A STABLE APPROACH TO THE EQUIVARIANT HOPF THEOREM

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ABSTRACT. Let G be a finite group. For semi-free G-manifolds which are oriented in the sense of Waner [20], the homotopy classes of G-equivariant maps into a G-sphere are described in terms of their degrees, and the degrees occurring are characterised in terms of congruences. This is first shown to be a stable problem, and then solved using methods of equivariant stable homotopy theory with respect to a semi-free G-universe.

KEYWORDS. G-manifold, degree, equivariant stable homotopy.

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INTRODUCTION

All manifolds considered will be smooth, closed, and connected. If M is an oriented n-manifold, any continuous map from M to an n-dimensional sphere has a degree, which is an integer. The Hopf theorem [11] says that the degree characterises the homotopy class of such a map, and that every integer occurs as the degree of such a map. Note that the degree map from homotopy classes to the integers factors through the group $\{M, S^n\}$ of *stable* homotopy classes of maps from M to S^n. The Hopf theorem follows from the facts that both the stabilisation and the *stable* degree map are isomorphisms. It is this approach to that result which will be generalised here to the equivariant category: it will first be shown that the problem, which at first sight seems to be unstable, is in fact stable, and then it will be solved using stable techniques.

The Hopf theorem has been generalised to equivariant contexts by a number of people, motivated by the problem to describe maps between representation spheres or homotopy representations. See Section 8.4. in [4], [10],

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Section II.4 in [5], or [3], [2], [7], and the references therein for recent contributions. All of these work unstably with a fairly elaborate obstruction machinery. Furthermore, they require the top-dimensional cohomology of the fixed point spaces to be cyclic, neglecting situations in which the fixed point space is not connected. The point of view taken here avoids elaborate notions of degree (and corresponding Lefschetz and Euler numbers) as in [6], [19], [16], [12], and the references therein, to name just a few. For the present purpose, the degree of a map is its stable homotopy class, and calculations of stable homotopy groups – which are much more accessible than their unstable relatives – process this into numerical information.

Throughout the paper, let G be a finite group. There will be occasions to assume that its order is a prime, but only for illustrative purposes. Let W be a real G-representation. Recall that a G-manifold M is called W-*dimensional*, if for every point x in M the tangential representation T_xM of the stabiliser G_x is isomorphic to the restriction of W from G to G_x. In particular, the components of the fixed point space

$$M^G = \coprod_\alpha M^G_\alpha$$

are manifolds of equal dimension. In addition, a W-dimensional manifold M is called *oriented* if W is oriented and there is a compatible choice of isomorphisms $T_xM \cong W$ which are orientation preserving. In particular, the components of the fixed point space inherit an orientation. (See [20] and the references therein for details.)

EXAMPLE 1. If $M \to N$ is a cyclic branched covering of complex manifolds, the representation W is a direct sum of trivial summands and a complex line on which the cyclic group acts in the natural way. □

The following example has been considered and applied in [18]. Note that contrary to the previous example, the fixed point space is not connected here.

EXAMPLE 2. For a prime order group G, and a complex G-representation V, let $M = \mathbb{C}P(V)$ be the associated projective space. The fixed point space is the disjoint union of the projective spaces of the isotypical summands, hence is equidimensional if V is a multiple of the regular G-representation. In that case, all tangential representations are isomorphic, as complex G-representations, to the complement of a trivial line in V. This we may take as our W. □

In this writing, we will be concerned with oriented W-dimensional semi-free G-manifolds M, and prove an equivariant Hopf theorem which gives a description of the set of G-homotopy classes of G-maps from M to S^W in the generic case when the dimension and codimension of the fixed point space are at least 2. (See Section 2 for free actions; trivial actions can be dealt with as in Example 1 below.) By orientability, the codimension of the fixed point space is always at least two if the action is non-trivial.

While the Hopf problem looks like an unstable problem, its turns out to be stable. Equivariant stable homotopy theory is more complicated than ordinary stable homotopy theory, as there is a stable homotopy category for each G-*universe*, which is an infinite-dimensional G-representation which contains the trivial representation, and contains each of its subrepresentation with infinite multiplicity. (Standard references for equivariant stable homotopy theory are [1], [15], [17], and [8].) The full-flavoured equivariant stable homotopy category corresponds to a *complete* G-universe, which contains every irreducible G-representation. A *semi-free* universe is obtained from a complete G-universe by restriction to the semi-free subrepresentations. The stable homotopy category depends only on the G-isomorphism class of the universe, and we will write $\{X,Y\}^G_{\mathrm{sf}}$ for the morphism groups from X to Y in an equivariant stable homotopy category corresponding to a semi-free G-universe. As the stable categories requires based objects, we will let M_+ denote the G-space which is M with a disjoint G-fixed basepoint added. By adjunction, there is a canonical bijection between the set $[M_+,S^W]^G$ of based G-homotopy classes and the set of (unbased) G-homotopy classes of maps from M to S^W. The proof of the following result is in Section 1. It justifies working stably afterwards.

THEOREM 1. *Let W be a G-representation, and let M be a W-dimensional semi-free G-manifold such that the dimension and codimension of the fixed point space are at least 2. Then the stabilisation map*

$$[M_+,S^W]^G \longrightarrow \{M_+,S^W\}^G_{\mathrm{sf}}$$

is bijective.

In order to study the stable groups $\{M_+,S^W\}^G_{\mathrm{sf}}$, the standard approach is to associate to a stable G-map f the set of all the degrees of the fixed point maps f^H for the various subgroups H of G. In the semi-free case, the only subgroups relevant are 1 and G, so that the fixed point maps can be organised into the *ghost map*

$$\{M_+, S^W\}^G_{\mathrm{sf}} \longrightarrow \{M_+, S^W\} \oplus \{M^G_+, S^{W^G}\} \tag{1}$$

which sends a stable G-map f to the pair (f, f^G). A priori, the first factor of the ghost map lies in the G-invariants of $\{M_+, S^W\}$, but the present assumptions on the G-actions on M and W ensure that G acts trivially on this group. Using the ordinary Hopf theorem, the image of an f under (1) can be interpreted in terms of degrees: an integer x and a sequence (y_α) of integers, indexed by the components of M^G. (See Section 8 for details.) One may show – using standard splitting and localisation techniques – that the ghost map (1) is an isomorphism away from the order of G, but this will also be shown directly in the course of this investigation. It means that the kernel and cokernel are finite abelian groups. The following summarises Propositions 7 and 8 of the main part.

THEOREM 2. *Let M be an oriented W-dimensional G-manifold. The source and the target of the ghost map* (1) *are free abelian. Their rank is one larger than the number of components of M^G. The map is injective and the cokernel is cyclic of order $|G|$.*

In other words, stable G-maps are determined by their degrees, and the numbers that occur satisfy a relation. The following result, which is proved in Section 8, says which.

THEOREM 3. *The image of the ghost map* (1) *is given by the subgroup of integers x and sequences (y_α) for which the congruence*

$$x \equiv \sum_\alpha y_\alpha \bmod |G|$$

is satisfied.

The chosen approach to the two theorems separates the homotopy theory from the geometry. Let me illustrate this by explaining how the well-known description of the Burnside rings of prime order groups can be deduced using these methods.

EXAMPLE 3. Let G be a prime order group, and let M be a point. Then M_+ and S^W are both 0-dimensional spheres. While the G-actions on M is trivial, of course, it will no longer be after stabilisation with a suitable oriented G-representation. Let us assume that this has been done without changing the notation. The group $\{S^0, S^0\}^G$ maps via the forgetful map to the group $\{S^0, S^0\} = \mathbb{Z}$. On the other hand, it maps to the group $\{(S^0)^G, (S^0)^G\} = \mathbb{Z}$ via the fixed point map. The product

$$\{S^0, S^0\}^G \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$

of the two maps is the ghost map. It is injective, and the image has index $|G|$. The image is not yet determined by this. There are still $|G| + 1$ choices. In order to give a description of the image, one needs to put in some new information. One can use the fact that the identity of S^0 is G-equivariant. It is mapped to the pair $(1, 1)$. This determines the image, which consists of those pairs (x, y) in $\mathbb{Z} \oplus \mathbb{Z}$ which satisfy the relation $x \equiv y \bmod |G|$. □

The homotopy theory for Theorem 2 is done in Sections 3 to 7. The method of proof is to deal with the G-trivial space M^G and with the G-space M/M^G, which is free away from the base point, separately. The latter uses a result from Section 2 which sometimes allows for a comparison of the group of G-maps out of a free G-space with the group of ordinary maps out of the quotient in the case when G does *not* act trivially on the target. In the final Section 8, the geometry of the situation is used to construct some maps whose images under the ghost map are easy to determine, leading to Theorem 3.

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1. STABILITY

In this section, we will prove the stability of the Hopf problem, and the proof given will also show that the semi-stable universe is the appropriate context in which to work.

Let us recall the classical situation. If M is an oriented n-manifold, by Freudenthal's suspension theorem, the stabilisation map

$$[M_+, S^n] \longrightarrow \{M_+, S^n\}$$

is bijective if $n \geqslant 2$. In the equivariant situation, we may use the equivariant extension of Freudenthal's theorem due to H. Hauschild, see [9] (or [1] for an exposition in English). This implies that the suspension map

$$[X, Y]^G \longrightarrow [\Sigma^V X, \Sigma^V Y]^G$$

is bijective for finite based G-CW-complexes X and Y, if two conditions are satisfied: the connectivity of Y^H has to be at least $\dim(X^H)/2$ for all subgroups H of G such that $V^H \neq 0$, and the connectivity of Y^K has to be at least $\dim(X^H)+1$ for all subgroups $K < H$ of G such that $V^H \neq V^K$. This will now be used to prove Theorem 1.

PROOF. The first condition corresponds to Freudenthal's condition in the non-equivariant case. It is therefore satisfied for all H – and for all V – in our situation $X = M_+$ and $Y = S^W$ as long as the dimension of M^G is at least 2.

The other condition refers to a genuinely equivariant phenomenon. Suppose first that $K = 1$ and H is a non-trivial subgroup of G. Then $M^H = M^G$ since M is semi-free, and the condition is satisfied for these K and H – and for all V – if and only if the codimension of M^G in M is at least 2.

If K and H are non-trivial subgroups of G such that $V^H \neq V^K$, the condition $\dim(M^G) \leqslant \dim(W^G) - 2$ needs to be satisfied, which does not seem to be the case. However, we always have $V^H = V^G = V^K$ if V is semi-free, so this does not occur, and the proof is finished. □

There is a change-of-universe natural transformation from $\{X, Y\}^G_{\mathrm{sf}}$ to the corresponding group with respect to a complete G-universe, see [13]. However, the preceding proof shows that one should not expect it to be bijective. It is clearly bijective if the order of G is a prime, since semi-free universes are complete for such G. In a similar vein, the proof above can easily be adapted to show that the stabilisation map from $[M_+, S^W]^G$ to the corresponding group of stable G-maps with respect to a complete G-universe is always bijective if M satisfies a suitable gap hypothesis: The fixed point space M^G has to be at least 2-dimensional, and the codimension of M^H in M^K has to be at least 2 for all subgroups $K < H$ of G.

2. A COMPARISON RESULT

Let G be a finite group. A based G-space is called *free* if the group G acts freely on the complement of the base point (which is fixed by G). Let F be a finite free based G-CW-complex. Since G is finite, this also yields an ordinary CW-structure on F. And it gives an ordinary CW-structure on the quotient $Q = F/G$.

If Y is a G-space, the group $\{F,Y\}^G_{\mathrm{sf}}$ is not obviously related to $\{Q,Y\}$ since the G-action on Y need not be trivial. If Y were a trivial G-space, there would be a tautological map from $\{Q,Y\}$ to $\{Q,Y\}^G_{\mathrm{sf}}$. One could then use a semi-free version of (5.3) from [1], which says that in this case the composition $\{Q,Y\} \to \{Q,Y\}^G_{\mathrm{sf}} \to \{F,Y\}^G_{\mathrm{sf}}$ with the map induced by the projection from F to Q would be an isomorphism. However, since the action on Y is non-trivial, the arrow on the left is not defined. Nevertheless, there sometimes is a way to compare $\{F,Y\}^G_{\mathrm{sf}}$ with $\{Q,Y\}$. This will be explained now.

By mapping the CW-filtrations into Y, one gets three spectral sequences, which converge to $\{F,Y\}^G_{\mathrm{sf}}$, $\{F,Y\}$ and $\{Q,Y\}$, respectively.

For an integer s, let $I(s)$ denote the (finite) set of s-dimensional G-cells in F. The filtration gives rise to a spectral sequence

$$E_1^{s,t} = \Big\{\bigvee_{I(s)} G_+, \Sigma^t Y\Big\}^G_{\mathrm{sf}} \Longrightarrow \{F, \Sigma^{s+t} Y\}^G_{\mathrm{sf}}. \tag{2}$$

In order to compute the groups on the E_2-page, the differentials on the E_1-page need to be discussed. The t-th row is obtained by applying the functor $\{?, \Sigma^t Y\}^G_{\mathrm{sf}}$ to the G-cellular complex

$$* \longleftarrow \bigvee_{I(0)} G_+ \longleftarrow \bigvee_{I(1)} G_+ \longleftarrow \ldots \longleftarrow \bigvee_{I(n)} G_+ \longleftarrow *$$

of F. Note that

$$\Big\{\bigvee_{I(s)} G_+, \Sigma^t Y\Big\}^G_{\mathrm{sf}} \cong \bigoplus_{I(s)} \{G_+, \Sigma^t Y\}^G_{\mathrm{sf}}, \tag{3}$$

and that the groups $\{G_+, \Sigma^t Y\}^G_{\mathrm{sf}}$ are right modules over $\{G_+, G_+\}^G_{\mathrm{sf}}$ via composition. Because of (3), the differentials can be identified with matrices, and the entries are given by right multiplication with elements of $\{G_+, G_+\}^G_{\mathrm{sf}}$. Therefore, let me pause to discuss this action in more detail.

Sending an element g of G to the G-map $G_+ \to G_+$ which sends an element x to xg induces an isomorphism of the group ring $\mathbb{Z}G$ with $\{G_+, G_+\}^G_{\mathrm{sf}}$ which reverses the order of the multiplication. Therefore, $\{G_+, \Sigma^t Y\}^G_{\mathrm{sf}}$ is a left $\mathbb{Z}G$-module. As such, it is isomorphic with the left $\mathbb{Z}G$-module $\{S^0, \Sigma^t Y\}$, the G-action being induced by the action on Y. (The adjunction isomorphism (5.1) in [1] is G-linear.) This finishes the digression on the $\mathbb{Z}G$-module structures.

One can now try to compare the spectral sequence (2) to the one

$$E_1^{s,t} = \Big\{\bigvee_{I(s)} S^0, \Sigma^t Y\Big\} \Longrightarrow \{Q, Y\}^{s+t} \tag{4}$$

obtained by the induced CW-filtration of Q. (This is of course the Atiyah-Hirzebruch spectral sequence.) As mentioned above, there is no reasonable map between the targets in sight, and I cannot offer a map of spectral sequences. However, note that the groups on the E_1-pages are always isomorphic: $\{G_+, \Sigma^t Y\}^G_{\mathrm{sf}} \cong \{S^0, \Sigma^t Y\}$, again by the adjunction isomorphism (5.1) in [1]. As for the differentials, the following is true.

PROPOSITION 1. *If $\{S^0, \Sigma^t Y\}$ is a trivial $\mathbb{Z}G$-module, the t-rows on the E_1-terms of the spectral sequences* (2) *and* (4) *are isomorphic as complexes. In particular, the groups on the t-rows of the E_2-pages are isomorphic.*

PROOF. The differentials on the E_1-page of the spectral sequence (4) are obtained by applying the functors $\{?, \Sigma^t Y\}$ to the cellular complex

$$* \longleftarrow \bigvee_{I(0)} S^0 \longleftarrow \bigvee_{I(1)} S^0 \longleftarrow \dots \longleftarrow \bigvee_{I(n)} S^0 \longleftarrow *$$

of Q. As for (2), they can be thought of as matrices. This time, the entries are obtained from the entries of those in (2) by passage to quotients, i.e. by applying the map $\epsilon : \{G_+, G_+\}^G_{\mathrm{sf}} \to \{S^0, S^0\},\ g \mapsto 1$. This means that the diagram

$$\begin{array}{ccc} \{\bigvee_{I(s)} G_+, \Sigma^t Y\}^G_{\mathrm{sf}} & \longrightarrow & \{\bigvee_{I(s+1)} G_+, \Sigma^t Y\}^G_{\mathrm{sf}} \\ \cong\uparrow & & \uparrow\cong \\ \{\bigvee_{I(s)} S^0, \Sigma^t Y\} & \longrightarrow & \{\bigvee_{I(s+1)} S^0, \Sigma^t Y\} \end{array}$$

(obtained from the isomorphisms above and the differentials) is only commutative if the elements in $\mathbb{Z}G$ which appear in the matrix of the top arrow act via ϵ. While at first sight it seems that one needs to know the details of the G-CW-structure to proceed, this is not the case: if $\{S^0, \Sigma^t Y\}$ is a trivial $\mathbb{Z}G$-module, the condition is fulfilled for all elements of $\mathbb{Z}G$. □

In nice situations, this result implies that $\{F, Y\}^G_{\mathrm{sf}}$ and $\{Q, Y\}$ are in fact isomorphic. One of these situations will be encountered in the following section.

3. FREE POINTS OF SPHERES

Let W be a orientable real G-representation which in non-trivial and semi-free. The results of the previous section will now be applied to the free G-space $F = S^W/S^{W^G}$ and $Y = S^W$. Let n and n^G be the real dimensions of S^W and S^{W^G}, respectively. Note that the number $n - n^G$ is positive and even.

PROPOSITION 2. *The three groups*

$$\{S^W/S^{W^G}, S^W\}, \quad \{(S^W/S^{W^G})/G, S^W\}, \quad \textit{and} \quad \{S^W/S^{W^G}, S^W\}^G_{\mathrm{sf}}$$

are all free abelian on one generator.

PROOF. As for $\{S^W/S^{W^G}, S^W\}$: Since the G-representation W is non-trivial, one knows that the quotient S^W/S^{W^G} is non-equivariantly equivalent to a wedge $S^n \vee S^{n^G+1}$. The map from the group $\{S^W/S^{W^G}, S^W\}$ to $\{S^W, S^W\} = \mathbb{Z}$ induced by the collapse map is an isomorphism. Note that $\{S^W/S^{W^G}, S^W\}$ is isomorphic to $\mathrm{H}^n(S^W/S^{W^G};\mathbb{Z})$ via the Hurewicz map.

As for $\{(S^W/S^{W^G})/G, S^W\}$: Also via the Hurewicz map, this group is isomorphic to the group $\mathrm{H}^n((S^W/S^{W^G})/G;\mathbb{Z})$. Therefore, one can use the spectral sequence

$$E_2^{s,t} = \mathrm{H}^s(G;\mathrm{H}^t(S^W/S^{W^G};\mathbb{Z})) \Longrightarrow \mathrm{H}^{s+t}((S^W/S^{W^G})/G;\mathbb{Z}) \tag{5}$$

for that. From the stable homotopy type of S^W/S^{W^G} it follows that the only non-trivial groups on the E_2-page are in two rows: $t = n^G + 1$ and $t = n$. Each of these contains the cohomology of the group G with coefficients in the trivial G-module $\mathbb{Z}$.

Thus there is only one page on which non-trivial differentials may occur. Since the dimension of $(S^W/S^{W^G})/G$ is at most $n = \dim_{\mathbb{R}}(W)$, all differentials between non-trivial groups must be non-trivial. This determines the E_∞-page. There are no extension problems. It follows that $\{(S^W/S^{W^G})/G, S^W\} \cong \mathbb{Z}$. But, notice the edge homomorphism of the spectral sequence, which is induced by the quotient map from S^W/S^{W^G} to $(S^W/S^{W^G})/G$. It can immediately be read off that the generator of $\mathrm{H}^n((S^W/S^{W^G})/G;\mathbb{Z})$ is mapped to $|G|$ times the generator of $\mathrm{H}^n(S^W/S^{W^G};\mathbb{Z})$. Of course, this just reminds us that the quotient map has degree $|G|$.

As for $\{S^W/S^{W^G}, S^W\}^G_{\text{sf}}$, Proposition 1 from the previous section can be used. The assumption on the action is satisfied for $Y = S^W$ and all t since the G-action on W preserves the orientation. We can thus deduce some of the groups on the E_2-page of the spectral sequence (2) from the previously calculated groups on the E_2-page of the spectral sequence (4): the groups $\mathrm{H}^s((S^W/S^{W^G})/G\,;\pi^t(S^W))$ vanish if $s > n$ or $t > 0$, and hence the only pair (s,t) with $s+t=n$ and $E_2^{s,t} \neq 0$ is the pair $(s,t)=(n,0)$. The corresponding group $\mathrm{H}^n((S^W/S^{W^G})/G\,;\mathbb{Z})$ has been shown to be isomorphic to $\mathbb{Z}$. There are no non-trivial differentials into and out of it, so it survives to E_∞. There are no extension problems. □

Now that the structure of those three groups is known, it is desirable to know the maps between them. The proof of the preceding proposition shows that the map

$$\{(S^W/S^{W^G})/G, S^W\} \to \{S^W/S^{W^G}, S^W\}$$

is injective with cyclic cokernel of order $|G|$. The same holds for the forgetful map:

PROPOSITION 3. *The forgetful map*

$$\{S^W/S^{W^G}, S^W\}^G_{\text{sf}} \longrightarrow \{S^W/S^{W^G}, S^W\}$$

is injective and has a cyclic cokernel of order $|G|$.

PROOF. Let us contemplate the following diagram.

$$\begin{array}{ccccc}
\{S^{W^G}, S^W\}^G_{\text{sf}} & \longleftarrow & \{S^W, S^W\}^G_{\text{sf}} & \longleftarrow & \{S^W/S^{W^G}, S^W\}^G_{\text{sf}} \\
 & & \downarrow & & \downarrow \\
 & & \{S^W, S^W\} & \xleftarrow{\cong} & \{S^W/S^{W^G}, S^W\}
\end{array}$$

The horizontal maps are induced by the obvious cofibre sequence. Therefore, the top row is exact. The vertical maps are the forgetful maps and make the diagram commute. Now if one picks a map in $\{S^W/S^{W^G}, S^W\}^G_{\text{sf}}$, its image in $\{S^W, S^W\}^G_{\text{sf}}$ has degree zero when restricted to the fixed point spheres. It follows that the degree of the image itself is a multiple of $|G|$ by following proposition. □

PROPOSITION 4. *Let $f\colon S^W \to S^W$ be G-equivariant for some finite group G. If the degree of the restriction of f to the fixed sphere is zero, then the degree of f itself is divisible by the order of G.*

PROOF. We will use Borel cohomology $b^*(X) = \mathrm{H}^*(EG_+ \wedge_G X; \mathbb{Z}/|G|)$ with co-efficients in $\mathbb{Z}/|G|$. The b^*-modules $b^*(S^{W^G})$ and $b^*(S^W)$ are free of rank 1 by the suspension theorem and the Thom isomorphism, respectively. The inclusion $S^{W^G} \subseteq S^W$ induces an inclusion $b^*(S^W) \subseteq b^*(S^{W^G})$ since the quotient S^W/S^{W^G} is free and therefore has b^*-torsion Borel cohomology. By hypothesis, the map f induces to zero map on $b^*(S^{W^G})$, and so it has to on $b^*(S^W)$. $\square$

4. FREE POINTS IN GENERAL

Let W be an orientable real G-representation which is non-trivial and semi-free, as before. The results of the previous section will now be generalised from S^W to oriented W-dimensional G-manifolds M.

PROPOSITION 5. *Source and target of the forgetful map*

$$\{M/M^G, S^W\}^G_{\mathrm{sf}} \longrightarrow \{M/M^G, S^W\}$$

are free abelian on one generator. The map is injective with a cyclic cokernel of order $|G|$.

PROOF. Choose an orientation preserving G-embedding of W onto a neighbourhood of a fixed point. Let A be the complement of the image. The collapse map from M to $M/A \cong S^W$ and the forgetful maps induce a commutative diagram

$$\begin{array}{ccc} \{M/M^G, S^W\}^G_{\mathrm{sf}} & \longleftarrow & \{S^W/S^{W^G}, S^W\}^G_{\mathrm{sf}} \\ \downarrow & & \downarrow \\ \{M/M^G, S^W\} & \longleftarrow & \{S^W/S^{W^G}, S^W\}. \end{array}$$

By Proposition 3 it is sufficient to show that the two horizontal maps are isomorphisms. The bottom row is isomorphic to

$$\{M_+, S^W\} \longleftarrow \{S^W, S^W\},$$

which is an isomorphism by the ordinary Hopf theorem. As for the top row, it is easy to see that it is surjective: the fibre of the inclusion of M/M^G in to S^W/S^{W^G} is A/A^G which is G-free and cohomologically at most $(n-1)$-dimensional. It follows that $\{A/A^G, S^W\}^G_{\text{sf}}$ is trivial. It does not follow, however, that $\{\Sigma(A/A^G), S^W\}^G_{\text{sf}}$ is trivial, too. But injectivity of the top arrow follows from injectivity of the right arrow, see Proposition 3, and injectivity of the bottom arrow, which has already been proven. □

5. FIXED POINTS

Let T be a trivial G-space, and let M be an oriented W-dimensional G-manifold as before. There is a map from $\{T, S^{W^G}\}$ to $\{T, S^{W^G}\}^G_{\text{sf}}$, using the fact that any map between trivial G-spaces is a G-map, and there is a map from $\{T, S^{W^G}\}^G_{\text{sf}}$ to $\{T, S^W\}^G_{\text{sf}}$ induced by the inclusion of S^{W^G} into S^W which is a G-map. The composition

$$\{T, S^{W^G}\} \longrightarrow \{T, S^W\}^G_{\text{sf}} \tag{6}$$

has a retraction, namely the fixed point map. The splitting theorem implies that a complement for the image of $\{T, S^{W^G}\}$ in $\{T, S^W\}^G_{\text{sf}}$ is isomorphic to $\{T, EG_+ \wedge_G S^W\}$. (See [14], Section 2, for the version for incomplete universes needed here.) One can use that to prove the following.

PROPOSITION 6. *The group $\{M^G_+, S^W\}^G_{\text{sf}}$ is isomorphic to the free abelian group $\{M^G_+, S^{W^G}\}$. The rank is the number of components of M^G.*

PROOF. Since S^W is $(n-1)$-connected, so is $EG_+ \wedge_G S^W$. The dimension of M^G_+ is smaller than n. It follows that the group $\{M^G_+, EG_+ \wedge_G S^W\}$ is zero. Thus, the map (6) is an isomorphism in the case $T = M^G_+$. This and the ordinary Hopf theorem imply that $\{M^G_+, S^W\}^G_{\text{sf}} \cong \{M^G_+, S^{W^G}\}$, which is free abelian of the indicated rank. □

6. ALL POINTS

Let M be an oriented W-dimensional G-manifold as before. We are now in the position to use the information gathered on the free points and on the fixed points in order to determine the structure of the source $\{M_+, S^W\}^G_{\text{sf}}$ of the ghost map.

PROPOSITION 7. *The group $\{M_+, S^W\}^G_{\text{sf}}$ is free abelian of rank one more than the number of components of M^G.*

PROOF. The starting point for the calculation of $\{M_+, S^W\}^G_{\text{sf}}$ is the cofibre sequence

$$M^G_+ \longrightarrow M_+ \longrightarrow M/M^G.$$

Mapping this into S^W, we get a long exact sequence

$$\cdots \longleftarrow \{M^G_+, S^W\}^G_{\text{sf}} \longleftarrow \{M_+, S^W\}^G_{\text{sf}} \longleftarrow \{M/M^G, S^W\}^G_{\text{sf}} \longleftarrow \cdots .$$

The group in the middle is to be computed. On the left side, since the G-space M/M^G is free and of smaller dimension than ΣS^W, the next group on the left $\{\Sigma^{-1}(M/M^G), S^W\}^G_{\text{sf}}$, which is isomorphic to $\{M/M^G, \Sigma S^W\}^G_{\text{sf}}$, is trivial. On the right side, the diagram

$$\begin{array}{ccc} \{M/M^G, S^W\}^G_{\text{sf}} & \longleftarrow & \{\Sigma M^G_+, S^W\}^G_{\text{sf}} \\ \big\downarrow{\scriptstyle \text{forget}} & & \big\downarrow{\scriptstyle \text{forget}} \\ \{M/M^G, S^W\} & \longleftarrow & \{\Sigma M^G_+, S^W\} \end{array}$$

shows that the top map from $\{\Sigma M^G_+, S^W\}^G_{\text{sf}}$ to $\{M/M^G, S^W\}^G_{\text{sf}}$ is zero: the left map is injective by Proposition 5, and the bottom right group is trivial by the dimension and connectivity of the spaces involved.

To sum up, there is a short exact sequence

$$0 \longleftarrow \{M^G_+, S^W\}^G_{\text{sf}} \longleftarrow \{M_+, S^W\}^G_{\text{sf}} \longleftarrow \{M/M^G, S^W\}^G_{\text{sf}} \longleftarrow 0.$$

By Proposition 6 again, $\{M^G_+, S^W\}^G_{\text{sf}} \cong \{M^G_+, S^{W^G}\}$ is free abelian of the indicated rank. Thus, the short exact sequence must be splittable. □

7. THE GHOST MAP

Let M be an oriented W-dimensional G-manifold as before. Now that the structure of the source and the target of the ghost map are known, it is time to study the map itself.

PROPOSITION 8. *The ghost map* (1) *is injective with a cyclic cokernel of order* $|G|$.

PROOF. One may compare the short exact sequence used in the proof of Proposition 7 to the short exact sequence

$$0 \leftarrow \{M^G_+, S^{W^G}\} \leftarrow \{M_+, S^W\} \oplus \{M^G_+, S^{W^G}\} \leftarrow \{M/M^G, S^W\} \leftarrow 0,$$

which is built by using the isomorphism between $\{M/M^G, S^W\}$ and the group $\{M_+, S^W\}$ discussed before and the identity on $\{M_+^G, S^{W^G}\}$. The two short exact sequences yield the rows in the diagram

$$\begin{array}{ccccc}
\{M_+^G, S^W\}_{\mathrm{sf}}^G & \longleftarrow & \{M_+, S^W\}_{\mathrm{sf}}^G & \longleftarrow & \{M/M^G, S^W\}_{\mathrm{sf}}^G \\
\downarrow & & \downarrow & & \downarrow \\
\{M_+^G, S^{W^G}\} & \longleftarrow & \{M_+, S^W\} \oplus \{M_+^G, S^{W^G}\} & \longleftarrow & \{M/M^G, S^W\},
\end{array}$$

which will now be used to compare both of them. The vertical arrow on the right is the forgetful map. This map was shown to be injective with a cyclic cokernel of order $|G|$ in Proposition 3. The vertical map in the middle is the ghost map: it sends a G-map f to the pair (f, f^G). The vertical arrow on the left is the isomorphism which sends f to f^G as discussed above. The diagram commutes, and the snake lemma implies the result. □

From what has already been shown, it is by now established that the image of the group $\{M_+, S^W\}_{\mathrm{sf}}^G$ under the ghost map is a subgroup of index $|G|$ in the direct sum $\{M_+, S^W\} \oplus \{M_+^G, S^{W^G}\}$. This subgroup contains $(|G|, 0)$ and projects onto $\{M_+^G, S^{W^G}\}$. But, this does not determine the image. Additional information from the geometric situation seems to be required. This will be supplied for in the following final section.

8. Examples of maps

Let M be an oriented W-dimensional G-manifold as before. In this final section, the group $\{M_+, S^W\}$, which is free abelian on one generator by the ordinary Hopf theorem, has a distinguished generator, namely the one that preserves the orientations. The elements of this group can hence be thought of as integers. Similarly, the fixed point space of M decomposes into components:

$$M^G = \coprod_\alpha M_\alpha^G.$$

The dimension of any of the components M_α^G agrees with that of S^{W^G}. Thus the group $\{(M_\alpha^G)_+, S^{W^G}\}$ has a distinguished generator, too. Collecting these together, the restriction of an element in $\{M_+, S^W\}_{\mathrm{sf}}^G$ to the fixed point space gives a family of integers, one for each α. Using these identifications, the ghost map sends an equivariant map to a pair (x, y) consisting of an integer x and a family $y = (y_\alpha)$ of integers.

With these preparations, we may now prove Theorem 3 from the introduction. The proof works as in Example 3 from the introduction.

PROOF. Let us fix an α. For any point in M^G_α, a neighbourhood in M is G-homeomorphic to the G-representation W. Collapsing the complement yields a G-map f_α from M_+ to S^W. Note that this map is the chosen generator of $\{M_+, S^W\}$, and the restriction to $(M^G_\alpha)_+$ is the chosen generator of the group $\{(M^G_\alpha)_+, S^W\}$. On the other hand, for $\beta \neq \alpha$, the collapse map sends the subspace $(M^G_\beta)_+$ to a point. Thus, the corresponding element in $\{(M^G_\beta)_+, S^W\}$ is zero. Thus if 1_α denotes the characteristic function of α, the element of $\{M_+, S^W\}^G_{\text{sf}}$ which is represented by f_α is mapped to $(1, 1_\alpha)$.

Now for each α there has been produced a G-map f_α in $\{M_+, S^W\}^G_{\text{sf}}$ which the ghost map sends to a pair $(x, y) = (1, 1_\alpha)$ satisfying the relation in the theorem. The subgroup of all pairs satisfying that relation is the unique subgroup of index $|G|$ which contains the pairs $(1, 1_\alpha)$ for all α. □

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