A simple introduction to AdS/CFT and its application to condensed matter physics.

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Abstract

The anti-de Sitter/ Conformal Field theory correspondence provides a unique novel perspective on critical phenomena at second order quantum phase transitions in systems with spatial dimensions $d > 1$. The first half of these lectures will provide technical background to apply the so called "holographic" techniques of the correspondence. The second half discusses the application to quantum phase transitions in condensed matter: how spontaneous symmetry breaking in a quantum critical system is similar and different to the standard case, the notion of semi-local quantum liquids and their connection to non-Fermi liquids and strange metals.

There are also many lecture notes available. A sample of references are:

- J. Erdmenger, Introduction to gauge gravity duality, Chapters 1,2,4,5,6.
- S.A. Hartnoll, Lectures on holographic methods for condensed matter physics, Class. Quant. Grav. 26, 224002 (2009).
- N. Iqbal, H. Liu and M. Mezei, Lectures on holographic non-Fermi liquids and quantum phase transitions.

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Lecture I

1. CONTEXT AND BACKGROUND

In QFT (merging of QM & SR) particles are localized excitations of fields. Localization =Assumes weak coupling. In strong coupling regime, wave functions start to overlap until indistinguishable. Point: there should be other (non-perturbative) excitations that correctly describe the physics in this regime. Can we identify them? If so we have an example of a duality. The difficult part is the identification. There are no hard and fast rules to do so. In some cases we know and we will illustrate this with a simple example of electric-magnetic duality in $d = 3+1$ dimensions. It is extremely simple as opposed to the topic of this course, AdS/CFT, which will be technically very involved. At its heart philosophically AdS/CFT is just another duality, however.

Example: Electromagnetic duality

$$S[A] = \int d^4x - \frac{1}{4g^2} F_{\mu\nu}^2 \tag{1.1}$$

Bianchi identity

$$\partial_{[\mu} F_{\nu\rho]} = 0 \quad \Rightarrow \quad F_{\nu\rho} = \partial_{[\nu} A_{\rho]} \tag{1.2}$$

EOM

$$\partial_\mu F^{\mu\rho} = J^\rho_{\text{electric}} \tag{1.3}$$

Introduce magnetic charge

$$\partial_{[\mu} F_{\nu\rho]} = \epsilon_{\mu\nu\rho\sigma} J^\sigma_{\text{magnetic}} \tag{1.4}$$

Symmetric. Really dual, means one can also think of Bianchi as e.o.m. Can be made manifest for $J = 0$ by using a Lagrange multiplier

$$S[F, \tilde{A}] = \int d^4x - \frac{1}{4g^2} F_{\mu\nu}^2 - \tilde{A}^\mu \epsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} \tag{1.5}$$

Note now $F_{\mu\nu}$ is the field. Integrating out (i.e. solving the e.o.m. for) the Lagrange multiplier $\tilde{A}_\mu$ gives us back the original action.

$$\frac{\partial S}{\partial \tilde{A}^\mu} = \epsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} = 0 \quad \Rightarrow \quad F_{\nu\rho} = \partial_{[\nu} A_{\rho]} \tag{1.6}$$

However, we can also integrate out $F_{\mu\nu}$. It is now algebraic. Completing squares

$$S[\tilde{A}] = \int -\frac{1}{4g^2}F_{\mu\nu}^2 + \partial^\nu \tilde{A}^\mu \epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} \tag{1.7}$$

$$= \int -\frac{1}{4g^2}(F_{\mu\nu} - g^2\epsilon_{\mu\nu\rho\sigma}\partial^\rho \tilde{A}^\sigma)^2 + \frac{g^2}{4}\epsilon_{\mu\nu\rho\sigma}(pa^{[\rho}\tilde{A}^{\sigma]})^2 \quad = \int -\frac{g^2}{4}(\partial_{[\rho}\tilde{A}_{\sigma]})^2 \tag{1.8}$$

where we used that $\epsilon_{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}\eta^{\rho\gamma}\eta^{\sigma\delta} = -2(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha})$. If we define $\tilde{g} = 1/g$ this is the original action. Note however that perturbation expansion in g is the *strong*-coupling expansion in $\tilde{g}$ and vice versa. This is the simplest example of a (strong-weak) duality. AdS/CFT is just an incredible more complicated version of the same story, where the dual theory is now given by a quantum-gravitational string theory in a curved space with one extra spatial dimension. Nevertheless a precise dictionary exists between these higher-dimensional gravitational theories and the original CFT. That dictionary and some of its uses is the goal of these lectures.

2. ANTI-DE-SITTER SPACE

The AdS in AdS/CFT stands for anti-de-Sitter space. Here we give a brief introduction to this spacetime. As always in physics, the best way to classify spacetimes is through their symmetries. For spacetimes symmetries are the isometries of the spacetime manifold i.e. coordinate transformations that leave the metric invariant. To each such symmetry we can therefore associate a distinct *Killing vector*. Recall that under general coordinate transformations $\delta x^\nu = \xi^\nu(x)$ the metric transforms as

$$\delta g_{\mu\nu} = D_{(\mu}\xi_{\nu)} \tag{2.1}$$

A Killing vector is a vector such that $D_{(\mu}\xi_{\nu)} = 0$.

The simplest spacetimes are those with the most symmetries. These will have the most Killing vectors. Since a d-dimensional symmetric metric has $d(d+1)/2$ components, there can be at most $d(d+1)/2$ independent Killing equations and thus at most $d(d+1)/2$ Killing vectors. Spacetimes with such amount of Killing vectors are called maximally symmetric. These Killing vectors will form a group. The simplest group that contains this number of generators of $SO(d+1)$, but this would not account for the Lorentzian signature. It turns out there are only three distinct maximally symmetric spacetimes. They are classified by the group formed by their Killing vectors. They are

Group	*Spacetime*
$SO(d,1)$	d-dimensional de Sitter space
$SO(d-2,2)$	d-dimensional anti-de Sitter space
$ISO(d-1,1) = SO(d-1,1) \propto Translations_d$	d-dimensional Minkowski space

The simplicity of maximally symmetric spacetimes reflects through in their curvature. Since essentially every point is similar to every other point the curvature cannot have a derivative dependence. Thus[1]

$$R_{\mu\nu\rho\sigma} = c(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \tag{2.2}$$

$$R_{\mu\nu} = c(d-1)g_{\mu\nu} \tag{2.3}$$

$$R = cd(d-1) \tag{2.4}$$

with c a constant. All these spacetimes are in fact solutions to the vacuum dynamical Einstein equations supplemented with a cosmological constant

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R - 2\Lambda) = 8\pi G T_{\mu\nu} \tag{2.5}$$

In vacuum $T_{\mu\nu} = 0$, then a contraction with the metric gives

$$R = \frac{2d}{d-2}\Lambda \tag{2.6}$$

and thus $c = 2\Lambda/(d-1)(d-2)$.

It turns out that $\Lambda > 0$ corresponds to de Sitter space, $\Lambda = 0$ to Minkowski space, and $\Lambda < 0$ to anti-de-Sitter space. In other words, anti-de-Sitter space is the maximally symmetric space that is the unique solution to the vacuum Einstein equations with a *negative* cosmological constant.

This may still not say much, but you are in fact very familiar with maximally symmetric Euclidean spaces. The maximally symmetric Euclidean spaces are the sphere ($\Lambda > 0$), flat space ($\Lambda = 0$), and the hyperboloid ($\Lambda < 0$). de Sitter space is thus the Lorentzian generalization of the sphere and anti-de Sitter space is the Lorentzian generalization of the hyperboloid. Just as the d-dimensional sphere is defined as the solution to the constraint

$$X_1^2 + X_2^2 + \ldots + X_{d+1}^2 = R^2 \tag{2.7}$$

[1] We are using conventions where $[D_\mu, D_\nu]V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma$ and $g_{\mu\nu}$ is mostly plus.

. and the hyperboloid is defined as

$$-X_{-1}^2 + X_1^2 + \ldots + X_d^2 = -R^2 \tag{2.8}$$

d-dimensional anti-de Sitter space is defined as the

$$-X_{-1}^2 - X_0^2 + X_1^2 + X_{d-1}^2 = -R^2 \tag{2.9}$$

We immediately see the $SO(d-1,2)$ symmetry. A solution to this defining relation is

$$X_1 = R_+ \cos\theta_1 \tag{2.10}$$

$$X_2 = R_+ \sin\theta_1 \cos\theta_2 \tag{2.11}$$

$$\vdots = \tag{2.12}$$

$$X_{d-2} = R_+ \sin\theta_1 \sin\theta_2 \ldots \cos\theta_{d-2} \tag{2.13}$$

$$X_{d-1} = R_+ \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{d-2} \tag{2.14}$$

$$X_0 = R_- \cos t \tag{2.15}$$

$$X_{-1} = R_- \sin t \tag{2.16}$$

where $R_+^2 = R^2 \sinh^2\tau,\ R_-^2 = R^2\cosh^2\tau$. The induced metric one gets is

$$ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 + \ldots dX_{d-1}^2 \tag{2.17}$$

$$= R^2(d\tau^2 - \cosh^2\tau dt^2 + \sinh^2\tau d\Omega_{d-2}^2) \tag{2.18}$$

where $d\Omega_{d-2}^2$ is the metric on the $d-2$-dimensional sphere S_{d-2}. Anti-de-Sitter space is the universal cover of this, where we unroll the periodic coordinate t. This coordinate system is quite special because it is a global coordinate system. It covers the whole of AdS (recall that generically one needs multiple coordinate patches to cover the whole space).

Now consider the coordinate transformation $\tau = \operatorname{arcsinh}\tan\rho$ with $0 \leq \rho \leq \pi/2$. This maps the hyperbolic direction τ to a finite range. In these coordinates the metric is

$$ds^2 = \frac{R^2}{\cos^2\rho}\left(d\rho^2 - dt^2 + \sin^2\rho d\Omega_{d-2}^2\right) \tag{2.19}$$

This metric allows us to understand the topology of AdS. For this we can ignore the overall conformal factor $R^2/\cos^2\rho$. The topologically equivalent spacetime

$$ds_{teq}^2 \sim \left(d\rho^2 - dt^2 + \sin^2\rho d\Omega_{d-2}^2\right) \tag{2.20}$$

To be filled

FIG. 1: Topology of anti-de Sitter space.

describes a cylinder with radial direction ρ, longitudinal direction t and each point in (ρ, t) is an S_{d-2}. See Fig.

We thus see that AdS has a (conformal) boundary at $\rho = \pi/2$. Note that in physical units this is an infinite distance away. This conformal boundary will play an important role in AdS/CFT.

There is a third convenient coordinate system for AdS. See exercises that it indeed solves the defining relation (2.9)

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 - dt^2 + dx_1^2 + \ldots + dx_{d-2}^2\right) \tag{2.21}$$

This coordinate system, the so-called Poincaré patch, covers only half of AdS. Relating it back to the global coordinate system through the defining relation (2.9) one can show that it covers the diamond depicted in Fig 1.

For a review see [106, 107].

3. EXERCISES

Problem 1:

In lecture we showed electromagnetic duality in $3+1$ dimensions. In $2+1$ dimensions the exact same exercise is called *particle-vortex* or Abelian-Higgs duality. Starting from a free $U(1)$ gauge theory in $2+1$ dimensions perform the same steps to dualize the theory. What is the theory you end up with, i.e. what type of theory is the vortex/Higgs theory?

Problem 2:

a. In lecture we claimed that the spacetime defined the embedding

$$-X_{-1}^2 - X_0^2 + X_1^2 + \ldots + X_{d-1}^2 = -R^2 \tag{3.1}$$

into $\mathbf{R}^{d-1,2}$ can be chosen to have local coordinates

$$X_1 = R_+ \cos\theta_1 \tag{3.2}$$

$$X_2 = R_+ \sin\theta_1 \cos\theta_2 \tag{3.3}$$

$$\vdots = \tag{3.4}$$

$$X_{d-2} = R_+ \sin\theta_1 \sin\theta_2 \ldots \cos\theta_{d-2} \tag{3.5}$$

$$X_{d-1} = R_+ \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{d-2} \tag{3.6}$$

$$X_0 = R_- \cos t \tag{3.7}$$

$$X_{-1} = R_- \sin t \tag{3.8}$$

where $R_+^2 = R^2 \sinh^2\tau,\ R_-^2 = R^2\cosh^2\tau$ gives the global AdS metric. Verify this.

b. There is another well-known choice to solve the AdS constraint as as an embedding. It is given by

$$X_i = Rx_i/z \qquad i = 0, \ldots, d-2 \tag{3.9}$$

$$X_{d-1} + X_{-1} = \frac{1}{z}(x_0^2 - x_1^2 - \ldots - x_{d-2}^2 - z^2) \tag{3.10}$$

$$X_{d-1} - X_{-1} = R^2\frac{1}{z} \tag{3.11}$$

What does the AdS metric look like for this choice?

Problem 3: In lecture we computed the two-point function of a scalar conformal operator from AdS using the AdS/CFT prescription. Do the same for a vector operator. There are a number of subleties. (1) As mentioned, in AdS the vector field is dynamical. The only way such a field makes sense is as a gauge field. We must therefore fix the gauge the solve for the Green's function. (2) Because of gauge invariance the vector field is massless. This will make the solution a bit simpler.

a. Write down the equation of motion for a vector field in an AdS_{d+1} background.

b. Verify that the *bulk-boundary* propagator

$$G_{\mu\nu} = \frac{z^{d-2}}{(z^2 + x_\mu x^\mu)^{d-1}}\eta_{\mu\nu} \tag{3.12}$$

$$G_{0\mu} = -\frac{z^{d-3}x_\mu}{(z^2 + x_\mu x^\mu)^{d-1}} \tag{3.13}$$

with all other components vanishing solves the equation of motion. Show also that it reduces to a delta-function on the boundary.

c. Consider an arbitrary boundary source $a_\mu(x^\nu)$ and its corresponding bulk solution $A_M = \int G_{M\nu} a^\nu$. Substitute this into the bulk action and use the AdS/CFT dictionary to determine the two-point function of conformal vector fields.

d. Verify that these conformal vector fields are conserved (global) currents.

A brief concluding comment. The gauge choice we have used here is not a standard one: it is one where the Green's function in position space is particularly simple. In more involved AdS/CFT computations, especially numerical, one normally chooses the where the components along the extra AdS direction vanish, i.e. $A_z = 0$.

4. THE ADS/CFT CORRESPONDENCE

4.1. CFT brush up

A CFT$_d$ is a special relativistic d-dimensional QFT which is invariant under $SO(d,2)$ instead of just the Lorentz group $SO(d-1,1)$.[2] Now think of renormalization of a standard QFT. This is a very deep statement that in the real world microscopic physics resonates through for macroscopic physics, but in a very predictable way that does not depend on the details of the microscopic model. A generic QFT is therefore scale dependent. CFTs by definition, however, are scale independent. This must mean that CFTs *do not renormalize*: all beta-functions and anomalous dimensions must be exactly zero.

4.2. AdS/CFT

$$\langle e^{\int d^dx J(x)\mathcal{O}(x)}\rangle_{\text{CFT}} = \int \mathcal{D}\phi \; e^{-S_{\text{AdS}}}\Big|_{\phi(x,\partial AdS)=J(x)} . \tag{4.1}$$

[2] Non-relativistic CFTs also exist, but we will not consider them here.

boundary: field theory	bulk: gravity
energy momentum tensor T^{ab}	metric field g_{ab}
global internal symmetry current J^a	Maxwell field A_a
order parameter/scalar operator $\mathcal{O}_{\rm b}$	scalar field ϕ
fermionic operator $\mathcal{O}_{\rm f}$	Dirac field ψ
spin/charge of the operator	spin/charge of the field
conformal dimension of the operator	mass of the field
source of the operator	boundary value of the field (leading part)
VEV of the operator	boundary value of radial momentum of the field (subleading part)
(Global aspects)	
global spacetime symmetry	local isometry
temperature	Hawking temperature
chemical potential/charge density	boundary values of the gauge potential
phase transition	Instability of black holes

TABLE I: The basic dictionary for AdS/CFT correspondance.

Lecture II

5. THE PHYSICS OF ADS/CFT CORRELATION FUNCTIONS

Let us now show how the GKPW rule precisely validates our ad hoc construction Eq.(**??**) that the two-point correlation function can be read off from the near-boundary asymptotics of AdS waves. The GKPW rule instructs us to consider the *on-shell* action with the boundary value of the field equal to the source in the dual field theory. For the simple scalar theory the action can be written in terms of a "bulk" and "boundary" contribution as,

$$S = -\frac{1}{2}\int_{\rm AdS} d^{D+1}x\sqrt{-g}\phi(\Box + m^2)\phi - \frac{1}{2}\oint_{\partial{\rm AdS}} d^D\xi\sqrt{-h}\phi\partial_n\phi \tag{5.1}$$

Here h is the determinant of the induced metric $h_{ab} = g_{\mu\nu}\partial_a\xi^\mu\partial_b\xi^\nu$ on the boundary which has the local coordinates $\xi^a(x)$. Only the bulk part contributes to the equations of motion. On-shell, i.e. when we substitute for $\phi(x)$ a solution for to the equation of motion, the "bulk" term integrated over the whole of the AdS space vanishes. We already learned that this solution — i.e. with appropriate boundary conditions in the interior — has a universal asymptotic behavior near the boundary $z = 0$ as,

$$\phi_{\text{sol}}(\omega, k, z) = A(\omega, k) z^{\Delta_-} + B(\omega, k) z^{\Delta_+} + \dots. \tag{5.2}$$

According to GKPW the boundary value of $\phi(\omega, k, z)$ is proportional to the source J. A priori we are now facing a problem since Δ_- will be generically negative and thus the boundary value of ϕ is not well defined. However, we know what the meaning of this divergence is in the boundary field theory. Approaching the boundary is like increasing the renormalization scale to infinity and here one typically encounters UV divergences. In other words, the theory has to be regulated and this can be done in a particularly elegant way using the bulk language. GKPW proposed that one should compute at an infinitesimal distance $z = \epsilon$ away from the formal boundary, and then modify the theory such that one can take an appropriate limit $\epsilon \to 0$. Let's do so. The "regulated" on-shell action equals

$$S_{\text{on-shell}}(\epsilon) = \frac{1}{2} \oint_{z=\epsilon} \frac{d\omega d^{D-1}k}{(2\pi)^D} z^{-D+1} \left(\Delta_- A^2 z^{2\Delta_- - 1} + (\Delta_- + \Delta_+) AB z^{\Delta_- + \Delta_+ - 1} + \dots \right). \tag{5.3}$$

The first term is formally divergent. The key to making the action well-defined is that adding an arbitrary boundary term to the action never changes the equation of motion. However, such an extra boundary term can be used to remove by hand the UV divergence. Adding a boundary counterterm of the form

$$\begin{aligned} S_{\text{counter}}(\epsilon) &= -\frac{1}{2}\Delta_- \oint_{z=\epsilon} \frac{d\omega d^{D-1}k}{(2\pi)^D} \sqrt{-h}\phi^2 \\ &= -\frac{1}{2}\Delta_- \oint_{z=\epsilon} \frac{d\omega d^{D-1}k}{(2\pi)^D} z^{-D} \left(A^2 z^{2\Delta_-} + 2AB z^{\Delta_+ + \Delta_-} + \dots \right) \end{aligned} \tag{5.4}$$

yields in combination with Eq. (5.3),

$$S_{\text{on-shell}}(\epsilon) + S_{\text{counter}}(\epsilon) = \frac{1}{2} \oint_{z=\epsilon} \frac{d\omega d^{D-1}k}{(2\pi)^D} z^{-D} \left((D - 2\Delta_-) AB z^D + \dots \right). \tag{5.5}$$

This is now all finite. We can now equate the leading behavior (A coefficient) of the ϕ_{sol} with the source J. Given that the above should coincide with the combination $iJ\langle O \rangle$ in the field theory, by taking the single derivative with respect to J this yields the expectation value $\langle O \rangle$ of the field theory operator sourced by J in AdS/CFT in the presence of the source. It is given by

$$\langle \mathcal{O}(\omega, k) \rangle_J = 2\nu B(\omega, k) \tag{5.6}$$

where we used $\Delta_\pm = \frac{D}{2} \pm \nu$. *Thus we see that equating the leading near-boundary behavior $A(\omega, k)$ of the solution ϕ_{sol} with the source implies that the subleading near-boundary behavior $B(\omega, k)$ is the corresponding response.*

One can similarly obtain the two-point correlation function by taking an additional derivative w.r.t. J and setting it to vanish. Linear response theory tells us already that $B(\omega, k)$ ought to be proportional to $A(\omega, k)$ and the proportionality is precisely the CFT Green's function,

$$\langle \mathcal{O}(-\omega, -k)\mathcal{O}(\omega, k)\rangle = 2\nu \frac{B(\omega, k)}{A(\omega, k)} \tag{5.7}$$

and we have demonstrated that the propagator rule Eq.(**??**) is indeed a consequence of the fundamental GKPW rule.

One can actually check that the GKPW rule encodes linear response theory by itself in a correct fashion. For this one has to realize that one in essence just needs to solve a simple Dirichlet boundary value problem. Recall that the equation of motion for ϕ has two independent solutions. Let us denote the solution with $A = 0$ as ϕ_B with boundary behavior $\phi_B(z) = Bz^{\Delta_+}(1 + \sum_n c_n z^n)$. This is the appropriate Dirichlet solution that vanishes at the boundary, and let us denote as $\phi_{int}(z)$ the solution with boundary behavior determined by regularity in the interior of AdS as in Eq. (**??**). The Dirichlet AdS Green's function obeying $\lim_{z\to 0} \mathcal{G}^{\text{AdS}} = 0$ thus equals

$$\mathcal{G}^{\text{AdS}}(z, z') = \frac{\phi_B(z)\phi_{int}(z')\theta(z - z') + \phi_{int}(z)\phi_B(z')\theta(z' - z)}{\phi_{int}\partial\phi_B - \phi_B\partial\phi_{int}}. \tag{5.8}$$

The Wronskian in the denominator assures the correct normalization and is independent of z, *i.e.* it may be evaluated for any preferred z. Then for a boundary source $J(\omega, k)$ the solution to the equation of motion is

$$\begin{aligned}\phi_{\text{sol}}(\omega_1, k_1, z) &= \lim_{\epsilon\to 0} \oint_{z'=\epsilon} \frac{d\omega d^{D-1}k}{(2\pi)^D} \partial_{z'}\mathcal{G}(z, \omega_1, k_1; z', \omega, k)J(\omega, k) \\ &= \lim_{\epsilon\to 0} \int \frac{d\omega d^{D-1}k}{(2\pi)^D} \frac{\partial\phi_B(\epsilon)\phi_{int}(z)}{\phi_{int}\partial\phi_B - \phi_B\partial\phi_{int}} J(\omega, k).\end{aligned} \tag{5.9}$$

By construction this obeys $\lim_{z\to 0} \phi_{\text{sol}}(\omega, k, z) = J(\omega, k)$, which can be seen by noting that the Wronskian reduces to $\phi_{int}\partial\phi_B$ for $z \to 0$. The normal derivative of the solution follows straightforwardly

$$\partial_z \phi_{\rm sol}(\omega_1, k_1, z) = \lim_{\epsilon\to 0} \oint_{z'=\epsilon} \frac{d\omega d^{D-1}k}{(2\pi)^D} \partial_z \partial'_z \mathcal{G}(z,z') J(\omega,k)$$
$$= \lim_{\epsilon\to 0} \int \frac{d\omega d^{D-1}k}{(2\pi)^D} \frac{\partial\phi_B(\epsilon)\partial\phi_{int}(z)}{\phi_{int}(\epsilon)\partial\phi_B(\epsilon)} J(\omega,k) = \lim_{\epsilon\to 0} \int \frac{d\omega d^{D-1}k}{(2\pi)^D} \frac{\partial\phi_{int}(z)}{\phi_{int}(\epsilon)} J(\omega,k) \tag{5.10}$$

Substituting this into the action one finds

$$S_{\text{on-shell}} + S_{\text{counter}} = \lim_{z\to 0} \left(\frac{1}{2} \int d^D x z^{-D+1} \phi_{\rm sol} \partial_z \phi_{\rm sol} - \frac{1}{2}\Delta_- \int d^D x z^{-D} \phi^2_{\rm sol} \right)$$
$$= \lim_{\epsilon\to 0} \int \frac{d\omega d^{D-1}k}{(2\pi)^D} \left(\epsilon^{-D+1} \frac{1}{2} J(-\omega,-k) \frac{\partial\phi_{int}(\epsilon)}{\phi_{int}(\epsilon)} J(\omega,k) - \frac{\Delta_-}{2} \epsilon^{-D} J(-\omega,-k) J(\omega,k) \right). \tag{5.11}$$

Near the boundary the solution ϕ_{int} has again the generic behavior $\phi_{int} = A_{int}(\omega,k) z^{\Delta_-} + B_{int}(\omega,k) z^{\Delta_+}$ and by taking two derivatives w.r.t. the source J one finds

$$\langle \mathcal{O}(-\omega,-k)\mathcal{O}(\omega,k)\rangle = \lim_{\epsilon\to 0} \epsilon^{-D+1} \frac{\partial\phi_{int}(\epsilon)}{\phi_{int}(\epsilon)} - \frac{1}{2}\Delta_-$$
$$= \epsilon^{-D+1} \frac{\partial_\epsilon(\epsilon^{-\Delta_-}\phi_{int}(\epsilon))}{\phi_{int}(\epsilon)} = 2\nu \frac{B(\omega,k)}{A(\omega,k)}. \tag{5.12}$$

Thus one is left with the same linear response answer (5.7).

6. ADS/CFT AT FINITE TEMPERATURE: BLACK HOLES, AND THE HAWKING PAGE TRANSITION AS THE HOLOGRAPHIC ENCODING OF CONFINEMENT/DECONFINEMENT

The Einstein equation for a $D+1$ dimensional Minkowski-signature space time with a negative cosmological constant is,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{D(D-1)}{2L^2} g_{\mu\nu} = 0. \tag{6.1}$$

This is solved by an AdS-Schwarzschild black hole background with the metric,

$$ds^2 = \frac{r^2}{L^2}(-f(r)dt^2 + dx_i^2) + \frac{L^2}{r^2 f(r)} dr^2, \quad i = 1, ..., D-1 \tag{6.2}$$

where L is the AdS radius, r is the radial direction, and the redshift factor

$$f(r) = 1 - r_0^D / r^D. \tag{6.3}$$

gives us the black horizon at r_0. To compute the temperature we use an insight from Gibbons and Hawking. Consider a general static black hole metric

$$ds^2 = -g_{tt}(r)dt^2 + \frac{dr^2}{g^{rr}(r)} + g_{xx}(r)d\vec{x}^2,$$

where $g_{tt}(r)$ and $g^{rr}(r)$ have a single zero at the horizon r_0, and Wick rotate to the Euclidean signature $\tau = it$ with the result

$$ds_E^2 = g_{tt}(r)d\tau^2 + \frac{dr^2}{g^{rr}(r)} + g_{xx}(r)(d\vec{x})^2. \tag{6.4}$$

Let's make the natural assumption that the properties of the black hole are reflected in the geometry near the horizon where g_{tt} and g^{rr} are vanishing. To focus in on this region we expand $g_{tt}(r) = g'_{tt}(r_0)(r - r_0) + ...$, $g^{rr}(r) = g^{rr\prime}(r_0)(r - r_0) + ...$ and $g_{xx}(r) = g_{xx}(r_0) + ...$ where the prefactors g''s are just numbers. Thus the near horizon (Euclidean) metric equals

$$ds_E^2 = g'_{tt}(r_0)(r - r_0)d\tau^2 + \frac{dr^2}{g^{rr\prime}(r_0)(r - r_0)} + g_{xx}(r_0)(d\vec{x})^2 + \dots \tag{6.5}$$

It is now convenient to re-parametrize the radial direction in terms of a new variable $R = 2\sqrt{r - r_0}/\sqrt{g^{rr\prime}}$ and the metric becomes

$$ds_E^2 = \frac{1}{4}R^2 g'_{tt} g^{rr\prime} d\tau^2 + dR^2 + g_{xx}(0)(d\vec{x})^2 + \dots$$

What matters is the plane spanned by this R and the imaginary time direction τ. This is just like the metric of a plane in polar coordinates with τ being the compact angular direction. Upon approaching the horizon $R \to 0$ one sees that the prefactor of $d\tau^2$ is vanishing: this means that the Euclidean time direction shrinks to a point! However, since the horizon is not a special point, we should not allow this point to be singular. Smoothness at the horizon can be achieved by insisting that $R = 0$ is the center of a Euclidean polar coordinate system and this implies that τ is periodic with period $\frac{4\pi}{\sqrt{g'_{tt}(r_0)g^{rr\prime}(r_0)}}$. This periodicity is directly identified with the inverse temperature of the black hole (as measured at $r = \infty$).

For the AdS-Schwarzschild black hole we thus find

$$T = \frac{\sqrt{g'_{tt} g^{rr\prime}}}{4\pi} = \frac{r_0^2}{4\pi L^2}\frac{df(r)}{dr}|_{r=r_0} = \frac{D r_0}{4\pi L^2}. \tag{6.6}$$

Let us illustrate the above with the most important equations for the bulk. As we emphasized, to introduce a second scale in addition to the temperature we make the spatial topology compact. All Euclidean isotropic solutions to AdS-Einstein's equations have the same generic form as Eqn. (**??**)

$$ds_2^2 = f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-1}^2, \tag{6.7}$$

The most general solution is

$$f(r) = 1 + \frac{r^2}{L^2} - \omega_D \frac{M}{r^{D-2}}, \qquad \omega_D = \frac{2\kappa^2}{(D-1)\mathrm{Vol}(S^{D-1})}. \tag{6.8}$$

For $M = 0$ we have "thermal AdS" (the time circle is still compact); for $M \neq 0$ we have the Euclidean (global-)AdS black hole.[3]

Note that $f(r)$ now has multiple zeroes. The outer most one, i.e. the larger solution of the equation

$$1 + \frac{r^2}{L^2} - \omega_D \frac{M}{r^{D-2}} = 0 \tag{6.10}$$

is the horizon r_+. Following the near-horizon prescription of the previous section, one can deduce the temperature

$$\beta = \frac{4\pi L^2 r_+}{D r_+^2 + (D-2)L^2}. \tag{6.11}$$

in units of $c = k_B = \hbar = 1$ and its inverse gives the temperature of the black hole $T = 1/\beta$.

We can compare which solution is thermodynamically favored by using the algorithm of the previous section to compute the free energies of the boundary corresponding with these two different bulk geometries. A crucial extra ingredient is that in order to compare these free energies, one has to insist that the bulk geometries describe precisely the same boundary space-time. Choosing a cut-off in the radial direction at $r = R$, one can express the imaginary time periodicity of the thermal AdS β' in terms of the black hole inverse temperature β such that both systems live in the same boundary space, sharing the same boundary time circle.At a cut-off radius R the periodicities in each geometry are

$$\beta'\big(1 + \frac{R^2}{L^2}\big)^{1/2} = \beta\big(1 + \frac{R^2}{L^2} - \frac{\omega_D M}{R^{D-2}}\big)^{1/2}. \tag{6.12}$$

[3] In the scaling limit

$$t = \lambda t, r = \lambda^{-1} r, d\Omega_{D-1}^2 = \lambda^2 d\vec{x}^2 \quad \text{with} \quad \lambda \to 0, \tag{6.9}$$

the metric (6.7) changes to the vacuum planar AdS_{D+1}. If we scale $M \to \lambda^{-D} M$ at the same time, one gets the planar AdS Schawartzshild black hole (6.2).

One now computes the difference in the boundary field theory free energy straightforwardly and it is related to the Euclidean action difference

$$\beta F_{ThAdS} - \beta'(R)F_{BH} = \frac{d}{\kappa^2 L^2} \lim_{R\to\infty} \left(\int_0^\beta dt \int_{r_+}^R dr \int_{S^{D-1}} d\Omega r^{D-1} - \int_0^{\beta'} dt \int_0^R dr \int_{S^{D-1}} d\Omega r^{D-1} \right)$$
$$= \frac{4\pi \mathrm{Vol}(S^{D-1}) r_+^{D-1}(L^2 - r_+^2)}{2\kappa^2 (D r_+^2 + (D-2)L^2)}. \tag{6.13}$$

where F_{ThAdS} and F_{BH} are the free energies for the black hole- and thermal AdS solutions. With the units restored, we have

$$\beta(F_{ThAdS} - F_{BH}) = c^3 \frac{4\pi \mathrm{Vol}(S^{D-1}) r_+^{D-1}(L^2 - r_+^2)}{2\kappa^2 (D r_+^2 + (D-2)L^2)}. \tag{6.14}$$

7. ADS/CFT AND HYDRODYNAMICS: MINIMAL VISCOSITIES

One step further then equilibrium thermodynamics is to consider small time-dependent fluctuations around a thermodynamic equilibrium. For very small energies and ultra-long wavelengths the state is still collective and the response of the system is described by *hydrodynamics*. A slightly more detailed definition would say that hydrodynamics applies when the length scale of interest is much larger than the mean-free-path between the microscopic constituents of the system. This is where the connection with AdS/CFT comes in. Since the mean-free-path is inversely proportional to the coupling constant, for strongly coupled theories hydrodynamics applies widely. And AdS/CFT can give a gravitational description of (matrix-valued) strongly coupled theories (in the large N limit).

The defining equations of relativistic hydrodynamics are extremely simple. They are just conservation of energy-momentum and charge

$$\partial_\mu T^{\mu\nu} = 0 \partial_\mu J^\mu = 0 \tag{7.1}$$

These are $d+1$ equations for $\frac{1}{2}d(d+1)+d$ unknowns. And the dynamical fluid equations must always be supplemented by additional constraints. The fluids you are probably familiar with are *perfect fluids*. These fluids are completely homogeneous and isotropic (rotationally and translationally invariant) and experience no force, i.e. they move at a constant velocity. Call this velocity u^μ, normalized such that $u^\mu u_\mu = -1$. Then the rest frame expression for the stress-tensor in terms of the energy density ϵ and pressure p

$$T^{\mu}_{\nu} = \begin{pmatrix} \epsilon & 0 & \dots & 0 \\ 0 & p & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & p \end{pmatrix} \tag{7.2}$$

together with the symmetries uniquely determine the stress-tensor of a perfect fluid to be

$$T_{\mu\nu} = (\epsilon + p)\, u_\mu u_\nu + p g_{\mu\nu} \tag{7.3}$$

The extra relation one must add is the equation of state which determines p in terms of ϵ.

In reality no perfect fluid exists. One should think of it as one thinks of an ideal gas. In reality interactions at the microscopic scale have an effect on the expression for the stress-tensor that causes it to deviates from the perfect fluid. Note that the symmetries played an essential role in the definition of the perfect fluid. So what is true is that the more and more these symmetries are manifested in the system, the better the perfect fluid description is.

Vice versa this means that deviation from the perfect fluid are captured by terms that break the perfect translational and rotational symmetries of the system. The fluid velocity thus has small variations in space and time. For simplicity let us consider the fluid in the rest frame, perform a lowest order expansion in gradient velocities and arrange these again according to representations of the rotational group. One has

$$T_{00} = \epsilon \ , \qquad T_{0i} = 0 \tag{7.4}$$

$$T_{ij} = \epsilon\delta_{ij} + \eta\left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\partial_\ell u^\ell\right) + \zeta\delta_{ij}\partial_\ell u^\ell \tag{7.5}$$

A relativistic expression exists, and can be found in many references (e.g. Baier). Here η and ζ are the *shear* and *bulk viscosities*. They are decay constants how rotations and decompressions relax.

For simplicity and physics let us consider the charge current rather than the energy-momentum current. The insight is that a small local excess of charge density should cause a small current to flow away from the excess until it is completely diluted into the system. Improving on the perfect fluid limit one would write down in the rest frame

$$J^0 = \rho \ , \qquad J^i = -D\partial_i\rho = -D\partial_i J^0 \tag{7.6}$$

with D the diffusion constant. This second constitutive equation is also known as Fick's law. Now take the divergence of both sides

$$\partial_i J^i = -D\Box J^0 \tag{7.7}$$

and use the dynamical equation (the charge conservation law) $\partial_\mu J^\mu = 0$ to write this as

$$\partial_0 J^0 = D\Box J^0 \tag{7.8}$$

Solving this in Fourier space (space only) one has

$$J^0(t,k) = e^{-Dk^2 t} J^0_{init}(k) \tag{7.9}$$

One sees the diffusion of charge happen, as the excess decays away over time. The shear and and bulk viscosities play the same role as the diffusion constant D with some more indices to carry around.

7.1. Fluctuation dissipation theorem and Kubo relations

Let's now also Fourier transform the charge diffusion equation for J^0 in time.

$$i\omega J^0(\omega,k) = Dk^2 J^0(\omega,k) \quad \Leftrightarrow (i\omega - Dk^2)J^0(\omega,k) = 0 \tag{7.10}$$

One should think of this dynamical equation as the equation of motion for the charge density. Suppose now one would like to solve a more complicated problem in this fluid, e.g. one with an external source for the charge. Then one would construct the Green's function from the equation of motion

$$G = \langle J^0 J^0 \rangle \simeq \frac{1}{i\omega - Dk^2} + \ldots \tag{7.11}$$

(Recall there are higher order terms in the gradient expansion. Since we are doing semiclassical physics, this has to be the retarded Green's function). This means that

$$\mathrm{Im} iG = \frac{Dk^2}{\omega^2 + D^2k^4} \tag{7.12}$$

and one can get D from by taking considering

$$\lim_{\omega\to 0} \mathrm{Im} iG = \frac{1}{Dk^2} \tag{7.13}$$

This *Kubo relation* is an example of the fluctuation-dissipation theorem. One can get macroscopic properties (the diffusion constant) by considering microscopic fluctuations. A manifestion of the fluctuation-dissipation theorem that you almost certainly know is the optical theorem: the total cross section is proportional to the imaginary part of the forward scattering amplitude.

Let us know use AdS/CFT to compute the diffusive properties of a strongly coupled (matrix)-theory. The example that we will use is the shear viscosity [48]. This is for two reasons (1) historically it was the first, (2) the finding that the viscosity is universal and bounded from below for all theories with a gravitational dual, (3) but, most importantly, in units of the entropy density it is the smallest viscosity known *and* it turned out to be in very close neighborhood to viscosities measured of the Quark-Gluon-Plasma in Relativistic Heavy Ion Collisions and Cold Fermi-gas experiments. We can do it here because it is among the simplest calculations for a linear response property one can imagine in the AdS/CFT context. The dynamics in the bulk is just pure Einstein gravity, while the geometry is that dual to thermal equilibrium: an AdS Schwarzschild black holeOne of the aspects this calculation will illustrate is that in AdS/CFT it is remarkably straightforward to compute directly in real time in thermal systems.

We will compute the viscosity from the Kubo formula

$$\eta = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G^R_{x_1x_2,x_1x_2}(\omega,0), \tag{7.14}$$

where, $G^R_{x_1x_2,x_1x_2}(\omega,0)$ is the retarded Green's function of the xy component of the energy momentum tensor, defined as

$$G^R_{\mu\nu,\alpha\beta}(\omega,\vec{k}) = -i\int dtd\vec{x}e^{i\omega t-i\vec{k}\cdot\vec{x}}\theta(t)\langle[T_{\mu\nu}(t,\vec{x}),T_{\alpha\beta}(0,0)]\rangle. \tag{7.15}$$

Using the GKPW formula the stress tensor perturbations are encoded in fluctuations of the metric. As usual in linearized gravity, one writes the metric as $g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$ where $g^0_{\mu\nu}$ is the background metric that solved for the AdS Schwarzschild black hole while $h_{\mu\nu}$ is the infinitesimal metric fluctuation, *i.e.* the gravitational wave/graviton.

The counting of the number of DOF for the massless graviton is a bit of a tricky affair: We focus on $4+1$ dimensional bulk. We first classify the graviton modes as follows. We take the spatial momentum to be in the x_3 direction $\vec{k}=(0,0,k)$ and this means that the perturbation $h_{\mu\nu} = h_{\mu\nu}(t,r,x_3)$. The system has an $SO(2)$ symmetry in the x_1x_2-plane and according to the behavior of the graviton modes under this symmetry we have three decoupled sets of graviton modes: the tensor mode (transverse mode) $h_{x_1x_2}$, the vector mode (shear mode), a linear combination of h_{tx_1}, $h_{x_3x_1}, h_{rx_1}$ together with h_{tx_2}, $h_{x_3x_2}, h_{rx_2}$ and two scalar modes (sound mode) from two linearly independent combinations of h_{tt}, h_{tx_3}, $h_{x_3x_3}$, $h_{x_1x_1} + h_{x_2x_2}$, h_{rr}, h_{tr} and h_{rx_3}. It directly follows from the linearized coordinate transformations on $h_{\mu\nu}$

$$\delta h_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \tag{7.16}$$

that $h_{x_1x_2}$ in this momentum configuration is a gauge-invariant, i.e. physical mode.

We now perturb the Einstein equation with a negative cosmological constant, eqn (6.1), around the solution of the AdS-Schwarzchild black hole geometry (6.2). Denoting the perturbation $h^{x_1}_{x_2}(t,r,x_3) \equiv g^{x_1\mu}h_{\mu x_2}$ with [47]

$$\delta g^{x_1}_{x_2} = h^{x_1}_{x_2}(t,r,x_3) = \int \frac{d\omega dk}{(2\pi)^2}\phi(r;\omega,k)e^{-i\omega t+ikx_3}, \tag{7.17}$$

one obtains linearized equation of motion for $\phi(r;\omega,k)$. One finds that it identical to the equation for a massless scalar

$$\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\phi(r;t,x_3) = 0 \tag{7.18}$$

Substituting the background metric eq. (6.2) and Fourier transforming we find

$$\phi(r;\omega,k)'' + \left(\frac{D+1}{r} + \frac{f'}{f}\right)\phi'(r;\omega,k) + \frac{(\omega^2 - k^2 f)L^4}{r^4 f^2}\phi = 0 \tag{7.19}$$

with $f = (1 - \frac{r_0^D}{r^D})$.

To obtain the retarded Green's function in the CFT we must now solve this equation with infalling boundary conditions at the horizon. The retarded Green's function is then given by the ratio of the subleading to the leading coefficient of the solution at the boundary:

$$\phi_{\text{sol}}(u,\omega,k) = A(\omega,k)r^{-\Delta_-} + B(\omega,k)r^{\Delta_- - D} + \dots \tag{7.20}$$

with $\Delta_- = 0$ in this case, and

$$G^{CFT}_{R,xy,xy}(\omega,k) \sim \frac{B(\omega,k)}{A(\omega,k)} \tag{7.21}$$

To obtain $A(\omega,k)$ and $B(\omega,k)$ is a relatively straightforward numerical exercise; in some special cases the answer is analytically known. Rather than using this brute force short-cut, we will give an alternate way [53, 54] to compute the solution which captures some of the essential physics. We first write the Green's function directly in terms of the solution $\phi_{\rm sol}$ following the GKPW construction

$$G_R^{CFT}(\omega,k) = \frac{1}{2\kappa^2}\lim_{r\to\infty} r^{-2\Delta_-}\sqrt{-g}g^{rr}\frac{\partial_r\phi_{\rm sol}(r)}{\phi_{\rm sol}(r)} \tag{7.22}$$

The overall factor $1/2\kappa^2$ follows from the normalization of the Einstein-Hilbert action.

The key step is that the viscosity (and all other transport coefficients) follow from the imaginary part of the retarded Green's function. Inserting $1 = \phi^*(r)/\phi^*(r)$ inside the limit one obtains for this imaginary part the expression

$$\mathrm{Im}G_R^{CFT}(\omega,k) = \lim_{r\to\infty} r^{-2\Delta_-}\sqrt{-g}g^{rr}\frac{\phi^*_{\rm sol}(r)\partial_r\phi_{\rm sol}(r) - \phi_{\rm sol}(r)\partial_r\phi^*_{\rm sol}(r)}{2i\phi^*_{\rm sol}(r)\phi_{\rm sol}(r)} \tag{7.23}$$

The numerator is readily recognized as the Wronskian which measures the flux density through a surface at fixed r. The physics is that for a field obeying a second order equation of the type

$$\phi'' + P(r)\phi' + Q(r)\phi = 0 \tag{7.24}$$

with $P(r)$ and $Q(r)$ real, the generalized Wronskian

$$W(r) = e^{\int^r P(r)}(\phi^*_{\rm sol}(r)\partial_r\phi_{\rm sol}(r) - \phi_{\rm sol}(r)\partial_r\phi^*_{\rm sol}(r)) \tag{7.25}$$

is conserved: $\partial_r W = 0$. Note that the combination $\sqrt{-g}g^{rr}$ is precisely this necessary prefactor: this is readily seen by acting with ∂_r and using the equation of motion (7.18). Rewriting the imaginary part of the retarded Green's function directly in terms of the conserved Wronskian we find

$$\mathrm{Im}G_R^{CFT}(\omega,k) = \lim_{r\to\infty} r^{-2\Delta_-}\frac{W(r)}{2i\phi^*_{\rm sol}(r)\phi_{\rm sol}(r)} \tag{7.26}$$

We will discuss the physics of this rewriting in a moment. Mathematically its computational power for frequency-independent transport coefficients is immediate. We can use the conservation of the Wronskian to evaluate the numerator at any point r. The most convenient is the horizon itself where the infalling boundary conditions are set. Near the horizon, near the single zero of $f(r)$, the equation of motion (7.19) reduces to

$$\phi'' + \frac{f'}{f}\phi' + \frac{L^4\omega^2}{r^4 f^2}\phi + \ldots = 0 \tag{7.27}$$

Writing $f(r) = (r - r_0)\frac{D}{r_0} + \ldots$,

$$\phi'' + \frac{1}{(r - r_0)}\phi' + \frac{L^4\omega^2}{D^2 r_0^2 (r - r_0)^2}\phi + \ldots = 0. \tag{7.28}$$

We can deduce the powerlaw dependence of the solution near the horizon by substituting the ansatz

$$\phi_{\rm sol}(r;\omega,k) = (r - r_0)^{\alpha}(1 + \ldots) \tag{7.29}$$

One finds

$$\alpha(\alpha - 1) + \alpha + \frac{L^4\omega^2}{D^2 r_0^2} = 0 \tag{7.30}$$

with solutions $\alpha = \pm\frac{i\omega L^2}{D r_0} = \pm\frac{i\omega}{4\pi T}$. In the last step we used the relation between the horizon location and the black hole temperature derived in eqn. (**??**). The choice $\alpha = -i\omega/4\pi T$ corresponds to the infalling solution. Thus near the horizon we may parametrize

$$\phi_{\rm sol}(z;\omega,k) = (r - r_0)^{-i\omega/4\pi T} F(r;\omega,k), \tag{7.31}$$

where $F(r;\omega,k)$ is regular at the horizon $r = r_0$. Evaluating now the conserved Wronskian near the horizon one finds

$$\begin{aligned} W(r_0) &= \lim_{r\to r_0} \sqrt{-g}g^{rr}\phi^* \overleftrightarrow{\partial} \phi \\ &= \lim_{r\to r_0} \frac{r^{D+1}}{L^{D+1}}\left(1 - \frac{r_0^D}{r^D}\right)\left(\frac{-2i\omega}{4\pi T}(r - r_0)^{-1}\right) F^*(1)F(1) + \ldots \\ &= \frac{r_0^{D+1}}{L^{D+1}}\frac{D}{r_0}\frac{-2i\omega}{4\pi T}F^*(1)F(1) \\ &= (\frac{4}{D}\pi T L)^{D-1}(-2i\omega)F^*(1)F(1). \end{aligned} \tag{7.32}$$

In the last line, we have again used the definition of the temperature $4\pi TL = Dr_0/L$. Substituting, one sees that the remaining unknown in the Green's function

$$\mathrm{Im}G_R^{CFT}(\omega) = \frac{1}{2\kappa^2}\lim_{r\to\infty}(\frac{4}{D}\pi T L)^{D-1}(-\omega)\frac{F^*(1)F(1)}{F^*_{\rm sol}(r)F_{\rm sol}(r)} \tag{7.33}$$

is the ratio of the absolute value of $F(r)$ at the horizon to $|F(r)|$ at the boundary. Formally one still needs to solve for $F(r)$ to determine this. However, in the limit $\omega \to 0, k \to 0$ the leading contribution will be the ω-independent solution for the remaining function F. From equation (7.19) this is readily seen to be the trivial constant function. This is the leading solution $\phi \sim Ar^{-\Delta_-}$ near $r \to \infty$ with our special case of $\Delta_- = 0$. Hence one obtains

$$\text{Im}G_R^{CFT}(\omega, k) = \frac{1}{2\kappa^2}(\frac{4}{D}\pi TL)^{D-1}(-\omega). \tag{7.34}$$

From the Kubo relation

$$\eta = -\lim_{\omega\to 0}\frac{1}{\omega}\text{Im}\langle T_{x_1x_2}(-\omega)T_{x_1x_2}(\omega)\rangle = -\lim_{\omega\to 0}\frac{1}{\omega}\text{Im}G_R^{CFT}(\omega, 0), \tag{7.35}$$

the shear viscosity therefore equals

$$\eta = \frac{1}{2\kappa^2}(\frac{4}{D}\pi TL)^{D-1}. \tag{7.36}$$

Recalling that $2\kappa^2 \equiv 16\pi G$ and comparing this to the entropy density (**??**) $s = \frac{4\pi}{2\kappa^2}(\frac{4}{D}\pi TL)^{D-1}$ for a $D+1$-dimensional AdS Schwarzschild black hole, we find the famous ratio

$$\frac{\eta}{s} = \frac{1}{4\pi}\frac{\hbar}{k_B}. \tag{7.37}$$

8. EXERCISES

Lecture III

9. CONDENSED MATTER IN A NUTSHELL

Condensed matter is about systems at finite density. When you put together a large amount of matter, how does it behave? For relatively dilute systems at finite temperature, one can readily apply the lessons of statistical physics. Many interesting phenomena occur, however, at low temperatures when quantum effects become important and/or when one considers densities large enough that the quantum wavefunctions of each of the constituents can overlap. This is the regime we will investigate here: we wish to know the macroscopic characteristics — density, pressure, equation of state, etc. — of $T/\mu \to 0$ low temperature (low energy) finite density quantum matter.

There is a large set of condensed matter wisdoms which we have acquired over decades that tell us what generically can happen when you study such quantum matter.

- The system can be *gapped*. This means that all excitations cost a finite amount of energy. In relativistic language, there is no massless state. Therefore at $T = 0$, i.e. when there is no energy density in the system, nothing (interesting) in fact happens. The system is in its groundstate, but is unable to respond to infinitesimal small (in energy) perturbations.

- The groundstate of the system can *spontaneously break a global symmetry*. This is the generic groundstate for a system of bosons. In the spontaneously broken or *ordered* state, the system has protected gapless/massless Goldstone modes. These modes then completely dominate the low temperature/low energy physics. The formalism to describe this is the Landau-Ginzburg free energy functional: a low-energy effective action for the order parameter of the symmetry breaking. From the free energy we can get all the other macroscopic equilibrium properties of the system.[4] The robustness of the

[4] Note that compared to particle physics, these Landau-Ginzburg functionals are often very simple. One only concentrates on the order parameter and not on any other degrees of freedom in the system. Thanks to Goldstone's theorem this is almost always sufficient at extremely low energies and macroscopic equilibrium properties. Particle physics Lagrangians are usually much more involved. One also wishes to know detail the behavior of excitations at moderate energies. In condensed matter one mostly wishes to connect the microscopics to the correct Landau-Ginzburg functional. In these broken groundstates, one therefore makes a moduli-space approximation.

results are guaranteed by Goldstone's theorem. In high dimensions the usual scaling arguments reduce the Landau-Ginzburg functional to a Gaussian (mean-field approximation). In lower dimensions quantum effects (fluctuations) are more important and one can get distinct non-mean field behavior.

- For a system with only fermions, one generically gets a *Landau Fermi liquid.* This is based on the Pauli principle. Recall that we are interested in matter at finite density and take a free fermion as the simplest example of a many-body-fermion system. In this free Fermi gas, two fermions cannot be in the same quantum state. They therefore must have different momentum. The occupied states of the fermions fill shells in momentum space outward from the origin as each shell has a slightly larger energy cost. The last fermion added to the system this way gives a preferred momentum scale, the fermi-momentum k_F (Fig 1.). The *Fermi liquid* is this Fermi gas state where one includes small interactions between the Fermions.

  The Fermi liquid has a well-defined stable fermionic quasiparticle excitation that costs no energy to excite. It is massless and it therefore controls the low-temperature/low-energy macroscopic behavior of the system. Its free energy is essentially that of a free non-interacting fermion. The robustness of the Fermi liquid follows from the Pauli principle. Intuitively this is very understandable. Strictly speaking, however, this is mathematically not as well understood as the Goldstone theorem for bosons. A way to phrase this is that there is no order parameter language for the Fermi liquid state. The closest mathematical explanation for the robustness is Luttinger's theorem. For charged fermions this states that *indepedent of the nature of the interactions* the total charge carried by the Fermi liquid state equals the volume of occupied states in momentum space expressed in terms of k_F We will discuss Luttinger's theorem in detail below.

This is what happens generically. The primary quest of quantum matter is to understand those systems that do *not* fall into one of these three classes.

A more refined version of this question is as follows. A *compressible* state is a finite density state of a system with a $U(1)$ symmetry, in which the charge Q as a function of the chemical potential μ behaves as

$$\langle Q\rangle \equiv -\frac{\partial}{\partial\mu}F \sim \mu^{\alpha} \quad , \; \alpha > 0 \tag{9.1}$$

In particular if a locally change the chemically potential, I will locally change the charge density. This state therefore conducts. In condensed matter nomenclature, it is therefore a *metal*. The amount of energy does not feature in this argument, so at infinitesimally small energies, this will still happen, and therefore a compressible state does not have a gap.

Recapitulating the lessons from above with one addition, the known compressible states are:

- A state with broken translational symmetry, i.e. a crystal. The state is a *solid*.
- A state with spontaneously broken $U(1)$ symmetry. One has a *superconductor* with a single massless Goldstone mode. Note that technically we have broken a global $U(1)$ and not the local $U(1)$ of Maxwell theory. In many condensed matter situations the electromagnetic interaction is so weak, however, that the mean free path of a photon is much larger than the sample size. This allows one to effectively think of the electromagnetic current as a global $U(1)$ current. At the end one can weakly gauge the global $U(1)$ to incorporate dynamical electromagnetic effects. From this viewpoint the spontaneously broken state is a superconductor, rather than a superfluid.
- The state is a *Fermi-liquid*. This is special in that it is not associated with a broken symmetry.

The question then is:

Is there a translationally invariant compressible state of matter without symmetry breaking that is not a Fermi liquid, and how do we characterize it?

This is where AdS/CFT will shed new light.

First, however, we will study in particular the Fermi liquid in some more detail. This will also explain why this question is so important. Before we do so, note that a finite density systems is manifestly associated with a $U(1)$ charge. This connection points to a strongly identifying macroscopic characteristic of the state, in addition to the pressure, density, etc.The expectation value of the current J^μ associated to this $U(1)$ current is an important macroscopic characteristic of this state. The timelike component J^0 is just the charge density. In equilibrium the expectation value of the spacelike component vanishes, of course, as it is odd under time reversal. But its fluctuations also contains very important information. To compute these, couple to an external source A_i^{external} as usual and add this to the action

$$\langle J^i \rangle = \langle \frac{\delta}{\delta A_i^{\text{external}}} S \rangle \tag{9.2}$$

We are already argued that in the absence of a driving source the expectation value vanishes. Thus for a small but finite source one finds

$$\langle J^i \rangle = \langle \frac{\delta}{\delta A_i^{\text{external}}} S \rangle = \langle J^i J^j \rangle A_j^{\text{external}} + \mathcal{O}(A) \tag{9.3}$$

One can think of this in extremely physical terms. We know that an external electric field can source a current. Specifically, in a gauge where $A_0 = 0$, $\partial_t A_i = E_i$. In Fourier space this becomes $A_i(\omega) = E_i(\omega)/(-i\omega)$. Now the amount of the current that this external electric field generates is controlled by the *conductivity* $\sigma(\omega)$ of the system. A larger conductivity results in a larger current etc. For a not-too-large electric field this is expressed through the defining relation $\langle J^i(\omega)\rangle \equiv \sigma^{ij}(\omega)E_i(\omega)$ where $\sigma^{ij}(\omega)$ is the conductivity tensor. The frequency dependent conductivity is also known as the optical or AC conductivity. The $\omega \to 0$ limit of the optical conductivity is the DC conductivity. Combining this defining relation with the current as a response to an external source, one obtains the expression for the conductivity

$$\begin{aligned}\sigma^{ij}(\omega) = \frac{\partial}{\partial E_j(\omega)} \langle J^i(\omega) \rangle &= \frac{1}{-i\omega} \frac{\partial}{\partial A_j^{\text{external}}(\omega)} \langle J^i J^k \rangle A_k^{\text{external}} \\ &= \frac{-1}{i\omega} \langle J^i(\omega) J^j(\omega) \rangle \end{aligned} \tag{9.4}$$

Thus for small fields one can read off the conductivity by measuring the two-point correlation of the current. Because the conductivity is closely related to the $U(1)$ charge of the chemical potential introduced to put the system at a finite density, we will encounter it extensively throughout these lectures.

9.1. The Landau Fermi Liquid Theory as an IR stable fixed point.

Before we start using AdS/CFT, for reference we present a modern view of Landau Fermi liquid theory. As we already alluded briefly, the Landau Fermi liquid is based on the zero temperature free Fermi gas. The momentum of the highest occupied state is called the Fermi momentum k_F and this momentum will characterize the macroscopic properties of the system. For charged free fermions the number density is equal to the $U(1)$ charge density. The finite density system is therefore obtained by introducing a chemical potential for the $U(1)$-charge. The low-energy (=low-frequency) dynamics are then readily seen to be described by expanding the free (non-relativistic spinless) electron action coupled to a $U(1)$ gauge field

$$\mathcal{S} = \int d^d x \Psi^\dagger(x)(i(\partial_t - ieA_t) + \frac{1}{2m}\nabla^2)\Psi(x) \tag{9.5}$$

in the presence of a finite chemical potential background

$$\begin{aligned} &A_t = \mu/e \\ &\mathcal{S} = \int d^d x \Psi^\dagger(x)(i\partial_t + \mu + \frac{1}{2m}\nabla^2)\Psi(x) \end{aligned} \tag{9.6}$$

One readily notes that there is a preferred momentum — the Fermi momentum $k_F = \sqrt{2m\mu}$ — for which the kinetic term action vanishes at $\omega = 0$, i.e. the state at $k = k_F$ costs zero energy to excite. The low energy effective action is the expansion around this state [112, 113].

$$\mathcal{S}_{\text{Fermi gas}} = \int \frac{d^d k}{(2\pi)^d} \Psi^\dagger(k)(i\partial_t - v_F(k - k_F))\Psi(k) + \dots \tag{9.7}$$

with $v_F = k_F/m$. Conventionally one invokes the Pauli principle that turning on small interactions cannot in any way change this occupation barrier and the existence of a momentum scale k_F, and postulates that the relevant degrees of freedom are still the fermions above. In the modern view this insight directly follows from the renormalization group [112, 113]. Assuming that the interactions between the fermions are controlled by a scale M, one can integrate out the interactions to arrive at an effective theory for the low-energy fermions. In perturbation theory this results in *analytic* higher order corrections to the dispersion relation plus induced interactions. As long as the scale M is much much larger than the chemical potential, rotational invariance uniquely determines

$$
\begin{aligned}
\mathcal{S}_{\text{Fermi liquid}} &= \int \frac{d^dk}{(2\pi)^d}\Psi^\dagger(k)\left(i\partial_t+\mu+\frac{\nabla^2}{2m}+\frac{\alpha}{M}(i\partial_t+\mu)^2+\frac{\beta}{M}(i\partial_t+\mu)\frac{\nabla^2}{2m}+\frac{\gamma}{M}\left(\frac{\nabla^2}{2m}\right)^2+\dots\right)\Psi(k) \\
&\quad+\frac{\xi}{M^d}(\Psi^\dagger\Psi)(\Psi^\dagger\Psi)+\dots \\
&= \int \frac{d^dk}{(2\pi)^d}\Psi^\dagger(k)\left(\omega-v_F(k-k_F)+\frac{\alpha}{M}\omega^2+\frac{\beta}{M}\omega(k-k_F)+\frac{\gamma}{M}(k-k_F)^2\dots\right)\Psi(k) \\
&\quad+\frac{\xi}{M^d}(\Psi^\dagger\Psi)(\Psi^\dagger\Psi)+\dots
\end{aligned}
\tag{9.8}
$$

where α,β,γ,ξ are dimensionless coupling constants generically of order $\mathcal{O}(1)$.

Straightforward dimensional analysis shows that all interactions between the electrons are irrelevant in any dimension $d>1$. The generic fixed point of an interacting finite density fermi system is thus a field theory of IR free fermionic *quasiparticles*. This generic effective low-energy theory is the Fermi liquid. Its tell-tale sign is that the two-point retarded correlation function of two such quasiparticles

$$
G_R(\omega,k) = \frac{1}{\omega-v_F(k-k_F)-\Sigma(\omega,k)} \tag{9.9}
$$

$$
= \frac{1}{\omega-v_F(k-k_F)-\Sigma'(\omega,k)-i\Sigma''(\omega,k)} \tag{9.10}
$$

has a self-energy $\Sigma(\omega,k)$ whose imaginary part $\text{Im}\Sigma(\omega,k)\equiv\Sigma''(\omega,k)$ behaves as $\Sigma''(\omega,k_F)\sim\frac{\alpha}{M}\omega^2+\dots$ for $k=k_F$. This is the reflection of the analyticity of the gradient expansion. The spectral density

$$
\mathcal{A} = -\frac{1}{\pi}\text{Im}G_R(\omega,k) \tag{9.11}
$$

$$
= -\frac{1}{\pi}\frac{\Sigma''(\omega,k)}{(\omega-v_F(k-k_F)+\Sigma'(\omega,k))^2+(\Sigma''(\omega,k))^2} \tag{9.12}
$$

therefore has a beautiful Lorentzian line shape around $\omega=0$ for $k=k_F$.

$$
\mathcal{A} = \frac{1}{\pi}\frac{\alpha\omega^2/M}{(\omega-v_F(k-k_F))^2+\alpha^2\frac{\omega^4}{M^2}}+\dots \tag{9.13}
$$

with total weight unity and width α/M. At finite temperature T the usual $t=-i\tau$, $\omega=i\omega_E$ with the Matsubara frequencies $\omega_E=nT$, means that "dimensionally" we can obtain a lot of information by just replacing $\omega\to\omega+iT$ in the *imaginary part of the self-energy*.

9.1.1. Spectral Functions

Single Fermion spectral functions play an important role in condensed matter physics. This is readily understandable as Fermi liquid theory is the foundation of band theory in crystals and as shown above the spectral function of a Fermi liquid has a distinct Lorentzian peak at $\omega = 0, k = k_F$. This peak can be directly measured with Angle Resolved Photo Emission Spectroscopy (ARPES). The probability of liberating an electron from a sample by shining light (photoemission) is directly proportional to the spectral function. Particularly in the last decade the resolution of energy and momentum of this final state electron has improved so much that extremely detailed spectral functions can be measured. The conventional way to express the single fermion spectral function is to first expand the Green's function (9.9) around $\omega = 0$ and $k = k_F$ as

$$G_R(\omega, k) = \frac{1}{\omega - v_F(k - k_F) - \Sigma'(0) - \omega\partial_\omega\Sigma'(0) - i\Sigma''(\omega) + \ldots} \tag{9.14}$$

$$= \frac{Z}{\omega - \frac{m}{m^\star}v_F(k - k_F) - i\Gamma} + G_{\text{incoherent}} \tag{9.15}$$

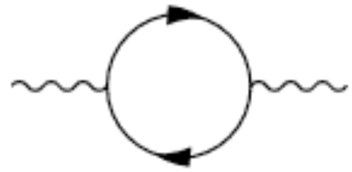
Here $Z \equiv 1/(1 - \partial_\omega\Sigma'(0))$ is known as the pole strength $\Gamma = Z\Sigma''(\omega)$ the width, and $m^\star = mv_Fk_F/(v_Fk_F - \Sigma'(0))$ the renormalized mass. [**Work out further**]

9.2. DC conductivity in Landau Fermi liquid theory

As an example of how the low-energy degrees of freedom determine the macroscopic properties, we will now compute the temperature dependence of the DC conductivity of the Fermi liquid. We will do so using the "Kubo" relation derived earlier

$$\sigma(\omega) = \frac{-1}{i\omega}\langle\vec{J}\vec{J}\rangle \tag{9.16}$$

In the effective action for the Fermi liquid, the current J^i is the composite operator $J^i = \frac{ie}{2m}\bar{\Psi}\partial^i\Psi$ (there are no gamma-matrices, as we are considering a spinless fermion for simplicity). The two point function of the currents to leading order is therefore the one-loop diagram



(9.17)

We will now simply state the result for the real part of this diagram. The real part can be powerfully rewritten in terms of Fermi-Dirac distributions $f(\omega)$ and the spectral function $\mathcal{A}(\omega, k)$. In d_S spatial dimensions one has

$$\begin{aligned} Re(\sigma_{DC}) &\sim \lim_{\omega\to 0} \frac{1}{i\omega} \mathrm{Im}\bar{\Psi}\Psi\bar{\Psi}\Psi \\ &\sim \lim_{\omega\to 0} \frac{1}{i\omega} \int d_S^d k \frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\frac{f(\omega_1)-f(\omega_2)}{\omega_1-\omega-\omega_2-i\epsilon}\mathcal{A}(\omega_1,k)\Lambda(\omega_1,\omega_2,\omega,-k)\mathcal{A}(\omega_2,k)\Lambda(\omega_2,\omega_1,\omega,k) \end{aligned} \tag{9.18}$$

Essentially this formula states that a photon with energy ω can create an electron pair with energies ω_1 and ω_2 pulled from the occupied states $f(\omega_i)\mathcal{A}(\omega_i, k)$. The relative minus sign between $f(\omega_1)$ and $f(\omega_2)$ follows from Fermi statistics and all the model depedent details are stuck in the transport vertex $\Lambda(\omega_1, \omega_2, \omega, k)$. We next make the assumption that this vertex is in fact momentum independent (at low energies). This is not entirely obvious, but can be verified to be so in most microscopic models. To leading order in ω Λ is then a constant, and we get

$$\begin{aligned} \sigma_{DC} &\sim \lim_{\omega\to 0} \frac{1}{i\omega} \int d^{d_S} k \frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\frac{f(\omega_1)-f(\omega_2)}{\omega_1-\omega-\omega_2-i\epsilon}\mathcal{A}(\omega_1,k)\mathcal{A}(\omega_2,k) \\ &\sim \int d^{d_S} k d\omega_1 \frac{df}{d\omega_1}\mathcal{A}(\omega_1,k)^2 \end{aligned} \tag{9.19}$$

One can get the dominant temperature-dependence in the spectral function by shifting the the imaginary part of the self-energy $\Sigma''(\omega) \to \Sigma''(\omega + iT)$. This lifts the pole at $\omega = 0$, but nevertheless the spectral function $\mathcal{A}(\omega, k)$ remains dominated by the Fermi-surface at $\omega \sim 0, k \sim k_F$.

One now notes that the Fermi surface is a $d_S - 1$-dimensional surface in momentum space with a single normal direction $k_\perp$. Because of the Pauli principle to lowest order all excitations can only carry momentum in the direction $k_\perp$. This means that $d_S - 1$ of the momentum integrals computing the DC conductivity localize on the Fermi surface. Doing so

$$\sigma_{DC} \sim k_F^{d_S-1} \int dk_\perp d\omega \frac{df}{d\omega} \mathcal{A}(\omega, k_\perp)^2 \tag{9.20}$$

we see that the system has reduced to an effective $d = 1$ system. Next we note that the derivative of the Fermi-Dirac distribution is essentially a delta-function at $\omega = 0$, so we can perform the ω-integral as well. We can now scale the temperature dependence out of integral by a redefinition $k_\perp \to T^2 k_\perp / M$ and we deduce characteristic temperature-squared dependence of the conductivity of a Fermi-liquid.[5] **[CHECK]**

$$\sigma_{DC} \sim k_F^{d_S-1} \int dk_\perp \mathcal{A}(0, k_\perp)^2|_{\Sigma''(0+iT)} \tag{9.21}$$

$$\sim k_F^{d_S-1} \int dk_\perp \frac{T^4/M^2}{\left((-v_F k_\perp)^2 + \frac{T^4}{M^2}\right)^2} \tag{9.22}$$

$$\sim \frac{T^2}{T^4} k_F^{d_S-1} M \sim \frac{1}{T^2} k_F^{d_S-1} M \tag{9.23}$$

Experimentally one usually measures the resistivity ρ. This is just the inverse of the conductivity. Thus for a Fermi liquid

$$\rho_{FL} \equiv \frac{1}{\sigma_{DC}^{FL}} \sim T^2 \tag{9.24}$$

9.2.1. Macroscopic features of FL follow from Quasi 1D

Completely following our general introduction, all the macroscopic properties of the Fermi liquid are essentially explained by this effective dimensional reduction of the massless quasi-particle to a $d_S = 1$ dimensional dynamics perpendicular to the Fermi surface.

For instance for a general massless system in d_S dimensions where there is no intrinsic scale the temperature scaling of the entropy density and the specific heat is simply determined by dimensional analysis.

$$s \sim \frac{1}{L^{d_S}} \sim \frac{T^{d_S}}{v^{d_S}} \tag{9.25}$$

$$c_V \sim \frac{T}{\partial S} \partial T \sim \frac{T^{d_S}}{v^{d_S}} \tag{9.26}$$

This is called *hyperscaling*. Here v is the characteristic IR velocity that appears in the dispersion relation of the massless excitation $\omega \sim vk$. For a relativistic system clearly $v = c$.

[5] Note that the DC-conductivity is dimensionless in $d_S = 2$ as it should.

For a Fermi liquid the d_S-dimensional system reduces to $d_S^{eff} = 1$ dimensional system due to the presence of the Fermi surface at k_F. As there is no other scale in the system, one can assume that hyperscaling holds in this reduced sense and argue that the temperature scaling of the entropy and specific heat behave as

$$s \sim \frac{T^{d_s-1}}{k_F} \sim k_F^{d_S} \frac{T}{E_F} \tag{9.27}$$

$$c_V \sim \frac{T}{E_F} \tag{9.28}$$

This very simple insight is indeed borne out by a detailed microscopic calculation.

9.2.2. Luttinger's theorem

Let us finally briefly discuss Luttinger's theorem. In the Fermi liquid it is very easy to establish a relation between the number density and the Fermi momentum k_F. The expectation value of the number density as a function of momentum is

$$n_F(k) = \int \frac{d\omega}{2\pi i} \langle \Psi^\dagger(\omega, k)\Psi(\omega, k)\rangle \tag{9.29}$$

Again the right hand side is a two point correlation function and to lowest order this therefore can be computed from the one-loop expression

$$\begin{aligned} n_F(k) &= \lim_{t\to 0_-} \int \frac{d\omega}{2\pi i} G_R && (9.30)\\ &= \lim_{t\to 0_-} \int \frac{d\omega}{2\pi i} \frac{Z}{\omega - v_F k_\perp - i\epsilon \mathrm{sign}(\omega)} && (9.31)\\ &= Z\theta(k_F - k) && (9.32) \end{aligned}$$

This is the classic step function profile of the zero-temperature occupation density of a free Fermi gas. The total number density for the free Fermi gas therefore equals

$$n_{total} = \int \frac{d^2k}{(2\pi)^2} Z\theta(k_F - k) = Z\frac{k_F^2}{4\pi} \tag{9.33}$$

(In the free system $\Sigma = 0$, but we need to be explicit about the $i\epsilon$ prescription in that case.) For this free system the number density is directly proportional to the total charge density in the system. In the conventional normalization

$$Q \equiv 4\pi^2 q n_{total} = \pi k_F^2 \tag{9.34}$$

This relation is *Luttinger's theorem*. Its power relies in the fact that Luttinger has proved that no perturbative or non-perturbative interactions can change this relation.[**Is this because conserved currents do not renormalize?**]. There is a caveat. The Fermi liquid groundstate must remain, e.g. turning on a BCS instability which causes pairing and condensation of the composite pair operator can violate this theorem.

As mentioned, Luttinger theorem is in some way the Fermi liquid equivalent of Goldstone's theorem. It explains the robustness of the Fermi surface and its associated massless quasiparticle. Nevertheless there is no fundamental understanding of Luttinger's theorem as spontaneous symmetry breaking provides for Goldstone's theorem.

9.3. Experimental non-Fermi-liquids

So far the story may seem quite academic. The importance of the question whether there are compressible states of matter without symmetry breaking that are not a Fermi liquid became immediate with the discovery of high temperature superconductors in 1985. These fall outside the standard BCS paradigm. Within BCS theory, one can derive a *gap equation*, which gives the value of the order parameter as a function of the temperature

$$\Delta(T) = \Lambda_{UV} e^{-\frac{1}{\lambda n_F(T)}} \tag{9.35}$$

[**CHECK**] Here Λ_{UV} is a UV cut-off, λ is the small perturbative coupling parameter that controls the binding energy of the Cooper pair and n_F is the number density. The identifying characteristic is the exponential suppression. This precludes BCS superconductivity at any high temperature. For normal materials one has, and one finds an upper bound for the critical temp

$$T^{\text{max}}_{\text{critical BCS}} = K \tag{9.36}$$

High temperature superconductors violate this bound by definition. The mechanism underlying superconductivity can therefore not be the conventional BCS mechanism. The superconducting phase itself is actually not the mystery. As explained superconductivity is just the spontaneous breaking of $U(1)$ symmetry and this can has no temperature limitations. Indeed experimentally one readily shows that one has (d-wave) Cooper paired electrons that Bose condense. The mystery is why the material *becomes* superconducting at such a high temperature. The failure of BCS theory means that it cannot be a regular Fermi liquid with a perturbative Cooper pairing interaction. Indeed experimentally the macroscopic characteristics of the *strange metal* state just above the critical temperature are *not* those of a regular Fermi liquid. The most notable of these is the fact that the resistivity scales linear in the temperature rather than quadratic.

The typical phase diagram of a high temperature superconductor is given in Fig. 2. At optimal doping — the doping value where T_c is maximal — the material is in this strange metal phase for $T > T_c$ and superconducts for $T < T_c$. For higher doping above T_c there is a crossover from the strange metal to a regular Fermi liquid regime. Very near the doping where the superconducting region vanishes, the onset of superconductivity can in fact be explained by standard BCS theory. For lower than optimal doping at temperatures higher than T_c there is a phase transition to another mysterious *pseudogap* phase[6] and at even lower doping a crossover to an antiferromagnetic phase.

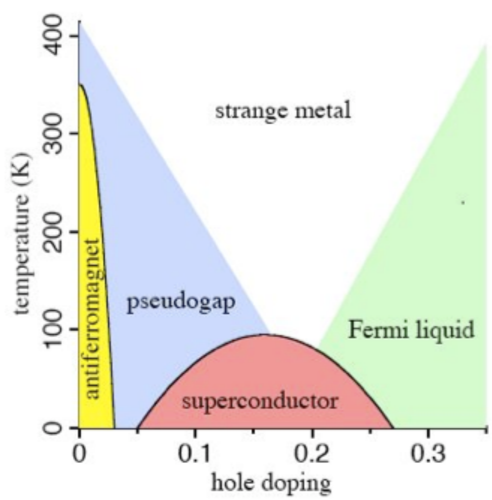


FIG. 2: The typical phase diagram of a high T_c superconductor.

Our focus here will be on the *strange metal* phase. Quite soon after the discovery of high T_c superconductivity, a very phenomenological proposal was made to explain the linear resistivity [135]. Suppose the full single fermion spectral function would be of the form

$$G = \frac{1}{\omega - v_F k_\perp + i\omega \ln \omega} \tag{9.37}$$

[6] Only very recently has it been settled that this is a true phase transition, see [? ?].

From our derivation of the scaling behavior of the resistivity, it is easy to say that if $\Sigma'' \sim \omega^a$, then the conductivity scales as $\sigma \sim T^{-a}$ and thus the resistivity as $\rho \sim T^a$. The logarithm is needed to differentiate it analytically from the true linear term in ω. This "model", or rather this Green's function, is clearly not that of a regular Fermi liquid. Collectively (phenomenological) single fermion spectral functions that do not have a width that scales as ω^2 are called *non-Fermi* liquids. And this particular one with $\Sigma'' \sim \omega \ln \omega$ is called the *marginal Fermi liquid*.

Although a tremendous effort has been made to understand the strange metal better and beyond the phenomenology of the marginal Fermi liquid, only in one qualitative aspect has progress been made. It is now believed that the barrier to understanding is explained by the fact that underlying the physics of the strange metal is the phenomenon of *Quantum Criticality*. This is the critical universal behavior that occurs in the vicinity of a *Quantum Phase transition*. The latter is a phase transition which happens at exactly $T = 0$; it is driven by quantum rather than thermal fluctuations. Now the relevant aspect is that if this phase transition is second order, the absence of a scale at the critical point (due to the diverging correlation length) means that the quantum field theory describing this point must be a conformal field theory. The special aspect of a quantum critical theory compared to a classical critical theory is that one now raises the temperature in the conformal field theory, the conformal constraints resonate through in the finite temperature physics. A CFT at finite T is still very special in that all its dynamics are still controlled by the $T = 0$ conformal symmetry and in general the only aspect that changes is that all dimensionfull quantities are now given in terms of the only present scale T. To get non-conformal (generic) behavior one needs at least two scales. This means that the phase diagram near a quantum critical point looks as in Fig. 9.3 The "fan-like" region is indeed very suggestive when compared to the phase diagram of high Tc superconductors. Although the idea that the physics underlying the strange metal is a finite T conformal field theory, in detail it is not so simple. In particular scale-invariance is only observed in terms of energy-temperature scaling. In spatial directions one still notes a distinct Fermi surface with ARPES data. This curious combination, scale-less in the "time-direction", but a distinct Fermi-momentum in the spatial directions has been coined "local quantum criticality".

The Fermi-surface and Fermions apparently continue to play a key role in the strange metal region. An approach to understanding the strange metal and its quantum criticality

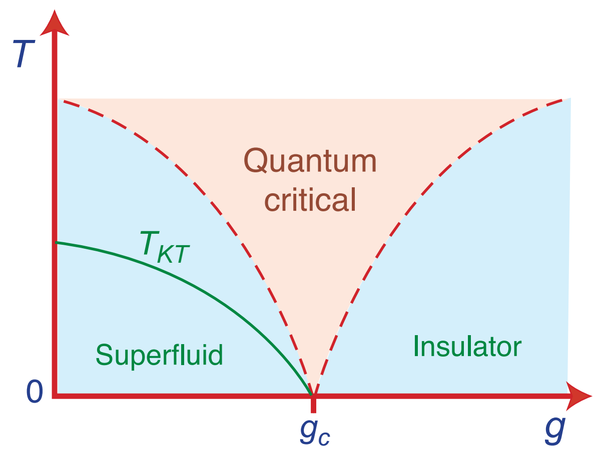


FIG. 3: Generic phase diagram of a quantum phase transition

is to understand the role of fermionic degrees of freedom in strongly interacting CFTs. That will be our ultimate goal. First we shall try to understanding strongly interacting CFTs, i.e. quantum critical systems, at finite density more generally.

10. FINITE DENSITY HOLOGRAPHY/SEMI-LOCAL QUANTUM LIQUID/ADS2-METAL

To describe a system at finite (charge) density we recall that *global* charges in the CFT, correspond to *local* symmetries on the AdS gravity side. Thus the minimal system we need to consider is AdS-Einstein gravity — dual to the energy-momentum sector — plus Maxwell theory — dual to a $U(1)$ global charge. Moreover, from statistical physics we recall that we can think of the chemical potential as the "source" for charge density. Since the expectation value of the current is given by the subleading part of the solution to Maxwell's equation, we deduce that an AdS system dual to a field theory at finite chemical potential will be a solution to the AdS equations of motion with the other *leading* non-normalizable solution to Maxwell equation non-vanishing.

The simplest such solution is the charged AdS black hole. In the remainder of this discussion of AdS/CFT we will consider AdS theories in $d = 3 + 1$ dimensions, describing $2 + 1$ dimensional strongly coupled systems such as can occur at interfaces in condensed matter experiments. In $d = 4$ the charged black hole has metric and timelike component of the gauge potential A_t

$$ds^2 = \frac{r^2}{L^2}(-fdt^2 + d\vec{x}^2) + \frac{L^2}{r^2}\frac{dr^2}{f}, \tag{10.1}$$

$$f(r) = 1 + \frac{Q^2}{r^4} - \frac{M}{r^3}, \quad A_t = \mu\left(1 - \frac{r_0}{r}\right) \tag{10.2}$$

where [**CHECK ESPECIALLY** g_F]

$$\mu = \frac{2Q}{r_0} \tag{10.3}$$

The differences in the geometry as compared to the AdS-Schwarzschild solution Eq. (6.2) are encapsulated by the altered redshift factor $f(r)$. The horizon is still determined by the largest root of this function, $f(r_0) = 0$. We have used the gauge freedom to shift the gauge field such that it vanishes at the horizon.

Using the recipe for the black hole thermodynamics of section **??**, the black hole temperature and entropy are

$$T = \frac{3r_0}{4\pi L^2}\left(1 - \frac{Q^2}{3r_0^4}\right) \;, \quad s = \frac{2\pi}{\kappa^2}\left(\frac{r_0}{L}\right)^2 \quad \mu = 2g_{EM}\frac{Q}{r_0} \tag{10.4}$$

where r_0 is related to M as

$$M = r_0^3 + \frac{Q^2}{r_0}. \tag{10.5}$$

The novel part of the charged black hole compared to the uncharged one is that one notices that for $r_0 = \frac{\sqrt{Q}}{\sqrt[4]{3}}$ the temperature vanishes, but the redshift factor stays non-trivial

$$T = 0 \Leftrightarrow \quad r_0 = \frac{\sqrt{Q}}{\sqrt[4]{3}} \tag{10.6}$$

$$f(z) = (1 - \frac{r_0}{r})^2(1 + 2\left(\frac{r}{r_0}\right)^2 + 3\left(\frac{r}{r_0}\right)^3) \tag{10.7}$$

One still has a black hole. This *extremal* black hole is the dual of a zero temperature finite density state.

We can already make a remarkable statement about the state in the CFT that this extremal black hole encodes. Note that the entropy remains finite even for $T = 0$. This means that the extremal AdS RN black hole encodes a groundstate with an *extensive* amount of degeneracy. This is very counterintuitive, and strongly suggests that this system is actually ridiculously unstable to any deformation that lifts the degeneracy.

10.1. AdS$_2$ near horizon geometry and emergent local quantum criticality

Let us study the extremal black hole some more. In particular, just as the case for the generic black hole, the characteristic physics is controlled by the horizon. In general, the horizon r_h is determined from the vanishing of $f(r_h) = 0$. To study what happens near the horizon we Taylor expand $f(r)$ near $r - r_h$ as

$$f(r) = f'(r_h)(r - r_h) + f''(r_h)(r - r_h)^2 + \dots \tag{10.8}$$

where the '...' are higher order terms in $r - r_h$. Following AdS/CFT it is precisely this near-horizon geometry that is representative for the IR physics in the field theory. In the case of a generic black hole, we learned that $f'(r_h) \propto T$. The finite temperature near horizon geometry is therefore quite different from the zero temperature near horizon geometry. It is governed by the next order coefficient $f''(r_h)$. This reflects the universal feature that extremal black holes have a double zero at the horizon. Specifically

$$f(r) = 6\frac{(r - r_0)^2}{r_0^2} + \dots \tag{10.9}$$

Inserting the near-horizon redshift factor in the full metric yields the near horizon geometry in the regime $\frac{r-r_0}{r_0} \ll 1$,

$$ds^2 = -\frac{6(r - r_0)^2 dt^2}{L^2} + \frac{L^2 dr^2}{6(r - r_0)^2} + \frac{r_0^2}{L^2} dx^2 + \dots, \tag{10.10}$$

while the gauge potential becomes

$$A_t = \frac{\mu}{r_0}(r - r_0) \tag{10.11}$$

Something surprising has happened! One infers that in terms of the near horizon coordinate $r - r_0$, the metric factors multiplying dt^2 and dr^2 acquire a similar structure as the bare AdS metric Eq. (**??**): this is an effective anti-de Sitter geometry, except that the space directions (dx^2) are multiplied by the constant $(r_0)^2$! Therefore, the space directions form just a flat space, while the effective anti de Sitter geometry is only realized in the two-dimensional t, r directions! To render this more explicit, let us re-parametrize the near-horizon metric in terms of a radial coordinate ζ which is the inverse of the distance from the horizon, and a radius L_2

$$\zeta = \frac{L_2^2}{r - r_0}, \quad L_2 = \frac{L}{\sqrt{6}}, \tag{10.12}$$

$$ds^2 = \frac{L_2^2}{\zeta^2}(-dt^2 + d\zeta^2) + \frac{r_0^2}{L^2} d\vec{x}^2, \quad A_t = \frac{1}{\sqrt{6}\zeta} \tag{10.13}$$

This is the metric of a space time with an $\text{AdS}_2 \times R^2$ geometry, where the AdS_2 part has a rescaled radius $L_2 = L/\sqrt{6}$ w.r.t. the original AdS_4 geometry.

The remarkable aspect of this near-horizon discovery is that the AdS_2 geometry precisely should encode a physical regime where time-like rescalings show self-similar behavior, but spatial rescalings can be sensitive to an underlying fundamental scale. This is precisely what the notion of *local* quantum criticality tries to capture. So the extremal AdS RN black hole is the *first* description of a theory which contains in it in a very concrete and quantitative way the notion of *local quantum criticality*. In the next lecture we will see how this manifests itself in more detail.

11. EXERCISE: COMPUTING THE CONDUCTIVITY OF THE ADS-RN METAL

Following our introduction one of the characteristic quantities to compute in a finite density system is the conductivity. Let us actually first compute the conductivity in the $\mu = 0$ AdS background. In the specific case of $d = 2 + 1$ dimensions, there is no need for a detailed computation. In d dimensions a conserved current has dimensions $d - 1$. Its Fourier transform therefore has dimensions -1. The two-point function without the momentum-conserving delta function therefore has dimension $d - 2$, and the conductivity $\sigma(\omega) = \frac{-1}{i\omega}\langle J(\omega)J(\omega)\rangle$ thus has dimension $d-3$. In our case $\sigma(\omega)$ is therefore dimensionless. Now in the pure AdS background dual to an exact CFT there is *no* scale in the problem. Therefore the conductivity can only be a pure number independent of ω.

What happens when we turn on a chemical potential? In the presence of a chemical potential, we now have an abundance of charge carriers at scales below μ. In the presence of a constant electric field this will cause a current to run. In fact if we change *nothing* else, this DC current will be infinite as the carriers start to accelerate. Combining this with a sum-rule, that the integrated density of states (as a function of ω) cannot change (this is unitarity/completeness of the underlying quantum theory), we see that between the DC current ($\omega = 0$) and the chemical potential-scale $\omega \sim \mu$, the conductivity (which is proportional to the number of states) must decrease compared to the $\mu = 0$ conductivity. As an exercise we will now see that this indeed happens in the conductivity of a finite density CFT system through a computation of its AdS dual.

We will only work at $T = 0$ and for computational convenience, we will use Poincaré like coordinates for the isotropic extremal charged AdS black hole. In such coordinates the background is

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu + g_{zz}dz^2 \\ &= \frac{L^2\mu^2}{12z^2}(-f(z)dt^2 + dx^2 + dy^2 + \frac{12dz^2}{\mu^2 f(z)}) \end{aligned} \tag{11.1}$$

The metric interpolates between an asympotically AdS space at $z = 0$ and an extremal horizon at the double zero of the redshift function $f(z)$

$$f(z) = (1-z)^2(1+2z+3z^2) \tag{11.2}$$

(Let us just mention for completeness that for the non-extremal finite temperature solution $f(z)$ has a single zero of course [7]

$$f(z) = (1-z)(1+z+z^2-q^2z^3), \quad T = \frac{\mu}{8q\pi}(3-q^2) \tag{11.3}$$

and one should replace $\mu^2/3$ with μ^2/q^2 in the metric.)

Together with an electrostatic potential

$$A_0 \equiv \Phi = \mu(1-z) \tag{11.4}$$

it is a solution to the equations of motion of the AdS-Einstein-Maxwell action

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R + \frac{6}{L^2} - \frac{L^2}{4}F_{MN}F^{MN}\right) \tag{11.5}$$

To obtain the optical conductivity at low-ω we will not use the Kubo formula, but Ohm's law directly. Recall that the conductivity is defined as $J^i = \sigma E_i$. Recall also that in AdS/CFT the dictionary computes the partition function in the presence of a source

$$Z(J^\mu)_{CFT} = \int DXe^{iS_{CFT}[X]+J^\mu\mathcal{O}_{CFT}(X)} \tag{11.6}$$

Thus to study the CFT in the presence of an external (electric) field, $E^i_{ext} = \partial_t A^i_{ext}$ (in the gauge $A^0_{ext} = 0$) we need that $J^i \sim A^i_{ext}$ is non-vanishing. Through the dictionary $J^i \overset{AdS/CFT}{\equiv} \lim_{z\to 0} A^i_{AdS}$ this means that leading source term to the equation of motion of A^i_{AdS} is non-vanishing. At the same time the *response* $J^i_{CFT} \overset{AdS/CFT}{\equiv} \lim_{z\to 0}\partial_z A^i_{AdS}$ (Here we use that we know the scaling dimension $\Delta = d-1$ for the operator dual to A^i). Putting

[7] For our reference: compared to Sean's lectures [1], we have set $\gamma = 2$, $r_+ = 2q/\mu$ and rescaled $r = r_+z$

this together we see that the conductivity just follows from solving the equation of motion for A^i_{AdS} in the background and then evaluating the quantity

$$\sigma(\omega) = \frac{J^x_{CFT}}{E^x_{CFT}} = \lim_{z\to 0} \frac{\partial_z A^x_{AdS}(\omega)}{-i\omega A^x_{AdS}(\omega)} \tag{11.7}$$

There is a tricky part to the computation. We need to evaluate the equation of motion for fluctuations A_x in the charged extremal black-hole background. As we will see, when we expand $A_\mu = A^{BH}_\mu + \delta A_\mu$ and $g_{\mu\nu} = g^{BH}_{\mu\nu} + h_{\mu\nu}$ the kinetic term will be off-diagonal in the fluctuations $\delta A_\mu,\ h_{\mu\nu}$. We will have to take this cross-coupling into account. So we take the action (11.5) and expand to second order

$$S = S_{BH} + \delta S|_{BH} + \delta^2 S|_{BH} + \dots \tag{11.8}$$

The first variation will vanish because the extremal BH is a solution to the equations of motion. For the second variation, let's consider the variation of the Maxwell term first. The first variation gives the stress-tensor for the metric variation and the Maxwell equation for the δA_μ and then we need to vary this one more time

$$\delta^2 S = \delta \int \sqrt{-g}\left(-\frac{1}{2}h^{MN}T^{Maxwell}_{MN} - L^2\partial_M \delta A_N F^{MN} + \dots\right) \tag{11.9}$$

By symmetry δA_x cannot mix with any of the other δA_M fluctuations, so we can set all those to vanish. Then from an expansion of the Maxwell stress tensor to first order in gauge-field fluctuations A_x around the background

$$\begin{aligned}
T^{Maxwell}_{MN} &= g^{RS}F_{MR}F_{NS} - \frac{1}{4}g_{MN}F_{RS}F^{RS} \\
&= \left(g^{RS}(\partial_{[M}\delta A_{R]})F^{(BH)}_{NS} + (M \leftrightarrow N)\right) - g_{MN}(\partial_R \delta A_S)F^{RS}_{(BH)} + \dots \qquad (11.10)\\
&= \left(g^{xx}(\partial_M \delta A_x)F^{(BH)}_{Nx} - \delta_{Mx}g^{RS}\partial_R\delta A_x F^{(bg)}_{NS} + (M \leftrightarrow N)\right) - g_{MN}(\partial_R \delta A_x)F^{Rx}_{(BH)} \qquad (11.11)\\
&= \delta_{Mx}\delta_{Nt}g^{zz}\partial_z\delta A_x\Phi' - \delta_{Mx}\delta_{Nz}g^{tt}\partial_t\delta A_x\Phi' + (M \leftrightarrow N) + \dots \qquad (11.12)
\end{aligned}$$

In the last line we substituted that the only non-vanishing part of $F^{(BH)}_{MN} = F_{z0} = -F_{0z} = \partial z A^{BH}_0 \equiv \Phi'$. Thus we see that the A_x mixes with the metric fluctuations h_{xt} and h_{xz}.

Now we need to compute $\delta^2 S_{Einstein-Hilbert}$ to compute their contribution. To do so, we use a trick. Again, because of the symmetries all components of h_{MN} where *neither* $M = x$ or $N = x$ can be set to zero consistently because of the symmetries of the system. Also we are only interested in frequency-dependent fluctuations and not momentum-dependent ones. Now, the trick is that an off-diagonal metric fluctuation with one component in a spatial direction $h_{xM}(z,t)$ that does not depend on the x-direction behaves effectively as a gauge-field with a spacetime dependent coupling [41, 52]

$$S^{EH}_{\text{eff}} = \int d^4x \sqrt{-g_{bg}} \left(-\frac{1}{4g_{\text{eff}}(z)^2} f^{(x)}_{MN} f^{(x)MN} \right) \tag{11.13}$$

with $g^2_{\text{eff}} = g^{xx}$, $\mathcal{A}^{(x)}_M = h^x_M$ and g_{bg} the background metric.

Analyzing this part by part, by symmetries again $\mathcal{A}_y$ decouples anc can be to zero. Next, the equation of motion for $\mathcal{A}_z$ is a constraint, and directly enforcing this constraint into the action the remaining two modes $\mathcal{A}_t$ and A_x decouple. To see this, note that for zero momentum, $\vec{k} = 0$, the only nonzero component of f_{MN} is f_{zt}. Fourier transforming all fields in the time-direction $\phi(t) = \int \frac{d\omega}{2\pi} \phi(\omega) e^{-i\omega t}$, the effective action therefore equals (as usual quadratic terms a_z^2 below should be read as $a_z(-\omega)a_z(\omega)$)

$$\begin{aligned} S_{\text{eff}} &= \delta^2 S^{EH}_{\text{eff}} + S^{int}_{\text{eff}} \\ &= \int \sqrt{-g_{bg}} \Big(-\frac{1}{2g^{xx}} g^{zz} g^{tt} \left(\partial_z \mathcal{A}_t^2 + \omega^2 \mathcal{A}_z^2 + 2i\omega (\partial_z \mathcal{A}_t) \mathcal{A}_z \right) \\ &\quad - \frac{1}{2} g^{zz} g^{tt} \mathcal{A}_t \Phi' \partial_z A_x + \frac{i\omega}{2} g^{zz} g^{tt} A_x \mathcal{A}_z \Phi' \Big) \end{aligned} \tag{11.14}$$

The second term S^{int}_{eff} is the first stress term obtained in eq. (11.9), where for simplicity A_x is what we called δA_x before.

The constraint that follows from varying w.r.t. $\mathcal{A}_z(\omega)$ equals

$$-\omega^2 \mathcal{A}_z - i\omega \partial_z \mathcal{A}_t + \frac{i\omega}{2} g^{xx} A_x \Phi' = 0 \tag{11.15}$$

Substituting the constraint back into the total action, including the second term in (11.9), one finds

$$\begin{aligned} S = \int \sqrt{-g_{bg}} \Big(&-\frac{1}{g^{xx}} g^{zz} g^{tt} \left((\partial_z \mathcal{A}_t)^2 \right) \Big) + \frac{1}{2} g^{zz} g^{tt} \partial_z \mathcal{A}_t A_x \Phi' - \frac{1}{8} g^{zz} g^{tt} g^{xx} A_x^2 (\Phi')^2 \\ &- \frac{1}{2} g^{zz} g^{tt} \Phi' (\partial_z A_x) \mathcal{A}_t \\ &- \frac{g^{xx} g^{zz}}{2} (\partial_z A_x)^2 - \frac{\omega^2 g^{xx} g^{tt}}{2} A_x^2 \Big) \end{aligned} \tag{11.16}$$

Integrating the term in the second line by parts and using the background equation of motion

$$\partial_z \sqrt{-g} g^{zz} g^{tt} \Phi' = 0 \tag{11.17}$$

one finds the decoupled system (after shifting $\partial_z \mathcal{A}_t$)

$$S = \int \sqrt{-g_{bg}} \left(-\frac{1}{g^{xx}} g^{zz} g^{tt} \left((\partial_z \mathcal{A}_t - g^{xx} A_x \Phi')^2 \right) \right) + \frac{1}{8} g^{zz} g^{tt} (\Phi' A_x)^2 g^{xx} \\ - \frac{g^{xx} g^{zz}}{2} (\partial_z A_x)^2 - \frac{\omega^2 g^{xx} g^{tt}}{2} A_x^2 \Bigg) \tag{11.18}$$

Using eq. (7.24) and eq. (5.12) we can now directly write down the equation of motion for A_x equals

$$\frac{1}{\sqrt{-g}} \partial_z \sqrt{-g} g^{xx} g^{zz} \partial_z A_x - \omega^2 g^{xx} g^{tt} A_x + \frac{1}{2} g^{tt} g^{xx} g^{zz} (\Phi')^2 A_x = 0 \tag{11.19}$$

Substituting the explicit metric of the extremal AdS-RN black hole, this simplifies to[8]

$$\partial_z f(z) \partial_z A_x + \frac{4q^2 \omega^2}{\mu^2 f(z)} A_x - \frac{4q^2 z^2}{2\mu^2} (\Phi')^2 A_x = 0 \tag{11.20}$$

The final step is to solve this equation numerically, and extract the objective of interest

$$\sigma = \lim_{z \to 0} \frac{i}{\omega} \frac{\partial_z A_x}{A_x} \tag{11.21}$$

We will do so with Mathematica in the exercise session. What one finds (Fig 4), is indeed the expected behavior of the conductivity, with an infinite peak at $\omega = 0$ (by Kramers-Kroenig this is $1/\omega$ in the imaginary part), a dip for $\omega < \mu$ and a constant asymptote for $\omega \gg \mu$.

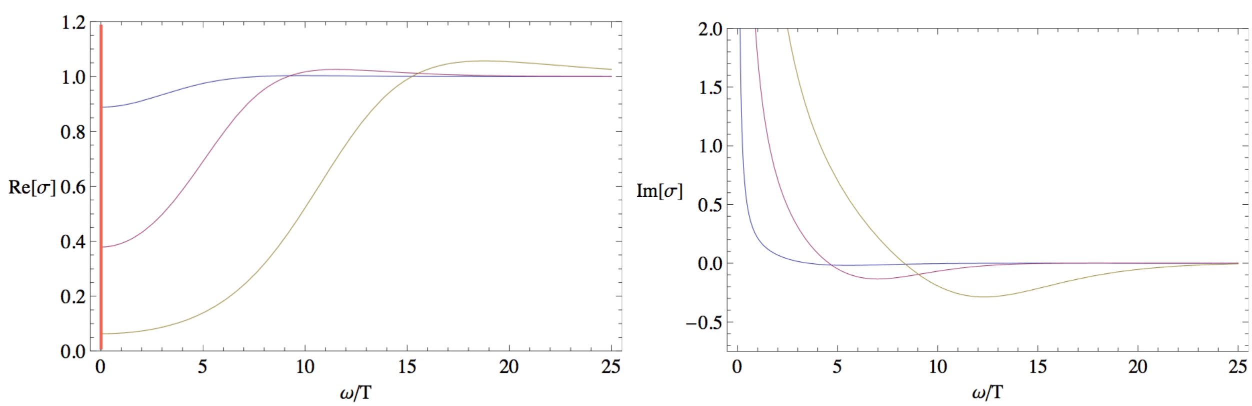


FIG. 4: The conductivity as a function of ω/T. When $\omega \to 0$, the imaginary of the optical conductivity behaves as $1/\omega$. This reflects the existence of a delta-function $\delta(\omega)$ in Reσ. In these plots, from top down, the chemical potential grows larger. These plots are taken from [1]

[8] This equals **[CHECK: There is a factor 2 difference in the Φ' term. This can be easily 'fixed' by adjusting the normalization of the stress-tensor.... It should be the factor of 2 just found in eqn (11.18)]** eqn (110) in [1] after the identification $z = \frac{\mu}{2q} r = r/r_+$, $r_+ = 2q/\mu$ and $\gamma = 2$.

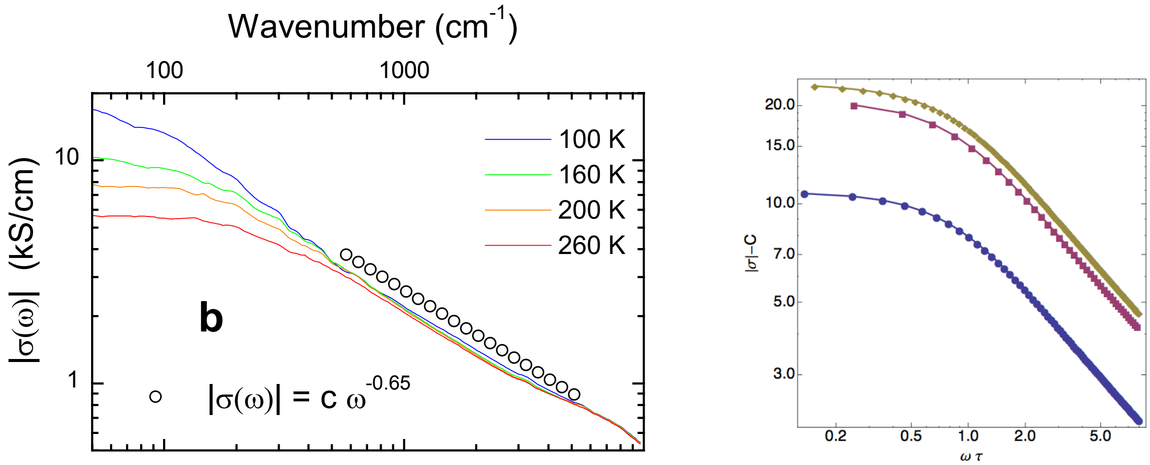


FIG. 5: Left: Observed power law $|\sigma(\omega)| = \frac{B}{\omega^{2/3}}$ in the optical conductivity measured in the strange metal phase of high Tc superconductors at optimal doping Experiments [73]. Right: Observed power law $|\sigma(\omega)| = \frac{B}{\omega^{2/3}} + C$ in the optical conductivity of a strongly coupled system in the presence of an ionic lattice computed numerically through AdS/CFT [72].

11.1. The promise of AdS/CFT

This very straightforward computation resonates in an extremely tantalizing current development. One of the mysteries of high-Tc superconductors is that the optical (= frequency dependence of the) conductivity in the strange metal phase shows an intermediate scaling behavior (Leftpanel in Fig. 5) with a power $\sigma \sim (i\omega)^{-2/3}$ [73]. The remarkable finding is that an AdS/CFT computation in the background of a "modulated" RN black hole with a chemical potential $\mu = \mu_0 + \mu_1 \cos(kx)$ to account for underlying atomic crystal lattice effects, has an almost identical $\sim (i\omega)^{-2/3}$ scaling regime [72]! This is super-tantalizing, and people are actively trying to understand if there is indeed a relation between this intricate numerical computation and experiment and what it is.

Lecture IV

12. HOLOGRAPHIC NON-FERMI LIQUIDS

We have seen that with a lattice perturbation the conductivity of the extremal AdS-RN black hole bears a remarkable resemblence to that measured in the strange metal phase of high T_c superconductors. We will now try to probe the strange metal more directly. Recall from our review of the Fermi liquid theory that a successful phenomenological model of the strange metal is the marginal Fermi liquid. This a special type of non-Fermi-liquid characterized by a self-energy $\Sigma(\omega) \sim \omega \ln \omega$ in the spectral function.

The spectral function is equivalent to the imaginary part of the retarded Green's function of a fermionic excitation. But (two-point) correlation functions is precisely what AdS/CFT is good at calculating. Let us therefore compute the spectral function of a fermionic operator in a strongly coupled CFT at finite density dual to the AdS-RN black hole.

12.1. Fermionic correlation functions from holography

The novelty compared to our previous computations is the spin 1/2 nature of a fermionic operator. Following the dictionary this is dual to a spin 1/2 field, and to ensure that it is the field whose condensed matter aspects we wish to study we make it charged. The minimal AdS action that describes the gravity dual of this system is the AdS-Einstein-Maxwell-Dirac action

$$S = \int d^{D+1}x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar{\Psi}\left[e^{\mu}_{a}\Gamma^{a}(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\Gamma^{ab} - iqA_\mu) - m\right]\Psi\right). \tag{12.1}$$

Because fermions transform non-trivially under the Lorentz group, they also couple in a special way to the spacetime background. This is encoded through both the coupling with a vielbein, the "square root" of the metric defined by $e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu}$ with η_{ab} the fixed Minkowski metric, and a spin connection $\omega_{\mu ab}$ defined through the demand that the vielbein should be covariantly constant

$$D_\mu e^a_\nu = 0 = \partial_\mu e^a_\nu - \Gamma^\sigma_{\mu\nu} e^a_\sigma + \omega^a_{\mu b} e^b_\nu. \tag{12.2}$$

This machinery ensures that the matrices Γ^a are the usual fixed Dirac-matrices obeying the anti-commutation relation $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}$.

Due to the first order nature of the Dirac action, there are some technical changes to the prescription of applying the AdS/CFT dictionary. That such changes are necessary is most readily apparent by counting components of fermions. In dimensions $d = 2n$ and $d = 2n+1$ a spinor has 2^n components. This means that if one has an even dimensional bulk with $d = 2n$ the spinor has double the components, 2^n than one would naively expect based on the dimensionality $d = 2n - 1$ of boundary. Due to the first order nature of the action this counting is wrong. The first order action simultaneously describes the fluctuation — half of the components — and its conjugate momentum — the other half. Clearly only the former should correspond to a boundary degree of freedom. The most ready way to do so is to use the extra direction to project the fermion into two distinct eigenstates $\Psi_\pm$ of Γ^r and call one the fluctuation, say Ψ_+. The other Ψ_- is then the conjugate momentum.[9] Under this projection the Dirac action reduces to

$$S = \int d^4x\sqrt{-g}\left(-\bar{\Psi}_+ \not{D}\Psi_+ - \bar{\Psi}_- \not{D}\Psi_- - m\Psi_+\Psi_- - m\bar{\Psi}_-\Psi_+\right). \tag{12.3}$$

The second issue is that the AdS/CFT correspondence instructs us to derive the CFT correlation functions from the on-shell action. The Dirac action, however, is proportional to its equation of motion. This reflects the inherent quantum nature of fermions.The action therefore appears to vanish on-shell. Recall, however, that in the case of scalars the full contribution came from a boundary term, and we did not specify this yet. In the case of the scalar the boundary term appeared naturally. Here we have to do some work to determine it. Having chosen Ψ_+ as the fundamental degree of freedom, we will choose a boundary source $\Psi^0_+ = \lim_{r\to\infty} \Psi_+(r)$. Then, because the dynamical equation is first order, the boundary value Ψ^0_- is not independent but related to that of Ψ^0_+ by the Dirac equation. We should therefore not include it as an independent degree of freedom when taking functional derivatives with respect to the source Ψ^0_+. Instead it should be varied to minimize the action. To ensure a well-defined variational system for Ψ_- we add a boundary

[9] Due to the first order nature of the action a bulk Dirac spinor in $d = 2n$ dimensions with 2^{n-1} (complex) degrees of freedom corresponds to 2^{n-1} component complex Dirac operator on the $d = 2n-1$ dimensional boundary. A bulk Dirac spinor in $d = 2n+1$ dimensions with 2^{n-1} complex degrees of freedom corresponds to a 2^{n-1} component Weyl spinor on the $d = 2n$ dimensional boundary.

action,

$$S_{\text{bdy}} = \int_{r=r_0} d^3x \sqrt{-h}\, \bar{\Psi}_+ \Psi_- \tag{12.4}$$

with $h_{\mu\nu}$ the induced metric. The variation of $\delta\Psi_-$ from the boundary action,

$$\delta S_{\text{bdy}} = \int_{r=r_0} d^3x \sqrt{-h}\, \bar{\Psi}_+ \delta\Psi_- \Big|_{\Psi_+^0 \text{ fixed}}, \tag{12.5}$$

now cancels the boundary term from variation of the bulk Dirac action

$$\begin{aligned} \delta S_{\text{bulk}} = & \int \sqrt{-g} \left(-\delta\bar{\Psi}_+ (\not{D}) \Psi_+ - \overline{((\not{D})\Psi_+)} \delta\Psi_+ - \delta\bar{\Psi}_- (\not{D}) \Psi_- - \overline{((\not{D})\Psi_-)} \delta\Psi_- \right) \\ & + \int_{z=z_0} \sqrt{-h} \left(-\bar{\Psi}_+ \delta\Psi_- - \bar{\Psi}_- \delta\Psi_+ \right) \Big|_{\Psi_+^0 \text{ fixed}}. \end{aligned} \tag{12.6}$$

The next complication is that the fermionic correlation function is in general a matrix between the various spin components. A completely covariant formulation exists [100, 123], but with some insight we can separate out each independent spin component [101]. This is very similar to a standard analysis of the Dirac equation in flat space as in any standard QFT course. Undoing the projection unto Γ^r, we first write the full covariant first order Dirac equation

$$\left(\Gamma^a e_a^\mu (\partial_\mu + \frac{1}{4} \omega_{\mu ab} \Gamma^{ab} - iqA_\mu) - m \right) \Psi = 0 \tag{12.7}$$

Note that the boundary term does not contribute. The first step is the insight that metrics whose components depend on only a single coordinate, such as AdS-black holes, have the property that by a redefinition of the Dirac fields,

$$\Psi = (-g g^{rr})^{-1/4} \psi \tag{12.8}$$

the spin connection $\omega_{\mu ab}$ can be removed. This simplifies the Dirac equation to the more standard form

$$\left(\Gamma^a e_a^\mu (\partial_\mu - iqA_\mu) - m \right) \psi = 0 \tag{12.9}$$

We shall also only be interested in background configurations with just $A_0 \equiv \Phi(r)$ non-zero and a function of the radial AdS direction r only.

In a flat spacetime we would now Fourier transform and project the 4-component Dirac spinor onto 2-component spin-eigenstates. Since the radial direction of anti-de-Sitter space breaks the four-dimensional Lorentz invariance, this cannot be done here. However, a similar projection exists onto *transverse helicities*, where the spin is always orthogonal to both the direction of the boundary momentum and the radial direction. We choose the basis for our $d = 3+1$ dimensional Dirac matrices

$$\Gamma^r = \begin{pmatrix} -\sigma_3 \mathbb{1} & 0 \\ 0 & -\sigma_3 \mathbb{1} \end{pmatrix}, \ \Gamma^t = \begin{pmatrix} i\sigma_1 \mathbb{1} & 0 \\ 0 & i\sigma_1 \mathbb{1} \end{pmatrix}, \ \Gamma^x = \begin{pmatrix} -\sigma_2 \mathbb{1} & 0 \\ 0 & \sigma_2 \mathbb{1} \end{pmatrix}, \ \Gamma^y = \begin{pmatrix} 0 & -\sigma^2 \mathbb{1} \\ -\sigma_2 \mathbb{1} & 0 \end{pmatrix} \tag{12.10}$$

and we Fourier transform only along the directions t, x, y parallel to the boundary $\psi(r,t,x,y) = \int \frac{d\omega}{2\pi} \frac{dk_x}{2\pi} \frac{dk_y}{2\pi} e^{ik_x x + ik_y y - i\omega t} \psi(r, \omega, k_x, k_y)$. Using rotational invariance we are free to choose the boundary momentum along the x-direction, $\vec{k} = (k_x, 0)$. With this choice for the momentum one can show that the operator $\Gamma^5 \Gamma^y$ commutes with the Dirac operator. We can therefore project onto its eigenstates $\chi_{1,2}$. This reduces the Dirac equation to an equation for the two-component t-helicities

$$\sqrt{g^{rr}} \, (-\sigma_3 \partial_r - \sqrt{g_{rr}} m) \, \chi_i(r; \omega, k) = -(i\sigma_2 \sqrt{g^{xx}} k_x + \sigma_1 \sqrt{-g^{tt}} (\omega + q\Phi)) \chi_i \tag{12.11}$$

Here we have use a shorthand that for a diagonal metric $g_{\mu\nu}|_{\mu \neq \nu} = 0$, the vielbein e_μ^m is literally the square root $e_\mu^m = \sqrt{g_{\mu\mu}} \delta_\mu^m$.

It suffices to consider only χ_1 from here on, as the results for χ_2 simply follow by changing $k_x \to -k_x$. [**I AM NOT SURE THIS IS CORRECT**]. To construct the CFT correlation function the essential part is the behavior of the solution near the AdS boundary. In that region $\sqrt{g^{rr}} = r/L + \ldots$, $\sqrt{g^{xx}} = \sqrt{g^{tt}} = L/r + \ldots$ thus to leading order in $r \to \infty$ the eqn (12.11) behaves as

$$\left(\partial_r + \frac{L}{r} \sigma_3 m \right) \chi_1(r; \omega, k) = 0 \tag{12.12}$$

Similar to scalar case near the AdS boundary the 2-component spinor χ_1 has the asymptotic behavior

$$\chi_1(r) = a \begin{pmatrix} 0 \\ 1 \end{pmatrix} r^{mL} + b \begin{pmatrix} 1 \\ 0 \end{pmatrix} r^{-mL} + \ldots \tag{12.13}$$

For a generic mass the first component is non-normalizable, but the second one is.

By acting with the operator $\left(\partial_r - i\sigma_2 \sqrt{g_{rr}} \sigma_1 m \right) (-i\sigma_2) \sqrt{\frac{g_{rr}}{g_{ii}}}$ on the equation for χ_1 one can obtain a second order equation for each of the individual components of χ_1, with two independent solutions. Given these homogenous solutions to the Dirac equation, the (bulk) Green's function for χ_+, still a two by two matrix, can be constructed. It equals

$$\mathcal{G}(\omega,k,r_1,r_2) = \frac{\psi_b(r)\otimes\bar{\psi}_{int}(r')\theta(r-r') - \psi_{int}(r)\otimes\bar{\psi}_b(r')\theta(r'-r)}{\frac{1}{2}\left(\bar{\psi}_{int}(r)\sigma^3\psi_b - \bar{\psi}_b(r)\sigma^3\psi_{int}\right)}. \tag{12.14}$$

Just as for the scalar case, $\psi_b(r)$ is the normalizable solution with leading coefficient $a=0$. and $\psi_{int}(r)$ is determined by the appropriate boundary conditions in the interior.

We can now use this bulk-Green's function to evaluate the on-shell Dirac action from which we can find the boundary correlation function. The on-shell Dirac action comes fully from the boundary term, we argued should be there in eqn (12.4). There we postulated that the fundamental field is $\Psi_+ \equiv \frac{1}{2}(1+\Gamma^r)\Psi$. For the t-helicity χ_1 this reduces to the projection with respect to $\frac{1}{2}(1-\sigma_3)$.

At the same time one should think of Ψ_- as dependent on the fundamental field Ψ_+. Since $\lim_{r\to\infty}\Psi_+(r)$ is the source, one can find the dependence of Ψ_- from the bulk Green's function

$$\Psi_-(r) = \lim_{r_2\to\infty}\oint\frac{1}{2}(1+\sigma^3)\mathcal{G}(r,r_2)\Psi_+(r_2) \tag{12.15}$$

Substituting this Green's function into the boundary action one obtains after projection on the independent t-helicity χ_1 **[MINUS SIGN MISSING]**

$$S = \lim_{r\to\infty}\int\sqrt{-h}\bar{\chi}_1^0\frac{1}{2}(1+\sigma_3)\frac{\psi_{int}(r)\otimes\bar{\psi}_b(\infty)}{\frac{1}{2}\left(\bar{\psi}_{int}(r)\sigma^3\psi_b - \bar{\psi}_b(r)\sigma^3\psi_{int}\right)}\chi_1^0. \tag{12.16}$$

In the t-helicity basis the boundary source (the non-normalizable coefficient) has only a lower component

$$\chi_1^0 = \begin{pmatrix} 0 \\ J \end{pmatrix}. \tag{12.17}$$

Writing also $\psi_{int}(r) = \begin{pmatrix} b_{int}r^{-m}+\ldots \\ a_{int}r^m+\ldots \end{pmatrix}$, $\psi_b(r) = \begin{pmatrix} br^{-m}+\ldots \\ 0+\ldots \end{pmatrix}$, one finds

$$S = \lim_{r\to\infty}\int\sqrt{-h}r^{-2m}\frac{(J^\dagger b_{int})(b^\dagger J)}{b^\dagger a_{int}+a_{int}^\dagger b}. \tag{12.18}$$

The final step is that an inspection of the Dirac equation reveals that *at zero density in pure AdS* one can always choose b and a_{int} real (but not b_{int}).

$$S = \lim_{r\to\infty}\int\sqrt{-h}r^{-2m}J^\dagger\frac{b_{int}}{a_{int}}J. \tag{12.19}$$

Differentiating the on-shell action w.r.t J and $J^\dagger$, and dropping both the $_{int}$ subscript and the overall r^{-2m} term gives the expression for the fermionic CFT correlation function

$$G_{\text{fermions}} = \frac{b}{a}. \tag{12.20}$$

12.1.1. On various Green's functions...

We mention now a subtlety that we brushed over earlier in the lectures. Recall that in general there are various types of Green's functions, Feynman, advanced, retarded etc. In principle which one you wish is determined by the apporpriate boundary conditions in the interior. However, note that the Green's function obtained by differentiating the boundary term in the action is always real (because the action is always real). Something appears amiss, because the advanced and retarded Green's functions are generically complex. A very careful finite temperature field theory analysis of AdS/CFT reveals that one in fact simply has to take the result (12.20) *in all cases* even when b and a are not manifestly real.

12.2. Non-Fermi liquids from holography

To study the spectral function of fermionic operators in the background of the holographic AdS-RN metal, we must therefore solve the Dirac equation (12.11) in the background of the extremal RN black hole and take the ratio of the subleading to leading fall-off near the AdS boundary.

A beautiful paper in the summer of 2009 showed how to do this semi-analytically using a standard trick from quantum mechanics: matching the asymptotic solutions near the boundary to the asymptotic solution near the horizon in an overlapping range of applicability. We already know what the solution looks like near the AdS boundary, see Eq. (12.13). Near the horizon $r - r_0 \ll 1$ the metric is given by the $\text{AdS}_2 \times R_2$ metric Eq. (10.10), and thus the Dirac equation becomes

$$\frac{\sqrt{6}}{L}(r-r_0)(-\sigma_3)\partial_r\chi_1 = -\left(i\sigma_2\frac{L}{r_0}k_x - m - \frac{L}{\sqrt{6}(r-r_0)}\sigma_1(\omega + q\frac{\mu}{r_0}(r-r_0))\right)\chi_1 \tag{12.21}$$

Changing coordinates to $\rho = \frac{\omega L}{6(r-r_0)}$ this equals

$$\rho\partial_\rho\chi_1 = -L\left(\sigma_1\frac{k_x L}{\sqrt{6}r_0} - i\sigma_2\rho - i\sigma_2\frac{q\mu L}{6r_0} - \sigma_3\frac{m}{\sqrt{6}}\right)\chi_1 \tag{12.22}$$

Acting with $\rho\partial_\rho$ on both sides and using the Dirac equation, this equals the second order equation

$$\rho^2\partial_\rho^2\chi_1 + \rho\partial_\rho\chi_1 = L^2\left(\sigma_1\frac{k_xL}{\sqrt{6}r_0} - i\sigma_2\rho - i\sigma_2\frac{q\mu L}{6r_0} - \sigma_3\frac{m}{\sqrt{6}}\right)^2\chi_1 + i\sigma_2 L\rho\chi_1 \tag{12.23}$$

$$= \left(\frac{k_x^2L^4}{6r_0^2} + \frac{m^2L^2}{6} - \frac{q^2\mu^2L^2}{36r_0^2}\right)\chi_1 + i\sigma_2L\rho\chi_1 \tag{12.24}$$

The region $\rho \gg 1$ is now the near-horizon region, whereas the region $\rho \ll 1$ is the boundary of the AdS_2 region where the near-horizon metric starts to break-down. This is where we should match to the asymptotic solution from the AdS_4 boundary. In this transition region, the boundary of AdS_2, the equation becomes diagonal

$$\rho^2\partial_\rho^2\chi_1 + \rho\partial_\rho\chi_1 = \left(\frac{k_x^2L^4}{6r_0^2} + \frac{m^2L^2}{6} - \frac{q^2\mu^2L^2}{36r_0^2}\right)\chi_1 + \dots \tag{12.25}$$

and can be solved in terms of a simple power of ρ

$$\chi \sim \alpha\rho^{-\nu_k} + \beta\rho^{\nu_k} + \dots \tag{12.26}$$

where $\nu_k = \sqrt{\frac{k_x^2L^4}{6r_0^2} + \frac{m^2L^2}{6} - \frac{q^2\mu^2L^2}{36r_0^2}}$. We are in particular interested in the ω-dependence of the solution. This can be made manifest by defining $\rho = \omega\zeta$.

$$\chi \sim \alpha\omega^{-\nu_k}\zeta^{-\nu_k} + \beta\omega^{\nu_k}\zeta^{\nu_k} + \dots \tag{12.27}$$

Note, however, that ζ is literally proportional to the natural coordinate on AdS_2. Thus following the definition of the Green's function as the ration of the subleading to the leading fall-off, the combination $\beta\omega^{\Delta_{AdS_2}}/\alpha$ is precisely the AdS_2 Green's function. Rescaling the solution, it can therefore be written in a very suggestive way as

$$\chi_1 \sim \eta_+\zeta^{-\nu_k} + \mathcal{G}_{AdS_2}(\omega)\eta_-\zeta^{\nu_k} + \dots \tag{12.28}$$

with $\eta_\pm$ a pair of ω-independent spinor s, $\mathcal{G}(\omega)_{AdS_2} \sim \omega^{2\nu_k}$.

This asymptotic expansion is valid in the regime where $r - r_0 \lesssim r_0$. The asymptotic expansion from the boundary on the other hand is valid roughly for any $r \gg \omega$. Thus there is an overlapping regime whenever $\omega < 2r_0$. For such small ω we can match at any point in the overlapping region. Generically each independent solution of the one expansion will be a linear combination of the independent solutions of the other expansion. Thus the asymptotic boundary behavior abstractly is of the form

$$a = a_+(\omega) + a_-\mathcal{G}(\omega) \tag{12.29}$$

$$b = b_+(\omega) + b_-\mathcal{G}(\omega) \tag{12.30}$$

where $a_\pm$ corresponds to the coefficient of the near-boundary solutions. This then allows us to express the (retarded) Green's function of the CFT for small ω in terms of the near boundary solutions

$$G_R(\omega) = \frac{b_+(\omega) + b_-\mathcal{G}(\omega)}{a_+(\omega) + a_-\mathcal{G}(\omega)} \tag{12.31}$$

In particular near $\omega = 0$ it will behave as

$$G_R(\omega,k) = \frac{b_+^{(0)} + \omega b_+^{(1)} + \mathcal{O}(\omega^2) + \mathcal{G}_k(\omega)\big(b_-^{(0)} + \omega b_-^{(1)} + \mathcal{O}(\omega^2)\big)}{a_+^{(0)} + \omega a_+^{(1)} + \mathcal{O}(\omega^2) + \mathcal{G}_k(\omega)\big(a_-^{(0)} + \omega a_-^{(1)} + \mathcal{O}(\omega^2)\big)}. \tag{12.32}$$

This is a remarkable result, because even though we do not know the exact expressions for $a_\pm^{(i)}(k)$ and $b_\pm^{(i)}(k)$, we can already deduce the nature of the Fermi-liquid quasiparticles, if there are any. The assumption that there are such particles is equal to the assumption that the Green's function has a pole at $\omega = 0$. This occurs if $a_+^{(0)}(k)$ vanishes for a specific k_F. Indeed near the Fermi surface, the retarded Green's function becomes of the canonical form

$$G_R(\omega,k) \simeq \frac{Z}{\omega - v_F(k-k_F) - \Sigma(k,\omega)} + \mathcal{O}(k-k_F,\omega), \quad , \tag{12.33}$$

with the self-energy $\Sigma(k,\omega) = \mathcal{G}(\omega)\left(\frac{a_-^{(}0)}{a_+^{(1)}} + \mathcal{O}(\omega)\right.$ The one aspect we do know already is the frequency dependence of the self-energy. It is $\Sigma \sim \omega^{2\nu_{k_F}}$. But this scaling is *precisely* what we wish to know, because the defining property of a Landau Fermi liquid is that $\Sigma \sim \omega^2$. Our calculation thus indeed shows that generically the fermionic excitations in the AdS-RN metal are *non-Fermi liquids.*

The precise details of the Fermi liquid — and whether there is indeed such a pole — depend sensitively on the parameters of the theory as ν_k depends directly on the charge and scaling dimension of the Fermionic operator. Explicit numerical calculation has shown that generically for large enough charge compared to the scaling dimension such a pole exists (Figs **??** and 6) with the specific properties listed in table II.

13. HOLOGRAPHIC SUPERCONDUCTORS

The CFT state dual to the AdS-RN black hole thus has remarkable properties, that for a specific choice of parameters can tantalizingly mimick the observed spectral functions in the strange metal phase of high T_c superconductors. The pure AdS-RN black hole has one

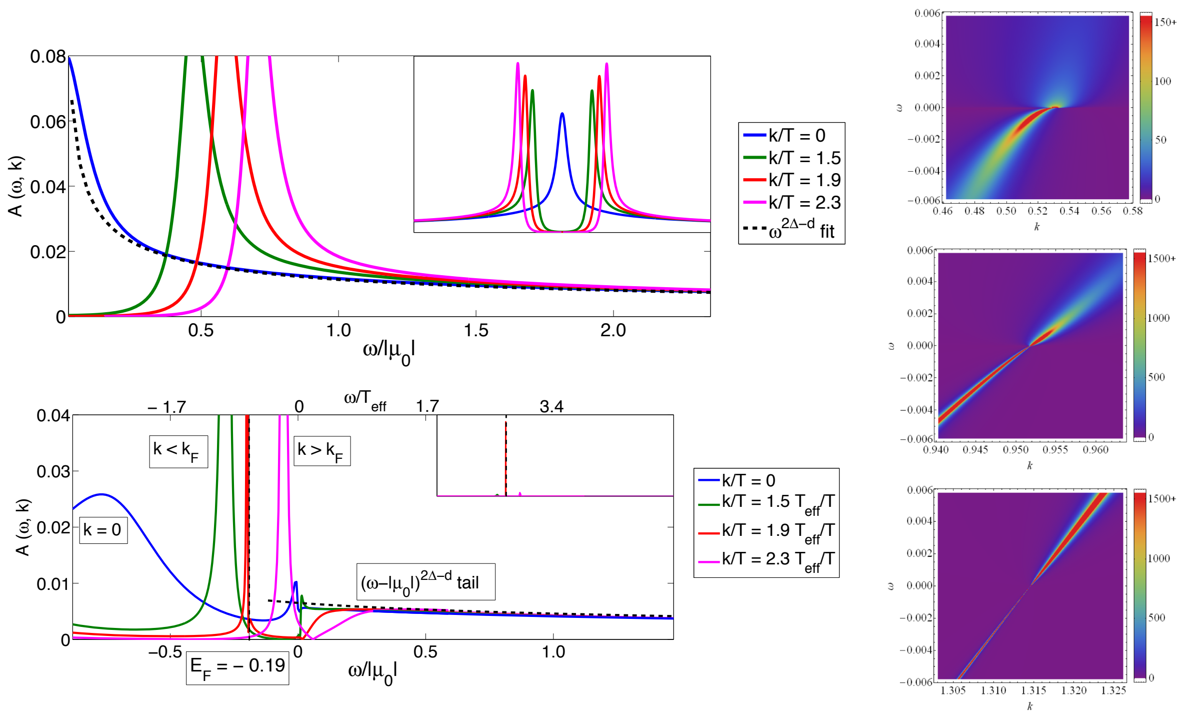


FIG. 6: **Left:** Emergence of a Fermi surface from a critical point as μ/T is increased [100] **Right:** The false color "ARPES" pictures of $T/\mu \ll 1$ holographic Fermi liquids: Top, $\nu_{k_F} < 1/2$, non-Fermi liquid; Middle, $\nu_{k_F} = 1/2$, marginal Fermi liquid; Bottom, $\nu_{k_F} > 1/2$, irregular Fermi liquid. Plots are taken from [102].

$\Sigma \sim \omega^{2\nu_{k_F}}$	**Fermi-system Phase**	**Quasiparticle Properties: (dispersion, peak)**
$2\nu_{k_F} > 1$	regular FL	$\omega_*(k) = v_F k_\perp, \quad \frac{\Gamma(k)}{\omega_*(k)} \propto k_\perp^{2\nu_{k_F}-1} \to 0, \quad Z = \text{constant}$
$2\nu_{k_F} = 1$	marginal FL	$G_R = \frac{h_1}{c_2\omega - v_F k_\perp + c_R \omega \ln \omega}, \quad Z \sim \frac{1}{\ln \omega_*} \to 0$
$2\nu_{k_F} < 1$	singular FL	$\omega_*(k) \sim k_\perp^{1/2\nu_k} \quad \frac{\Gamma(k)}{\omega_*(k)} \to \text{const}, \quad Z \propto k_\perp^{\frac{1-2\nu_{k_F}}{2\nu_{k_F}}} \to 0$

TABLE II: The zoo of non-Fermi liquids one can find in AdS/CFT. .

in-your-face problem, however: its extensive groundstate entropy. Strictly speaking the RN-black hole therefore describes the typical state amongst a set of degenerate states that *scales as the volume of the system*. This suggests that the system is in fact immensely unstable.

Let's see if we can understand these instabilities. When we considered only the Einstein-Maxwell sector in the AdS theory, we concentrated on only the energy-momentum and electromagnetic current sector of the dual CFT but ignored all other excitations in the theory. The most plausible explanation is that the instabilities are driven by these ignored excitations.[10] In the previous section we considered adding fermions to the AdS-Einstein-Maxwell theory. For the purpose of instabilities, we'll make it even simpler and consider an additional charged scalar field. The action we consider is thus

$$S = \int d^{D+1}x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (\partial^\mu + iqA^\mu)\bar{\phi}(\partial_\mu - iqA_\mu)\phi - m^2\bar{\phi}\phi\right). \quad (13.1)$$

13.1. (In)stability analysis

Let us first perform a standard stability analysis and consider the field ϕ as a small fluctuation around the extremal AdS-RN background. After Fourier transforming along the boundary directions the equation of motion for ϕ reads

$$\frac{1}{\sqrt{-g}}\partial_r\sqrt{g}g^{rr}\partial_r\phi + g^{tt}(\omega + qA_t)^2 - g^{ii}k^2 - m^2L^2\phi = 0 \quad (13.2)$$

Substituting the background for the extremal AdS_4-RN black hole, we have

$$\frac{1}{r^2}\partial_r r^4 f(r)\partial_r + \frac{1}{r^2 f(r)}(\omega + qA_t)^2 - \frac{1}{r^2}(k^2) - m^2L^2\phi = 0 \quad (13.3)$$

Since we are considering ϕ as a fluctuation, and since any instability should require very little energy, i.e. we can concentrate on small ω fluctuations, we can perform the same matching analysis as for the fermions to study the solutions to the equation of motion.

Near the AdS-boundary we had $\phi = Ar^{\Delta-d} + Br^{-\Delta} + \ldots$ with

$$\Delta = \frac{3}{2} + \sqrt{\frac{9}{4} + m^2L^2} \quad (13.4)$$

Now note that the scaling dimension stays real for all values $m^2L^2 > -9/4$. It appears the equation makes sense for negative m^2. This sounds strange, because normally we associated a negative m^2 of a scalar fluctuation as a tachyonic instability where the scalar sits at the top of a local maximum of the potential. What happens is that in AdS is that there is an

[10] The precise way of saying this is that the Einstein-Maxwell theory is not a complete quantum theory of gravity. Other fields are necessary to have a completely consistent theory.

additional gravitational contribution to the potential which modifies the tachyonic instability precisely to the location where Δ turns imaginary. This is easily seen by taking the flat space limit $L \gg 1$ which connects the stability boundary $\Delta = 0$ directly to $m^2 = 0$. So in pure AdS we may consider slightly negative scalar m^2 and still have a stable system.

In the AdS2-region (10.12) on the other hand the equation becomes

$$\zeta^2 \partial_\zeta^2 \phi + \zeta^2 (\omega + \frac{q}{6\zeta})^2 \phi - \frac{k^2}{r_0^2} - m^2 L_2^2 \phi = 0 \qquad (13.5)$$

Its asymptotic behavior near the boundary is

$$\phi \sim \alpha \zeta^{1-\Delta_{AdS_2}} + \beta \zeta^{\Delta_{AdS_2}} + \dots \qquad (13.6)$$

where $\Delta_{AdS_2} = \frac{1}{2} + \nu_k$ with $\nu_k = \sqrt{\frac{k^2}{r_0}^2 + m^2 L_2^2 - \frac{q^2}{6}}$. From these we read off that near the horizon in the AdS_2 region the scalar is unstable when $m^2 L_2^2 = m^2 L^2/6 > -\frac{1}{4}$. We have indeed found an instability. A zero-momentum scalar mode therefore has a mass-window

$$-\frac{6}{4} \gg m^2 L^2 \gg -\frac{9}{4}. \qquad (13.7)$$

where the system has a tachyonic instability near the horizon, but well-defined as a theory. The next question is what the new grounstate is that arises from this instability.

13.2. The holographic superconductor solution

Before we construct this new groundstate, let us discuss its physics. The resulting solution will be a gravitational system with a non-zero electric field partially if not fully sourced by a non-vanishing scalar profile. Since this profile must extend to the AdS boundary, there is a directly translatable consequence in the CFT. The groundstate has a nonvanishing charged expectation value for the scalar field, i.e. the $U(1)$ symmetry is spontaneously broken. A more precise way of saying it is that we have Higgsed the dynamical gauge field in the bulk and this corresponds to global symmetry breaking on the dual boundary. It is in this Landau-Ginzburg sense (where the $U(1)$ of electromagnetism is considered a global symmetry) that the new groundstate is a superconductor.

The full holographic superconductor solution involves solving the coupled AdS-Einstein-

Maxwell-Scalar system

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R-2\Lambda) = \kappa(T^{EM}_{\mu\nu} + T^{scalar}_{\mu\nu}) \tag{13.8}$$

$$D_\mu F^{\mu\nu} = iq(\bar{\phi}\partial^\mu\phi - \phi\partial^\mu\bar{\phi}) + 2q^2 A^\mu \bar{\phi}\phi \tag{13.9}$$

$$\frac{1}{\sqrt{g}}(\partial_\mu - iqA_\mu)g^{\mu\nu}\sqrt{-g}(\partial_\nu - iqA_\nu)\phi - m^2\phi = 0 \tag{13.10}$$

One can solve the whole system, see e.g. [78, 129, 130], but the essential physics is already captured in a simple limit. The dynamics of interest is in the $U(1)$ sector and to emphasize this we can take the limit $q \ll \kappa/L^2$. Since we want to keep the dynamics of the scalar field, this means we should do so after rescaling $A_\mu \to \hat{A}_\mu/q$. We also wish to keep the $A^\mu\bar{\phi}\phi$ interaction, so we also first rescale $\phi \to \hat{\phi}/q$. After this rescalings the stress-energy tensor becomes $T_{\mu\nu} \to \frac{1}{q}\hat{T}_{\mu\nu}$, so in the limit $q \to \infty$, the dynamics decouples from the gravity sector completely. One has the effective equations of motion (dropping the hatted notation)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R-2\Lambda) = 0 \tag{13.11}$$

$$D_\mu F^{\mu\nu} = 2A^\mu\bar{\phi}\phi \tag{13.12}$$

$$\frac{1}{\sqrt{-g}}(\partial_\mu - iA_\mu)g^{\mu\nu}\sqrt{-g}(\partial_\nu - iA_\nu)\phi - m^2\phi = 0 \tag{13.13}$$

The solution to the vacuum AdS-Einstein equation is just the AdS-Schwarzschild black hole 6.2. We now make the ansatz that the solution will maintain the translational and rotational invariance of the black hole. We posit

$$\phi = \phi(r)\ , \quad A_0 = \Phi(r)\ , \quad A_{\mu\neq 0} = 0 \tag{13.14}$$

Moreover we can take $\phi(r)$ to be purely real without any loss. The Maxwell-scalar sector then becomes

$$\frac{1}{r^2}\partial_r r^2 \partial_r \Phi = \frac{2}{f(r)r^2}\Phi\phi^2 \tag{13.15}$$

$$\frac{1}{r^2}\partial_r f(r) r^4 \partial_r \phi - m^2\phi = -\frac{1}{f(r)r^2}\Phi^2\phi \tag{13.16}$$

The groundstate we seek is then the solution to these equations subject to the appropriate boundary conditions. We wish to study the system at a finite chemical potential. Thus near the AdS boundary $\Phi(r) = \mu + \ldots$. Moreover, we do not want an explicit source for the scalar field, thus it must fall-off near the boundary. With these two conditions, it is easy to see that the near-AdS boundary behavior is exactly the same as for the fluctuations, with the condition that we only consider the normalizable mode for ϕ.

$$\Phi = \mu - \frac{\rho}{2r} + \dots \tag{13.17}$$

$$\phi = \mathcal{O} r^{\Delta} + \dots \tag{13.18}$$

We can therefore immediately interpret $\mathcal{O}$ as the gravitational encoding of the expectation value of the CFT scalar operator dual to ϕ. It will be the order parameter in the dual theory.

Now μ is fixed, but the charge density ρ and the order parameter vev $\mathcal{O}$ are unknown. The system is still underdetermined, since we need two more boundary conditions. These are determined at the horizon of the black-hole. This is the location where the redshift factor $f(r)$ vanishes. Consistency of the Maxwell equation (13.17) then indicates that either Φ or ϕ has to vanish at the horizon as well. If one chooses the latter, the only reasonable solution will be $\phi = 0$ everywhere. Therefore one boundary condition is that $\Phi(r_h) = 0$. Substituting this in the scalar field equation one finds that $\phi(r)$ at the horizon must satisfy $\partial_r \phi = m^2 \phi / (f'(r_h) + 4/r_h)$.

We now have a uniquely determined system and it remains to find the exact solution. In practice the only way to do so is numerically. The subtle part is that one has to impose boundary conditions on both boundaries. There are several numerical recipes to do so. The most common is a shooting method, where one fixes two of the boundary conditions at the horizon and varies the remaining two until one finds a solution that obeys the asymptotically AdS-boundary conditions. With modern computers this can be done quite fast. One finds that for high T/μ or high m/q there is no non-trivial solution. But for a $q/m \gtrsim 1$ there is a critical temperature/chemical potential ratio, for which a non-trivial solution exists; see Fig. 7.

13.2.1. The macroscopic properties of the holographic superconductor

The characteristic feature of a superconductor is of course the vanishing of the resistance. Using the tools we developed earlier in computing conductivities with AdS/CFT we can also compute the conductivity of the holographic superconductor. The results, compared to the RN black hole, are given in the right panel in Fig. 7. One notes in fact that for $T < T_c$ there is a remnant conductivity. However as this keeps decreasing as T is lowered one sees

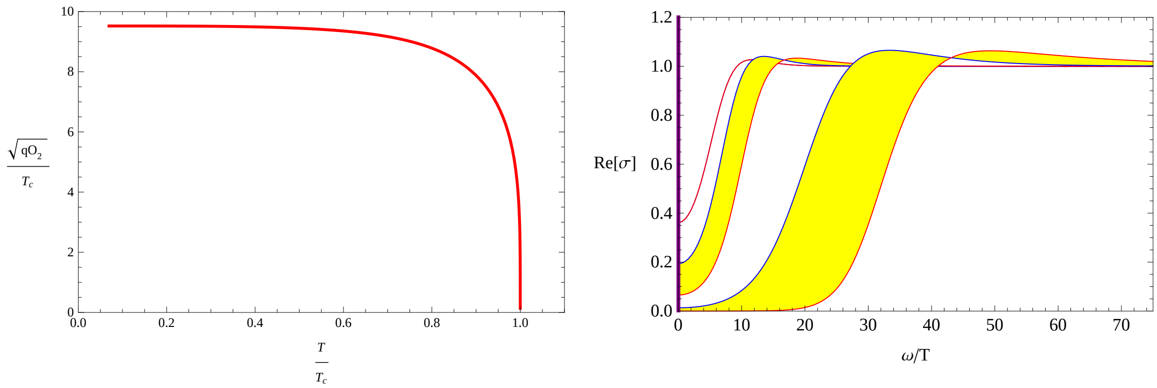


FIG. 7: **Left:** The condensate $\mathcal{O}_2$ as a function of temperature for $q = 3, m^2 = -2$. Near the critical temperature, $\mathcal{O} \propto (T_c - T)^{1/2}$. **[FIGURE SOURCE?] Right:** Electrical conductivity for holographic superconductors of $\mathcal{O}_2$ with $q = 3, m^2 = -2$ (red line) and RN black holes (blue line): $T/T_c = 1, 0.775, 0.274$ from top to bottom.The curves at $T = T_c$ coincide. There is a delta function at the origin in all cases. **FIG. Source[IS THIS AN ORIGINAL FIGURE]**

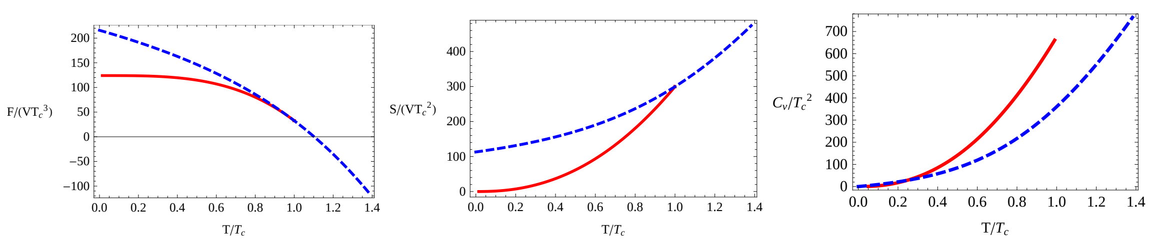


FIG. 8: The free enengy, entropy and specific heat for the RN black hole (blue dashed line) and hairy black hole (red solid line) of $\mathcal{O}_2$ as a function of temperature for $q = 3, m^2 = -2$. It is easy to see that below T_c the hairy black hole is a more stable state. **FIGURE SOURCE**

the resistivity shrinking. For any finite T there is always a tiny resistivity left.

Similarly we can compute other macroscopic properties of the holographic superconductor. They are shown in Fig. 8.

Even though the qualitative physics of the holographic superconductor should simply be seen as standard spontaneous symmetry in the language of the dual gravitational theory, we should remark that *quantitatively* the holographic superconductor differs from spontaneous symmetry breaking in weakly coupled non-critical theories. Specifically in weakly coupled quantum field theories the canonical scaling dimension Δ of a scalar field is always infinitesimally close to the free scaling dimension $\Delta_{free} = (d-2)/2$. The scaling dimension of the holographic superconductor, however, is a completely free parameter encoded in its AdS mass $m^2L^2 = \Delta(\Delta - d)$. One consequence is that the dynamical onset of superconductivity measured through its susceptibility differs [94].

Let us end with a brief observation about the revolutionary nature of the holographic superconductor. Conventionally the viewpoint about black holes is that they are the most stable objects around. Matter can only fall in, and classically never come out. The intuition is that Because gravity is attractive, once you have a black hole, it will always be there. Quantum mechnically the black hole will radiate, but this is not a strong instability of the system. The astonishing aspect of the holographic superconductor solution is that it shows that for AdS-black holes this wisdom does not hold. In AdS space black holes *can* be unstable in the classical sense. The holographic superconductor is simply the first such solution. By now we have legions of examples. There are p-wave and d-wave holographic superconductors, where spin-1 and spin-2 fields condense; there are striped superconductors, where the groundstate is not translationally invariant, etc.

14. EXERCISES

In the final exercise of this session we will discuss what happens if the instability is driven by spin 1/2 fermions.

Exercise 1: Construct the electron star background and compute its macroscopics.

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