Fractional Level WZW Models as Logarithmic CFTs

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A Brief History of CFT

- Conformal field theory (CFT) is one of the success stories of modern physics, finding application in both statistical mechanics and string theory.
- It is founded on the work of Belavin, Polyakov and Zamolodchikov on the minimal models [NPB 241 (1984)], Knizhnik and Zamolodchikov on current algebras [NPB 247 (1984)] and Witten on 2D bosonisation [CMP 92 (1984)].
- This led to the Wess-Zumino-Witten (WZW) models as archetypal examples of CFTs. These describe strings propagating on a (compact? simply-connected?) connected Lie group G.
- Much of their study reduces to studying the representation theory of their chiral algebra, the corresponding untwisted affine Kac-Moody algebra $\hat{\mathfrak{g}}$.

Fractional Level WZW Models

- One success was to use the unitarity of the WZW models to prove the unitarity of certain minimal models.
- This used the coset construction of Goddard, Kent and Olive [PLB 152 (1985)] to construct these unitary minimal models as cosets of WZW models.
- Standard WZW models are parametrised by a non-negative integer k, the level. For other k, the action does not define a consistent quantum field theory.
- But, the coset construction would give the remaining (non-unitary) minimal models if we were allowed to use certain fractional values for k.
- Might there exist consistent "fractional level WZW models" which need not correspond to strings on a group?

Example: The $\widehat{\mathfrak{sl}}(2)_k$ WZW Model

- This model describes strings on $\mathsf{SU}(2)$, has $\widehat{\mathfrak{sl}}(2)$ for a chiral algebra, and its space of states is

$$\mathscr{H} = \bigoplus_{\lambda=0}^{k} \widehat{\mathscr{L}_\lambda} \otimes \widehat{\mathscr{L}_\lambda} \qquad (k \in \mathbb{N}),$$

where $\widehat{\mathscr{L}_\lambda}$ is the irreducible $\widehat{\mathfrak{sl}}(2)$-module generated by a highest weight state of $\mathfrak{sl}(2)$-weight λ.

- The irreps $\widehat{\mathscr{L}_\lambda}$, $\lambda = 0, 1, \ldots, k$
  1. are integrable and unitary,
  2. carry a representation of the modular group $\mathsf{SL}(2;\mathbb{Z})$,
  3. are closed under fusion:

$$\widehat{\mathscr{L}_\lambda} \times \widehat{\mathscr{L}_\mu} = \widehat{\mathscr{L}}_{|\lambda-\mu|} \oplus \widehat{\mathscr{L}}_{|\lambda-\mu|+2} \oplus \cdots \oplus \widehat{\mathscr{L}}_{\min\{\lambda+\mu, 2k-\lambda-\mu\}}.$$

- Moreover, fusion and the modular properties are related by the Verlinde formula.

- We'd like similar properties to hold for the (posited) fractional level WZW models.
- Kac and Wakimoto discovered [Adv. Math. 70 (1988)] that at the required fractional levels k, there are a finite number of admissible irreps whose characters carry a rep of $\mathrm{SL}(2;\mathbb{Z})$.
- Led to many attempts to "construct" fractional level models from these irreps [Koh-Sorba, Bernard-Felder, Mathieu-Walton, Awata-Yamada, Ramgoolam, Feigin-Malikov, Andreev, ...].
- There were a few problems:
  1. The Verlinde formula gave negative fusion coefficients.
  2. The admissible irreps did not close under conjugation.
  3. Other methods of computing fusion rules gave different fusion coefficients (with their own problems).
- Many "solutions" proclaimed — none universally agreed upon. CFT textbooks regarded fractional level theories as "intrinsically sick".

Logarithmic CFT to the Rescue!

- Gaberdiel [NPB 618 (2001)] reanalysed the fusion rules at fractional level. Found that the problem was the assumption that fusion closes on admissible reps.
- At $k = -\frac{4}{3}$, fusion of admissibles generates an infinite number of distinct irreducibles. It also generates indecomposables, implying a logarithmic CFT.
- Lesage, Mathieu, Rasmussen and Saleur [NPB 647 (2002)] later showed that for $k = -\frac{1}{2}$, fusion also generates an infinite number of distinct irreducibles, but no indecomposables in this case.
- However, they did propose a “logarithmic lift” in which indecomposables contribute.
- Partial resolution to the fractional level puzzle, but modular properties still unexplained.

Motivation (Why do we care?)

- WZW models are supposed to be fundamental building blocks for rational unitary CFTs.
- Fractional level WZW models were supposed to be fundamental building blocks for rational non-unitary CFTs.
- Perhaps they are actually fundamental building blocks for quasi-rational non-unitary CFTs, logarithmic ones included.
- Logarithmic CFTs describe the continuum limit of non-local observables in statistical models, SLE processes and AdS/CFT-duals to topological gravity models.
- WZW models on supergroups are unlikely to behave like integer level WZW models in general — fractional level models may be expected to capture more of their features.
- Non-compact WZW studies should benefit from fractional level results, *e.g.* that indecomposables are difficult to avoid in general.

$\mathfrak{sl}(2)$ and its Representations

This is the Lie algebra of traceless 2×2 matrices. A convenient basis is $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, so

$$\left[h,e\right] = 2e, \qquad \left[e,f\right] = h, \qquad \left[h,f\right] = -2f.$$

The eigenvalue of h acting on a state is the state's weight.

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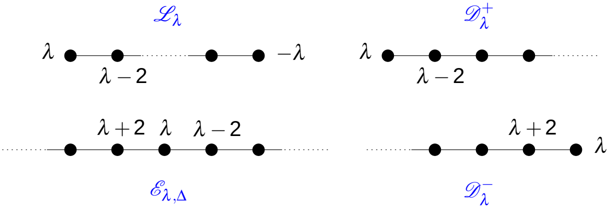
The (weight) representations fall into four classes: Those with a highest weight state ($e\left|v\right\rangle = 0$), those with a lowest weight state ($f\left|w\right\rangle = 0$), those with both and those with neither.



The Affine Kac-Moody Algebra $\widehat{\mathfrak{sl}}(2)$

This is the Lie algebra $\widehat{\mathfrak{sl}}(2) = \mathfrak{sl}(2) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$, where K is central and

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m\kappa(x, y)\,\delta_{m+n=0} K\,.$$

Here, $\kappa(x, y) = \mathrm{tr}(xy)$ is the Killing form of $\mathfrak{sl}(2)$. The eigenvalue of K on a cyclic representation is its level k. We always write x_n instead of $x \otimes t^n$.

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This is usually supplemented with the element

$$L_0 = \frac{1}{2(k+2)} \sum_{r \in \mathbb{Z}} : \frac{1}{2} h_r h_{-r} + e_r f_{-r} + f_r e_{-r} :$$

of the universal enveloping algebra. We have

$$\left[L_0, x_n\right] = -n x_n.$$

Representations of $\widehat{\mathfrak{sl}}(2)$

The affine weight of a state in a representation of $\widehat{\mathfrak{sl}}(2)$ is the triple (λ, k, Δ) giving its eigenvalues under h_0, K and L_0. Δ is the state's conformal dimension.

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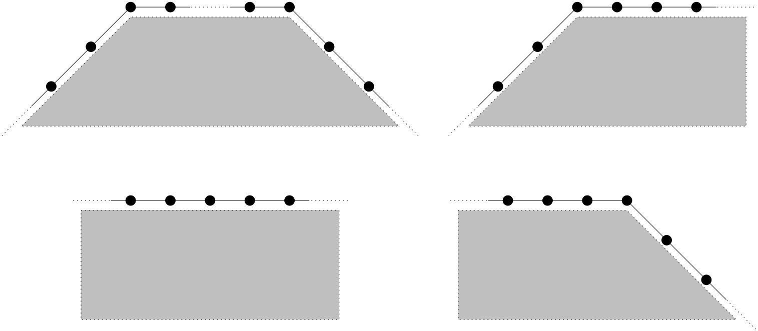
Useful $\widehat{\mathfrak{sl}}(2)$-reps for CFT are obtained from $\mathfrak{sl}(2)$-reps via the induced module construction:



Automorphisms of $\widehat{\mathfrak{sl}}(2)$

The only automorphism of $\mathfrak{sl}(2)$ which preserves the Cartan subalgebra $\mathbb{C}h$ is the Weyl reflection w:

$$\mathsf{w}(e) = f, \qquad \mathsf{w}(h) = -h, \qquad \mathsf{w}(f) = e.$$

This lifts to conjugation on $\widehat{\mathfrak{sl}}(2)$ as follows:

$$\mathsf{w}(e_n) = f_n, \quad \mathsf{w}(h_n) = -h_n, \quad \mathsf{w}(f_n) = e_n, \quad \mathsf{w}(K) = K.$$

The automorphisms of $\widehat{\mathfrak{sl}}(2)$ which preserve $\mathbb{C}h_0 \oplus \mathbb{C}K \oplus \mathbb{C}L_0$ are generated by w and the spectral flow γ:

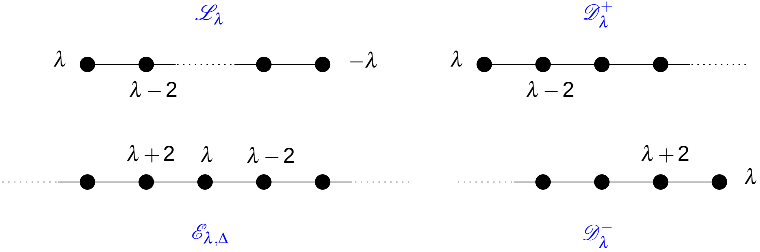
$$\gamma(e_n) = e_{n-1}, \quad \gamma(h_n) = h_n + \frac{1}{2}\delta_{n,0}, \quad \gamma(f_n) = f_{n+1}, \quad \gamma(K) = K.$$

Note that $\mathsf{w}(L_0) = L_0$, but $\gamma(L_0) = L_0 - \frac{1}{2}h_0 + \frac{1}{4}K$.

Twisted Representations

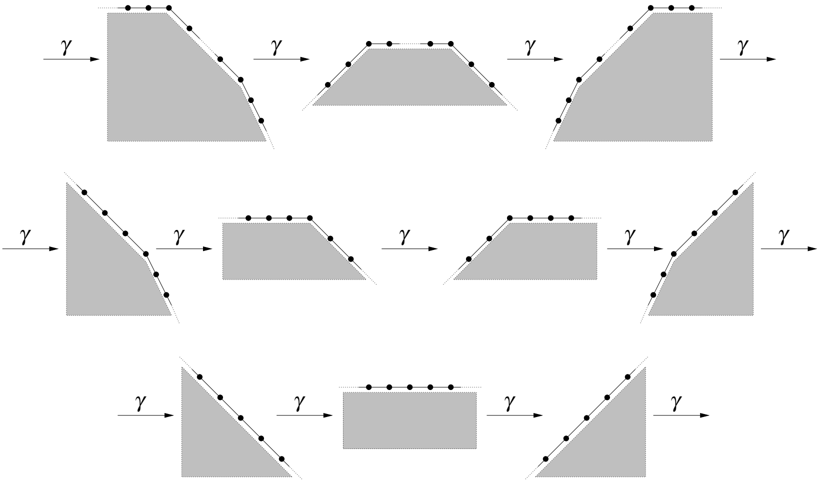
Twisting a representation by w amounts to taking the conjugate representation. For $\mathfrak{sl}(2)$, this gives

$$\mathscr{L}_\lambda \longleftrightarrow \mathscr{L}_\lambda, \qquad \mathscr{E}_{\lambda,\Delta} \longleftrightarrow \mathscr{E}_{-\lambda,\Delta}, \qquad \mathscr{D}^+_\lambda \longleftrightarrow \mathscr{D}^-_{-\lambda}.$$



The induced $\widehat{\mathfrak{sl}}(2)$-modules behave identically.

Twisting our induced $\widehat{\mathfrak{sl}}(2)$-modules by γ is far less trivial!



We get infinitely many distinct representations.

Example: Integrable $\widehat{\mathfrak{sl}}(2)$-Modules

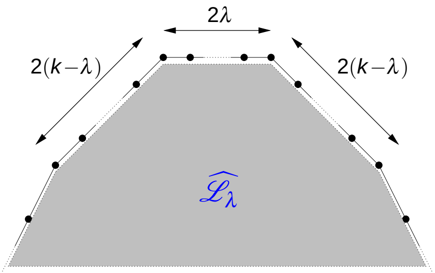
Recall that for $k \in \mathbb{N}$, the $\widehat{\mathfrak{sl}}(2)_k$ WZW model is constructed from the integrable modules $\widehat{\mathscr{L}}_0, \widehat{\mathscr{L}}_1, \ldots, \widehat{\mathscr{L}}_k$. These are the irreducible quotients of the modules induced from the $\mathfrak{sl}(2)$-modules $\mathscr{L}_0, \mathscr{L}_1, \ldots, \mathscr{L}_k$.

Amazingly,

$$\widehat{\mathscr{L}}_\lambda \longleftrightarrow \widehat{\mathscr{L}}_{k-\lambda}$$

under γ.

But this is far from typical...



These irreducibles are also self-conjugate.

Constructions at $k = -\frac{1}{2}$ $(c = -1)$

This level is interesting because the $\beta\gamma$ ghost system has $\widehat{\mathfrak{sl}}(2)_{-1/2}$ symmetry. Let $\widehat{\mathscr{L}_\lambda}$, $\widehat{\mathscr{D}}_\lambda^+$, $\widehat{\mathscr{D}}_\lambda^-$ and $\widehat{\mathscr{E}}_{\lambda,\Delta}$ denote the irreducible quotients induced from $\mathscr{L}_\lambda$, $\mathscr{D}_\lambda^+$, $\mathscr{D}_\lambda^-$ and $\mathscr{E}_{\lambda,\Delta}$.
$\widehat{\mathscr{L}_0}$ is the vacuum module. Its irreducibility means that

$$\left(156e_{-3}e_{-1}-71e^2_{-2}+44e_{-2}h_{-1}e_{-1}-52h_{-2}e^2_{-1}+16f_{-1}e^3_{-1}-4h^2_{-1}e^2_{-1}\right)\left|0\right\rangle=0.$$

Using the state-field correspondence (or Zhu's algebra), this restricts the "allowed modules" to the irreducibles

$$\widehat{\mathscr{L}_0}, \quad \widehat{\mathscr{L}_1}, \quad \widehat{\mathscr{D}}^+_{-1/2}, \quad \widehat{\mathscr{D}}^+_{-3/2}, \quad \widehat{\mathscr{D}}^-_{1/2}, \quad \widehat{\mathscr{D}}^-_{3/2}, \quad \widehat{\mathscr{E}}_{\lambda,-1/8}.$$

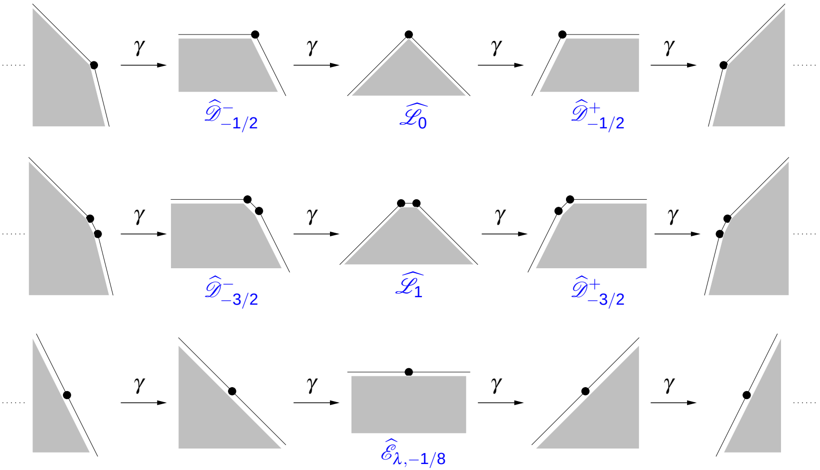
For the $\widehat{\mathscr{E}}_{\lambda,-1/8}$, any λ is allowed. However, $\lambda = \frac{1}{2}, \frac{3}{2}$ do not give irreducibles. Rather, one gets four allowed indecomposables corresponding to the four ways of coupling $\widehat{\mathscr{D}}^\pm_{\mp 1/2}$ with $\widehat{\mathscr{D}}^\pm_{\mp 3/2}$.

The conformal dimensions of the zero-grade states of $\widehat{\mathscr{L}_0}$ and $\widehat{\mathscr{L}_1}$ are 0 and $\frac{1}{2}$. For the other modules, such states have conformal dimension $-\frac{1}{8}$. We remark that:

- The allowed highest weight modules, $\widehat{\mathscr{L}_0}$, $\widehat{\mathscr{L}_1}$, $\widehat{\mathscr{D}}^+_{-1/2}$ and $\widehat{\mathscr{D}}^+_{-3/2}$, are precisely the admissible modules of Kac and Wakimoto when $k = -\frac{1}{2}$.
- The set of allowed modules is closed under conjugation.
- The set of allowed modules does not close under spectral flow! But,

$$\widehat{\mathscr{D}}^-_{1/2} \xrightarrow{\gamma} \widehat{\mathscr{L}_0} \xrightarrow{\gamma} \widehat{\mathscr{D}}^+_{-1/2} \quad \text{and} \quad \widehat{\mathscr{D}}^-_{3/2} \xrightarrow{\gamma} \widehat{\mathscr{L}_1} \xrightarrow{\gamma} \widehat{\mathscr{D}}^+_{-3/2},$$

  suggesting that the other spectral flow images should also be allowed modules.



A schematic illustration of the “allowed modules” in a $k = -\frac{1}{2}$ fractional level WZW model showing the induced action of the spectral flow automorphism γ.

A Minimal Theory

We can try to construct a minimal CFT generated by the admissible representations of Kac and Wakimoto. Requiring closure under conjugation gives all the "allowed modules" except the $\widehat{\mathscr{E}}_{\lambda,-1/8}$.
Any CFT spectrum must closed under the fusion operation $\times$.
We compute (carefully) that

$$\widehat{\mathscr{L}_0} \times \widehat{\mathscr{L}_0} = \widehat{\mathscr{L}_0}, \qquad \widehat{\mathscr{L}_0} \times \widehat{\mathscr{L}_1} = \widehat{\mathscr{L}_1}, \qquad \widehat{\mathscr{L}_1} \times \widehat{\mathscr{L}_1} = \widehat{\mathscr{L}_0}.$$

This gives all fusion rules (if spectral flow behaves itself), *eg.*

$$\widehat{\mathscr{D}}^+_{-3/2} \times \widehat{\mathscr{D}}^-_{3/2} = \gamma\big(\widehat{\mathscr{L}_1}\big) \times \gamma^{-1}\big(\widehat{\mathscr{L}_1}\big) = \widehat{\mathscr{L}_1} \times \widehat{\mathscr{L}_1} = \widehat{\mathscr{L}_0},$$
$$\widehat{\mathscr{D}}^+_{-1/2} \times \widehat{\mathscr{D}}^+_{-1/2} = \gamma\big(\widehat{\mathscr{L}_0}\big) \times \gamma\big(\widehat{\mathscr{L}_0}\big) = \gamma^2\big(\widehat{\mathscr{L}_0} \times \widehat{\mathscr{L}_0}\big) = \gamma^2\big(\widehat{\mathscr{L}_0}\big).$$

Closure under fusion therefore requires that all spectral flow images contribute to the theory.

Modular Properties

The minimal spectrum generated by the admissible modules under fusion is then the set of spectral flow images of $\widehat{\mathscr{L}}_0$ and $\widehat{\mathscr{L}}_1$. We want the admissibles for their modular properties. Their characters may be expressed in terms of Jacobi theta functions and Dedekind's eta function:

$$\chi_{\widehat{\mathscr{L}}_0} = \frac{1}{2}\left[\frac{\eta(q)}{\vartheta_4(z;q)} + \frac{\eta(q)}{\vartheta_3(z;q)}\right] \qquad \chi_{\widehat{\mathscr{L}}_1} = \frac{1}{2}\left[\frac{\eta(q)}{\vartheta_4(z;q)} - \frac{\eta(q)}{\vartheta_3(z;q)}\right]$$

$$\chi_{\widehat{\mathscr{D}}^+_{-1/2}} = \frac{1}{2}\left[\frac{-\mathrm{i}\eta(q)}{\vartheta_1(z;q)} + \frac{\eta(q)}{\vartheta_2(z;q)}\right] \quad \chi_{\widehat{\mathscr{D}}^+_{-3/2}} = \frac{1}{2}\left[\frac{-\mathrm{i}\eta(q)}{\vartheta_1(z;q)} - \frac{\eta(q)}{\vartheta_2(z;q)}\right].$$

These characters form a (reducible) rep of $\mathrm{SL}(2;\mathbb{Z})$:

$$S = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & \mathrm{i} & \mathrm{i} \\ -1 & -1 & \mathrm{i} & \mathrm{i} \end{pmatrix} \qquad T = \begin{pmatrix} e^{\mathrm{i}\pi/12} & 0 & 0 & 0 \\ 0 & -e^{\mathrm{i}\pi/12} & 0 & 0 \\ 0 & 0 & e^{-\mathrm{i}\pi/6} & 0 \\ 0 & 0 & 0 & e^{-\mathrm{i}\pi/6} \end{pmatrix}.$$

What about the spectral flow images?

It turns out that we have a periodicity of the form

$$\cdots \xrightarrow{\gamma} -\chi_{\widehat{\mathscr{L}}_1} \xrightarrow{\gamma} -\chi_{\widehat{\mathscr{D}}^+_{-3/2}} \xrightarrow{\gamma} \chi_{\widehat{\mathscr{L}}_0} \xrightarrow{\gamma} \chi_{\widehat{\mathscr{D}}^+_{-1/2}} \xrightarrow{\gamma} -\chi_{\widehat{\mathscr{L}}_1} \xrightarrow{\gamma} \cdots$$

$$\cdots \xrightarrow{\gamma} -\chi_{\widehat{\mathscr{L}}_0} \xrightarrow{\gamma} -\chi_{\widehat{\mathscr{D}}^+_{-1/2}} \xrightarrow{\gamma} \chi_{\widehat{\mathscr{L}}_1} \xrightarrow{\gamma} \chi_{\widehat{\mathscr{D}}^+_{-3/2}} \xrightarrow{\gamma} -\chi_{\widehat{\mathscr{L}}_0} \xrightarrow{\gamma} \cdots$$

at the level of modular functions. There are only four linearly independent characters! As power series,

$$\chi_{\gamma^\ell(\widehat{\mathscr{L}}_\lambda)}(z;q) = \operatorname{tr}_{\gamma^\ell(\widehat{\mathscr{L}}_\lambda)} z^{h_0} q^{L_0+1/24}$$

converges for $|q|<1$ and $|q|^{(-\ell+1)/2} < |z| < |q|^{(-\ell-1)/2}$.
Equating the character of $\widehat{\mathscr{D}}^+_{-3/2} = \gamma(\widehat{\mathscr{L}_1})$ with minus that of $\widehat{\mathscr{D}}^-_{1/2} = \gamma^{-1}(\widehat{\mathscr{L}_0})$ is analogous to

$$\sum_{n=0}^{\infty} z^{\lambda-2n} = \frac{z^\lambda}{1-z^{-2}} = -\frac{z^{\lambda+2}}{1-z^2} = -\sum_{n=1}^{\infty} z^{\lambda+2n}.$$

Formally, the map from the modules to the characters (as meromorphic theta functions) is not 1–1:

Fusion Ring $\xrightarrow{\text{projection}}$ Character Ring .

Its kernel is spanned by the modules $\gamma^{\ell\pm1}\big(\widehat{\mathscr{L}_0}\big) \oplus \gamma^{\ell\mp1}\big(\widehat{\mathscr{L}_1}\big)$ and these form an ideal in the fusion ring. Fusion then descends to the character ring.

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Recall fusion and modular *S*-matrix should be related by the Verlinde formula (but negative coefficients!).

Resolution: The modular properties determine only the character ring. *eg.* S^2 is conjugation:

$$S^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \Leftrightarrow \quad \begin{cases} \chi_{\mathsf{w}\left(\widehat{\mathscr{L}}_0\right)} = \chi_{\widehat{\mathscr{L}}_0} \quad \chi_{\mathsf{w}\left(\widehat{\mathscr{L}}_1\right)} = \chi_{\widehat{\mathscr{L}}_1} \\ \chi_{\mathsf{w}\left(\widehat{\mathscr{D}}^+_{-1/2}\right)} = \chi_{\widehat{\mathscr{D}}^-_{1/2}} = -\chi_{\widehat{\mathscr{D}}^+_{-3/2}} \\ \chi_{\mathsf{w}\left(\widehat{\mathscr{D}}^+_{-3/2}\right)} = \chi_{\widehat{\mathscr{D}}^-_{3/2}} = -\chi_{\widehat{\mathscr{D}}^+_{-1/2}}. \end{cases}$$

Augmenting the Theory

- It seems that we have a good spectrum. It is modular invariant and closed under fusion, strong evidence that one can construct a consistent CFT. But we lost the $\widehat{\mathscr{E}}_{\lambda,-1/8}$.
- We can probe the CFT by using it as a "fundamental building block" to construct new theories. One simple example is to consider its coset by the subalgebra generated by the h_n and K. These generate the affine Kac-Moody algebra $\widehat{\mathfrak{u}}(1)$.
- The coset algebra contains the Virasoro algebra of central charge $c = -2$, but can be shown to be bigger. In fact, it can be identified as the triplet algebra $\mathfrak{W}(2,3,3,3)$ of Kausch. [PLB 259 (1991)]
- However, our spectrum reduces to only two $\mathfrak{W}$-irreducibles under the coset mechanism. The triplet model needs four...

We should therefore augment our spectrum by whichever $\widehat{\mathfrak{sl}}(2)_{-1/2}$-modules (if any) reduce to the remaining two $\mathfrak{W}$-irreducibles under the coset mechanism. The only ones which do the job turn out to be the self-conjugate irreducibles

$$\widehat{\mathscr{E}}_{0,-1/8} \qquad \text{and} \qquad \widehat{\mathscr{E}}_{1,-1/8},$$

and their images under spectral flow.

We now have to check the fusion rules of the augmented spectrum. We find that

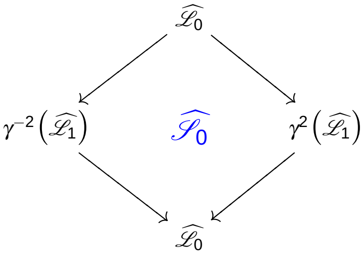
$$\begin{aligned} \widehat{\mathscr{L}_0} \times \widehat{\mathscr{E}}_{0,-1/8} &= \widehat{\mathscr{E}}_{0,-1/8} & \qquad \widehat{\mathscr{L}_1} \times \widehat{\mathscr{E}}_{0,-1/8} &= \widehat{\mathscr{E}}_{1,-1/8} \\ \widehat{\mathscr{L}_0} \times \widehat{\mathscr{E}}_{1,-1/8} &= \widehat{\mathscr{E}}_{1,-1/8} & \qquad \widehat{\mathscr{L}_1} \times \widehat{\mathscr{E}}_{1,-1/8} &= \widehat{\mathscr{E}}_{0,-1/8}. \end{aligned}$$

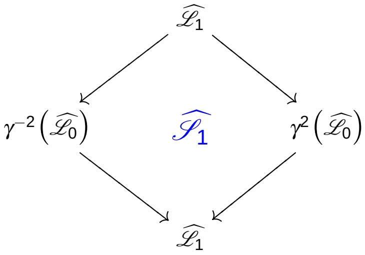
However, we do not expect that the fusion rules will close on this augmented spectrum (the four $\mathfrak{W}$-irreducibles are not closed under fusion in the triplet model).

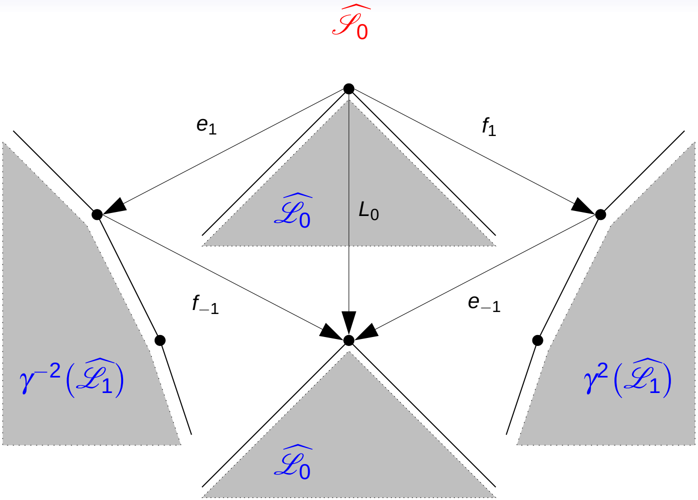
The fusion rules of the $\widehat{\mathscr{E}}_{\lambda,-1/8}$ among themselves are significantly more delicate to compute. Nevertheless, we find

$$\widehat{\mathscr{E}}_{0,-1/8} \times \widehat{\mathscr{E}}_{0,-1/8} = \widehat{\mathscr{S}_0} \qquad \widehat{\mathscr{E}}_{1,-1/8} \times \widehat{\mathscr{E}}_{1,-1/8} = \widehat{\mathscr{S}_0}$$
$$\widehat{\mathscr{E}}_{0,-1/8} \times \widehat{\mathscr{E}}_{1,-1/8} = \widehat{\mathscr{S}_1},$$

where $\widehat{\mathscr{S}_0}$ and $\widehat{\mathscr{S}_1}$ are new indecomposable modules. They are formed from four irreducibles coupled together:







A schematic illustration of the indecomposable $\widehat{\mathscr{S}_0}$ showing how its constituent irreducibles are glued together.

Conclusions

- We have seen that at $k = -\frac{1}{2}$, there is a subset of the allowed $\widehat{\mathfrak{sl}}(2)$-modules which is modular invariant and closed under fusion.
- The problematic negative integers given by conjugation and the Verlinde formula have been explained as describing the character ring rather than the fusion ring.
- Fractional level theories are built using an infinite number of unfamiliar irreducible modules whose conformal dimensions are not bounded below. Spectral flow allows us to control this.
- Consistency may require augmenting with further modules, and these modules generate indecomposables under fusion. Fractional level WZW models will then be logarithmic CFTs.

Outlook

This leads to many questions, *eg*:

- Can we construct consistent (logarithmic) CFTs in the bulk from these modules? If so, what are the boundary CFTs?
- Do the indecomposables encountered have a structure theory?
- Is the story similar for the other fractional levels?
- Is it similar for other affine (super)algebras?
- What other interesting CFTs can be constructed from these models?
- Can we realise the non-unitary minimal models as cosets if the consistent fractional level WZW models turn out to be logarithmic?
- Can we use fractional level WZW models to study logarithmic versions of the minimal models?