APPENDIX D

Stable complex and Spin^c-structures

In this book, G-manifolds are often equipped with a stable complex structure or a Spin^c structure. Specifically, we use these structures to define quantization. In this appendix we review the definitions and basic properties of these structures. We refer the reader to [**ABS, Du, Fr**] and [**LM**, Appendix D] for alternative introductions to Spin^c-structures. The reader may also consult the early works [**ASi, BH, Bot2**]. In this chapter, all G-actions are assumed to be proper.

We introduce two equivalence relations, that we call *bundle equivalence* and *homotopy*, both for stable complex structures and for Spin^c structures. In the literature, a stable complex structure is usually taken up to bundle equivalence, and a Spin^c-structure is usually taken up to homotopy. This choice leads to problems which we avoid by keeping track of both equivalence relations for both structures. (See Section 3.)

Finally, we note that the notion of a Spin^c structure is essentially equivalent to the notion of a "quantum line bundle" [**Ve4**] and, when the underlying manifold is symplectic, to the notion of an Mp^c ("metaplectic^c") structure [**RR**].

1. Stable complex structures

In this section we discuss stable complex structures and their equivalences.

1.1. Definitions.

Definition D.1. A *stable complex structure* on a real vector bundle E is a fiberwise complex structure on the Whitney sum $E \oplus \mathbb{R}^k$ for some k, where $\mathbb{R}^k$ denotes the trivial bundle with fiber $\mathbb{R}^k$. A stable complex structure on a manifold is a stable complex structure on its tangent bundle.

Suppose that a Lie group G acts on E by bundle automorphisms. For instance, if $E = TM$ is the tangent bundle of a G-manifold M, we take the natural induced action unless stated otherwise. An *equivariant* stable complex structure on E is a stable complex structure such that G acts on $E \oplus \mathbb{R}^k$ by complex bundle automorphisms. Here, $\mathbb{R}^k$ is equipped with the trivial G-action.

Example D.2. An *almost complex structure* on M is a fiberwise complex structure on the tangent bundle TM, i.e., an automorphism of real vector bundles $J \colon TM \to TM$ such that $J^2 =$ identity. This is a special case of a stable complex structure. However, not every stable complex structure arises in this way. For instance, S^5 admits a stable complex structure via $TS^5 \oplus \mathbb{R} = S^5 \times \mathbb{R}^6 = S^5 \times \mathbb{C}^3$, whereas an almost complex manifold must be even dimensional.

Example D.3. A complex manifold is almost complex, and hence stable complex. (Let us recall why: holomorphic coordinates identify each tangent space with $\mathbb{C}^n$; holomorphic transition functions transform these $\mathbb{C}^n$'s by complex linear transformations, by the Cauchy-Riemann equations.)

1.2. Equivalence relations. One usually works with an *equivalence class* of stable complex structures. We have two notions of equivalence: *bundle equivalence* and *homotopy.* For the sake of brevity, we only give the definitions in the presence of a group action. Let E_0 and E_1 be G-equivariant vector bundles over a G-manifold M. Let J_0 and J_1 be stable complex structures, given by G-invariant fiberwise complex structures on $E_0 \oplus \mathbb{R}^k$ and $E_1 \oplus \mathbb{R}^l$.

Definition D.4. The structures J_0 and J_1 are *bundle equivalent* if there exist a and b such that $E_0 \oplus \mathbb{R}^k \oplus \mathbb{C}^a$ and $E_1 \oplus \mathbb{R}^l \oplus \mathbb{C}^b$ are isomorphic as G-equivariant complex vector bundles, where $\mathbb{C}^a$ and $\mathbb{C}^b$ are equipped with trivial G-actions.

Bundle equivalence of stable complex structures is sufficient for most purposes and is sometimes referred to as *equivalence.* (In our papers [**CKT**] and [**GGK2**], a "stable complex structure" meant a bundle equivalence class.) For stable complex structures on the *same* vector bundle we have a second notion of equivalence:

Definition D.5. Suppose that $E_0 = E_1 = E$. The structures J_0 and J_1 are *homotopic* if there exist a and b such that $k+2a = l+2b$ and such that the resulting complex structures on the vector bundle $E \oplus \mathbb{R}^m$, where $m = k+2a = l+2b$, obtained from its identifications with $(E \oplus \mathbb{R}^k) \oplus \mathbb{C}^a$ and with $(E \oplus \mathbb{R}^l) \oplus \mathbb{C}^b$, are homotopic through a family of G-invariant fiberwise complex structures.

Stable complex structures that arise from geometric constructions, such as restricting to a boundary or reducing with respect to a group action are usually defined up to homotopy. See Section 1.3.

The *quantization* of a stable complex manifold (M, J) equipped with a complex line bundle $\mathbb{L}$ is invariant under *both* equivalence relations: homotopy and bundle equivalence. The fact that homotopic J's give the same quantization is a direct consequence of the Fredholm homotopy invariance of the index. This fact is used to show that the quantization of a reduced space is well defined; see Example D.13. On the other hand, to prove that bundle-equivalent stable complex structures have the same quantization, one must invoke the index theorem. See Section 3 of this appendix or Section 7 of Chapter 6 for more details.

Proposition D.6. *Homotopic stable complex structures are bundle equivalent.*

Proof. It is enough to show that if J_t is a smooth family of invariant fiberwise complex structures on a G-equivariant real vector bundle E, then (E, J_0) and (E, J_1) are isomorphic as G-equivariant complex vector bundles.

We view J_t as a fiberwise complex structure on the pull-back $\tilde{E}$ of E to $M \times [0,1]$. This turns $\tilde{E}$ into a G-equivariant complex vector bundle over $M \times [0,1]$ which restricts to (E, J_0) and (E, J_1) on the components of the boundary. Fix a G-invariant connection on $\tilde{E}$. The vector field on $\tilde{E}$ obtained as the horizontal lift of the vector field $\partial/\partial t$ on $M \times [0,1]$ integrates to a family of isomorphisms $(E, J_0) \to (E, J_t)$, $t \in [0,1]$. □

Remark D.7. The same argument applies to reductions of the structure group to any closed subgroup, not necessarily $\mathrm{GL}(n, \mathbb{C}) \subseteq \mathrm{GL}(2n, \mathbb{R})$.

Bundle equivalent stable complex structures need not be homotopic:

EXAMPLE D.8. Consider $M = \mathbb{C}$ with the complex structures $\sqrt{-1}$ and $-\sqrt{-1}$. As stable complex structures, they are bundle equivalent: the tangent bundle $T\mathbb{C}$ with either of these structures is isomorphic to the trivial complex line bundle $M \times \mathbb{C}$. However, these structures are not homotopic; for instance, they induce opposite orientations. (See Section 1.4.)

Moreover, bundle equivalent stable complex structures need not be homotopic even if they induce the same orientation. See Example D.25.

Notice that in Definition D.4 the equivalence relation is taken *with* the group action. It would make no sense to first consider a (non-equivariant) bundle equivalence class and then let a group act. The reason is that a group action on a manifold (or on a vector bundle) induces a group action on the set of stable complex structures but not on the set of their bundle equivalence classes or on a particular class. There is an analogy here with the notion of an equivariant vector bundle over a G-manifold; a lift of a G-action to a bundle is not natural and might not exist.

EXAMPLE D.9. On the manifold $M = \mathbb{C}$, the stable complex structures $\sqrt{-1}$ and $-\sqrt{-1}$ are bundle equivalent. However, if we let $G = S^1$ act by rotations, these structures become non-equivalent. (The isotropy weight at the origin is equal to 1 for the structure $\sqrt{-1}$ and is equal to -1 for the structure $-\sqrt{-1}$. See Proposition D.17.)

Unless stated otherwise, in the presence of a G-action all stable complex structures are assumed to be invariant and all homotopies or bundle equivalences are assumed to be equivariant.

1.3. Geometric constructions. In this section we discuss geometric constructions which gives rise to homotopy classes of stable complex structures: the almost complex structure compatible with a symplectic form, the reduction of a stable complex structure, the restriction to a boundary, and the restriction to a fixed point set and to its normal bundle.

The first important source of homotopy classes of stable complex structures is symplectic manifolds:

DEFINITION D.10. An almost complex structure $J\colon TM \to TM$ is *compatible* with a symplectic form ω if $\langle u, v\rangle := \omega(u, Jv)$ defines a Riemannian metric.

REMARK D.11. It is not hard to see that, if J is an almost complex structure, a (real valued) two-form ω that satisfies any one of the following conditions satisfies them all:

1. $\langle u, v\rangle := \omega(u, Jv)$ is symmetric;
2. $\omega(Ju, Jv) = \omega(u, v)$ for all u, v;
3. ω is a differential form of type $(1, 1)$ with respect to J.

Compatibility means, in addition, that $\langle u, v\rangle$ is positive definite. It implies that ω is non-degenerate.

EXAMPLE D.12. On a symplectic G-manifold (M, ω) there exists an equivariant almost complex structure J compatible with ω, unique up to homotopy.

See [**Ste**, Section 41], [**Wei1**, Lecture 2], or [**McDSa**, Sections 2.5 and 4.1].

SKETCH OF PROOF. Given an arbitrary Riemannian metric on M, define an operator A by the condition $\langle u, v\rangle = \omega(u, Av)$. Then $J = A(-A^2)^{-\frac{1}{2}}$ is an almost complex structure compatible with ω. In this way, we obtain a continuous map $\langle\, ,\, \rangle \mapsto J$ from the space of Riemannian metrics onto the space of almost complex structures compatible with ω. The former is non-empty and convex, and hence connected. As a consequence, the latter is non-empty and connected, which proves the claim. In the presence of a G-action preserving the symplectic structure, we use the same argument, but starting with the space of G-invariant metrics. □

Stable complex structures arise when we consider reduction:

EXAMPLE D.13. Let $\Psi\colon M \to \mathfrak{g}^*$ be an (abstract) moment map. An invariant almost complex structure J on M generally does not induce an almost complex structure on the reduced space $M_{\mathrm{red}} = \Psi^{-1}(0)/G$, unless Ψ is a moment map for an genuine symplectic form ω and J is compatible with ω. (See [**CKT**] for conditions under which J descends to M_{red}, when $G = S^1$.) However, J does induce a *stable complex structure* on M_{red}, unique up to homotopy. See Section 2.3 of Chapter 5 for details. Moreover, homotopic or bundle equivalent structures on M induce, respectively, homotopic, or bundle equivalent, structures on M_{red}. Finally, because quantization is defined for a *homotopy* class (or, more generally, for a bundle equivalence class) of stable complex structures, the quantization of the reduced space is well defined.

Another important reason to consider stable complex structures is that the notion of a *cobordism* of such structures is particularly simple. This notion relies on the crucial observation that a stable complex structure on a manifold induces one on its boundary:

PROPOSITION D.14. *Let M be a G-manifold with boundary ∂M. An equivariant stable complex structure on M induces one on ∂M, which is canonical up to homotopy. Homotopic or bundle equivalent, equivariant stable complex structures on M induce, respectively, homotopic or bundle equivalent, equivariant stable complex structures on ∂M.*

PROOF. We have a short exact sequence of vector bundles over the boundary,

$$0 \to T(\partial M) \to TM|_{\partial M} \to N(\partial M) \to 0, \tag{D.1}$$

where $T(\partial M)$ is the tangent bundle to the boundary and where

$$N(\partial M) = (TM|_{\partial M})\,/T(\partial M)$$

is the normal bundle to the boundary.

The normal bundle to the boundary is a one-dimensional real vector bundle, oriented by choosing the "outward" direction to be positive. This determines an isomorphism with the trivial vector bundle, $N(\partial M) \cong \partial M \times \mathbb{R}$, and the isomorphism is unique up to homotopy.

The sequence (D.1) splits, and the splitting is unique up to homotopy. (A short exact sequence of equivariant vector bundles $0 \to A \to B \to C \to 0$ always splits, and any two splittings are homotopic: the space of all splittings can be (non-canonically) identified with the space of sections of the vector bundle $\mathrm{Hom}_G(C, A)$, and this space is connected.)

We obtain an isomorphism

$$TM|_{\partial M} = \mathbb{R} \oplus T(\partial M),$$

which is canonical up to homotopy. The lemma follows. □

REMARK D.15. One can also define cobordisms of *almost complex* structures on manifolds. The structure required on a cobording $(2n+1)$-dimensional manifold is then a reduction ("tangent" to the boundary) of the structure group of the tangent bundle to $\mathrm{GL}(n, \mathbb{C})$. As usual, all manifolds are assumed to be oriented. In the non-equivariant case, the resulting cobordism ring is in fact equal to the cobordism ring of stable complex structures, as follows, e.g., from the results of [**Bak, Gin2, Mor**]. However, stable complex cobordisms are considerably more tracktable and for many purposes more natural objects than their almost complex counterparts.

In this book we need to consider fixed point sets and their normal bundles.

Let an abelian Lie group G act properly on a manifold M. Let F be a connected component of the fixed point set M^H for some closed subgroup H of G. Recall that F is a closed submanifold of M, on which G acts (non-effectively, with H acting trivially), and the normal bundle $NF = (TM|_F)/TF$ is a G-equivariant real vector bundle over F.

PROPOSITION D.16. *An equivariant stable complex structure on M induces an equivariant stable complex structure on F and a fiberwise complex structure on the normal bundle NF such that G acts on NF by complex bundle automorphisms.*

Homotopic, or bundle equivalent, structures on M induce homotopic, or bundle equivalent, structures on F and NF.

COROLLARY D.17. Let G be a torus and let M be a stable complex G-manifold. At a fixed point for the G-action, the non-zero isotropy weights are well defined, even if the stable complex structure is defined only up to equivalence.

PROOF OF COROLLARY D.17. Let $F \subseteq M^G$ be a connected component of the fixed point set. The non-zero isotropy weights at a point $p \in F$ carry exactly the same information as the G-action on the complex vector space N_pF. □

PROOF OF PROPOSITION D.16. Let J be a fiberwise complex structure on $TM \oplus \mathbb{R}^k$. Then $TM|_F \oplus \mathbb{R}^k$ is a G-equivariant complex vector bundle over F. The sub-bundle $TF \oplus \mathbb{R}^k$ is G-invariant and complex, because it consists precisely of those vectors that are fixed by H. The normal bundle $NF = TM|_F/TF$ is naturally isomorphic to the quotient $(TM|_F \oplus \mathbb{R}^k)/(TF \oplus \mathbb{R}^k)$, making it into a complex vector bundle. Because an equivariant homotopy or bundle equivalence of complex structures must preserve the sub-bundle of H-fixed vectors, the effects of these equivalences on the structures on F and on NF are as stated. □

In particular, let
$$\pi \colon E \to F$$
be a G-equivariant real vector bundle, and suppose that there exists a subgroup $H \subseteq G$ whose fixed point set E^H is precisely the zero section F. An equivariant stable complex structure J on the total space of E gives rise to an equivariant stable complex structure J_F on F and a fiberwise complex structure J_f on E. The converse is also true:

PROPOSITION D.18. *An equivariant stable complex structure J_F on F and an invariant fiberwise complex structure J_f on E determine an equivariant stable complex structure J on the total space E, defined up to homotopy, which, in turn, induces the structures J_F and J_f. If J_f and J_F are defined up to homotopy, or bundle equivalence, so is J.*

PROOF. We have a short exact sequence of bundles over E,

$$0 \to \pi^* E \to TE \to TF \to 0.$$

We have a complex structure on $\pi^* E$ coming from J_f, and a stable complex structure on TF coming from J_F. A splitting of the sequence gives an isomorphism

$$TE \cong \pi^* E \oplus TF,$$

unique up to homotopy. We set $J = J_f \oplus J_F$. □

Propositions D.16 and D.18 immediately imply the following result, which is used to specify the right-hand side of the stable complex "Linearization Theorem" in Chapter 4.

PROPOSITION D.19. *Let J be an equivariant stable complex structure on M and let F be a connected component of the set M^H of fixed points for some subgroup H of G. Then there is a unique up to homotopy stable complex structure J_F on the total space NF which induces the same stable complex structure on F and fiberwise complex structure on NF as those induced by J. Homotopic or bundle equivalent, structures on M induce homotopic, or bundle equivalent, structures on NF.*

1.4. Orientations and Chern numbers. A stable complex structure induces an orientation, obtained as the "difference" of the complex orientation on $E \oplus \mathbb{R}^k$ and the standard orientation on $\mathbb{R}^k$. Homotopic stable complex structures determine the same orientation. However, bundle equivalent stable complex structures need not induce the same orientation. So, for instance, a manifold equipped with a stable complex structure up to bundle equivalence is *orientable*, but is not naturally *oriented*.

For instance, the complex structures $\sqrt{-1}$ and $-\sqrt{-1}$ on $\mathbb{C}$ induce opposite orientations although they are bundle equivalent (cf. Example D.8). More generally,

LEMMA D.20. *Every stable complex structure J is bundle equivalent to a stable complex structure J' which induces the opposite orientation.*

PROOF. Let J be a fiberwise complex structure on $E \oplus \mathbb{R}^k$. Let $J' = J \oplus -\sqrt{-1}$ on $E \oplus \mathbb{R}^k \oplus \mathbb{C}$. Then J' is isomorphic to $J \oplus \sqrt{-1}$ (via conjugation of the last factor), but J and J' induce opposite orientations. □

An *oriented bundle equivalence* between stable complex structures J_0 and J_1 defined on $E \oplus \mathbb{R}^k$ and on $E \oplus \mathbb{R}^l$ is an isomorphism of G-equivariant complex vector bundles $E \oplus \mathbb{R}^k \oplus \mathbb{C}^a$ and $E \oplus \mathbb{R}^l \oplus \mathbb{C}^b$ which is fiberwise orientation preserving. (This is well defined even if E is not a priori oriented.) If J_0 and J_1 induce opposite orientations on E, an oriented bundle equivalence between them does not exist. If they induce the same orientation, an oriented bundle equivalence between them is the same thing as a bundle equivalence. In particular, if J_0 and J_1 are homotopic, there exists an oriented bundle equivalence between them.

From Lemma D.20 we conclude that an oriented bundle equivalence class carries exactly the same information as a bundle equivalence class plus an auxiliary orientation.

Bundle equivalent (equivariant) stable complex structures have the same (equivariant) Chern classes. To integrate these classes and obtain characteristic *numbers*, one also needs an *orientation of the manifold*. We specialize to $E = TM$, so that an orientation of M is the same thing as a fiberwise orientation of E. It is then convenient to fix an orientation $\mathcal{O}$ of M as an additional structure. This is the approach often taken in complex cobordism theory. (See, e.g., [**Sto, Rudy**], and [**May**].)

EXAMPLE D.21. In Example 5.5 of Chapter 5 we described non-standard stable complex structures on $\mathbb{CP}^n$ whose Chern classes are different from those of any almost complex structure. We deduce that these stable complex structures are not even bundle equivalent to almost complex structures.

Let M be an oriented manifold. Let $\mathcal{J}$ denote the set of stable complex structures on M. Let $\mathcal{J}_0$ denote the set of stable complex structures that induce the given orientation on M. Then the natural map

$$\mathcal{J}_0/\text{oriented bundle equivalence} \to \mathcal{J}/\text{bundle equivalence}$$

is a bijection, as follows from the above discussion. The natural map

$$\mathcal{J}_0/\text{homotopy} \to \mathcal{J}/\text{bundle equivalence} \tag{D.2}$$

is well defined and onto, but is not always one-to-one. See Example D.25.

From now on we will consider oriented manifolds, equipped with compatible stable complex structures, up to oriented bundle equivalence.

1.5. Oriented complex cobordisms. We now define cobordisms of oriented stable complex G-manifolds. In this context, stable complex structures are taken up to bundle equivalence.

DEFINITION D.22 (Oriented complex cobordism). Let M_0 and M_1 be oriented stable complex G-manifolds. An *oriented complex cobordism* between these manifolds is an oriented stable complex G-manifold with boundary W and a diffeomorphism of ∂W with $-M_0 \bigsqcup M_1$ which transports the stable complex structure on ∂W to structures on M_0 and M_1 that are bundle equivalent to the given ones. (The minus sign indicates that the diffeomorphism reverses the orientation of M_0 and preserves the orientation of M_1.)

Cobordant oriented stable complex manifolds have the same characteristic numbers. This follows immediately from Stokes's theorem. The converse is also true: if two oriented stable complex manifolds have the same characteristic numbers, then there exists an oriented complex cobordism between them, by a theorem of Milnor and Novikov. See [**MiSt, Sto**].

1.6. Relations with other definitions of a stable complex structure. In the literature one encounters several notions of stable complex structures. For example, weak complex structures from [**BH**] are bundle equivalence classes, whereas weakly complex structures from [**CF1, CF2**] are homotopy classes. Stable complex structures are sometimes also referred to as stable almost complex structures [**May, Rudy**], weakly almost complex structures [**BH**], or U-structures. Cobordisms of stable complex structures are called complex cobordisms, or unitary cobordisms.

In algebraic topology, a stable complex structure is usually defined on the *stable normal* bundle to M (see, e.g., [**Sto**] and [**Rudy**]). We, as, e.g., in [**May**], define stable complex structures on G-manifolds to be on *stable tangent* bundles. In the non-equivariant setting, the two approaches are equivalent. This is no longer true in the equivariant case. (See [**May**], p. 337 for details.) An equivariant stable complex structure on M gives rise to an equivariant complex structure on the stable normal bundle to M, but the converse is not true. The key feature of Definition D.1 is that the group action on the "stabilizing" part of the stable tangent bundle is required to be trivial. Without this requirement, the notions of equivariant stable complex structures on the tangent and stable normal bundles would be essentially equivalent. However, some of the properties of equivariant stable complex structures would be lost; for instance, Proposition D.16 and Corollary D.17 would not hold for "normal" equivariant stable complex structures, and many applications, such as in [**GGK2**, Section 4], would not be true for manifolds with stable complex structures on the normal bundle.

1.7. Stable complex structures on spheres. In conclusion we give some examples of classification results for stable complex structures on manifolds. In the non-equivariant case these classification questions are handled using homotopy theory and, in particular, obstruction theory (see, e.g., [**Hu**]). Below we outline the solutions in the simplest case, where the manifold is a sphere. We restrict our attention to stable complex structures compatible with a fixed orientation, as explained in Section 1.4.

Let us first recall some general facts and set notation to be used later on.

Let V_n be the space of complex structures on $\mathbb{R}^{2n}$ compatible with a fixed orientation. It is easy to see that $V_n = \mathrm{SO}(2n)/\mathrm{U}(n)$. The stable homotopy groups of V_n are as follows (see [**Mas1**]). (Here, $q > 0$ and n is large enough so that $2n-1 > q$.)

$$\pi_q(V_n) = \begin{cases} \mathbb{Z} & \text{for } q \equiv 2 \mod 4; \\ \mathbb{Z}_2 & \text{for } q \equiv 0,\ 7 \mod 8; \\ 0 & \text{otherwise.} \end{cases}$$

The natural inclusion $V_n \to V_{n+1}$ induces an isomorphism of these stable homotopy groups.

As usual, let $B\,\mathrm{SO}(2n)$ and $B\,\mathrm{U}(n)$ denote the classifying spaces for $\mathrm{SO}(2n)$ and $\mathrm{U}(n)$. The inclusion $\mathrm{U}(n) \to \mathrm{SO}(2n)$ induces a map $B\,\mathrm{U}(n) \to B\,\mathrm{SO}(2n)$ which is a fiber bundle with fiber V_n. The inclusion of V_n into $B\,\mathrm{U}(n)$ as a fiber induces a map $\pi_q(V_n) \to \pi_q(B\,\mathrm{U}(n))$. Denote the image of this map by Γ_q. Here, as above, we assume that n is large enough for a fixed q so that Γ_q is independent of n.

PROPOSITION D.23. *Stable complex structures on S^q compatible with a fixed orientation are classified by the elements of $\pi_q(V_n)$ up to homotopy and by the elements of Γ_q up to bundle equivalence. The projection $\pi_q(V_n) \to \Gamma_q$ is the natural map between equivalence classes (cf. Proposition D.6).*

PROOF. For any q, the tangent bundle TS^q is stably trivial. In fact, $TS^q \oplus \mathbb{R} = S^q \times \mathbb{R}^{q+1}$. It follows that homotopy classes of stable complex complex structures on S^q, compatible with a given orientation, are in a one-to-one correspondence with $\pi_q(V_n)$, where n is large enough.

Isomorphism classes of real oriented $2n$-dimensional vector bundles over M are classified by homotopy classes of maps $M \to B\,\mathrm{SO}(2n)$, [**Hus**]. Likewise, complex n-dimensional vector bundles are classified by the homotopy classes of maps $M \to B\,\mathrm{U}(n)$. Forgetting the complex structure on a vector bundle corresponds to the map $B\,\mathrm{U}(n) \to B\,\mathrm{SO}(2n)$. Fiber bundle equivalence classes of complex structures on an oriented real vector bundle are described by the homotopy classes of lifts of $M \to B\,\mathrm{SO}(2n)$ to $M \to B\,\mathrm{U}(n)$. Specializing this to the case of stable complex structures on S^q we note that the classifying map $S^q \to B\,\mathrm{SO}(2n)$ is contractible. In other words, this map represents $0 \in \pi_q(B\,\mathrm{SO}(2n))$. As follows from the long exact sequence

$$\ldots \to \pi_q(V_n) \to \pi_q(B\,\mathrm{U}(n)) \to \pi_q(B\,\mathrm{SO}(2n)) \to \ldots$$

of the homotopy groups, the lifts of a contractible map are classified by the image Γ_q of $\pi_q(V_n)$.

We leave the proof of the last assertion to the reader as an exercise. □

Example D.24 (Stable complex structures on S^2). Stable complex structures on S^2, compatible with a fixed orientation, are classified by $\mathbb{Z}$ up to either homotopy or bundle equivalence, and these two equivalence relations are identical on S^2. Indeed, for $q = 2$, the natural map $\mathbb{Z} = \pi_2(V_n) \to \Gamma_2$ is an isomorphism.

Example D.25 (Stable complex structures on S^7 and S^8). The first sphere for which the two classifications are inequivalent is S^7. Applying Proposition D.23 to this sphere, we see that up to homotopy the stable complex structures on S^7 are classified by $\pi_7(V_n) = \mathbb{Z}_2$. Up to bundle equivalence, stable complex structures on S^7 are classified by Γ_7 which is zero, for $\pi_7(B\,\mathrm{U}(n)) = \pi_6(\mathrm{U}(n)) = 0$. Similarly, the two classifications are different for S^8, where $\pi_8(V_n) = \mathbb{Z}_2$ and $\Gamma_8 = 0$. (Indeed, Γ_8 is the image of $\mathbb{Z}_2$ in $\pi_8(B\,\mathrm{U}(n)) = \pi_7(\mathrm{U}(n)) = \mathbb{Z}$.)

Remark D.26. Calculations similar to Examples D.24 and D.25 also show that the two classifications are equivalent for all S^q with $q \leq 6$. Combining this with elementary obstruction theory (see, e.g., [**Hu**]), one can show that the two classifications of non-equivariant stable complex structures on M (compatible with a fixed orientation) are equivalent when $\dim M \leq 6$.

Classification problems for stable complex structures become more subtle in the equivariant setting.

Example D.27 (Equivariant stable complex structures on S^2). Consider the standard action of $G = S^1$ on S^2 by rotations about the z-axis. Then any two G-equivariant stable complex structures on S^2, compatible with a fixed orientation, are equivariantly homotopy equivalent and hence also bundle equivalent.

Proof. Let J be a G-equivariant complex structure on $TS^2 \oplus \mathbb{R}^k$ compatible with the orientation. If p is the north or the south pole, we have an equivariant decomposition of the stable tangent space as $T_pS^2 \oplus \mathbb{R}^k$, where $\mathbb{R}^k$ is fixed by G, and both components are J-complex subspaces. Since J is compatible with the orientation, J is equivariantly homotopic to a stable complex structure which has some standard form near the poles. Hence, without loss of generality, we can assume that J has such a form near the poles.

Let us fix a trivialization of $TS^2 \oplus \mathbb{R}^k$ along the arc C of a meridian connecting the two poles. Then J is completely determined by its values along C. Now it is easy to see that equivariant homotopy classes of stable complex structures on S^2 are in a one-to-one correspondence with homotopy classes of the maps of C to V_n with fixed end-points. This set is essentially $\pi_1(V_n) = 0$, which implies the assertion of Example D.27. □

2. Spin^c-structures

2.1. The definition of Spin^c-structures. Recall that the group $\mathrm{Spin}(n)$ is the connected double covering of the group $\mathrm{SO}(n)$. (In fact, $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$, so $\mathrm{Spin}(n)$ is the universal covering of $\mathrm{SO}(n)$, when $n > 2$.) Let

$$q\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$$

denote the covering map, and let ϵ denote the non-trivial element in the kernel of this map. Then

$$\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} \mathrm{U}(1)$$

is the quotient of $\mathrm{Spin}(n) \times \mathrm{U}(1)$ by the two element subgroup generated by $(\epsilon, -1)$. In other words, elements of $\mathrm{Spin}^c(n)$ are equivalence classes $[s, c]$, where $s \in \mathrm{Spin}(n)$, $c \in \mathrm{U}(1)$, and $[s\epsilon, c] = [s, -c]$. The projections to the two factors give rise to two natural homomorphisms,

$$\pi\colon \mathrm{Spin}^c(n) \to \mathrm{SO}(n), \qquad [s, c] \mapsto q(s), \tag{D.3}$$

and

$$\det\colon \mathrm{Spin}^c \to \mathrm{U}(1), \qquad [s, c] \mapsto c^2, \tag{D.4}$$

which give rise to short exact sequences

$$1 \to \mathrm{U}(1) \to \mathrm{Spin}^c(n) \xrightarrow{\pi} \mathrm{SO}(n) \to 1$$

and

$$1 \to \mathrm{Spin}(n) \to \mathrm{Spin}^c(n) \xrightarrow{\det} \mathrm{U}(1) \to 1.$$

Let $E \to M$ be a real vector bundle of rank n. On the conceptual level, a Spin^c-structure on E is an "extension" of its structural group to $\mathrm{Spin}^c(n)$. More precisely, let $\mathrm{GL}(E)$ denote the frame bundle of E, that is, the principal $\mathrm{GL}(n)$-bundle whose fiber over $p \in M$ is the set of bases of the vector space E_p. The group $\mathrm{Spin}^c(n)$ acts on $\mathrm{GL}(E)$ through the composition $\mathrm{Spin}^c(n) \xrightarrow{\pi} \mathrm{SO}(n) \hookrightarrow \mathrm{GL}(n)$.

Definition D.28. A Spin^c-structure on E is a principal $\mathrm{Spin}^c(n)$-bundle

$$P \to M$$

together with a $\mathrm{Spin}^c(n)$-equivariant bundle map

$$p\colon P \to \mathrm{GL}(E).$$

A Spin^c structure on a manifold M is a Spin^c structure on its tangent bundle $E = TM$.

Although a Spin^c-structure is more than just a principal bundle P, we will often refer to a Spin^c-structure (P, p) as simply P.

The map p determines an isomorphism of vector bundles,

$$P \times_\pi \mathbb{R}^n \cong E, \tag{D.5}$$

and vice versa: an isomorphism (D.5) determines an equivariant map $p\colon P \to \mathrm{GL}(E)$. The standard metric and orientation on $\mathbb{R}^n$ transport to E through (D.5), such that the bundle $\mathrm{SO}(E)$ of oriented orthogonal frames is precisely the image of the map $p\colon P \to \mathrm{GL}(E)$. In particular, a vector bundle that admits a Spin^c structure is *orientable*.

In the literature, a Spin^c structure is commonly defined for a vector bundle which is *a priori* equipped with a fiberwise metric and orientation. We will consider this notion under a slightly different name:

DEFINITION D.29. Let $E \to M$ be a vector bundle. If E is equipped with a fiberwise orientation, an *oriented* Spin^c-*structure* on E is a Spin^c structure which induces the given orientation. If E is equipped with a fiberwise metric, a *metric* Spin^c-*structure* on E is a Spin^c structure which induces the given metric.

Hence, Spin^c structures as are usually referred to in the literature are what we call *oriented metric* Spin^c structures.

One may also discard the metric and orientation completely; this is achieved through the notion of an ML^c ("metalinear^c") structure, which we discuss in Section 2.6. This notion is superior to others in that it requires the least *a priori* structure. *Our* definition of a Spin^c structure is an intermediate notion, which, on one hand, avoids an *a priori* choice of metric and orientation, but does give rise to these structures. One advantage of this notion is purely psychological, in that it allows us to keep the common name "Spin^c". In Sections 2.4, 2.5, and 2.6, we will see that these different notions of Spin^c structures are essentially equivalent. In practice, one may work with any one of these notions.

The *determinant line bundle* associated with the Spin^c-structure is the complex line bundle

$$\mathbb{L}_{\det} = P \times_{\det} \mathbb{C}$$

over M, associated through the homomorphism (D.4).

We may think of P as a circle bundle over $\mathrm{SO}(E)$, with $p\colon P \to \mathrm{SO}(E)$ being the projection map.

LEMMA D.30. *The associated line bundle, $P \times_{\mathrm{U}(1)} \mathbb{C} \to \mathrm{SO}(E)$, is a square root of the pullback to $\mathrm{SO}(E)$ of the determinant line bundle $\mathbb{L}_{\det}$.*

We leave the proof as an exercise to the reader.

Because the subgroup $\mathrm{U}(1)$ is the center of $\mathrm{Spin}^c(n)$, its right action on P commutes with the entire principal $\mathrm{Spin}^c(n)$ action. Therefore, P is a $\mathrm{Spin}^c(n)$-*equivariant* circle bundle over $\mathrm{SO}(E)$. Conversely, a $\mathrm{Spin}^c(n)$ equivariant circle bundle over $\mathrm{SO}(E)$ is a Spin^c structure on E if $K := \ker \pi \cong \mathrm{U}(1)$ acts by the principal action.

PROPOSITION D.31. *An oriented vector bundle E over M admits a Spin^c-structure if and only if its second Stiefel-Whitney class $w_2(E)$ is integral, i.e., lies in the image of the homomorphism*

$$\rho\colon H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}_2).$$

We refer the reader to [**FF**] or [**Fr**] for the proof. The definitions of the Stiefel-Whitney class $w_2(E)$ and the Chern class $c_1(\mathbb{L})$ can be found in [**MiSt**].

One may attempt to define an equivalence of Spin^c structures (P, p) and (P', p') to be a principal bundle map $F\colon P \to P'$ which respects the maps to $\mathrm{GL}(E)$. Spin^c structures that are equivalent in this sense must induce the same metric and orientation. In fact, this will be our notion of equivalence of *metric* Spin^c structures in Section 2.4. We have formulated our definition of a Spin^c structure in such a way as to *avoid* fixing a metric and an orientation, and we will work with notions of equivalence which allow *different* metrics or orientations. In the next sections we introduce two such notions, which we call *bundle equivalence* and *homotopy* of Spin^c structures. In the literature, it is often unclear what equivalence relation is taken, and this results in seemingly contradictory classification results for Spin^c-structures. We outline the homotopy classification of Spin^c structures in Section 2.7.

Finally, we note that the *quantization* of a Spin^c structure (see Section 3 of this appendix or Section 7 of Chapter 6) is invariant under *both* equivalence relations: homotopy and bundle equivalence. The fact that homotopic Spin^c structures give the same quantization is a direct consequence of the Fredholm homotopy invariance of the index. On the other hand, to prove that bundle-equivalent Spin^c structures have the same quantization one must invoke the index theorem.

2.2. Bundle equivalence of Spin^c-structures. The simplest equivalence relation between Spin^c-structures is the equivalence of principal bundles:

DEFINITION D.32. Let (P, p) and (P', p') be Spin^c-structures on vector bundles E and, respectively, E' over M. These structures are *bundle equivalent* if P and P' are equivalent as Spin$^c(n)$-principal bundles.

REMARK D.33. An isomorphism of principal bundles $F\colon P \to P'$ gives rise to a unique isomorphism of principal bundles $f\colon \mathrm{GL}(E) \to \mathrm{GL}(E')$ such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{F} & P' \\ {\scriptstyle p}\downarrow & & {\scriptstyle p'}\downarrow \\ \mathrm{GL}(E) & \xrightarrow{f} & \mathrm{GL}(E'). \end{array} \tag{D.6}$$

Indeed, f is determined by

$$\mathrm{GL}(E) \overset{p}{\cong} P \times_{\mathrm{Spin}^c(n)} \mathrm{GL}(n) \xrightarrow{F} P' \times_{\mathrm{Spin}^c(n)} \mathrm{GL}(n) \overset{p'}{\cong} \mathrm{GL}(E').$$

Similarly, if two Spin^c-structures are bundle equivalent, the underlying vector bundles E and E' are isomorphic, via

$$E \cong P \times_{\mathrm{Spin}^c(n)} \mathbb{R}^n \to P' \times_{\mathrm{Spin}^c} \mathbb{R}^n \cong E'.$$

Note that we do not need E and E' to be the *same* vector bundle, and that we impose no restriction on the isomorphism f in (D.6). We will later encounter stricter notions of equivalence, which apply to the case $E = E'$, and in which we insist that f be equal to, or homotopic to, the identity map.

REMARK D.34. A Spin$^c(n)$-principal bundle can also be described in terms of transition functions in the usual way. Fix a cover of M by contractible open sets U_i. A Spin^c-principal bundle P is given by a collection of functions $\varphi_{ij}\colon U_{ij} = U_i \cap U_j \to \mathrm{Spin}^c(n)$ such that the cocycle condition holds, i.e., $\varphi_{ij}\varphi_{jk}\varphi_{ki} = 1$. Then the vector bundle E is determined by the $\mathrm{GL}^+(n)$-valued cocycle $\pi\varphi_{ij}$. Two cocycles φ_{ij} and ψ_{ij} give rise to bundle equivalent Spin^c-structures on E if and only if these cocycles differ by a coboundary: $\psi_{ij} = f_i \varphi_{ij} f_j^{-1}$ for some collection of functions $f_i\colon U_i \to \mathrm{Spin}^c(n)$. The determinant line bundle of P is given by the U(1)-cocycle $\det \varphi_{ij}$.

REMARK D.35. Recall that every Spin$^c(n)$-bundle over a manifold is a pullback of the universal Spin$^c(n)$-bundle, $E\,\mathrm{Spin}^c(n) \to B\,\mathrm{Spin}^c(n)$, through a map to the classifying space $B\,\mathrm{Spin}^c(n)$, and this map is unique up to homotopy. Two Spin^c-structures are bundle equivalent exactly if the corresponding maps to the classifying space are homotopic. This notion of homotopy is different (in fact, weaker) than the notion of homotopy that will be introduced in the next section.

Remark D.36. The real (or, equivalently, rational) characteristic classes of $\mathrm{Spin}^c(n)$-structures are described by the cohomology ring $H^*(B\,\mathrm{Spin}^c(n))$. For n even, which is the case of most interest for us, this ring (over $\mathbb{R}$) is the polynomial ring with generators c_1 (of degree two), e of degree $n = 2m$, and $p_1, \dots, p_m$ with $\deg p_j = 4j$; see, e.g., [**FF**]. The class c_1 corresponds to the first Chern class of $\mathbb{L}_{\det}$, the class e corresponds to the Euler class of E, and the classes p_j are the Pontrjagin classes. As a consequence, for Spin^c-structures on an even–dimensional manifold, all characteristic classes but c_1 are determined by the topology of M and hence are independent of the Spin^c-structure.

The notion of bundle equivalence is most natural in the topological context. The characteristic classes of (P, p) are entirely determined by P and thus are invariants of bundle equivalence.

2.3. Homotopy of Spin^c-structures. For Spin^c structures over a *fixed* vector bundle E one has the following equivalence relation, which is finer than bundle equivalence:

Definition D.37. Spin^c structures (P, p) and (P', p') over a vector bundle E are *homotopic* if there exists a principal bundle isomorphism

$$F\colon P \to P'$$

and a smooth family of Spin^c-equivariant maps

$$p_t\colon P \to \mathrm{GL}(E),$$

for $t \in [0, 1]$, such that $p_0 = p$ and $p_1 = p' \circ F$:

$$\begin{array}{ccc} P & \xrightarrow{F} & P' \\ \Big\downarrow{\scriptstyle p\sim p'\circ F} & & \Big\downarrow{\scriptstyle p'} \\ \mathrm{GL}(E) & & \mathrm{GL}(E). \end{array}$$

We have the following useful characterization of homotopy:

Proposition D.38. *Let E be a vector bundle. Two Spin^c structures (P, p) and (P', p') over E are homotopic if and only if there exists a principal bundle isomorphism $F\colon P \to P'$ such that the induced automorphism $f\colon \mathrm{GL}(E) \to \mathrm{GL}(E)$ is homotopic to the identity through bundle automorphisms:*

$$\begin{array}{ccc} P & \xrightarrow{F} & P' \\ {\scriptstyle p}\Big\downarrow & & {\scriptstyle p'}\Big\downarrow \\ \mathrm{GL}(E) & \xrightarrow{f\sim identity} & \mathrm{GL}(E). \end{array}$$

Moreover, it is enough to assume that f is homotopic to the identity through $\mathrm{SO}(n)$-equivariant maps.

Proof. Let $F\colon P \to P'$ be an isomorphism of principal bundles. Let f be the unique automorphism of $\mathrm{GL}(E)$ such that $f \circ p = p' \circ F$, as in (D.6).

Suppose that there exists a family of Spin^c-equivariant maps $p_t\colon P \to \mathrm{GL}(E)$ such that $p_0 = p$ and $p_1 = p' \circ F$. Let $f_t\colon \mathrm{GL}(E) \to \mathrm{GL}(E)$ be the unique automorphism such that the following diagram commutes:

$$\text{(D.7)} \qquad \begin{array}{ccc} P & \xrightarrow{F} & P' \\ {\scriptstyle p_t}\big\downarrow & & {\scriptstyle p'}\big\downarrow \\ \mathrm{GL}(E) & \xrightarrow{f_t} & \mathrm{GL}(E). \end{array}$$

(See Remark D.33.) Then, by uniqueness, $f_0 = f$ and $f_1 =$ identity.

On the other hand, suppose that there exists a family of $\mathrm{SO}(n)$-equivariant maps f_t of $\mathrm{GL}(E)$ such that $f_0 = f$ and $f_1 =$ identity. Let $p_t = f_t^{-1} \circ p' \circ F$. Then $p_t\colon P \to \mathrm{GL}(E)$ is $\mathrm{Spin}^c(n)$-equivariant, $p_0 = p$, and $p_1 = p' \circ F$. □

Clearly, if two Spin^c structures on E are homotopic, they are also bundle equivalent.

Finally, we note that the definitions of Spin^c structures and their equivalences naturally extend to equivariant structures in the presence of a proper G-action. Various claims that we have made regarding these objects have equivariant analogues. In fact, the same proofs work, with the word "equivariant" or "invariant" inserted wherever appropriate. Note that some of these proofs rely on the existence of an invariant fiberwise metric on a G-equivariant vector bundle E, or the existence of an equivariant connection on a G-equivariant principal bundle. Properness of the action is needed to guarantee that such metrics or connections do exist. See Section 3.2 of Appendix B for how to "average" with respect to a proper group action.

2.4. Metric Spin^c structures. Let $E \to M$ be a vector bundle. Recall that a Spin^c structure on E is a principal Spin^c bundle $P \to M$ together with a $\mathrm{Spin}^c(n)$ equivariant bundle map $p\colon P \to \mathrm{GL}(E)$. If E is *a priori* equipped with a fiberwise orientation or metric, recall that an oriented, resp., metric Spin^c structure is one which induces the given orientation, resp., the given metric.

Let E be a vector bundle with a fiberwise metric and let $\mathrm{O}(E)$ be its orthonormal frame bundle. An *equivalence* of metric Spin^c structures (P, p) and (P', p') over E is an isomorphism $F\colon P \to P'$ which lifts the identity map on $\mathrm{O}(E)$:

$$\begin{array}{ccc} P & \xrightarrow{F} & P' \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p'}\big\downarrow \\ \mathrm{O}(E) & \xrightarrow{\text{identity}} & \mathrm{O}(E) \end{array}$$

Spin^c structures up to homotopy are the same as metric Spin^c structures up to equivalence:

PROPOSITION D.39. *Let $E \to M$ be a vector bundle with a fiberwise metric. Then:*

(1) *Every* Spin^c*-structure is homotopic to a metric* Spin^c*-structure.*
(2) *Two metric* Spin^c*-structures are equivalent if and only if they are homotopic.*

Because homotopy preserves orientation, and so does equivalence of metric Spin^c structures, it is enough to prove, for any pre-chosen orientation and metric on E,

(1') Every oriented Spin^c-structure is homotopic to an oriented metric Spin^c-structure.
(2') Two oriented metric Spin^c-structures are equivalent if and only if they are homotopic.

Proof of (1'): Let $\mathrm{GL}^+(E)$ denote the oriented frame bundle and $\mathrm{SO}(E)$ the oriented orthogonal frame bundle of E. Let (P, p') be an oriented Spin^c structure and $\mathrm{SO}(E')$ the oriented orthonormal frame bundle for the induced metric. Metrics on E exactly correspond to sections of $\mathrm{GL}^+(E)/\mathrm{SO}(n)$. Because this bundle has contractible fibers, there exists a smooth family of bundle maps

$$\begin{array}{ccc} \mathrm{GL}^+(E)/\mathrm{SO}(n) & \xrightarrow{\varphi_+} & \mathrm{GL}^+(E)/\mathrm{SO}(n) \\ \downarrow & & \downarrow \\ M & = & M \end{array}$$

such that φ_0=identity and φ_1 sends the section $\mathrm{SO}(E')/\mathrm{SO}(n)$ to the section $\mathrm{SO}(E)/\mathrm{SO}(n)$.

Considering $\mathrm{GL}^+(E)$ as a principal $\mathrm{SO}(n)$-bundle over $\mathrm{GL}^+(E)/\mathrm{SO}(n)$, the family φ_t lifts to a family of $\mathrm{SO}(n)$-equivariant maps,

$$\begin{array}{ccc} \mathrm{GL}^+(E) & \xrightarrow{f_t} & \mathrm{GL}^+(E) \\ \downarrow & & \downarrow \\ \mathrm{GL}^+(E)/\mathrm{SO}(n) & \xrightarrow{\varphi_t} & \mathrm{GL}^+(E)/\mathrm{SO}(n) \end{array}$$

such that f_0 = identity and f_1 sends $\mathrm{SO}(E')$ to $\mathrm{SO}(E)$. We then have maps

$$\begin{array}{ccc} P & = & P \\ {\scriptstyle p'}\downarrow & & {\scriptstyle p}\downarrow \\ \mathrm{SO}(E') & \xrightarrow{f_1} & \mathrm{SO}(E) \end{array}$$

where $p = f_1 \circ p'$. Then (P, p) is an oriented metric Spin^c structure. The structures (P, p) and (P, p') are homotopic by Proposition D.38. □

Proof of (2'): Let (P, p) and (P', p') be oriented metric Spin^c structures. Suppose that there exists an isomorphism $F\colon P \to P'$ of principal bundles, inducing an isomorphism $f\colon \mathrm{GL}^+(E) \to \mathrm{GL}^+(E)$, and a family $f_t\colon \mathrm{GL}^+(E) \to \mathrm{GL}^+(E)$ of bundle isomorphisms such that $f_1 = f$ and f_0 = identity. These descend to maps

$$\varphi_t\colon \mathrm{GL}^+(E)/\mathrm{SO}(n) \to \mathrm{GL}^+(E)/\mathrm{SO}(n)$$

such that φ_0 = identity and φ_1 sends $\mathrm{SO}(E)/\mathrm{SO}(n)$ to itself. Because the bundle $\mathrm{GL}^+(E)/\mathrm{SO}(n) \to M$ has contractible fibers, the homotopy φ_t can be deformed, through homotopies $\varphi_{t,s}$ with the same endpoints, to a homotopy which sends $\mathrm{SO}(E)/\mathrm{SO}(n)$ to itself for all t. The lifting of φ_t to f_t extends to a family of $\mathrm{SO}(n)$-equivariant maps

$$\begin{array}{ccc} \mathrm{GL}^+(E) & \xrightarrow{f_{t,s}} & \mathrm{GL}^+(E) \\ \downarrow & & \downarrow \\ \mathrm{GL}^+(E)/\mathrm{SO}(n) & \xrightarrow{\varphi_{t,s}} & \mathrm{GL}^+(E)/\mathrm{SO}(n). \end{array}$$

such that $f_{0,s}$ = identity and $f_{1,s} = f$ for all s. The map $f_{t,1}$ sends $\mathrm{SO}(E)$ to itself for all t.

Considering P and P' as $\mathrm{Spin}^c(n)$-equivariant $\mathrm{U}(1)$-bundles over $\mathrm{SO}(E)$, the family $f_{t,1}$ lifts to a family of automorphisms,

$$\begin{array}{ccc} P & \xrightarrow{F_t} & P' \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p'}\big\downarrow \\ \mathrm{SO}(E) & \xrightarrow{f_{t,1}} & \mathrm{SO}(E). \end{array}$$

such that $F_1 = F$. The automorphism $F_0 \colon P \to P'$ descends to the identity map on $\mathrm{SO}(E)$. This gives the required equivalence of metric Spin^c structures. $\square$

2.5. Spin^c structures and Pin^c structures. A Pin^c structure is a "Spin^c structure without an orientation". To define it, consider the group

$$\mathrm{Pin}^c(n) = \mathrm{Pin}(n) \times_{\mathbb{Z}_2} \mathrm{U}(1),$$

where $\mathrm{Pin}(n)$ is the non-trivial double covering of $\mathrm{O}(n)$. A Pin^c-*structure* on a vector bundle E is a principal $\mathrm{Pin}^c(n)$-bundle P together with a $\mathrm{Pin}^c(n)$-equivariant map

$$p \colon P \to \mathrm{GL}(E),$$

where $\mathrm{Pin}^c(n)$ acts on $\mathrm{GL}(E)$ through the homomorphisms $\mathrm{Pin}^c(n) \to \mathrm{O}(n) \hookrightarrow \mathrm{GL}(n)$.

A Pin^c structure induces a fiberwise metric on E (but not a fiberwise orientation).

The group $\mathrm{Pin}^c(n)$ contains two connected components. The connected component that contains the identity element is $\mathrm{Spin}^c(n)$. Let D denote the other connected component of $\mathrm{Pin}^c(n)$. We have the following properties:

(1) If $a, b \in D$, then $ab \in \mathrm{Spin}^c(n)$.
(2) $\mathrm{Spin}^c(n)$ acts on D by multiplication from the right, freely and transitively.
(3) $\mathrm{Spin}^c(n)$ acts on D by multiplication from the left, freely and transitively.

These properties imply that we have an isomorphism of spaces with left and right $\mathrm{Spin}^c(n)$-actions

$$D \times_{\mathrm{Spin}^c(n)} D \xrightarrow{\cong} \mathrm{Spin}^c(n) \tag{D.8}$$

given by

$$[a, b] \mapsto ab.$$

Consider the map $\pi \colon D \to \mathrm{GL}(n)$ obtained as the composition

$$\pi : \quad D \hookrightarrow \mathrm{Pin}^c(n) \to \mathrm{O}(n) \hookrightarrow \mathrm{GL}(n).$$

For each Spin^c structure (P, p), we obtain another Spin^c structure (P', p') by setting

$$P' = P \times_{\mathrm{Spin}^c(n)} D$$

and

$$p'([u, a]) = p(u)\pi(a).$$

(Note that $\pi(a) \in \mathrm{GL}(n)$ acts on $p(u) \in \mathrm{GL}(E)$ from the right.)

This defines an orientation reversing involution

$$\tau \colon (P, p) \mapsto (P', p')$$

on the set of homotopy classes of Spin^c structures. (The fact that τ is an involution follows from (D.8).)

Let us now consider a vector bundle E equipped with a fiberwise orientation $\mathcal{O}$. Then we have a natural bijection between the three sets:

(1) Pin^c structures on E;
(2) Spin^c structures on E which are compatible with $\mathcal{O}$;
(3) Spin^c structures on E which are incompatible with $\mathcal{O}$.

Indeed, given a Pin^c structure (Q, q), we get Spin^c structures (P, p) and (P', p') by $P = \pi^{-1}(\mathrm{GL}^+(E))$, $P' = \pi^{-1}(\mathrm{GL}^-(E))$, $p = q|_P$, and $p' = q|_{P'}$, where $\mathrm{GL}^-(E)$ is the bundle of disoriented frames. Conversely, given a Spin^c structure (P, p) (either compatible or incompatible with $\mathcal{O}$) we get a Pin^c structure (Q, q) with $Q = P \times_{\mathrm{Spin}^c(n)} \mathrm{Pin}^c(n)$. Finally, the involution τ interchanges oriented Spin^c structures and disoriented Spin^c structures.

Therefore, on a vector bundle E (without a pre-chosen orientation), a Spin^c structure contains the same information as a Pin^c structure together with an orientation.

2.6. Pin^c structures and ML^c structures. To eliminate metrics and orientations altogether, we may work with the "metalinear^c" group

$$\mathrm{ML}^c(n) = \mathrm{ML}(n) \times_{\mathbb{Z}_2} \mathrm{U}(1),$$

where $\mathrm{ML}(n)$ is the non-trivial double covering of $\mathrm{GL}(n)$. The group $\mathrm{ML}^c(n)$ contains the group $\mathrm{Pin}^c(n)$ as a maximal compact subgroup.

An ML^c-*structure* on a vector bundle E is a principal $\mathrm{ML}^c(n)$-bundle $\tilde{P}$, together with an ML^c-equivariant map $\tilde{p}\colon \tilde{P} \to \mathrm{GL}(E)$, where $\mathrm{ML}^c(n)$ acts on the principal $\mathrm{GL}(n)$-bundle $\mathrm{GL}(E)$ through the homomorphism $\pi\colon \mathrm{ML}^c(n) \to \mathrm{GL}(n)$.

The pair $(\tilde{P}, \tilde{p})$ plays the role of a "Spin^c-structure without the metric or orientation". When $E = TM$, and after fixing an orientation, the concept of an ML^c structure is equivalent to Duflo and Vergne's notion of a "quantum line bundle" on M; see [**Ve4**].

To a Pin^c structure (P, p) we associate an ML^c structure $(\tilde{P}, \tilde{p})$ by

$$\tilde{P} = P \times_{\mathrm{Spin}^c(n)} \mathrm{ML}^c(n) \quad \text{and} \quad \tilde{p}([u, a]) = p(u) \cdot \pi(a).$$

Conversely, from an ML^c structure $(\tilde{P}, \tilde{p})$ and a metric on E we get a Pin^c structure by taking P to be the preimage of $\mathrm{O}(E)$ in $\tilde{P}$.

Therefore, on a vector bundle E, a Pin^c structure contains the same information as an ML^c structure together with a fiberwise metric.

An *equivalence* of ML^c structures $(\tilde{P}, \tilde{p})$ and $(\tilde{P}', \tilde{p}')$ is an isomorphism $\tilde{F}\colon \tilde{P} \to \tilde{P}'$ which lifts the identity map on $\mathrm{GL}(E)$:

$$\text{(D.9)} \qquad \begin{array}{ccc} \tilde{P} & \xrightarrow{\tilde{F}} & \tilde{P}' \\ {\scriptstyle \tilde{p}}\downarrow & & {\scriptstyle \tilde{p}'}\downarrow \\ \mathrm{GL}(E) & \xrightarrow{\text{identity}} & \mathrm{GL}(E) \end{array}$$

On a vector bundle with a fiberwise metric, it is easy to see that ML^c structures up to equivalence are the same as metric Pin^c structures up to equivalence. Combining this with the results of the previous sections yields the following result:

PROPOSITION D.40. *Let $E \to M$ be an oriented vector bundle. There exists a natural one to one correspondence between the following sets of structures:*

(1) *Oriented* $\mathrm{Spin}^{\mathrm{c}}$ *structures on E, up to homotopy;*
(2) Pin^{c} *structures on E, up to homotopy;*
(3) ML^{c} *structures on E, up to equivalence.*

Hence, the homotopy classification of oriented $\mathrm{Spin}^{\mathrm{c}}$ structures is the same as the classification of each of the structures (1)–(3). See Proposition D.43 below.

Finally, on a symplectic vector bundle, an ML^{c} structure is the same as a Mp^{c} (metaplectic^{c}) structure, as we now explain. The metaplectic group $\mathrm{Mp}(2n)$ is the non-trivial double covering of the (real) symplectic group $\mathrm{Sp}(2n)$. The metaplectic^{c} group is

$$\mathrm{Mp}^{\mathrm{c}}(2n) = \mathrm{Mp}(2n) \times_{\mathbb{Z}_2} \mathrm{U}(1).$$

A metaplectic^{c} structure on a symplectic vector bundle E of rank $2n$ is an extension of its structure group from the symplectic group $\mathrm{Sp}(2n)$ to the metaplectic^{c} group $\mathrm{Mp}^{\mathrm{c}}(2n)$. A symplectic structure on a vector bundle $E \to M$ determines a reduction of structure group of any ML^{c} structure to an Mp^{c} structure: Let $\mathrm{Sp}(E)$ denote the bundle of symplectic frames (whose fiber over $m \in M$ is the set of linear symplectomorphisms of E_m with R^{2n}). If $(\tilde{P}, \tilde{p})$ is an ML^{c} structure, the preimage of $\mathrm{Sp}(E)$ in $\tilde{P}$ is a principal bundle with structure group Mp^{c} and provides us with a metaplectic^{c} structure on E.

2.7. Classification of Spin$^{\mathrm{c}}$-structures, and the distinguishing line bundle. We will now classify the $\mathrm{Spin}^{\mathrm{c}}$-structures on a vector bundle E, up to homotopy.

Recall that $\mathrm{Spin}^{\mathrm{c}}(n) = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} \mathrm{U}(1)$ and there is a natural projection $\pi\colon \mathrm{Spin}^{\mathrm{c}}(n) \to \mathrm{SO}(n)$. We identify $K = \ker \pi$ with $\mathrm{U}(1)$ by $[1, c] \mapsto c$. Note that

$$\mathrm{Spin}^{\mathrm{c}}(n) \times_K \mathrm{U}(1) = \mathrm{Spin}^{\mathrm{c}}(n).$$

Given a complex Hermitian line bundle $\mathbb{L}$ over M, we can "twist" (P, p) by $\mathbb{L}$ and obtain another $\mathrm{Spin}^{\mathrm{c}}$-structure, (P', p'), by

$$P' = P \times_K \mathrm{U}(\mathbb{L}),$$

where $\mathrm{U}(\mathbb{L}) \subset \mathbb{L}$ is the unit circle bundle. The projection $p'\colon P' \to \mathrm{GL}(E)$ and the principal $\mathrm{Spin}^{\mathrm{c}}(n)$-action are induced from (P, p). The fact that these are well defined follows from the fact that $K = \ker \pi$ and that K is the center of $\mathrm{Spin}^{\mathrm{c}}(n)$. Note that (P', p') induces the same metric and orientation as (P, p).

LEMMA D.41. *Let (P, p) be a* $\mathrm{Spin}^{\mathrm{c}}$ *structure. Let (P', p') and (P'', p'') be the results of twisting (P, p) by line bundles $\mathbb{L}'$ and $\mathbb{L}''$. If the line bundles $\mathbb{L}'$ and $\mathbb{L}''$ are equivalent, then the* $\mathrm{Spin}^{\mathrm{c}}$ *structures (P', p') and (P'', p'') are homotopic.*

PROOF. An isomorphism $\mathbb{L}' \to \mathbb{L}''$ naturally extends to an isomorphism $P \times_K \mathrm{U}(\mathbb{L}') \to P \times_K \mathrm{U}(\mathbb{L}'')$ which respects the maps p' and p''. □

Because line bundles up to equivalence are classified by $H^2(M; \mathbb{Z})$, we get an action of $H^2(M; \mathbb{Z})$ on the set of homotopy classes of $\mathrm{Spin}^{\mathrm{c}}$-structures:

DEFINITION D.42. Let $\delta \in H^2(M; \mathbb{Z})$, We define the action of δ on (P, p) to give the $\mathrm{Spin}^{\mathrm{c}}$ structure with $P' = P \times_K \mathrm{U}(\mathbb{L})$, where $\mathbb{L}$ is a complex Hermitian line bundle with $c_1(\mathbb{L}) = \delta$.

In exactly the same way we can twist a Pin^c structure or an ML^c structure. For each of these structures, we replace the principal bundle P by the bundle $P \times_K \mathrm{U}(\mathbb{L})$ where $K \cong \mathrm{U}(1)$ is in the center of the structure group $\mathrm{Pin}^c(n)$ or $\mathrm{ML}^c(n)$.

PROPOSITION D.43. *Let $E \to M$ be an oriented vector bundle which admits a* Spin^c*-structure.*

(1) *The action of $H^2(M;\mathbb{Z})$ on the set of homotopy classes of oriented* Spin^c*-structures on E is effective and transitive. In particular, this set is (non-canonically) in one-to-one correspondence with the cohomology group $H^2(M;\mathbb{Z})$.*
(2) *Let (P,p) be an oriented* Spin^c*-structure on E with determinant line bundle $\mathbb{L}_{\mathrm{det}}$, and let $\mathbb{L}'_{\mathrm{det}}$ be the determinant line bundle of the* Spin^c*-structure obtained by the action of $u \in H^2(M;\mathbb{Z})$ on (P,p). Then*
$$c_1(\mathbb{L}'_{\mathrm{det}}) = c_1(\mathbb{L}_{\mathrm{det}}) + 2u.$$
(3) *A line bundle $\mathbb{L}$ is the determinant line bundle for some* Spin^c*-structure on E if and only if $c_1(\mathbb{L}) = w_2(E) \mod 2$.*

REMARK D.44. This proposition shows that the determinant line bundle $\mathbb{L}_{\mathrm{det}}$ does not determine the homotopy class of a Spin^c-structure on E uniquely. Namely, $\mathbb{L}_{\mathrm{det}}$ determines a Spin^c-structure up to elements of the kernel of the homomorphism $H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z})$ induced by multiplication by 2.

The proof of Proposition D.43 follows from a construction which is reverse, in a certain sense, to the $H^2(M;\mathbb{Z})$ action of Definition D.42. Namely, we will give a recipe which associates to any two oriented Spin^c-structures (P,p) and (P',p') on E a "distinguishing" complex line bundle $\mathbb{L}$ over M, such that twisting (P,p) by $\mathbb{L}$ will give (P',p'). The Chern class $\delta = c_1(\mathbb{L}) \in H^2(M;\mathbb{Z})$ will only depend on the homotopy classes of the Spin^c-structures.

By Proposition D.39, we may restrict our attention to oriented metric Spin^c structures, and will need to show that equivalent structures give isomorphic line bundles.

Recall that P and P' can be viewed as principal K-bundles over $\mathrm{SO}(E) = P/K = P'/K$, where $K = \ker \pi \cong \mathrm{U}(1)$. Since K commutes with all elements of $\mathrm{Spin}^c(n)$, we can view $P \to \mathrm{SO}(E)$ and $P' \to \mathrm{SO}(E)$ as $\mathrm{Spin}^c(n)$-equivariant K-bundles. Denote by $\mathbb{F}$ and $\mathbb{F}'$ the associated equivariant complex line bundles. Then $\mathrm{Spin}^c(n)$ also acts on $(\mathbb{F})^* \otimes \mathbb{F}'$ so that the action of K is trivial. (Here we view K as a subgroup of $\mathrm{Spin}^c(n)$ and not as the structural group.) Thus the $\mathrm{Spin}^c(n)$-action on $(\mathbb{F})^* \otimes \mathbb{F}'$ factors through an $\mathrm{SO}(n)$-action. It follows that this line bundle descends to (i.e., is a pull-back of) a line bundle $\mathbb{L}_\delta$ over $\mathrm{SO}(E)/\mathrm{SO}(n) = M$. We set δ to be the first Chern class of the resulting line bundle on M.

DEFINITION D.45. The line bundle $\mathbb{L}_\delta = \mathbb{L}_\delta(P,P')$ constructed above is called the *distinguishing line bundle* of the Spin^c-structures (P,p) and (P',p'). The cohomology class $\delta(P,P') = c_1(\mathbb{L})$ is called the *distinguishing cohomology class*.

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REMARK D.46. If we replace (P,p) and (P',p') by metric Spin^c structures that are homotopic to them, the resulting distinguishing line bundle is isomorphic to $\mathbb{L}_\delta$. By Proposition D.39, we get a line bundle $\mathbb{L}_\delta$, defined canonically up to equivalence of complex line bundles, for every two Spin^c structures (P,p) and (P',p'), which only depends on their homotopy class.

REMARK D.47. We have $\mathbb{L}_\delta(P', P) \cong \mathbb{L}_\delta(P, P')^*$ and

$$\mathbb{L}_\delta(P, P'') \cong \mathbb{L}_\delta(P, P') \otimes \mathbb{L}_\delta(P', P'') \tag{D.10}$$

for Spin^c-structures P, P', and P''. Likewise, $\delta(P', P) = -\delta(P, P')$ and $\delta(P, P'') = \delta(P, P') + \delta(P', P'')$. We emphasize that $\delta(P, P')$ and, up to isomorphism, $\mathbb{L}_\delta(P, P')$ are invariants of homotopy but not of bundle equivalence of Spin^c-structures. However, $2\delta(P, P')$ and $\mathbb{L}_\delta(P, P')^{\otimes 2}$ are invariants of bundle equivalence.

The class δ depends only on the equivalence class of the metric Spin^c structures (P, p) and (P', p').

PROOF OF PROPOSITION D.43. To prove the first assertion, it is enough to observe that the distinguishing line bundle for the Spin^c-structures P and $P' = P \times_K \mathrm{U}(\mathbb{L})$ is $\mathbb{L}_\delta = \mathbb{L}$.

The second assertion follows immediately from the fact that the composition $\mathrm{U}(1) \cong K \hookrightarrow \mathrm{Spin}^c(n) \stackrel{\det}{\to} \mathrm{U}(1)$ is the map $c \mapsto c^2$.

The third statement follows from Proposition D.31. □

REMARK D.48. In practice we will need to compare Spin^c structures which may induce opposite orientations: on a vector bundle without a pre-chosen orientation, two Spin^c structures are distinguished by

(1) the distinguishing line bundle $\mathbb{L}_\delta$;
(2) whether or not they induce the same orientation.

This follows immediately from the results of Section 2.5. Note that the distinguishing line bundle can be defined even if (P, p) and (P', p') induce opposite orientations. For instance, we may define it by passing to the associated Pin^c structures and noting that the above construction of the distinguishing line bundle works word-for-word for a pair of Pin^c structures, with $\mathrm{Spin}^c(n)$ replaced by $\mathrm{Pin}^c(n)$ and $\mathrm{SO}(E)$ replaced by $\mathrm{O}(E)$ everywhere. Equivalently, we may replace (P, p) by $\tau(P, p)$ (see Section 2.5) and then carry out the above construction of $\mathbb{L}_\delta$.

All the above remains valid in the presence of a proper G-action. In particular, we can twist a G-equivariant Spin^c structure by a G-equivariant line bundle, and, for any two G-equivariant Spin^c structures P and P', the distinuishing line bundle $\mathbb{L}_\delta(P, P')$ is a G-equivariant line bundle.

3. Spin^c-structures and stable complex structures

In this section we associate a Spin^c structure (P, p) to any stable complex structure J, and we compare the Spin^c structures coming from different stable complex structures.

REMARK D.49. In the literature, one usually takes stable complex structures up to bundle equivalence and Spin^c structures up to homotopy. With this approach, the map $J \mapsto (P, p)$ is not well defined: if a stable complex structure is only defined up to bundle equivalence, so is the resulting Spin^c-structure, and one does not get a well defined homotopy class. One of the reasons that we keep track of both equivalence relations is to avoid this problem.

3.1. The canonical Spin^c-structure on a complex vector bundle. In the context of this book, the main sources of Spin^c-structures are complex and stable complex structures.

The anti-canonical line bundle of a complex vector bundle E is the top wedge $K^* = \Lambda^m_{\mathbb{C}} E$, where $m = \mathrm{rk}_{\mathbb{C}} E$. (The symbol K is usually reserved for the canonical line bundle; the two are dual to each other.) Similarly, if E is equipped with a stable complex structure, its anti-canonical line bundle is the anti-canonical line bundle of the complex vector bundle $(E \oplus \mathbb{R}^k, J)$ through which the stable complex structure on E is defined. The isomorphism class of the anti-canonical line bundle only depends on the bundle equivalence class of the stable complex structure.

Proposition D.50. *A complex vector bundle E and complex line bundle $\mathbb{L}$ over M determine a* Spin^c*-structure on E uniquely up to homotopy. The determinant line bundle of this* Spin^c*-structure is isomorphic to $K^* \otimes \mathbb{L}^{\otimes 2}$ where K^* is the anti-canonical line bundle of E. Homotopic (resp., bundle equivalent) complex structures on E give rise to homotopic (resp., bundle equivalent)* Spin^c *structures.*

Proof. We first construct an inclusion of $\mathrm{U}(n)$ into $\mathrm{Spin}^c(2n)$. We start with the standard inclusion $\mathrm{U}(n) \hookrightarrow \mathrm{SO}(2n)$ and lift it to an inclusion $\overline{\mathrm{U}}(n) \hookrightarrow \mathrm{Spin}(2n)$ for the connected double covering $\overline{\mathrm{U}}(n)$ of $\mathrm{U}(n)$. Likewise, the homomorphism $\det\colon \mathrm{U}(n) \to \mathrm{U}(1)$ lifts to a homomorphism $\overline{\mathrm{U}}(n) \to \overline{\mathrm{U}}(1) \cong \mathrm{U}(1)$. These homomorphisms give rise to an inclusion $\overline{\mathrm{U}}(n) \to \mathrm{Spin}(2n) \times \overline{\mathrm{U}}(1)$ which descends to the required inclusion $\mathrm{U}(n) \hookrightarrow \mathrm{Spin}(2n) \times_{\mathbb{Z}_2} \overline{\mathrm{U}}(1) = \mathrm{Spin}^c(2n)$.

Set $P = \mathrm{U}(E) \times_{\mathrm{U}(n)} \mathrm{Spin}^c(2n)$ with its natural projection p to $\mathrm{SO}(E) = \mathrm{U}(E) \times_{\mathrm{U}(n)} \mathrm{SO}(2n)$, where $\mathrm{U}(E)$ is the principal $\mathrm{U}(n)$-bundle of unitary frames in E with respect to some Hermitian metric, and where the projection p arises from the homomorphism $\pi\colon \mathrm{Spin}^c(2n) \to \mathrm{SO}(2n)$. Then (P, p) is a Spin^c-structure on E, and, up to homotopy, it is independent of the choice of the Hermitian metric. We get a Spin^c-structure with the required determinant line bundle by twisting P by $\mathbb{L}$ as in Definition D.42.

Suppose now that we have a *family* J_t of complex structures on E, parametrized by $t \in [0, 1]$. This makes the product $E \times [0, 1]$ into a complex vector bundle over $M \times [0, 1]$. The previous construction gives rise to a Spin^c structure $(\tilde{P}, \tilde{p})$ on $E \times [0, 1]$, whose restriction to $E \times \{t\}$ is the Spin^c structure associated with J_t. The trivial connection on the fibration $\mathrm{GL}^+(E) \times [0, 1] \to [0, 1]$ lifts to a connection on $\tilde{P} \to [0, 1]$ whose parallel transport provides us with a homotopy (see Definition D.37) between the Spin^c structures associated with J_0 and J_1.

An equivalence of complex line bundles $E_0 \to E_1$ gives an equivalence of principal $\mathrm{U}(n)$ bundles $\mathrm{U}(E_0) \to \mathrm{U}(E_1)$ and, further, an equivalence of the associated Spin^c bundles. □

Combining Proposition D.50 with the results of Section 2.7, we get the following result.

Lemma D.51. *Any two complex structures J, J' on a real vector bundle E determine a "distinguishing" complex line bundle*

$$\mathbb{L}_\delta = \mathbb{L}_\delta(J, J')$$

uniquely up to isomorphism. We have

$$K' = K \otimes \mathbb{L}_\delta^{\otimes 2}$$

where K' and K are the anti-canonical line bundles associated to J and J'. If J'' is a third complex structure,

$$\mathbb{L}_\delta(J, J') \otimes \mathbb{L}_\delta(J', J'') = \mathbb{L}_\delta(J, J'').$$

Applying this to the bundles $E \oplus \mathbb{R}^k$, we get a similar result for stable complex structures:

LEMMA D.52. *Any two stable complex structures J, J' on a real vector bundle E determine a "distinguishing" complex line bundle*

$$\mathbb{L}_\delta = \mathbb{L}_\delta(J, J')$$

uniquely up to isomorphism. We have

$$K' = K \otimes \mathbb{L}_\delta^{\otimes 2}$$

where K' and K are the anti-canonical line bundles associated to J and J'. For any line bundle $\mathbb{L}$, the Spin^c *structure associated to J' and $\mathbb{L}$ is homotopic to the* Spin^c *structure associated to J and $\mathbb{L} \otimes \mathbb{L}_\delta$. If J'' is a third stable complex structure,*

$$\mathbb{L}_\delta(J, J') \otimes \mathbb{L}_\delta(J', J'') = \mathbb{L}_\delta(J, J'').$$

We will work out in detail one important special case:

EXAMPLE D.53. Consider $\mathbb{C}^d = \mathbb{C}^r \times \mathbb{C}^{d-r}$ with the standard complex structure $J = J_r \oplus J_{d-r}$ and the non-standard complex structure $J^\# = (-J_r) \oplus J_{d-r}$, where J_r and J_{d-r} are, respectively, the standard complex structures on $\mathbb{C}^r$ and $\mathbb{C}^{d-r}$. Let $(S^1)^d$ act by rotating the coordinates. Consider the spin^c structures associated to J and $J^\#$. Their (equivariant) distinguishing line bundle is the top wedge $\bigwedge^r \mathbb{C}^r$.

Note that J and $J^\#$ induce the same orientation if and only if r is even but that their distinguishing line bundle is always defined by Remark D.48.

REMARK D.54. Every line bundle over $\mathbb{C}^d$ is trivial. The example is meaningful when we consider all structures as G-equivariant with $G = (S^1)^d$.

PROOF OF EXAMPLE D.53. It is enough to consider the case $r = d = 1$; the general case follows by flipping the complex structure in stages, one coordinate at a time. Because orientation is flipped, we must work with Pin^c, not Spin^c structures.

Groups. We identify $\mathrm{U}(1) = \mathrm{SO}(2)$ and denote

$$e^{i\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

An element of $\mathrm{O}(2) = \mathrm{SO}(2) \rtimes \mathbb{Z}_2$ can be written uniquely as either $e^{i\theta}$ or $e^{i\theta}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. In multiplying two such elements, note that

$$e^{i\theta_1}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)e^{i\theta_2} = e^{i(\theta_1 - \theta_2)}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right).$$

We write an element of Pin(2) formally as either $e^{\frac{i\theta}{2}}$ or $e^{\frac{i\theta}{2}}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with $\theta \in \mathbb{R}/4\pi\mathbb{Z}$. The homomorphism

$$q \colon \mathrm{Pin}(2) \to \mathrm{O}(2)$$

sends $e^{\frac{i\theta}{2}}$ to $e^{i\theta}$ and $e^{\frac{i\theta}{2}}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ to $e^{i\theta}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

The inclusion map

$$\mathrm{U}(1) \hookrightarrow \mathrm{Pin}^c(2) := \mathrm{Pin}(2) \times_{\mathbb{Z}_2} \mathrm{U}(1)$$

is

$$e^{i\theta} \mapsto [e^{\frac{i\theta}{2}}, e^{\frac{i\theta}{2}}].$$

Frame bundles. Consider $V = \mathbb{R}^2 = \mathbb{C}$ with the standard circle action and with the distinguished element $e = 1$. Its orthogonal frame bundle is

$$\mathrm{O}(V) = \{(w, v_1, v_2) \mid w \in V, v_1, v_2 \in V, \|v_1\| = \|v_2\| = 1, v_2 \perp v_1\}.$$

The principal $\mathrm{O}(2)$ action is $B\colon (v_1, v_2) \mapsto (v_1, v_2)B$ for $B \in \mathrm{O}(2)$. The circle action on V lifts to the left circle action on $\mathrm{O}(V)$ given by

$$\lambda \cdot (w, v_1, v_2) = (\lambda w, \lambda v_1, \lambda v_2).$$

Fix a complex structure J on V. Concretely, we will consider either $J_1 = \sqrt{-1}$ or $J_2 = -\sqrt{-1}$. The unitary frame bundle is

$$\mathrm{U}(V) = \{(w, u) \mid w \in V, u \in V, \|u\| = 1\}.$$

The principal $\mathrm{U}(1)$ action is

$$e^{i\theta}\colon (w, u) \mapsto (w, e^{J\theta}u)$$

where $e^{J\theta}u = \cos\theta u + \sin\theta Ju$. The left circle action is

$$\lambda \cdot (w, u) = (\lambda w, \lambda u).$$

Note that if $\lambda = e^{i\alpha}$, then

$$\lambda u = e^{\pm J\alpha}u \tag{D.11}$$

according to whether $J = \sqrt{-1}$ or $J = -\sqrt{-1}$.

The associated $\mathrm{Pin}^c(2)$ bundle over V is

$$P = \{[w, u, A, a] \mid (w, u) \in \mathrm{U}(V), [A, a] \in \mathrm{Pin}^c(2)\}$$

with

$$[w, e^{J\theta}u, A, a] = [w, u, e^{\frac{i\theta}{2}}A, e^{\frac{i\theta}{2}}a]. \tag{D.12}$$

The map to $\mathrm{O}(V)$ sends $[w, u, A, a]$ to (w, v_1, v_2) by

$$(v_1, v_2) = (u, Ju)q(A).$$

The left circle action, $\lambda \cdot [w, u, A, a] = [\lambda w, \lambda u, A, a]$, can, by (D.11) and (D.12), be written as $\lambda \cdot [w, u, A, a] = [\lambda w, u, \lambda^{\pm\frac{1}{2}}A, \lambda^{\pm\frac{1}{2}}a]$, according to whether $J = \sqrt{-1}$ or $J = -\sqrt{-1}$.

Line bundles. We now take the associated line bundle over $\mathrm{O}(V)$. An element of $\mathbb{L} = P \times_{\mathrm{U}(1)} \mathbb{C}$ is written as $[w, u, A, a, z]$. However, since $\{u = e\}$ is a trivialization of $\mathrm{U}(V)$ and $\mathrm{Pin}^c(2) \times_{\mathrm{U}(1)} \mathbb{C} = \mathrm{Pin}(2) \times_{\mathbb{Z}_2} \mathbb{C}$, we can set $u = e$ and $a = 1$, and get that

$$\mathbb{L} = \{[w, A, z] \mid w \in V, [A, z] \in \mathrm{Pin}(2) \times_{Z_2} \mathbb{C}\}.$$

The map to $\mathrm{O}(V)$ sends $[w, A, z]$ to (w, v_1, v_2), where

$$(v_1, v_2) = (e, Je)q(A).$$

The left circle action is

$$\lambda \cdot [w, A, z] = [\lambda w, \lambda^{\pm\frac{1}{2}}A, \lambda^{\pm\frac{1}{2}}z],$$

according to whether $J = \sqrt{-1}$ or $J = -\sqrt{-1}$. Let $\mathbb{L}_1$ and $\mathbb{L}_2$ denote the bundles obtained from $J_1 = \sqrt{-1}$ and $J_2 = -\sqrt{-1}$.

We need to show that the distinguishing line bundle is $V \times \mathbb{C}$ with the circle action $\lambda(w, \zeta) = (\lambda w, \lambda\zeta)$. Let $\pi^*\mathbb{L}_\delta$ denote its pullback to $\mathrm{O}(V)$. We define the required isomorphism

$$\mathbb{L}_2 \otimes \pi^*\mathbb{L}_\delta \to \mathbb{L}_1$$

by

$$[w, A, z \otimes \zeta] \mapsto [w, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) e^{\frac{\pi i}{4}} A, z\zeta].$$

The fact that this respects the maps to $\mathrm{O}(V)$ follows easily from the facts that $(e, J_2 e) = (e, -J_1 e)$ and

$$q\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) e^{\frac{\pi i}{4}} A\right) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) e^{\frac{\pi i}{2}} q(A) = \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\, q(A).$$

Equivariance with respect to the left circle action follows easily from the facts that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) e^{\frac{\pi i}{4}} \lambda^{-\frac{1}{2}} A = \lambda^{\frac{1}{2}} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) e^{\frac{\pi i}{4}} A$ and $\left(\lambda^{-\frac{1}{2}} z\right)(\lambda\zeta) = \lambda^{\frac{1}{2}}(z\zeta)$. ☐

3.2. Destabilization of Spin^c-structures. By Proposition D.50, every almost complex manifold equipped with a complex line bundle has a canonical Spin^c-structure. Likewise, a stable complex structure on M and a complex line bundle give rise to a "stable" Spin^c-structure, i.e., a Spin^c-structure on the stable tangent bundle. It turns out, however, that Spin^c-structures naturally destabilize: every "stable" Spin^c-structure induces a genuine Spin^c-structure.

PROPOSITION D.55 (Cannas da Silva, [**CKT**, Lemma 2.4]). *Let E be a real n-dimensional vector bundle. Every* Spin^c*-structure on the Whitney sum $E \oplus \mathbb{R}^k$ canonically induces a* Spin^c*-structure on E with the same determinant line bundle. Homotopic* Spin^c*-structures on $E \oplus \mathbb{R}^k$ induce homotopic* Spin^c*-structures on E.*

PROOF. Let P' be a principal $\mathrm{Spin}^c(n+k)$ bundle for a Spin^c structure on $E \oplus \mathbb{R}^k$ and let

$$p' \colon P' \to \mathrm{SO}(E \oplus \mathbb{R}^k)$$

be the corresponding map of principal bundles. Recall that this map makes P' into a $\mathrm{Spin}^c(n+k)$ equivariant $\mathrm{U}(1)$ bundle over $\mathrm{SO}(E \oplus \mathbb{R}^k)$. The inclusion $E \hookrightarrow E \oplus \mathbb{R}^k$ induces an $\mathrm{SO}(n)$-invariant inclusion $\mathrm{SO}(E) \hookrightarrow \mathrm{SO}(E \oplus \mathbb{R}^k)$. Let P denote the preimage of $\mathrm{SO}(E)$ in P' and let $p = p'|_P$. Then we have a pull-back diagram

$$\begin{array}{ccc} P & \longrightarrow & P' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ \mathrm{SO}(E) & \longrightarrow & \mathrm{SO}(E \oplus \mathbb{R}^k) \end{array}$$

which is equivariant with respect to the natural homomorphisms

$$\begin{array}{ccc} \mathrm{Spin}^c(n) & \longrightarrow & \mathrm{Spin}^c(n+k) \\ \downarrow & & \downarrow \\ \mathrm{SO}(n) & \longrightarrow & \mathrm{SO}(n+k). \end{array}$$

The pair (P, p), which is a $\mathrm{Spin}^c(n)$ equivariant principal $\mathrm{U}(1)$ bundle over $\mathrm{SO}(E)$, provides the required Spin^c structure on E.

Destabilization of ML^c structures is defined in a similar way. It is easy to see that equivalent ML^c structures on $E \oplus \mathbb{R}^k$ destabilizer to equivalent ML^c structures on E. Because ML^c structures up to equivalence are the same as oriented Spin^c structures up to homotopy (see Proposition D.40), homotopic Spin^c structures on $E \oplus \mathbb{R}^k$ destabilize to homotopic Spin^c structures on E. ☐

Remark D.56. This construction gives a one-to-one correspondence between the sets of homotopy classes of Spin^c-structures on $E \oplus \mathbb{R}^k$ and on E. (The inverse comes from the natural inclusion $\mathrm{Spin}^c(n) \to \mathrm{Spin}^c(n+k)$.)

Spin^c-structures are analogous, in some sense, to $\mathrm{O}(n)$-structures on real vector bundles (i.e., fiberwise Euclidean metrics). Proposition D.55 is well known to also hold for $\mathrm{SO}(n)$ structures or $\mathrm{O}(n)$ structures; the proof is similar to the one given above. Furthermore, the determinant complex line bundle $\mathbb{L}_{\mathrm{det}}$ is the Spin^c-analogue of the determinant real line bundle $\mathbb{D} = \mathrm{O}(E) \times_{\mathrm{det}} \mathbb{R}$. From this point of view, the last assertion of Proposition D.43, which gives a necessary and sufficient condition for the existence of a Spin^c-structure with a given determinant bundle, is similar to the condition $w_1(E) = w_1(\mathbb{D})$ for E to have an $\mathrm{O}(n)$-structure with real determinant line bundle $\mathbb{D}$. (In particular, E is orientable if and only if $w_1(E) = 0$.) Recall in this connection that $w_2(E) = 0$ is the necessary and sufficient condition for E to admit a Spin-structure.

3.3. Stable complex structures and Spin^c-structures; the shift formula. We recall some definitions from Section 1. A stable complex structure on a vector bundle E is a complex structure on a Whitney sum $E \oplus \mathbb{R}^k$. Two notions of equivalence for stable complex structures are of interest for us. Stable complex structures J_0 and J_1 on vector bundles E_0 and, respectively, E_1 are said to be bundle equivalent if the complex vector bundles $E_0 \oplus \mathbb{R}^{k_0}$ and $E_1 \oplus \mathbb{R}^{k_1}$ on which J_0 and J_1 are actually defined become isomorphic after addition of some numbers of trivial complex vector bundles to each of them. On the other hand, stable complex structures J_0 and J_1 on the *same* vector bundle E are said to be homotopic if the complex structures on $E \oplus \mathbb{R}^{k_0}$ and $E \oplus \mathbb{R}^{k_1}$ become homotopic after addition of some numbers of trivial complex vector bundles.

Combining Propositions D.50 and D.55, we obtain

Proposition D.57. *A stable complex vector bundle E and complex line bundle $\mathbb{L}$ over M determine a* Spin^c*-structure on E uniquely up to homotopy. The determinant line bundle of this* Spin^c*-structure is isomorphic to $\mathbb{L}_{\mathrm{det}} = K^* \otimes \mathbb{L}^{\otimes 2}$, where K^* is the anti-canonical line bundle of the stable complex structure. Homotopic stable complex structures determine homotopic* Spin^c*-structures on E.*

In particular, every stable complex structure on M and complex line bundle over M determine a Spin^c-structure on M, up to homotopy. Furthermore, we also have the following "Spin^c Shift Formula":

Proposition D.58. *Let J_0 and J_1 be stable complex structures on E and $\mathbb{L}_0$ and $\mathbb{L}_1$ be complex line bundles over M. Denote by $\mathbb{L}_\delta$ the distinguishing line bundle for the* Pin^c*-structures determined by $(J_0, \mathbb{L}_0)$ and $(J_1, \mathbb{L}_1)$ (cf. Remark D.48). Then the pairs $(J_0, \mathbb{L}_0 \otimes \mathbb{L}_\delta)$ and $(J_1, \mathbb{L}_1)$ give rise to homotopic* Pin^c*-structure on E. If J_0 and J_1 induce the same orientation, we get homotopic* Spin^c *structures on E.*

Remark D.59. It follows from Propositions D.50 and D.58 that

$$\mathbb{L}_0^{\otimes 2} \otimes \mathbb{K}_0^* \otimes \mathbb{L}_\delta^{\otimes 2} = \mathbb{L}_1^{\otimes 2} \otimes \mathbb{K}_1^*,$$

where $\mathbb{K}_0^*$ and $\mathbb{K}_1^*$ are the anti-canonical bundles of J_0 and J_1. In particular, if $\mathbb{L}_0 = \mathbb{L}_1$, we have $\mathbb{L}_\delta^{\otimes 2} = \mathbb{K}_0 \otimes \mathbb{K}_1^*$. Note also that, as is clear from the proof given below, $\mathbb{L}_\delta$ is completely determined by J_0 and J_1 when $\mathbb{L}_0 = \mathbb{L}_1$.

Proof of Proposition D.58. By adding, if necessary, a trivial bundle, we may ensure that all complex structures are defined on the same vector bundle, which we denote again by E. Denote by P_i, $i = 0, 1$, the Spin^c-structures determined by $(J_i, \mathbb{L}_i)$ and by P' the Spin^c-structure arising from $(J_0, \mathbb{L}_0 \otimes \mathbb{L}_\delta)$. Then

$$\mathbb{L}_\delta(P_0, P') = \mathbb{L}_\delta = \mathbb{L}_\delta(P_0, P_1).$$

By the cocycle equation (D.10) of Remark D.46,

$$\mathbb{L}_\delta(P_1, P') = \mathbb{L}_\delta(P_0, P_1)^* \otimes \mathbb{L}_\delta(P_0, P') = \mathbb{L}_\delta^* \otimes \mathbb{L}_\delta$$

is trivial. We finish the proof by the classification of Section 2.7. □

3.4. Equivariant Spin^c-structures and reduction. On a G-equivariant vector bundle $E \to M$, we define a G-equivariant Spin^c-structure by requiring $P \to M$ to be G-equivariant principal Spin$^c(n)$-bundle and $p\colon P \to \mathrm{GL}^+(E)$ to be a $G \times \mathrm{Spin}^c(n)$-equivariant map. The definitions and results of Sections 3.1–3.3 extend immediately to the equivariant case. (In Proposition D.43, one needs to use Theorem C.47 to identify the group of G-equivariant complex line bundles with $H^2_G(M; \mathbb{Z})$.) Since all constructions are canonical, Propositions D.50, D.55, D.57, and D.58 hold literally for equivariant Spin^c-structures. In same vein, these results extend to Spin^c-structures on orbifolds.

The reduction procedure for Spin^c-structures is very similar to reduction of stable complex structures (see Section 2.3 of Chapter 5). As in Section 2 of Chapter 5, let G be a torus, M a G-manifold, and $\Psi\colon M \to \mathfrak{g}^*$ an abstract moment map. For a regular value $\alpha \in \mathfrak{g}^*$, the reduced space is the orbifold $M_\alpha = \Psi^{-1}(\alpha)/G$.

Proposition D.60 (Reduction of Spin^c-structures). *A G-equivariant* Spin^c*-structure (P, p) on M gives rise to a* Spin^c*-structure (P_α, p_α) on the reduced space M_α, unique up to homotopy. Homotopic* Spin^c*-structures reduce to homotopic* Spin^c*-structures on M_α.*

Proof. Let $Z = \Psi^{-1}(\alpha)$. Recall that $T_Z M$ decomposes into

$$T_Z M = \pi^*(TM_\alpha) \oplus \mathfrak{g} \oplus \mathfrak{g}^*$$

as an equivariant vector bundle, where $\pi\colon Z \to M_\alpha = Z/G$ is the natural projection, and that this decomposition is canonical up to homotopy. (See equation (5.6).) The quotient P/G is an (orbifold) Spin^c-structure on the vector bundle $TM_\alpha \oplus \mathfrak{g} \oplus \mathfrak{g}^*$ over M_α. Applying the destabilization procedure of Proposition D.55, we obtain the reduced Spin^c-structure on the reduced space M_α.

The second assertion follows easily from the definitions. □

Now let J be a G-equivariant stable complex structure on M and P the associated Spin^c-structure. Reducing J as in Section 2.3 of Chapter 5, we obtain a stable complex structure J_α on M_α, and reducing P, we obtain the reduced Spin^c-structure P_α on M_α.

Proposition D.61. *P_α is the* Spin^c*-structure associated with J_α.*

We leave the proof as an exercise to the reader.

3.5. Quantization. In this book, we use a Spin^c-structure to make sense of "quantization". Let us recall how this is done.

A Spin^c-structure (P, p) on a manifold X, together with the choice of a connection on the bundle $P \to X$, give rise to an elliptic operator D acting on sections of certain vector bundles over X. (See below.) We define the quantization to be the index (kernel minus cokernel) of this operator. We will see below that this index is well defined.

Let us recall in greater detail how to get an elliptic operator from a Spin^c-structure (P, p) and a connection on P. (See [**Bot2, LM, Du, Fr**].) We assume that the underlying manifold X is $2n$-dimensional, so that P is a principal $\mathrm{Spin}^c(2n)$ bundle and p gives a vector bundle isomorphism

$$TX \cong P \times_{\mathrm{Spin}^c(2n)} \mathbb{R}^{2n}.$$

The group $\mathrm{Spin}^c(2n)$ has two famous complex linear representations, called the spinor representations, and denoted M_+ and M_-, and a $\mathrm{Spin}^c(2n)$-equivariant linear map

$$\mathbb{R}^{2n} \otimes M_+ \to M_-. \tag{D.13}$$

Let S_+ and S_- denote the bundles associated to P with fibers M_+ and M_-:

$$S_+ = P \times_{\mathrm{Spin}^c(2n)} M_+ \quad \text{and} \quad S_- = P \times_{\mathrm{Spin}^c(2n)} M_-.$$

From (D.13) we get a bundle map

$$TX \otimes S_+ \to S_-. \tag{D.14}$$

Choose a connection on P. This induces a connection on S_+. Covariant differentiation defines a map

$$\nabla\colon \text{ sections of } S_+ \to \text{ sections of } T^*X \otimes S_+. \tag{D.15}$$

The metric allows us to identify $T^*X = TX$. Then we can compose the maps (D.14) and (D.15) and get the Dirac Spin^c operator

$$D\colon \text{ sections of } S_+ \to \text{ sections of } S_-.$$

The above definition of the Dirac Spin^c operator depends on the choice of a connection on the principal bundle $P \to X$. However, the space of such connections is connected. This implies that different choices of a connection give rise to homotopic Dirac operators. By the Fredholm homotopy invariance of the index, these operators have the same index. Hence, the index associated to a Spin^c structure is well defined and does not depend on the choice of connection.

Homotopic Spin^c-structures also give rise to homotopic Dirac operators. Therefore, the index associated to a Spin^c-structure only depends on the homotopy class of the Spin^c-structure. One applications of this is in showing that the quantization of a reduced space is well defined. See Example D.13.

This argument fails to apply to Spin^c structures that are merely bundle equivalent. However, in this case we can involve the Atiyah–Singer index theorem, [**ASi**]. By this theorem, the index of the Spin^c Dirac operator is determined by the characteristic classes of the Spin^c structure. Thus we conclude that the index is also an invariant of bundle equivalence. Explicitly, the Atiyah Singer index theorem gives

$$\dim \mathcal{Q}(M) = \int_M e^{c_1(\mathbb{L}_{\det})} \hat{A}(TM),$$

where we integrate with respect to the orientation induced by the Spin^c structure. Notice that $\hat{A}(TM)$ is independent of the Spin^c structure and that $e^{c_1(\mathbb{L}_{\det})}$ only depends on the determinant line bundle. So $\mathcal{Q}(M)$ is defined for any Pin^c structure and orientation. If we keep the Pin^c structure but flip the orientation, this has the effect of flipping the sign of $\mathcal{Q}(M)$. Concretely, $\mathcal{Q}(M)$ is the index (kernel minus co-kernel) of an elliptic operator, and flipping the orientation has the same effect as switching the kernel and co-kernel.

In Section 3.1 we have associated a Spin^c structure to a pair $(J, \mathbb{L})$ consisting of a stable complex structure J and a line bundle $\mathbb{L}$ on a manifold M. We define the quantization $\mathcal{Q}(M, J, \mathbb{L})$ to be the index associated to this Spin^c structure.

By the results of Sections 3.5 and 2.6, one can also associate a Dirac operator to a Pin^c structure and orientation, or to an ML^c structure and orientation. The operator itself depends on certain auxiliary choices, but its index is independent of these choices. This index is, by definition, the quantization of the given orientation and Pin^c, or ML^c, structure.

We have also defined a distinguishing line bundle $\mathbb{L}_\delta = \mathbb{L}_\delta(J_2, J_1)$ associated to any two stable complex structures J_1 and J_2, such that the Spin^c structure associated to J_1 and $\mathbb{L}$ is homotopic to the Spin^c structure associated to J_2 and $\mathbb{L} \otimes \mathbb{L}_\delta$. Therefore,

$$\mathcal{Q}(M, J_2, \mathbb{L} \otimes \mathbb{L}_\delta) = \pm \mathcal{Q}(M, J_1, \mathbb{L}),$$

where the sign is according to whether or not J_2 and J_1 induce the same orientation.

When ω is a symplectic form on M, one usually defines the quantization $\mathcal{Q}(M, \omega)$ to be the index of the Dirac Spin^c operator D associated a the pre-quantization line bundle $\mathbb{L} \to M$ and an almost complex structure J compatible with ω. (See Section 3.) However, there is a slightly different definition which has some advantages: one may take the index of D to define the quantization of (M, ω) where ω is *half the curvature of the determinant bundle* associated to the Spin^c structure. (This leads to a different notion of whether (M, ω) is at all quantizable.) In this appendix we do not work with two-forms; we refer the reader to Section 7.3 of Chapter 6 for a further discussion of this notion of Spin^c-quantization.