The Laughlin Wavefunction

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Contents

Constants and Conventions

Useful constants and quantities which will be used in this summary:

- $\omega_B = \frac{eB}{m}$, the cyclotron frequency,
- $l_B = \sqrt{\frac{\hbar}{eB}}$, the magnetic length,
- $\Phi_0 = \frac{2\pi\hbar}{e}$, the quantum of flux,
- $E_i = \hbar\omega_B \left(n + \frac{1}{2}\right)$, the energy of states in the i-th Landau level,
- $N_{LL} = \frac{AB}{\Phi_0}$, the number of states per Landau level.

Holomorphic coordinates are defined as

$$z = x - iy\,, \quad \bar{z} = x + iy, \tag{1}$$

such that the angular momentum operator reads

$$L = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}. \tag{2}$$

1 Introduction

The integer quantum Hall effect as explained in the previous lecture is not sufficient for a complete description of the physics in the Quantum Hall System. In certain experiments, with especially clean samples, plateaus in the resistivity graph were measured at values not corresponding to integer filling fractions ν (see Figure 1). Instead, they match fractional values with small denominators. A few values at which additional plateaus were measured are $\nu = \frac{1}{5}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}$ in the first Landau Level as well as $\nu = \frac{4}{3}, \frac{5}{3}, \frac{7}{5}, \frac{12}{5}$ in higher Levels. This is called the fractional quantum Hall Effect.

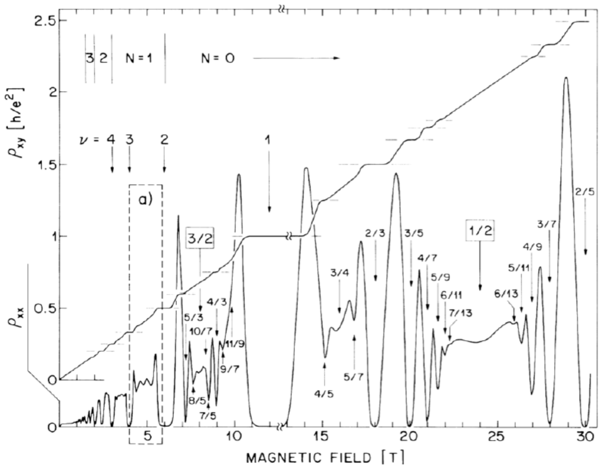

Figure 1: Measurement of the Hall resistivity, with plateaus at fractional values. Note that the plateaus are less prominent compared to the integer case. Data and figure taken from [4].

The appearance of these new phenomena should not be too surprising as until now we have neglected interactions between electrons completely. Explicitly, the Coulomb potential

$$V = \frac{e^2}{4\pi\epsilon_0 \left|\vec{x_i} - \vec{x_j}\right|^2} \tag{3}$$

has to be added to the Hamiltonian for each pair of electrons. This new Hamiltonian is very hard to solve, therefore we will later just write down an Ansatz: The famous Laughlin Wavefunction. But prior to that we still have to investigate an aspect that is also relevant to the integer quantum Hall physics, which is impurities of the sample.

The main reference for this lecture were the notes of David Tong on the quantum Hall effect [3].

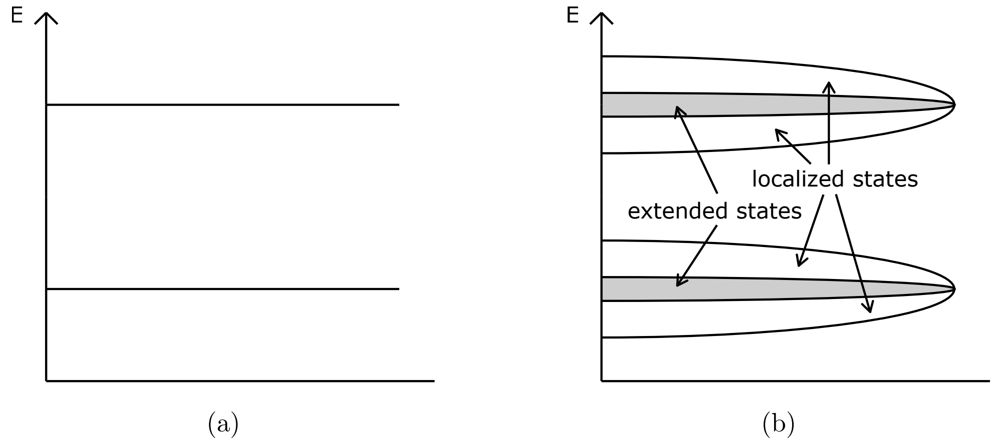

Figure 2: The quantum Hall spectrum neglecting interactions, (a) without and (b) with disorder in the sample.

2 Role of Disorder

2.1 Without Interactions

We have yet to consider a very important aspect of the quantum Hall effect, which are impurities in the sample. In fact, we will see that the quantization of the conductivity only happens in the presence of some disorder.

We will model the impurities by adding some potential V to the Hamiltonian: $H \to H + V(x)$. The only restrictions on V that we demand are it to be small versus the energy difference between Landau levels (so we can use perturbation theory), and it to vary slowly compared to the cyclotron radius:

$$V \ll \hbar\omega_B\,, \qquad |\nabla V| \ll \frac{\hbar\omega_B}{l_B} \tag{4}$$

Without assuming more about V, we can still say two things about the spectrum of the modified Hamiltonian. One is that it is no longer discrete and the degeneracy of states gets lifted. The second is that some states get localized, in particular those close to the energetic extremes in each Landau level (see Figure 2). The reason is that these states lie close to maxima or minima of $V(x)$. Semiclassically, they can be thought of as superpositions of electrons whose centers of cyclotron motion moves along the equipotentials near those extreme points. This argument was the reason to demand the second part of (4).

Now we will see how this effect leads to the quantization of conductivity. Assume that the magnetic field B is tuned such that exactly n Landau levels are completely filled. If one now increases B, the number of states per Landau level, $N_{LL} = \frac{AB}{\Phi_0}$, is increased, and the n-th level is no longer completely filled. But the states that get emptied first are those with the highest energies, and those are localized! Hence, them no longer being

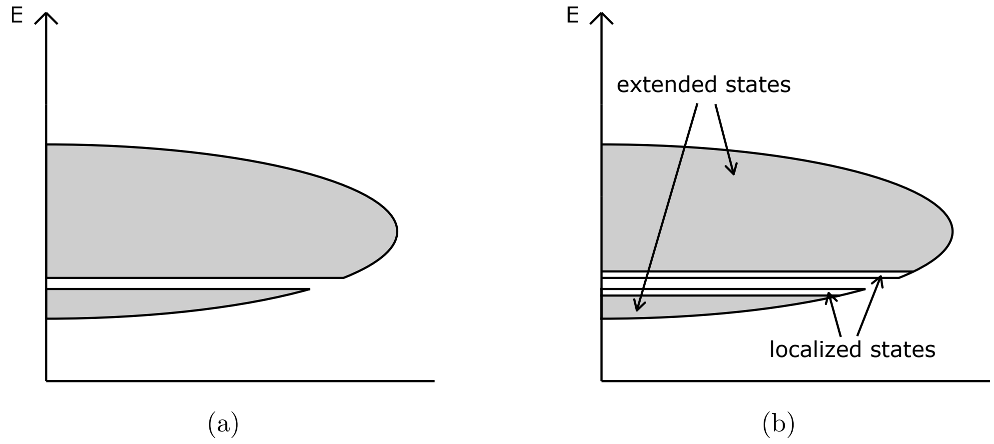

Figure 3: The first Landau Level of the quantum Hall spectrum with interactions between electrons, (a) without and (b) with small disorder in the sample. Only one gap and only the localized states relevant for the plateau corresponding to that gap are shown.

filled has no effect on the conductivity. Decreasing B leads to a similar story, where the lowest states in the next level are filled first, which are again localized. Hence, the conductivity graph has plateaus around the values of B where an integer number of landau levels are completely filled. The more states get localized, that is (at least until some point) the more disorder is in the system, the more prominent the plateaus are.

2.2 With Interactions

The argument above gives us a simple possibility to also explain the plateaus in the fractional quantum Hall effect, taken one crucial assumption: The spectrum of the Hamiltonian including interactions has gaps at the filling fractions ν where the plateaus are observed. If the disorder in the sample additionally is on a smaller scale than the width of these gaps, the argument from above can be rerun to explain the observed plateaus (see figure 3 for a demonstration with one plateaus).

The rest of this lecture should give some intuition why such gaps might arise.

3 Laughlins Ansatz

The main idea to explain the additional plateaus is to take particle interaction into account. Before writing down the full N-particle wave function we will have a look at the two particle case. We neglect mixing between Landau levels ($\hbar\omega_B \gg V$), and hence the wave function in holomorphic coordinates of a state in the lowest Landau level (LLL) takes the form

$$\psi \propto f(z_1, z_2)\, e^{-(|z_1|^2+|z_2|^2)/4l_B^2}, \tag{5}$$

with f holomorphic in z_1 and z_2. For two particles, centre of mass angular momentum M and relative angular momentum m are good quantum numbers, hence we can write down the eigenstates of the Hamiltonian:

$$\psi \propto (z_1 + z_2)^M (z_1 - z_2)^m \, e^{-(|z_1|^2+|z_2|^2)/4l_B^2} \tag{6}$$

The fact that this was possible without even taking into account the shape of V is due to the restriction of lying in the LLL.

Now for the $N-$particle case: If all particles lie in the LLL, it again follows on general grounds that the wave function has the shape

$$\psi(z_i) = f(z_i)\, e^{-\sum_{i=1}^{N} \frac{|z_i|^2}{4l_B^2}}, \tag{7}$$

with f now being holomorphic in all the z_i. But now we cannot restrict f further immediately, apart from demanding it to be totally antisymmetric. Laughlins Ansatz, valid for N particles and a filling fraction of $\nu = \frac{1}{m}$ with an odd integer m reads:

$$\psi(z_i) = \prod_{i<j} (z_i - z_j)^m \, e^{-\sum_{i=1}^{N} \frac{|z_i|^2}{4l_B^2}} \tag{8}$$

A few properties of the wave function which can be seen immediately are

- m is odd, hence ψ is totally antisymmetric. An even m would lead to a wave function for bosons.
- The amplitude goes to zero wherever two electrons come very close to each other.
- For big distances from the origin the wave function falls off exponentially.

The last two observations tell that there is a balance between electrons coming close to each other and them being far apart.

A not quite as obvious observation is that the Laughlin wave function has indeed the postulated filling fraction of $\nu = \frac{1}{m}$. An easy but somewhat heuristic way of seeing that is looking at the angular momentum of one particle, for example the one at position z_1. In 8, the part of the pre factor depending on z_1 is $\prod_{i=2}^{N}$. Thus, the maximum angular momentum of that particle is $m(N-1)$ which corresponds to a radius of $R \approx \sqrt{2mN} l_B$. The area hence is $A = \pi R^2 \approx 2\pi m N l_B^2$, and the total number of states in the first Landau level is evaluates to be

$$\frac{AB}{\Phi_0} = \frac{A}{2\pi l_B^2} = \frac{2\pi m N l_B^2}{2\pi l_B^2} = mN, \tag{9}$$

which indeed gives the expected filling fraction of $1/m$.

A last thing to note in this section is that the Laughlin wavefunction is not an exact solution. But for a small number of particles it has been numerically shown to have a large ($> 99\%$) overlap with the true solution. While that is most likely not the case for a macroscopic number of particles, Laughlins Ansatz is still believed to capture a lot of qualitative physical properties of the system very well.

To gain a better understanding of what the Laughlin wavefunvtion describes, an analogy to a 2D plasma will be introduced in the next section.

4 The Plasma Analogy

While the Laughlin wavefunction is easy to write down, actual computations with it can be very difficult. For example, trying to compute the local density, as described by the operator

$$n(z) = \sum_{i=1}^{N} \delta(z - z_i) \tag{10}$$

$$\langle\psi|\, n(z)\, |\psi\rangle = \frac{\int \prod_i d^2 z_i n(z) P[z_i]}{\int \prod_i d^2 z_i P[z_i]} \tag{11}$$

with the unnormalized probability density $P[z]$ defined by

$$P[z_i] = \prod_{i<j} \frac{|z_i - z_j|^{2m}}{l_B^{2m}}\, e^{-\sum_{i=1}^{N} \frac{|z_i|^2}{2l_B^2}}. \tag{12}$$

The critical observation to make at this point is that these expressions are very reminiscent of computations in statistical physics. To expand on that comparison we just write the probability density as a Boltzmann distribution function:

$$P[z_i] = e^{-\beta U(z_i)} \quad \Rightarrow \quad \beta U(z_i) = -2m \sum_{i<j} \log\left(\frac{|z_i - z_j|}{l_B}\right) + \frac{1}{2l_B^2} \sum_{i=1}^{N} |z_i|^2 \tag{13}$$

choosing, for now arbitrarily, $\beta = 2/m$ gives

$$U(z_i) = -m^2 \sum_{i<j} \log\left(\frac{|z_i - z_j|}{l_B}\right) + \frac{m}{4l_B^2} \sum_{i=1}^{N} |z_i|^2. \tag{14}$$

Remarkably, this is precisely the potential energy of a two dimensional plasma of particles witch charge $q = -m$. Both terms in (14) can be interpreted accordingly:

- The two dimensional Poisson equation reads $-\nabla^2 \phi = 2\pi q \delta^2(\vec{r})$ and yields

$$\phi = -q \log\left(\frac{r}{l_B}\right) \tag{15}$$

  as the solution. Multiplying with the charge q to get a potential, and summing over all pairs of particles gives the first term in (14), for $q = -m$.

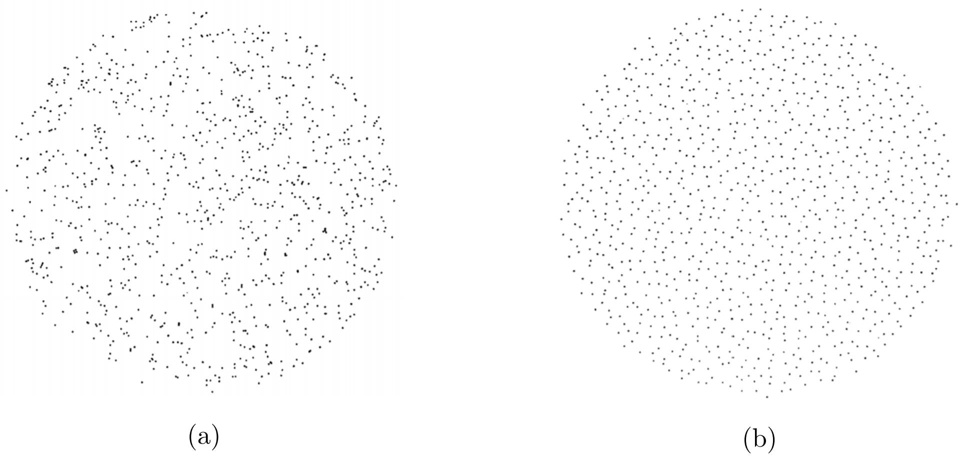

Figure 4: Comparison of samples drawn from (a) a distribution with the positions of particles being independent of each other and (b) the distribution corresponding to the Laughlin wavefunction. Figures from [1]

- The second term can be interpreted as a neutralizing background of constant charge density ρ_0. The Poisson equation for such a density reads: $-\nabla\phi = 2\pi\rho_0$. By plugging the contribution of one particle to the second term in 14 in, one gets:

$$-\nabla^2\left(\frac{|z|^2}{4l_b^2}\right) = -\frac{1}{l_B^2} \quad \Rightarrow \quad \rho_0 = -\frac{1}{2\pi l_B^2} \tag{16}$$

It is crucial to note that the interpretation above does not mean that the Laughlin wavefunction has translational invariance: We have seen before that it is spatially confined and the second sum in (14) also adds higher energy cost for particles far from the origin. The plasma analogy only holds in a disk around the origin with a certain radius.

Now one can apply all knowledge about two dimensional plasma to the Laughlin wavefunction.

First of all, the plasma analogy gives us an easy way to again confirm the filling fraction of the Laughlin wavefunction. The particles in the plasma will want to neutralize the background charge density, hence we get for the number density:

$$-qn = mn = \rho_0 \qquad \Rightarrow \qquad n = \frac{1}{2\pi l_B^2 m} \tag{17}$$

which is one mth of the density in the fully filled Landau Level .

Another aspect to look at is the phase of the plasma: At low Temperatures, n is low too, and the plasma forms a solid crystal (where the particles are arranged in a triagonal lattice). At higher temperatures, the plasma becomes a liquid. Numerical computations have shown that the solid state arises approximately for $m \geq 70$. A typical electron configuration in the liquid phase can be seen in figure 4 (b).

Again, this is only what the Laughlin wavefunction describes, not what actually happens in the quantum Hall system. In reality the solid phase, the so called Wigner Crystal, is preferred for values of ν under approximately 1/7 ([3]).

In the next lecture, the plasma analogy will be revisited and extended to excitations of the Laughlin Wavefunction.

5 Toy Hamiltonians

The Laughlin wavefunction is no exact solution to the Hamiltonian including Coulomb interactions. Hence it would be nice to write down a somewhat similar Hamiltonian that is exactly solved by Laughlins Ansatz, to further justify transferring its properties to the real physical system. This will be done in this section, beginning with the simplest case of just two particles.

5.1 Two Particles

As mentioned in section 3, the LLL two particle wavefunction is

$$|M, m\rangle \propto (z_1 + z_2)^M (z_1 - z_2)^m \, e^{-(|z_1|^2 + |z_2|^2)/4l_B^2}, \tag{18}$$

with M the center of mass and m the relative angular momentum. From these eigenstates one we can compute the quantities

$$v_m = \frac{\langle M, m| \, V \, |M, m\rangle}{\langle M, m| \, |M, m\rangle}. \tag{19}$$

They are sometimes called the "Haldane pseudopotentials" and do not depend on M. Because the wavefuntions are peaked at certain realtive distances r_m, the v_m approximately capture the spatial structure of V:

$$r \approx \sqrt{2m} l_B \quad \Rightarrow \quad v_m \approx V(r = r_m = \sqrt{2m} l_B). \tag{20}$$

Because the v_m are discrete the states $|M, n\rangle$ are bound, even for a repulsive interaction. This is due to the magnetic field, without it there is only a continuous spectrum of scattering states.

From the v_m we can easily construct a Hamiltonian with the same dynamics as the original one (in the LLL where the kinetic energy vanishes):

$$H = \sum_{m'} v_{m'} \mathcal{P}_{m'}. \tag{21}$$

The operators $\mathcal{P}_m$ project onto states where the two particles have relative angular momentum m, that is eigenstates of the operator L_{12} with L_{ij} defined by

$$L_{ij} = (z_i - z_j)\frac{\partial}{\partial(zi - zj)} - (\overline{z_i} - \overline{z_j})\frac{\partial}{\partial(\overline{z_i} - \overline{z_j})}. \tag{22}$$

Now we can easily design Hamiltonians by choosing our own values for v_m. For example

$$v_{m'} = \begin{cases} 1, & m' < m \\ 0, & m' \geq m \end{cases} \tag{23}$$

assigns a constant energy cost to relative angular momenta smaller than some value m and no cost for bigger ones. It can be though of as a primitive model for a repulsive potential since smaller relative angular momenta correspond to smaller distances.

5.2 Many Particles

In order to use the same method to construct N-particle Hamiltonians, we have to introduce operators $\mathcal{P}_{m'}(ij)$, projecting on eigenstates of L_{ij}, that is states where particles i and j have relative angular momentum m. The we can define our Hamiltonian:

$$H = \sum_{m'=1}^{\infty} \sum i < j v_{m'} \mathcal{P}_{m'}(ij). \tag{24}$$

In general, these many particle Hamiltonians are hard to solve because the $\mathcal{P}_{m'}(ij)$ do not commute with each other:

$$[\mathcal{P}_{m'}(ij), \mathcal{P}_{m'}(jk)] \neq 0. \tag{25}$$

But by choosing the v_m again as in (23) we see that $H\psi = 0$ for all ψ of the form

$$\psi(z_i) = s(z_i) \prod_{i<j} (z_i - z_j)^m \, e^{-\sum_{i=1}^{N} \frac{|z_i|^2}{4l_B^2}} \tag{26}$$

with any symmetric polynomial s. For $s = 1$ we get back the Laughlin state. If we want it to be the only ground state we can either add a confining potential or a term punishing higher total angular momentum to the Hamiltonian.

The Laughlin wavefunction is the "smallest" state with zero energy. Squeezing it further would intruduce relative angular momenta smaller than m and hence lead to a finite energy cost. This is known as the incompressibility of the quantum Hall liquid and responsible for the gaps in the spectrum.

6 Outlook

It turns out that non-constant choices for s in (26) lead to interesting physics as well. For example, choosing $s(z_i) = \prod_{i=1}^{z_i - \eta}$ gives the wave function of a so-called "quasi-hole":

$$\psi_{hole}(z_i, \eta) = \prod_{i=1}^{N}(z_i - \eta) \prod_{i<j}(z_i - z_j)^m \, e^{-\sum_{i=1}^{N} \frac{|z_i|^2}{4l_B^2}}. \tag{27}$$

It will turn out that the quasi-holes and their partners, the quasi-particles, have fractional charge and lead to the broader field of anyons, a type of particle which is neither bosonic nor fermionic. They will be introduced in the next lecture.

References

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