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EQUIVARIANT STABLE HOMOTOPY THEORY

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Equivariant maps between spheres.

Let G be a finite group.

A finite-dimensional real vector-space V on which G acts linearly will be called a *G-module.* Its one-point compactification S^V is a sphere with G-action, in which we shall regard ∞ as a base-point. Our object is to describe the homotopy-classes of equivariant maps between such spheres.

For each G-module V there is a concept of *suspension* : if X is a G-space with base-point x_0 we define the suspension $S^V X$ as

$$S^V \wedge X = (S^V \times X)/((\infty \times X) \cup (S^V \times x_0)).$$

If X and Y are G-spaces with base-points then $[X ; Y]_G$ denotes the set of homotopy-classes of base-point-preserving G-maps $X \to Y$. There is a suspension-map $[X ; Y]_G \to [S^V X ; S^V Y]_G$ for any G-module V. One can order the isomorphism-classes of G-modules by

$$V \leqslant V' \Longleftrightarrow V \text{ is isomorphic to a submodule of } V' ;$$

then one defines the set of stable equivariant maps

$$\{X ; Y\}_G = \varinjlim_{V} [S^V X ; S^V Y],$$

which is an abelian group. (Strictly speaking the limit is taken over the category of G-modules and embeddings).

PROPOSITION 1. — $[S^{V\oplus W} ; S^W]_G$ is independent of W if W is sufficiently large, and can be identified with the set of cobordism-classes of compact V-framed G-manifolds.

The terminology in the proposition is explained by

DEFINITION 1. — If V is a G-module, a compact G-manifold M is called *V-framed* if there is given a stable G-isomorphism φ_M of its tangent bundle T_M with $M \times V$, i.e. if there is given a G-module W and an isomorphism of G-vector-bundles $T_M \oplus (M \times W) \cong M \times (V \oplus W)$. Such a manifold is said to bound if there is a G-manifold N with boundary M and a stable isomorphism of T_N with $N \times (V \oplus \mathbf{R})$ wich induces φ_M.

The proof of Proposition 1 depends essentially on the concept of "consistent transversality" introduced by Wasserman [2]. Details can be found in [1].

Proposition 1 makes clear in particular what plays the role of the *degree* of a map in the equivariant theory. Recall that one defines for any group G its *Burnside* ring $A(G)$ as the Grothendieck group of the category of finite G-sets, i.e. $A(G)$ is the free abelian group on the set of conjugacy-classes of subgroups of finite index in G. Then we have

COROLLARY. – For large W, $[S^W ; S^W]_G \cong A(G)$ as rings, where the multiplication in $[S^W ; S^W]_G$ is composition of maps, and that in $A(G)$ corresponds to forming the product of G-sets.

Thus the equivariant homotopy class of a map $S^W \to S^W$ is determined by the degrees of its restrictions to the fixed-point subsets of the subgroups H of G ; and the diagram

$$\begin{array}{ccc} [S^W ; S^W]_G & \longrightarrow & A(G) \\ \downarrow & & \downarrow \epsilon_H \\ [(S^W)^H ; (S^W)^H] & \xrightarrow{\text{degree}} & \mathbf{Z} \end{array}$$

commutes, where ϵ_H assigns to a G-set S the cardinal of S^H.

PROPOSITION 2. – If $V = \mathbf{R}^n$ with trivial G-action then

$$[S^{V \oplus W} ; S^W]_G \cong \bigoplus_H \pi_n^S (BW_H) ,$$

where the sum is taken over the conjugacy classes of subgroups H of G, π_*^S denotes stable homotopy, and $W_H = N_H/H$, where N_H is the normalizer of H in G.

Proof. – If M is a V-framed G-manifold then the isotropy-group must be constant on each component of M, for if g is an element of the isotropy group at x then g acts trivially on the tangent-space to M at x, and so leaves fixed all the geodesics through x. But if M has all its isotropy-groups conjugate to H one can write it as $(G/H) \times_{W_H} M^H$, where M^H, the H-invariant part of M, is a free W_H space. Thus a general V-framed manifold can be written $\coprod_H (G/H) \times_{W_H} M_H$, where M_H is a V-framed free W_H-manifold. The cobordism-classes of such M_H can be identified with $\pi_n^S (BW_H)$, and Proposition 2 follows.

Equivariant stable cohomology theory

For any pair $Y \subset X$ of compact G-spaces and any virtual G-module α (i.e. any $\alpha \in RO(G)$) let us define

$$\omega_G^\alpha (X , Y) = \varinjlim_V \, [S^V(X/Y) ; S^{V+\alpha}] = \{X/Y ; S^\alpha\}.$$

This is a generalized cohomology theory in the sense that it satisfies obvious homotopy, exactness, and excision axioms (for any pair (X , Y) there is a boundary homomorphism $\omega_G^\alpha(Y) \to \omega_G^{\alpha+1} (X , Y)$). It has the additional stability property that $\tilde{\omega}_G^\alpha(X) \cong \tilde{\omega}_G^{\alpha+V}(S^V X)$ for any X and V. Furthermore it is universal among cohomology theories with those four properties.

On free G-spaces and trivial G-spaces one can express ω_G^α in terms of ordinary stable homotopy, at least when $\alpha \in Z \subset RO(G)$, as follows.

PROPOSITION 3. – If X is a free compact G-space, then

$$\omega_G^n(X) \cong \omega^n(X/G) = \pi_S^n(X/G).$$

PROPOSITION 4. – If G acts trivially on X then

$$\omega_G^n(X) \cong \bigoplus_H \{X\,;\,S^n BW_H^+\},$$

where BW_H^+ is the union of BW_H with a disjoint base-point.

As ordinary stable homotopy coincides with homology when tensored with the rationals, and as classifying-spaces for finite groups have trivial rational homology, one deduces from Proposition 4 that $\omega_G^n(X) \otimes \mathbf{Q} \cong A(G) \otimes H^n(X\,;\mathbf{Q})$ when G acts trivially on X. More generally one has

PROPOSITION 5. – For any compact G-space X, and any $\alpha \in RO(G)$,

$$\omega_G^\alpha(X) \otimes \mathbf{Q} \cong \bigoplus_H H^{\alpha_H}(X^H\,;\,\mathbf{Q})^{W_H}$$

where, if $\alpha = V - W \in RO(G)$, $\alpha_H = \dim V^H - \dim W^H$.

It is easy to see that $\{S^V; S^W\}_G$ is a finitely generated abelian group, so Proposition 5 implies the

COROLLARY. – $\{S^V; S^W\}_G$ is finite unless $\dim V^H = \dim W^H$ for some subgroup H of G.

The equivariant J-homomorphism (1).

The relationship between equivariant stable cohomotopy as defined here and equivariant K-theory is precisely analogous to that in the classical case. There is a J-homomorphism

$$J : KO_G^{-1}(X) \to \omega_G^0(X)$$

(from the additive group KO_G^{-1} to the multiplicative group of ω_G^0) defined by the usual Hopf construction. Its image can be determined in the following way.

The Adams operations ψ^k act on $KO_G(X)$, and hence on the profinite completion $KO_G(X)^\wedge$. They define an action of $\mathbf{Z}$ on $KO_G(X)^\wedge$ which is continuous when $\mathbf{Z}$ is given the profinite topology, and so the action extends to an action of the profinite completion $\hat{Z}$. The group of units $\hat{Z}^*$ of this ring is the product of the subgroup (± 1) with a topologically cyclic group Γ. Let α be a generator of Γ. Then $\psi^\alpha : KO_G(X)^\wedge \to KO_G(X)^\wedge$ extends to a transformation of multiplicative cohomology theories, and so one can define a new multiplicative cohomology theory J_G^* with a multiplicative transformation $J_G^* \to KO_G^{*\wedge}$ fitting into an exact triangle

$$\cdots \to J_G^* \to KO_G^{*\wedge} \xrightarrow{\psi^\alpha - 1} KO_G^{*\wedge} \to \cdots .$$

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(1) The proofs of the results in this section depend on the work of Sullivan on the Adams conjecture.

Thus there is a short exact sequence

$$0 \to \operatorname{coker}(\psi^a - 1) \to J_G^* \to \ker(\psi^a - 1) \to 0. \qquad \dots (\dagger)$$

In terms of the theory J_G^* one can describe the J-homomorphism as follows. The Hurewicz homomorphism $\omega_G^* \to KO_G^{*\wedge}$ factorizes through J_G^*, giving a multiplicative transformation $h : \omega_G^* \to J_G^*$. In view of the exact sequence (†) one sees that this assigns to an element of stable cohomotopy its d- and e-invariants in the sense of Adams. The J-homomorphism $J : KO_G^{-1}(X) \to \omega_G^0(X)$ factorizes through $\widetilde{J}_G^0(X)$ to give an exponential map $J : \widetilde{J}_G^0(X) \to \omega_G^0(X)$. If G is a p-group the composite $hJ : \widetilde{J}_G^0(X) \to J_G^0(X)$ is an isomorphism between the additive group $\widetilde{J}_G^0(X)$ and the multiplicative group $1 + \widetilde{J}_G^0(X)$. Then $\widetilde{J}_G^0(X)$ is a direct summand in the multiplicative group $1 + \widetilde{\omega}_G^0(X)$.

The definition of an equivariant cohomology theory.

In conclusion I shall mention two facts which tend to support the use of all real representations for suspending in equivariant stable homotopy theory, and the indexing of equivariant cohomology theories by $RO(G)$.

The first is the generalization of the construction of Eilenberg-Maclane spaces as the infinite symmetric products of spheres.

PROPOSITION 6. – If A is a topological abelian group with G-action, and V is a G-module, there is a G-space $B^V A$ and a G-homotopy-equivalence

$$A \to \operatorname{Map}(S^V ; B^V A) ;$$

and if $A = \mathbf{Z}$ with trivial G-action one can take $B^V A = F(S^V)$, the free abelian group on S^V.

The second is the generalization of a theorem of Barratt and Quillen. If S is a finite G-set let us write Σ_S for the group of G-automorphisms of S. One can form an associative monoid $\Gamma_G = \coprod_S B\Sigma_S$, the sum being over all finite G-sets S. The monoid can also be written $\prod_H \coprod_{n \in \mathbf{N}} B(\Sigma_n \int W_H)$, where H runs through the conjugacy-classes of subgroups G, and $\Sigma_n \int W_H$ denotes the semi-direct product

$$\Sigma_n \widetilde{\times} (\overleftarrow{W_H \times \overset{n}{\dots} \times W_H}).$$

PROPOSITION 7. – The classifying-space for Γ_G is homotopy-equivalent to that of $\varinjlim_V \operatorname{Map}_G(S^V ; S^V)$, where V runs through all G-modules.

The theorem of Barratt and Quillen tells one that

$$B(\coprod_n B(\Sigma_n \int W_H)) \simeq B(\Omega^\infty S^\infty (BW_H^+)),$$

so one deduces.

COROLLARY. – $\varinjlim_V \operatorname{Map}_G(S^V ; S^V) \simeq \prod_H \Omega^\infty S^\infty (BW_H^+)$.

This is of course just a restatement of Proposition 4, but it provides a completely different proof of it.

REFERENCES

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