Fuzzy geometry of membranes in M theory.

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INTRODUCTION

D-Branes in String Theory

D-branes are described by

- Solutions of Supergravity
- Boundary conditions of Strings
- Worldvolume effective theories

For N Dp-branes in Type IIA/B superstring theory

The worldvolume description is p+1 dimensional $U(N)$ SYM theory.

$$\int d^{p+1}x \; tr(F_{\mu\nu}^2 + (\partial\Phi_i)^2 + [\Phi_i, \Phi_j]^2 + \cdots)$$

M theory is an eleven dimensional theory which unifies diverse ten dimensional string theories.

It contains membranes (M2-branes) and five-branes (M5-branes).

What is the worldvolume theory of N membranes or of N fivebranes in flat space ? in a low energy limit ?

Almost nothing is known about the above question !

We do know about the case $N = 1$. i.e for single 5-brane or single membrane.

There are reasons, from supergravity description of the branes, that they are rather exotic theories.

Unlike D-branes, where entropy calculated from supergravity scales like N^2 in accordance with expectation from $U(N)$, in these cases, we have $N^{3/2}$ and N^3 respectively for M2- and M5-branes.

We know that the single 5-brane admits solutions which describe a deformation of the 5-brane into a 5-brane/2-brane system, a funnel

Figure

The 2-brane carries charge $N > 1$. So it should be describable by a multi-2-brane theory.

This is indeed well understood in the case of D-branes.
A D3-brane can form a D1-funnel.
The same system can be described by the $U(N)$ theory of the $D1-$branes.

The D1-D3 system teaches us an important lesson. The S^2 cross section becomes a fuzzy 2-sphere.
The M2-M5 system would contain some kind of fuzzy 3-sphere.

Recent developments have resulted in a proposal (ABJM) for a membrane theory in a quotient of eleven dimensional space. Solutions of the membrane theory have been indeed claimed to describe fuzzy 3-spheres.

In Nastase-Papageorgakis-Ramgoolam (arXiv:0903.3966) we showed that the claimed fuzzy 3-spheres are not fuzzy 3-spheres. They are fuzzy 2-spheres.

They are a new realization of fuzzy 2-spheres not known before – using matrices in bifundamental of gauge group rather than adjoint.

This is as expected from the physical picture.

Quotient of flat space.

OUTLINE

- The Matrix geometry of D1-D3 system :
  Fuzzy 2-sphere.

- The Matrix geometry of M2-M5 system in ABJM membrane theory :
  New construction of Fuzzy 2-sphere
  (from bifundamental matter)
  Not a Fuzzy 3-sphere

- The spacetime picture :
  Origins of ABJM
  Quotients and consequences.

- How would we see the fuzzy 3D geometry of M2-M5 ?

The D1-D3 system

The worldvolume description of a single D1-brane contains 8 scalars $\Phi^{\mu}(x_0, x_1)$ which describe the position of each point of the $D1$-brane in the 8 directions.

Naively we might expect that for N D1-branes we would have $\Phi^{\mu}_{a}(x_0, x_1)$ with a running from 1 to N.

But in fact, the stringy nature of the D-brane excitations implies that we have $\Phi^{\mu}_{ab}(x_0, x_1)$ which can be viewed as matrix elements of 8 hermitian matrices $\Phi^{\mu}(x_0, x_1)$.

Static funnel configurations are described by

$$\partial_{x_1}^2 \Phi^i = [\Phi^j, [\Phi^i, \Phi^j]]$$

For any N, $\Phi_i = f(x_1) X_i$ can solve the equations, for appropriate f when

$$[X^i, X^j] = i\epsilon^{ijk} X^k$$

Relations of the $SU(2)$ algebra. In fact N-dimensional representation matrices of the $SU(2)$ algebra.

Focus on the irreducible matrix representations of $SU(2)$. We have spin J representation related to N by

$$N = 2J + 1$$

For these representations

$$X_i X_i = J(J+1) = \frac{(N^2 - 1)}{4}$$

Defining $x_i = \frac{X_i}{\sqrt{J(J+1)}}$ we have the equation of a unit sphere

$$x_i x_i = 1$$

In some sense, the solutions are describing a spherical geometry.

In quantum physics we are interested not just in the solution, but also in the fluctuations of the solution.

The $SO(1,1) \times SO(8)$ symmetry of the D1-brane theory has been broken to $SO(1,1) \times SO(3) \times SO(5)$. The X_i are in the vector (spin 1) of $SO(3)$ (which is isomorphic to $SU(2)$).

The general fluctuations, i.e general matrices, can be organized according to the symmetries, in particular the $SO(3)$.

$$\delta M = a_0 + a_i X_i + a_{ij} X_i X_j + \cdots$$

We are getting a sequence of representations of $SU(2)$.

$$Mat_N = \bigoplus_{\ell=0}^{N-1} Mat_\ell$$

As $N \to \infty$ we get

$$Mat_N = \bigoplus_{\ell=0}^{\infty} Mat_\ell$$

Where else have we seen an infinite sequence of representations of $SU(2)$?

The space of spherical harmonics on S^2, i.e the $Y_{l,m}$.

This is not a coincidence. This sequence of matrix realizations of the sphere equations is the "fuzzy sphere" construction.

A lot of geometry : multiplication of functions on a manifold, Laplacians, derivatives, integrals etc. can be generalized to the set-up of these Matrix (fuzzy) geometries.

Applying this fuzzy geometry technology, allows us to recover, in the large N limit, a field theory of fluctuations on an emergent S^2.

At finite N we have a non-commutative sphere.

The classical symmetries

$$L_i \longrightarrow i\epsilon_{ijk}x_j\partial_k$$

They act on functions. In this matrix geometry, general functions are replaced by general matrices.

$$\begin{aligned}
&f \longrightarrow M\\
&L_i(f) \longrightarrow [X_i, M]\\
&\Box(f) \longrightarrow [X_i, [X_i, M]]\\
&\int f \longrightarrow TrM
\end{aligned}$$

Action of symmetries, Laplacians, integral etc. all have counterparts in the matrix geometry.

Figure of D1-D3.

We started with a D1-theory with $\Phi^i(x_0, x_1), \Phi^a(x_0, x_1)$ on 1+1 dimensional Minkowski space $\mathbb{R}^{1,1}$

Fuzzy geometry technology allows us to derive a $U(1)$ gauge theory $\mathbb{R}^{1,1} \times S^2$.

The $\Phi^a_{pq}(x_0, x_1)$ which were matrices become ordinary scalars $\phi^a(x_0, x_1, \theta, \phi)$.

Matrix degrees of freedom have been converted to functions on S^2. The Φ_i give rise to 2 gauge fields and a scalar.

$$\Phi_i = fX_i + K_i^a A_a + x_i \phi$$

where K_i^a are some KIlling vectors on the sphere.

The same system can be described from the point of view of the D3-brane.

Expand D3-brane in the presence of a monopole and a transverse scalar diverging at the core of the funnel.

Energy per unit length agrees.

The fluctuation action should agree. (The agreement was demonstrated in detail in a related D0-D2 time dependent system PRT)

M2-M5 system in ABJM membrane theory

ABJM theory is Chern-Simons + matter theory with $U(N) \times U(N)$ gauge group.

$$
\begin{aligned}
S_{\mathrm{ABJM}} &= \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\lambda} \mathrm{Tr}\Big(A_\mu^{(1)} \partial_\nu A_\lambda^{(1)} + \frac{2i}{3} A_\mu^{(1)} A_\nu^{(1)} A_\lambda^{(1)} \\
&\quad - A_\mu^{(2)} \partial_\nu A_\lambda^{(2)} - \frac{2i}{3} A_\mu^{(2)} A_\nu^{(2)} A_\lambda^{(2)} \Big) + \cdots
\end{aligned}
$$

The CS levels are k and $-k$. This is proposed to be the theory of N membranes for a spacetime where the $\mathbb{R}^8$ transverse space is replaced by $\mathbb{R}^8/Z_k = \mathbb{C}^4/Z_k$.

For $k = 1$ we have flat space.

There are matter fields in the bi-fundamental of the gauge group $(N, \bar{N})$ and $(\bar{N}, N)$. There is an $SU(4)$ global symmetry which is broken to $SU(2) \times SU(2)$ by the solutions of interest.

In $SU(2) \times SU(2)$ language we have fields $(Q^{\dot{\alpha}}, R^{\alpha})$ which combine into a 4 of $SU(4)$ with $C^I = (Q^{\dot{\alpha}}, R^{\alpha})$.

The ABJM action for these matter fields contains a sextic potential term.

$$
\begin{aligned}
S_{\mathrm{ABJM}} &= \cdots + \frac{k}{4\pi}\int d^3x \mathrm{Tr}\left(D_\mu C_I^\dagger D^\mu C^I\right) \\
&+\frac{4\pi^2}{3k^2}\mathrm{Tr}\Big(C^I C_I^\dagger C^J C_J^\dagger C^K C_K^\dagger + C_I^\dagger C^I C_J^\dagger C^J C_K^\dagger C^K \\
&+4C^I C_J^\dagger C^K C_I^\dagger C^J C_K^\dagger - 6C^I C_J^\dagger C^J C_I^\dagger C^K C_K^\dagger\Big) + \cdots
\end{aligned}
$$

Covariant derivatives contain the gauge fields. CS terms given before. Fermion terms not written here.

The funnel solution involves matrices

$$R^{\alpha} = G^{\alpha}$$
$$Q^{\dot{\alpha}} = 0$$

The α is in the 2-dimensional rep. of $SU(2)$. The equation of motion when the matrices solve

$$G^{\alpha} = G^{\beta} G^{\dagger}_{\beta} G^{\alpha} - G^{\alpha} G^{\dagger}_{\beta} G^{\beta}$$

This is the analog, in this context, of the matrix equations

$$[X_i, X_j] = i\epsilon_{ijk} X_k$$

which were the fuzzy 2-sphere equations relevant to the D1-D3 intersection.

Explicit forms of matrices :

$$
\begin{aligned}
(G^1)_{m,n} &= \sqrt{m-1}\;\delta_{m,n} \\
(G^2)_{m,n} &= \sqrt{(N-m)}\;\delta_{m+1,n} \\
(G_1^\dagger)_{m,n} &= \sqrt{m-1}\;\delta_{m,n} \\
(G_2^\dagger)_{m,n} &= \sqrt{(N-n)}\;\delta_{n+1,m}
\end{aligned}
$$

given in Gomis,Rodriguez-Gomez,Van Rammsdonk, Verlinde.

They satisfy

$$\sum_\alpha G^\alpha G_\alpha^\dagger = N-1$$

In D-branes, each transverse direction x_i corresponds to X_i. In this case the fields correspond to complex cocordinates

$$\begin{aligned} R^1 &\to x_1 + ix_2 \equiv z_1 \\ R^2 &\to x_3 + ix_4 \equiv z_2 \\ Q^1 &\to x_5 + ix_6 \equiv z_3 \\ Q^2 &\to x_7 + ix_8 \equiv z_4 \end{aligned}$$

The equation for G's looks, after rescaling, like a matrix version of $\sum_{i=1}^{4} x_i x_i = 1$, suggestive of a 3-sphere.

However there is an additional reality condition obeyed by the matrices

$$G^1 = G_1^\dagger$$

This would correspond to

$$z_1 = \bar{z}_1$$

This is not what we expect from 3-sphere.

It is more like a 2-sphere equation.

So are these matrices G just describing a fuzzy 2-sphere in disguise ?

If we define

$$J_i = (\tilde{\sigma}_i)^{\alpha}_{\beta} G^{\beta} G^{\dagger}_{\alpha}$$

We find they obey the fuzzy 2-sphere equations

$$[J_i, J_j] = i\epsilon^{ijk} J_k$$
$$J_i J_i = \frac{(N^2 - 1)}{4}$$

These live in the Lie algebra of $U(N)$.

We can also find a copy of the fuzzy 2-sphere equations in the $U(\bar{N})$ (the other factor in the product gauge group).

$$
\begin{aligned}
\bar{J}_i &= (\tilde{\sigma}_i)^{\alpha}_{\beta} G^{\dagger}_{\alpha} G^{\beta} \\
[\bar{J}_i, \bar{J}_j] &= i\epsilon^{ijk} \bar{J}_k \\
\bar{J}_i \bar{J}_i &= \frac{((N-1)^2 - 1)}{4}
\end{aligned}
$$

We can combine the two into $2N \times 2N$ matrices.

$$
\mathbf{J}_i = \begin{pmatrix} J_i & 0 \\ 0 & \bar{J}_i \end{pmatrix}
$$

Relations between J's and G's

$$
\begin{aligned}
[\mathbf{J}_i, \mathbf{G}^\alpha] &= \begin{pmatrix} 0 & J_i G^\alpha - G^\alpha \bar{J}_i \\ \bar{J}_i G^\dagger_\alpha - G^\dagger_\alpha J_i & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & (\tilde{\sigma}_i)^\alpha_\beta G^\beta \\ -G^\dagger_\beta (\tilde{\sigma}_i)^\beta_\alpha & 0 \end{pmatrix}
\end{aligned}
$$

BY expanding the most general $2N \times 2N$ matrices in terms of spherical harmonics (representations of $SU(2)$), we are ready to derive the field theory of the fluctuations.

Fluctuation : SOME DETAILS

$$
\begin{aligned}
R^{\alpha} &= fG^{\alpha} + r^{\alpha} \,, \qquad & R^{\dagger}_{\alpha} &= fG^{\dagger}_{\alpha} + r^{\dagger}_{\alpha} \\
Q^{\dot{\alpha}} &= q^{\dot{\alpha}} \,, & Q^{\dagger}_{\dot{\alpha}} &= q^{\dagger}_{\dot{\alpha}} \\
A^{(i)}_{\mu} &= A^{(i)}_{\mu} \,, & \psi^{\dagger I} &= \psi^{\dagger I}
\end{aligned}
$$

We further expand

$$
\begin{aligned}
r^{\alpha} &= rG^{\alpha} + s^{\alpha}_{\beta} G^{\beta} \\
r^{\dagger}_{\alpha} &= G^{\dagger}_{\alpha} r + G^{\dagger}_{\beta} s^{\beta}_{\alpha}
\end{aligned}
$$

We extract a vector of $SO(3)$

$$
s_i = s^{\beta}_{\alpha} (\tilde{\sigma}_i)^{\alpha}_{\beta} \,, \qquad s^{\alpha}_{\beta} = \frac{1}{2} s_i (\tilde{\sigma}_i)^{\alpha}_{\beta}
$$

$$
s_i = K^a_i A_a + x_i \phi
$$

The theory we get is a $U(1) \times U(1)$ theory on an emergent S^2 coupled to charged matter.

With the $\mathbb{R}^{2,1}$ base space we already had, we have a $U(1) \times U(1)$ Chern-Simons field theory on $\mathbb{R}^{2,1} \times S^2$.

Is this a 4-brane theory ? If so, we need a $U(1)$ gauge group and dynamical gauge fields.

A similar problem of relating $U(1) \times U(1)$ CS theory to ordinary $U(1)$ gauge theory was solved in the $k = 1$ theory (for $SU(2)$ gauge group) and generally in ABJM.

Mukhi and Papageorgakis
Pang and Wang

This was done in the case of fluctuations around the SUSY configurations of the ABJM model, corresponding to separating the N branes, which form the space $(\mathbb{C}^4/Z_k)^N/S_N$

For example, expand around $X_1 = x_1, X_2 = X_3 = \cdots = 0$. The identifications $Z_k \sim e^{\frac{2\pi i}{k}} Z_k$ around this background lead to

$$x_2 \sim x_2 + \frac{2\pi x_1}{k}$$

at large k.

Large k is where the semiclassical analysis of fluctuations is valid. The above quotient implies that we are reducing to Type IIA. We expect to recover $D2$-brane with dynamical gauge field.

The expected D2-action was indeed found.

Form linear combinations

$$A^{(1)} + A^{(2)} = A$$
$$A^{(1)} - A^{(2)} = B$$

and write fluctuation action in terms of these variables.

The CS terms imply a mass term for B. Action for B is quadratic. Solving its equation of motion yields $B \sim F(A) + \partial\phi_2$. Integrating out B leaves

$$\int F(A)^2$$

the desired Yang Mills action.

Similar manipulations work here and allow us to convert the $U(1)\times U(1)$ 2+1 Chern-Simons gauge field action, to a 2+1 dynamical Yang Mills. The emergent gauge field on S^2 that comes from the matrix degrees of freedom combines to give an action with local $4+1$ Lorentz invariance on $\mathbb{R}^{2,1}\times S^2$

$$
\begin{aligned}
S_{4+1} = \frac{1}{g_{YM}^2}\int d^3x\ \mu^{-2} d\Omega\Big[& -\frac{1}{4}F_{AB}F^{AB} - \frac{1}{2}\partial_A\Phi\partial^A\Phi - \frac{\mu^2}{2}\Phi^2 \\
& -\partial_\mu Q^{\dot\alpha}_{\alpha}(\delta^{\alpha}_{\beta} + x_i(\tilde\sigma_i)^{\alpha}_{\beta})\partial^\mu Q^{\beta}_{\dot\alpha} \\
& +\Big((\nabla_a)^{\alpha}_{\gamma} Q^{\dot\alpha}_{\alpha}\Big)(\delta^{\gamma}_{\beta} + x_i(\tilde\sigma_i)^{\gamma}_{\beta})\Big((\nabla^a)^{\beta}_{\mu} Q^{\mu}_{\dot\alpha}\Big) + \frac{\mu}{2}\,\omega^{ab}F_{ab}\Phi\Big]
\end{aligned}
$$

The A, B indices run over $4+1$ dimensions.

Comments/Intepretation

The semiclassical fluctuation analysis is valid at large k. This is very far from the flat space limit. The quotient compactifies the circle fibre of the Hopf fibration of S^3 inside $\mathbb{C}^2$ described by Z_1, Z_2.

Effectively the action for small fluctuations sees an M-IIA reduction along this circle.

A derivation of the same fluctuation action from a D4-brane in an appropriate IIA background is an open problem.

The geometry of the Hopf fibration sheds light on the construction.

In usual fuzzy 2-sphere

$$x_i \sim \frac{X_i}{N}$$

constructions of D-brane physics

Deriving a gauge theory on S^2 described by x_i is very natural because transverse coordinates are matrices

$$\Phi_i \sim X_i$$

Here the matrix coordinates corresponding to spacetime are

$$R^{\alpha} \sim G^{\alpha}$$

but the coordinates of the emergent S^2 are

$$x_i \sim \frac{G(\tilde{\sigma}_i)G^{\dagger}}{N}$$

So we want both G and x to be geometrical. Is the quadratic relation something known in classical geometry ?

YES ! They describe the projection of S^3 to S^2 base of the Hopf fibration.

Describe the embedding in $R^4 = \mathbb{C}^2$ of S^3 by

$$z_1 \bar{z}_1 + z_2 \bar{z}_2 = 1$$

The $U(1)$ action on S^3 is

$$\begin{aligned} z_1 &\longrightarrow e^{i\phi} z_1 \\ z_2 &\longrightarrow e^{i\phi} z_2 \end{aligned}$$

This is the $U(1)$ of the Hopf fibration. The projection to the sphere is given by

$$x_i = (\tilde{\sigma}_i)^{\alpha}_{\beta} z^{\beta} \bar{z}_{\alpha}$$

with $x_i x_i = 1$.

In fact the structure of the ABJM action, the need for bifundamental fields can be guessed from the idea of a matrix version of the Hopf projection equation.

$$X_i|v> \sim |w>$$

This is possible for $|v>$ and $|w>$ in the same spin J because $J \otimes 1$ contains J.

On the other hand,

$$G^{\alpha}|v> \sim |w>$$

is possible for $|v>$ in spin J if $|w>$ is spin $J-1/2$. This means the simplest gauge theory supporting the $SU(2)$ is a $U(N) \times U(N-1)$ with $SU(2)$ transformations $V_N \oplus V_1$.

Requiring a Z_2 symmetry exchanging the gauge groups we have $U(N) \times U(N)$ with the transformation under $SU(2)$ being $V_N \oplus (V_{N-1} \oplus V_1)$.

Sextic terms in potential are just what are needed for Laplacians on the S^2.

$$\begin{aligned}
\int \phi \Box \phi \quad & \sim Tr\phi[X_i, [X_i, \phi]] \\
& \sim Tr(\phi G G^\dagger G G^\dagger \phi) \\
& \sim Tr(C C^\dagger C C^\dagger C C^\dagger)
\end{aligned}$$

How do we see the fibre ? Non-perturbative effects. E.g Instantons on 4-branes are related to $D4 - D0$ which carry momentum in the 11'th direction.

Evidence of S^3 at $k = 1$?

Is there some description of multi-membrane theory in flat space which can demonstrate the S^3 geometry of M2-M5 intersections ?

Matrix constructions of $SO(4)$ covariant S^3 require projections, non-associativity, but haven't been shown to be solutions of any candidate M2-M5 actions (although they have been shown to solve time-dependent BFSS equations !)

Infinite-dimensional realisations of Bagger-Lambert 3-algebra can be based on 3-spheres. Physics of membranes from this ?