University of Pennsylvania.

First, however, here are some historical background and brief comments on some of the other topics listed.

Although Maxwell's equations were for a long time expressed only in coordinate dependent form, they were recast in a particularly attractive way in terms of differential forms on Minkowski space [Max73].

More subtle differential geometry and implicitly characteristic classes occurred visibly in Dirac's magnetic monopole (1931) [Dir31], which lived in a $U(1)$ bundle over $\mathbf{R}^3 - 0$. The magnetic charge was given by the first Chern number; for magnetic charge 1, the monopole lived in the Hopf bundle, introduced that same year 1931 by Hopf [Hop31], though it seems to have taken some decades for that coincidence to be recognized [GP75]. Thus were characteristic classes (and by implication the cohomology of Lie algebras and of Lie groups) introduced into physics.

1931 - it was a very good year. It also saw Birkhoff's proof of the Ergodic Theorem [Bir31], Borsuk's theory of retracts [Bor31], Chandrasekhar's description of stellar collapse to white dwarfs [Cha31], de Rham's description of his (and Elie Cartan's) cohomology [dR31], Gödel's incompleteness theorem [G3̈1] and Hopf and Rinow's results on complete Riemannian manifolds [HR31] - it was a *very* good year!

The chart below lists some of the major themes and papers in cohomological physics that I am aware of; suggestions for further entries would be appreciated. There should also be included group theoretic cohomology which appeared in the work of Bargmann [Bar47, Bar54] on extensions of the Galilean, Lorentz and de Sitter groups, using explicit 2-cocycles. Corresponding Lie algebra 2-cocycles appear more recently in the study of W-algebras, extensions of the Virasoro algebras [**?**]. This in turn is related to deformation theory whcih first appeared in a physically context as Wigner's *contractions* of the Lorentz group to the Poincaré group [Wig85].

Year	DG and Knot Theory	$H^*(Lie)$ and Char. Classes	Homological Algebra	String Field Theory	Deformation Quantization	Variational Bicomplex
1833	Gauss					
1931		Dirac Hopf				
1947						
1954						
1959	Calugareanu					
1967		Fade'ev-Popov				
1968	Pohl					
1971	Fuller					
1975		BRST				
1975		BFV				
1978					BFFLS	
1983		BV				
1984						Vinogradov
1985			Henneaux			
1986					DeWilde- -Lecomte	
1987			Browning- -McMullen			
1988			Stasheff			
1989			FHST			
1991					Fedosov	
1993				Zwiebach		
1994	Bott-Taubes					
1996					Donin	
1997	Bott-Cattaneo				Kontsevich	
1998				Zwiebach		
1999				Chas-Sullivan	Cattaneo- -Felder	
2000						

In the above table,
BRST = Becchi, Rouet, Stora and Tyutin
BFFLS = Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer
BFV = Batalin, Fradkin and Vilkovisky

BV = Batalin and Vilkovisky
FHST = Fisch, Henneaux, Stasheff and Teitelboim

2. From Gauss to Vasiliev and beyond

A considerable body of work in the XXth century has its roots in the work of Gauss, with essentially no intermediaries in over 200 years. Gauss explicitly defined the linking number of two circles imbedded in 3-space by an integral defined in terms of the electromagnetic effect of a current circulating in one of the circles.

> Of *Geometria Situs*, that Leibnitz guessed and of which only a pair of geometers (Euler and Vandermonde) were privilieged to have had a weak sight, we know not much more that nothing after a century and a half.
>
> A major task from the boundary of *Geometria Situs* and *Geometria Magnitudinus* would be to count the linking number of two closed or infinite curves.
>
> Let the coordinates of an arbitrary point on the first curve be x, y, z; on the second x', y', z' and
>
> $$\int\int \frac{(x'-x)(dydz'-dzdy')+(y'-y)(dzdx'-dxdz')+(z-z')(dxdy'-dydx')}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{3/2}}$$
>
> then the integral extended over both curves equals $4\pi m$ and m is the linking number.
>
> The value is symmetric, i.e. it remains the same, if the two curves are interchanged. 1833, Jan. 22.

A revisionist view of his integral would see it as the integral of a 2-form on $S^1 \times S^1$, the configuration space of ordered pairs of points, the first on one circle and the second on the other.

A successful generalization to a knot invariant waited until 1959: Calugareanu [Cal59] (later improved by Pohl [Poh68] and B. Fuller [Ful71]). One difficulty was that configurations of ordered pairs of distinct points on a single circle do NOT give a compact manifold

In the past decade, there has been a major renaissance in knot theory, both mathematical and physical, one aspect of which is the use of compactifications of such configuration spaces. It was Kontsevich's definition [Kon93] of Vassiliev knot invariants via integrals that led to Bott and Taubes description [BT94] in terms of compactifications of such configuration spaces as manifolds with corners. In particular, for ordered pairs of points on a circle, the compactification is just a cyclinder, but for ordered triples, the manifold is $S^1 \times W_3$ where W_3 is a hexagon. Here are pictures of W_3 and W_4:

For those of you who know my work, the resemblance to the associahedra is manifest. With Kimura and Voronov [**?**], I revisited the associahedra as a compactified moduli space of ordered n-tuples of distinct points on the real line.

The compactification of importance here is the real non-projective version of the Fulton-Mac Pherson compactification from algebraic geometry [FM94]. which is described in terms of 'blow-ups'. The real non-projective version amounts to removing successively smaller tubular neighborhoods of the various sub-diagonals in Δ beginning with the thin diagonal where all x_i are equal. The special cases of the associahedra and cyclohedra can thus be seen as truncated simplices; for full details, see [SS96]. The associahedra can also be seen as truncated products of simplices, as seen and illustrated beautifully by Devadoss [**?**].

Bott and Cattaneo [BC97] extend the same approach to integral invariants of 3-manifolds.

I will return to consideration of configuration spaces and moduli spaces in section **??**.

3. Homological reduction of constrained Poisson algebras

Cohomological physics had a major break through with the 'ghosts' introduced by Fade'ev and Popov [FP67]. These were incorporated into what came to be known as BRST cohomology (Becchi-Rouet-Stora [BRS75] and Tytutin [Tyu75]) and which was applied to a variety of problems in mathematical physics. There the ghosts were reinterpreted by Stora [Sto77] and others in terms of the Maurer-Cartan forms in the case of a finite dimensional Lie group and more generally as generators of the Chevalley-Eilenberg cochain complex [CE48] for Lie algebra cohomology.

Thanks to Henneaux [Hen85] and Browning and McMullen [BM87], I became aware of the use of this ghost technology for the cohomological reduction of constrained Poisson algebras. The motivating physical systems are described as differential equations of motion or evolution involving smooth functions on a manifold W, but the true 'physical degrees of freedom' correspond to a quotient of $V \subset W$ by a foliation $\mathcal{F}$. Homological reduciton refers tgo describing the appropriate algebra for $C^\infty(M)$ as H^0 of a differntial graded algebra given by homological algebra techniques in terms of $V \subset W$ and $\mathcal{F}$. Specifically the Batalin-Fradkin-Vilkovisky approach [BF83, FF78, FV75] extended BRST by reinventing the Koszul-Tate resolution of the ideal of constraints describing V and producing a synergistic combination of both Chevalley-Eilenberg and resolution cohomology. Here it was that I saw the essential features of a strong homotopy Lie algebra (sh-Lie algebra or L_∞-algebra), the Lie analog of the A_∞-algebras of my thesis under John Moore [**?**].

The setting is the following:

The original manifold W is assumed to be symplectic. This means there is a 2-form ω such that $d\omega = 0$ and $\omega^{dimW} \neq 0$. Equivalently, ω induces an isomorphism

$$TW \to T^*W.$$

Think of R^{2n} with coordinates $(q^1, \ldots, q^n; p_1, \ldots, p_n)$ and 2-form $dq^i \wedge dp_i$.

From an algebra point of view, the crucial point is two-fold: For any function $f \in C^\infty(W)$, there is a Hamiltonian vector field X_f defined by $\omega(X_f, \ ) = df$. For two functions $f, g \in C^\infty(W)$, their Poisson bracket $\{f, g\} \in C^\infty(W)$ is defined by

$$\{f, g\} = \omega(X_f, X_g) = df(X_g) = -dg(X_f).$$

This bracket makes $C^\infty(W)$ into a Poisson algebra, that is, a commutative algebra P (with product denoted fg) together with a bracket $\{\ ,\ \} : P \otimes P \to P$ forming a Lie algebra such that $\{f, \ \}$ is a derivation of P as a commutative algebra: $\{f, gh\} = \{f, g\}h + g\{f, h\}$, called the Leibniz rule.

A **Hamiltonian system with constraints** means we have functions $\phi_\alpha : W \to \mathbf{R}, 1 \leq \alpha \leq r$, the constraints. Solutions of the system are constrained to lie in a subspace $V \subset W$ given as the zero set of $\phi : W \to \mathbf{R}^r$ with components ϕ_α. The algebra $C^\infty(V)$ is given by $C^\infty(W)/I$ where I is the ideal generated by the ϕ_α. Dirac calls the constraints **first class** if I is closed under the Poisson bracket. In this case, the Hamiltonian vector fields X_{ϕ_α} determined by the constraints are tangent to V (where V is smooth) and give a foliation $\mathcal{F}$ of V. Similarly, $C^\infty(W)/I$ is an I-module with respect to the bracket. (In symplectic geometry, the corresponding variety is called **coisotropic** [W].) The true physics of the system is the induced system on the space of leaves $V/\mathcal{F}$.

In this context, the classical BRST construction, at least as developed by Batalin-Fradkin-Vilkovisky in the case of regular constraints, is a homological model for $C^\infty(V/\mathcal{F})$ or rather for the full de Rham complex $\Omega(V, \mathcal{F})$ consisting of forms on vertical vector fields, those tangent to the leaves. If we think of $\mathcal{F}$ as an involutive sub-bundle of the tangent bundle to V, then $\Omega(V, \mathcal{F})$ consists of sections of $\Lambda^*\mathcal{F}$, the exterior algebra bundle on the dual of $\mathcal{F}$. In adapted local coordinates $(x^1, ..., x^{r+s})$ with $(x^1, ..., x^r)$ being coordinates on a leaf, a typical longitudinal form is

$$f_J(x)dx^J \quad \text{where} \quad J = (j_1, ..., j_q) \quad \text{with} \quad 1 \leq j_1 < ...j_q \leq r.$$

The usual exterior derivative of differential forms restricts to determine the vertical exterior derivative because $\mathcal{F}$ is involutive.

By "model" for $\Omega(V, \mathcal{F})$, I mean in the sense of rational homotopy theory [Sul77], that is, a free graded commutative algebra with a derivation differential weakly homotopy equivalent to $\Omega(V, \mathcal{F})$.

The de Rham complex $\Omega^*(M)$ of a smooth manifold M can be described in terms of the Chevalley-Eilenberg complex of the Lie algebra of vector fields on M with coefficients in $C^\infty(M)$. Rinehart [**?**] generalized this complex to the context of a pair L, A where A is a Lie module over L and L is a commutative module over A with conditions on the relation between the two module structures. In particular, $\Omega(V, \mathcal{F})$ can be described as the Rinehart complex for the Lie algebra of vertical vector fields with coefficients in P/I. As an algebra, this can be described as $Alt_P(I, P/I)$, denoting alternating P-multilinear functions on I with values in P/I. This description is useful for comparison with the BFV complex. The latter was crucially a Poisson algebra extension of the Poisson algebra $C^\infty(W)$ and its differential contained a piece which reinvented the Koszul complex for the ideal I. The differential also contained a piece which looked like the Cartan-Chevalley-Eilenberg differential; this followed from the physical motivation for seeking a differential of BRST-type.

This model is constructed as follows: Let $P = C^\infty(W)$ and Φ be the vector space spanned by the ϕ_α. Construct the Koszul complex for the ideal I in terms of the generators ϕ_α. That is, let $s\phi_\alpha$ denote a copy of ϕ_α but regarded as having degree -1. Let δ be the derivation of $P \otimes \Lambda s\Phi$ determined by $\delta\phi = s\phi$ for $\phi \in \Phi$. In other words, $P \otimes \Lambda s\Phi$ is the Koszul complex [Kos50] for the ideal I in the commutative algebra P. If I is what is now known as a regular (at one time: Borel) ideal, the Koszul complex $(P \otimes \Lambda s\Phi, \delta)$ is a model for P/I. For more general ideals, this fails, i.e., $H^i(P \otimes \Lambda s\Phi, \delta) \neq 0$ for some $i \neq 0$. The Tate resolution [Tat57] (which, as inspired by John Moore, mimics a Postnikov system) kills this homology by systematically enlarging $s\Phi$ to a graded vector space Ψ and gives a model $(P \otimes \Lambda\Psi, \delta)$ as desired. We refer to this model as K_I for brevity. It is graded by the grading on Ψ extended multiplicatively, δ being still of degree 1.

Now we wish to replace P/I by K_I in $Alt_P(I, P/I)$ with the Rinehart generalization of the Cartan-Chevalley-Eilenberg differential d and further alter it to a model which is itself a (graded) Poisson algebra.

Theorem 3.1. *If I is a first class ideal, there is a differential graded Poisson algebra, called the BFV complex,*

$$\pi : ((\Lambda\Psi)^* \otimes P \otimes \Lambda\Psi), \partial) \to (Alt_P(I, P/I), d)$$

with ∂ of the form $\delta + d +$ "terms of higher order". If the ideal is regular or close to it, this provides a model for $Alt_P(I, P/I)$.

The existence of the terms of higher order is nowadays usually obtained by standard methods in Homological Perturbation Theory (HPT) [Gug82, GL89, GLS90, GS86, Hue84, HK91] using the contractability of the Koszul-Tate complex.

By the way, in physpeak, the generators $s\phi_\alpha$ of $\Lambda\Psi$ are called *anti-ghosts* and the generators $s\phi^\alpha$ of $\Lambda(\Psi)^*$ are called *ghosts*. For those who prefer such language, an interpretation in terms of super-manifolds is possible, as we indicate in section 4 for the corresponding Lagrangian formalism.

But what do these terms of higher order signify? The derivation ∂ is in fact of the form $\{Q, \ \}$ for Q in the BFV complex, which Q can be expanded as $\sum Q_i$ were i denotes the number of ghost factors and runs from 1 on. The first terms Q_1 gives the Koszul-Tate differential and part of the Chevalley-Eilenberg, the rest being given by Q_2. What is Q_3 telling us? Since Φ is the vector space span of the constraints, the bracket encoded in Q_2 does not satisfy the Jacobi identity, but does up to a homotopy provided by Q_3. The homotopy can be denoted as a tri-linear $[\ ,\ ,\ ]$ on $\Phi^{\otimes 3}$. And so it goes. The total structure includes that of an L_∞-algebra. L_∞-algebras are also known variously as *strong homotopy Lie algebras* or *sh-Lie algebras.* The defining identities for an L_∞-algebra [LS93] are:

$$\begin{aligned} & d[v_1, \ldots, v_n] + \sum_{i=1}^{n} \epsilon(i)[v_1, \ldots, dv_i, \ldots, v_n] \\ (1) \qquad & = \sum_{p+q=n+1} \sum_{\text{unshuffles } \sigma} \epsilon(\sigma)[[v_{i_1}, \ldots, v_{i_p}], v_{j_1}, \ldots, v_{j_q}], \end{aligned}$$

where $\epsilon(\sigma)$ denotes an appropriate sign. An *unshuffle* σ is a permutation such that $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_q$; think of separating a deck of cards into two decks, the order within each deck being as in the original deck.

L_∞-algebras have occurred elsewhere on the boundary between mathematical physics and mathematics, notably in deformation quantization [Kon97] and in Zwiebach's string field theories ([**?**]).

4. The Batalin-Vilkovisky complex for gauge symmetries of Lagrangians

Soon after Batalin, Fradkin and Vilkovisky had handled the Hamiltonian case, Batalin and Vilkovisky applied similar techniques to quantizing Lagrangians with symmetries. Their method turned out to be of considerable interest in the classical setting.

We start with with a bundle $E \to M$ and the associated jet bundles. A *Lagrangian* is a function L on some finite jet bundle J^nE, giving an *action* by integrating over M after pulling L down to

M by the jet of a section $\sigma : M \to E$. *Symmetries* refer to variations in σ which leave the integral unchanged.

The Batalin-Vilkovisky approach [BV83, BV84, BV85] to quantizing particle Lagrangians and subsequently applied to string field theory, both classical and quantum [Zwi93], can, with hindsight, be recognized as that of homological algebra [HT92]. It is the analog of their construction with Fradkin for the Hamiltonian problem, the Koszul complex now being that of (the ideal generated by) the Euler-Lagrange equations (Equations of Motion). The most striking difference is that the Poisson bracket is replaced by an 'anti'-bracket of degree 1 (like a Whitehead product or the Browder bracket for loop space homology) and begins with a pairing of the Koszul generators with the original fibre coordinates. The 'quantum' Batalin-Vilkovisky master equation has the form of the Maurer-Cartan equation for a flat connection, while the 'classical' version has the form of the integrability equation of deformation theory. These are more than analogies; the master equations are the integrability equations of the deformation theory of, respectively, differential graded commutative algebras and graded commutative algebras. Just as the Maurer-Cartan equation makes sense in the context of Lie algebra cohomology, so the Batalin-Vilkovisky master equation has an interpretation in terms of L_∞-algebras.

Under the rubric of the anti-field, anti-bracket formalism, physicists reinvented and then extended homological algebra. Here the 'standard construction' is the Batalin-Vilkovisky complex.

4.1. **The jet bundle setting for Lagrangian field theory.** Let us begin with a space Φ of **fields** regarded as the space of sections of some bundle $\pi : E \to M$. For expository and coordinate computational purposes, I will assume E is a trivial vector bundle and will write a typical field as $\phi = (\phi^1, \ldots, \phi^k) : M \to \mathbf{R}^k$. In terms of local coordinates, we start with a trivial vector bundle $E = F \times M \to M$ with base manifold M, locally R^n, with coordinates $x^i, i = 1, \ldots, n$ and fibre R^k with coordinates $u^a, a = 1, \ldots, k$. We 'prolong' this bundle to create the associated jet bundle $J = J^\infty E \to E \to M$ which is an infinite dimensional vector bundle with coordinates u^a_I where $I = i_1 \ldots i_r$ is a symmetric multi-index (including, for $r = 0$, the empty set of indices, meaning just u^a). The notation is chosen to bring to mind the mixed partial derivatives of order r. Indeed, a section of J is the (infinite) jet $j^\infty\phi$ of a section ϕ of E if, for all r, we have $\partial_{i_1}\partial_{i_2}...\partial_{i_r}\phi^a = u^a_I \circ j^\infty\phi$ where $\phi^a = u^a \circ \phi$ and $\partial_i = \partial/\partial x^i$.

Definition 4.1. *A* **local function** *"on E" $L(x, u^{(p)})$ is a smooth function in the coordinates x^i and the coordinates u^a_I, where the order $|I| = r$ of the multi-index I is less than or equal to some integer p.*

Thus a local function is in fact the pullback to J of a smooth function on some finite jet bundle J^pE, i.e. a composite $J \to J^pE \to R$.

The **space of local functions** will be denoted $Loc\, E$.

Definition 4.2. *A* **local functional**

$$\mathcal{L}[\phi] = \int_M L(x, \phi^{(p)}(x))dvol_M = \int_M (j^\infty\phi)^* L(x, u^{(p)})dvol_M \tag{2}$$

is the integral over M of a local function evaluated for sections ϕ of E. (Of course, we must restrict M and ϕ or both for this to make sense.)

The variational approach is to seek the critical points of such a local functional. More precisely, we seek sections ϕ such that $\delta\mathcal{L}[\phi] = 0$ where δ denotes the variational derivative corresponding to an 'infinitesimal' variation: $\phi \mapsto \phi + \delta\phi$. The condition $\delta\mathcal{L}[\phi] = 0$ is equivalent to the Euler-Lagrange equations on the corresponding local function L as follows: Let

$$D_i = \frac{\partial}{\partial x^i} + u^a_{Ii}\frac{\partial}{\partial u^a_I} \tag{3}$$

be the total derivative acting on local functions and

$$E_a = (-D)_I \frac{\partial}{\partial u^a_I} \tag{4}$$

the Euler-Lagrange derivatives. Here the notation $(-D)_I$ means $(-1)^r D_{i_1} \cdots D_{i_r}$. The Euler-Lagrange equations are then

$$E_a(L) = 0. \tag{5}$$

A Lagrangian $\mathcal{L}$ determines a **stationary surface** or **solution surface** or **shell** $\Sigma \subset J$ such that ϕ is a solution of the variational problem (equivalently, the Euler-Lagrange equations) if $j^\infty \phi$ has its image in Σ. The corresponding algebra is the **stationary ideal** $\mathcal{I}$ of local functions which vanish 'on shell', i.e. when resticted to the solution surface Σ.

The Euler-Lagrange equations generate $\mathcal{I}$ as a differential ideal, but this means we may have not only **Noether identities**

$$r^a_\alpha E_a(L) = 0 \text{ with } r^a_\alpha \in Loc\, E \tag{6}$$

but also

$$r^{aI}_\alpha D_I E_a(L) = 0 \text{ with } r^{aI}_\alpha \in Loc\, E. \tag{7}$$

Of course we have 'trivial' identities of the form

$$D_J E_b(L) \mu^{bJaI}_\alpha D_I E_a(L) = 0, \tag{8}$$

since we are dealing with a commutative algebra of functions. We now assume we have a set of indices $\{\alpha\}$ such that the above identities generate all the non-trivial relations in $\mathcal{I}$. According to Noether [Noe18], each such identity corresponds to an **infinitesimal gauge symmetry**, i.e. an infinitesimal variation that preserves the space of solutions or, equivalently, a vector field tangent to Σ. For each Noether identity indexed by α, we denote the corresponding vector field by δ_α. We denote by Ξ, the **space of gauge symmetries**, considered as a vector space but also as a module over $Loc\, E$. We can regard δ_α as a (constant) vector field on the space of fields Φ and hence δ as a linear map

$$\delta : \Xi \to Vect\ \Phi.$$

Since the bracket of two such vector fields $[\delta_\alpha, \delta_\beta]$ is again a gauge symmetry, it agrees with something in the image of δ when acting on solutions. If we denote that something as $[\alpha, \beta]$, one says this bracket 'closes on shell'. It is not in general a Lie bracket, since the Jacobi identity may hold only 'on shell'.

To make this more explicit, write

$$[\delta_\alpha, \delta_\beta] = \delta_{[\alpha,\beta]} + \nu^a_{\alpha\beta} \frac{\delta L}{\delta u^a}. \tag{9}$$

The possible failure of the Jacobi identity results from those last terms which vanish only on shell and, especially, the fact that we are working in a module over $Loc\, E$. (For example, we have structure *functions* rather than structure constants in terms of our generators.)

All of this, including these latter subtleties, are incorporated into the remarkable complex due to Batalin and Vilkovisky [BV83, BV84, BV85] using anti-field and ghost technology and the anti-bracket of Zinn-Justin [ZJ75]. Let me take you 'through the looking glass' and present a 'bi-lingual' (math and physics) dictionary.

We first extend $Loc\, E$ by adjoining generators of various degrees to form a free graded commutative algebra over $Loc\, E$, that is, even graded generators give rise to a polynomial algebra and odd graded generators give rise to a Grassmann (= exterior) algebra. The generators (and their products) are, in fact, bigraded (p, q); the graded commutativity is with respect to the total degree $p - q$.

For each variable u^a, adjoin an anti-field u^*_a and for each r_α, adjoin a corresponding ghost C^α and a corresponding anti-ghost C^*_α. Here is a table showing the corresponding math terms (explained below) and the bidegrees.

Physics Term	Math Term	Ghost Degree	Anti-ghost Degree	Total Degree
field	section	0	0	0
anti-field	Koszul generator	0	1	-1
ghost	Cartan-Eilenberg generator	1	0	1
anti-ghost	Tate generator	0	2	-2

Note that the anti-field coordinates depend on E alone, but the ghosts and anti-ghosts depend also on the specific Lagrangian. Again an interpretation in the language of super-manifolds is possible: Just as an ordinary manifold can be thought of (almost entirely) in terms of the algebra $C^\infty(M)$ of smooth functions on M, so a *s*upermanifold $\mathcal{M}$ can be thought of in terms its Z_2-graded algebra of smooth functions. The difference is that the oddly graded functions anti-commute with each other and all other pairs commute. In terms of local coordinates, $(x_1, \ldots, x_p; \eta_1, \ldots, \eta_q)$, they form a graded commutative algebra with the x_i of degree 0 and the η_i of degree 1. (More generally, one could consider the situation more familiar in algebraic topology where the coordinates would be Z-graded.) It is worthwhile to think of the x_i as coordinates on a base manifold M and the η_i as fibre coordinates for a bundle over the base. Two particularly important examples are the tangent TM and cotangent bundles T^*M of M but *with the parity of the fibre coordinates reversed,* i.e. redefined as odd. After reversal, these are denoted as ΠTM or $T[1]M$, respectively ΠT^*M or $T^*[1]M$. The (smooth) functions on $T[1]M$ can be identified with differential forms on M, while the (smooth) functions on $T^*[1]M$ can be identified with (alternating) multi-vector fields on M.

[**?**, **?**, AKSZ97, **?**, **?**]

From this point of view of the Batalin-Vilkovisky machinery, we think of the supermanifold $\mathcal{M}$ as $T[1]E$ where E is a G-bundle over M. Later (see below) we will adjoin further variables so that we can think of the supermanifold $\mathcal{M}$ as $T[1]J^\infty E$. This may provide some 'geometric intuition' for what is essentially a process in homological algebra.

The algebra constructed so far can in turn be given an **anti-bracket** (,) of degree -1 which, remarkably, combines with the product we began with to produce precisely an analog of a Gerstenhaber algebra [LZ93, KVZ96], though this was not recognized until much later. (In the super-geometric language, the anti-bracket corresponds to an odd symplectic structure.)

Definition 4.3. *A* **Gerstenhaber algebra** *is a graded commutative and associative algebra A together with a bracket $[\cdot, \cdot] : A \otimes A \to A$ of degree -1, such that for all homogeneous elements x, y, and z in A,*

$$[x, y] := -(-1)^{(|x|-1)(|y|-1)}[y, x],$$

$$[x, [y, z]] = [[x, y], z] + (-1)^{(|x|-1)(|y|-1)}[y, [x, z]],$$

and

$$[x, yz] = [x, y]z + (-1)^{(|x|-1)|y|}y[x, z].$$

The anti-bracket is defined on generators and then extended to polynomials by applying the graded Leibniz identity so that $(\psi,\ )$ is a graded derivation for any ψ in this algebra. The only non-zero anti-brackets of generators are

$$(u^a, u_b^*) = \delta_b^a \quad \text{and} \quad (C^\alpha, C_\beta^*) = \delta_\beta^\alpha.$$

Now we further extend $Loc\, E$ with corresponding jet coordinates u_a^{I*}, C_I^α and C_α^{I*} with the corresponding pairings giving the extended anti-bracket. The resulting Batalin-Vilkovisky algebra we denote $\mathcal{BV}$.

4.2. **Differentials on the graded algebra $\mathcal{B}V$.** Define an operator s_0 of degree -1 on $\mathcal{BV}$ as $(L_0,\ )$.

We call the antifields Koszul generaters because

$$s_0 u_a^* = \frac{\delta L_0}{\delta u^a} \tag{10}$$

as in the Koszul complex for the idealcorresponding to the Euler-lagrange equations, so that $H^{0,0} \subset \Phi$ is given by $\frac{\delta L_0}{\delta u^a} = 0$, but

$$s_0(r_\alpha^a u_a^*) = r_\alpha^a \frac{\delta L}{\delta u^a} = 0, \tag{11}$$

which may give a non-trivial $H^{0,1}$.

Now consider the extended Lagrangian

$$L_1 = L_0 + u_a^* r_\alpha^a C^\alpha \tag{12}$$

and $s_1 = (L_1,\ )$, so that

$$s_1 C_\alpha^* = u_a r_\alpha^a \tag{13}$$

and $H^{0,1}$ is now 0, as in Tate's extension of the Koszul complex of the ideal to produce a resolution [Tat57]. That is why we refer to the anti-ghosts as Tate generators. (If needed, Tate tells us to add further generators in bidegree $(0, q)$ for $q > 2$ so that $H^{0,q} = 0$ for $q > 0$.)

Further extend L_1 to

$$L_2 = L_1 + C_\alpha^* c_{\beta\ggg}^\alpha C^\beta C^\ggg, \tag{14}$$

so that

$$s_2 C^\alpha = c_{\beta\ggg}^\alpha C^\beta C^\ggg$$

$$s_2 u^a = r_\alpha^a C^\alpha$$

which is how the Chevalley-Eilenberg coboundary looks in terms of bases for a Lie algebra and a module and corresponding structure constants. However, we may not have $(s_2)^2 = 0$ since r_α^a and $c_{\beta\ggg}^\alpha$ are functions. Batalin and Vilkovisky prove that all is not lost. First, they add to L_2 a term involving the functions $\nu_{\alpha\beta}^a$.

L versus integrate L aka S

Theorem 4.1. *L_2 can be further extended by terms of higher degree in the anti-ghosts to L_∞ so that $(L_\infty, L_\infty) = 0$ and hence the corresponding s_∞ will have square zero.*

With hindsight, we can see that the existence of these terms of higher order is guaranteed because the antifields and antighosts provide a resolution of the ideal.

We refer to this complex $(\mathcal{BV}, s_\infty)$ as the **Batalin-Vilkovisky complex**.

4.3. **The Classical Master Equation and Higher Homotopy Algebra.** What is the significance of $(s_\infty)^2 = 0$ in our Lagrangian context, or, equivalently, of the **Classical Master Equation** $(L_\infty, L_\infty) = 0$? There are three answers: in higher homotopy algebra, in deformation theory and in mathematical physics. It is the deformation theory that provides the transition betweeen the other two.

If we expand $s = s_0 + s_1 + ...$ where the subscript indicates the change in the ghost degree p, the individual s_i do not correspond to $(x,\ )$ for any term x in L_∞ but do have the following description:

so that we see the $\mathcal{BV}$-complex as a multi-complex. The differential s_1 gives us the Koszul-Tate differential d_{KT} and part of s_2 looks like that of Chevalley-Eilenberg. That is, $C^*_\alpha c^\alpha_{\beta\ggg} C^\beta C^\ggg$ describes the (not-quite-Lie) bracket on Ξ. Further terms with one anti-ghost C^*_α and three ghosts $C^\beta C^\ggg C^\delta$ describe a tri-linear $[\ ,\ ,\ ]$ and so on for multi-brackets of possibly arbitrary length. Moreover, the graded commutativity of the underlying algebra of the $\mathcal{BV}$-complex implies appropriate symmetry of these multi-brackets. The condition that $s^2_\infty = 0$ translates the identities (1) for an L_∞-algebra.

4.4. **The Quantum Master Equation.** ADD Q-ME
include def of BV-alg

5. String field theory

String field theory (SFT) deals with functions on a space of strings, either the space of maps M^I called open strings or the space of maps $\mathcal{L}M = Maps(S^1, M)$ or, rather, its quotient by rotation of S^1, $\mathcal{S}M := \mathcal{L}M/S^1$. The functions, known as em fields may be vector valued or may be sections of a bundle, e.g. differential forms; usually they will form an algebra. Notice that $M^{S^1} \subset M^I$ via the usual quotient map $I \to S^1$ which identifies the end points.

In order to do Lagrangian (or Hamiltonian) field theory, we want the fields to form an algebra reflecting the string structure, namely, as a convolution algebra determined by the decomposition of a string into two, in all possible ways. For example, for
$\phi, \psi \in Fun\mathcal{S}M$, define $\phi \star \psi$ by

$$(\phi \star \psi)(X) = \int \phi(Y)\psi(Z)$$

where the integral is over all $Y, Z \in \mathcal{S}M$ such that $X = Y \star Z$ where $\star$ is the chosen string composition, Poincaré, Moore, Lashof-Witten or HIKKO.

6. Deformation quantization

Although I was unaware of it for decades, at the same time that I was beginning my own research, Murray Gerstenhaber [Ger63] developed a cohomological theory for the deformation of algebras. This provides a bridge between the kind of homotopy theory I have done and mathematical physics via deformation quantization (Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer [BFF$^+$78a, BFF$^+$78b]), though the cohomological aspects were of minor importance in that application originally.

By a formal deformation of an algebra A, we mean an algebra structure on the formal power series ring $A[[t]]$ with coefficients in A which reduces to the algebra structure on A when t is set equal to 0. Thus we can write

$$a * b = \sum t^i m_i(a, b)$$

with $m_0(a, b)$ the original ab.

If the star product is to be associative, m_1 must be a 2-cocycle in the Hochschild cochain complex $C^\bullet(A < A)$ of A with coefficients in itself. Whether or not such a 2-cocyle extends to a full derformation depends on the cohomology class $[m_1] \in H^2(A, A)$. Gerstenhaber introduced his bracket in showing that the Hochschild complex had the structure of a differential graded Lie algebra, if the gradings are shifted by one, and, hence, the total cohomology $H^*(A, A)$ had the structure of a graded Lie algebra, - in other words, again the analog of the Whitehead product algebra. The first obstruction was given by the bracket of $[m_1]$ with itself and higher obstructions by iterated n-ary brackets $[\ , \dots ,\ ]$, the Lie analogs [Ret93] of Massey products [Mas58].

Now back on the physics side, the problem of quantization of $C^\infty(M)$, the algebra of smooth functions on a symplectic manifold, interacted with deformation theory as follows: The Leibniz rule for the Poisson bracket implies that $\{\ ,\ \}$ is a Hochschild 2-cocycle and a candidate infinitesimal deformation. Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer showed that if the Poisson bracket extended to a full associative deformation or *star-product* $\star$ of the commutative product, then one form of 'quantization' could be achieved. Over the past decade, a successful attack leading to the existence of star products on all symplectic manifolds was completed first by DeWilde and Lecomte [DWL83] and refined further by Fedosov [Fed94] and Donin [Don97]. Meanwhile, attention had extended to Poisson manifolds, ones where the smooth functions admitted a bracket satisfying the same formal algebraic properties but without coming from a symplectic 2-form on the manifold. For example, the Lie bracket on a Lie algebra $\mathfrak{g}$ induces a Poisson bracket on the algebra $C^\infty(\mathfrak{g}^*)$. Finally, Kontsevich [Kon97] succeeded in showing that the existence of a star-product for any Poisson manifold (for more general Poisoon algebras, there is a counterexample due to Mathieu [**?**]) and remarkably he succeeded by using integrals on certain compactified configuration spaces (very similar to those used by Zwiebach in his open-closed string field theory [Zwi98]) as well as ideas first developed in rational homotopy theory.

The relevance of L_∞-algebras is the following: As indicated, the Hochschild cochain complex $C(A, A)$ of multilinear maps of $A^{\otimes n}$ to A admits the structure of a dg Lie algebra and hence the Hochschild cohomology admits the structure of a dg Lie algebra with $d = 0$. If there were a map of dg Lie algebras between cohomology and cochains inducing an isomorphism in cohomology, deformation quantization would always be possible once the primary obstruction vanished. No such map is known for a general Poisson bracket even on R^n, but there is a complicated alternative which works well enough. That is, consider a dg Lie algebra as a special case of an L_∞-algebra and allow L_∞-maps. This is what Kontsevich, by a remarkable tour de force, carries out.

Consider a chain map $f : L \to K$ of dg Lie algebras which induces an isomorphism in cohomology but does not respect the brackets strictly but only 'mod an exact term', i.e.

$$f[x, y] = [fx, fy] + dh(x, y) + h(dx, y) + h(x, dy).$$

In other words, h is a homotopy from $f \circ \{\ ,\ \}$ to $\{\ ,\ \} \circ (f \otimes f)$. An L_∞-map consists of a whole family of higher homotopies $h_n : L^{\otimes n} \to K$ which fit together compatibly [**?**]. Kontsevich shows that, for the graded Lie algebra of polyvector fields, $L = \Gamma(M, \Lambda^* TM)$, and K, the graded Lie algebra of polydiffernetial operators (the sub dg Lie algebra of the Hochschild cochain complex

$Hoch(C^\infty(M), C^\infty(M))$ spanned over $C^\infty(M)$ by cochains of the form $\partial_{I_0} \otimes \cdots \otimes \partial_{I_p}$), there is such an L_∞-map. This suffices to show that deformation quantization is always possible once the primary obstruction vanishes. Moreover, the maps h_n can be used used to give a precise formula for the higher order terms m_n. The maps h_n are constructed using differential forms on the compactified moduli spaces of configurations of ordered tuples $(x_1, \ldots, x_p; y_1, \ldots, y_q)$ where the x_i are on the real axis and the y_j on the upper half plane.

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Although Kontsevich's proof does not involve a physicist's vision directly, it is possible to view it that way, as has been worked out by Cattaneo and Felder [CF99], providing a very specific realization of the Batalin-Vilkovisky machinery.

Kontsevich's approach can be stated effectively in terms of a homological algebraic concept called *formality* which arose in rational homotopy theory [**?**, **?**]. It applies to any type of differential graded algebra, but preferably to an algebra A such that $H(A)$ is of the same type, e.g. associative, associative commutative or Lie. Such an algebra is called *formal* if there is an algebra map or even a strongly homotopy mulitplicative map $H(A) \to A$ inducing an isomorphism in homology. The precursor example is provide by symmetric? spaces with A the deRham algebra of differential forms, for then there is such a map given by choosing a harmonic representative in each class (in general, the product of harmonic forms is not harmonic).

Kontsevich proved that arbitrary Poisson manifolds admitted star products by showing that the Hochschild cohomology of the Poisson algebra A of $\mathbb{R}^n$ was formal and then such local star products could be patched together to give a global star product. His work lead to further conjectures and theorems in related mathemaical contexts, though I am unaware of nay with direct physical relevance.

7. Configuration, moduli spaces and operads

Spaces of maps, such as

occur in physics in *sigma models* as do closely related configuration and moduli spaces.

For a general space, I will denote by $Config_n(X)$ the space

$$X^n - \Delta = \{(x_1, \ldots, x_n) | x_i \neq x_j \ if \ i \neq j\}.$$

'Moduli space' will refer to a quotient of a configuration space or to a quotient of a configuration space with 'decorations', e.g. local coordinates at the points. We will have maps $\mathcal{M}_n(X) \to Config_n(X)/\sim$.

A major branch of contemporary mathematical physics, including conformal field theory and topological quantum field theory, is based on algebraic structures parameterized by moduli spaces $\mathcal{M}_{n,g}$ of ordered configurations of points on a Riemann surface of genus g. Of crucial importance is the operation of sewing two such surfaces together, meaning forming a topologist's connected sum but with attention to matching the complex structure in the overlap. If one of the points in a configuration is distinguished as outgoing and sewing is resticted to the outgoing point on the first surface with any of the incoming points on the other surface, the mathematical structure of an *operad* results [**?**, **?**]. This is comparatively straightforward and related to classical algebraic topology for genus 0, i.e. $\mathcal{M}_{n+1} = \mathcal{M}_{n+1,0} = \mathcal{M}_{n+1}(S^2)$ (known in the physics literature as being at *tree level*). Passing to homology of the moduli space gives a clear description of an algebraic structure which is then represented on a 'physical' state space $\mathcal{S}$.

$$H(\mathcal{M}_{n+1}) \to Hom(\mathcal{S}^{\otimes n}, \mathcal{S})$$

That physical state space $\mathcal{S}$ is in turn the homology of a complex $\mathcal{C}$ and the chain level representation (or perhaps *strong homotopy representation* [**?**])

$$C(\mathcal{M}_{n+1}) \to Hom(\mathcal{C}^{\otimes n}, \mathcal{C})$$

is of physical significance, perhaps hidden at the homology level. Particularly striking examples occur in Zwiebach's closed string field theory and more recently in his combined open-closed string field theory [**?**].

OTHER EXAMPLE? ksv

8. Coda

Thus we see a rather intricate interweaving of several kinds of cohomology, including especially that of configuration spaces and moduli spaces, being brought to bear on problems in physics. In turn, physics has provided not only new applications for existing mathematics but also novel new concepts (e.g. Batalin-Vilkovisky algebras) and problems to enrich mathematics.

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