$$\text{background fields in} \left.\begin{array}{l}\text{differential}\\ \text{twisted}\\ \text{nonabelian}\end{array}\right\} \text{cohomology}$$

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based on joint work with John Baez, Thomas Nikolaus, Hisham Sati, Zoran Škoda, Jim Stasheff, Danny Stevenson, Konrad Waldorf

goal:

- understand background fields for σ-model QFTs structurally:
- such that quantization to extended QFTs can be understood structurally by extension

$$\begin{tikzcd}
\Pi_n(X) \arrow[r, "\nabla"] \arrow[d, hook] & A \\
\mathrm{Bord}_n(X) \arrow[ur, "\exp(iS_\nabla(-))"'] \arrow[d] & \\
\mathrm{Bord}_n \arrow[uur, dashed, "\int^{\phi:\Sigma\to X} \exp(S(-))d\mu(\phi)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\begin{array}{c}\text{target}\\ \text{space}\end{array} \arrow[r, "\text{backgr. field}", "\text{diff. cocycle}"'] \arrow[d, hook] & A \\
\begin{array}{c}\text{field}\\ \text{space}\end{array} \arrow[ur, "\text{action}"] \arrow[d] & \\
\text{worldvolume} \arrow[uur, dashed, "\text{QFT}"'] &
\end{tikzcd}$$

background fields are:

- bundles of sorts: nonabelian cocycles
- with two extra bits of structure
  - with smooth higher degree form connection: differential cocycles
  - twisted by charges: twisted cocycles

Smooth nonabelian cohomology. **claim:** twists with higher connection find natural home in *differential nonabelian cohomology*

setup:

- take "spaces" not in Top, but in the $(\infty,1)$-topos $\infty\mathsf{Grpd}^\infty$ of *smooth ∞-groupoids* (∞-stacks on Diff)
- nonabelian generalization of sheaf cohomology: cohomology is hom

$$\mathbf{H}(X,A) := \mathrm{Maps}_{\infty\mathsf{Grpd}^\infty}(X,A)\,.$$

twists: for $A \to \hat{B} \to B$ a fibration sequence of smooth ∞-groupoids, for $c \in \mathbf{H}(X,B)$ a B-cocycle, the <u>c-twisted A-cohomology</u> is the homotopy pullback

$$\begin{tikzcd}
\mathbf{H}^c(X,A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & {*} \arrow[d, "c"] \\
\mathbf{H}(X,\hat{B}) \arrow[r] & \mathbf{H}(X,B)
\end{tikzcd}$$

Examples.

fibration sequence	**twist**	**twisted cocycles**	**remarks**
$\mathbf{B}U(n) \to \mathbf{B}PU(n) \to \mathbf{B}^2U(1)$	lifting gerbe	twisted vector bundle gerbe module	Freed-Witten anomaly cancellation
$\mathbf{B}\mathrm{String}(n) \to \mathbf{B}\mathrm{Spin}(n) \to \mathbf{B}^3U(1)$	Chern-Simons 2-gerbe	twisted String-bundles	Green-Schwarz anomaly cancellation
$\mathbf{B}\mathrm{Fivebrane}(n) \to \mathbf{B}\mathrm{String}(n) \to \mathbf{B}^7U(1)$	Chern-Simons 6-gerbe	twisted Fivebrane-bundles	dual Green-Schwarz anomaly cancellation
$\mathbf{B}^2U(1) \to \mathbf{B}(U(1) \to \mathbb{Z}_2) \to \mathbf{B}\mathbb{Z}_2$	orientifold	orientifold gerbe	

connections and parallel transport

- for every X there is smooth fundamental path ∞-groupoid $X \hookrightarrow \Pi(X)$

define nonabelian differential cohomology to be twisted relative cohomology for that

fibration sequence	**twist**	**twisted cocycles**	**remarks**
$\begin{array}{ccccc} A & \longrightarrow & A & \longrightarrow & \mathbf{E}A \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{E}A & \longrightarrow & \mathbf{B}A \end{array}$	curvature char. forms	higher connections	relative cohomoloy for $X \hookrightarrow \Pi(X)$: connection is "curvature-twisted" flat cocycle

unwrapping shows that differential cocycles are given by diagrams as:

$$\begin{tikzcd}
& X \arrow[rr, "g"] \arrow[d] \arrow[ddl, "="'] & & A \arrow[d] \arrow[ddl] \\
& \Pi(X) \arrow[rr, "\nabla"] \arrow[dd] & & \mathbf{E}A \arrow[dd] \\
X \arrow[rr] \arrow[dr] & & \mathbf{E}A \arrow[dr] & \\
& \Pi(X) \arrow[rr, "F"] & & \mathbf{B}A
\end{tikzcd}$$

underlying A-cocycle .

connection

curvature

volume holonomy is evaluation of $\Pi(X) \to \mathbf{B}A$ on chains

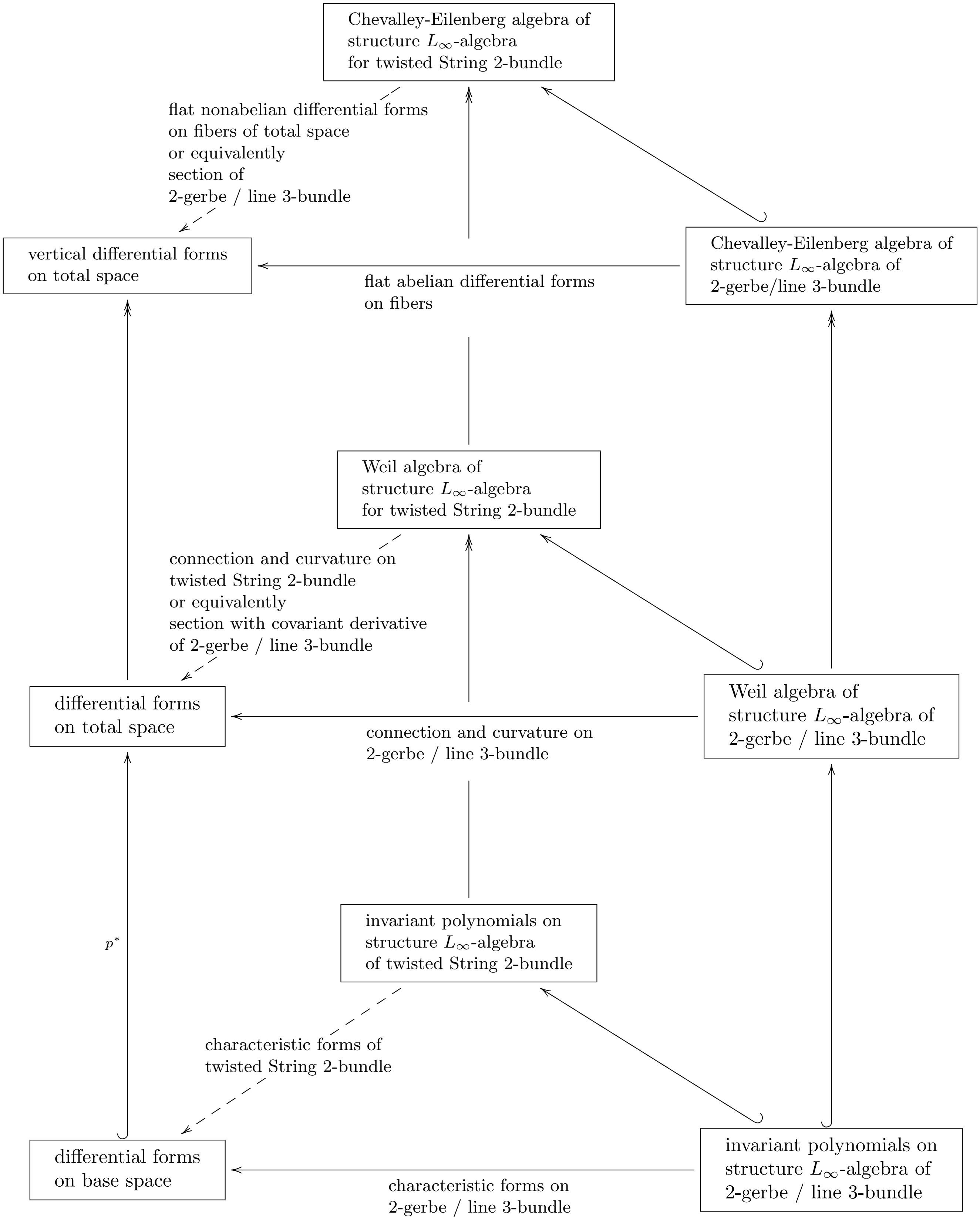
- check: differential refinement of twisted nonabelian cocycles does produce the desired background fields

here: example of computation for twisted String-bundles with connection:

Connection data in terms of local L_∞-algebra valued forms This diagram of smooth ∞-groupoids translates into the following kind of diagram of L_∞-algebroids; (for the case of twisted String connection with μ the 3-cocycle on $\mathfrak{g} = \mathfrak{so}(n)$).

$$
\begin{tikzcd}
 & \mathrm{CE}(b\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \arrow[dl, dashed, "{\{A_{\mathrm{vert}}, B_{\mathrm{vert}}, (C_3)_{\mathrm{vert}}\}}"'] & \\
\Omega^\bullet_{\mathrm{vert}}(P) & & \mathrm{CE}(b^2\mathfrak{u}(1)) \arrow[ll, "(C_3)_{\mathrm{vert}}"'] \arrow[ul, hook] \\
 & \begin{array}{c}\mathrm{CE}(\mathrm{inn}(b\mathfrak{u}(1)) \hookrightarrow \mathrm{cs}_P(\mathfrak{g})) \\ \simeq \mathrm{W}(b\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu)\end{array} \arrow[uu, two heads] \arrow[dl, dashed, "{\{A, F_A, B, \nabla B, C_3, G_4\}}"'] & \\
\Omega^\bullet(P) \arrow[uu, two heads] & & \mathrm{W}(b^2\mathfrak{u}(1)) \arrow[ll, "{(C_3, G_4)}"'] \arrow[ul, hook] \arrow[uu, two heads] \\
 & \mathrm{inv}(\mathrm{inn}(b\mathfrak{u}(1)) \hookrightarrow \mathrm{cs}_P(\mathfrak{g})) \arrow[uu] \arrow[dl, dashed, "{\{H_3, G_4, P(F_A)\}}"'] & \\
\Omega^\bullet(X) \arrow[uu, hook, "p^*"] & & \mathrm{inv}(b^2\mathfrak{u}(1)) \arrow[ll, "G_4"'] \arrow[ul, hook] \arrow[uu, hook]
\end{tikzcd}
$$

The following diagram is a labeled map for the above one.


Chevalley-Eilenberg algebra of structure L_∞-algebra for twisted String 2-bundle
flat nonabelian differential forms on fibers of total space or equivalently section of 2-gerbe / line 3-bundle
vertical differential forms on total space
flat abelian differential forms on fibers
Chevalley-Eilenberg algebra of structure L_∞-algebra of 2-gerbe/line 3-bundle
Weil algebra of structure L_∞-algebra for twisted String 2-bundle
connection and curvature on twisted String 2-bundle or equivalently section with covariant derivative of 2-gerbe / line 3-bundle
differential forms on total space
connection and curvature on 2-gerbe / line 3-bundle
Weil algebra of structure L_∞-algebra of 2-gerbe / line 3-bundle
p^*
invariant polynomials on structure L_∞-algebra of twisted String 2-bundle
characteristic forms of twisted String 2-bundle
differential forms on base space
characteristic forms on 2-gerbe / line 3-bundle
invariant polynomials on structure L_∞-algebra of 2-gerbe / line 3-bundle

Now, chasing the generators of the graded-commutative algebras through this diagram and recording the condition imposed by the respect of the morphisms of DGCAs for differentials, one finds that in components the commutativity of this diagram encodes the following differential form data and the following relations on that.

$$
\begin{tikzcd}[column sep=large]
 & \boxed{\begin{array}{l} dt^a = -\frac{1}{2}C^a{}_{bc}t^b \wedge t^c \\ db = \mu - k \\ dk = 0 \end{array}} \arrow[dl, dashed, "\begin{array}{l} t^a \mapsto A^a_{\mathrm{vert}} \\ b \mapsto B_{\mathrm{vert}} \\ k \mapsto (C_3)_{\mathrm{vert}} \end{array}"'] & \\
\boxed{\begin{array}{l} F^a_{A_{\mathrm{vert}}} = 0 \\ dB_{\mathrm{vert}} = \mu_{A_{\mathrm{vert}}} - (C_3)_{\mathrm{vert}} \\ d(C_3)_{\mathrm{vert}} = 0 \end{array}} & & \boxed{dk = 0} \arrow[ul, "k \mapsto k"'] \arrow[ll, "k \mapsto (C_3)_{\mathrm{vert}}"'] \\
 & \boxed{\begin{array}{l} dt^a = -\frac{1}{2}C^a{}_{bc}t^b \wedge t^c + r^a \\ dr^a = -C^a{}_{bc}t^b \wedge r^c \\ db = \mathrm{cs} + c - k \\ dc = l - P \\ dk = l \end{array}} \arrow[uu] \arrow[dl, dashed, "\begin{array}{l} t^a \mapsto A^a \\ r^a \mapsto F^a_A \\ b \mapsto B \\ c \mapsto \nabla B \\ k \mapsto C_3 \\ l \mapsto \mathcal{G}_4 \end{array}"'] & \\
\boxed{\begin{array}{l} H_3 := \nabla B = dB + C_3 - \mathrm{CS}(A, F_A) \\ dH_3 = \mathcal{G}_4 - \langle F_A \wedge F_A \rangle \\ d\mathcal{G}_4 = 0 \end{array}} \arrow[uu, "i^*"] & & \boxed{\begin{array}{l} dk = l \\ dl = 0 \end{array}} \arrow[uu, "\begin{array}{l} k \mapsto k \\ l \mapsto 0 \end{array}"'] \arrow[ul, "\begin{array}{l} k \mapsto k \\ l \mapsto l \end{array}"'] \arrow[ll, "\begin{array}{l} k \mapsto C_3 \\ l \mapsto \mathcal{G}_4 \end{array}"'] \\
 & \boxed{\begin{array}{l} dc = l - P \\ dl = 0 \\ dP = 0 \end{array}} \arrow[uu] \arrow[dl, dashed, "\begin{array}{l} c \mapsto \nabla B := H_3 \\ l \mapsto \mathcal{G}_4 \\ P \mapsto \langle F_A \wedge F_A \rangle \end{array}"'] & \\
\boxed{\begin{array}{l} dH_3 = \mathcal{G}_4 - \langle F_A \wedge F_A \rangle \\ d\mathcal{G}_4 = 0 \end{array}} \arrow[uu, "p^*"] & & \boxed{dl = 0} \arrow[uu, "l \mapsto l"'] \arrow[ul, "l \mapsto l"'] \arrow[ll, "l \mapsto \mathcal{G}_4"']
\end{tikzcd}
$$

Here, as usual, $P \in W(\mathfrak{g})$ is the invariant polynomial on $\mathfrak{g}$ in transgression with with the cocycle $\mu \in \mathrm{CE}(\mathfrak{g})$. With $\{t^a\}$ a fixed chosen basis of $\mathfrak{g}^*$ in degree 1 and $\{r^a\}$ the corresponding basis in degree 2, we have $P = P_{ab} r^a \wedge r^b$ and $\mu = \mu_{abc} t^a \wedge t^b \wedge t^c$ and $\mathrm{cs} = P_{ab} t^b \wedge r^a + \frac{1}{6}\mu_{abc} t^a \wedge t^b \wedge t^c$. So we have found in particular

curvature	$H_3 := dB + C_3 - \mathrm{CS}(A, F_A)$
Bianchi identity	$dH_3 = \mathcal{G}_4 - \langle F_A \wedge F_A \rangle$

This is the familiar twisted Bianchi identity from the Green-Schwarz mechanism.