On the Finiteness of Quasi-alternating Links

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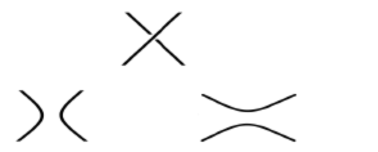
Definition of Quasi-alternating Links

Definition (Ozsváth and Szabó (2005))

The set $\mathcal{Q}$ of quasi-alternating links is the smallest set satisfying the following properties:

- The unknot belongs to $\mathcal{Q}$.
- If L is link with a diagram containing a crossing c such that
  1. both smoothings of the diagram of L at the crossing c, L_0 and L_1 as in the figure below belong to $\mathcal{Q}$,



  2. $\det(L_0), \det(L_1) \geq 1$,
  3. $\det(L) = \det(L_0) + \det(L_1)$.

Then L is in $\mathcal{Q}$ and in this case we say L is quasi-alternating at the crossing c.

Remark

Any non-split alternating link is quasi-alternating at any crossing of any reduced alternating diagram. Therefore, quasi-alternating links can be considered as a natural generalization of alternating links.

Remark (Some known facts of alternating links)

1. *There are only finitely many alternating links of a crossing number less than or equal to some positive integer. This is true since there are finitely many links of a crossing number less than or equal to some positive integer.*
2. *The determinant of any alternating link is an upper bound of its crossing number. This is true since the determinant is equal to the number of spanning trees of the Tait graph of the corresponding alternating diagram.*
3. *The breadth of the Jones polynomial of any alternating link is equal to the crossing number of the given link.*

Lemma (Finiteness Properties of Alternating Links)

1. *There are only finitely many alternating links of a given determinant.*
2. *There are only finitely many values of the Jones polynomial of alternating links of a given determinant.*
3. *There are only finitely many alternating links of a given Jones polynomial.*
4. *There are only finitely many alternating links of a given breadth of the Jones polynomial.*

Proof.

1. The given determinant is a positive integer and we know that there are only finitely many alternating links of crossing number less than or equal to this positive integer and thus the result follows.
2. The result follows directly by the previous result.
3. The given Jones polynomial determines a value for determinant and thus the result follows directly by applying the first result.
4. The given breadth is a nonnegative integer and we know that there are only finitely many alternating links of crossing number equal to this nonnegative integer and thus the result follows.

□

Questions and Motivation of this Work

Question

Is it true that there are only finitely many quasi-alternating links of a given determinant?

Question

Is it true that there are only finitely many values of the Jones polynomial of quasi-alternating links of a given determinant?

Question

Is it true that there are only finitely many quasi-alternating links of a given Jones polynomial?

Question

Is it true that there are only finitely many quasi-alternating links of a given breadth of the Jones polynomial?

Conjectures

Conjecture (Green (2010))

There are only finitely many quasi-alternating links of a given determinant.

Conjecture (Q., Qublan and Jaradat (2013))

For any quasi-alternating link L, we have $c(L) \leq \det(L)$.

Remark

It is easy to see that establishing the second conjecture not only implies the first conjecture but also implies affirmative answers for the first three questions.

Remark

It has been proven by independent work of different people that there are only finitely many quasi-alternating links of determinant less than or equal to seven.

Theorem

There are only finitely many values of the Jones polynomial of quasi-alternating links of a given determinant.

Definition

The Jones polynomial $V_L(t)$ of an oriented link L is the Laurent polynomial in $t^{\pm 1/2}$ with integer coefficients defined by

$$V_L(t) = ((-A)^{-3w(L)} \langle L \rangle)_{t^{1/2}=A^{-2}} \in \mathbb{Z}[t^{-1/2}, t^{1/2}],$$

where $\langle L \rangle$ is the Kauffman bracket of the unoriented link and $w(L)$ is the writhe of the oriented link diagram defined by $w(L) = y(L) - x(L)$ with $x(L)$ and $y(L)$ denote the number of negative and positive crossings, respectively in L according to the scheme in the following Figure.

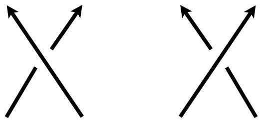


Figure: Negative and positive crossings respectively.

Lemma

The Jones polynomial of the link L at the crossing c and up to mirror image satisfies one of the following skein relations:

1. *If c is a positive crossing, then* $V_L(t) = -t^{\frac{1}{2}} V_{L_0}(t) - t^{\frac{3e}{2}+1} V_{L_1}(t)$.
2. *If c is a negative crossing, then* $V_L(t) = -t^{\frac{-3e}{2}-1} V_{L_0}(t) - t^{\frac{-1}{2}} V_{L_1}(t)$.

where e denotes the difference between the number of negative crossings in L_1 and the number of negative crossings in L.

Lemma

The absolute value of any coefficient of any quasi-alternating link is bounded above by the determinant.

Definition

We say that a polynomial has a gap of length s if this polynomial has two monomials t^i and t^{i+s+1} of nonzero coefficients and all the monomials t^m have zero coefficients for $m = i+1, i+2, \ldots, i+s$.

Lemma

If L is a quasi-alternating link, then the length of any gap in the Jones polynomial $V_L(t)$ is one.

Definition

For any link invariant, we define an equivalence relation in the class of quasi-alternating links by setting two quasi-alternating links to be equivalent in this invariant relation if they have the same value of this link invariant.

Proof of the Main Theorem.

We apply induction on the determinant of the given quasi-alternating link. If $\det(L) = 1$, then the result holds since the only quasi-alternating link of determinant one is the unknot. Now, suppose that the result holds when the determinant is less than or equal to n. In particular, there is a finite list $\{K_1, K_2, \ldots, K_l\}$ of Jones polynomial equivalence classes of quasi-alternating links of determinant less than or equal to n. In other words, if K is a quasi-alternating link of determinant less than or equal to n, then $V_K(t) = V_{K_i}(t)$ for some $1 \leq i \leq l$. The fact of L being quasi-alternating of determinant $n + 1$ implies that the links L_0 and L_1 have determinant less than $n + 1$. By a direct application of the induction hypothesis on the links L_0 and L_1, we have only finitely many values of $V_{L_0}(t)$ and $V_{L_1}(t)$ since L_0 and L_1 represent two Jones polynomial equivalence classes in the above list.

□

Continuation.

We let $m = \min\{\operatorname{mindeg} V_{K_i}(t) \mid 1 \leq i \leq l\}$ and $M = \max\{\operatorname{maxdeg} V_{K_i}(t) \mid 1 \leq i \leq l\}$ for the following two cases of the proof:

(1) For the first case, we have $V_L(t) = -t^{\frac{1}{2}} V_{L_0}(t) - t^{\frac{3e}{2}+1} V_{L_1}(t)$ with only finitely many values of the number e. Otherwise this will force consecutive coefficients of $V_L(t)$ to vanish and this contradicts with the result of one of the previous lemmas. This argument implies that there will be only finitely many values of $V_L(t)$ since $\operatorname{mindeg} V_L(t) \geq m - \left|\frac{3e}{2} + 1\right|$ and $\operatorname{maxdeg} V_L(t) \leq M + \left|\frac{3e}{2} + 1\right|$ and the absolute value of the coefficients of $V_L(t)$ are bounded above by $n + 1$ as a result of a previous lemma.

□

Continuation.

(2) For the second case, we have $V_L(t) = -t^{\frac{-3e}{2}-1}V_{L_0}(t) - t^{\frac{-1}{2}}V_{L_1}(t)$ with only finitely many values of the number e. Otherwise this will force consecutive coefficients of $V_L(t)$ to vanish and this contradicts with the result of one of the previous lemmas. This argument implies that there will be only finitely many values of $V_L(t)$ since mindeg $V_L(t) \geq m - \left|\frac{-3e}{2} - 1\right|$ and maxdeg $V_L(t) \leq M + \left|\frac{-3e}{2} - 1\right|$ and the absolute value of the coefficients of $V_L(t)$ are bounded above by $n + 1$ as a result of previous lemma. $\square$

Consequences of the Main Theorem

Corollary

There are only finitely many Jones polynomial equivalence classes of quasi-alternating links of determinant less than or equal to any positive integer. Moreover, we conclude that there are only finitely many Khovanov homology equivalence classes of quasi-alternating links of determinant less than or equal to any positive integer.

Proof.

There is only finitely many Jones polynomial equivalence classless of a given determinant as a result of the main theorem. The first result follows since there is only finitely many positive integers less than or equal this determinant. The second result follows directly since the graded Euler characteristic of the Khovanov homology is equal to the normalized Jones polynomial. □

Remark

One might wonder if every Jones polynomial equivalence classes of quasi-alternating links is a finite set. It is easy to see that some of them are finite. For example, the Jones polynomial equivalence classes of $1, -t^{\frac{-1}{2}} - t^{\frac{-5}{2}}, -t^{-4} + t^{-3} + t^{-1}, -t^{\frac{-9}{2}} - t^{\frac{-5}{2}} + t^{\frac{-3}{2}} - t^{\frac{-1}{2}}, t^{-2} - t^{-1} + 1 - t + t^{2}, -t^{-7} + t^{-6} - t^{-5} + t^{-4} + t^{-2}, -t^{\frac{-17}{2}} + t^{\frac{-15}{2}} - t^{\frac{-13}{2}} + t^{\frac{-11}{2}} - t^{\frac{-9}{2}} - t^{\frac{-5}{2}}, -t^{-10} + t^{-9} - t^{-8} + t^{-7} - t^{-6} + t^{-5} + t^{-3}$ *and* $-t^{-6} + t^{-5} - t^{-4} + 2t^{-3} - t^{-2} + t^{-1}$ *are all finite since there are only finitely many quasi-alternating links of determinant* $1, 2, 3, 4, 5, 6$ *and* 7*, respectively.*

Corollary

In the class of quasi-alternating links, every determinant equivalence class is finite iff every Jones polynomial equivalence class is finite.

Proof.

The first implication follows directly as a result of the fact that every Jones polynomial equivalence class consists of links of the same determinant. Now the second implication follows since each determinant equivalence class is a finite union of Jones polynomial equivalence classes as a result of having only finitely many values of the Jones polynomial of quasi-alternating links of a given determinant. □

Green's conjecture can be restated as follows:

Conjecture (Green)

There are only finitely many quasi-alternating links of a given Jones polynomial.

Corollary

In the class of quasi-alternating links, every breadth of the Jones polynomial equivalence class is finite implies every determinant equivalence class is finite.

Proof.

It is not too hard to see that every determinant equivalence class is a finite union of breadth of the Jones polynomial equivalence classes as a result of the fact that the breadth of the Jones polynomial is bounded above by twice of the determinant. □

Examples and Further Discussion

Example

Kanenobu constructed an infinite family of knots $K(p,q)$ for integers p and q. It is known that $\det(K(p,q)) = 25$ for any p and q and $V_{K(p,q)}(t) = (-t)^{p+q}((t^{-2} - t^{-1} + 1 - t + t^2)^2 - 1) + 1)$. Therefore, this implies that the result of main theorem does not hold in the class of all links. Moreover, we can conclude that questions number one and two have negative answers in the class of all links.

Example

It is clear that the subfamily of all Kanenobu knots $K(p, q)$ for $p > 0$ and $q < 0$ and fixed $p + q$ consists of knots of the same Jones polynomial. Also, it is not too hard to see that this family is infinite as a result of the known fact that $c(K(p, q)) = |p| + |q| + 7$ for any Kanenobu knot in this subfamily. This implies that question number three has a negative answer in the class of all links.

Example

It is easy to see the subfamily of all Kanenobu knots $K(p, q)$ for $p > 0$ and $q < 0$ and fixed $|p + q| \leq 4$ consists of knots of breadth eight of the Jones polynomial. Also, it is not too hard to see that this family is infinite as a result of the known fact that $c(K(p, q)) = |p| + |q| + 7$ for any Kanenobu knot in this subfamily. This implies that question number four has a negative answer in the class of all links.

Corollary

Let L be a link such that $V_{L_0}(t)$ and $V_{L_1}(t)$ are not both zero. If $\det(L_1) = 0$, *then the infinite family* $\{L^{\frac{1}{[n]}} \mid n \in \mathbb{N}\}$ *contains only finitely many quasi-alternating links. Also if* $\det(L_0) = 0$, *then the infinite family* $\{L^{[n]} \mid n \in \mathbb{N}\}$ *contains only finitely many quasi-alternating links.*

Proof.

It is easy to see that $\det(L^*) = \det(L)$ in either case as a result of the assumption. Now if the crossing c is positive, we use the first formula in Lemma 15 to show that

$$V_{L^{\frac{1}{[n]}}}(t) = \begin{cases} (-t^{\frac{1}{2}})^n V_{L_0}(t) - t^{\frac{3e}{2}+\frac{3}{2}} V_{L_1}(t) \sum_{i=1}^{n} \left((-t)^{i-1}\right), \text{if } n \text{ is even} \\ (-t^{\frac{1}{2}})^n V_{L_0}(t) - t^{\frac{3e}{2}+2} V_{L_1}(t) \sum_{i=1}^{n} \left((-t)^{i-1}\right), \text{if } n \text{ is odd.} \end{cases}$$

This formula shows that $V_{L^{\frac{1}{[i]}}}(t) \neq V_{L^{\frac{1}{[j]}}}(t)$ for $i \neq j$. This infinite family of links consists of links of the same determinant with infinitely many values of the Jones polynomial. As a consequence of the main theorem, only finitely many links of this family are quasi-alternating. The case if the crossing is negative can be treated in a similar manner. A similar argument can be used also to prove that the second set contains only finitely many quasi-alternating links. □

Lemma

Let L be a quasi-alternating link of $\text{breadth}(V_L(t)) \leq 3$, *then L is the unknot or the Hopf link or the trefoil knot.*

Proof.

We prove this result by induction on the determinant of the link L. It is known that the result holds if the determinant is one since the only quasi-alternating link of determinant one is the unknot as a consequence of the definition. Now we assume that the result holds for any quasi-alternating link of determinant less than the determinant of the link L. We assume that the result holds for the quasi-alternating links L_0 and L_1 of determinant less than the determinant of L. □

Continuation of the Proof.

It is easy to see that there is no cancellation in the two terms in the formulas for the skein relations for $V_L(t)$ in the case of L being quasi-alternating link at the crossing c as a result of the facts that the polynomials $V_L(t), V_{L_0}(t)$ and $V_{L_1}(t)$ are all alternating, $\det(L) = \det(L_0) + \det(L_1)$ and $\det(L) = |V_L(-1)|$. Thus According to the formulas, we must have $\text{breadth}(V_{L_0}(t)) \leq 3$ and $\text{breadth}(V_{L_1}(t)) \leq 3$. Also, the set of monomials of nonzero coefficients in the two terms in these formulas must be either equal or one of them is a subset of the other or disjoint with the maximum difference in their degrees less than or equal to three otherwise we have $\text{breadth}(V_L(t)) > 3$. This implies from the induction hypothesis that L_0 is either the unknot, the Hopf link or the trefoil knot and the same applies for the link L_1. Therefore, we obtain $\det(L) = \det(L_0) + \det(L_1) \leq 6$. It is known that there are only finitely many quasi-alternating links of determinant less than or equal to six and among them only the unknot, the Hopf link or the trefoil has breadth less than or equal to three. □

We like to enclose this talk by the following conjecture

Conjecture

There are only finitely many quasi-alternating links of a given breadth of the Jones polynomial.

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Thank You