Plane-wave limits and homogeneous M-theory backgrounds

Simon Philip

Doctor of Philosophy
University of Edinburgh
2005

Abstract

In this thesis we study plane-wave limits and M-theory vacua. We consider several hereditary properties of the plane-wave limit but focus on that of homogeneity. We show that a sufficient condition for a plane-wave limit along a particular geodesic of any spacetime to be homogeneous is that the geodesic be homogeneous. On reductive homogeneous spacetimes we reduce the calculation to a set of algebraic formulae by two different methods; the first uses the covariant description of the plane-wave limit [Blau,O'Loughlin,Papadopoulos. JHEP,01:047,2002] and the second employs a non-adapted coordinate description of the plane-wave limit. We study how the homogeneous structure on a reductive homogeneous spacetime behaves under the plane-wave limit and apply our formulae to many relevant examples.

We then consider supersymmetric M-theory vacua and the Lie supersymmetry superalgebra on these backgrounds. We show that those backgrounds which preserve more than 24 of the supersymmetries are necessarily homogeneous and provide some evidence that this boundary is sharp. The symmetric square of the spinor bundle of an 11-dimensional spacetime is isomorphic to a particular bundle of differential forms, this can be used to interpret Killing spinors as differential forms satisfying a system of first order equations [Gauntlett,Gutowski,Pakis. JHEP,12:049,2003]. We use this technique to investigate both the geometric and algebraic nature of the 24+ supergravity solutions, in particular those which are plane-waves. Finally we consider some more general homogeneous supergravity solutions, including homogeneous 5-dimensional supergravity.

Declaration

I declare that this thesis was composed by myself and that the work therein is my own except where explicitly stated otherwise in the text.

The work leading to this thesis was carried out in the School of Mathematics at the University of Edinburgh, between October 2001 and June 2005.

Some of the work in this thesis has been developed in collaboration with J. M. Figueroa-O'Farrill and P. Meessen and reported in both *Supersymmetry and homogeneity of M-theory backgrounds*, Class.Quant.Grav.**22** (2005) 207-226, and *Homogeneity and plane-wave limits*, to appear in JHEP (2005). Also, some of the work has been presented in the article *Penrose limits of homogeneous spaces* that has been submitted to J.Geom.Phys (2005).

The work has not been submitted for any other degree or professional qualification other than those journal submissions detailed above.

16/9/05

(*Simon Philip*)

Acknowledgements

First of all, I owe a huge debt to my supervisor José Figueroa-O'Farrill, whose patience and encouragement made this PhD possible. I have enjoyed many interesting and helpful discussions with him, and I owe much of my understanding of the topics in this thesis to him. I would particularly like to thank him for his insight and ideas, and for pushing me in the right directions.

I am deeply grateful to Patrick Meessen for many helpful discussions during our collaboration. I would also like to thank Michael Singer, David Calderbank, Harry Braden, José Antonio Oubiñia and Hannu Rajaniemi for useful conversations, and their generosity in educating me.

I am grateful for the financial support of EPSRC.

Finally, many thanks to my parents Barbara and John Philip, and to other family and friends; in particular Teresa Mayer for putting up with me for most of the last four years.

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Chapter 1

Introduction

The principal objects of study of this thesis are M-theory backgrounds with (super)symmetries, and in particular homogeneous backgrounds. In this introduction we shall endeavor to explain the numerous reasons for studying such backgrounds, and also place them in context of M-theory in general.

1.1 Supergravity and supersymmetry

Since the mid-nineties, evidence has been accumulating for the existence of an 11-dimensional quantum theory, called **M-theory**, which underlies all the known 10-dimensional string theories. The low energy limit of M-theory, when energy levels are way below the string scale of 10^{19} GeV, is a classical theory called **11-dimensional supergravity**. In this sense, we can identify M-theory backgrounds with **11-dimensional supergravity** solutions. Discovered [1, 2] in 1976, 11-dimensional supergravity is fundamentally Einstein's theory of gravity together with a non-linear generalisation of Maxwell's theory of electromagnetism in an 11-dimensional spacetime, and incorporates both the Kaluza-Klein idea of gravity theories in dimensions higher than 4 and supersymmetry. The data for an 11-dimensional supergravity bosonic background is a triple (M, g, F) where M is an 11-dimensional lorentzian spin manifold with metric g and F is a 4-form subject to the following field equations:

$$\text{Maxwell's equations:} \quad \begin{cases} dF &= 0 \\ d * F &= F \wedge F \end{cases} \tag{1.1.1}$$

$$\text{Einstein's equations:} \quad \mathrm{Ric}_{ij} = \frac{1}{2}F^2_{ij} + \frac{1}{2}(s - \frac{1}{2}|F|^2)g_{ij} \ , \tag{1.1.2}$$

where $F^2_{ij} = \frac{1}{6}F_{ipqr}F_j^{pqr}$ and $|F|^2 = \frac{1}{24}F_{ijkl}F^{ijkl}$. Here Ric and s are the Ricci and scalar curvatures respectively and we are using the Einstein summation convention. Notice that if we take the trace of equation (1.1.2) we find

$$s = \frac{1}{6}|F|^2 \ .$$

It follows that if $F = 0$, so that the background is purely gravitational, then it must be Ricci flat.

Writing $F = dA$ (locally) for some 3-form A, the action of such a theory is given by

$$\int_M \left(\frac{1}{2} s \text{ vol} - \frac{1}{4} F \wedge *F + \frac{1}{12} F \wedge F \wedge dF \right)$$

where vol is the signed volume form:

$$\text{vol} = \sqrt{\det g}\; dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{10}.$$

There are many supergravity theories in dimensions lower than 11. The well known type IIA supergravity, which is a dimensional reduction of 11-dimensional supergravity, and IIB supergravity theories. Also, various other theories in dimensions 4 to 9, such as the 5 and 6 dimensional supergravity theories (see for example [3] and [4].)

The full 11-dimensional supergravity theory has a bosonic sector which comprises of the dynamical fields g and A where $F = dA$, and a fermionic sector which contains the gravitino Ψ: a section of the tensor product of the spinor and cotangent bundles $\mathcal{S} \otimes T^*M$. Infinitesimal supersymmetry variation of the gravitino with respect to a spinor ε defines a super covariant derivative $\mathcal{D} : \Gamma(\mathcal{S}) \longrightarrow \Gamma(T^*M \otimes \mathcal{S})$,

$$\delta_\varepsilon \Psi_X = \mathcal{D}_X \varepsilon \ ,$$

which we may expand in terms of the Levi-Cività connection and F,

$$\mathcal{D}_X = \nabla_X + \frac{1}{6} \iota_X F + \frac{1}{12} X^\flat \wedge F \ . \tag{1.1.3}$$

For a bosonic background we set the fermionic sector to zero and require that this is preserved by a supersymmetry transformation. Thus the geometric realization of supersymmetry on (M, g, F) is the existence of **Killing spinors**, that is, the existence of at least one spinor ε which is parallel with respect to $\mathcal{D}$. Each Killing spinor is completely determined by its value at a point p, for then parallel transport determines its value everywhere else.

An important invariant of the theory is the amount of supersymmetry, or to be more precise, the number of linearly independent Killing spinors. This number is usually recorded as a fraction ν of the maximal number of Killing spinors, which is 32 for an 11-dimensional theory[1]. It offers two complementary refinements: the holonomy representation of the super covariant derivative $\mathcal{D}$ on the one hand, and the supersymmetry superalgebra on the other. The fraction ν can be recovered

[1]However, we reserve the right to give the amount of supersymmetry in either form; as a fraction ν or as the integer number of linearly independent Killing spinors

as the dimension of the invariant subspace in the holonomy representation, or the dimension of the odd subspace of the superalgebra.

The table below summarizes some of the known supergravity backgrounds with $\nu \geq \frac{1}{2}$.

ν	M-theory background
$\frac{1}{2}$	generic M-wave [5], M-branes [6, 7], Kaluza-Klein monopole [8, 9, 10]
$\frac{9}{16}$ to $\frac{3}{4}$	discrete cyclic quotients of $AdS_4 \times S^7$[11], Gödel type backgrounds [3, 12], plane-waves both symmetric [13, 14, 15, 16] and time dependent [17]
$\frac{13}{16}$	symmetric plane-wave [18]
1	flat, $AdS_4 \times S^7$ and $AdS_7 \times S^4$[19], Kowalski-Glikman wave [20, 21]

(1.1.4)

As the table shows, other than the maximal ones, there are not many solutions known with $\nu > \frac{3}{4}$, and no known solutions in the region $\frac{27}{32} < \nu < \frac{31}{32}$. At the time of writing it is not known whether all the possible fractions $0, \frac{1}{32}, \ldots, \frac{31}{32}, 1$ can occur.

Supersymmetry is a strong constraint on the geometry of a background, which is a considerable help when solving the field equations. Indeed, the known supergravity classification results make use of extra symmetries imposed on the backgrounds, such as supersymmetries or isometries. For example, the classification given in [22] of the maximally supersymmetric solutions:

Theorem 1.1.1. *([22]) Let (M, g, F) be a maximally supersymmetric solution of 11-dimensional supergravity. Then it is locally isometric to one of the following:*

1. $AdS_7(-7s) \times S^4(8s)$

   with $F = \sqrt{6s}\ vol(S^4)$ *where* $s > 0$ *is the scalar curvature of* M*,*

2. $AdS_4(8s) \times S^7(-7s)$

   with $F = \sqrt{6s}\ vol(AdS_4)$ *where* $s < 0$ *is the scalar curvature of* M*,*

3. $CW_{11}(H)$

   with $H = -\frac{\mu^2}{36} diag\ (4,4,4,1,1,1,1,1,1)$ *and* $F = \mu\ dx^- \wedge dx^1 \wedge dx^2 \wedge dx^3$.

Above, $\mathrm{AdS}_n(s)$ and $S^n(s)$ denote the n-dimensional anti de-Sitter space and sphere of constant scalar curvature s respectively, and $CW_{11}(H)$ is the Cahen-Wallach ([23]) symmetric plane-wave with metric

$$2dx^+dx^- + \left(\sum_{i,j=1}^{n-2} H_{ij}x^ix^j\right)(dx^-)^2 + \sum_{i=1}^{n-2}(dx^i)^2, \tag{1.1.5}$$

where $H = (H_{ij})$ is a constant symmetric bilinear form.

Other classifications include the 11-dimensional Freund-Rubin solutions where the 4-dimensional factor is anti de-Sitter space and the 7-dimensional factor is homogeneous [24, 25] and, in dimensions 5 and 6, the recent classifications of supersymmetric solutions in [3] and [4] respectively.

1.2 Plane-waves

Much of our understanding of closed string theory is based on a few particular examples of exactly solvable models. Solvability in this context means that it is possible to find solutions to the classical string equations explicitly, perform a canonical quantization, determine the spectrum of the Hamiltonian operator and possibly compute some of the simplest scattering amplitudes. Many of these models are found by specifying a closed string theory on a background geometry together with p-form field strengths and a dilaton. In superstring theory, these backgrounds usually define a solution to some supergravity theory carrying a large fraction of supersymmetry. Broadly speaking, there are three classes of examples of exactly solvable models:

- Strings on flat space and its various orbifolds, as well as models related to flat space by T-duality transformations.
- Strings on WZW models and their orbifolds.
- Strings on plane-wave backgrounds.

The large fraction supersymmetry carried by such models is reflected in the number of 11-dimensional supergravity plane-wave backgrounds listed in table (1.1.4).

Plane-wave metrics are a special subclass of the pp-wave metrics which carry a parallel null vector field. The generic plane-wave metric is of the form

$$dx^+dx^- + A(x^+)(\boldsymbol{x},\boldsymbol{x})(dx^+)^2 + |d\boldsymbol{x}|^2 \ , \tag{1.2.1}$$

where $A(x^+)$ is a symmetric bilinear form, but there are refinements to special subclasses including homogeneous and symmetric plane-waves.

In [26] Penrose introduced a method for taking a continuous limit of any spacetime to a plane wave. The method effectively involves "zooming in" on a null geodesic in such a way that the metric stays nondegenerate. In [27] Güven extended the method to that of supergravity theories where it is a useful tool for generating new solutions to the supergravity equations from known ones. Since then several papers have investigated the properties of these plane-wave limits, [28, 29, 30, 17, 31].

Plane-wave limits have been used as evidence for the celebrated $AdS/\ CFT$ correspondence. The plane-wave limits of the $AdS_5 \times S^5$ type IIB superstring background were calculated in [30], one of which was shown to be the BFHP maximally supersymmetric plane wave background [29]. String theory in this background is exactly solvable [32, 33] giving rise to an explicit form of the $AdS/\ CFT$ correspondence [34] in which both the gauge theory and the gravity sides are weakly coupled, allowing many perturbative checks albeit for a restricted class of observables.

It has been shown [30] that the fraction of supersymmetry preserved by a solution never decreases under the plane-wave limit. In particular, the plane-wave limit of a maximally supersymmetric solution is a maximally supersymmetric plane-wave. The plane-wave limits of the maximally supersymmetric Freund-Rubin type solutions $AdS_4 \times S^7$ and $AdS_7 \times S^4$ have also been calculated in [30], and are found to be either flat Minkowski spacetime or the Kowalski-Glikman solution KG, depending on the null geodesic chosen. These limits fit into a commutative diagram:

$$\begin{tikzcd}
AdS_4 \times S^7 \arrow[r] \arrow[dr] & KG \arrow[d] & AdS_7 \times S^4 \arrow[l] \arrow[dl] \\
& \text{Flat} &
\end{tikzcd} \tag{1.2.2}$$

which displays all of the maximally supersymmetric solutions to 11-dimensional supergravity.

Similarly, the dimension of the isometry algebra never decreases under the plane-wave limit. Given this, one may postulate that the plane-wave limit of a homogeneous background is always a homogeneous plane-wave. Indeed, plane-wave limits onto homogeneous plane waves have been investigated, such as the plane-wave limits of the Gödel-like spacetimes [17]. However, in [31] it was shown that the riemannian product of the homogeneous Kaigorodov spacetime with the sphere has a plane-wave limit which is not itself homogeneous. We shall show that a sufficient condition for a plane-wave limit to be homogeneous is that the geodesic along which the limit is taken is a **homogeneous geodesic**, that is the orbit of a one-parameter subgroup of isometries. On a reductive homogeneous space, we shall give algebraic formulae for the plane-wave limit along a homogeneous geodesic and give a necessary condition for the plane-wave limit to be homogeneous.

1.3 Homogeneous backgrounds

On a (reductive) homogeneous space, the geometry can be completely described by the value of the metric at a point. This allows one to reduce difficult to solve systems of differential equations, such the Einstein condition, to more tractable algebraic equations. This has clear benefits for the business of solving the supergravity equations of motion such as (1.1.1) and (1.1.2), so it is natural to consider **homogeneous supergravity** solutions; those solutions where knowledge of both the metric and field strength F at a point is enough to specify the background completely.

All of the $\nu > \frac{1}{2}$ solutions listed in table (1.1.4) are homogeneous. Of course, the existence of Killing spinors is not unrelated to the existence of Killing vectors; the spinor inner product induces a map

$$\xi : \mathcal{S} \odot \mathcal{S} \to TM \tag{1.3.1}$$

which maps Killing spinors to Killing vectors. This map can be extended to an isomorphism between the symmetric square of the spinor bundle and a bundle of differential forms. This allows one to write the Killing spinor equation as an equation on forms, something which has clear advantages. In particular, on a homogeneous space these equations become algebraic.

Therefore, it is natural to ask how much supersymmetry must be preserved to guarantee that the background is homogeneous. It is known [17] that plane-wave backgrounds with $\nu > \frac{1}{2}$ are necessarily homogenous. This and the lack of non-homogeneous solutions in (1.1.4) with $\frac{1}{2} < \nu < \frac{13}{16}$ make it tempting to conjecture that a background with more than half of the supersymmetries will be homogeneous. However, we shall show that in fact we need $\nu > \frac{3}{4}$ to guarantee homogeneity and give evidence to suggest that there are backgrounds with $\nu = \frac{3}{4}$ which are not homogeneous.

Similarly, all of the solutions in table (1.1.4) with $\nu > \frac{5}{8}$ are symmetric. We will use the formulation of the Killing spinor equation in terms of differential forms together with symplectic linear algebra to show that plane-wave backgrounds with ν greater than $\frac{3}{4}$ are symmetric.

Since $\nu > \frac{1}{2}$ plane-waves are homogeneous, preservation of supersymmetries under the plane-wave limit means that the plane-wave limit of a $\nu > \frac{1}{2}$ background is necessarily homogeneous. In particular this is of interest for the $\nu > \frac{3}{4}$ backgrounds because, as mentioned above, homogeneity is not necessarily inherited by the plane-wave limit. This provides a potential method for studying the $\nu > \frac{3}{4}$ backgrounds.

1.4 Thesis outline

In chapter 2 we give the background needed on homogeneous spaces, homogeneous structures and Killing vectors. We consider homogeneous geodesics, lorentzian g.o. spaces and review a lorentzian version of Kaplan's 6-dimensional g.o. space. We also examine Komrakov's classification of 4-dimensional pseudo-riemannian homogeneous spaces, in particular those that are lorentzian which we list in appendix B.

In chapter 3 we give some background on plane-wave metrics and plane-wave limits. Then we consider some hereditary properties of plane-wave limits including Güven's extension to supergravity and some submanifold geometry. The chapter is finished with some examples, including the Hamilton-Jacobi method for taking the plane-wave limit.

In chapter 4 we consider plane-wave limits onto homogeneous plane-waves and, in particular, along homogeneous geodesics. We give two derivations of algebraic formulae for calculating such plane-wave limits, the first uses the covariant description of the plane-wave limit [28] and the second employs a non-adapted coordinate system description of the plane-wave limit. We also examine the type of homogeneity inherited by the limiting metric under special circumstances. We conclude the chapter by applying these formulae to several examples, including the Kaigorodov space, Gödel like universes and Kaplan's g.o. space.

In chapter 5 we start by constructing the supersymmetry superalgebra and illustrate with some examples. We then examine the isomorphism induced by squaring spinors to construct differential forms, and use it to write down a curvature formula. We end the chapter by calculating the amount of supersymmetry required to guarantee homogeneity and provide some evidence that the bound we discover is sharp.

Chapter 6 contains a discussion of homogeneous supergravity, in particular invariant forms. We take a closer look at homogeneous plane-wave backgrounds and supersymmetries on them, and calculate how much supersymmetry guarantees that the plane-wave is symmetric. Then we look at homogeneous five and six dimensional supergravity theories whose Maxwell forms are constructed from homogeneous structures.

In appendix A we discuss geometric Killing spinors and repeat some of the constructions of chapter 5 for them. Appendix B contains the aforementioned table of 4-dimensional lorentzian homogeneous spaces from Komrakov's classification.

1.5 Notation

Most of the notation used in this thesis will be explained at point of use, with earlier explanations either referred to or repeated if notation is used in different sections/chapters. However, there is some notation and conventions that we will use consistently which we shall make clear now.

- $\bigwedge^k T^*M$: the bundle of differential k-forms, sometimes shortened to $\wedge^k$.
- ∇: the Levi-Cività connection.
- $\mathcal{L}$: the Lie derivative.
- We will sometimes abbreviate vector fields $\frac{\partial}{\partial u}$ to ∂_u.
- We shall use the Einstein summation convention unless stated otherwise; for example $g^{ij}e_i = \sum_i g^{ij}e_i$.
- Unless stated otherwise, all manifolds have dimension n.
- We will denote sets of coordinates or vectors such as $(y_1, \dots, y_n)$ by $\boldsymbol{y}$.
- We call an orthogonal basis $e_1, \dots, e_n$ with $|e_i|^2 = 1$ for $i = 1, \dots, p$ and $|e_i|^2 = -1$ for $i = p+1, \dots, n$ a **pseudo-orthonormal** basis.
- We call a basis $e_+, e_-, e_1, \dots e_{n_2}$ with $e_1, \dots e_{n_2}$ orthonormal and orthogonal to e_+, e_-, and with e_+, e_- null and $\langle e_+, e_- \rangle = 1$ a **lightcone-orthonormal** basis.

Chapter 2

Homogeneous spaces

In this chapter we will give the definitions and results we need in relation to homogeneous spaces: Killing vector fields, homogeneous structures, homogeneous geodesics and g.o. spaces. When dealing with supergravity, most of the time we are happy to restrict ourselves to studying local solutions; that is data (U, g, F) where U is an open neighborhood and we can ignore global topological issues. For this reason, after a comparison of the global and local versions of homogeneity, we shall focus on results of particular relevance to local homogeneity. But first we shall take a brief look at Killing vectors and the Killing transport.

2.1 Killing vectors and Killing transport

Let X be a vector field on a connected pseudo-riemannian manifold (M, g). Define $A_X : TM \to TM$ by

$$A_X(Y) = -\nabla_Y X \ . \tag{2.1.1}$$

Then X is a **Killing vector** if A_X is skew-symmetric with respect to g. As is well-known, a vector field X is Killing if and only if $\mathfrak{L}_X g = 0$, which shows that Killing vectors are infinitesimal generators of isometries. Each Killing vector ξ satisfies **Killing's identity**:

$$\nabla_X A_\xi = R(X, \xi) \tag{2.1.2}$$

where

$$R(X,Y)Z = \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z \ .$$

Proof. Using the identity

$$\begin{aligned}(\nabla_X A_\xi)Y &= \nabla_X A_\xi Y - A_\xi \nabla_X Y \\ &= -\nabla_X \nabla_Y \xi + \nabla_{\nabla_X Y}\xi \ ,\end{aligned}$$

we have the following equality

$$
\begin{aligned}
(\nabla_X A_\xi)Y - (\nabla_Y A_\xi)X &= -\nabla_X\nabla_Y\xi + \nabla_{\nabla_X Y}\xi + \nabla_Y\nabla_X\xi - \nabla_{\nabla_Y X}\xi \\
&= -\nabla_X\nabla_Y\xi + \nabla_Y\nabla_X\xi + \nabla_{[X,Y]}\xi \\
&= R(X,Y)\xi \\
&= R(X,\xi)Y - R(Y,\xi)X \ ,
\end{aligned}
$$

where we have used the algebraic Bianchi identity

$$
R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \ . \tag{2.1.3}
$$

This shows that

$$
(\nabla_X A_\xi)(Y) - R(X,\xi)Y
$$

is symmetric in $X \leftrightarrow Y$. On the other hand

$$
g((\nabla_X A_\xi)(Y) - R(X,\xi)Y, Z) = -g((\nabla_X A_\xi)(Z) - R(X,\xi)Z, Y) \ .
$$

whence $(\nabla_X A_\xi)(Y) - R(X,\xi)Y = 0$. □

Consider the bundle

$$
\mathcal{E} \doteq TM \oplus \mathfrak{so}(TM) \ ,
$$

where $\mathfrak{so}(TM)$ is the bundle of skew-symmetric endomorphisms (relative to g) of the tangent bundle. If we define a covariant derivative D on $\mathcal{E}$ by

$$
D_X(\xi, A) := (\nabla_X\xi + A(X), \nabla_X A + R(X,\xi)) \ ,
$$

then the parallel sections of $\mathcal{E}$ with respect to D are precisely the Killing vectors of g. Thus a Killing vector is completely determined by its value at an initial point and also the value of its first derivative:

$$
(\zeta(p), A_\xi(p)) \ ,
$$

with the full Killing vector given by parallel transport by the covariant derivative D. We call parallel transport with respect to D **Killing transport** and shall see in the next section that this has a natural generalisation on reductive homogeneous spaces.

Let $\mathfrak{k}$ denote the space of parallel sections of $\mathcal{E}$ with respect to D. Then the Lie bracket on $\mathfrak{k}$ inherited from the Lie bracket of Killing vectors is

$$
[(\xi, A), (\eta, B)] = (A\eta - B\xi, [A,B] + R(\xi,\eta)) \ . \tag{2.1.4}
$$

Proof. By definition we have

$$[(\xi, A), (\eta, B)] = ([\xi, \eta], -\nabla[\xi, \eta]) \; .$$

Using the torsion free property of ∇ and the definitions of A and B we have

$$[\xi, \eta] = \nabla_\eta \xi - \nabla_\xi \eta = A\eta - B\xi \; .$$

Similarly

$$\begin{aligned} -\nabla_X[\xi, \eta] &= -\nabla_X(A\eta - B\xi) \\ &= -(\nabla_X A)\eta - A\nabla_X \eta + (\nabla_X B)\xi + B\nabla_X \xi \\ &= -R(\xi, X)\eta + ABX + R(\eta, X)\xi - BAX \\ &= [A, B]X + R(\xi, \eta)X \; , \end{aligned}$$

where we have used Killing's identity (2.1.2) and the algebraic Bianchi identity (2.1.3). □

Now the bundle $\mathcal{E}$ has a natural Lie bracket given by

$$[(\xi, A), (\eta, B)]_{\mathcal{E}} = ([\xi, \eta], [A, B]) \; . \tag{2.1.5}$$

Thus the curvature $R(\xi, \eta)$ measures the failure of $[-,-]_{\mathcal{E}}$ to agree with the Lie bracket on $\mathfrak{t}$. The bracket on $\mathfrak{t}$ extends to arbitrary sections of $\mathcal{E}$, but the Jacobi identity will fail precisely because of the curvature term.

2.2 Reductive homogeneous spaces

Let $\mathrm{Iso}(M, g)$ be the group of isometries of the pseudo-riemannian space (M, g).

Definition 2.2.1. A connected lorentzian space (M, g) is **homogeneous** if its group of isometries acts transitively on M. That is, for every pair of points $p, q \in M$ there exists an isometry $h \in \mathrm{Iso}(M, g)$ such that $q = h \cdot p$.

If $G \subset \mathrm{Iso}(M, g)$ is a subgroup which acts transitively on M, then the map

$$\phi_o : G \to M \; , \tag{2.2.1}$$

which sends an isometry $g \in G$ to the point $g \cdot o \in M$, for some fixed point $o \in M$, is a surjection. The subgroup $H \subset G$ which fixes the point o is called the **isotropy subgroup** of o. The map ϕ_o induces a diffeomorphism $M \cong G/H$.

The local version of this is

Definition 2.2.2. A lorentzian space (M, g) is **locally homogeneous** if for every pair of points $p, q \in M$, there exist neighborhoods U, V of p and q respectively and a local isometry $h : U \to V$ such that $q = h \cdot p$.

As mentioned above, in supergravity we usually work with local metrics. In this context, the relevant concept is that of **local transitivity** rather than homogeneity. We say (M, g) is locally transitive if every point $p \in M$ has a neighborhood U such that for all $q \in U$ there exists a local isometry h with $q = h \cdot p$. That is to say that the neighborhood U is locally homogeneous. Folklore arguments (see for example page 237 in [35]) show that this implies the existence, at any point p, of a set of n Killing vectors $\{X_i\}$ such that the vectors $\{X_i(p)\}$ form a basis for T_pM. For the converse, given a finite set of Killing vectors on an open neighborhood U of p we may exponentiate at the point p to obtain the action of a Lie group G_p. The orbit of p under G_p is locally of the same dimension as M, and thus contains a subneighborhood U' of p. Therefore, since M is connected, G must act transitively on U. Local transitivity is clearly implied by local homogeneity, and is in fact equivalent to it:

Proof. Since M is connected there exists a continuous path $\gamma : \mathrm{I} \to M$ from any point p to any point q. Any point $\gamma(t)$ has a neighborhood $U(t)$ such that for any point $r \in U(t)$ there is a local isometry taking $\gamma(t)$ to r. The sets $U(t) \cap \gamma(\mathrm{I})$ form an open cover of $\gamma(\mathrm{I})$ and thus, by pulling back to the interval via γ, we have an open cover of I, namely $V(t) = \gamma^{-1}(U(t) \cap \gamma(\mathrm{I}))$. The interval is compact, so we can obtain a finite subcover $V_i = V(t_i)$ where $\{0 = t_0 < t_1 < \cdots < t_N = 1\}$ is some partition of the interval I such that $V_i \cap V_{i+1} \neq \emptyset$. Choose $r_i \in \gamma^{-1}(V_{i-1}) \cap \gamma^{-1}(V_i)$. By definition there exist local isometries f_i and h_i such that $r_{i+1} = h_i \cdot \gamma(t_i)$ and $r_i = f_i \cdot \gamma(t_i)$. The desired local isometry is given by

$$\psi := f_N^{-1} \circ h_{N-1} \cdots \circ f_2^{-1} \circ h_1 \circ f_1^{-1} \circ h_0 \ .$$

Finally let V be an open neighborhood of q so that $U = \psi^{-1}(V)$ is defined. The open set U is a neighborhood of p and clearly $\psi : U \to V$. $\square$

The crucial difference between local transitivity and global homogeneity is that locally transitive metrics need not be complete. For example, the sphere S^2 is a homogeneous space; however, if we remove the north pole p then $S^2 \backslash p$ is only locally homogeneous. The isometries which are defined on the whole of $S^2 \backslash p$ are those of the sphere which fix p and have orbits given by parallels to the equator.

Differentiating the map ϕ_o we obtain a linear map

$$d\phi_0 : \mathfrak{g} \to T_oM \ , \tag{2.2.2}$$

where $\mathfrak{g}$ is the Lie algebra of G. The kernel of this map is the Lie algebra $\mathfrak{h}$ and thus forms part of an exact sequence

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \xrightarrow{d\phi_o} T_oM \longrightarrow 0 \,. \tag{2.2.3}$$

This is an exact sequence of H-modules where H acts on $\mathfrak{h}$ and $\mathfrak{g}$ via the adjoint representation and on T_oM via the **linear isotropy representation**:

Definition 2.2.3. Let o denote the coset of H in M and fix a frame $u_0 : \mathbb{R}^n \to T_oM$ of the frame bundle F. Define the **linear isotropy representation** $\rho : H \to GL(n, \mathbb{R})$ by

$$\rho(h) := u_o^{-1} \circ h_* \circ u_o \,, \tag{2.2.4}$$

where $h \in H$, $h_* : T_oM \to T_oM$ denotes the differential of h at o.

The metric g defines an inner product $\langle -, - \rangle$ on T_oM. Invariance of g under G is equivalent to invariance of $\langle -, - \rangle$ under H, whence the linear isotropy representation is a Lie algebra homomorphism $\rho : H \to \mathfrak{so}(n, \mathbb{R})$.

We can give an explicit formulation of the isotropy representation by taking a complement $\mathfrak{m}$ to $\mathfrak{h}$ in $\mathfrak{g}$, so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathfrak{m}$ is isomorphic to T_oM. Then the isotropy representation is given by

$$\rho(h)X = (\mathrm{Ad}\,(h)X)_{\mathfrak{m}} \qquad \text{for } X \in \mathfrak{m} \,, \tag{2.2.5}$$

where the subscript $\mathfrak{m}$ means projection to the subspace $\mathfrak{m}$ and where the identification $\mathfrak{m} \cong T_oM \cong \mathbb{R}^n$ is implicit.

Definition 2.2.4. A pair $(\mathfrak{g}, \mathfrak{h})$ of a Lie algebra and subalgebra is **reductive**[1] when there exists a subspace $\mathfrak{m} \cong T_pM \subset \mathfrak{g}$ such that

1. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$,
2. $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.

This is equivalent to $\mathfrak{m}$ being stable under the isotropy representation (and also to the splitting of the exact sequence (2.2.3) in the homological sense.)

We will often say that a space is "reductive", but this is an abuse of notation as reductivity is not a geometric property of (M, g) but of the linear isotropy representation and hence of the description of M as a coset space G/H. A space (M, g) may admit different coset descriptions G_1/H_1 and G_2/H_2, one of which

[1] Strictly speaking this means that G/H is **weakly reductive**, with reductive reserved for those splits $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ such that $\mathfrak{m}$ is stable under the action of Ad (H) rather than ad $(\mathfrak{h})$. However if H is connected, which we shall assume for the remainder of this thesis, then they are the same thing.

is reductive and the other is not. For example, the Kaigorodov space which we will consider in section 4.4.2. For this space the coset presentation G/H given by taking $G = \mathrm{Iso}(M, g)$ is not reductive, but it does admit a subgroup G' such that G'/H' is reductive. Nevertheless, we will say that a homogeneous pseudo-riemannian space (M, g) is **reductive** if there exists a transitively acting subgroup of isometries G, with isotropy H for which the pair $(\mathfrak{g}, \mathfrak{h})$ is reductive.

It was shown in [36] that a necessary and sufficient condition for a coset presentation G/H of a pseudo-riemannian homogeneous space (M, g) to be reductive is that the restriction to $\mathfrak{h}$ of the Cartan-Killing form K for $\mathfrak{g}$ is non-degenerate. If (e_i) is an orthonormal frame for TM, then we can write K as

$$K_p(X, Y) = -\sum_i g(A_X \circ A_Y(e_i), e_i)|_p \tag{2.2.6}$$

where A_X is defined in equation (2.1.1). It is not difficult to see that this defines an Ad (H)-invariant inner product. So if we let $\mathfrak{m} = \mathfrak{h}^{\perp}$ be the perpendicular complement of $\mathfrak{h}$ then Ad (H)-invariance and non-degeneracy imply that this defines a reductive split. If g is riemannian then K is positive definite and therefore automatically non-degenerate when restricted to $\mathfrak{h}$, whence all coset presentations G/H for riemannian homogeneous spaces are reductive. However, if g has indefinite signature then reductivity is not an empty condition as illustrated by the Kaigorodov space.

A pair $(\mathfrak{g}, \mathfrak{h})$ is **symmetric** if it is reductive and also satisfies

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} \ . \tag{2.2.7}$$

This is equivalent to the existence of the usual symmetric space symmetry; that is an isometry $f_x : M \longrightarrow M$ which satisfies $f_x(x) = x$ and $d(f_x)_x = -I_{T_xM}$ for some $x \in M$.

The above definitions of reductive and symmetric spaces have generalisations to the locally transitive case. A locally transitive space is reductive if for all $p \in M$, there exists a coset description of the associated open neighborhood $U(p) = G/H$ which is reductive. Similarly, a locally homogeneous space is locally symmetric if there exists a coset description of each open neighborhood which is symmetric. This is equivalent to

$$\nabla R = 0 \ , \tag{2.2.8}$$

where R is the Riemann curvature tensor of g.

We can study the coset descriptions G/H of a space (M, g) by studying the G-invariant connections on M. For example, the Levi-Cività connection of g is invariant under the full isometry group $\mathrm{Iso}(M, g)$ of g. The following theorem gives a description of the space of G-invariant connections.

Theorem 2.2.5. *Let $\mathfrak{F}$ be the frame bundle of $M = G/H$ a reductive homogeneous space of dimension n with decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then there is a one-to-one correspondence between the set of G-invariant connections on $\mathfrak{F}$ and the set of linear maps $\Lambda_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{gl}(n, \mathbb{R})$ such that*

$$\Lambda_{\mathfrak{m}}(ad\ h(X)) = ad\ (\rho(h))(\Lambda_{\mathfrak{m}}(X)) \ , \tag{2.2.9}$$

for $X \in \mathfrak{m}$ and $h \in H$.

The correspondence is given by

$$\omega_{u_o}(\tilde{X}) = \begin{cases} \rho(X) & \text{if } X \in \mathfrak{h} \ , \\ \Lambda_{\mathfrak{m}}(X) & \text{if } X \in \mathfrak{m} \ , \end{cases} \tag{2.2.10}$$

where ω is the connection one-form, $\tilde{X}$ is the natural lift of $X \in \mathfrak{g}$ to $\mathfrak{F}$ and ρ is not only as above $H \to GL(n, \mathbb{R})$ but also the induced Lie algebra homomorphism $\mathfrak{h} \to \mathfrak{gl}(n, \mathbb{R})$.

Proof. See chapter X, Theorem 2.1 in [37]. □

Definition 2.2.6. The connection obtained by taking $\Lambda_{\mathfrak{m}} = 0$ is called the **canonical connection** .

The canonical connection can also be described in the following way. Let θ be the left-invariant Maurer-Cartan form of G

$$\theta_g(X) := (L_{g^{-1}})_*(X) \ , \tag{2.2.11}$$

where $L_{g^{-1}}$ denotes left multiplication by g^{-1} and $*$ denotes differentiation. Let $\sigma : U \to G$ be a local coset representative. Then the pull back of θ by σ splits as

$$\sigma^*(\theta) = \theta_{\mathfrak{h}} + \theta_{\mathfrak{m}} \ , \tag{2.2.12}$$

where $\theta_{\mathfrak{h}}(X) \in \mathfrak{h}$ and $\theta_{\mathfrak{m}}(X) \in \mathfrak{m}$ for all $X \in TM$. The one-form $\theta_{\mathfrak{h}}$ defines the connection one-form for the canonical connection.

The geodesics of the canonical connection are curves $\gamma(t)$ of the form

$$\exp(tX), \ t \in \mathbb{R}, X \in \mathfrak{g} \ . \tag{2.2.13}$$

For a globally homogeneous space this shows that the canonical connection is always geodesically complete, since the exponential is defined for all t.

The importance of the canonical connection can be seen from the following theorem whose original form is due to Ambrose-Singer.

Theorem 2.2.7 ([38, 39, 36]). *Let (M, g) be a pseudo-riemannian manifold with Levi-Cività connection ∇. Then (M, g) is locally reductive homogeneous if and only if there exists a $(2,1)$ tensor S defining a metric connection $\tilde{\nabla} := \nabla - S$ with curvature $\tilde{R}$ such that $\tilde{\nabla} S = \tilde{\nabla}\tilde{R} = 0$.*

Proof. Write $M = G/H$, with decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and canonical connection $\tilde{\nabla}$ and let $S = \nabla - \tilde{\nabla}$. As G acts by isometries we have ∇ is also G-invariant, whence S and $\tilde{R}$ are both G-invariant. Therefore, see [37], they are both parallel with respect to $\tilde{\nabla}$. For the converse see [37]. □

(The first version of this theorem for riemannian signature appeared in [38]. This was re-interpreted in terms of the canonical connection in [39] and extended to the pseudo-riemannian case in [36].)

By adding the hypothesis that M be connected and simply connected to theorem 2.2.7 we may replace *locally homogeneous* with *globally homogeneous.*

As the proof shows, the metric connection $\tilde{\nabla}$ in the theorem is the canonical connection defined above. The tensor S, which is called a **homogeneous structure**, is not necessarily the torsion of $\tilde{\nabla}$ (and not necessarily skew-symmetric in its lower indices.) Indeed, the torsion $\tilde{\tau}$ is given by the skew-symmetrization of S:

$$
\begin{aligned}
\tilde{\tau}(X,Y) &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] \\
&= \nabla_X Y - S_X Y - \nabla_Y X + S_Y X - [X,Y] \\
&= -S_X Y + S_Y X \ ,
\end{aligned}
\tag{2.2.14}
$$

since the Levi-Cività connection is torsion free. However, the theorem can be rewritten (due to Kostant [39]) in terms of the torsion of the canonical connection in the following way:

Theorem 2.2.8. *Let (M, g) be a pseudo-riemannian manifold. Then (M, g) is locally reductive homogeneous if and only if there exists a complete affine metric connection $\tilde{\nabla}$ with torsion τ and curvature $\tilde{R}$ such that $\tilde{\nabla}\tau = \tilde{\nabla}\tilde{R} = 0$.*

It is not difficult to write down the curvature of the canonical connection in terms of the curvature of Levi-Cività connection R and the homogeneous structure:

$$
\tilde{R}(X,Y)Z = R(X,Y)Z + [S_X, S_Y]Z - S_{S_X Y - S_Y X} Z \ , \tag{2.2.15}
$$

for $X, Y, Z \in TM$. This shows that we can in fact replace $\tilde{R}$ in both theorems 2.2.7 and 2.2.8 with R.

The Ambrose-Singer theorem is a generalisation of the locally symmetric condition (2.2.8), and also the promised *generalisation* of the Killing transport discussed in section 2.1. Like the Killing transport, the Ambrose-Singer theorem describes Killing vectors X by parallel transport along the geodesic curves of the canonical connection (2.2.13). It will play a pivotal role in our discussion of plane-wave limits and homogeneous supergravity.

2.3 Reductive homogeneous structures

Since both the canonical and Levi-Cività connections preserve the metric we have

$$g(S_X Y, Z) = -g(Y, S_X Z) \ , \tag{2.3.1}$$

whence $S : TM \to \mathfrak{so}(TM)$. Each such tensor is a section of the vector bundle $T^*M \otimes \mathfrak{so}(TM)$ associated to the orthonormal frame bundle. By using the metric, this can equivalently be thought of as the subbundle $\mathcal{T} = TM \otimes \Lambda^2 T^*M \subset \otimes^3 TM$ and S as the trilinear map

$$S(X, Y, Z) = g(S_X Y, Z) \ .$$

The bundle $\mathcal{T}$ splits up into the Whitney sum of three bundles

$$\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3 \ ,$$

each one corresponding to an irreducible representation of the orthogonal group. In terms of Young tableaux, this decomposition is given by

$$T^* \otimes \Lambda^2 T^* = \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$$
$$\ydiagram{1} \otimes \ydiagram{1,1} = \ydiagram{1} \oplus \ydiagram{2,1} \oplus \ydiagram{1,1,1} \ .$$

More explicitly, the bundles $\mathcal{T}_i$ can be described as follows[2]:

1. $\mathcal{T}_1 = \{S \in \mathcal{T} \mid S(X,Y,Z) = g(X,Y)\phi(Z) - g(X,Z)\phi(Y), \phi \in T^*M\}$.

2. $\mathcal{T}_2 = \{S \in \mathcal{T} \mid S(X,Y,Z) + S(Y,Z,X) + S(Z,X,Y) = 0 \text{ and } C_{12}(S) = 0\}$ where $C_{12} : \otimes^3 T^*M \to T^*M$ is defined by $C_{12}(S)(X) = \sum_i S(e_i, e_i, Z)$ where (e) is a pseudo-orthonormal frame.

3. $\mathcal{T}_3 = \{S \in \mathcal{T} \mid S(X,Y,Z) + S(Y,X,Z) = 0\}$.

It is possible to write down the explicit expressions for each of the components of S. We will write $S_{abc} = S(e_a, e_b, e_c)$ relative to a pseudo-orthonormal frame. Then

$$S_{abc} = S^{\ydiagram{1}}_{abc} + S^{\ydiagram{2,1}}_{abc} + S^{\ydiagram{1,1,1}}_{abc} \ ,$$

[2]We abuse notation slightly and identify the bundles $\mathcal{T}_i$ with their sheaves of sections, whence $S \in \mathcal{T}_i$ means that S is a section of $\mathcal{T}_i$, etc

where

$$
\begin{aligned}
S^{\square}_{abc} &= g_{ab}\xi_c - g_{ac}\xi_b \\
S^{\boxminus}_{abc} &= \tfrac{1}{3}\left(S_{abc} + S_{bca} + S_{cab}\right) \\
S^{\boxplus}_{abc} &= S_{abc} - S^{\square}_{abc} - S^{\boxminus}_{abc}\ ,
\end{aligned}
$$

where

$$\xi_c = \tfrac{1}{n-1} g^{ab} S_{abc}\ ,$$

with $n = \dim M$.

Following Tricerri and Vanhecke [40] we can use these three bundles to distinguish 8 types of homogeneous structures. Since S is parallel with respect to $\tilde{\nabla}$, its type under the orthogonal group does not change under parallel transport by $\tilde{\nabla}$. Thus it is enough know the type of S at the origin coset $o \in M$.

1. $S = 0$: the locally symmetric spaces;

2. $S \in \mathcal{T}_1$: there is a vector ξ such that

$$S(X,Y) = g(X,Y)\xi - g(X,\xi)Y.$$

   In riemannian signature, Tricerri and Vanhecke proved that (M,g) is locally isometric to hyperbolic space. In lorentzian signature there are two cases to distinguish: the norm of ξ is zero or non-zero. In the latter case Gadea and Oubiñia [41] proved that (M,g) is locally isometric to anti de-Sitter space, whereas if ξ is null, then Montesinos Amilibia [42] showed that (M,g) is a singular homogeneous plane-wave

$$g = 2dx^+dx^- + \frac{1}{(x^+)^2}H_0(\boldsymbol{x},\boldsymbol{x})(dx^+)^2 + |d\boldsymbol{x}|^2\ ,$$

   with H_0 a constant bilinear form;

3. $S \in \mathcal{T}_2$;

4. $S \in \mathcal{T}_3$: homogeneous spaces which admit a homogeneous structure of this type are called **naturally reductive**. We shall say more about naturally reductive spaces in the next section 2.4;

5. $S \in \mathcal{T}_1 \oplus \mathcal{T}_2$: we have $S(X,Y,Z) + S(Y,Z,X) + S(Z,X,Y) = 0$;

6. $S \in \mathcal{T}_1 \oplus \mathcal{T}_3$: there is a vector ξ such that

$$S(X,Y) + S(Y,X) = 2g(X,Y)\xi - g(X,\xi)Y - g(Y,\xi)X.$$

It is shown in [43] that if ξ has non-zero norm, then the underlying geometry is again that of a symmetric space. Whereas, if ξ is null, then it is a generic singular homogeneous plane-wave [44]:

$$g = 2dx^+dx^- + H_0(e^{-f\log x^+}\boldsymbol{x}, e^{-f\log x^+}\boldsymbol{x})\frac{(dx^+)^2}{(x^+)^2} + |d\boldsymbol{x}|^2 ,$$

with H_0 a constant bilinear form and f a skew symmetric matrix;

7. $S \in \mathcal{T}_2 \oplus \mathcal{T}_3$: we have $C_{12}(S) = 0$, and finally;

8. $S \in \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$: no conditions.

It must be stressed that a given homogeneous space can admit more than one homogeneous structure. We can understand this as follows. There is a one-to-one correspondence between homogeneous structures S and reductive splits $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. In principle, different choices of $\mathfrak{h}$ and $\mathfrak{m}$ give rise to different homogeneous structures. Indeed, given $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, the homogeneous structure S at the identity coset o is given by

$$S(X,Y,Z) = g\left(\nabla_Y X\big|_o, Z\right) , \tag{2.3.2}$$

where X, Y, Z are Killing vectors in $\mathfrak{m}$.

Now suppose that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive split, and let $\mathfrak{g}' \subseteq \mathfrak{g}$ be a subalgebra such that the restriction of the map $\mathfrak{g} \longrightarrow T_oM$ to $\mathfrak{g}'$ is still surjective. Let $\mathfrak{h}' = \mathfrak{g}' \cap \mathfrak{h}$ and let $\mathfrak{m}' = \mathfrak{g}' \cap \mathfrak{m}$, then surjectivity implies that $\mathfrak{m}' = \mathfrak{m}$, whence $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}$ is still a reductive split. Suppose we can pick a subspace $\mathfrak{m}' \subset \mathfrak{g}'$ such that $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$ is still a reductive split. This means that $\mathfrak{m}'$ is the graph of an $\mathfrak{h}'$-equivariant linear map $\varphi : \mathfrak{m} \longrightarrow \mathfrak{h}'$; that is,

$$\mathfrak{m}' = \{\varphi(X) + X \mid X \in \mathfrak{m}\} .$$

The larger $\mathfrak{h}'$ is, the more linear maps $\mathfrak{m} \to \mathfrak{h}'$ there are. But simultaneously the $\mathfrak{h}'$-equivariance condition becomes stronger. It is therefore not inconceivable that this method of restricting to subalgebras should exhibit nontrivial φ's. Moreover, if we can find a **maximal** $\mathfrak{g}$, then this method will allow us to calculate all the homogeneous structures on a homogeneous space M. Of course, the largest possible $\mathfrak{g}$ is the full isometry algebra $\mathfrak{iso}(M,g)$, however this may not be reductive. Any subalgebra of $\mathfrak{iso}(M,g)$ may not contain all other subalgebras, but there will be a finite collection of maximal reductive subalgebras $(\mathfrak{g}_i)$ such that any smaller subalgebra is contained in some $\mathfrak{g}_i$. Observe that conjugate subalgebras yield isomorphic homogeneous structures.

For a concrete example of a homogeneous space admitting many different coset presentations $M = G/H$, consider the 7-sphere. It can be written as

$S^7 = SO(8)/SO(7) = \mathrm{Spin}(7)/G_2 = Sp(2)/Sp(1)$, with each presentation corresponding to a different reductive split and a different homogeneous structure. The characterisation of the non-degenerate $\mathcal{T}_1$ class gives another example. We will see more examples of this in section 4.4.

Conversely, given a homogeneous structure S we can reconstruct the Lie bracket restricted to the subspace $\mathfrak{m}$ by the following formula

$$[X, Y] = S_X Y - S_Y X + \widetilde{R}(X, Y) , \tag{2.3.3}$$

where $X, Y \in \mathfrak{m}$ and S and $\widetilde{R}$ are evaluated at the point o. This defines the subspace $\mathfrak{m} \oplus [\mathfrak{m}, \mathfrak{m}]$, from which we may define the full reductive split $\mathfrak{m} \oplus \mathfrak{h}$ to be the algebraic closure of this subspace under the Lie bracket (2.3.3) together with

$$[A, X] = A(X) \quad \text{and} \quad [A, B] = AB - BA , \tag{2.3.4}$$

where $X \in \mathfrak{m}$ and $A, B \in \mathrm{End}(\mathfrak{m})$. Notice that not all elements of $\mathfrak{h}$ need appear in $\widetilde{R}$, in fact the holonomy algebra $\mathrm{hol}(\widetilde{\nabla})$ must be an ideal of $\mathfrak{h}$.

2.4 Calculating on reductive spaces

Let X, Y, Z be Killing vectors on $M = G/H$. Then one sees that

$$g(\nabla_X Y, Z) = \frac{1}{2} g([X, Y], Z) + \frac{1}{2} g([X, Z], Y) + \frac{1}{2} g(X, [Y, Z]) . \tag{2.4.1}$$

At the point $o \in M$ we deduce

$$\nabla_X Y = \Lambda_{\mathfrak{m}}(Y)(X) = -\frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y) , \tag{2.4.2}$$

where $\Lambda_{\mathfrak{m}}$ is as in theorem 2.2.5 and U is the symmetric bilinear mapping of $\mathfrak{m} \times \mathfrak{m}$ into $\mathfrak{m}$ defined by[3]

$$2\langle U(X, Y), Z \rangle = \langle X, [Z, Y]_{\mathfrak{m}} \rangle + \langle [Z, X]_{\mathfrak{m}}, Y \rangle , \tag{2.4.3}$$

where $X, Y, Z \in \mathfrak{m}$. It should be remarked that (2.4.2) is only valid at $o \in M$ as otherwise $\nabla_X Y$ is not necessarily a Killing vector. However, since ∇ is G-invariant, one can determine the $(\nabla_X Y)|_p$ at any other point $p \in M$ by acting by an isometry which takes o to p.

The formula (2.3.2) for the corresponding homogeneous structure (at o) can now be written explicitly:

$$S(X, Y, Z) = \frac{1}{2} \langle [X, Y]_{\mathfrak{m}}, Z \rangle + \frac{1}{2} \langle [Z, X]_{\mathfrak{m}}, Y \rangle + \frac{1}{2} \langle [Z, Y]_{\mathfrak{m}}, X \rangle , \tag{2.4.4}$$

[3]The apparent difference in sign between equation (2.4.1) and equations (2.4.2) and (2.4.3) stems from the fact that Killing vectors on G/H generate left translations on G, whence they are right-invariant. Thus the map $\mathfrak{g} \to$ Killing vectors is an anti-homomorphism.

for $X, Y, Z \in \mathfrak{m}$.

The U tensor is not generally invariant under the linear isotropy representation. Indeed, if $Z \in \mathfrak{h}$ and $X, Y \in \mathfrak{m}$ we have

$$(Z \cdot U)(X,Y) = [[Z,X]_{\mathfrak{h}}, Y]_{\mathfrak{m}} + [[Z,Y]_{\mathfrak{h}}, X]_{\mathfrak{m}} \; ; \tag{2.4.5}$$

although it clearly does when G/H is reductive.

Recall that a homogeneous space (M, g) is called **naturally reductive** if it admits a homogeneous structure S of type $\mathcal{T}_3$. This is equivalent to admitting a homogeneous structure S with $U = 0$. While reductivity is a property of the isotropy representation, natural reductivity is also a property of the metric. The canonical connection of a naturally reductive coset description G/H has the same geodesic structure as that of the Levi-Cività connection: S is totally skew-symmetric so

$$\nabla_X X = \tilde{\nabla}_X X + S(X,X) = \tilde{\nabla}_X X \; , \tag{2.4.6}$$

for any vector field X and therefore the geodesic equations for the two connections are the same.

The Riemann curvature of the Levi-Cività connection is given by

$$R(X,Y,Z,W) = g(-\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z, W). \tag{2.4.7}$$

It is G-invariant and can be calculated at o using equation (2.4.2). Let X, Y, W, Z be Killing vectors, then

$$\begin{aligned} R(X,Y,Z,W) =& \langle U(X,W), U(Y,Z) \rangle - \langle U(X,Z), U(Y,W) \rangle \\ &+ \frac{1}{12} \langle [X,[Y,Z]]_{\mathfrak{m}}, W \rangle - \frac{1}{12} \langle [X,[Y,W]]_{\mathfrak{m}}, Z \rangle \\ &- \frac{1}{6} \langle [X,[Z,W]]_{\mathfrak{m}}, Y \rangle - \frac{1}{12} \langle [Y,[X,Z]]_{\mathfrak{m}}, W \rangle \\ &+ \frac{1}{12} \langle [Y,[X,W]]_{\mathfrak{m}}, Z \rangle + \frac{1}{6} \langle [Y,[Z,W]]_{\mathfrak{m}}, X \rangle \\ &- \frac{1}{6} \langle [Z,[X,Y]]_{\mathfrak{m}}, W \rangle - \frac{1}{12} \langle [Z,[X,W]]_{\mathfrak{m}}, Y \rangle \\ &+ \frac{1}{12} \langle [Z,[Y,W]]_{\mathfrak{m}}, X \rangle + \frac{1}{6} \langle [W,[X,Y]]_{\mathfrak{m}}, Z \rangle \\ &+ \frac{1}{12} \langle [W,[X,Z]]_{\mathfrak{m}}, Y \rangle - \frac{1}{12} \langle [W,[Y,Z]]_{\mathfrak{m}}, X \rangle \\ &- \frac{1}{2} \langle [X,Y]_{\mathfrak{m}}, [Z,W]_{\mathfrak{m}} \rangle - \frac{1}{4} \langle [X,Z]_{\mathfrak{m}}, [Y,W]_{\mathfrak{m}} \rangle \\ &+ \frac{1}{4} \langle [X,W]_{\mathfrak{m}} \, , [Y,Z]_{\mathfrak{m}} \rangle. \end{aligned} \tag{2.4.8}$$

Let (E_j) be an orthonormal basis for $\mathfrak{m}$ and let $\mathcal{Z} = \sum_i U(E_i, E_i)$. Then, by taking trace of the formula above, we recover a formula for the Ricci curvature:

$$
\begin{aligned}
r(X,Y) = &-\frac{1}{2}\sum_j \langle [X,E_j]_{\mathfrak{m}}, [Y,E_j]_{\mathfrak{m}}\rangle - \frac{1}{4}\sum_j \langle [X,[Y,E_j]_{\mathfrak{m}}]_{\mathfrak{m}}, E_j\rangle \\
&-\frac{1}{4}\sum_j \langle [Y,[X,E_j]_{\mathfrak{m}}]_{\mathfrak{m}}, E_j\rangle - \frac{1}{2}\sum_j \langle [X,[Y,E_j]_{\mathfrak{h}}]_{\mathfrak{m}}, E_j\rangle \\
&-\frac{1}{2}\sum_j \langle [Y,[X,E_j]_{\mathfrak{h}}]_{\mathfrak{m}}, E_j\rangle + \frac{1}{4}\sum_{s,t,j} \langle [E_s,E_j]_{\mathfrak{m}}, X\rangle \langle [E_t,E_j]_{\mathfrak{m}}, Y\rangle \langle E_s, E_t\rangle \\
&-\frac{1}{2}\langle [\mathcal{Z},X]_{\mathfrak{m}}, Y\rangle - \frac{1}{2}\langle [\mathcal{Z},Y]_{\mathfrak{m}}\ , X\rangle
\end{aligned}
\tag{2.4.9}
$$

and, by taking the trace of this, for the Scalar curvature:

$$
\begin{aligned}
s = &-\frac{1}{2}\sum_{j,k} |[E_k,E_j]_{\mathfrak{m}}|^2 - \frac{1}{2}\sum_{j,k}\langle [E_k,[E_k,E_j]_{\mathfrak{m}}]_{\mathfrak{m}}, E_j\rangle \\
&-\sum_{j,k}\langle [E_k,[E_k,E_j]_{\mathfrak{h}}]_{\mathfrak{m}}, E_j\rangle - |\mathcal{Z}|^2 \\
&+\frac{1}{4}\sum_{s,t,j,k}\langle [E_s,E_j]_{\mathfrak{m}}, E_k\rangle \langle [E_t,E_j]_{\mathfrak{m}}, E_k\rangle \langle E_s,E_t\rangle
\end{aligned}
\tag{2.4.10}
$$

One can also write down expressions for the curvature and torsion of the canonical connection:

$$
\begin{aligned}
\tilde{R}(X,Y)Z &= [[X,Y]_{\mathfrak{h}}, Z]_{\mathfrak{m}} \\
\tau(X,Y) &= [X,Y]_{\mathfrak{m}}\ ,
\end{aligned}
\tag{2.4.11}
$$

for $X,Y,Z \in \mathfrak{m}$.

Given the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and an invariant tensor $F_o(-,\dots,-)$ at the point o, such as the metric $\langle -,-\rangle$ or the curvature $R(-,-,-,-)$, one can use the left invariant Maurer-Cartan form θ to re-construct the full tensor:

$$F = F_o(\theta_{\mathfrak{m}}, \theta_{\mathfrak{m}}, \dots, \theta_{\mathfrak{m}})\ .$$

To calculate the Maurer-Cartan form directly, one chooses a local coset representative

$$\sigma : U \longrightarrow G\ ,$$

then the pull back of θ by σ is of the form

$$\sigma^*(\theta) = \sigma^{-1} d\sigma\ .$$

For example, one may choose

$$\sigma(x_1,\dots,x_n) = \exp(x_1X^1)\exp(x_2X^2)\dots\exp(x_nX^n)\ ,$$

where (X^n) is a basis for $\mathfrak{m}$. Then

$$\begin{aligned}\sigma^{-1}d\sigma =&\exp(-x_nX^n)\dots\exp(-x_1X^1)dx_1X^1\exp(x_1X^1)\dots\exp(x_nX^n)\\ &+\exp(-x_nX^n)\dots\exp(-x_2X^2)dx_2X^2\exp(x_2X^2)\dots\exp(x_nX^n)+\dots .\end{aligned}$$

This can be evaluated by noticing

$$\begin{aligned}\exp(-x_iX^i)X^j\exp(x_iX^i) &= \exp([-x_iX^i,-])\cdot X^j\\ &= X^j - x_i[X^i,X^j]+\frac{(x_i)^2}{2}[X^i,[X^i,X^j]]-\dots ,\end{aligned}$$

where $\cdot$ means the action of the matrix exponential. Calculation of this matrix exponential can be difficult and one may have to make use of the following formula [45]:

$$de^H = \int_0^1 (e^{xH}dHe^{(1-x)H})dx . \tag{2.4.12}$$

One can also reconstruct the Killing vectors. Let $g \in G$, $x \in M$ and define $h : G \times M \to G$ by

$$g\cdot\sigma(x) = \sigma(g\cdot x)\cdot h(g,x) . \tag{2.4.13}$$

Now take $g = e^{tX}$ with $t \in \mathbb{R}$, $X \in \mathfrak{g}$ and differentiate the above equation with respect to t at $t = 0$. This gives ξ_X, the Killing vector in the X direction.

2.5 Homogeneous geodesics

For our study of plane-wave limits of homogeneous spaces we will find that it is important to consider the null **homogeneous geodesics** of the background spacetime. These are null geodesics γ that are the orbit of a 1-parameter subgroup of isometries. A curve given by $\gamma(t) = \exp(tX)_p$, for some $X \in \mathfrak{iso}(M;g)$ and $p \in M$, is a homogeneous geodesic if it satisfies the self parallel equation $\nabla_{\gamma'}\gamma' = c\gamma'$. For if it solves the self parallel equation we may reparameterise the geodesic $\gamma' \mapsto h\gamma'$ and solve the equation

$$\nabla_{h\gamma'}h\gamma' = hdh(\gamma')\gamma' + ch^2\gamma' = 0 ,$$

for h to obtain the usual geodesic equation. When this is the case we call $X \in \mathfrak{iso}(M,g)$ a **geodetic vector**. On a homogeneous space G/H we may use an isometry to take $p = o$ above. It may not be the case that $X \in \mathfrak{g}$, for example we will see in 2.5.2 that this is the case for the Kaplan space if we take $\mathfrak{g} = \mathfrak{m}$. If the homogeneous space is reductive then we may apply the Koszul formula (2.4.1) and find that $X \in \mathfrak{g}$ is geodetic if and only if

$$\langle X_{\mathfrak{m}}, [Z,X]_{\mathfrak{m}}\rangle = c\langle X_{\mathfrak{m}}, Z_{\mathfrak{m}}\rangle \tag{2.5.1}$$

for all $Z \in \mathfrak{g}$ and some $c \in \mathbb{R}$. If X in equation (2.5.1) belongs to $\mathfrak{m}$, then we say that the geodesic is **canonically homogeneous** since then γ is also a geodesic of the canonical connection.

If we input $Z = X$ into equation (2.5.1) then we find two cases to consider: if the norm of X is non-zero then $c = 0$, or if the norm of X is zero then c may not be zero. If $c = 0$ then we say that the geodesic is **absolutely homogeneous**.

Another equivalent formulation of the above definition of a homogeneous geodesic, which is relevant for non-homogeneous spaces, is that a geodesic γ is homogeneous if there exists a Killing vector field ξ which is aligned with the geodetic vector field γ' along the geodesic; that is $\xi|_\gamma = h\gamma'$ for some function h. In terms of the Killing transport, this is equivalent to the existence of a solution (γ, A) to the Killing transport equations with $A(\gamma') = 0$.

The existence of homogeneous geodesics in the riemannian setting is guaranteed by a theorem of Kowalski and Szenthe [46, 47]. It states that every homogeneous riemannian manifold admits at least one homogeneous geodesic through every point. The same result is also true for reductive lorentzian manifolds; however, it gives no guarantee about the existence of null homogeneous geodesics.

In fact all lorentzian homogeneous examples known to the author (and this includes all 4-dimensional homogeneous spaces appearing on Komrakov's classification [48],) contain at least one null homogeneous geodesic. However we shall consider an example below 2.5.1 of an algebra $K1.1^2.1$ taken from Komrakov's classification which demonstrates that not all homogeneous spaces contain an absolutely homogeneous null geodesic.

To the author's knowledge there are no known results about the existence of homogeneous geodesics in the nonreductive case.

At the other extreme, homogeneous spaces in which all geodesics are homogeneous are known as **geodesic orbit spaces** or, as they are often abbreviated to, **g.o. spaces**. Once upon a time, all g.o. spaces were thought to be naturally reductive. In fact, this is only true for g.o spaces in which all geodesics are canonically homogeneous. Kaplan [49] constructed a 6-dimensional riemannian g.o. space which is not naturally reductive and we shall review the lorentzian version of this space below 2.5.2. The following theorem gives some useful tools for working with g.o. spaces.

Theorem 2.5.1. *([50, 51]) Let $M = G/H$ be a pseudo-riemannian g.o. space and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ a reductive split. Then*

1. *There exists at least one canonical ad (H)-equivariant map $\phi : \mathfrak{m} \longrightarrow \mathfrak{h}$ (a*

geodesic graph) such that, for any $X \in \mathfrak{m}/\{0\}$, the curve

$$\exp t(X + \phi(X))(o)$$

is a geodesic.

2. *A geodesic graph is either linear (which is equivalent to natural reductivity with respect to some reductive split $\mathfrak{g} = \mathfrak{m}' \oplus \mathfrak{h}$,) or it is non-differentiable at the origin of $\mathfrak{m}$.*

Conversely, property 1. implies that G/H is a g.o. space. The geodesic graph is uniquely determined by fixing an ad (H)-invariant inner product on $\mathfrak{h}$.

In [52], Kowalski and Vanhecke have proved that up to dimension 5, every riemannian g.o. space is, or can be made, naturally reductive. Further, in dimension 6 they classified all riemannian g.o. spaces which are in no way naturally reductive.

Before considering the two examples mentioned, we shall first investigate what the eight different distinguished types of homogeneous structures can tell us about the existence of homogeneous geodesics, and in particular those that are null.

For example suppose that S is a section of $\mathcal{T}_1 \oplus \mathcal{T}_3$, then for a null geodesic γ of the $\widetilde{\nabla}$ connection we have

$$0 = \widetilde{\nabla}_{\gamma'}\gamma' = \nabla_{\gamma'}\gamma' - g(\gamma', \gamma')\xi + g(\gamma', \xi)\gamma' = \nabla_{\gamma'}\gamma' + g(\gamma', \xi)\gamma' \, .$$

Now if we reparameterise $\gamma(u)$ to $\overline{\gamma}(s)$, such that $\gamma' = \partial_u = h(s)\partial_s = h(s)\overline{\gamma}'$, we find that

$$0 = h^2 \nabla_{\overline{\gamma}'}\overline{\gamma}' + h\left(\nabla_{\overline{\gamma}'}h\right)\overline{\gamma}' + h^2 g(\overline{\gamma}' \, , \xi)\overline{\gamma}' \, .$$

So that a solution to

$$\frac{\partial h}{\partial s} + g(\overline{\gamma}' \, , \xi)h = 0$$

maps a null geodesic of $\widetilde{\nabla}$ to a null geodesic of ∇. Conversely, given a null geodesic for ∇ we can perform the inverse transformation and obtain a null geodesic for $\widetilde{\nabla}$. Thus, every null geodesic in a spacetime which admits a homogeneous structure of type $\mathcal{T}_1 \oplus \mathcal{T}_3$ is canonically homogeneous with respect to this structure. This also follows from the characterisation of lorentzian homogeneous spaces admitting a homogeneous structure of type $\mathcal{T}_1 \oplus \mathcal{T}_3$ given in [43].

The other classes of homogeneous structures listed in 2.3 (other than $0, \mathcal{T}_1, \mathcal{T}_3$ and $\mathcal{T}_1 \oplus \mathcal{T}_3$,) say little about the existence of homogeneous geodesics. For example, in section 3.6.3 we shall consider a homogeneous space $K1.4^6$ which admits a homogeneous structure of type $\mathcal{T}_1 \oplus \mathcal{T}_2$ which contains non-homogeneous geodesics. However, we shall see that the Kaplan space 2.5.2, which is g.o. and hence every geodesic is homogeneous, also admits a homogeneous structure of type $\mathcal{T}_1 \oplus \mathcal{T}_2$.

2.5.1 An example from Komrakov's classification

The algebra $K1.1^2.1$ has a parameter λ, we shall only consider $\lambda = 0$ so that the resulting homogeneous space admits a lorentzian metric. If λ is non-zero the metric is either riemannian or hyperbolic. The isometry algebra is the semi-direct product $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m}$ with $\mathfrak{h}$ a one dimensional Lie algebra spanned by e_1 and $\mathfrak{m}$ is a four dimensional algebra spanned by (u_1, u_2, u_3, u_4). The algebra is given by

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_3	0	$-u_1$	0
u_1	$-u_3$	0	0	$-u_2$	u_1
u_2	0	0	0	0	$2u_2$
u_3	u_1	u_2	0	0	u_3
u_4	0	$-u_1$	$-2u_2$	$-u_3$	0

Up to homothety (and Lie algebra homomorphism) there is a two-parameter family of $\mathfrak{h}$-invariant lorentzian inner products given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix} \tag{2.5.2}$$

with $\alpha\beta < 0$.

This algebra is reductive with split $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, and the homogeneous structure corresponding to this split is given by equation (2.4.4) and has (nonzero) components $S_{ijk} = S(u_i, u_j, u_k)$ given by

$$S_{123} = S_{213} = S_{312} = \frac{1}{2}\alpha \qquad S_{224} = -2\alpha \qquad S_{114} = S_{334} = -1 \ ,$$

which is of generic type $\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$.

It is possible to deform this homogeneous structure by choosing a different reductive split $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}'$ with $\mathfrak{m}'$ the graph of an $\mathfrak{h}$-equivariant linear map $\varphi : \mathfrak{m} \longrightarrow \mathfrak{h}$. We find that there is a 2-parameter family of such maps, and hence a 2-parameter family of such splits. Indeed, let $\mathfrak{m}'$ denote the span of the following vectors

$$u_1 \ , \qquad u_2 + c_2 e_1 \ , \qquad u_3 \ , \qquad \text{and} \qquad u_4 + c_4 e_1 \ ,$$

with resulting homogeneous structure

$$S_{123} = S_{312} = \frac{1}{2}\alpha \qquad S_{213} = c_2 + \frac{1}{2}\alpha \qquad S_{224} = -2\alpha.$$
$$S_{114} = S_{334} = -1 \qquad S_{413} = c_4 \ .$$

For generic values of c_2, c_4 this is again of type $\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$, but there is a point, $c_2 = \frac{1}{2}\alpha$ and $c_4 = 0$, for which the $\mathcal{T}_3$ component is absent.

Up to the action of the isotropy, a null vector (at the identity coset) can be written as

$$U^1 u_1 + U^2 u_2 + U^4 u_4 \ ,$$

where $(U^1)^2 + \alpha(U^2)^2 + \beta(U^4)^2 = 0$. We must distinguish between two cases: $\alpha < 0, \beta > 0$ and $\alpha > 0, \beta < 0$. In either case, the timelike component can be set to 1 (for future-pointing null rays) without loss of generality.

1. $\alpha < 0, \beta > 0$.
   In this case, the null vector is $u_2 + pu_4 + qu_1$, with $q = \sqrt{-\alpha - \beta p^2}$. We find that the geodetic equation (2.5.1) has a unique solution, with geodetic vector
   $$u_2 + pu_4 \qquad \text{with } p^2 = -\alpha/\beta \text{ and } c = -2p.$$
   This geodesic is canonically but not absolutely homogeneous.

2. $\alpha > 0, \beta < 0$.
   In this case, the null vector is $u_4 + pu_2 + qu_1$, with $q = \sqrt{-\beta - \alpha p^2}$. Here we find two homogeneous geodesics:
   $$\begin{aligned} &u_4 + pu_2 \qquad \text{with } p^2 = -\beta/\alpha \text{ and } c = -2, \\ &u_4 + qu_1 \qquad \text{with } q^2 = -\beta \text{ and } c = -1. \end{aligned}$$
   This geodesic is also canonically but not absolutely homogeneous.

Therefore, the homogeneous space derived from the $K1.1^2$ algebra has no null absolutely homogeneous geodesics. (This is in fact effectively the only 4-dimensional lorentzian homogeneous space without any null absolutely homogeneous geodesics.)

2.5.2 Kaplan's lorentzian g.o. space

The lorentzian version of Kaplan's g.o. space (see for example [51]) is a 2-step nilpotent Lie group with a left-invariant metric. The Lie algebra $\mathfrak{m}$ is spanned by (X_i) for $i = 1, \ldots, 6$ subject to the nonzero Lie brackets:

$$\begin{aligned} [X_1, X_3] &= X_5 \qquad [X_1, X_4] = X_6 \\ [X_2, X_4] &= -X_5 \qquad [X_2, X_3] = X_6 \end{aligned} \tag{2.5.3}$$

and the left invariant metric is induced from the inner product for which (X_i) is a pseudo-orthonormal frame with X_6 timelike. Notice that this inner product is not ad-invariant

$$1 = \langle [X_1, X_3], X_5 \rangle \neq \langle X_1, [X_3, X_5] \rangle = 0 \ .$$

If we choose the local coset representative to be

$$\sigma(x_1, \ldots, x_6) = \exp(x_1 X_1) \ldots \exp(x_6 X_6) \ ,$$

then the Maurer-Cartan form is given by

$$\begin{aligned} \sigma^{-1} d\sigma = & X_1 dx_1 + X_2 dx_2 + X_3 dx_3 + X_4 dx_4 \\ & + X_5 (dx_5 + x_3 dx_1 - x_4 dx_2) \\ & + X_6 (dx_6 + x_3 dx_2 + x_4 dx_1) \ . \end{aligned}$$

Whence, the metric is given by

$$\sum_{i=1}^{4} dx_i^2 + (dx_5 + x_3 dx_1 - x_4 dx_2)^2 - (dx_6 + x_3 dx_2 + x_4 dx_1)^2 \ , \qquad (2.5.4)$$

which exhibits M as an $\mathbb{R}^2$-bundle over flat $\mathbb{R}^4$, or as a real line bundle over the five-dimensional Gödel metric of [3]:

$$\sum_{i=1}^{4} dx_i^2 - (dx_6 + x_3 dx_2 + x_4 dx_1)^2 \ . \qquad (2.5.5)$$

We shall consider this Gödel metric in more detail in section 4.4.1.

The Lie algebra of isometries is a semi-direct product $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ consists of those (outer) derivations of $\mathfrak{m}$ which are skew-symmetric with respect to the inner product $\langle -, - \rangle$. (In the riemannian case this can be seen as a consequence of a theorem of Gordon [53].) Let δ be an outer derivation, then it preserves the center $\mathfrak{c}$, which is the span of X_5 and X_6. As δ is skew-symmetric, it also preserves the orthogonal complement $\mathfrak{a}$ of the center, which is the span of the (X_i) with $i = 1, \ldots, 4$. The Lie bracket on $\mathfrak{m} = \mathfrak{c} \oplus \mathfrak{a}$ defines a map

$$\Lambda^2_+ \mathfrak{a} \to \mathfrak{c} \ ,$$

which is equivariant under the action of δ. It is not hard to show that δ must act trivially on both parts of $\mathfrak{m}$. Hence $\mathfrak{h} = \mathfrak{so}(\mathfrak{a})_- \subset \mathfrak{so}(\mathfrak{a})$ comprises of anti-self dual rotations in $\mathfrak{a}$ and therefore $\mathfrak{h} \cong \mathfrak{sp}(1)$. Let Y_a, $a = 1, 2, 3$, denote a basis for $\mathfrak{h}$. Then the non-zero Lie brackets are given by (2.5.3) together with

$$\begin{array}{lll} [Y_1, X_1] = X_3 & [Y_2, X_1] = X_4 & [Y_3, X_1] = X_2 \\ [Y_1, X_2] = X_4 & [Y_2, X_2] = -X_3 & [Y_3, X_2] = -X_1 \\ [Y_1, X_3] = -X_1 & [Y_2, X_3] = X_2 & [Y_3, X_3] = -X_4 \\ [Y_1, X_4] = -X_2 & [Y_2, X_4] = -X_1 & [Y_3, X_4] = X_3 \\ [Y_1, Y_2] = -2Y_3 & [Y_2, Y_3] = -2Y_1 & [Y_3, Y_1] = -2Y_2 \ . \end{array} \qquad (2.5.6)$$

To calculate the Killing vector fields explicitly, we can use the method (2.4.13). Using the coset representative chosen above, the Killing vectors are given by

$$
\begin{aligned}
\xi_{X_i} =&\partial_i \quad \text{for } i = 1, 2, 5, 6. \\
\xi_{X_3} =&\partial_3 - x_1\partial_5 - x_2\partial_6 \\
\xi_{X_4} =&\partial_4 + x_2\partial_5 - x_1\partial_6 \\
\xi_{Y_1} =& - x_3\partial_1 - x_4\partial_2 + x_1\partial_3 + x_2\partial_4 - \frac{1}{2}(x_1^2 - x_2^2 - x_3^2 + x_4^2)\partial_5 - (x_1x_2 - x_3x_4)\partial_6 \\
\xi_{Y_2} =& - x_4\partial_1 + x_3\partial_2 - x_2\partial_3 + x_1\partial_4 - (x_1x_2 + x_3x_4)\partial_5 + \frac{1}{2}(x_1^2 - x_2^2 + x_3^2 - x_4^2)\partial_6 \\
\xi_{Y_3} =& - x_2\partial_1 + x_1\partial_2 + x_4\partial_3 - x_3\partial_4 \, .
\end{aligned}
$$

The geodesic orbit structure of this space is easy to calculate. Remember that this requires finding a map $\phi : \mathfrak{m} \to \mathfrak{h}$ such that for all $0 \neq X \in \mathfrak{m}$ we have $X + \phi(X)$ is geodetic. If $X = \sum_i v_i X_i$, then one finds that $\phi(X) = \sum_i \phi_i X_i$ where

$$
\begin{aligned}
\phi_1 =&(v_1^2 - v_2^2 + v_3^2 - v_4^2)\frac{v_5}{|v_\perp|^2} - 2(v_1v_2 + v_3v_4)\frac{v_6}{|v_\perp|^2} \\
\phi_2 =& - 2(v_1v_2 - v_3v_4)\frac{v_5}{|v_\perp|^2} - (v_1^2 - v_2^2 - v_3^2 + v_4^2)\frac{v_6}{|v_\perp|^2} \\
\phi_3 =&2(v_1v_2 + v_3v_4)\frac{v_5}{|v_\perp|^2} + 2(v_1v_2 - v_3v_4)\frac{v_5}{|v_\perp|^2}
\end{aligned}
\tag{2.5.7}
$$

where $|v_\perp|^2 = \sum_i |v_i|^2$. Notice that this function is non-linear and hence M is not naturally reductive.

Starting with the reductive split $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ with $\mathfrak{g}$ the full isometry algebra, the homogeneous structure calculated using equation (2.4.4) is given in components $S_{ijk} = S(X_i, X_j, X_k)$ by

$$
\begin{gathered}
S_{135} = S_{326} = S_{416} = S_{425} = S_{524} = S_{614} = S_{623} = \frac{1}{2} \\
S_{146} = S_{236} = S_{245} = S_{315} = S_{513} = -\frac{1}{2} \, .
\end{gathered}
$$

This can be seen to be of type $\mathcal{T}_2 \oplus \mathcal{T}_3$.

As explained at the end of section (2.4), we can search for other homogeneous structures by restricting to subalgebras $\mathfrak{g}' \subseteq \mathfrak{g}$ and looking for reductive splits $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$, where $\mathfrak{h}' = \mathfrak{g}' \cap \mathfrak{h}$ and $\mathfrak{m}'$ is the graph of an $\mathfrak{h}'$-equivariant linear map $\mathfrak{m} \to \mathfrak{h}'$.

Let us decompose $\mathfrak{h}$ and $\mathfrak{m}$ into irreducible $\mathfrak{h}$-modules: $\mathfrak{h}$ is three-dimensional and simple, whence irreducible, whereas $\mathfrak{m}$ breaks up into two one-dimensional trivial submodules and an irreducible four-dimensional submodule. It follows that there are no nontrivial $\mathfrak{h}$-equivariant linear maps $\mathfrak{m} \to \mathfrak{h}$, since such a map would restrict to an isomorphism on irreducible submodules, but the decomposition above shows they have no isotypical submodules in common. Therefore we must consider proper subalgebras $\mathfrak{g}' \subsetneq \mathfrak{g}$ in order to obtain other homogeneous structures. Conjugate subalgebras lead to isomorphic homogeneous structures, it follows that there is only one possibility: any one-dimensional subalgebra $\mathfrak{h}' \subset \mathfrak{h}$. We will consider the one spanned by Y_1, any other choice is related by conjugation.

Decomposing $\mathfrak{m}$ and $\mathfrak{h}'$ into irreducible representations of $\mathfrak{h}'$ we find

$$\mathfrak{m} = \mathbb{R}_0 \oplus \mathbb{R}_0 \oplus \mathbb{R}^2_1 \oplus \mathbb{R}^2_1 \qquad \text{and} \qquad \mathfrak{h}' = \mathbb{R}_0 \ ,$$

where the subscripts indicate the highest weight of the representation. The trivial representations in $\mathfrak{m}$ are spanned by X_5 and X_6, respectively, whereas the two-dimensional representations are spanned by (X_1, X_3) and (X_2, X_4), respectively. We therefore have a two-parameter family of $\mathfrak{h}'$-equivariant linear maps $\varphi : \mathfrak{m} \longrightarrow \mathfrak{h}'$, given by

$$\varphi(v^i X_i) = (\alpha v^5 + \beta v^6) Y_1 \ .$$

The graph of φ is then the subspace $\mathfrak{m}' \subset \mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}$ spanned by

$$X_1 \ , \quad X_2 \ , \quad X_3 \ , \quad X_4 \ , \quad X_5 + \alpha Y_1 \ , \quad \text{and} \quad X_6 + \beta Y_1 \ .$$

This means that the $[-,-]_{\mathfrak{m}'}$ brackets are different from those for $\mathfrak{m}$, the differences given by

$$\begin{aligned}
[X_5 + \alpha Y_1, X_1]_{\mathfrak{m}'} &= \alpha X_3 & \qquad [X_6 + \beta Y_1, X_1]_{\mathfrak{m}'} &= \beta X_3 \\
[X_5 + \alpha Y_1, X_2]_{\mathfrak{m}'} &= \alpha X_4 & \qquad [X_6 + \beta Y_1, X_2]_{\mathfrak{m}'} &= \beta X_4 \\
[X_5 + \alpha Y_1, X_3]_{\mathfrak{m}'} &= -\alpha X_1 & \qquad [X_6 + \beta Y_1, X_3]_{\mathfrak{m}'} &= -\beta X_1 \\
[X_5 + \alpha Y_1, X_4]_{\mathfrak{m}'} &= -\alpha X_2 & \qquad [X_6 + \beta Y_1, X_4]_{\mathfrak{m}'} &= -\beta X_2 \ .
\end{aligned}$$

We can now compute the corresponding homogeneous structure using formula (2.4.4) and we obtain a two-parameter family of $\mathcal{T}_2 \oplus \mathcal{T}_3$ structures:

$$\begin{aligned}
S_{326} = S_{416} = S_{614} = S_{623} &= \frac{1}{2} & \qquad S_{146} = S_{246} &= -\frac{1}{2} \\
S_{316} = S_{426} = S_{613} = S_{624} &= \frac{1}{2}\beta & \qquad S_{136} = S_{246} &= -\frac{1}{2}\beta \\
S_{315} = S_{513} &= -\frac{1}{2}(1+\alpha) & \qquad S_{135} &= \frac{1}{2}(1+\alpha) \\
S_{425} = S_{524} &= \frac{1}{2}(1-\alpha) & \qquad S_{245} &= -\frac{1}{2}(1-\alpha) \ .
\end{aligned}$$

Naturally, when $\alpha = \beta = 0$ we recover the homogeneous structure of the maximal reductive split.

2.6 Komrakov's classification

B. Komrakov Jnr has compiled a complete classification of 4-dimensional pseudo-riemannian homogeneous spaces [48], which is a useful source of examples on which to test conjectures. He considers the isotropy representation $\rho : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ of a homogeneous space G/H and classifies first all the complex forms and then the real forms of the subalgebra $(\rho(\mathfrak{h}))^{\mathbb{C}} \subset \mathfrak{so}(4, \mathbb{C})$. He then uses this to classify all pairs $(\mathfrak{g}, \mathfrak{h})$ up to isomorphism and list them as algebras.

The algebras have been labelled in the form $Ka.b^c.d$ where a is the dimension of the isotropy subalgebra, b and c label the different isotropy representations for each dimension and d labels the different algebras for each isotropy representation.

For each pair, we can calculate which inner products $\langle -,- \rangle$ on $\mathfrak{m}$ are $\mathfrak{h}$-invariant, in the sense that the isotropy representation is skew-symmetric with respect to $\langle -,- \rangle$. Then we can use the Maurer-Cartan form θ to recover the full metric from $\langle -,- \rangle$. We have calculated all the inner products for all of the algebras in the classification and computed the metrics for all those with lorentzian signature. Then we have used the GRTensor package for Maple to calculate the Ricci tensor and decide which are Einstein, Ricci flat or flat, and have compiled the data in a large table in appendix B. We have only done these calculations for $\dim \mathfrak{h} \geq 1$, and not included the 15 non-isomorphic Lie algebras because there are no restrictions given by the isotropy representation on which inner products they admit. So the family of metrics on each Lie algebra is isomorphic to the whole of SO(4), which is too large to reasonably include in the table in the appendix. We have made no attempt to weed out any redundancy in the list in terms that there may be isometries between some entries; for example, the many flat metrics listed. In [54] Komrakov classified all 4-dimensional lorentzian homogeneous (M, g) and invariant F solutions to the four-dimensional Einstein-Maxwell equations

$$dF = d * F = 0 \tag{2.6.1}$$

$$\mathrm{Ric}_{ij} + F_{ik}F_j^k = kg_{ij} \ , \tag{2.6.2}$$

where $k \in \mathbb{R}$. He did this by studying the equations algebraically using equations (2.4.9) and (2.4.10). Obviously the results of this classification coincide with those given in the appendix B, but we think that the list of fully expanded forms of the metrics (which are not given in [48] or [54]) is a useful list to have.

Below are some statistics to give an idea of the size and makeup of the classification:

- Number of isotropy representations admitting riemannian metrics: 6

- Number of isotropy representations admitting lorentzian metrics: 14
- Number of isotropy representations admitting metrics of $(2,2)$ signature: 30

  (There is some overlap in these cases where a representation admits metrics of different signatures.)
- The number of symmetric/reductive algebras admitting a riemannian metric: 21/29
- The number of symmetric/reductive/nonreductive algebras admitting a lorentzian metric: 35/64/6
- The number of symmetric/reductive/non-reductive algebras admitting a metric of $(2,2)$ signature: 57/123/9

As already mentioned, a look at the 6 lorentzian non-reductive algebras reveals that they all have reductive subalgebras.

Chapter 3

Plane-wave limits

The purpose of this chapter is to introduce the plane-wave limit of a supergravity background and the notion of a hereditary property. We begin by describing the plane-wave metrics and the geometric aspect of the plane-wave limit. Then we consider various hereditary properties including Güven's extension to supergravity. We finish the chapter with a look at some examples illustrating some of the known methods for taking the plane-wave limit.

3.1 *pp*-waves and plane-waves

It is first convenient to introduce the widely known class of plane-fronted gravitational waves with parallel rays, or as they are more commonly referred to ***pp*-waves**. These lorentzian metrics are characterised by the existence of a parallel null vector field. A coordinate system $(x^-, x^+, \boldsymbol{x})$ can always be found so that the metric takes the form

$$2dx^+dx^- + H(x^+, \boldsymbol{x})(dx^+)^2 + 2K(x^+, \boldsymbol{x}, d\boldsymbol{x})dx^+ + |d\boldsymbol{x}|^2 \ , \tag{3.1.1}$$

where $K(x^+, \boldsymbol{x}, -)$ is a linear map. Clearly the parallel null vector field is ∂_-.

The plane-waves are those *pp*-waves whose components are the same at every point of the wave surface, in this sense they are said to have 'plane symmetry'. The metric of a plane-wave in **Brinkmann** coordinates is given by

$$2dx^+dx^- + H(x^+, \boldsymbol{x}, \boldsymbol{x})(dx^+)^2 + |d\boldsymbol{x}|^2 \ , \tag{3.1.2}$$

where $H(x^+, -, -)$ is a symmetric bilinear form dependent only on x^+. A plane-wave can also be given in a **Rosen** coordinate system:

$$2dudv + C(u, d\boldsymbol{y}, d\boldsymbol{y}) \ , \tag{3.1.3}$$

where $C(u, -, -)$ is a non-degenerate symmetric bilinear form dependent only on u.

To change coordinates from Rosen to Brinkmann we set up so called harmonic coordinates $(x^-, x^+, \boldsymbol{x})$ defined by

$$u = x^- \quad v = x^+ + \tfrac{1}{4}C'_{ij}(u)Q^i{}_a(u)Q^j{}_b(u)y^a y^b \quad x^i = Q^i{}_a y^a \ . \tag{3.1.4}$$

Here the prime $'$ represents differentiation with respect to u. The matrix $Q^i{}_a$ is such that

$$C_{ij}Q^i{}_a Q^j{}_b = \delta_{ab} \quad \text{and} \quad C_{ij}((Q')^i{}_a Q^j{}_b - Q^i{}_a (Q')^j{}_b) = 0 \tag{3.1.5}$$

Defining the bilinear form $H(u, -, -)$ by the matrix

$$H_{ab}(u) = (C'_{ij}(Q')^i{}_a + C_{ij}(Q'')^i{}_a)Q^j{}_b \ ,$$

the metric takes the form of equation (3.1.2). There is a similar formulation of inverse coordinate change from Brinkmann to Rosen [44].

One situation that occurs in our calculations with plane-wave metrics is the following. Suppose we have a natural *pp*-wave coordinate system (3.1.1) in which $H(x^+, -, -)$ is a symmetric bilinear form as it is for a plane-wave, but $K = K(\boldsymbol{x}, d\boldsymbol{x})$ is a bilinear form independent of x^+ . Although at first it appears that this metric is not a plane-wave, in fact several simple coordinate changes show that it is. We can split K into its symmetric and skew-symmetric parts and consider them separately. If K is symmetric, then it can be absorbed into the rest of the metric by a change in the x^+ coordinate:

$$x^- \mapsto x^- - K(\boldsymbol{x}, \boldsymbol{x}) \ .$$

If K is skew-symmetric it can also be absorbed by a coordinate change to $\boldsymbol{x}$:

$$\boldsymbol{x} \mapsto e^{-x^+ K}\boldsymbol{x} \ . \tag{3.1.6}$$

Under this transformation,

$$\begin{aligned} H(x^+, \boldsymbol{x}, \boldsymbol{x}) &\mapsto H(x^+, e^{-x^+K}\boldsymbol{x}, e^{-x^+K}\boldsymbol{x}) \\ K(\boldsymbol{x}, d\boldsymbol{x}) &\mapsto K(\boldsymbol{x}, d\boldsymbol{x}) + K^2(\boldsymbol{x}, \boldsymbol{x})dx^+ \\ |d\boldsymbol{x}|^2 &\mapsto |d\boldsymbol{x}|^2 - 2K(\boldsymbol{x}, d\boldsymbol{x})dx^+ - K^2(\boldsymbol{x}, \boldsymbol{x})(dx^+)^2 \ , \end{aligned}$$

where K^2 is the bilinear form associated to the square of the matrix for K. So we see this allows us to cancel the non-plane-wave like $2K(\boldsymbol{x}, d\boldsymbol{x})dx^+$ term in exchange for gaining an extra $K^2(\boldsymbol{x}, \boldsymbol{x})(dx^+)^2$ term. A similar coordinate transformation can be used to deal with such linear terms in Rosen coordinates.

Within the class of plane-waves there are two important refinements: the **homogeneous plane-waves**[1] and the **symmetric plane-waves**. Every plane-wave is of **cohomogeneity one**; that is the orbit of any point $p \in M$ under the isometry group is a hypersurface in M. This can easily be seen from the Rosen coordinate description (3.1.3), since it is clear that the $n-1$ vector fields $\partial_v, \partial_{\boldsymbol{y}}$ are Killing vectors. Thus for a plane-wave to be homogeneous it is sufficient that there is one more Killing vector in the ∂_u or ∂_+ direction.

In [44], Blau and O'Loughlin have classified all homogeneous plane-waves into two classes. The first class consists of complete metrics and the second class incomplete metrics:

Theorem 3.1.1 ([44]). *There are two classes of homogeneous plane-waves:*

1. ***Regular waves:***

$$g = 2dx^+dx^- + H_0(e^{-x^+f}\boldsymbol{x}, e^{-x^+f}\boldsymbol{x})(dx^+)^2 + |d\boldsymbol{x}|^2 \ ,$$

2. ***Singular waves:***

$$g = 2dx^+dx^- + H_0(e^{-f\log x^+}\boldsymbol{x}, e^{-f\log x^+}\boldsymbol{x})\frac{(dx^+)^2}{(x^+)^2} + |d\boldsymbol{x}|^2 \ ,$$

where f is a skew-symmetric matrix and H_0 is a symmetric bilinear form.

The isometry algebra of the generic homogeneous plane-wave is given by:

$$\begin{aligned}
&[e_i, Y_j] = \delta_{ij}Z, \qquad [e_i, X] = -Y_i, \\
&[Y_i, Y_j] = 2f_{ij}Z, \qquad [X, Z] = cZ \\
&[X, Y_i] = (c\delta_{ij} + 2f_{ij})Y_j + (c(H_0)_{ij} - cf_{ij} - f_{ik}f_{kj})e_j
\end{aligned} \tag{3.1.7}$$

The isotropy subalgebra has basis (e_i). From this it is clear that homogeneous plane-waves are reductive. The c is as in equation (2.5.1) for the geodetic vector X. Each metric in the first class of regular plane-waves has an isometry algebra with $c = 0$, and is naturally reductive as is evidenced by the $\mathcal{T}_3$ structure associated to the above algebra:

$$S = \frac{1}{2}f_{ij}dx^+ \wedge dx^i \wedge dx^j \ . \tag{3.1.8}$$

When $[f, H_0] = 0$, the f drops out of the metric and the plane-wave is symmetric. These symmetric plane-waves are often called the **Cahen-Wallach spaces** (see [23] for the original paper or [20].)

[1] These are sometime called the Hpp-waves in the literature.

Each metric in the second class of singular plane-waves has an isometry algebra with $c = 1$. The homogeneous structure associated to this algebra is of type $\mathcal{T}_1 \oplus \mathcal{T}_3$ and given by

$$S_{++-} = \frac{1}{x^+} , \quad S_{+ij} = \frac{1}{x^+} f_{ij} , \quad S_{i+j} = \frac{1}{x^+}(\delta_{ij} - f_{ij}) , \tag{3.1.9}$$

When $[f, H_0] = 0$, the singular plane-wave admits a homogeneous structure of type $\mathcal{T}_1$. In fact, as was mentioned in 2.3, the singular homogeneous plane-waves with $f = 0$ are the only spacetimes admitting a degenerate $\mathcal{T}_1$ structure. By solving the Ambrose-Singer equations it can be shown that these singular homogeneous plane-waves do not admit a homogeneous structure of type $\mathcal{T}_3$. Performing the coordinate transformation

$$x^+ \longmapsto e^{x^+} ,$$

changes the form of the singular plane-wave to

$$2e^{x^+} dx^+ dx^- + H_0(e^{-x^+ f}\boldsymbol{x}, e^{-x^+ f}\boldsymbol{x})(dx^+)^2 + |d\boldsymbol{x}|^2 .$$

Hence, the bilinear form $H(x^+, -, -)$ which determines the plane-wave is of the same form as that of the non-singular plane-waves. Note that we can write the matrix associated to $H(x^+, -, -)$ as,

$$e^{x^+ f} H_0 e^{-x^+ f} = \exp(x^+[f, -]) \cdot H_0, \tag{3.1.10}$$

where $\cdot$ denotes the action of the exponential on matrices. Written this way, it is apparent that $H(x^+, -, -)$ is the solution to the differential equation

$$\frac{dH(x^+)}{dx^+} = [f, H(x^+)], \tag{3.1.11}$$

with initial condition $H(0) = H_0$.

3.2 The plane-wave limit

Let (M, g) be a lorentzian manifold of dimension n and let γ be a null geodesic of (M, g). Then given a point $x \in \gamma$ there exists a coordinate neighborhood (U, μ), $\mu : U \to \mathbb{R}^n$, of x defining **adapted coordinates** $\mu(y) = (u(y), v(y), \boldsymbol{y}(y))$, where u is a coordinate along γ, such that in U the metric may be written as

$$g = 2dudv + \alpha dv^2 + \beta(d\boldsymbol{y})dv + C(d\boldsymbol{y}, d\boldsymbol{y}) . \tag{3.2.1}$$

Here α, β and C are smooth functions, $\beta(-)$ is linear and $C(-, -)$ is a positive definite bilinear form.

To choose such coordinates one chooses a one-parameter family of hypersurfaces parameterized by v and foliated by null geodesics. The coordinate along the prescribed geodesics is given by u and γ is given by $(u,0,0)$. In other words, one chooses a local embedding of the null geodesic γ into a **twist-free** null geodesic congruence with tangent vector field ∂_u; that is a null geodetic vector field such that

$$d\iota_{\partial_u} g = 0 \ .$$

Then one chooses $(n-2)$-submanifolds on which the restricted metric is riemannian and allows v to be the parameter labelling these submanifolds.

Let $\Omega \in (0,\infty)$ and consider the linear map

$$\begin{aligned} \psi_\Omega : \mathbb{R}^n &\to \mathbb{R}^n \\ : (u,v,\boldsymbol{y}) &\mapsto (u,\Omega^2 v,\Omega \boldsymbol{y}) \ . \end{aligned} \tag{3.2.2}$$

This map induces the following change of coordinates:

$$\phi_\Omega^U = \mu^{-1} \circ \psi_\Omega \circ \mu : U \to U \ . \tag{3.2.3}$$

(If necessary, to make this well defined, we may need to shrink U so that it does not contain any "holes".) By patching together such coordinate neighborhoods along γ we may think of ϕ_Ω as a diffeomorphism from a tubular neighborhood of γ to a tubular subneighborhood. If we apply this change of coordinates to g, rescale the result by Ω^{-2} and then take the limit as $\Omega \to 0$ we obtain a well defined plane-wave metric:

$$\begin{aligned} g_{pl} &= \lim_{\Omega\to 0} \Omega^{-2} \phi_\Omega^* \, g \\ &= dudv + C(u,0,0)(d\boldsymbol{y},d\boldsymbol{y}) \ . \end{aligned} \tag{3.2.4}$$

We call g_{pl} together with the tubular neighborhood of γ the **plane-wave limit** of (M,g) along γ, and call ϕ_Ω the **plane-wave limit map**. The existence of adapted coordinates and the plane-wave limit was first noticed by Penrose in [26]. Notice that at $\Omega = 0$, ϕ_Ω is no longer a diffeomorphism.

It is not difficult to see that this plane-wave limit is well defined, in the sense that its definition is independent of the choice of adapted coordinates (3.2.1). Indeed, let $(r,s,\boldsymbol{x})$ be a different choice of coordinates such that

$$g = drds + \rho ds^2 + \psi(d\boldsymbol{x})ds + \Theta(d\boldsymbol{x},d\boldsymbol{x}) \ , \tag{3.2.5}$$

where ρ,ψ,Θ are functions of $(r,s,\boldsymbol{x})$. As both u and r are parameters along the geodesic γ we may as well choose them equal $u=r$. An easy check shows that when restricted to the geodesic γ the change of coordinates matrix must be of the form

$$\begin{pmatrix} dr \\ ds \\ dx^i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c^i & e^i_k \end{pmatrix} \begin{pmatrix} du \\ dv \\ dy^k \end{pmatrix}$$

and that under this

$$\Theta_{ij} e^i_k e^j_l = C_{kl} \; . \tag{3.2.6}$$

In fact c^i must also be zero because the second row in the matrix equation above shows that $s = v + K$, K a constant, and the change of basis matrix for the dual basis to the one-forms above is the inverse transpose:

$$\begin{pmatrix} \partial_u \\ \partial_v \\ \partial_{y^i} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -c^i (e^{-1})^i_k \\ 0 & 0 & (e^{-1})^i_k \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_s \\ \partial_{x^k} \end{pmatrix} \tag{3.2.7}$$

As e^i_k is nondegenerate we must have $c^i = 0$. Putting this into the plane-wave limit metric (3.2.4) we find

$$\begin{aligned} drds + \Theta(r,0,0)(d\boldsymbol{x}, d\boldsymbol{x}) &= dudv + \Theta(r,0,0)(ed\boldsymbol{y}, ed\boldsymbol{y}) \\ &= dudv + C(u,0,0)(d\boldsymbol{y}, d\boldsymbol{y}) \; , \end{aligned} \tag{3.2.8}$$

which shows that the plane-wave limit derived from the two different adapted coordinate systems is the same. Notice that this is really a statement about the choice of twist-free null geodesic congruence.

A sufficient condition for telling when two plane-wave limits will be isometric is the following (the statement of this theorem appeared in [30] although the proof did not).

Theorem 3.2.1 (Covariance of the plane-wave limit). *Let $(M, g), (M', g')$ both be lorentzian manifolds. Let γ and γ' be two null geodesics inside M and M' respectively. Let $f : M_\gamma \longrightarrow M'_{\gamma'}$ be an isometry of tubular neighborhoods of γ and γ' which maps γ onto γ'. Then the plane-wave limits of (M, g) and (M', g') along γ and γ' respectively are isometric.*

Proof. Let $(U, \mu = (u, v, \boldsymbol{y}))$ be a coordinate neighborhood of a point x on γ such that the metric g takes the form (3.2.1). Define a coordinate neighborhood $(f(U), \mu' = (u', v', \boldsymbol{y}'))$ about $f(x)$ by

$$\mu'(f(x)) = \mu(x) \; , \tag{3.2.9}$$

so that $u' = u \circ f^{-1}$ is a coordinate along γ'. As $g = f^* g'$, then g' also takes the form of (3.2.1) in this neighborhood.

Now consider $f \circ \phi_\Omega^U : U \to U'$. We have

$$\begin{aligned} f \circ \phi_\Omega^U &= f \circ \mu^{-1} \circ \psi_\Omega \circ \mu \\ &= f \circ (\mu' \circ f)^{-1} \circ \psi_\Omega \circ (\mu' \circ f) \\ &= \mu'^{-1} \circ \psi_\Omega \circ \mu' \circ f \\ &= \phi_\Omega^{U'} \circ f \ . \end{aligned} \tag{3.2.10}$$

Therefore

$$\begin{aligned} g_{pl} &= \lim_{\Omega \to 0} \Omega^{-2} (\phi_\Omega^U)^* g \\ &= \lim_{\Omega \to 0} \Omega^{-2} (\phi_\Omega^U)^* f^* g' \\ &= \lim_{\Omega \to 0} \Omega^{-2} (f \circ \phi_\Omega^U)^* g' \\ &= \lim_{\Omega \to 0} \Omega^{-2} (\phi_\Omega^{U'} \circ f)^* g' \\ &= \lim_{\Omega \to 0} \Omega^{-2} f^* \circ (\phi_\Omega^{U'})^* g' \\ &= f^* g'_{pl} \ . \end{aligned} \tag{3.2.11}$$

□

Notice that the plane-wave limit of the plane-wave metric

$$dudv + C(u, 0, 0)(dy, dy) \ , \tag{3.2.12}$$

along the geodesic ∂_u does not change the metric, whereas along ∂_v it leads to flat space. This shows that the covariance condition certainly isn't necessary for plane-wave limits to be isometric. For example, consider the plane-wave limit of any lorentzian space not isometric to a plane-wave, and compare it to the trivial plane-wave limit of a plane-wave along ∂_u.

3.3 The space of lorentzian metrics

The proofs of some of the hereditary properties we give below make use of continuity arguments. To make such arguments concrete we shall briefly consider the topology of the space of metrics, in order to specify the sense in which the plane-wave limit is a continuous limit.

We can consider the space of lorentzian metrics $\mathcal{M}_N$ on the tubular neighborhood N of γ as a smooth infinite-dimensional manifold. $\mathcal{M}_N$ is modelled on the set

$$C^\infty(\mathrm{GL}(TN, T^*N)) \tag{3.3.1}$$

of sections of invertible linear maps from the tangent space of N to the cotangent space together with a topology induced from the Whitney-C^∞ topology (see [55].)

Then

$$\{\Omega^{-2}(\phi_\Omega^{-1})^* g \mid \Omega \in [0,1]\} \tag{3.3.2}$$

is a continuous path in $\mathcal{M}_N$ with one end point g and the other g_{pl} and so

$$\lim_{\Omega\to 0} \Omega^{-2}\phi_\Omega^* \, g \tag{3.3.3}$$

certainly converges to g_{pl} in this topology.

It is interesting to consider $\mathcal{M}_N$ as a pseudo-riemannian manifold. The tangent bundle to $\mathcal{M}_N$ is modelled on

$$T\mathcal{M}_N \cong C^\infty(\mathrm{GL}(TN, T^*N)) \times C^\infty(\mathrm{L}_c(TN, T^*N)) \ , \tag{3.3.4}$$

where $C^\infty(\mathrm{L}_c(TN, T^*N))$ is the set of compactly supported sections of linear maps from TN to T^*N. Let G be the following pseudo-riemannian metric on $\mathcal{M}_N$:

$$G_b(g,h) := \int_N \mathrm{tr}\ (b^{-1}gb^{-1}h)\mathrm{vol}(b) \ , \tag{3.3.5}$$

where $g, h \in C^\infty(\mathrm{L}_c(TN, T^*N))$, $b \in C^\infty(\mathrm{GL}(TN, T^*N))$ and $\mathrm{vol}(b)$ is the volume form associated to b.

We can use this metric to calculate the length of the path from g to g_{pl}. If we let $g(t) := e^{2t}\phi_{e^{-t}}^* \, g$ so that $\Omega = e^{-t}$ in (3.2.4) and g takes the form (3.2.1), then

$$\begin{aligned}
\frac{\partial g(t)}{\partial t} &= \sum_{i=1}^{n-2}\left[-e^{-t}\beta_i(t) - 2e^{-3t}v\frac{\partial \beta_i}{\partial v}(t) - \sum_{j=1}^{n-2} e^{-2t}y^j\frac{\partial \beta_i}{\partial y^j}(t)\right] dy^i dv \\
&\quad + \sum_{i,j=1}^{n-2}\left[\sum_{k=1}^{n-2} -e^{-t}y^k\frac{\partial C_{ij}}{\partial y^k}(t) - 2e^{-2t}v\frac{\partial C_{ij}}{\partial v}(t)\right] dy^i dy^j \\
&\quad + \left[-2e^{-2t}\alpha(t) - \sum_{i=1}^{n-2} e^{-3t}y^i\frac{\partial \alpha}{\partial y^i}(t) - 2e^{-4t}v\frac{\partial \alpha}{\partial v}(t)\right] dv^2 \\
&= \sum_{i=1}^{n-2}[A]_i dy^i dv + \sum_{i,j=1}^{n-2}[B]_{ij}dy^i dy^j + [C]dv^2 \ .
\end{aligned} \tag{3.3.6}$$

In order to calculate $G_g(g_t, g_t)$, where $g_t := \frac{\partial g(t)}{\partial t}$, we need to find tr $(g^{-1}g_t g^{-1}g_t)$. If we let

$$\epsilon := \sum_{j=1}^{n-2}\left(C^{ij}(t)\frac{\partial}{\partial y^j} - t\beta_j C^{ij}(t)\frac{\partial}{\partial u}\right)$$

and apply (3.3.6) we find

$$
\begin{aligned}
g^{-1}g_t g^{-1}g_t\left(\frac{\partial}{\partial u}\right) &= 0\,,\\
g^{-1}g_t g^{-1}g_t\left(\frac{\partial}{\partial v}\right) &= \sum_{i,j,k=1}^{n-2}[A]_i\left(C^{ij}(t)[A]_j\frac{\partial}{\partial u}+C^{ij}(t)[B]_{jk}\epsilon_k\right),\\
g^{-1}g_t g^{-1}g_t\left(\frac{\partial}{\partial y^i}\right) &= \sum_{i,j,k,l=1}^{n-2}[B]_{ij}\left(C^{jk}(t)[A]_k\frac{\partial}{\partial u}+C^{jk}(t)[B]_{kl}\epsilon_l\right)\,,
\end{aligned}
$$

and therefore

$$
\operatorname{tr}\,(g^{-1}g_t g^{-1}g_t) = \sum_{i,j,k,l=1}^{n-2}[B]_{ij}[B]_{kl}C^{jk}(t)C^{il}(t)\,. \tag{3.3.7}
$$

So if Γ is the path from g to g_{pl} in $\mathcal{M}_N$, then with respect to the metric (3.3.5) the length of Γ is

$$
\begin{aligned}
L(\Gamma) &= \int_\Gamma (G_g(s)(g_t(s),g_t(s)))^{\frac{1}{2}}ds\\
&= \int_\Gamma\int_N\sum_{i,j,k,l=1}^{n-2}[B]_{ij}[B]_{kl}C^{jk}(s)C^{il}(s)\mathrm{vol}(g(s))\,.
\end{aligned} \tag{3.3.8}
$$

Equation (3.3.6) shows us that $[B]_{ij}\to 0$ as $t\to\infty$ and therefore the length of Γ (unsurprisingly) tends to zero. It also shows that if C is independent of v and $\boldsymbol{y}$ then Γ is null.

3.4 Hereditary properties

We say that a property of the metric g is **hereditary** if the plane-wave limit g_{pl} has the same property. For example,

Proposition 3.4.1. *Suppose (M,g) is locally symmetric/conformally flat. Then (M_γ, g_{pl}) is locally symmetric/conformally flat. If (M,g) is Einstein then (M_γ, g_{pl}) is Ricci flat, in particular it is Einstein.*

Proof. Let ∇_Ω, R_Ω denote the connection and curvature of $g_\Omega := \Omega^{-2}\phi_\Omega^*\,g$ respectively. As ϕ_Ω is a diffeomorphism, if $\nabla R=0$ then $\nabla_\Omega R_\Omega = 0$ for $\Omega>0$. By continuity, we see that $\nabla_{pl}R_{pl}=0$. Similarly if the Weyl tensor W of g vanishes then $W_\Omega=0$ for $\Omega>0$, and continuity ensures that $W_{pl}=0$.

If $\mathrm{Ric}(g)=\lambda g$ then

$$
\mathrm{Ric}(g_\Omega) = \mathrm{Ric}(\Omega^{-2}\phi_\Omega^*\,g) = \mathrm{Ric}(\phi_\Omega^*\,g) = \lambda\phi_\Omega^*\,g\,. \tag{3.4.1}
$$

This gives

$$\mathrm{Ric}(g_\Omega) = \Omega^2 \lambda g_\Omega \ , \tag{3.4.2}$$

and by continuity we see that $\mathrm{Ric}(g_{pl}) = 0$. $\square$

These hereditary properties can be used to easily compute the plane-wave limits of anti de-Sitter space AdS. Anti de-Sitter space is Einstein and conformally flat, hence any plane-wave limit is Ricci flat and conformally flat and thus flat.

The heritability of these curvature properties, such as the Einstein condition, may lead one to suspect that more complicated gravity equations could be preserved too. This is indeed the case; Güven [27] has shown how to extend the plane-wave limit to supergravity in such a way that solutions are mapped to plane-wave solutions. We shall describe this extension now. Let $(M, g, A_1, \ldots, A_n)$ be a local solution to a supergravity theory, that is a lorentzian manifold (M, g) together with a collection of differential form fluxes A_k which define field strengths $F_k = dA_k$ and together satisfy some equations of motion. For example, an 11-dimensional supergravity theory (M, g, F) which locally satisfies $F = dA$ and the Einstein-Maxwell type equations (1.1.1) and (1.1.2). We think of the field strengths F_k as being the fundamental quantities, consequently the fluxes are only defined up to a gauge transformation $A_k \mapsto A_k + d\Lambda_k$, which fixes F. Using these gauge transformations it is possible to manipulate a flux A such that

$$\iota_{\partial_u} A = 0 \ , \tag{3.4.3}$$

where ∂_u is the null geodesic vector field for the adapted coordinates (3.2.1). Specifically let $\Lambda = \int du \wedge \iota_{\partial_u} A$, which exists locally.

Now, we pull back the form A_k using the plane-wave limit map ϕ_Ω and rescale:

$$\Omega^{-k} \phi_\Omega^* \ A_k \ .$$

Letting Ω tend to zero, the choice of gauge ensures that the limit is well defined

$$(A_k)_{pl} := \lim_{\Omega \to 0} \Omega^{-k} \phi_\Omega^* \ A_k \ .$$

This also defines $(F_k)_{pl} := d(A_k)_{pl}$. Since the limiting process is continuous any system of differential equations will be preserved, so the resulting plane-wave limit supergravity data

$$(M_{pl}, g_{pl}, (A_1)_{pl}, \ldots, (A_n)_{pl})$$

will still solve the equations of motion.

More generally, an argument of Geroch [56] shows that the plane-wave limit preserves parallel sections of any connection. In particular, it was shown in [30] that neither the dimension of the isometry algebra nor the number of linearly independent Killing spinors ever decreases under the limit.

Although the plane-wave limit is fundamentally local in nature, we may consider the heritability of global properties such as completeness. Indeed, if (M, g) is a geodesically complete lorentzian manifold, then the plane-wave limit along any null geodesic is also geodesically complete:

Proof. Let $\gamma(t)$ be a geodesic with respect to ∇_{pl} for $t \in [a, b]$. Without loss we may assume that γ is contained in a normal coordinate neighborhood of some point on γ, so that there is a unique geodesic from $\gamma(a)$ to $\gamma(b)$ with respect to ∇_Ω for $\Omega \in [0, 1]$ (which is possible because ∇_Ω varies continuously with respect to Ω and $[0, 1]$ is compact.) Let γ_Ω be the unique geodesic with respect to ∇_Ω between $\gamma(a)$ and $\gamma(b)$. Then $\gamma_\Omega(t)$ may be extended to $(-\infty, \infty)$ as ∇_1 is geodesically complete and ϕ_Ω is a diffeomorphism. Continuity implies that the sequence of geodesics $\gamma(\Omega)$ for $\Omega = \frac{1}{k}$ converges to γ in the following sense. Any neighborhood of any point on γ intersects all but a finite number of geodesics of the sequence. Therefore, by continuity of the geodesic equation with respect to Ω, we have that γ may be extended beyond (a, b). $\square$

In the next section we shall consider heritability of some submanifold properties and in the next chapter we consider homogeneity. But before finishing this section we prove one last heritability result that will be useful later:

Proposition 3.4.2. *Let (M, g) be a lorentzian manifold and let γ be a null geodesic. Then at any point $x \in M$ there exists an orthonormal basis $\{e_i(\Omega)\}_{i=1}^n$ for T_xM with respect to g_Ω, varying continuously with respect to Ω, such that*

$$\{\lim_{\Omega \to 0} e_i(\Omega)\}_{i=1}^n \tag{3.4.4}$$

is a well-defined orthonormal basis for T_xM with respect to g_{pl}.

Proof. Let $(U, \mu = (u, v, \boldsymbol{y}))$ be a coordinate neighborhood of a point x such that g takes the form (3.2.1). Take the set $\{\frac{\partial}{\partial y^k}\}_{k=1}^{n-2}$ and apply the Gramm-Schmidt process with respect to g_Ω to obtain an orthonormal set $\{e_k\}_{k=1}^{n-2}$. Note that this set will be independent of Ω as this is a basis for T_xM and x is the zero point in the choice of coordinate (see (3.2.1).) Apply Gramm-Schmidt to the additional vectors $\frac{\partial}{\partial u} - \frac{\partial}{\partial v}, \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$ to obtain

$$\begin{aligned} e_u(\Omega) &= \frac{1}{(2 - \Omega^2\alpha - \sum_{i=1}^{n-1}\Omega^2(\eta^i)^2)}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v} + \Omega\sum_{i=1}^{n-1}\eta^i e_i\right), \\ e_v(\Omega) &= \frac{1}{(2 + \Phi(\Omega))}\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v} + \Omega^2\alpha e_u(\Omega) - \Omega\sum_{i=1}^{n-1}(\eta_i e_i + \Omega^2(\eta^i)^2 e_u)\right), \end{aligned} \tag{3.4.5}$$

where $\eta^i := g(\frac{\partial}{\partial v}, e_i)$ and $\Phi(\Omega)$ is some function of (Ω, u, v, y) such that $|e_v|^2 = 1$ and which tends to zero as $\Omega \rightarrow 0$. Letting $\Omega \rightarrow 0$, we obtain an orthonormal basis with respect to g_{pl}:

$$\left\{e_1, \ldots, e_{n-1}, e_u = \frac{1}{2}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right), e_v = \frac{1}{2}\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\right\}. \tag{3.4.6}$$

□

It is worth noting that given an arbitrary orthonormal basis $\{e_i\}_{i=1}^n$ for T_xM, the limit of $\{(\phi_\Omega)_* e_i\}_{i=1}^m$ as $\Omega \rightarrow 0$ is not necessarily a well-defined basis. For example, start with a basis containing the element $\frac{1}{\alpha}\frac{\partial}{\partial v}$. In the limit this element will blow up.

3.5 Plane-wave limits and submanifold geometry

Let X be a vector field on M and expand it in the adapted coordinates (3.2.1):

$$X = X^u \frac{\partial}{\partial u} + X^v \frac{\partial}{\partial v} + X^i \frac{\partial}{\partial y^i}.$$

If we apply the derivative of the plane-wave limit map ϕ_Ω to X we obtain a vector field X_Ω. But if we take the $\Omega \rightarrow 0$ limit of X_Ω it may blow up. However, we can rescale X_Ω by some power of Ω before taking the limit so that the limit is well defined. If $p(X)$ is the least such power, then

$$\lim_{\Omega \rightarrow 0} \Omega^{p(X)} \phi_\Omega^* X = X_{pl}. \tag{3.5.1}$$

We call X_{pl} the **plane-wave limit** of X along γ. It is easy to check that this is well-defined using the coordinate transformations in the discussion of the metric (3.2.5).

Consider a distribution D containing vector fields X and Y. Let D_{pl} be the distribution made up of the plane-wave limit of the vector fields in D. The plane-wave limit of vector fields induces a map on the Lie bracket:

$$[X, Y] \mapsto [X_{pl}, Y_{pl}], \tag{3.5.2}$$

which is an Inönü-Wigner contraction [57]. It follows that if D is involutive then so is D_{pl}. We can use this to define the plane-wave limit N_{pl} of a submanifold N by taking its involutive distribution of tangent vector fields. This is consistent with the notion of the plane-wave limit of the whole ambient space (M, g) along a null geodesic since the plane-wave limit of M, by the above definition, is equal to M. Notice that the dimension of N is not necessarily equal to the dimension of N_{pl}.

Three natural types of submanifold to consider are the totally geodesic, the minimal and the calibrated submanifolds. We will consider how these types of submanifold behave under the plane-wave limit, however first we need to consider immersions. Suppose that $h : N \to M$ is an immersion of a submanifold N into a lorentzian space (M, g), then the induced metric h^*g on N is non-degenerate. A problem when considering the plane-wave limit of immersed submanifolds N is that N_{pl} is not necessarily immersed. For example, the two-dimensional submanifold with tangent bundle spanned by (∂_v, ∂_1) may well have a non-degenerate induced metric, however its plane-wave limit, which is given by the same distribution, is a degenerate submanifold of the plane-wave limit of the ambient space. There are in fact three classes of immersed submanifold to consider:

- **Transversal** submanifolds N for which the tangent bundle is spanned by vectors of the form $X^i\partial_i + X^u\partial_u$, but does not contain ∂_u. If N is immersed then so is N_{pl}.

- **Lorentzian** submanifolds N which contain the null geodesic generated by ∂_u and a complementary null vector field such that the induced metric is non-degenerate. If N is immersed then so is N_{pl}.

- **Degenerative** submanifolds N for which the tangent bundle includes vectors of the form $\partial_v + X^i\partial_i + X^u\partial_u$, but does not contain ∂_u. N_{pl} is not immersed even if N is immersed.

The first two classes will be important in the following.

Let (M, g) be a lorentzian manifold, N a totally geodesic submanifold and γ be a null geodesic of M not necessarily contained in N. N is totally geodesic in M if and only if it is an immersed submanifold and the second fundamental form $II(X, Y) := (\nabla_Y X)^\perp$ vanishes on N where $\perp$ is the projection to the orthogonal complement of TN. Let $(u, v, \boldsymbol{y})$ be adapted coordinates (3.2.1) for M with respect to γ which define the map ϕ_Ω (3.2.3).

Since ϕ_Ω is a diffeomorphism we have that

$$[(\phi_\Omega^*\nabla)_{(\phi_{\Omega*}^{-1}X)}(\phi_{\Omega*}^{-1}Y)]^\perp = 0 \ . \tag{3.5.3}$$

If we multiply by $\Omega^{p(X)+p(Y)}$ and take the limit as $\Omega \to 0$ we find it is well-defined and continuity ensures it is zero:

$$[(\nabla_{pl})_{X_{pl}}Y_{pl}]^\perp = \lim_{\Omega\to 0}[\Omega^{p(X)+p(Y)}(\phi_\Omega^*\nabla)_{(\phi_\Omega)_*X}(\phi_\Omega)_*Y]^{\perp_\Omega} = 0 \ , \tag{3.5.4}$$

Therefore the submanifold N_{pl} is a totally geodesic submanifold of (M, g_{pl}) if it is immersed. The discussion above shows that this is the case if N is either transversal or lorentzian.

We can see how this works algebraically in symmetric spaces. Totally geodesic submanifolds in symmetric spaces are characterised by **Lie triple systems**. A Lie triple system is a subspace $\mathfrak{s} \subset \mathfrak{g}$ of the isometry algebra such that $X, Y, Z \in \mathfrak{s}$ implies $[X, [Y, Z]] \in \mathfrak{s}$. Given a symmetric space M with reductive split $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, there is a one to one correspondence between Lie triple systems $\mathfrak{s} \subset \mathfrak{m}$ and totally geodesic submanifolds N of M, with $N = \exp(\mathfrak{s})$. The plane-wave limit is therefore given by $N_{pl} = \exp(\mathfrak{s}_{pl})$ with the Lie bracket on $\mathfrak{s}_{pl}$ the contraction (3.5.2). As it is a contraction, $\mathfrak{s}_{pl}$ is also a Lie triple system subspace of $\mathfrak{g}_{pl}$ and hence N_{pl} is totally geodesic in (M, g_{pl}).

Now suppose that N is a minimal submanifold. N is minimal in M if and only if it is immersed and the mean curvature vector $\mathcal{H}$ vanishes. The mean curvature vector is given by $\mathcal{H}_x = \operatorname{tr} \mathrm{II}_x = \sum_{i=1}^{n} [\nabla_{e_i} e_i]^{\perp}$, where $\{e_i\}$ is an orthonormal basis for $T_x N$. We will consider the transversal and lorentzian immersed submanifolds separately.

First suppose N is transversal and let $(e_1, \ldots, e_m)$ be an orthonormal basis for N. Then to obtain $(e_i)_{pl}$ we must scale $(\phi_\Omega)_* e_i$ by Ω before taking the $\Omega \to 0$ limit. This precisely compensates for the Ω^{-2} scaling of the metric when taking the plane-wave limit, and so $((e_1)_{pl}, \ldots, (e_m)_{pl})$ is an orthonormal basis for TN_{pl}. Since ϕ_Ω is a diffeomorphism

$$0 = (\mathcal{H}_\Omega)_x = \sum_{i=1}^{n} [(\phi_\Omega^* \nabla)_{(\Omega \phi_\Omega^{-1})_* e_i} (\Omega \phi_\Omega^{-1})_* e_i]^{\perp} \to \sum_{i=1}^{n} [(\nabla_{pl})_{(e_i)_{pl}} (e_i)_{pl}]^{\perp} (\mathcal{H}_{pl})_x \ , \tag{3.5.5}$$

and continuity ensures that $(\mathcal{H}_{pl})_x = 0$. Therefore the plane-wave limit of N is minimal.

Now suppose that the submanifold N is lorentzian and choose adapted coordinates $(u, v, y^1, \ldots, y^m)$ for N with respect to the null geodesic and extend the transversal part to adapted coordinates $(u, v, y^1, \ldots, y^n)$ for the whole of M. As ϕ_Ω is a diffeomorphism we have $\mathcal{H}_\Omega = 0$ for Ω non-zero. Now we can use the orthonormal basis $\{e_i(\Omega)\}$ for TM constructed in proposition 3.4.2 to take the limit of

$$0 = \mathcal{H}_\Omega = \sum_{i=1}^{n} [(\phi_\Omega^{-1*} \nabla)_{e_i(\Omega)} e_i(\Omega)]^{\perp} \to \mathcal{H}_{pl} \ , \tag{3.5.6}$$

as $\Omega \to 0$ and continuity ensures that $\mathcal{H}_{pl} = 0$. Therefore the plane-wave limit of N is minimal.

Finally, let N be a calibrated submanifold with calibrating form θ. A p-form

θ is a calibration on a lorentzian space M if it is closed and

$$\theta(\zeta) \geq \mathrm{vol}(\zeta) \tag{3.5.7}$$

for all tangent p-planes ζ. (See for example [58] for more details.) In local coordinates the volume form can be given by

$$\mathrm{vol}_x = \sqrt{\det(g_x)} dx^1 \wedge \cdots \wedge dx^n \ . \tag{3.5.8}$$

Choose adapted coordinates $(u, v, y^1, .., y^{n-2})$ for M which define the map ϕ_Ω, then we may use the restriction of this map to pull back the volume form of a submanifold N. When taking the plane-wave limit one also scales the metric by Ω^{-2}, this will also scale the induced metric on N and hence the volume form:

$$\begin{aligned}\sqrt{\det(\Omega^{-2}\phi_\Omega^* g)}\phi_\Omega^*(dx^1 \wedge \cdots \wedge dx^n) &= \Omega^{-p}\sqrt{\det \phi_\Omega^* g}\phi_\Omega^*(dx^1 \wedge \cdots \wedge dx^n) \\ &= \Omega^{-p}\phi_\Omega^* \mathrm{vol} \ .\end{aligned}$$

For the calibrating form we may do the same trick as for Güven's extension and consider θ locally as $d\epsilon$. Then by changing gauge $\epsilon \mapsto \epsilon + d\Theta$ we may manipulate ϵ so that $\iota_{\partial_u}\epsilon = 0$. This guarantees that the limit of

$$\Omega^{-p}\phi_\Omega^*\epsilon \to \epsilon_{pl} \tag{3.5.9}$$

as $\Omega \to 0$ is well defined, and defines the closed form

$$\theta_{pl} = d\epsilon_{pl} \ . \tag{3.5.10}$$

Continuity also ensures that equation (3.5.7) holds in the limit, so θ_{pl} defines a calibrating form.

An immersed submanifold $N \subset M$ is calibrated by θ if $\theta|_N = \mathrm{vol}_N$ (see [58] for more details.) Therefore, the plane-wave limit N_{pl} of either a transverse or lorentzian submanifold is calibrated by θ_{pl} if N is calibrated by θ.

3.6 Examples

Something hidden by the definition of the plane-wave limit given in 3.2 is that calculating it by finding an adapted twist-free coordinate system can be quite difficult. This is principally because not every null geodesic vector field defines a twist-free geodesic congruence. However the Hamilton-Jacobi method from symplectic geometry provides a method for picking out a geodesic vector field which defines a twist-free congruence. We shall describe this method and illustrate with an example below, but first we will review some examples of plane-wave limits from the literature.

3.6.1 Anti de-Sitter space

As already mentioned, all plane-wave limits of the anti de-Sitter space AdS_n are flat space. In [30] the plane-wave limits of the $AdS_n \times S^m$ supergravity backgrounds are also considered. Using the covariance property 3.2.1, we only need to consider the isometry classes of null geodesics. There are two such classes: the set of geodesics which are contained completely in the AdS factor and the set which are not. In both cases it is not too difficult to write down an adapted coordinate system and we find that the plane-wave limit of a geodesic in the first class is flat and the plane-wave limit of a geodesic in the second is a Cahen-Wallach space.

In particular, as mentioned in the introduction, the 11-dimensional Minkowski space occurs as a plane-wave limit of the Kowalski-Glikman maximally supersymmetric plane-wave [21] and the BFHP maximally supersymmetric plane-wave [59] occurs as a plane-wave limit of the $AdS_5 \times S^5$ solution of IIB-string theory.

In [60] the plane-wave limit of the $AdS_3 \times S^3$ is exhibited as an Inönü-Wigner group contraction [57]. The space $AdS_3 \times S^3$ is isometric to the Lie group $SU(1,1) \times SU(2)$ with a bi-invariant metric. The geodesics of the bi-invariant metric are 1-parameter subgroups, that is to say that they are all homogeneous geodesics. This is used to describe the plane-wave limit as a group contraction. Such special cases of plane-wave limits were also considered in [61]. In the next chapter we shall show that this is a special case of a more general phenomenon, when taking plane-wave limits along homogeneous geodesics.

3.6.2 Branes

The paper [30] also considered plane-wave limits of the many different supergravity brane solutions. The typical metric and field strength F for an n-dimensional supergravity brane solution is

$$g = A^2(r)\eta + B^2(r)\delta \quad \text{and} \quad F_{p+2} = \mathrm{vol}(\mathbb{E}^{(1,p)}) \wedge dC(r) \ , \tag{3.6.1}$$

where η is the Minkowski metric on $\mathbb{R}^{1,p}$ and δ is the euclidean metric on $\mathbb{R}^{n-p-1}$. The isometry group of this generic brane metric $G = \mathrm{ISO}(1,p) \times \mathrm{SO}(n-p)$ acts with cohomogeneity one. The generic orbit is diffeomorphic to $\mathbb{R}^{p+1} \times S^{n-p-1}$. There are three isometry classes of null geodesics:

1. **tangential geodesics** that are tangent to the brane world volume,
2. **radial geodesics** which have no component tangent to the sphere part of the orbit structure,

3. **generic geodesics** which are neither of the above.

Plane-wave limits of tangential geodesics are flat, and those of the radial geodesics lead to a variety of plane-waves depending on the type of brane. For example, the plane-wave limit of the D3 brane is Ricci flat, plane-wave limits of the D3, NS5, M2 and M5 branes are flat in the near horizon limit and the plane-wave limit of the fundamental string is homogeneous in the near horizon limit. The plane-wave limits of the generic geodesics are more complicated. The paper [30] also considers intersecting branes among other things.

3.6.3 Hamilton-Jacobi

In this section we shall review how the Hamilton-Jacobi method can be used to compute adapted coordinates and calculate an example. For further references see either M. Blau's lecture notes [62] or [63] for more on the Hamilton-Jacobi equation in symplectic geometry.

That the Hamilton-Jacobi formalism can be used to find adapted coordinates first appeared in [31], although no formal proof appeared there. The following description appeared in [62]. One starts with an energy action defined by the lagrangian $L : TM \to \mathbb{R}$ given by

$$L(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{2}|\boldsymbol{x}'|^2 \ . \tag{3.6.2}$$

The geodesic equations for a null curve are given by the Euler-Lagrange equations together with the constraint that L vanishes. Let

$$H : T^*M \to \mathbb{R}, \quad (\boldsymbol{x}, \boldsymbol{q}) \mapsto \langle(\boldsymbol{x}, \boldsymbol{q}), \boldsymbol{x}'\rangle - L(\boldsymbol{x}, \boldsymbol{x}')$$

be the associated hamiltonian, where the bracket $\langle -, - \rangle$ is the obvious pairing. The associated hamiltonian vector field X_H defines a Jacobi field when restricted to a geodesic γ. Let $v : M \to \mathbb{R}$ be a solution to the Hamilton-Jacobi equation

$$H \circ dv = 0 \ , \tag{3.6.3}$$

then a null geodesic satisfies

$$|dv|^2 = 0 \ . \tag{3.6.4}$$

Now consider a neighborhood of a null geodesic which contains no conjugate points. Suppose we embed the geodesic γ into a twist-free congruence of null geodesics. Let $\gamma(p)$ denote the unique null geodesic of this congruence passing through the point p and v the Hamilton-Jacobi solution (3.6.3) for the geodesic congruence $\gamma(p)$. Consider the coordinates $(u, v, \boldsymbol{y})$ where u is an affine parameter along $\gamma(p)$ and $\boldsymbol{y}$ are some transverse coordinates. The definition of v gives $g(dv, -) = \partial_u$. Using this we have

$$\begin{aligned} g_{uu} &= 0 \\ g^{vv} &= |dv|^2 = 0 \\ g^{uv} &= g(du, dv) = du(\partial_u) = 1 \\ g^{iv} &= g(dy^i, dv) = dy^i(\partial_u) = 0 \ . \end{aligned}$$

The calculation of g_{ui} from g^{-1} involves the determinant of the $(n-1) \times (n-1)$ minor where the u^{th}-row and the i^{th}-column have been removed from g^{-1}. The v^{th} column of this minor is zero, hence $g_{ui} = 0$. It then follows that $g_{uv} = 1$ and therefore, putting all of the above together, we find that $(u, v, \boldsymbol{y})$ defines an adapted coordinate system.

As an example of an application of the Hamilton-Jacobi method, we will calculate the plane-wave limits of a homogeneous space taken from Komrakov's classification 2.6. Consider the algebra (Komrakov number 1.4^6)

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	u_1
u_2	$-u_1$	0	0	0	u_2
u_3	$-u_2$	0	0	0	$u_1 + u_3$
u_4	0	$-u_1$	$-u_2$	$-u_1 - u_3$	0

This isometry algebra is the semi-direct product $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m}$ of a one dimensional Lie algebra $\mathfrak{h}$ spanned by e_1 and a four dimensional Lie algebra $\mathfrak{m}$ spanned by $\{u_1, u_2, u_3, u_4\}$. Up to homothety (and Lie algebra automorphism) there is a unique $\mathfrak{h}$-invariant inner product given by

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

There is a two-parameter family of $\mathfrak{h}$-equivariant linear maps $\mathfrak{m} \to \mathfrak{h}$ which we will label with α and β. The graph $\mathfrak{m}'$ of a map in this family is the subspace of $\mathfrak{g}$ spanned by

$$u_1 + \alpha e_1 \ , \qquad u_2 \ , \qquad u_3 \ , \qquad \text{and} \qquad u_4 + \beta e_1 \ .$$

The subspace $\mathfrak{m}'$ is no longer a Lie subalgebra, but projecting the brackets to $\mathfrak{m}'$

we obtain

$$[\boldsymbol{u}_1+\alpha\boldsymbol{e}_1,\boldsymbol{u}_4+\beta\boldsymbol{e}_1]_{\mathfrak{m}'}=\boldsymbol{u}_1+\alpha\boldsymbol{e}_1 \qquad [\boldsymbol{u}_2,\boldsymbol{u}_4+\beta\boldsymbol{e}_1]_{\mathfrak{m}'}=\boldsymbol{u}_2-\beta(\boldsymbol{u}_1+\alpha\boldsymbol{e}_1)$$
$$[\boldsymbol{u}_1+\alpha\boldsymbol{e}_1,\boldsymbol{u}_3]_{\mathfrak{m}'}=\alpha\boldsymbol{u}_2 \qquad [\boldsymbol{u}_3,\boldsymbol{u}_4+\beta\boldsymbol{e}_1]_{\mathfrak{m}'}=\boldsymbol{u}_1+\boldsymbol{u}_3-\beta\boldsymbol{u}_2 \ .$$
$$[\boldsymbol{u}_1+\alpha\boldsymbol{e}_1,\boldsymbol{u}_2]_{\mathfrak{m}'}=\alpha(\boldsymbol{u}_1+\alpha\boldsymbol{e}_1)$$

The resulting homogeneous structure has components $S_{ijk}=S(u_i,u_j,u_k)$ given by

$$S_{134}=S_{314}=S_{334}=1 \qquad S_{123}=-\alpha \qquad S_{423}=-\beta \qquad S_{224}=-1\ ,$$

which is generically of type $\mathcal{T}_1\oplus\mathcal{T}_2\oplus\mathcal{T}_3$, but of type $\mathcal{T}_1\oplus\mathcal{T}_2$ when $\alpha=\beta=0$. Taking the local coset representative to be

$$\sigma=\exp(u\boldsymbol{u}_1)\exp(y_1\boldsymbol{u}_2)\exp(v\boldsymbol{u}_3)\exp(y_2\boldsymbol{u}_4):M\to G\ ,$$

the Maurer-Cartan form $\sigma^{-1}d\sigma$ is given by

$$(e^{y_2}du+y_2e^{y_2}dv)\boldsymbol{u}_1+e^{y_2}dy_1\boldsymbol{u}_2+e^{y_2}dv\boldsymbol{u}_3+dy_2\boldsymbol{u}_4\ .$$

Thus the induced metric is

$$ds^2=\langle(\sigma^{-1}d\sigma)_{\mathfrak{m}},(\sigma^{-1}d\sigma)_{\mathfrak{m}}\rangle=e^{2y_2}(-2dudv-2y_2dv^2+(dy_1)^2)+(dy_2)^2\ .$$

Using 2.4.13 to reconstruct the Killing vectors we find

$$\begin{aligned}\zeta_{u_1}&=\frac{\partial}{\partial u}\\ \zeta_{u_2}&=\frac{\partial}{\partial y_1}\\ \zeta_{u_3}&=\frac{\partial}{\partial v}\\ \zeta_{u_4}&=(-v-u)\frac{\partial}{\partial u}-y_1\frac{\partial}{\partial y_1}-v\frac{\partial}{\partial v}+\frac{\partial}{\partial y_2}\\ \zeta_{e_1}&=y_1\frac{\partial}{\partial u}-v\frac{\partial}{\partial y_1}\ .\end{aligned}$$

To determine the plane-wave limits, we first determine the null directions up to the action of isometries. Let $U=\sum_i U^iu_i\in\mathfrak{m}$ be a null vector. Then

$$2U^1U^3=(U^2)^2+(U^4)^2\ . \qquad (3.6.5)$$

The action of the isotropy is obtained by exponentiating the adjoint action of $e_1\in\mathfrak{h}$:

$$\begin{pmatrix}U^1\\U^2\\U^3\\U^4\end{pmatrix}=\begin{pmatrix}1&t&\frac{1}{2}t^2&0\\0&1&t&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}U^1\\U^2\\U^3\\U^4\end{pmatrix}=\begin{pmatrix}U^1+tU^2+\frac{1}{2}t^2U^3\\U^2+tU^3\\U^3\\U^4\end{pmatrix}\ .$$

We must distinguish between two cases:

1. If $U^3 = 0$ then so are U^2 and U^4 by (3.6.5), whereas $U^1 \neq 0$. Therefore the null vector can be chosen to be $\boldsymbol{u}_1$.

2. If $U^3 \neq 0$ then we can use the isotropy action to put $U^2 = 1$ and rescale the null vector to make $U^3 = 1$, so that the null vector is then $\boldsymbol{u}_3 + \alpha \boldsymbol{u}_4 + \frac{1}{2}(1+\alpha^2)\boldsymbol{u}_1 + \boldsymbol{u}_2$ for some $\alpha \in \mathbb{R}$.

We shall consider only the second case here, leaving the first to the end of chapter 4 when we have more machinery. Suppose that $S(u, v, y_1, y_2)$ is a solution to the Hamilton-Jacobi equation (3.6.3), such that $g(dS, dS) = 0$. Introducing momenta p_u, p_v, p_1 we find

$$S(u, v, x, y) = p_u u + p_v v + p_1 y_1 + \phi \ ,$$

where

$$\phi(y_2, p_u, p_v, p_1) = \int \sqrt{2p_u p_v - 2p_u^2 y_2 - p_1^2 e^{y_2}}dy_2 = \int f'(y_2) dy_2 \ .$$

Now changing coordinates such that

$$\begin{aligned}
ds &= p_u du + p_v dv + p_1 dy_1 + f'(y_2) dy_2 \\
dr &= \frac{dy_2}{f'(y_2)} \\
dz_1 &= \frac{dv}{p_u} - \frac{e^{2y_2}}{f'(y_2)} dy_2 \\
dz_2 &= \frac{dy_1}{p_1} - \frac{e^{2y_2}}{f'(y_2)} dy_2
\end{aligned}$$

we can rewrite this metric in the following adapted form:

$$2dsdr + 2e^{-2y_2} dsdz_1 + (2p_u p_v e^{-2y_2} - 2p_u^2 y_2 e^{-2y_2}) dz_1^2 - 2e^{-2y_2} p_1^2 dz_1 dz_2 + e^{-2y_2} p_1^2 dz_2^2 \ ,$$

where y_2 is a function of r defined above. This is the adapted form with r the coordinate along the geodesic. Taking the plane-wave limit of this metric we obtain

$$2dsdr + (2p_u p_v e^{-2y_2} - 2p_u^2 y_2 e^{-2y_2}) dz_1^2 - 2e^{-2y_2} p_1^2 dz_1 dz_2 + e^{-2y_2} p_1^2 dz_2^2 \ .$$

The discussion in case 2 above means we only need to consider $p_u = 1, p_v = \frac{1}{2}(1+\alpha^2), p_1 = 1$. We make the change to Brinkmann co-ordinates (3.1.4) with

$$Q_{Ak}(r) = \begin{pmatrix} \frac{e^{y_2}}{\sqrt{-2y_2+\alpha^2}} & 0 \\ \frac{e^{y_2}}{\sqrt{-2y_2+\alpha^2}} & e^{y_2} \end{pmatrix} ,$$

and obtain

$$2dx^+dx^- + \left(-2e^{2y_2}(x^1)^2 + e^{2y_2}(x^2)^2\right)(dx^+)^2 + |dx|^2 \ . \tag{3.6.6}$$

where y_2 is a function of x^+ which solves the equation:

$$y_2'' - (y_2')^2 = -\exp(2y_2) \ . \tag{3.6.7}$$

Now, if this metric is homogeneous then it must be of the form of the plane-wave in theorem 3.1.1. But the solution to equation (3.6.7) is non-polynomial, whereas for a homogeneous plane-wave equation (3.1.10) shows that $H(x^+)$ is polynomial in x^+. Therefore this plane-wave limit is not homogeneous.

We will return to this homogeneous metric at the end of the next chapter, after we have constructed some tools for dealing with plane-wave limits of homogeneous spaces.

Chapter 4

Plane–wave limits of homogeneous spaces

In this chapter we will consider the heritability of homogeneity. As already noted in the introduction, the plane-wave limit preserves the amount of symmetry of a background in the sense that neither the dimension of the isometry algebra [56, 30] nor the number of linearly independent Killing spinors [30] ever decrease in the plane-wave limit. The plane-wave limits of the Kaigorodov space, computed in [31] using the Hamilton-Jacobi method, show this does not necessarily imply that homogeneity is hereditary. So a natural question to ask is: "given an arbitrary spacetime (M, g), along which null geodesics γ is the plane-wave limit homogeneous?". In the first section of this chapter we will show that a sufficient condition for the plane-wave limit to be homogeneous is that the geodesic be homogeneous. Then, using the algebraic machinery for calculating on reductive spaces, we give two different derivations of formulae for the plane-wave limit of a reductive homogeneous space along a homogeneous geodesic. We conclude the chapter with several examples including another look at the Kaigorodov space in the light of our new formulae.

The results in section 4.1 have been reported in [64]. Most of the results and calculations in sections 4.2, 4.3 and 4.4 are the fruits of the collaboration with J. M. Figueroa-O'Farrill and P. Meessen and were reported in [65].

4.1 Plane-wave limits along homogeneous geodesics

We have already seen that the generic plane-wave is of cohomogeneity one and g_{pl} is locally homogeneous if and only if it has a Killing vector which agrees with γ' at any point $p \in \gamma$. So if the twist-free geodesic congruence ∂_u which γ is a member of defines a Killing vector field then the plane-wave limit will be homogeneous.

Indeed, let $(u, v, \boldsymbol{y})$ be adapted coordinates, and suppose that the null geodetic vector field ∂_u is a Killing vector field. Then we have

$$
\begin{aligned}
0 = \mathfrak{L}_{\frac{\partial}{\partial u}} g &= d(i_{\frac{\partial}{\partial u}} g) + i_{\frac{\partial}{\partial u}} dg \\
&= d(dv) + \frac{\partial \alpha}{\partial u} dv^2 + \frac{\partial \beta}{\partial u}(d\boldsymbol{y})dv + \frac{\partial C}{\partial u}(d\boldsymbol{y}, d\boldsymbol{y}) \,.
\end{aligned}
$$

Therefore C is independent of u and hence g_{pl} is flat.

Of course, requiring ∂_u to be a Killing vector is a very strong condition. A reasonable weakening of this is to suppose the geodesic γ is homogeneous, which means that there exists a Killing vector ξ such that $\xi|_\gamma = h\partial_u|_\gamma$ for some $h \in C^\infty(M)$. Then $\xi|_\gamma$ is generated by Killing transport of $(\xi(p), A_\xi(p))$ along γ. Now by definition,

$$
(A_\xi h\gamma')\,|_\gamma = (A_\xi \xi)\,|_\gamma = 0,
$$

where by $|_\gamma$ we mean restriction to $\gamma \in M$, not restriction of the tangent bundle. Therefore, if we write A_ξ in components:

$$
A_\xi = \sum_{i,j} (A_\xi)_i^j \, dx^i \otimes \frac{\partial}{\partial x^j},
$$

we see that

$$
(A_\xi)_u^{y^i}\,|_\gamma = (A_\xi)_u^v\,|_\gamma = 0.
$$

Also, as ξ is a Killing vector, we have

$$
g\left(A_\xi \frac{\partial}{\partial y^i}, \frac{\partial}{\partial u}\right)|_\gamma = -g\left(\frac{\partial}{\partial y^i}, A_\xi \frac{\partial}{\partial u}\right)|_\gamma = 0 \,.
$$

Therefore,

$$
(A_\xi)_{y^i}^v\,|_\gamma = 0 \,.
$$

Now consider the pull-back of the Killing transport covariant derivative under the plane-wave limit map ϕ_Ω;

$$
(\phi_\Omega^{-1})^* D_\xi(X)|_\gamma = (\phi_\Omega^{-1})^* \nabla_\xi(X)|_\gamma - (\phi_\Omega^{-1})^* A_\xi(X)|_\gamma \,.
$$

The components of $A_\xi|_\gamma$ scale under the plane-wave limit map in the following way:

$$
\begin{aligned}
(A_\xi)_u^{y^i}\,|_\gamma &\mapsto \Omega^{-1}\,(A_\xi)_u^{y^i}\,|_\gamma \\
(A_\xi)_u^v\,|_\gamma &\mapsto \Omega^{-2}\,(A_\xi)_u^v\,|_\gamma \\
(A_\xi)_{y^i}^v\,|_\gamma &\mapsto \Omega^{-1}\,(A_\xi)_{y^i}^v\,|_\gamma
\end{aligned}
$$

and other components which either stay constant or tend to zero as $\Omega \to 0$. Taking the limit as $\Omega \to 0$, we see from above that the three components of A_ξ that could blow up are in fact zero. Therefore

$$(D_{pl})_\xi(X)(u,v,y) := \lim_{\Omega\to 0}[D_\xi(X)(u,0,0)]$$

is well-defined and along with

$$(D_{pl})_\xi(A) := (\nabla_{pl})_\xi A - R_{pl}(\xi, X) ,$$

defines a Killing transport covariant derivative on along γ with respect to g_{pl}. Therefore parallel translation by D_{pl} along γ generates the remaining Killing vector needed, and g_{pl} is homogeneous.

We can immediately see a couple of things from the above. First, if γ is an (absolutely) homogeneous geodesic of g, then it is also an (absolutely) homogeneous geodesic of the plane-wave limit of g along γ. Consequently, the classification of homogeneous plane-waves 3.1.1 tells us that the plane-wave limit along an absolutely homogeneous geodesic ($c = 0$) is a regular homogeneous plane-wave, and along a non-absolutely homogeneous geodesic ($c \neq 0$) is a singular homogeneous plane-wave.

Second, if g is geodesically complete then we saw in the last chapter that the plane-wave limit is complete, so 3.1.1 tells us that the plane-wave limit must be a regular homogeneous plane-wave.

The above gives a sufficient condition on a null geodesic, in a generic space-time, for the plane-wave limit along it to be homogeneous. It is however not a necessary condition as the following example shows. Consider the metric

$$2dudv + udv^2 + \sqrt{u}\sum_i (dx^i)^2 .$$

This is an incomplete and nonhomogeneous metric, with no Killing vector in the ∂_u direction. Therefore the null geodesic given by ∂_u is not homogeneous. The plane-wave limit along this geodesic,

$$2dudv + \sqrt{u}\sum_i (dx^i)^2 ,$$

is however a singular homogeneous plane wave [66].

We will also see in section 4.4 examples of reductive spaces which have plane-wave limits along non-homogeneous geodesics onto homogeneous plane-waves. However, we will give a necessary and sufficient criteria for the case of reductive spaces in section 4.2.1.

4.2 Plane-wave limits of reductive spaces

As we saw in the last chapter, calculating plane-wave limits and in particular finding adapted coordinates, can often be difficult. Sometimes one can use the Hamilton-Jacobi method, but more often than not there is no known method for finding a twist-free geodetic vector field.

We have already seen that the usual machinery of differential geometry and supergravity can be described algebraically on a reductive space. We need only knowledge of the metric at the point $o \in M$ and the Lie algebra to reconstruct the whole metric. Thus, one might suspect that an operation such as the plane-wave limit of a homogeneous geodesic in a reductive space should have a completely algebraic description. In light of the difficulties one often encounters when calculating plane-wave limits, such an algebraic formulation would be a useful device.

4.2.1 The covariant method

Let g be a lorentzian metric and γ a null geodesic of g. Consider g to be written in an adapted coordinate system (3.2.1) and let $(\partial_u, \partial_v, \partial_i)$ denote the dual frame to (du, dv, dy^i).

In [28], the following covariant formulation of the plane-wave limit is given. We say that a local frame (E_+, E_-, E_a) is **adapted to a null geodesic** γ, if the following conditions are satisfied:

1. E_+ is a geodesic vector field such that $E_+|_\gamma$ is proportional to $\partial_u|_\gamma$;
2. $\nabla_u E_- = \nabla_u E_a = 0$ along γ; and
3. the metric takes the form
$$g = 2\varepsilon^+\varepsilon^- + \sum_a \varepsilon^a\varepsilon^a$$
where the ε's are the dual coframe.

Let (E_+, E_-, E_a) be such an adapted frame. We can write E_a in the form

$$E_a = E_a^i\partial_i + E_a^u\partial_u + E_a^v\partial_v \ .$$

By taking its inner product with E_+ and with E_b we see that restricted to the geodesic γ we have

$$E_a^v = 0$$

and

$$E_{ai}E_b^i = C_{ij}E_a^jE_b^i = \delta_{ab} \ .$$

Calculating the covariant derivative of E_a we have

$$(E^i_a)' + E^j_a \Gamma^i_{ju} = 0$$

and the dual equation

$$(E_{ai})' - E_{aj}\Gamma^j_{iu} = 0 \ .$$

Thus

$$(E_{ai})' E^i_b = -E_{ai}(E^i_b)' = E_{ai}E^j_b\Gamma^i_{ju} = E^i_a E_{bj}\Gamma^i_{ju} = E^i_a(E_{bj})' \ . \tag{4.2.1}$$

Now consider the plane-wave limit g_{pl} of the metric g. A frame E_M satisfying equation (4.2.1) defines a change of coordinates from the Rosen coordinate description of g_{pl} to a Brinkmann coordinate description

$$2dx^+dx^- + H_{ij}(x^+)x^ix^j(dx^+)^2 + \sum_i (dx^i)^2,$$

where

$$H_{ab}(x^+) = -R(E_+, E_a, E_+, E_b)|_\gamma = -R(E_+, \partial_i, E_+, \partial_j)|_\gamma E^i_a|_\gamma E^j_b|_\gamma \ .$$

This covariant description of the plane-wave limit illustrates that the limit is really an invariant of the null geodesic and not just a remnant of a special coordinate system. However, it is not much easier to apply than the usual plane-wave limit as finding a parallel frame can be difficult. On the other hand, on reductive spaces it is a fruitful approach.

Indeed, suppose that (M, g) is a locally reductive homogeneous space with a homogeneous structure S. Let M be locally isomorphic to the quotient G/H and let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ be the reductive split of the Lie algebra of G associated to S. Let $U \in \mathfrak{g}$ be the geodetic vector that determines γ as homogeneous. Let $V \in \mathfrak{m}$ be the dual null vector and complete to a basis with orthonormal elements $Y_i \in \mathfrak{m}$. The classification of homogeneous plane waves [44] states that the plane-wave limit in Brinkmann coordinates will be of the form:

$$H(x^+) = e^{x^+ f} H_0 e^{-x^+ f} \quad \text{or} \quad H(x^+) = e^{\log(cx^+) f} H_0 e^{-\log(cx^+) f}/(cx^+)^2 \ ,$$

where H_0 is a nondegenerate symmetric bilinear form, f is a skew-symmetric bilinear form and $c \neq 0$ is the constant in (2.5.1). The first case corresponds to the regular plane-waves and the second to the singular waves. We shall take the origin o for the regular waves to be the point $(0, 0, 0)$, while for the singular waves we take $(1/c, 0, 0)$.

We will now use the above covariant description and the algebraic description of the curvature tensor on such a background to write down an algebraic formula for both H_0 and f.

Let E_M be an adapted frame to the geodesic γ which when restricted to o corresponds to the basis (U, V, Y_i). For a regular homogeneous plane-wave limit g_{pl} we have

$$\exp(x^+[f,-]) \cdot H_0 = H_{ab}(x^+) = -R(E_+, E_a, E_+, E_b)|_\gamma \,. \tag{4.2.2}$$

Thus, evaluating at o,

$$\begin{aligned}(H_0)_{ab} &= -R(E_+, E_a, E_+, E_b)|_0 \\ &= -R(U_\mathfrak{m}, Y_a, U_\mathfrak{m}, Y_b) \,,\end{aligned} \tag{4.2.3}$$

where $U_\mathfrak{m}$ is the projection to $\mathfrak{m}$ of $U \in \mathfrak{g}$ and $Y_a = E_a(0) \in \mathfrak{m}$. Similarly, we find that (4.2.3) holds for the singular plane-waves.

Now, if we differentiate the left hand side of equation (4.2.2) and evaluate at o we obtain

$$\frac{\partial}{\partial x^+}\left(H_{ab}(x^+)\right)\Big|_o = -2cH_0 + [f, H_0] \,.$$

Differentiating the right hand side yields,

$$\begin{aligned}\frac{\partial}{\partial x^+}\left(H_{ab}(x^+)\right)|_o &= -\frac{\partial}{\partial x^+}R(E_+, E_a, E_+, E_b)|_o \\ &= -\nabla_U(R(E_+, E_a, E_+, E_b)|_\gamma) \\ &= -\left(\nabla_U R(E_+, E_a, E_+, E_b)\right)|_\gamma \\ &= -\left(\nabla_U R\right)(E_+, E_a, E_+, E_b)|_\gamma \,,\end{aligned}$$

where we have used the fact that U is a vector field tangent to γ and that the frame E_M is parallel to U.

The object ∇R is tensorial, that is

$$(\nabla R)(\cdot, \dots, hX, \dots, \cdot) = h(\nabla R)(\cdot, \dots, X, \dots, \cdot) \,,$$

for any $h \in C^\infty(M)$. Whence, by passing the restriction to 0 through the curvature, we have

$$\frac{\partial}{\partial x^+}\left(H_{ab}(x^+)\right)|_0 = -\left(\nabla_{U_\mathfrak{m}} R\right)(U_\mathfrak{m}, Y_a, U_\mathfrak{m}, Y_b) \,.$$

As $U_\mathfrak{m}$ is a Killing vector [67]

$$(\nabla_{U_\mathfrak{m}} - S_{U_\mathfrak{m}} \cdot)R = \mathfrak{L}_{U_\mathfrak{m}} R = 0 \,.$$

Hence we can replace the differential action of the covariant derivative with the algebraic action of the linear map $S_{U_\mathfrak{m}}$,

$$\begin{aligned}\frac{\partial}{\partial x^+}\left(H_{ab}(x^+)\right)|_0 &= -\left(S_{U_\mathfrak{m}} \cdot R\right)(U_\mathfrak{m}, Y_a, U_\mathfrak{m}, Y_b), \\ &= -S_{U_\mathfrak{m}} \cdot R(U_\mathfrak{m}, Y_a, U_\mathfrak{m}, Y_b) + R(S_{U_\mathfrak{m}} U_\mathfrak{m}, Y_a, U_\mathfrak{m}, Y_b) \\ &\quad + R(U_\mathfrak{m}, S_{U_\mathfrak{m}} Y_a, U_\mathfrak{m}, Y_b) + R(U_\mathfrak{m}, Y_a, S_{U_\mathfrak{m}} U_\mathfrak{m}, Y_b) \\ &\quad + R(U_\mathfrak{m}, Y_a, U_\mathfrak{m}, S_{U_\mathfrak{m}} Y_b)\end{aligned}$$

where we have used that the action of $S_{U_\mathfrak{m}}$ annihilates functions. Therefore we obtain the formula

$$\begin{aligned} -2c(H_0)_{ab} + [f, H_0]_{ab} = R(S_{U_\mathfrak{m}} U_\mathfrak{m}, Y_a, U_\mathfrak{m}, Y_b) + R(U_\mathfrak{m}, S_{U_\mathfrak{m}} Y_a, U_\mathfrak{m}, Y_b) \\ + R(U_\mathfrak{m}, Y_a, S_{U_\mathfrak{m}} U_\mathfrak{m}, Y_b) + R(U_\mathfrak{m}, Y_a, U_\mathfrak{m}, S_{U_\mathfrak{m}} Y_b) \ . \end{aligned} \tag{4.2.4}$$

Similarly, differentiating a second time and evaluating at zero, we find that $(6c^2 H_0 - 3c[f, H_0] + [f, [f, H_0]])_{ab}$ is given by

$$\begin{aligned} & R(S_{U_\mathfrak{m}} S_{U_\mathfrak{m}} U_\mathfrak{m}, Y_a, U_\mathfrak{m}, Y_b) + R(U_\mathfrak{m}, S_{U_\mathfrak{m}} S_{U_\mathfrak{m}} Y_a, U_\mathfrak{m}, Y_b) \\ & + R(U_\mathfrak{m}, Y_a, S_{U_\mathfrak{m}} S_{U_\mathfrak{m}} U_\mathfrak{m}, Y_b) + R(U_\mathfrak{m}, Y_a, U_\mathfrak{m}, S_{U_\mathfrak{m}} S_{U_\mathfrak{m}} Y_b) \\ & + 2R(S_{U_\mathfrak{m}} U_\mathfrak{m}, S_{U_\mathfrak{m}} Y_a, U_\mathfrak{m}, Y_b) + 2R(S_{U_\mathfrak{m}} U_\mathfrak{m}, Y_a, S_{U_\mathfrak{m}} U_\mathfrak{m}, Y_b) \\ & + 2R(S_{U_\mathfrak{m}} U_\mathfrak{m}, Y_a, U_\mathfrak{m}, S_{U_\mathfrak{m}} Y_b) + 2R(U_\mathfrak{m}, S_{U_\mathfrak{m}} Y_a, S_{U_\mathfrak{m}} U_\mathfrak{m}, Y_b) \\ & + 2R(U_\mathfrak{m}, S_{U_\mathfrak{m}} Y_a, U_\mathfrak{m}, S_{U_\mathfrak{m}} Y_b) + 2R(U_\mathfrak{m}, Y_a, S_{U_\mathfrak{m}} U_\mathfrak{m}, S_{U_\mathfrak{m}} Y_b) \\ & + R(S_{S_{U_\mathfrak{m}} U_\mathfrak{m}} U_\mathfrak{m}, Y_a, U_\mathfrak{m}, Y_b) + R(U_\mathfrak{m}, S_{S_{U_\mathfrak{m}} U_\mathfrak{m}} Y_a, U_\mathfrak{m}, Y_b) \\ & + R(U_\mathfrak{m}, Y_a, S_{S_{U_\mathfrak{m}} U_\mathfrak{m}} U_\mathfrak{m}, Y_b) + R(U_\mathfrak{m}, Y_a, U_\mathfrak{m}, S_{S_{U_\mathfrak{m}} U_\mathfrak{m}} Y_b) \ . \end{aligned} \tag{4.2.5}$$

Similar expressions can be obtained for higher order brackets between f and H_0. By calculating enough terms of the form $[f, \dots, [f, H_0]]$, one can solve for the skew-symmetric matrix f, but in fact, it is not difficult to write down a general solution.

First we note that since U is geodetic, we have

$$S_{U_\mathfrak{m}} U_\mathfrak{m} + S_{U_\mathfrak{h}} U_\mathfrak{m} = S_U U_\mathfrak{m} = -c \, U_\mathfrak{m} \ ,$$

where we are extending[1] the definition (2.4.4) of S to the whole of $\mathfrak{g}$ by $S_Y X = \nabla_X Y$. Together with invariance of the curvature, this allows one to manipulate (4.2.4)

$$\begin{aligned} [f, H_0]_{ab} &= R(U_\mathfrak{m}, (S_{U_\mathfrak{m}} + S_{U_\mathfrak{h}}) Y_a, U_\mathfrak{m}, Y_b) + R(U_\mathfrak{m}, Y_a, U_\mathfrak{m}, (S_{U_\mathfrak{m}} + S_{U_\mathfrak{h}}) Y_b) \\ &= \langle R(U_\mathfrak{m}, Y_b) U_\mathfrak{m}, S_U Y_a \rangle + \langle R(U_\mathfrak{m}, Y_a) U_\mathfrak{m}, S_U Y_b \rangle \ . \end{aligned} \tag{4.2.6}$$

Recall that $(H_0)_{ab} = -R(U_\mathfrak{m}, Y_a, U_\mathfrak{m}, Y_b)$, therefore, we can take f to be

$$f_{ab} = -\langle S_U(Y_a), Y_b \rangle = S(U, Y_b, Y_a)$$

where we have used that

$$\langle S_U Y_a, U_\mathfrak{m} \rangle = -c \langle Y_a, U_\mathfrak{m} \rangle = 0$$

[1]This is clearly consistent with its definition on $\mathfrak{m}$, as the canonical connection vanishes there. In this way it denotes the skew-symmetric endomorphism $-A_X$ of TM associated to a Killing vector, as described in chapter 2. Notice, though, that strictly speaking this is an abuse of notation since S is tensorial, so that $S(\mathfrak{h})$ should vanish at o but here it clearly does not.

and thus

$$\langle S_U Y_a, U_{\mathfrak{m}}\rangle\langle V, R(U_{\mathfrak{m}}, Y_b)U_{\mathfrak{m}}\rangle = \langle S_U Y_a, V\rangle\langle U_{\mathfrak{m}}, R(U_{\mathfrak{m}}, Y_b)U_{\mathfrak{m}}\rangle = 0 \ . \tag{4.2.7}$$

In summary, the plane-wave limit is given by

$$g_{pl} = dx^+\left(2e^{-2cx^+}dx^- + H_0(e^{-x^+f}\boldsymbol{x}, e^{x^+f}\boldsymbol{x})dx^+\right) + |d\boldsymbol{x}|^2 \ ,$$

where

$$\begin{aligned} c &= -S(U,U,V) \\ f_{ab} &= -S(U,Y_a,Y_b) \\ (H_0)_{ab} &= -R(U_{\mathfrak{m}},Y_a,U_{\mathfrak{m}},Y_b) \ , \end{aligned} \tag{4.2.8}$$

with the curvature given by (2.4.8) and the extension of S to $\mathfrak{g}$ given by

$$S(X,Y,Z) = \frac{1}{2}\langle [X,Y]_{\mathfrak{m}}, Z_{\mathfrak{m}}\rangle + \frac{1}{2}\langle [Z,X]_{\mathfrak{m}}, Y_{\mathfrak{m}}\rangle + \frac{1}{2}\langle [Z,Y]_{\mathfrak{m}}, X_{\mathfrak{m}}\rangle \ ,$$

so that the cumbersome enterprise of taking a plane-wave limit along a homogeneous geodesic is reduced to some algebraic and straightforward calculations. The result is a regular plane-wave if $c = 0$ and a singular plane-wave if $c \neq 0$.

We can also apply some of the above discussion to a non-homogeneous geodesic with initial direction $U_{\mathfrak{m}}$. For then, the relation

$$(H_0)_{ab} = -R(U_{\mathfrak{m}},Y_a,U_{\mathfrak{m}},Y_b) \tag{4.2.9}$$

still holds. As done above, we can take the derivative of the curvature tensor and if there exists a solution f to equation (4.2.4) then the plane-wave limit is homogeneous. However, since there is no $U_{\mathfrak{h}}$ such that $U_{\mathfrak{m}} + U_{\mathfrak{h}}$ is geodetic we need to deal with $S_{U_{\mathfrak{m}}}U_{\mathfrak{m}} = cU_{\mathfrak{m}} + c^V V + c^i Y_i$ where not all the c^i, c^v vanish. From the definition of S above we can easily see that $c^V = 0$. Evaluating the righthand side of (4.2.6) with $h_{ab} = -\langle S_{U_{\mathfrak{m}}}(Y_a), Y_b\rangle$ we find

$$\begin{aligned} [f,H_0]_{ab} = [h,H_0]_{ab} &+ c^a R(U_{\mathfrak{m}},Y_b,U_{\mathfrak{m}},V) + c^b R(U_{\mathfrak{m}},Y_a,U_{\mathfrak{m}},V) \\ &+ R(c^i Y_i,Y_a,U_{\mathfrak{m}},Y_b) + R(c^i Y_i,Y_b,U_{\mathfrak{m}},Y_a) \ , \end{aligned} \tag{4.2.10}$$

and thus the plane-wave limit is homogeneous if and only if we can solve this equation for f.

When M is 4-dimensional, so that H_0 is a 2×2-matrix, it is not difficult to see that there exists a solution f to $P = [f, H_0]$ if and only if P is of the form

$$\begin{pmatrix} 2f_{12}(H_0)_{12} & f_{12}((H_0)_{22} - (H_0)_{11}) \\ f_{12}((H_0)_{22} - (H_0)_{11}) & -2f_{12}(H_0)_{12} \end{pmatrix} . \tag{4.2.11}$$

In other dimensions, one way to formulate a necessary and sufficient condition for the existence of such a solution is to suppose that H_0 is diagonal:

$$H_0 = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{n-2} \end{pmatrix} . \tag{4.2.12}$$

We can always arrange this by choosing a basis of eigenvectors Y_i of H_0 with eigenvalues λ_i. Then there exists a solution f to $P = [f, H_0]$ if and only if $\lambda_i = \lambda_j$ implies that $P_{ij} = 0$.

This gives a method for deciding when the plane-wave limit is homogeneous, however a given solution to equation (4.2.10) may not lead you to the correct plane-wave limit since such solutions are not necessarily unique. In order to specify the unique f for the plane-wave limit one may need to consider higher derivatives of the curvature tensor such as (4.2.5).

4.2.2 The nearly-adapted method

One thing the covariant approach to plane-wave limits teaches us is that the limit does not care about such details as the embedding of the null geodesic [28]. In particular, this means that one should be able to use a not necessarily twist-free coordinate system, which in many cases is the natural starting point, since generically a geodesic vector will not generate a twistfree congruence.

Let γ be a null homogeneous geodesic generated by a geodetic vector $U \in \mathfrak{g}$ so that equation (2.5.1) holds. Let $V \in \mathfrak{m}$ be the complimentary null vector to $U_{\mathfrak{m}}$ and complete with $(Y_i) \in \mathfrak{m}$ to a lightcone-orthonormal frame.

Let our local coset representative σ be

$$\sigma = e^{\sum_i y^i Y_i} e^{vV} e^{uU} . \tag{4.2.13}$$

Then the Maurer-Cartan form θ can be expanded as

$$\sigma^*\theta = \theta^U U + \theta^V V + \theta^i Y_i + \theta^\alpha e_\alpha$$

where Greek indices are reserved for the isotropy and (e_α) is a basis for $\mathfrak{h}$. The metric can then be expanded as

$$g = 2\theta^U \theta^V + \sum_i (\theta^i)^2 . \tag{4.2.14}$$

Calculating the Maurer-Cartan form using σ gives

$$\sigma^*(\theta) = \sigma^{-1} d\sigma = e^{-uU} e^{-vV} e^{-\sum_i y^i Y_i} d\left(e^{\sum_i y^i Y_i}\right) e^{vV} e^{uU} + e^{-uU} V dv e^{uU} + U du .$$

where we can calculate the first term using formula (2.4.12). A few things are clear; first du can only appear in θ^U and thus ∂_u is null. This also tells us that the isomorphism from the set of left invariant vector fields to the Lie algebra $\mathfrak{g}$ that is determined by θ maps ∂_u to U. We will denote the inverse of this isomorphism as $\mathfrak{g} \ni X \mapsto X^*$ in the following. Secondly,

$$\partial_u \theta^V = \partial_u \langle \theta_{\mathfrak{m}}, U_{\mathfrak{m}} \rangle = U^* g(\theta^*_{\mathfrak{m}}, U^*_{\mathfrak{m}}) = g(\nabla_{U^*}\theta^*_{\mathfrak{m}}, U^*_{\mathfrak{m}}) + g(\theta^*_{\mathfrak{m}}, \nabla_{U^*} U^*_{\mathfrak{m}}) \ ,$$

where $\theta^*_{\mathfrak{m}} = \theta^U U^* + \theta^V V^* + \theta^i Y^*_i$. Now applying the identity (2.4.1) we have,

$$\partial_u \theta^V = g([U^*, \theta^*_{\mathfrak{m}}], U^*_{\mathfrak{m}}) = -\langle [U, \theta_{\mathfrak{m}}]_{\mathfrak{m}}, U_{\mathfrak{m}} \rangle = -c\langle \theta_{\mathfrak{m}}, U_{\mathfrak{m}} \rangle = -c\theta^V \ ,$$

where we have used that U is geodetic. This shows that the only dependence on u in θ^V is a multiplicative factor of e^{-cu}. In particular, since the dv part of θ is only dependent on u, the $dudv$ part of the metric is of the form e^{-cu}. This can be absorbed into the rest of the metric by a coordinate change:

$$u \mapsto -\frac{1}{c}\log u \ ,$$

however, this is not necessary since u is not rescaled in the plane-wave limit. Also, it is important to note that this coordinate system is not necessarily a twist-free adapted coordinate system of the form (3.2.1). We will see that this is not important and one can still take a plane-wave limit.

We can expand out the Maurer-Cartan form further and then take the plane-wave limit.

$$\theta^U = du + \langle e^{-uU} V e^{uU}, V \rangle dv + \theta^U_i dy^i \ ,$$

where θ^U_i is a function of u, v and (y^i). Applying the plane-wave limit rescaling $(u, v, y^i) \mapsto (u, \Omega^2 v, \Omega y^i)$ to θ^U and taking the limit $\Omega \to 0$ we see that $\theta^U \to du$.

$$\begin{aligned}\theta^V &= e^{-cu}(dv + \langle e^{-vV} e^{-\sum_i y^i Y_i} d\left(e^{\sum_i y^i Y_i}\right) e^{vV}, U_{\mathfrak{m}} \rangle dy^i) \\ &= e^{-cu}(dv + (\langle -y^j [Y_j, Y_i]_{\mathfrak{m}} + \ldots, U_{\mathfrak{m}} \rangle dy^i)\end{aligned}$$

where $\ldots$ are terms involving v and higher order terms in y^j. If we rescale by Ω^{-2}, apply the plane-wave limit rescaling and take the limit $\Omega \to 0$ we find that all the terms in $\ldots$ go to zero and we are left with,

$$\theta^V_{pl} = e^{-cu}(dv - y^j \langle [Y_j, Y_i]_{\mathfrak{m}}, U_{\mathfrak{m}} \rangle dy^i) \ .$$

Similarly for θ^i we have

$$\begin{aligned}\theta^i &= \langle e^{-vV} e^{-\sum_i y^i Y_i} d\left(e^{\sum_i y^i Y_i}\right) e^{vV}, Y_j \rangle dy^j \\ &= \langle (e^{-uU} Y_j e^{uU})_{\mathfrak{m}} + \ldots, Y_i \rangle dy^j \ ,\end{aligned}$$

where ... are terms which involve v and higher order terms in y^i. Re-scaling by Ω^{-1} and taking the plane-wave limit we are left with

$$\theta^i_{pl} = \langle (e^{-uU} Y_j e^{uU})_{\mathfrak{m}}, Y_i \rangle dy^j \ .$$

Therefore the plane-wave limit of the metric in this coordinate system is well defined:

$$g_{pl} = 2\theta^V_{pl} du + \sum_i (\theta^i_{pl})^2 \ .$$

Expanding this we find that the metric is nearly a plane wave in Rosen coordinates (as one would expect if this was the standard plane-wave limit) but it has an additional $dudy^i$ term with a coefficient which is linear in y^j:

$$g_{pl} = 2e^{-cu} du(dv - y^j \langle [Y_j, Y_i]_{\mathfrak{m}}, U_{\mathfrak{m}} \rangle dy^i) + \langle (e^{-uU} Y_i e^{uU})_{\mathfrak{m}}, (e^{-uU} Y_j e^{uU})_{\mathfrak{m}} \rangle dy^i dy^j \ .$$

Note that we have had to use that U is geodetic in the calculation of the last term. We can make the change to a Brinkmann type coordinate system irrespective of this extra term. If we let

$$Q^i_a(u) = \langle (e^{uU} Y_a e^{-uU})_{\mathfrak{m}}, Y_i \rangle \ ,$$

then under the coordinate change defined by (3.1.4), we find the metric is

$$\begin{aligned} g_{pl} =& 2e^{-2cx^+} dx^- dx^+ + (\langle [U, Y_a]_{\mathfrak{m}}, Y_b \rangle - \langle [U, Y_b]_{\mathfrak{m}}, Y_a \rangle - \langle [Y_a, Y_b]_{\mathfrak{m}}, U_{\mathfrak{m}} \rangle) x^b dx^a dx^+ \\ &+ (\langle [U, Y_a]_{\mathfrak{m}}, [U, Y_b]_{\mathfrak{m}} \rangle - \langle [Y_a, [U, Y_b]]_{\mathfrak{m}}, U \rangle) x^a x^b (dx^+)^2 + \sum_i (dx^i)^2 \ . \end{aligned}$$

Notice that

$$\langle [Y_a, [U, Y_b]]_{\mathfrak{m}}, U \rangle$$

is symmetric in a and b because of the Jacobi identity and the geodetic vector property (2.5.1). In light of the above, we also define

$$f_{ab} = \frac{1}{2} \langle [U, Y_a]_{\mathfrak{m}}, Y_b \rangle - \frac{1}{2} \langle [U, Y_b]_{\mathfrak{m}}, Y_a \rangle - \frac{1}{2} \langle [Y_a, Y_b]_{\mathfrak{m}}, U_{\mathfrak{m}} \rangle \ . \qquad (4.2.15)$$

To show this is a plane wave and bring it to the proper Brinkmann form we make the change of coordinates (3.1.6)

$$y_a \mapsto e^{-f_{ab} x^+} y_b \ .$$

This leaves the metric in the form

$$2e^{-2cx^+} dx^- dx^+ + (e^{x^+ f} H_0 e^{-x^+ f})_{ab} x^a x^b (dx^+)^2 + \sum_i (dx^i)^2 \ , \qquad (4.2.16)$$

where

$$(H_0)_{ab} = \langle [U, Y_a]_{\mathfrak{m}}, [U, Y_b]_{\mathfrak{m}} \rangle - \langle [Y_a, [U, Y_b]]_{\mathfrak{m}}, U \rangle + f^2_{ab} \ . \tag{4.2.17}$$

An easy check shows that these formulae do indeed coincide with those derived by the covariant method.

However, since we have not worked with an adapted coordinate system, at no stage in the above have we proved that the formula we have obtained is actually for the usual plane-wave limit of the geodesic γ. At least, not a proof which is independent of the covariant method. We will provide one now. Consider a metric of the form

$$2dudv + \alpha dv^2 + \beta_i dy^i dv + K_{ij} y^i dy^j du + C_{ij} dy^i dy^j \ ,$$

such that ∂_u is a null geodesic and K_{ij} is skew-symmetric. Up to a coordinate transformation in u this is the form of the metric in equation (4.2.14). An easy calculation shows that the R_{uiuj} component of the curvature of this metric:

$$R(\partial_u, \partial_i)\partial_u = -\nabla_{\partial_u} \nabla_{\partial_i} \partial_u + \nabla_{\partial_i} \nabla_{\partial_u} \partial_u + \nabla_{[\partial_u, \partial_i]} \partial_u \ ,$$

is independent of K_{ij}. If we apply the plane-wave limit rescaling, multiply by Ω^{-2} and take the limit as $\Omega \to 0$ we get

$$2dudv + K_{ij} y^i dy^j du + C_{ij}(u) dy^i dy^j \ .$$

This metric is a plane-wave, as we can change to Brinkmann coordinates and then absorb the linear term into the rest of the metric (as we did above). Since a plane-wave is completely determined by the R_{uiuj} part of its curvature, the metric (4.2.16) must be isometric to the usual plane-wave limit of the geodesic ∂_u.

We can relate this nearly-adapted method for taking the plane-wave limit to the Hamilton-Jacobi method described in 3.6.3. The local coset representative (4.2.13) at $\boldsymbol{y} = \boldsymbol{0}$:

$$\sigma(u, v, \boldsymbol{0}) = e^{vV} e^{uU} \ ,$$

defines a geodesic variation of $\gamma(u)$ and hence defines the geodesic congruence in which γ is embedded. The Jacobi-field associated to this variation is the restriction of the Killing vector ξ_V associated to $V \in \mathfrak{m}$ to the geodesic γ [68], this coincides with the left-invariant vector field V^* associated to V restricted to γ. The metric dual of V^*, which is the V component of the Maurer-Cartan form θ^V, is therefore a solution to the Hamilton-Jacobi type equation

$$H \circ \theta^V = 0 \ . \tag{4.2.18}$$

The usual Hamilton-Jacobi equation (3.6.3) defines a twist-free geodesic congruence whereas the congruence considered here is not necessarily twist-free because θ^V is not necessarily exact.

4.3 Homogeneous structures under the plane-wave limit

More than just being able to say that the plane-wave limit is homogeneous, in some circumstances we can say something about the type of homogeneous structure inherited by the plane-wave limit along a homogeneous geodesic. It is clear from (4.2.8) that the homogeneous structure of the plane-wave limit along a homogeneous geodesic is inherited from the original metric g in some sense, since the whole plane-wave limit metric is defined in terms of algebraic data. In fact, (4.2.8) for f can be interpreted as the Ambrose-Singer formula $\nabla_{pl} R_{pl} = S_{pl} \cdot R_{pl}$ on the plane-wave limit. However, this inheritance is not in a continuous fashion, so it is difficult to make conclusions about the type of homogeneous structure inherited under the plane-wave limit. To study this situation we may consider a stronger form of inheritance of the homogeneous structure that is continuous.

Let (M, g) be a reductive homogeneous space with a null homogeneous geodesic γ. The Ambrose-Singer theorem gives us a connection $\tilde{\nabla}$ such that $\tilde{\nabla} S = \tilde{\nabla} R = 0$. Let M_γ be a tubular neighborhood of γ and consider $\phi_\Omega(M_\gamma)$. Now ϕ_Ω is a diffeomorphism for $\Omega \neq 0$ so $\phi_\Omega(M_\gamma)$ is reductive homogeneous for $\Omega > 0$. This defines the metric connection

$$\tilde{\nabla}_\Omega := (\phi_\Omega^{-1})^* \tilde{\nabla} = (\phi_\Omega^{-1})^* \nabla - (\phi_\Omega^{-1})^* S \, . \tag{4.3.1}$$

We may choose adapted coordinates (3.2.1) for g with respect to γ and expand S in these coordinates

$$S = \nabla - \tilde{\nabla} = \sum_{i,j,k=1}^{n} S_{ij}^k dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k} \, . \tag{4.3.2}$$

The components of S scale in the following way under the plane-wave limit map ϕ_Ω

$$S_{uy^i}^v \mapsto \Omega^{-1} S_{uy^i}^v|_\gamma \qquad S_{uu}^v \mapsto \Omega^{-2} S_{uu}^v|_\gamma \qquad S_{uu}^{y^i} \mapsto \Omega^{-1} S_{uu}^{y^i}|_\gamma \, ,$$

and other terms which either remain the same or tend to zero in the limit $\Omega \to 0$.

If γ is canonically homogeneous then there is a Killing vector ξ such that $\xi|_\gamma = h\gamma' = h\partial_u|_\gamma$ and $\xi|_\gamma$ is generated by parallel transport of $\xi(p)$ along γ by the canonical connection. Now by definition,

$$(\nabla_{\partial_u} \partial_u)|_\gamma = 0 \quad \text{and} \quad (\tilde{\nabla}_\xi \xi)|_\gamma = 0$$

where by $|_\gamma$ we mean restriction to $\gamma \in M$ not restriction of the tangent bundle. Thus

$$0 = (\tilde{\nabla}_{h\partial_u} h\partial_u)|_\gamma = (\nabla_{h\partial_u} h\partial_u)|_\gamma - S(h\partial_u, h\partial_u)|_\gamma = h dh(\partial_u)\partial_u|_\gamma - h^2 S(\partial_u, \partial_u)|_\gamma \, ,$$

and therefore,

$$S^{y^i}_{uu}|_\gamma = S^v_{uu}|_\gamma = 0 \; .$$

In fact it is clear that γ is canonically homogeneous if and only if these components vanish. Using metric compatibility of $\tilde{\nabla}$ as in (2.3.1) and the adapted coordinates (3.2.1) we also see $S^v_{uy^i}|_\gamma = 0$. The Levi-Cività connection ∇_{pl} of the plane-wave limit along γ is equal to

$$\lim_{\Omega \to 0} (\phi_\Omega^{-1})^* \nabla \; , \tag{4.3.3}$$

and the above shows that the limit $S_{pl}|_\gamma := \lim_{\Omega\to 0}(\phi_\Omega^{-1})^* S|_\gamma$ is well defined on γ. Thus, by (4.3.1), the limit $\tilde{\nabla}_{pl}|_\gamma := \lim_{\Omega \to 0} \tilde{\nabla}_\Omega|_\gamma$ is well defined. Now

$$\{\tilde{\nabla}_\Omega|_\gamma g_\Omega|_\gamma \mid \Omega \in [0,1]\}$$

is a continuous path in the space of tensors of type $(3,0)$ on γ, whence continuity shows $\tilde{\nabla}_{pl}|_\gamma g_{pl}|_\gamma = 0$. Similarly we have

$$\tilde{\nabla}_{pl}|_\gamma g_{pl}|_\gamma = \tilde{\nabla}_{pl}|_\gamma S_{pl}|_\gamma = \tilde{\nabla}_{pl}|_\gamma R_{pl}|_\gamma = 0 \; . \tag{4.3.4}$$

Let us define $\tilde{\nabla}_{pl}(u,v,y^i) := \tilde{\nabla}_{pl}|_\gamma(u,0,0)$. Since g_{pl} is independent of v, y^i, it follows that

$$\tilde{\nabla}_{pl} g_{pl} = \tilde{\nabla}_{pl} S_{pl} = \tilde{\nabla}_{pl} R_{pl} = 0 \; . \tag{4.3.5}$$

Therefore theorem 2.2.7 implies that the plane-wave limit is homogenous, and moreover the homogeneous structure has been inherited by g_{pl} in a continuous manner. We shall call this inherited homogeneous structure S_{pl} the **plane-wave limit** of S.

As a corollary of the above discussion, we see that a homogeneous structure S has a well-defined plane-wave limit along a null geodesic $\gamma(t)$ if and only if $\gamma(t)$ can be re-parameterized to a geodesic of the canonical connection with respect to S. Bearing in mind the discussion around equation (2.3.3), one must conclude that the plane-wave limit along a canonical geodesic is equivalent to an Inönü-Wigner contraction [57], where the extra isometries that can arise through the plane-wave limit will be elements of the isotropy subalgebra.

Now let us consider the plane-wave limit along canonically homogeneous geodesics for each of the 8 different classes of homogeneous structures individually;

1. If a metric g admits a vanishing homogeneous structure then its plane-wave limit is a symmetric plane-wave.
2. Suppose the metric g admits a homogeneous structure S of type $\mathcal{T}_1$. Then either it is isometric to anti de-Sitter space and all plane-wave limits are

flat, or it is a singular homogeneous plane-wave. In this case, introducing $\alpha(Z) = g(\xi, Z)$ as in equation (4.2.8),

$$c = -S(U, U, V) = \alpha(U) \ .$$

There are two scenarios to consider, i) $\alpha(U) = 0$ and ii) $\alpha(U) \neq 0$. Comparing this with the classification of homogeneous plane-waves reviewed in Section 3.1, we must conclude that in case i) the resulting spacetime admits a pure $\mathcal{T}_3$ structure and must be a regular homogeneous plane-wave, whereas in case ii) the resulting spacetime is a singular homogeneous plane-wave.

3. Suppose the metric g admits a homogeneous structure S of type $\mathcal{T}_2$ and let $e_1, \ldots, e_n$ be an orthonormal frame with respect to g. Then $\Omega(\phi_\Omega^{-1})_* e_i$ is an orthonormal frame with respect to $\Omega^{-2}\phi_\Omega^* g$ for $\Omega > 0$. Thus

$$\begin{aligned} 0 &= C_{12}(S)((\phi_\Omega)_* Z) = \sum_i S(e_i, e_i, (\phi_\Omega)_* Z) \\ &= \sum_i \Omega^{-2}\phi_\Omega^* S(\Omega(\phi_\Omega^{-1})_* e_i, \Omega(\phi_\Omega^{-1})_* e_i, Z) = C_{12}(S_\Omega)(Z) \ . \end{aligned} \tag{4.3.6}$$

Thus if the limit of $C_{12}(S_\Omega)$ as $\Omega \to 0$ is well defined we must have $C_{12}(S_{pl})(Z) = 0$.

Now, using the basis from proposition 3.4.2 we find

$$\begin{aligned} 0 &= C_{12}(S_{pl})(\partial_v) = S_{pl}(e_u, e_u, \partial_v) + S_{pl}(e_v, e_v, \partial_v) + S_{pl}(e_i, e_i, \partial_v) \\ &= \frac{1}{2} S^u_{uu}(u), \end{aligned} \tag{4.3.7}$$

where we have used $(S_{pl})^u_{vv} = (S_{pl})^u_{e_i e_i} = 0$. Thus we find that the null homogeneous geodesic ∂_u is absolutely homogeneous on g_{pl} and hence the plane-wave limit is a regular homogeneous plane-wave.

4. If a metric g admits a homogeneous structure S of type $\mathcal{T}_3$ then all plane-wave limits are regular homogeneous plane-waves.

5. If a metric g admits a homogeneous structure S of type $\mathcal{T}_1 \oplus \mathcal{T}_2$ then all plane-wave limits along canonical geodesics again admit a $\mathcal{T}_1 \oplus \mathcal{T}_2$ structure. Note that this tells us little about whether g_{pl} is a singular or regular homogeneous plane-wave since the inherited homogeneous structure S_{pl} is not necessarily the same as those given after theorem 3.1.1.

6. Suppose the metric g admits a homogeneous structure S of type $\mathcal{T}_1 \oplus \mathcal{T}_3$, then we can apply the same argument as in the $\mathcal{T}_1$ case above to consider two cases: i) $\alpha(U) = 0$ and ii) $\alpha(U) \neq 0$. In case i) the plane-wave limit

admits a pure $\mathcal{T}_3$ structure and must be a regular homogeneous plane wave, whereas in case ii) the resulting spacetime is a singular homogeneous plane-wave.

7. If a metric g admits a homogeneous structure S of type $\mathcal{T}_2 \oplus \mathcal{T}_3$ then we can apply the same argument as given for the $\mathcal{T}_2$ case to see that any plane-wave limit along a canonical geodesic is a naturally reductive plane-wave.

8. If a metric g admits a homogeneous structure of type $\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$ nothing more can be said.

4.4 Examples

Any reductive homogeneous space may be expressed in terms of the following data: a reductive Lie algebra of isometries $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with an $\mathfrak{h}$-invariant lorentzian inner product $\langle -, - \rangle$ on $\mathfrak{m}$. We will use the following recipe to compute all the plane-wave limits along homogeneous geodesics of such spaces:

1. First we determine the possible null directions up to isometry by decomposing the projectivised light-cone of $\mathfrak{m}$ under the orbits of the exponentiated action of $\mathfrak{h}$. We label each orbit by giving a null direction in the light-cone.

2. Next we distinguish those null directions $U_\mathfrak{m} \in \mathfrak{m}$ for which the geodesic with initial direction $U_\mathfrak{m}$ is homogeneous and those which are not. This amounts to determining whether there is some $U_\mathfrak{h} \in \mathfrak{h}$ for which $U := U_\mathfrak{m} + U_\mathfrak{h}$ is geodetic; that is, whether U obeys (2.5.1) for some value of c. If it does, then the plane-wave limit along U will be homogeneous: regular if $c = 0$ and singular otherwise.

3. For the geodetic vectors U we choose a frame $U_\mathfrak{m}, V, Y_a$ for $\mathfrak{m}$ such that $\langle U_\mathfrak{m}, V \rangle = 1$ and $\langle Y_a, Y_b \rangle = \delta_{ab}$. Then we determine the explicit form of the plane-wave metric by computing the matrices f and H_0 using formulae (4.2.15) and (4.2.17), respectively.

4. For the non-geodetic directions $U_\mathfrak{m}$ we use the criterion set out at the end of section 4.2.1 to establish whether the plane-wave limit is homogeneous. If it is homogeneous, then we use equation (4.2.10) to calculate f and (4.2.8) to calculate H_0.

The final calculations of f and H_0 can be implemented using one's favorite computer algebra software.

4.4.1 Higher dimensional Gödel universes

The five-dimensional Gödel universe is a reductive homogeneous space and also a maximally supersymmetric solution of minimal five-dimensional supergravity, whose lift to M-theory in 11-dimensions preserves 20 supersymmetries [3]. The plane-wave limit of the five-dimensional Gödel universe is the five-dimensional maximally supersymmetric plane-wave [69]. The plane-wave limits of the M-theory Gödel universe were investigated in [17] and shown to form a family of time-dependent plane-waves interpolating between two symmetric plane-waves, one of which corresponds to the lift to M-theory of the five-dimensional maximally supersymmetric plane-wave. In this subsection, we will rederive these results using our Lie algebraic formalism.

4.4.1.1 The five-dimensional Gödel universe

We start with the five-dimensional Gödel universe, which is defined on a circle bundle over flat euclidean space:

$$g = -(dt + A)^2 + \sum_{i=1}^{4}(dx^i)^2 \ , \tag{4.4.1}$$

where the connection one-form A is given by

$$A = \frac{1}{2}(x^1 dx^2 - x^2 dx^1) - \frac{1}{2}(x^3 dx^4 - x^4 dx^3) \ . \tag{4.4.2}$$

The two-form F which makes the Gödel universe a five dimensional supergravity solution is given simply by

$$F = dA = dx^1 \wedge dx^2 - dx^3 \wedge dx^4 \ , \tag{4.4.3}$$

which is clearly an anti-selfdual two-form on $\mathbb{E}^4$ with respect to the natural orientation. Clearly the form $F = dA$ is left unchanged if a closed 1-form is added to $A \mapsto A + d\mu$. This allows one to promote any infinitesimal symmetry of F to an isometry by adding a compensating gauge transformation. The two-form F is invariant under both a subgroup $\mathrm{U}(2) \ltimes \mathbb{R}^4$ of the group of isometries of $\mathbb{E}^4$, and by the $\mathrm{U}(1)$ group which acts by translation along the fibre and is generated infinitesimally by ∂_t. The $\mathrm{U}(2)$ and the $\mathrm{U}(1)$ acting on the fibre still leave the metric invariant, but the $\mathbb{R}^4$ translations do not because they do not leave A invariant. Nevertheless a gauge transformation can be added to make $dt + A$ and hence the metric invariant. Doing so one finds the following Killing vectors leaving g and

F invariant:

$$\begin{gathered}
\partial_t \qquad \partial_1 - \frac{1}{2}x^2\partial_t \qquad \partial_2 + \frac{1}{2}x^1\partial_t \qquad \partial_3 + \frac{1}{2}x^4\partial_t \qquad \partial_4 - \frac{1}{2}x^3\partial_t \\
x^1\partial_2 - x^2\partial_1 \qquad x^3\partial_4 - x^4\partial_3 \\
x^1\partial_3 - x^3\partial_1 + x^2\partial_4 - x^4\partial_2 \qquad x^1\partial_4 - x^4\partial_1 - x^2\partial_3 + x^3\partial_2 \;.
\end{gathered} \tag{4.4.4}$$

Notice that at any point (t, x^i) of M, the five Killing vectors in the first line span the tangent space, so that M is indeed a homogeneous space.

The isometry algebra $\mathfrak{g}$ is isomorphic to the semidirect product

$$\mathfrak{g} \cong (\mathfrak{su}(2) \times \mathfrak{u}(1)) \ltimes \mathfrak{h}(2) \;,$$

where $\mathfrak{h}(2)$ is the two-dimensional Heisenberg algebra

$$[P_i, P_j] = \Omega_{ij} P_0 \;,$$

generated by $P_0 = \partial_t$ and $P_i = \partial_i - \frac{1}{2}\sum_j \Omega_{ij} x^j \partial_t$, where Ω_{ij} is the symplectic form with nonzero entries $\Omega_{12} = 1 = -\Omega_{21}$ and $\Omega_{34} = -1 = -\Omega_{43}$. The $\mathfrak{su}(2) \times \mathfrak{u}(1) \subset \mathfrak{so}(4)$ in the expression for $\mathfrak{g}$ acts on $\mathfrak{h}(2)$ by restricting the natural action of $\mathfrak{so}(4)$ on the P_i. The corresponding isometry group G is given by

$$G \cong \mathrm{U}(2) \ltimes \mathrm{H}(2) \;,$$

with $\mathrm{U}(2) \subset \mathrm{SO}(4)$ acting on $\mathrm{H}(2)$ in the natural way.

Let $o \in M$ be the origin coset with coordinates $(t = x^i = 0)$. The vectors $P_0, P_1, \ldots, P_4$ form a pseudo-orthonormal frame for T_oM, with P_0 timelike. The isotropy subgroup H which fixes o is precisely the above $\mathrm{U}(2)$ subgroup of G, and therefore $M \cong G/\mathrm{U}(2)$. A calculation of ∇R shows that M is not symmetric.

The decomposition of the full isometry algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}(2)$ is reductive. Using equation (2.4.4) we find that the components $S_{ijk} = S(P_i, P_j, P_k)$ of the homogeneous structure at o are given by

$$S_{0ij} = S_{i0j} = -S_{ij0} = \frac{1}{2}\Omega_{ij} \;,$$

which can be seen to be of type $\mathcal{T}_2 \oplus \mathcal{T}_3$.

We can deform this homogeneous structure by considering a reductive split $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}'$ where $\mathfrak{m}'$ is the graph of an $\mathfrak{h}$-equivariant linear map $\mathfrak{m} \longrightarrow \mathfrak{h}$. Decomposing $\mathfrak{m}$ and $\mathfrak{h}$ into irreducibles we find that there is a one-parameter map of such linear maps $\varphi_\alpha(v^i P_i) = \alpha v^0 Y_0$, where $Y_0 \in \mathfrak{h}$ is the Killing vector $Y_0 = x^1\partial_2 - x^2\partial_1 + x^3\partial_4 - x^4\partial_3$. Its graph $\mathfrak{m}'$ is spanned by

$$P_1 \;, \quad P_2 \;, \quad P_3 \;, \quad P_4 \;, \quad \text{and} \quad P_0 + \alpha Y_0 \;.$$

This modifies the $[-,-]_{\mathfrak{m}'}$ brackets:

$$[P_i, P_j]_{\mathfrak{m}'} = \Omega_{ij}(P_0 + \alpha Y_0) \quad \text{and} \quad [P_0 + \alpha Y_0, P_i]_{\mathfrak{m}'} = \alpha \Omega_{ij} P_j \ .$$

We can now compute the corresponding homogeneous structure using formula (2.4.4) and we obtain a one-parameter family of $\mathcal{T}_2 \oplus \mathcal{T}_3$ structures:

$$S_{0ij} = (\frac{1}{2} + \alpha)\Omega_{ij} \quad \text{and} \quad S_{i0j} = -S_{ij0} = \frac{1}{2}\Omega_{ij} \ . \tag{4.4.5}$$

Naturally, when $\alpha = 0$ we recover the earlier homogeneous structure. For generic α this homogeneous structure is of type $\mathcal{T}_2 \oplus \mathcal{T}_3$, but there are two special values of α: for $\alpha = -1$ it is of type $\mathcal{T}_3$ and for $\alpha = \frac{1}{2}$ it is of type $\mathcal{T}_2$. This shows that the Gödel universe is naturally reductive, and in particular a g.o. space.

One can obtain more homogeneous structures by considering smaller subalgebras, but we will not do so here.

In order to determine all the plane-wave limits of the Gödel universe we will exploit the covariance property of the plane-wave limit 3.2.1. A null geodesic γ in M is locally determined by an initial point $\gamma(0) \in M$ and an initial direction $\gamma'(0)$, which is a point on the future-pointing, say, celestial sphere at $\gamma(0)$. Since M is homogeneous, we can let $\gamma(0)$ be any convenient point; we will choose the origin o and retain the freedom of using the isotropy subgroup of o. The (future) celestial sphere at o, which consists of those vectors $v = v^\mu P_\mu$ such that $\langle v, v\rangle = 0$ and $v^0 = 1$, is the unit three-sphere in $\mathbb{E}^4 = \langle P_0\rangle^\perp$. The isotropy group $\mathrm{U}(2)$ acts on $\mathbb{E}^4$ by restricting the natural representation of $\mathrm{SO}(4)$, whence it acts transitively on the spheres. Therefore we see that the isometry group of (M, g, F) acts transitively on the space of null geodesics and hence all plane-wave limits are isometric.

Let us choose our geodesic to have initial direction $P_0 + P_1$. This vector is not geodetic, however we may modify it by adding a vector $U_{\mathfrak{h}} \in \mathfrak{h}$ in such a way that (2.5.1) is satisfied. A quick calculation shows that $P_0 + P_1 - Y_0$ is geodetic with $c = 0$, which means that the plane-wave limit is a regular homogeneous wave. Moreover, this geodesic is canonically homogeneous with respect to the reductive split $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}'$ with $\mathfrak{m}'$ spanned by $P_0 - Y_0$, P_1, P_2, P_3 and P_4.

In fact the limit is the symmetric plane wave discovered in [69]. To determine the limit we employ the formulae (4.2.15) and (4.2.17). We find that

$$f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad \text{and} \qquad H_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix} ,$$

in agreement with the results of [17].

4.4.1.2 The Gödel universe in M-theory

The five-dimensional Gödel universe can be lifted to a supersymmetric M-theory background $(\widetilde{M}, g, G)$ preserving 20 supersymmetries [3] simply by taking its riemannian product with a flat six-dimensional space. If we think of this six-dimensional space as $\mathbb{C}^3$ with its standard Kähler structure ω, then the M-theory four-form (1.1.1) is given by $G = F \wedge \omega$. It follows that the symmetry group of this M-theory background is

$$(\mathrm{U}(2) \ltimes \mathrm{H}(2)) \times \left(\mathrm{U}(3) \ltimes \mathbb{R}^6\right) ,$$

which still acts transitively, making $(\widetilde{M}, g, G)$ into a homogeneous background. Let z^α denote local coordinates on $\mathbb{C}^3$ and let o be the origin with coordinates $t = x^i = z^\alpha = 0$. The isotropy subgroup which fixes this point is $\mathrm{U}(2) \times \mathrm{U}(3)$, and this defines the reductive split $(\mathfrak{h}(2) \oplus \mathbb{R}^6) \oplus \mathfrak{h}$.

The isotropy subgroup H acts with cohomogeneity one on the (future) celestial sphere in $T_o\widetilde{M}$. Indeed, we can decompose a tangent vector into $\boldsymbol{v} = \boldsymbol{v}_G + \boldsymbol{v}'$, with $\boldsymbol{v}_G$ the component tangent to the five-dimensional Gödel universe and $\boldsymbol{v}'$ the component tangent to $\mathbb{C}^3$. The action of H preserves the norms $|\boldsymbol{v}_G|^2$ and $|\boldsymbol{v}'|^2$ separately. Let $\boldsymbol{v}$ be a future-pointing null vector. By further rescaling, we can take the P_0 component to be 1, whence $\boldsymbol{v}_G = P_0 + \boldsymbol{v}_\perp$ where $|\boldsymbol{v}_\perp|^2 + |\boldsymbol{v}'|^2 = 1$. Fix an angle $\vartheta \in [0, \frac{\pi}{2}]$ and let $|\boldsymbol{v}_\perp| = \cos\vartheta$ and $|\boldsymbol{v}'| = \sin\vartheta$. The isotropy subgroup cannot change ϑ, but it acts transitively on these spheres, whence we can make $\boldsymbol{v}_\perp$ and $\boldsymbol{v}'$ point in any desired direction. Letting T_i denote the translation generators for the $\mathbb{R}^6$ subgroup of the isometries of $\mathbb{C}^3$, we can write the null vector as

$$P_0 + \cos\vartheta P_1 + \sin\vartheta T_1 \ .$$

This vector is not geodetic unless we add $-Y_0$, as in the five-dimensional Gödel universe. Doing so we see that

$$P_0 + \cos\vartheta P_1 + \sin\vartheta T_1 - Y_0$$

does obey equation (2.5.1) with $c = 0$. This means that the plane-wave limits will again be regular.

Indeed, using equation (4.2.15), we find that the only nonzero components of f are

$$f_{14} = -\frac{1}{2}\sin\vartheta \qquad \text{and} \qquad f_{23} = -\frac{1}{2} \ .$$

Similarly, using equation (4.2.17) the matrix H_0 has nonzero components

$$(H_0)_{11} = -1 + \tfrac{3}{4}\sin^2\vartheta \ , \quad (H_0)_{22} = A_{33} = -\tfrac{1}{4} \ , \qquad \text{and} \qquad (H_0)_{44} = -\tfrac{1}{4}\sin^2\vartheta \ .$$

Notice that since $[H_0, f] \neq 0$ this is not a symmetric plane wave.

4.4.2 Kaigorodov space

The Kaigorodov space K is an $(n+3)$-dimensional lorentzian manifold with metric [70]

$$-(\varepsilon^0)^2 + \sum_{i=1}^{n+2} (\varepsilon^i)^2$$

where

$$\varepsilon^0 = e^{(4+n)\ell\rho} dt \ , \quad \varepsilon^i = e^{2\ell\rho} dy^i \ , \quad \varepsilon^{n+1} = e^{-n\ell\rho} dx + e^{(4+n)\ell\rho} dt \ , \quad \varepsilon^{n+2} = d\rho \ ,$$

where, here and in the sequel, the indices $i, j, \ldots$ run from 1 to n. This spacetime can be seen to have a pp-wave singularity and is not geodesically complete [71]. Up to homothety, we can (and will) set $\ell = 1$ from now on.

The Killing vector fields of this metric can be seen to be

$$\begin{aligned}
X_0 &= \frac{\partial}{\partial t}, \qquad X_{n+1} = \frac{\partial}{\partial x}, \qquad X_i = \frac{\partial}{\partial y^i} \ , \\
X_{n+2} &= \frac{\partial}{\partial \rho} - (n+4)t\frac{\partial}{\partial t} + nt\frac{\partial}{\partial x} - 2y^i \frac{\partial}{\partial y^i} \ , \\
L_i &= x\frac{\partial}{\partial y^i} - y^i \frac{\partial}{\partial t}, \qquad L_{ij} = y^i \frac{\partial}{\partial y^j} - y^j \frac{\partial}{\partial y^i} \ .
\end{aligned}$$

These determine the full isometry Lie algebra as a semi-direct product $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m}$ where the isotropy subalgebra $\mathfrak{h} = \mathfrak{iso}(n)$ is spanned by the L_i's and L_{jk}'s, and $\mathfrak{m}$ is spanned by X_0, X_{n+1}, X_{n+2} and the X_i's. The algebra is given by

$[,]$	L_r	L_{uv}	X_0	X_i	X_{n+1}	X_{n+2}
L_r	0	$\delta_{vr}X_u - \delta_{ur}X_v$	0	$-\delta_{ri}X_0$	X_r	$(n+2)L_r$
L_{st}	$\delta_{sr}L_t - \delta_{tr}L_s$	L_{stuv}	0	$\delta_{is}X_t - \delta_{ti}X_s$	0	0
X_0	0	0	0	0	0	$(n+4)X_0$
X_i	$\delta_{ri}X_0$	$\delta_{iv}X_u - \delta_{iu}X_v$	0	0	0	$2X_i$
X_{n+1}	$-X_r$	0	0	0	0	$-nX_{n+1}$
X_{n+2}	$-(n+2)L_r$	0	$-(n+4)X_0$	$-2X_i$	nX_{n+1}	0

where $L_{stuv} = \delta_{su}L_{tv} - \delta_{tu}L_{sv} + \delta_{jl}L_{ik} - \delta_{il}L_{jk}$.

The metric induces an inner product on $\mathfrak{m}$ with non-zero terms given by

$$\begin{aligned}
\langle X_i, X_j \rangle &= \delta_{ij} \ , \qquad \langle X_{n+1}, X_{n+1} \rangle = 1 \ , \\
\langle X_{n+2}, X_{n+2} \rangle &= 1 \ , \quad \langle X_0, X_{n+1} \rangle = 1 \ .
\end{aligned} \tag{4.4.6}$$

It is clear that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ is not a reductive split. However, $\mathfrak{m}$ is a Lie algebra in its own right, and therefore exhibits the Kaigorodov space as a Lie group. The corresponding homogeneous structure $S_{abc} = S(X_a, X_b, X_c)$, from equation (2.4.4), is given by

$$\begin{aligned}
S_{n+2,0,n+1} &= -(2+n) \qquad & S_{n+1,0,n+2} &= S_{0,n+1,n+2} = -2 \\
S_{n+1,n+1,n+2} &= n & S_{i,j,n+2} &= -\delta_{ij} \ .
\end{aligned} \tag{4.4.7}$$

It is not hard to see that it has generic type $\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$.

We now determine the action of the isotropy group $\mathrm{ISO}(n)$ on the celestial sphere in T_oK. Relative to the basis $(X_1, \ldots, X_{n+2}, X_0)$, an element $(A, \boldsymbol{b})$ of $\mathrm{ISO}(n) = \mathrm{SO}(n) \ltimes \mathbb{R}^n$ has matrix

$$\begin{pmatrix} A & A\boldsymbol{b} & 0 & 0 \\ \mathbf{0} & 1 & 0 & 0 \\ \mathbf{0} & 0 & 1 & 0 \\ -\boldsymbol{b}^t & -\frac{1}{2}|\boldsymbol{b}|^2 & 0 & 1 \end{pmatrix} ,$$

which has been obtained as the product

$$\begin{pmatrix} A & \\ & \mathbb{1}_3 \end{pmatrix} \exp(b^i L_i) .$$

Acting on a tangent vector $v = (\boldsymbol{v}, v^{n+1}, v^{n+2}, v^0) \in T_oK$, we find

$$(A, \boldsymbol{b}) \cdot \begin{pmatrix} \boldsymbol{v} \\ v^{n+1} \\ v^{n+2} \\ v^0 \end{pmatrix} = \begin{pmatrix} A\boldsymbol{v} + v^{n+1} A\boldsymbol{b} \\ v^{n+1} \\ v^{n+2} \\ v^0 - \boldsymbol{b}^t\boldsymbol{v} - \frac{1}{2}|\boldsymbol{b}|^2 v^{n+1} \end{pmatrix} .$$

Its not hard to check that this indeed preserves the inner product on T_oK. Let v have zero norm, so that

$$(v^{n+1})^2 + (v^{n+2})^2 + |\boldsymbol{v}|^2 = -2v^0 v^{n+1} .$$

Since $v \neq 0$, it follows that $v^0 \neq 0$. We must therefore distinguish two cases, according to whether v^{n+1} does or does not vanish.

- If $v^{n+1} = 0$, then also $v^{n+2} = 0$ and $\boldsymbol{v} = \mathbf{0}$. We can then choose $v^0 = \pm 1$, whence $v = \pm X_0$.

- If $v^{n+1} \neq 0$, then we can choose $\boldsymbol{b} = -\boldsymbol{v}/v^{n+1}$ to bring v to the form

$$\begin{pmatrix} \mathbf{0} \\ v^{n+1} \\ v^{n+2} \\ -\frac{1}{2v^{n+1}}((v^{n+1})^2 + (v^{n+2})^2) \end{pmatrix} ,$$

  where we have used that v is null. We can choose $v^{n+1} = \pm 1$, $v^{n+2} = \alpha$ so that finally

$$v = \pm \left(X_{n+1} + \alpha X_{n+2} - \frac{1}{2}(1 + \alpha^2) X_0 \right) .$$

Choosing, for definiteness, future-pointing null geodesics, the action of the isotropy subgroup leaves two non-isometric null directions, one of them parametrised by a real number α:

$$X_0 \qquad \text{and} \qquad X_{n+1} + \alpha X_{n+2} - \frac{1}{2}(1 + \alpha^2) X_0 . \tag{4.4.8}$$

It is not difficult to check that X_0 is a geodetic vector with $c = 0$, so that the corresponding plane-wave limit will be a regular homogeneous plane-wave. The null geodesic along $X_{n+1} + \alpha X_{n+2} - \frac{1}{2}(1+\alpha^2)X_0$ is only homogeneous when $\alpha^2 = 1$, in which case $X_{n+1} + \alpha X_{n+2} - X_0$ is geodetic with $c = -\alpha(4+n)$ and the limit will be a singular homogeneous plane wave.

It is not difficult to see that in both cases the skew-symmetric matrix f given by equation (4.2.15) vanishes. It is easy to show that when the geodetic vector is X_0, the symmetric matrix $H_0 = 0$, whence the plane-wave limit is flat. When the geodetic vector is $X_{n+1} + \alpha X_{n+2} - X_0$, a calculation shows that the nonzero components of H_0 are

$$(H_0)_{ij} = 4\delta_{ij} \qquad \text{and} \qquad (H_0)_{n+1,n+1} = n^2 \ . \tag{4.4.9}$$

In [31] all the plane-wave limits of both the Kaigorodov space and the product space $K_{n+3} \times S^p$ have been calculated using the Hamilton-Jacobi method 3.6.3. It is shown that the plane-wave limits of $K_{n+3} \times S^p$ along the non-homogeneous null geodesics which have a non-zero component in the tangent space to the sphere are non-homogeneous plane-waves. We can check the non-homogeneous geodesics of the Kaigorodov space using the necessary and sufficient condition derived at the end of section 4.2.1. There are two cases to distinguish:

1. $\alpha = 0$: here $U_m = X_{n+1} - \frac{1}{2}X_0$ and we can take $V = X_0$, $Y_i = X_i$ and $Y_{n+1} = X_{n+2}$. A simple calculation shows that

$$R(U_m, Y_a, U_m, V) = R(Y_i, Y_a, U_m, Y_b) = 0$$

for all a, b, i. Therefore, equation (4.2.10) is solved by $f_{ab} = S(U_m, Y_b, Y_a)$ which makes the plane-wave limit homogeneous. We calculate both c and f to be zero and H_0 to have non-zero components

$$(H_0)_{ij} = -2(n+2)\delta_{ij} \quad \text{and} \quad (H_0)_{n+1,n+1} = 2n(n+2) \ . \tag{4.4.10}$$

Therefore the plane-wave limit is a symmetric plane-wave. Notice that we had to use formulae (4.2.8) for H_0, since equation (4.2.17) holds only for geodetic U.

2. $\alpha \neq 0$: here $U_m = X_{n+1} + \alpha X_{n+2} - \frac{1}{2}(1+\alpha^2)X_0$ and we can take $V = X_{n+1} - \alpha X_{n+2} - \frac{1}{2}(1+\alpha^2)X_0$, $Y_i = X_i$ and $Y_{n+1} = X_{n+1} + \frac{1}{2}(\alpha^2 - 1)X_0$. In this case it is not difficult to see that $R(U_m, Y_a, U_m, Y_b)$ is diagonal but

$$c^{n+1}R(U_m, Y_{n+1}, U_m, V) + R(c^{n+1}Y_{n+1}, Y_a, U_m, Y_b)$$

is non-zero. Therefore the criterion explained at the end of section 4.2.1 shows that the plane-wave limit can not be homogeneous.

4.4.3 Kaplan's g.o. space

We have considered the homogeneous structures of Kaplan's space in section 2.5.2. To calculate the plane-wave limits we first determine the null geodesics up to isometry. Homogeneity means we only need to consider geodesics passing through the origin o of M. A null vector at this point is given by $U_{\mathfrak{m}} = \sum_{i=1}^{6} U^i X_i \in \mathfrak{m}$, with $\sum_{i=1}^{5}(U^i)^2 = (U^6)^2$. Without loss of generality we can choose $U^6 = \pm 1$, depending on whether it is future- or past-pointing, respectively. The isotropy group $\mathrm{SU}(2) \subset \mathrm{SO}(4)$ leaves X_5 invariant and acts transitively on spheres in the four-dimensional space spanned by (X_1, X_2, X_3, X_4). This means that up to isometry, there is a (quarter-)circle family of past- and future-pointing null geodesics, with tangent vectors

$$U_{\mathfrak{m}} = \sin\vartheta X_1 + \cos\vartheta X_5 \pm X_6 \ , \tag{4.4.11}$$

for $\vartheta \in [0, \frac{\pi}{2}]$.

Applying the geodetic vector equations (2.5.7) to $U_{\mathfrak{m}}$ we find

$$\phi_1 = U^5 = \cos\vartheta \qquad \phi_2 = -U^6 = \mp 1 \qquad \phi_3 = 0 \ .$$

This restriction of the geodesic graph $\phi : \mathfrak{m}\backslash\{0\} \to \mathfrak{h}$ to the null vectors is linear, whereas the graph as a whole is nonlinear (showing that M is not naturally reductive), so the space is in some sense like a naturally reductive space when restricted to (certain) null geodesics. These equations tell us the vector we need to add to $U_{\mathfrak{m}}$ to make it geodetic:

$$\sin\vartheta X_1 + \cos\vartheta X_5 \pm X_6 + \cos\vartheta Y_1 \mp Y_2 \ .$$

Using equation (4.2.15), we find that

$$f = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & \pm\frac{1}{2}\cos\vartheta \\ 0 & 0 & \mp\frac{3}{2} & -\frac{3}{2}\cos\vartheta \\ \frac{1}{2} & \pm\frac{3}{2} & 0 & 0 \\ \mp\frac{1}{2}\cos\vartheta & \frac{3}{2}\cos\vartheta & 0 & 0 \end{pmatrix} ,$$

and, using (4.2.17), that

$$H_0 = \begin{pmatrix} \frac{1}{8}(-3-\cos 2\vartheta) & \mp\frac{3}{4}\sin^2\vartheta & 0 & 0 \\ \mp\frac{3}{4}\sin^2\vartheta & \frac{1}{8}(-3-\cos 2\vartheta) & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\cos 2\vartheta & 0 \\ 0 & 0 & 0 & \frac{1}{4}(-3+2\cos 2\vartheta) \end{pmatrix} .$$

It is easy to see that $[H_0, f] = 0$ if and only if $\vartheta = 0$, in which case the resulting spacetime is a conformally flat symmetric plane wave.

4.4.4 Komrakov K1.4[6]

Recall that we used this metric to demonstrate the Hamilton-Jacobi method in section 3.6.3, where we have determined that up to the action of isometries the null directions U_{m} at the origin fall into two families:

1. $U_{\mathsf{m}} = \boldsymbol{u}_1$: a simple calculation shows that in this case u_1 satisfies equation (2.5.1) with $c = 0$ and thus the plane-wave limit along that geodesic will be a regular homogeneous plane-wave. Using equations (4.2.15) and (4.2.17), we find that the plane-wave limit along u_1 is flat.

2. $U_{\mathsf{m}} = \boldsymbol{u}_3 + \alpha \boldsymbol{u}_4 + \frac{1}{2}(1+\alpha^2)\boldsymbol{u}_1 + \boldsymbol{u}_2$ for some $\alpha \in \mathbb{R}$: in this case there is no value of α for which the corresponding geodesic is homogeneous. This was to be expected because our calculation of the plane-wave limit using the Hamilton-Jacobi method in section 3.6.3 showed that it is nonhomogeneous for all α. Indeed, it is not difficult to show that the 2×2-matrix

$$c^a R(U_{\mathsf{m}}, Y_b, U_{\mathsf{m}}, V) + R(c^i Y_i, Y_a, U_{\mathsf{m}}, Y_b) + a \leftrightarrow b$$

   does not satisfy the criterion (4.2.11) which shows that the plane-wave limit is not homogeneous.

4.4.5 Komrakov K1.1^2.1

Recall that we examined this example in section 2.5.1, where we have already determined the null geodetic vectors for this homogeneous space. For ease of exposition we will take $|\alpha| = |\beta| = 1$ in the metric (2.5.2) from now on. We consider the two cases $\alpha = -1, \beta = 1$ and $\alpha = 1, \beta = -1$ and their geodetic vectors described in 2.5.1:

- For $\alpha = -1$, $\beta = 1$ we have $U = u_2 + pu_4 + qu_1$, $q^2 = 1 - p^2$;
  The 2×2-matrix

$$c^a R(U_{\mathsf{m}}, Y_b, U_{\mathsf{m}}, V) + R(c^i Y_i, Y_a, U_{\mathsf{m}}, Y_b) + a \leftrightarrow b \tag{4.4.12}$$

  is of the form (4.2.11) only when $p = \pm 1, q = 0$ or $p = 0, q = \pm 1$ and in fact completely vanishes in both these cases. Whence the plane-wave limit is homogeneous if and only if either $p = 0$ or $q = 0$:

  - For $q = 0, p = \pm 1$ we have the geodetic vector $U = u_2 + pu_4$ for which we find $c = \mp 2$. The skew-symmetric matrix f has components $f_{12} = -\frac{1}{2}$, whereas the symmetric matrix H_0 is given by

$$H_0 = \begin{pmatrix} \frac{3}{4} & \pm 1 \\ \pm 1 & \frac{3}{4} \end{pmatrix} .$$

It is clear that $[f, H_0] \neq 0$. Indeed,

$$e^{zf} H_0 e^{-zf} = \begin{pmatrix} \frac{3}{4} \pm \sin z & \pm \cos z \\ \pm \cos z & \frac{3}{4} \mp \sin z \end{pmatrix} .$$

- For $p = 0, q = \pm 1$ we have $U = u_2 + q u_1$ with $c = 0$. This vector is *not* geodetic, however the plane-wave limit in its direction is homogeneous. The skew-symmetric matrix f vanishes and H_0 is given by

$$H_0 = \begin{pmatrix} 1 \pm 1 & \mp\frac{3}{2} \\ \mp\frac{3}{2} & 3 \end{pmatrix} .$$

Therefore the plane-wave limit is symmetric.

- For $\alpha = 1$, $\beta = -1$ we have $U_m = u_4 + p u_2 + q u_1$, $q^2 = -1 + p^2$;
For this case the matrix (4.4.12) is of the form (4.2.11) if and only if $p = 0, q = \pm 1$ or $p = \pm 1, q = 0$, therefore these are the directions with homogeneous plane-wave limit:

  - For $p = 0, q = \pm 1$ we have $U = u_4 \pm u_2$ is geodetic with $c = -2$;
    In this case, the skew-symmetric matrix f has components $f_{12} = \pm\frac{1}{2}$, whereas the symmetric matrix H_0 is given by

$$H_0 = \begin{pmatrix} \frac{3}{4} & \mp 1 \\ \mp 1 & \frac{3}{4} \end{pmatrix} .$$

    It is clear that $[f, H_0] \neq 0$.

  - For $p = \pm 1, q = 0$ we have $U = u_4 \pm u_1$ with $c = 1$;
    Finally, in this case, the skew-symmetric matrix f has components $f_{12} = \pm\frac{1}{2}$, whereas the symmetric matrix H_0 is given by

$$H_0 = \begin{pmatrix} \frac{15}{4} & \pm 1 \\ \pm 1 & \frac{7}{4} \end{pmatrix} .$$

    Again $[H_0, f] \neq 0$ and indeed

$$e^{zf} H_0 e^{-zf} = \begin{pmatrix} \frac{11}{4} + \cos z - \sin z & \pm(\cos z + \sin z) \\ \pm(\cos z + \sin z) & \frac{11}{4} - \cos z + \sin z \end{pmatrix} .$$

Chapter 5

Supersymmetry and homogeneity

In this chapter we study the relation between supersymmetry and homogeneity. The supersymmetry superalgebra is a natural invariant of a supergravity background whose even and odd subspaces are spanned by the Killing vectors and the Killing spinors respectively. The bracket on the odd subspace is a symmetric bilinear map from the spinor bundle to the tangent bundle which maps Killing spinors to Killing vectors. Under this map, the square of the spinor bundle can naturally be thought of as an extension of the Killing transport bundle $\mathcal{E}$. Given this bracket, it is natural to ask what fraction ν of supersymmetry is required for a background to be necessarily homogeneous? We shall see that if $\nu > \frac{3}{4}$ then the background is indeed homogeneous, and shall provide some evidence that there are non-homogeneous backgrounds with $\nu = \frac{3}{4}$. But we shall start with some of the underlying Clifford algebra.

The results of sections 5.2, 5.3, 5.5 and 5.6 are the products of the collaboration with J. M. Figueroa-O'Farrill and P. Meessen reported in [72].

5.1 Clifford algebras

The Clifford algebra conventions we use mostly follow the books [73] and [74] but for completeness we shall review them here.

Let $\mathbb{E}^{r,s}$ denote the real $(r+s)$-dimensional vector space together with an inner product $\langle\ ,\ \rangle$ defined by the following norm:

$$|x|^2 = x_1^2 + \ldots . + x_r^2 - x_{r+1}^2 - \ldots . - x_{r+s}^2 \,.$$

Let $\bigotimes \mathbb{E}^{r,s}$ denote the tensor product algebra on $\mathbb{E}^{r,s}$ with the inner product obtained by extending $\langle\ ,\ \rangle$ in the usual way. The **Clifford Algebra** $\mathrm{Cl}(r,s)$ is defined as the quotient

$$\frac{\bigotimes \mathbb{E}^{r,s}}{\{x \otimes x = -|x|^2 \mathbb{1}\}}\,, \tag{5.1.1}$$

where $\{x \otimes x = -|x|^2 \mathbb{1}\}$ is the ideal generated by the relation. **Clifford multiplication**, denoted by $\cdot$, on $\mathrm{Cl}(r,s)$ is inherited from tensor product. The Clifford algebra $\mathrm{Cl}(r,s)$ is isomorphic as a vector space (not as an algebra) to the exterior algebra $\bigwedge \mathbb{E}^{r,s}$.

We are particularly interested in eleven dimensional lorentzian signature $\mathbb{E}^{1,10}$. The Clifford algebra $\mathrm{Cl}(1,10)$ is isomorphic to two copies of 32×32 real matrices $M_{32} \times M_{32}$. Therefore up to isomorphism there are two real **Pinor** representations

$$\mathrm{Cl}(r,s) \cong \mathrm{End}_{\mathbb{R}}(P_+) \oplus \mathrm{End}_{\mathbb{R}}(P_-) \ , \tag{5.1.2}$$

where the vector space $P = P_+ \oplus P_-$ is called the space of pinors. These two representations are distinguished geometrically by the volume form vol of $\mathbb{R}^{1,10}$. The isomorphism to the exterior algebra allows elements of $\wedge^n \mathbb{E}^{1,10}$ to act on $P_\pm$.

Now let (M,g) be an 11-dimensional lorentzian manifold with tangent bundle TM and co-tangent bundle T^*M. A choice of co-frame for T^*M gives an isomorphism from each fiber to $\mathbb{E}^{1,10}$. Thus we may construct a Clifford algebra $\mathrm{Cl}(1,10)_x$ above each point $x \in M$. These algebras patch together smoothly to form a **Clifford Bundle** $\mathrm{Cl}(T^*M)$. The isomorphism of the Clifford algebra to the exterior algebra extends to a bundle isomorphism $\mathrm{Cl}(T^*M) \cong \bigwedge T^*M$.

Moreover, if (M,g) is spin, we can form the (not necessarily unique) bundles $\mathcal{S}_\pm$ associated to each of the irreducible representations $P_\pm$ of $\mathrm{Cl}(1,10)$. Differential forms act naturally on sections of $\mathcal{S}_\pm$ via the isomorphism $\bigwedge T^*M \to \mathrm{Cl}(T^*M)$ and the pointwise action of $\mathrm{Cl}(1,10)$ on $\mathcal{S}_\pm$.

Given a pseudo-riemannian manifold (M,g), define the musical isomorphisms $\flat : TM \to T^*M$ and $\sharp : T^*M \to TM$ by

$$X^\flat(Y) = g(X,Y) \quad \text{and} \quad g(\mu^\sharp, X) = \mu(X) \ ,$$

where $X, Y \in TM$ and $\mu \in T^*M$. If $\omega \in \wedge^p T^*M$ then the Clifford product is given by

$$X^\flat \cdot \omega = X^\flat \wedge \omega - \iota_X \omega \ , \tag{5.1.3}$$

and

$$\omega \cdot X^\flat = (-1)^p (X^\flat \wedge \omega + \iota_X \omega) \ . \tag{5.1.4}$$

Iterating these identities we find for example,

$$(X^\flat \wedge Y^\flat) \cdot \omega = X^\flat \wedge Y^\flat \wedge \omega + \iota_X \iota_Y \omega - X^\flat \wedge \iota_Y \omega + Y^\flat \wedge \iota_X \omega \ , \tag{5.1.5}$$

and

$$\omega \cdot (X^\flat \wedge Y^\flat) = X^\flat \wedge Y^\flat \wedge \omega + \iota_X \iota_Y \omega + X^\flat \wedge \iota_Y \omega - Y^\flat \wedge \iota_X \omega \ . \tag{5.1.6}$$

If ω is a p-form and $*\omega$ its Hodge dual, then their Clifford actions are related by

$$* \omega = (-1)^{p(p+1)} \omega \cdot \mathrm{vol} . \tag{5.1.7}$$

The bundles $\mathcal{S}_\pm$ inherit from $P_\pm$ a symplectic structure which is compatible with the action of the Clifford algebra; that is,

$$(\psi, X^\flat \cdot \phi) = -(X^\flat \cdot \psi, \phi) . \tag{5.1.8}$$

This identity implies that the bilinear form

$$\beta_X(\psi, \phi) = (\psi, X^\flat \cdot \phi) , \tag{5.1.9}$$

associated with the vector X is symmetric.

More generally, if ω is a p-form and ω^* denotes its adjoint with respect to this symplectic structure; that is

$$(\omega \cdot \psi, \phi) = (\psi, \omega^* \cdot \phi) . \tag{5.1.10}$$

One finds that

$$\omega^* = (-1)^{p(p+1)/2} \omega , \tag{5.1.11}$$

whence 1-forms, 2-forms and 5-forms preserve the symplectic structure. Indeed, $\mathfrak{sp}(32, \mathbb{R}) = \Lambda^1 \oplus \Lambda^2 \oplus \Lambda^5$ under $\mathfrak{so}(1, 10)$.

As a principle for this chapter, indeed the whole thesis, we will try to work invariantly whenever possible. However, sometimes the physics notation of expressing Clifford products with indices can simplify calculations significantly. We will therefore sometimes use an explicit basis of gamma matrices (Γ^i) with $i = 0, \dots, 10$ for $\mathrm{Cl}(1, 10)$ which satisfy

$$\Gamma^i \Gamma^j + \Gamma^j \Gamma^i = -\eta^{ij} \mathbb{1} , \tag{5.1.12}$$

where η is the symmetric constant bi-linear form with non-zero components $\eta^{00} = -1$ and $\eta^{ii} = 1$ for $i = 1, \dots, 10$. With respect to this basis the symplectic structure is given by

$$(\psi, \phi) = \psi^T \Gamma^0 \phi . \tag{5.1.13}$$

Of course calculations of Clifford products using gamma matrices can also be a messy business, often with indices "all over the place". To make some of these complicated expressions more succinct and easier to read we shall make use of the following invariant notation for the Clifford product. Let ω and ν be differential forms and let

$$\omega *_k \nu$$

denote k-contractions between ω and ν and wedge product the rest together, so that if $\omega \in \bigwedge^n T^*M$ and $\nu \in \bigwedge^m T^*M$ then

$$\omega \cdot \nu = \sum_{k=0}^{\min(m,n)} \omega *_k \nu \ .$$

For example, if $\omega, \nu \in \bigwedge^2 T^*M$, then

$$\begin{aligned} \omega *_0 \nu &= \omega \wedge \nu \\ \omega *_1 \nu &= (\omega_{ij}\nu^i_k - \nu_{ij}\omega^i_k) dx^j \wedge dx^k \\ \omega *_2 \nu &= (\omega_{ij}\nu^{ij}) \ . \end{aligned}$$

This product satisfies

$$\omega *_k \nu = (-1)^{(m-k)(n-k)} \nu *_k \omega \ , \tag{5.1.14}$$

and is not necessarily associative. Not only does this notation allow one to write down Clifford type equations succinctly, it is also a useful calculational tool as it helps to keep track of which degrees of differential forms vanish in a calculation.

5.2 The Killing super algebra

The Killing spinors and the F-preserving Killing vectors of a supergravity background (M, g, F) define a Lie superalgebra, which we call the **Killing superalgebra** of the background. Special cases of this construction have appeared in [75, 76, 77, 78, 79, 80, 20, 59], but here we treat the general case.

We write the Killing superalgebra as the sum $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where the even subspace $\mathfrak{g}_0$ is the algebra of F-preserving Killing vectors and the odd subspace $\mathfrak{g}_1$ is the of the algebra of Killing spinors. The grading means that we must distinguish three types of brackets.

Firstly we have the bracket $[-,-] : \mathfrak{g}_0 \otimes \mathfrak{g}_0 \to \mathfrak{g}_0$. This corresponds to the Lie bracket of Killing vectors defined in (2.1.4). It clearly satisfies the Jacobi identity and thus $\mathfrak{g}_0$ is a Lie algebra.

Next we have the bracket $[-,-] : \mathfrak{g}_0 \otimes \mathfrak{g}_1 \to \mathfrak{g}_1$. This corresponds to the action of Killing vectors on Killing spinors via the **spinorial Lie derivative** [81]. If $\rho : \mathfrak{so}(TM) \to \mathrm{End}(\mathbb{S})$ denotes the spin representation, then the spinorial Lie derivative $\mathcal{L}_\xi \varepsilon$ is given by

$$[(\xi, A_\xi), \varepsilon] = \nabla_\xi \epsilon + \rho(A_\xi)\varepsilon \ , \tag{5.2.1}$$

where $(\xi, A_\xi) \in \mathfrak{t}$ and $\varepsilon \in \mathfrak{g}_1$. Note that $\mathcal{L}_\xi \varepsilon$ is only defined when ξ is a Killing vector. If $(\xi, A_\xi) \in \mathfrak{g}_0$, then the right-hand side will again be in $\mathfrak{g}_1$ since for all vector fields X, we have

$$[\mathcal{L}_\xi, \mathcal{D}_X] = \mathcal{D}_{[\xi,X]} \ . \tag{5.2.2}$$

The spinorial Lie derivative satisfies

$$\mathcal{L}_X \mathcal{L}_Y \epsilon - \mathcal{L}_Y \mathcal{L}_X \epsilon = \mathcal{L}_{[X,Y]} \epsilon \ , \tag{5.2.3}$$

which is equivalent to the $(\mathfrak{g}_0, \mathfrak{g}_0, \mathfrak{g}_1)$ Jacobi identity.

Proof. Applying (5.2.1) we have

$$\begin{aligned}
[\mathcal{L}_X, \mathcal{L}_Y]\epsilon &= \mathcal{L}_X(\nabla_Y \epsilon + \rho(A_Y)\epsilon) - \mathcal{L}_Y(\nabla_X \epsilon + \rho(A_X)\epsilon) \\
&= \nabla_X \nabla_Y \epsilon + \rho(A_X)\nabla_Y \epsilon + \nabla_X(\rho(A_Y)\epsilon) + \rho(A_X A_Y)\epsilon - (X \leftrightarrow Y) \\
&= \nabla_X \nabla_Y \epsilon - \nabla_Y \nabla_X \epsilon + [\rho(A_X), \rho(A_Y)]\epsilon + \rho(\nabla_X A_Y)\epsilon - \rho(\nabla_Y A_X)\epsilon \ .
\end{aligned} \tag{5.2.4}$$

We now use

$$[\nabla_X, \nabla_Y]\epsilon = \nabla_{[X,Y]}\epsilon - \rho(R(X,Y))\epsilon \tag{5.2.5}$$

and Killing's identity (2.1.2) repeatedly to arrive at

$$\begin{aligned}
[\mathcal{L}_X, \mathcal{L}_Y]\epsilon &= \nabla_{[X,Y]}\epsilon + \rho([A_X, A_Y])\epsilon + \rho(R(X,Y))\epsilon \\
&= \nabla_{[X,Y]}\epsilon + \rho(A_{[X,Y]})\epsilon \\
&= \mathcal{L}_{[X,Y]}\epsilon \ .
\end{aligned} \tag{5.2.6}$$

□

The third bracket $[-,-] : \mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathfrak{g}_0$ corresponds to the map

$$\xi : \mathcal{S} \otimes \mathcal{S} \to TM \ , \tag{5.2.7}$$

which takes two spinors ψ and ϕ and produces a vector field $\xi[\psi, \phi]$ such that for all vector fields Y we have

$$g(\xi[\phi, \psi], Y) = (\psi, Y^\flat \cdot \phi) \ . \tag{5.2.8}$$

In fact, the compatibility condition (5.1.8) means we can reduce the domain of ξ to the symmetric product of the spinor bundle $\xi : \mathcal{S} \odot \mathcal{S} \to TM$. The vector field $\xi[\psi, \psi]$, which we call the **square** of ψ, is necessarily causal: $g(\xi[\psi,\psi], \xi[\psi,\psi]) \leq 0$. The simplest argument for this causality condition makes use of a contradiction: suppose this were not the case and $\xi[\psi,\psi]$ were spacelike. Then one should be able to choose a pseudo-orthonormal frame for TM so that the timelike component of $\xi[\psi,\psi]$, which we will label ξ^0, vanishes. This can be written as

$$0 = \xi^0 = (\psi, \Gamma^0 \cdot \psi) = \psi^T (\Gamma^0)^2 \psi = \psi^T \psi \neq 0 \ , \tag{5.2.9}$$

which is clearly a contradiction.

The map (5.2.7) is defined on all spinors, but when restricted to Killing spinors the vector ξ is a Killing vector which in addition preserves F [82].

Proof.

$$\begin{aligned} g(\nabla_X \xi[\psi,\phi], Y) &= X g(\xi[\psi,\phi], Y) - g(\xi[\psi,\phi], \nabla_X Y) \\ &= X(\psi, Y^\flat \cdot \phi) - (\psi, \nabla_X Y^\flat \cdot \phi) \\ &= (\nabla_X \psi, Y^\flat \cdot \phi) + (\psi, Y^\flat \cdot \nabla_X \phi) \ . \end{aligned}$$

Using that $\mathcal{D}\psi = \mathcal{D}\phi = 0$ we have,

$$g(\nabla_X \xi[\psi,\phi], Y) = (\psi, \Omega_X^* \cdot Y^\flat \cdot \phi) + (\psi, Y^\flat \cdot \Omega_X \cdot \phi) \ ,$$

where

$$\Omega_X = -\frac{1}{12} X^\flat \wedge F - \frac{1}{6}\iota_X F, \quad \text{and} \quad \Omega_X^* = \frac{1}{12} X^\flat \wedge F - \frac{1}{6}\iota_X F \ ,$$

is its symplectic adjoint. Using equations (5.1.3) and (5.1.4) we arrive at,

$$g(\nabla_X \xi[\psi,\phi], Y) = -\frac{1}{3}(\psi, \iota_X \iota_Y F \cdot \phi) + \frac{1}{6}(\psi, X^\flat \wedge Y^\flat \wedge F \cdot \phi) \ , \tag{5.2.10}$$

which is skew-symmetric in X and Y, thus $\xi[\psi,\phi]$ is a Killing vector.

Now define a 2-form B by

$$B(X,Y) = (\psi, X^\flat \wedge Y^\flat \cdot \phi) \ . \tag{5.2.11}$$

We shall compute its covariant derivative:

$$\begin{aligned} (\nabla_Z B)(X,Y) &= (\nabla_Z \psi, X^\flat \wedge Y^\flat \cdot \phi) + (\psi, X^\flat \wedge Y^\flat \cdot \nabla_Z \phi) \\ &= (\Omega_Z \psi, X^\flat \wedge Y^\flat \cdot \phi) + (\psi, X^\flat \wedge Y^\flat \cdot \Omega_Z \phi) \\ &= (\psi, \Omega_Z^* \cdot (X^\flat \wedge Y^\flat) \cdot \phi) + (\psi, (X^\flat \wedge Y^\flat) \cdot \Omega_Z \phi) \ . \end{aligned} \tag{5.2.12}$$

Using equations (5.1.5) and (5.1.6) we arrive at,

$$\begin{aligned} (\nabla_Z B)(X,Y) &= \frac{1}{6} g(Y,Z)(\psi, X^\flat \wedge F \cdot \phi) - \frac{1}{6} g(X,Z)(\psi, Y^\flat \wedge F \cdot \phi) \\ &+ \frac{1}{6}(\psi, Y^\flat \wedge Z^\flat \wedge \iota_X F \cdot \phi) + \frac{1}{6}(\psi, X^\flat \wedge Z^\flat \wedge \iota_Y F \cdot \phi) \\ &- \frac{1}{3}(\psi, X^\flat \wedge Y^\flat \wedge \iota_Z F \cdot \phi) - \frac{1}{3}(\psi, \iota_X \iota_Y \iota_Z F \cdot \phi) \ . \end{aligned} \tag{5.2.13}$$

We now alternate this identity to obtain dB:

$$\begin{aligned} dB(X,Y,Z) &= (\nabla_X B)(Y,Z) + (\nabla_Y B)(Z,X) + (\nabla_Z B)(X,Y) \\ &= -(\psi, \iota_X \iota_Y \iota_Z F \cdot \phi) \ . \end{aligned}$$

Noting that

$$(\psi, \iota_X \iota_Y \iota_Z F \cdot \phi) = F(\xi(\psi,\phi), X, Y, Z) \ ,$$

we have that

$$\iota_{\xi[\psi,\phi]} F = dB \ . \tag{5.2.14}$$

Now

$$\begin{aligned}\mathcal{L}_{\xi[\psi,\phi]}F &= \iota_{\xi[\psi,\phi]}dF + d(\iota_{\xi[\psi,\phi]})F \\ &= \iota_{\xi[\psi,\phi]}dF + d(dB)\end{aligned}$$

which, since F is closed, implies that the vector field $\xi[\psi, \phi]$ leaves F invariant. □

Recalling the bundle $\mathcal{E}$ defined in section 2.1, we can extend the map ξ to a map

$$\mathcal{A} : \mathcal{S} \odot \mathcal{S} \to \mathcal{E} \ , \tag{5.2.15}$$

given explicitly by

$$\mathcal{A}[\psi, \phi] = (\xi[\psi, \phi], -\nabla\xi[\psi, \phi]) \ ,$$

where $\xi[\psi, \phi]$ is given by equation (5.2.7). This map sends parallel sections (with respect to $\mathcal{D}$) of $\mathcal{S} \odot \mathcal{S}$ to parallel sections (with respect to D) of $\mathcal{E}$. If we also let $\mathcal{A}$ denote its restriction to these $\mathcal{D}$ parallel sections, then $-\nabla\xi[\psi, \phi]$ is given by (5.2.10).

The fundamental property of $\mathcal{A}$ is its equivariance under the action of $\mathfrak{g}_0$. In other words,

$$[(X, A_X), \mathcal{A}[\psi, \phi]] = \mathcal{A}[\mathcal{L}_X\psi, \phi] + \mathcal{A}[\psi, \mathcal{L}_X\phi] \ . \tag{5.2.16}$$

Equivalently, for all vector fields Y (not necessarily Killing),

$$g(\mathcal{L}_X\xi[\psi, \phi], Y) = (\mathcal{L}_X\psi, Y^\flat \cdot \phi) + (\psi, Y^\flat \cdot \mathcal{L}_X\phi) \ .$$

Proof. We compute the left-hand side:

$$\begin{aligned}g(\mathcal{L}_X\xi[\psi, \phi], Y) &= g(\nabla_X\xi[\psi, \phi] - \nabla_{\xi[\psi,\phi]}X, Y) \\ &= (\nabla_X\psi, Y^\flat \cdot \phi) + (\psi, Y^\flat \cdot \nabla_X\phi) + (\psi, \nabla_Y X^\flat \cdot \phi) \ .\end{aligned}$$

Next, we compute the right-hand side:

$$\begin{aligned}(\mathcal{L}_X\psi, Y^\flat \cdot \phi) + (\psi, Y^\flat \cdot \mathcal{L}_X\phi) &= (\nabla_X\psi, Y^\flat \cdot \phi) + (A_X \cdot \psi, Y^\flat \cdot \phi) \\ &\quad + (\psi, Y^\flat \cdot \nabla_X\phi) + (\psi, Y^\flat \cdot A_X \cdot \phi) \ .\end{aligned}$$

The difference is therefore

$$(\psi, \nabla_Y X^\flat \cdot \phi) + (\psi, A_X \cdot Y^\flat \cdot \phi) - (\psi, Y^\flat \cdot A_X \cdot \phi) \ ,$$

which can easily be seen to vanish as a consequence of the identity

$$[A_X, Y] = A_X(Y) = -\nabla_Y X. \tag{5.2.17}$$

□

Equation (5.2.16) is equivalent to the $[\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_1]$-Jacobi identity. It also implies that $[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0$ is an ideal, in other words, $\mathfrak{g}_1$ generates an ideal $[\mathfrak{g}_1, \mathfrak{g}_1] \oplus \mathfrak{g}_1 \subset \mathfrak{g}$.

The final Jacobi identity to consider is the $[\mathfrak{g}_1, \mathfrak{g}_1, \mathfrak{g}_1]$ identity. This is equivalent to the vanishing of a $\mathfrak{g}_0$-equivariant symmetric trilinear map $J : S^3\mathfrak{g}_1 \longrightarrow \mathfrak{g}_1$, defined by

$$J(\psi, \phi, \varepsilon) := \mathcal{L}_{\xi[\psi,\phi]}\epsilon + \mathcal{L}_{\xi[\phi,\varepsilon]}\psi + \mathcal{L}_{\xi[\varepsilon,\psi]}\phi \ . \tag{5.2.18}$$

Polarization implies that the vanishing of J is equivalent to

$$\mathcal{L}_{\xi[\varepsilon,\varepsilon]}\varepsilon = 0 \ , \tag{5.2.19}$$

for all Killing spinors ε. In other words the Jacobi equation is equivalent to every Killing spinor being left invariant under the Killing vector made by squaring itself.

Equation (5.2.19) does not involve any derivatives. Indeed, it is equivalent to

$$(2\iota_\xi F + \xi^\flat \wedge F + B \wedge *F + C \wedge F) \cdot \varepsilon = 0 \ , \tag{5.2.20}$$

where $\xi^\flat, B$ and C are made by squaring the spinor ε, respectively:

$$\begin{aligned} \xi^\flat(X) &= (\varepsilon, X^\flat \cdot \varepsilon) \\ B(X,Y) &= (\varepsilon, X^\flat \wedge Y^\flat \cdot \varepsilon) \\ C(X_1, \ldots, X_5) &= (\varepsilon, X_1^\flat \wedge \ldots . \wedge X_5^\flat \cdot \varepsilon) \ . \end{aligned}$$

Equation (5.2.19) is clearly linear in F and cubic in ε and it is equivariant under the action of $\mathrm{Spin}(1, 10)$. As a consequence it only needs to be checked for one (F, ε) in each of the (projectivised) $\mathrm{Spin}(1, 10)$-orbits of the relevant representation space. Rather than working out the orbit decomposition of this large space, one can try to prove the statement for *all* F and one ε in each of the $\mathrm{Spin}(1, 10)$ orbits in the spinor representation. There are two such orbits; when the Killing vector associated to ε is null and when it is timelike. This was checked in [72] by both *JMF* using an explicit representation of $\mathrm{Cl}(1, 10)$ and the computer package Mathematica and by *PM* with the Maple package.

5.3 Examples

5.3.1 Gravitational backgrounds

Consider those supergravity backgrounds where $F = 0$ and the fermionic sector is set to zero, so that the background is purely gravitational. Then the super covariant derivative (1.1.3) reduces to the Levi-Cività connection and Killing spinors are just parallel spinors. This means that the Killing vectors in $[\mathfrak{g}_1, \mathfrak{g}_1]$

are also parallel, so their action on $\mathfrak{g}_1$ is trivial, whence $[\mathfrak{g}_1, \mathfrak{g}_1]$ is Abelian and consists of translations.

Examples of such backgrounds include flat space, the M-wave [5], the Kaluza-Klein monopole [8, 9, 10] as well as their generalizations [83]. For flat space, $[\mathfrak{g}_1, \mathfrak{g}_1]$ is the translation ideal. For the M-wave, $[\mathfrak{g}_1, \mathfrak{g}_1]$ is a 1-dimensional ideal spanned by the parallel null vector ∂_v of the pp-wave. Indeed, suppose that ∂_u is a locally complementary null vector to ∂_v such that $\partial_u^\flat \cdot \partial_v^\flat + \partial_v^\flat \cdot \partial_u^\flat = \mathbb{1}$. The M-wave admits 16 linearly independent Killing spinors which are characterised by the condition $\partial_v^\flat \cdot \varepsilon = 0$. This means that

$$\begin{aligned} \varepsilon &= \partial_u^\flat \cdot \partial_v^\flat \cdot \varepsilon + \partial_v^\flat \cdot \partial_u^\flat \cdot \varepsilon \\ &= \partial_v^\flat \cdot \partial_u^\flat \cdot \varepsilon \ . \end{aligned}$$

Let ψ and ϕ be Killing spinors. If X is perpendicular to ∂_v, then

$$\begin{aligned} g(\xi[\psi,\phi],X) &= (\psi, X^\flat \cdot \phi) \\ &= (\psi, X^\flat \cdot \partial_v^\flat \cdot \partial_u^\flat \cdot \phi) \\ &= -(\psi, \partial_v^\flat \cdot X^\flat \cdot \partial_u^\flat \cdot \phi) \\ &= (\partial_v^\flat \cdot \psi, X^\flat \cdot \partial_u^\flat \cdot \phi) \\ &= 0 \ . \end{aligned}$$

So $\xi[\psi,\phi]$ is perpendicular to all X which are perpendicular to ∂_v, whence ξ is collinear with ∂_v. Now, ξ and ∂_v are parallel, hence $\xi[\psi,\phi] = c\partial_v$ for some constant c.

For the Kaluza-Klein monopole, $[\mathfrak{g}_1, \mathfrak{g}_1]$ is the translations ideal in the flat factor. Indeed, the geometry is $\mathbb{R}^{1,10-n} \times X^n$ where X^n is an n-dimensional riemannian manifold admitting parallel spinors but with no parallel vectors. The list of possible holonomy groups of X has been compiled in [84] and are given by $\mathrm{SU}(5)$ for $n = 10$, any of $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \subset \mathrm{Sp}(2) \subset \mathrm{SU}(4) \subset \mathrm{Spin}(7)$ for $n = 8$, G_2 for $n = 7$, $\mathrm{SU}(3)$ for $n = 6$ and $\mathrm{Sp}(1) = \mathrm{SU}(2)$ for $n = 4$. In all cases we obtain that $[\mathfrak{g}_1, \mathfrak{g}_1]$ is the translation ideal of $\mathbb{R}^{1,10-n}$.

5.3.2 Branes

For the elementary half-supersymmetric M2- and M5-brane backgrounds one also finds that $[\mathfrak{g}_1, \mathfrak{g}_1]$ is the translation ideal $\mathbb{R}^{1,p}$ on the brane. Both of these backgrounds are geometrically a warped product

$$g = H^\alpha \eta + H^\beta \delta \ ,$$

where η is the Minkowski metric on $\mathbb{R}^{1,p}$, $p = 2, 5$; δ is the euclidean metric on $\mathbb{R}^q$, $q = 8, 5$, respectively; and H is a harmonic function on $\mathbb{R}^q$ such that the metric is asymptotically flat. The coefficients α and β are functions of p, but we do not need their explicit form. The Killing spinors are given by

$$\varepsilon = H^{\alpha/4}\varepsilon_\infty ,$$

where ε_∞ is a parallel spinor in the asymptotically flat geometry which obeys the algebraic condition

$$\nu_\eta \cdot \varepsilon_\infty = \varepsilon_\infty ,$$

where ν_η is the volume form of the Minkowski metric η. Notice that the same identity is satisfied by ε itself.

For the M2-brane, the volume form ν_η is a 3-form and hence is self-adjoint relative to the symplectic structure on the spinor bundle. If X is perpendicular to the brane world-volume, then $X \cdot \nu_\eta = -\nu_\eta \cdot X$, and hence if ε_1 and ε_2 are Killing spinors,

$$\begin{aligned}
(\varepsilon_1, X \cdot \varepsilon_2) &= (\varepsilon_1, X \cdot \nu_\eta \cdot \varepsilon_2)\\
&= (\varepsilon_1, -\nu_\eta \cdot X \cdot \varepsilon_2)\\
&= -(\nu_\eta \cdot \varepsilon_1, X \cdot \varepsilon_2)\\
&= -(\varepsilon_1, X \cdot \varepsilon_2) .
\end{aligned}$$

Therefore $\xi[\varepsilon_1, \varepsilon_2]$ is tangent to the world-volume.

A similar calculation shows the analogous result for the M5-brane. Here the volume form ν_η is a 6-form, whence it is skew-adjoint with respect to the symplectic structure. If X is perpendicular to the brane world-volume, now $X \cdot \nu_\eta = \nu_\eta \cdot X$, but a calculation almost identical to the one above shows that $\xi[\varepsilon_1, \varepsilon_2]$ is again tangent to the brane world-volume.

If X is tangent to the brane world-volume and ε is a Killing spinor, a quick calculation shows that

$$\nabla_X \varepsilon = \frac{1}{4}\alpha \, d\log H \cdot X \cdot \varepsilon .$$

Let $Y = Y_T + Y_\perp$ be the decomposition of any vector field Y into tangent and perpendicular components relative to the brane world-volume, and let $\varepsilon_1, \varepsilon_2$ be

Killing spinors. Then

$$
\begin{aligned}
\langle \nabla_X \xi[\varepsilon_1, \varepsilon_2], Y \rangle &= \langle \nabla_X \xi[\varepsilon_1, \varepsilon_2], Y_T \rangle \\
&= \langle \xi[\nabla_X \varepsilon_1, \varepsilon_2], Y_T \rangle + \langle \xi[\varepsilon_1, \nabla_X \varepsilon_2], Y_T \rangle \\
&= (\nabla_X \varepsilon_1, Y_T \cdot \varepsilon_2) + (\varepsilon_1, Y_T \cdot \nabla_X \varepsilon_2) \\
&= \frac{1}{4}\alpha \; (d \log H \cdot X \cdot \varepsilon_1, Y_T \cdot \varepsilon_2) + \frac{1}{4}\alpha \; (\varepsilon_1, Y_T \cdot d \log H \cdot X \cdot \varepsilon_2) \\
&= \frac{1}{4}\alpha \; (d \log H \cdot \varepsilon_1, X \cdot Y_T \cdot \varepsilon_2) + \frac{1}{4}\alpha \; (d \log H \cdot \varepsilon_1, Y_T \cdot X \cdot \varepsilon_2) \\
&= \frac{1}{2}\alpha \langle X, Y \rangle (\varepsilon_1, d \log H \cdot \varepsilon_2) \\
&= \frac{1}{2}\alpha \langle X, Y \rangle \langle \xi[\varepsilon_1, \varepsilon_2], d \log H \rangle \\
&= 0 \;,
\end{aligned}
$$

where we have used that $d \log H$ is perpendicular to the brane world-volume repeatedly. In other words, the Lorentz component of $\xi[\varepsilon_1, \varepsilon_2]$ vanishes, whence it is a translation.

5.4 The square of the spinor bundle

We have already introduced the differential forms

$$
\begin{aligned}
\xi^\flat[\psi, \phi](X) &= (\psi, X^\flat \cdot \phi) \;, \\
B[\psi, \phi](X_1, X_2) &= (\psi, X_1^\flat \wedge X_2^\flat \cdot \phi) \;, \\
C[\psi, \phi](X_1, X_2, \ldots, X_5) &= (\psi, X_1^\flat \wedge \ldots . \wedge X_5^\flat \cdot \phi) \;.
\end{aligned}
\tag{5.4.1}
$$

Each of $\xi^\flat$, B and C defines a map from the symmetric tensor-square of the spinor bundle $\mathcal{S} \odot \mathcal{S}$ to the bundles of $1, 2$ and 5-forms respectively. We may put these together to form one map

$$
\mathcal{B} : \mathcal{S} \odot \mathcal{S} \to T^*M \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M \tag{5.4.2}
$$

given by

$$
\mathcal{B}(\psi \odot \phi) = (\xi[\psi, \phi], B[\psi, \phi], C[\psi, \phi]) \;.
$$

Then $\mathcal{B}$ extends linearly to a vector bundle isomorphism $\mathcal{B} : S \odot S \cong \wedge^1 \oplus \wedge^2 \oplus \wedge^5$ since $S \odot S$ and $\wedge^1 \oplus \wedge^2 \oplus \wedge^5$ have the same fiber dimension: $32^2/2 = 528 = 11 + 55 + 462$. This allows us to identify the two bundles.

These forms ξ, B and C satisfy a number of algebraic relations. For example, for the square of a spinor ψ we have

$$
\begin{aligned}
\iota_{\xi[\psi,\psi]} C[\psi, \psi] &= \frac{1}{2} B[\psi, \psi] \wedge B[\psi, \psi] \\
\iota_{\xi[\psi,\psi]} B[\psi, \psi] &= 0 \;,
\end{aligned}
\tag{5.4.3}
$$

and this may be polarised to obtain a relation for $\psi \odot \phi$. There are many other relations, such as those listed in [82].

Consider a bispinor $\sum_i \psi_i \odot \phi_i \in \mathcal{S} \odot \mathcal{S}$. Extending the definition of Killing in the natural way, we call $\sum_i \psi_i \odot \phi_i$ **Killing** if it is parallel with respect to $\mathcal{D}$. This is equivalent to

$$\begin{aligned}\sum_i ((\nabla_X \psi_i) \odot \phi_i + \psi_i \odot (\nabla_X \phi_i)) &= \sum_i \Omega_X \cdot (\psi_i \odot \phi_i) \\ &= \sum_i ((\Omega_X \cdot \psi_i) \odot \phi_i + \psi_i \odot (\Omega_X \cdot \phi_i)) \ .\end{aligned} \tag{5.4.4}$$

Let us consider the case where a Killing bispinor is simple, that is of the form $\psi \odot \phi$. Results about decomposable bispinors will follow from linearity. If ψ and ϕ are linearly dependent, so that $\phi = k\psi$, then it is clear that the bispinor is Killing if and only if the spinors ψ and ϕ are. So let us suppose that ψ and ϕ are linearly independent. We can expand the covariant derivatives of the spinors ψ and ϕ as

$$\nabla_X \psi = \sum_i \mu_i^\psi(X) s_i \tag{5.4.5}$$

$$\nabla_X \phi = \sum_i \mu_i^\phi(X) s_i \ , \tag{5.4.6}$$

where (s_i) is a local frame for the spinor bundle S with $s_1 = \psi$ and $s_2 = \phi$. Substituting this into equation (5.4.4) and equating with the right hand side we find that $\mu_i^\psi = \mu_i^\phi = 0$ for $i \geq 3$, also $\mu_2^\psi = \mu_1^\phi = 0$ and $\mu_1^\psi = -\mu_2^\phi$. Therefore a decomposable bispinor $\psi \odot \phi$ is Killing if and only if the spinors satisfy

$$\begin{aligned}\mathcal{D}_X \psi &= \nabla_X \psi - \Omega_X \cdot \psi = \mu(X)\psi \\ \mathcal{D}_X \phi &= \nabla_X \phi - \Omega_X \cdot \phi = -\mu(X)\phi \ ,\end{aligned} \tag{5.4.7}$$

where we have dropped the upper indices on μ.

This extra term in the Killing spinor equation does not obstruct the fundamental property of Killing spinors; that the vector field ξ is a Killing vector field. Indeed the proof of this is almost identical to that given in 5.2 with the μ contribution cancelling and leaving the same result (5.2.10).

If μ is closed then locally there exists a function f such that $\mu = df$. Then, by the re-scaling $\psi \odot \phi = e^{-f}\psi \odot e^f\phi$, we can gauge away μ:

$$\nabla_X(e^{-f}\psi) = -e^{-f}df(X)\psi + e^{-f}(\Omega_X \cdot \psi + \mu(X)\psi) = e^{-f}\Omega_X \cdot \psi \ ,$$

and similarly for ϕ.

The action of the curvature tensor of the supergravity connection $\mathcal{D}$ on ψ may be expressed as,

$$
\begin{aligned}
R^{\mathcal{D}}(X,Y)\cdot\psi &= -\mathcal{D}_X\mathcal{D}_Y\psi + \mathcal{D}_Y\mathcal{D}_X\psi + \mathcal{D}_{[X,Y]}\psi \\
&= d\mu(X,Y)\psi - \mu(X)\mathcal{D}_Y\psi + \mu(Y)\mathcal{D}_X\psi\ .
\end{aligned}
$$

A Clifford contraction together with the first Bianchi identity gives

$$
\mathrm{tr}\ R^{\mathcal{D}}(X)\cdot\psi = d\mu(X)\cdot\psi - \mu(X)\,\not{\mathcal{D}}\,\psi + \mu\cdot\mathcal{D}_X\psi\ , \tag{5.4.8}
$$

for any $X \in TM$. Similarly, for ϕ we obtain

$$
\mathrm{tr}\ R^{\mathcal{D}}(X)\cdot\phi = -d\mu(X)\cdot\phi + \mu(X)\,\not{\mathcal{D}}\,\phi - \mu\cdot\mathcal{D}_X\phi\ ,
$$

where $\not{\mathcal{D}}$ is the Dirac operator of the supergravity connection $\mathcal{D}$:

$$
\not{\mathcal{D}}\,\psi = \sum_i e_i\cdot\mathcal{D}_{e_i}\psi\ ,
$$

and (e_i) is a pseudo-orthonormal frame for TM. The left hand side of these two equations is the sum of the equations of motion acting on the spinor [85], and thus vanishes on a supergravity solution. Indeed, if one were to expand out the gamma-trace of the curvature $R^{\mathcal{D}}$ as done in [86] one would find:

$$
\begin{aligned}
\mathrm{tr}\ R^{\mathcal{D}}\cdot\psi &= (\mathrm{Ric}(X) - \frac{1}{2}F^2(X) - \frac{1}{2}(s - \frac{1}{2}|F|^2)g(X))\cdot\psi \\
&\quad - \frac{1}{6}(*(d*F - F\wedge F)\cdot X - 6\iota_X*(d*F - F\wedge F))\cdot\psi \\
&\quad - \frac{1}{6}(dF\cdot X - \iota_X dF)\cdot\psi\ .
\end{aligned} \tag{5.4.9}
$$

We will see in the next section that this implies that a spacetime which admits more than 24 Killing spinors and a 4-form F which satisfies the Maxwell equations (1.1.1), automatically satisfies the Einstein equation (1.1.2). Substituting this fact and equation (5.4.7) into (5.4.8) we find the integrability condition

$$
\begin{aligned}
0 &= d\mu(X)\cdot\psi - \mu(X)\left(\sum_i \mu(e_i)e_i\cdot\psi\right) + \mu(X)\mu\cdot\psi \\
&= d\mu(X)\cdot\psi\ ,
\end{aligned}
$$

for all $X \in TM$. Taking the symplectic inner product with ψ, and also separately a Clifford contraction with $d\mu$ we find

$$
\text{(a)}\quad (\psi, d\mu(X)\cdot\psi) = 0 \quad\text{and}\quad \text{(b)}\quad (d\mu)_{ij}(d\mu)_i^{\ j} = 0\ , \tag{5.4.10}
$$

where there is no sum over i. We have already seen that at a point p the vector $\xi[\psi,\psi]$ is either timelike or null. First suppose that it is timelike and choose an orthonormal frame (e^i) so that $e^0 = \xi$, then (a) shows that $(d\mu)_{0i} = 0$. Applying (b) we find that $(d\mu)_{ij} = 0$, whence $d\mu = 0$. Now suppose that ξ is null and choose a lightcone-orthonormal frame (e^+, e^-, e^i) with $e^+ = \xi$ and e^- its complementary null vector. Then (a) implies that $(d\mu)_{-a} = 0$ for any a, while (b) gives $(d\mu)_{+i} = (d\mu)_{ij} = 0$ which is enough to show that $d\mu = 0$ for this case as well.

It follows that $d\mu = 0$, in which case μ can be gauged away. Therefore for supergravity backgrounds, a bispinor $\psi \odot \phi$ is Killing if and only if both ψ and ϕ are Killing spinors. Linearity ensures that the same is true for a decomposable bispinor.

Under the identification (5.4.2) the Killing bispinor equation $\mathcal{D}(\phi \odot \psi) = 0$ becomes the three equations

$$\begin{aligned}
\nabla_X \xi[\psi \odot \phi] &= \xi[\Omega_X \cdot (\psi \odot \phi)] \\
\nabla_X B[\psi \odot \phi] &= B[\Omega_X \cdot (\psi \odot \phi)] \\
\nabla_X C[\psi \odot \phi] &= C[\Omega_X \cdot (\psi \odot \phi)] \, .
\end{aligned}$$

We can calculate the righthand side of each of these equations in terms of ξ, B and C to represent the Killing bispinor equation on the bundle of one, two and five-forms. In fact we have already calculated this for both ξ and B in (5.2.10) and (5.2.13) respectively.

For simplicity, let

$$\alpha(X) = -\frac{1}{12} X^\flat \wedge F \quad \text{and} \quad \beta(X) = -\frac{1}{6} \iota_X F \, ,$$

and define two families of differential forms H_k and G_k by

$$\begin{aligned}
H_k(X, Y_1, \ldots, Y_{7-k}) = &- 2\alpha(X) *_{6-k} (Y_1^\flat \wedge \cdots \wedge Y_{7-k}^\flat) \\
&+ 2\beta(X) *_{5-k} (Y_1^\flat \wedge \cdots \wedge Y_{7-k}^\flat)
\end{aligned} \tag{5.4.11}$$

$$\begin{aligned}
G_k(X, Y_1, \ldots, Y_{7-k}) = &- 2\alpha(X) *_{4-k} (Y_1^\flat \wedge \cdots \wedge Y_{7-k}^\flat) \\
&+ 2\beta(X) *_{3-k} (Y_1^\flat \wedge \cdots \wedge Y_{7-k}^\flat) \, ,
\end{aligned} \tag{5.4.12}$$

so that for each ordered set of tangent vectors $(X, Y_1, \ldots, Y_{7-k})$ we have that $H_k(X, Y_1, \ldots, Y_{7-k})$ is a k-form, and similarly for G_k which is a $k+4$-form. For example, the fully expanded $H_5(X, Y, Z)$ form is

$$\begin{aligned}
H_5(X, Y, Z) = &- \frac{1}{6} g(X, Z) F \wedge Y^\flat + \frac{1}{6} g(X, Y) F \wedge Z^\flat + \frac{1}{6} X^\flat \wedge \iota_Z F \wedge Y^\flat \\
&- \frac{1}{6} X^\flat \wedge \iota_Y F \wedge Z^\flat - \frac{1}{3} \iota_X F \wedge Z^\flat \wedge Y^\flat \, ,
\end{aligned}$$

which is precisely the right hand side of (5.2.13). A similar calculation to that of (5.2.13) for C yields

Theorem 5.4.1. *$(\xi^\flat, B, C)$ is a Killing bispinor if and only if*

$$\begin{aligned}
(\nabla_X \xi^\flat)Y &= -\frac{1}{3}B(\iota_X\iota_Y F^\sharp) - \frac{1}{6}C(\iota_X\iota_Y * F^\sharp) \\
(\nabla_X B)(Y,Z) &= -\frac{1}{3}\xi^\flat(\iota_X\iota_Y\iota_Z F^\sharp) + C(H_5^\sharp(X,Y,Z)) \\
(\nabla_X C)(Y_1,\dots,Y_5) &= -\frac{1}{6}\xi^\flat(\iota_X\iota_{Y_1}\dots\iota_{Y_5} * F^\sharp) - B(H_2^\sharp(X,Y_1,\dots,Y_5)) \\
&\quad - C(*G_2^\sharp(X,Y_1,\dots,Y_5)) \, .
\end{aligned} \tag{5.4.13}$$

These equations were originally calculated in [82] by squaring a single Killing spinor. One way to think of these equations is as a generalisation of the Killing transport equations. If we define a connection $\hat{D}$ on the bundle $\wedge^1 \oplus \wedge^2 \oplus \wedge^5$ by

$$\hat{D}_X \begin{pmatrix} \xi^\flat \\ B \\ C \end{pmatrix} = \nabla_X \begin{pmatrix} \xi^\flat \\ B \\ C \end{pmatrix} + \begin{pmatrix} -\frac{1}{3}B(\iota_X F^\sharp) - \frac{1}{6}C(\iota_X * F^\sharp) \\ -\frac{1}{3}\xi^\flat(\iota_X F^\sharp) + C(H_5^\sharp(X)) \\ -\frac{1}{6}\xi^\flat(\iota_X * F^\sharp) - B(H_2^\sharp(X)) - C(*G_2^\sharp(X)) \end{pmatrix} ,$$

where we have suppressed some of the notation on the righthand side, then Killing bispinors are parallel sections of $\hat{D}$. However, notice that the 2-form B is not equal to the 2-form A from the Killing transport. We could of course correct this by changing the definition of $\hat{D}$, but then it would not be so natural from the view point of the isomorphism (5.4.2).

The bundle $\mathcal{S} \odot \mathcal{S}$ however does not naturally inherit a Lie bracket in the same way the bundle $\mathcal{E}$ does. For example, one may attempt to form a bracket by noting that the symplectic structure provides an isomorphism between the bundle $\mathcal{S} \odot \mathcal{S}$ and the bundle $\mathrm{SEnd}(\mathcal{S})$ of symmetric endomorphisms of $\mathcal{S}$. Explicitly the isomorphism is given by

$$\psi \odot \phi \mapsto (\psi, -)\phi + (\phi, -)\psi \, . \tag{5.4.14}$$

However the natural Lie bracket on $\mathrm{End}(\mathcal{S})$, given by $[P,Q] = PQ - QP$, is not closed on $\mathrm{SEnd}(\mathcal{S})$:

$$[P,Q]^T = (PQ - QP)^T = QP - PQ = -[P,Q] \quad \text{if } P,Q \in S\mathrm{End}(\mathcal{S}) \, . \tag{5.4.15}$$

In fact, in relation to the grading $\mathrm{End}(\mathcal{S}) = \mathrm{SEnd}(\mathcal{S}) \oplus \mathfrak{so}(\mathcal{S})$ of endomorphism bundle into symmetric and skew-symmetric parts, the Lie bracket satisfies

$$\begin{aligned}
[\mathrm{SEnd}(\mathcal{S}), \mathrm{SEnd}(\mathcal{S})] &\subset \mathfrak{so}(\mathcal{S}) \\
[\mathfrak{so}(\mathcal{S}), \mathrm{SEnd}(\mathcal{S})] &\subset \mathrm{SEnd}(\mathcal{S}) \\
[\mathfrak{so}(\mathcal{S}), \mathfrak{so}(\mathcal{S})] &\subset \mathfrak{so}(\mathcal{S}) \, .
\end{aligned} \tag{5.4.16}$$

The isomorphism (5.4.15) extends naturally to define an isomorphism between the full endomorphism bundle $\mathrm{End}(\mathcal{S})$ and the tensor product bundle $\mathcal{S} \otimes \mathcal{S}$, and this in turn is isomorphic to the bundle $\oplus_{i=0}^5 \Lambda^i$ of differential forms up to degree 5 via the natural extension to the map (5.4.2) so that it includes 3 and 4-forms. Then the above grading and bracket on End($\mathcal{S}$) are equivalent to the grading $\oplus_{i=1}^5 \Lambda^i = (\Lambda^1 \oplus \Lambda^2 \oplus \Lambda^5) \oplus (\mathbb{R} \oplus \Lambda^3 \oplus \Lambda^4)$ and wedge product of forms.

Killing's identity (2.1.2) relates the curvature tensor to a second derivative of ξ:

$$R(X, \xi)Y = (\nabla_X \nabla \xi)Y \ . \tag{5.4.17}$$

We can use this to formulate the sectional curvature for a two-plane where one of the generators is $\xi[\psi, \phi]$. The easiest way to calculate the right hand side in this case is another spinor calculation as done for theorem 5.4.1. It is equivalent to the Λ^1 part of the double derivative of the bispinor $\psi \odot \phi$:

$$\begin{aligned} \xi^\flat[\nabla_X \nabla(\psi \odot \phi)Y] &= \xi^\flat[\nabla_X \Omega \cdot (\psi \odot \phi)Y] \\ &= \xi^\flat[(\nabla_X \Omega) \cdot (\psi \odot \phi)Y + \Omega_Y \cdot \Omega_X \cdot (\psi \odot \phi)] \ . \end{aligned} \tag{5.4.18}$$

We will take $Y = \xi$ and evaluate this on X in order to get the sectional curvature $R(\xi, X, \xi, X)$. The second term in this last equality can be expanded as

$$\begin{aligned} \xi^\flat[\Omega_\xi \cdot \Omega_X(\psi \odot \phi)](X) =& (\psi, \Omega_X^* \cdot (\Omega_\xi^* \cdot X + X \cdot \Omega_\xi) \cdot \phi) \\ &+ (\psi, (\Omega_\xi^* \cdot X + X \cdot \Omega_\xi) \cdot \Omega_X \cdot \phi) \ . \end{aligned} \tag{5.4.19}$$

We have already calculated $\Omega_\xi^* \cdot X + X \cdot \Omega_\xi$ in equation (5.2.10):

$$\Omega_\xi^* \cdot X + X \cdot \Omega_\xi = -\frac{1}{3}\iota_\xi \iota_X F + \frac{1}{6}\xi^\flat \wedge X^\flat \wedge F = 2\xi^\flat \wedge \alpha + 2\iota_\xi \beta \ .$$

After a similar calculation to that which lead to equation (5.2.10) we find that the right hand side of (5.4.19) is given by

$$2(\alpha + \beta) \cdot (\xi^\flat \wedge \alpha + (2\xi^\flat \wedge \alpha + 2\iota_\xi \beta)\iota_\xi \beta) + 2(\xi^\flat \wedge \alpha + \iota_\xi \beta) \cdot (-\alpha + \beta) \ ,$$

which may be expanded and separated into 1, 5 and 9 form parts:

$$\begin{aligned} K_1(\xi, X) =& \frac{1}{9}\iota_X F *_2 \iota_\xi \iota_X F - \frac{1}{36} X^\flat \wedge F *_5 \xi^\flat \wedge X^\flat \wedge F \\ K_5(\xi, X) =& \frac{1}{9}\iota_X F \wedge \iota_\xi \iota_X F - \frac{1}{18}\iota_X F *_2 \xi^\flat \wedge X^\flat \wedge F \\ &+ \frac{1}{18} X^\flat \wedge F *_1 \iota_\xi \iota_X F - \frac{1}{36} X^\flat \wedge F *_3 \xi^\flat \wedge X^\flat \wedge F \\ K_9(\xi, X) =& \frac{1}{9}\iota_X F \wedge \xi^\flat \wedge X^\flat \wedge F + \frac{1}{18} X^\flat \wedge F *_1 \xi^\flat \wedge X^\flat \wedge F \ . \end{aligned}$$

Similarly, for the first part of equation (5.4.18) with $Y = \xi$ and evaluated at X we have

$$\Omega_\xi(\nabla_X F)^* \cdot X + X \cdot \Omega_\xi(\nabla_X F) = -\frac{1}{3}\iota_\xi \iota_X \nabla_X F + \frac{1}{6}\xi^\flat \wedge X^\flat \wedge \nabla_X F \ . \tag{5.4.20}$$

Consequently, the sectional curvature is given by

$$R(\xi, X, \xi, X) = -\xi^\flat(K_1^\sharp) + A(*K_9^\sharp) - C(K_5^\sharp) + \frac{1}{3}B(\iota_\xi \iota_X \nabla_X F) + \frac{1}{6}C(\iota_\xi \iota_X * \nabla_X F) \, . \tag{5.4.21}$$

If $F \wedge F = 0$ then K_9 vanishes, and if F is simple so that it may be written as $F dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$ then both K_5 and K_9 vanish.

5.5 Local homogeneity of 24+ backgrounds

Let us fix a point $p \in M$ and denote the lorentzian inner product space induced by the metric restricted to T_pM as V with inner product $\langle -, - \rangle$ and with associated norm $| - |^2$.

Let $W \subset \mathcal{S}$ denote the subspace of Killing spinors. The map (5.2.7) defines a symmetric bilinear map

$$\xi : \mathcal{S} \odot \mathcal{S} \to V \, . \tag{5.5.1}$$

The aim of this section is to show that if there is enough supersymmetry then the background must be homogeneous. Equivalently, we want to show that if $\dim W$ is large enough, then the restriction

$$\xi|_W : W \odot W \to V \tag{5.5.2}$$

of ξ to W is surjective. This means that T_pM will be spanned by Killing vectors. Since p is arbitrary, this will be the case for all p and thus the background will be locally homogeneous.

If $W = S$ then the representation theory of the spin group ensures that ξ is surjective. On the other hand there are examples with $\dim = 16$ which are not locally homogeneous: for example the generic M-wave [5], the M-branes [6, 7] and the Kaluza-Klein monopole [8, 9, 10]. Therefore there has to be a minimal $16 < N \leq 32$ such that whenever $\dim W \geq N$ the map $\xi|_W$ is surjective. Using the symplectic linear algebra on W we will show that $N = 25$. The proof comprises of two parts: in this section we prove that N is at most 25, and in the next section we give an example with $\dim W = 24$ where ξ is not surjective. This second part we call the **24+ conjecture**, because the argument is purely linear algebra and does not take into account the supergravity equations of motion.

Let us start by introducing some notation for symplectic linear algebra. The subspace symplectically perpendicular to W is defined by

$$W^\perp = \{\varepsilon \in \mathcal{S} \mid (\varepsilon, w) = 0 \text{ for all } w \in W\} \, .$$

From the rank-nullity theorem we have that

$$\dim W^\perp + \dim W = \dim \mathcal{S} \, , \tag{5.5.3}$$

even though W and $W^\perp$ may not be disjoint. For example, a 1-dimensional subspace is always contained in its perpendicular space. The relationship between W and $W^\perp$ can be used to define special types of subspaces. If $W \subset W^\perp$ then we call W *isotropic.* The dimension of an isotropic subspace is at most half the dimension of the ambient vector space $\mathcal{S}$. When the dimension is precisely half, so that $W = W^\perp$, then W is called *lagrangian.* At the other extreme, if W and $W^\perp$ are disjoint then we call W *symplectic.*

We can assume that $\dim W > 16$ because of the known examples mentioned above. Now, $\xi|_W$ will be surjective if and only if the perpendicular space to its image is trivial. That is, if $v \in V$ and

$$(\psi, v \cdot \phi) = 0 \ , \tag{5.5.4}$$

then we have $v = 0$. Suppose that $v \in V$ satisfies (5.5.4) and consider the Clifford endomorphism defined by v:

$$(v\cdot) : \mathcal{S} \longrightarrow \mathcal{S}.$$

Restricting this endomorphism to the subspace W, equation (5.5.4) is equivalent to $(v\cdot)$ mapping into the perpendicular space of W

$$(v\cdot)|_W : W \longrightarrow W^\perp \ . \tag{5.5.5}$$

Under our assumptions we have that $\dim W > \frac{1}{2}\dim \mathcal{S} = 16$. The rank-nullity theorem tells us that $\dim W > \dim W^\perp$. Hence, on dimensional grounds, $(v\cdot)$ must have kernel.

On the other hand, the Clifford relation gives $v^2 = -|v|^2\mathbb{1}$. Thus $(v\cdot)$ has kernel if and only if $|v|^2 = 0$, in other words v must be null. A null subspace of any lorentzian vector space is at most 1-dimensional. Hence the perpendicular space to the image of ξ is at most 1-dimensional and if it is 1-dimensional, then it is spanned by v. It follows that a supergravity background with greater than 16 supersymmetries is of cohomogeneity-one.

Now, from the relation $v^2 = 0$ we see that $\mathrm{Im}\ (v\cdot) \subset \ker(v\cdot)$. To prove the other inclusion $\ker(v\cdot) \subset \mathrm{Im}\ (v\cdot)$, let u be a complementary null vector to v such that

$$u \cdot v + v \cdot u = \mathbb{1} \ , \tag{5.5.6}$$

and let $\varepsilon \in \ker(v\cdot)$. Then applying (5.5.6) to ε

$$\varepsilon = u \cdot v \cdot \varepsilon + v \cdot u \cdot \varepsilon = v \cdot u \cdot \varepsilon \ ,$$

and thus $\varepsilon \in \mathrm{Im}\ (v\cdot)$. Therefore $\mathrm{Im}\ (v\cdot) = \ker(v\cdot)$. A similar argument shows that $\ker(u\cdot) = \mathrm{Im}\ (u\cdot)$. Using equation (5.1.8), we *claim* that $\ker u\cdot$ and $\ker v\cdot$ are complementary lagrangian subspaces, and therefore rank $(v\cdot) = \dim \mathrm{Im}\ (v\cdot) = 16$.

Proof. (*of claim*) If $\varepsilon \in \ker(v\cdot) \cap \ker(u\cdot)$, then by applying equation (5.5.6) we have

$$\varepsilon = u \cdot v \cdot \varepsilon + v \cdot u \cdot \varepsilon = 0 .$$

Hence $\ker(u\cdot)$ and $\ker(v\cdot)$ are complementary.

To see they are lagrangian, first consider $\varepsilon = v \cdot \psi \in \ker(v\cdot) = \text{Im }(v\cdot)$. Let $\phi \in \ker(v\cdot)$, then

$$(\varepsilon, \phi) = (v \cdot \psi, \phi) = -(\psi, v \cdot \phi) = 0 .$$

Hence $\varepsilon \in (\ker v\cdot)^{\perp}$ and thus $\ker v\cdot \subset (\ker v\cdot)^{\perp}$. A similar argument shows the same thing for $u\cdot$.

Next suppose that $\varepsilon \in (\ker v\cdot)^{\perp} = (\text{Im } v\cdot)^{\perp}$. Then $(\varepsilon, \phi) = 0$ for all $\phi = v\cdot\psi \in \text{Im }(v\cdot)$. Which gives

$$-(v \cdot \varepsilon, \psi) = (\varepsilon, v \cdot \psi) = 0 ,$$

for all $\psi \in \mathcal{S}$. As $(-,-)$ is non-degenerate, we see that $\varepsilon \in \ker(v\cdot)$ and thus $(\ker v\cdot)^{\perp} \subset \ker(v\cdot)$. Again, a similar argument shows the same thing for $u\cdot$. □

Let U be a complementary subspace to W, that is $W \oplus U = \mathcal{S}$. With respect to this split, the matrix of the linear map β defined by

$$\beta_v(\psi, \phi) = (\psi, v^\flat \cdot \phi) ,$$

is of the form

$$\begin{pmatrix} 0 & A \\ A^t & B \end{pmatrix} \tag{5.5.7}$$

where $A : U \to W$, $A^t : W \to U$ and $B : U \to U$. We know that this matrix has rank 16 since $(-,-)$ is non-degenerate and $v\cdot$ has rank 16. We will now estimate the maximum possible rank of this matrix in terms of the dimension of the subspace W.

The kernel of β consists of vectors $(w, u) \in W \oplus U$ such that $Au = 0$ and $A^t w + Bu = 0$. Since $\dim W > 16$, we have $\dim U < \dim W$. Which gives $\text{rank } A = \dim \text{Im } A \leq \dim U$ as $A : U \to W$.

Now, suppose the rank of A is maximal. Then if $Au = 0$ then $u = 0$. In which case the kernel of β is of the form $(w, 0) \in W \oplus U$ with $w \in \ker A^t$. The rank of A^t is equal to the rank of A, so A^t is surjective. Hence

$$\begin{aligned} \dim \ker A^t &= \dim W - \dim \text{Im } A^t \\ &= \dim W - \dim U . \end{aligned}$$

So the rank of β is *at most*

$$\begin{aligned} \text{rank } A^t &= 32 - (\dim \ker A^t) \\ &= 32 - (\dim W - \dim U) \\ &= 2 \dim U , \end{aligned}$$

since $32 = \dim S = \dim W + \dim U$.

But we know that rank $\beta = 16$. Hence, $16 \leq 2 \dim U$. That is $\dim U \geq 8$ or equivalently $\dim W \leq 24$. This means that if $\dim W > 24$ no such v can exist and the map $\xi|_W$ is surjective.

As a corollary of the proof given above, it is not difficult to see that if a spacetime admits more than 24 supersymmetries and a four-form F which satisfies the Maxwell equations (1.1.1), then the Einstein equations come for free. Indeed, from (5.4.9) we have

$$\mathrm{tr}\ R^{\mathfrak{D}} \cdot \psi = (\mathrm{Ric}(X) - \frac{1}{2}F^2(X) - \frac{1}{2}(s - \frac{1}{2}|F|^2)g(X)) \cdot \psi \,. \tag{5.5.8}$$

For a pair of Killing spinors ψ and ϕ we have $\mu = 0$ in equation (5.4.8), so equation (5.5.8) vanishes. Writing the Einstein expression on the righthand side of (5.5.8) as $E(X) \cdot \psi$, we may take the symplectic inner product with ϕ so that

$$(\phi, E(X) \cdot \psi) = 0 \,, \tag{5.5.9}$$

for all $\psi, \phi \in W$ and $X \in V$. Therefore if $\dim W > 24$ the vector $E(X)^\sharp$ must vanish.

5.6 24+ conjecture

It is not clear that this result is sharp since we have not taken $v^2 = 0$ into account in the matrix for β. We will show that it is by exhibiting a 24-dimensional subspace $W \subset S$ such that $\xi|_W$ is not surjective.

First we choose a basis for S adapted to the Clifford endomorphism v. Since $v^2 = 0$ and $\ker v = \mathrm{Im}\ v$ we may write the matrix for v with respect to this basis as

$$N = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$$

where I is a 16×16 identity matrix. Relative to this split the symplectic inner product has matrix

$$\Omega = \begin{pmatrix} A & -B^t \\ B & C \end{pmatrix}$$

where A and C are skew-symmetric. We can restrict this matrix Ω further as v is skew-symmetric with respect to the symplectic form:

$$\Omega N + N^t \Omega = 0 \quad \Longrightarrow \quad \Omega = \begin{pmatrix} 0 & -B \\ B & C \end{pmatrix}$$

with B now symmetric. By choosing a complementary subspace to $\ker v$ appropriately, say choose it to be $\ker u$ where u is a complementary null vector as above, we can take $C = 0$. B must then be a non-degenerate symmetric matrix for the symplectic form to be non-degenerate. Relative to this basis the bilinear form β has matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} .$$

The symmetric matrix B defines an inner product on $\ker u$. Take this inner product to have signature $(8,8)$ so that we can split $\ker u = K_+ \oplus K_-$ as a direct sum of maximally isotropic (relative to B) subspaces. The 24-dimensional subspace $W = \ker v \oplus K_+$ is β-isotropic, that is for every $w_1, w_2 \in W$ we have $\beta(w_1, w_2) = (w_1, v \cdot w_2) = 0$. This proves that $\xi|_W$ is not surjective and hence our result is sharp.

In fact, by taking B to have signature $(n, 16-n)$ for $n = 1, 2, \ldots, 8$, and letting K be an n-dimensional isotropic subspace of $\ker u$, we can arrive at $W = \ker v \oplus K$ of dimension $16 + n$ for which $\xi|_W$ is not surjective. Hence this provides counter examples for $16 < \dim W \leq 24$.

However, this only shows that the result is sharp on purely *algebraic* grounds. Geometrically the subspace W is characterised by more than its dimension. It is the subspace of invariants of the holonomy representation of the connection $\mathcal{D}$ on S at the point p, and it is not clear that every subspace $W \subset S$ can appear. Indeed, as we mentioned in the introduction 1, all known backgrounds with $\nu > \frac{1}{2}$ are (locally) homogeneous. Nevertheless we believe that this is evidence in favor of the conjecture that $\nu_c = \frac{3}{4}$ and hence that non-homogeneous M-theory backgrounds with 24 supercharges should indeed exist.

Chapter 6

Homogeneous supergravity backgrounds

As mentioned in the introduction 1 there are many different supergravity theories in different dimensions from four to eleven. The data for a bosonic supergravity background is a lorentzian spacetime (M, g) together with a collection of differential forms F_i which satisfy some equations of motion such as those for 11-dimensional supergravity (1.1.1) and (1.1.2). We have already seen that supersymmetries generate Killing vectors which not only leave g invariant but F as well. So it is natural to consider the subgroup $\mathrm{Iso}(M, g, F)$ of the isometries of g which also preserve the form F, i.e.

$$h \in \mathrm{Iso}(M, g) \quad \text{such that} \quad h^* F_i = F_i \ ,$$

and call a supergravity theory (M, g, F_i) **homogeneous** if there is a subgroup G of $\mathrm{Iso}(M, g, F_i)$ which acts transitively on M. More specifically, we will focus on reductive homogeneous supergravity backgrounds $(G/H, g, F_i)$ with reductive split $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and homogeneous structure S. In the remainder of this chapter we will drop the i index from F, but bear in mind that there may be more than one field strength F in a given supergravity background.

6.1 Equations of motion

Typically the field strengths F satisfy equations of the form

$$dF = 0 \quad \text{and} \quad d * F = \lambda F \wedge F \ , \tag{6.1.1}$$

where λ can be taken to be either 0 or 1. Since F is invariant under the subgroup of isometries G, we can describe it as a constant multilinear form on $\mathfrak{m}$, which we will also denote F, which is $\mathfrak{h}$-invariant:

$$F([Y, X_1]_{\mathfrak{m}}, X_2, \ldots, X_i) + F(X_1, [Y, X_2]_{\mathfrak{m}}, \ldots, X_i) + F(X_1, X_2, \ldots, [Y, X_i]_{\mathfrak{m}}) = 0 \ ,$$

where $Y \in \mathfrak{h}$ and $X_j \in \mathfrak{m}$ for $j = 1, \ldots, i$. Then we obtain the full form of F by applying it to the projection to $\mathfrak{m}$ of the Maurer-Cartan form θ:

$$F = F(\theta_{\mathfrak{m}}, \ldots, \theta_{\mathfrak{m}}) \ . \tag{6.1.2}$$

The connection 1-form for the canonical connection is $\theta_{\mathfrak{h}}$, therefore

$$\tilde{\nabla}\theta_{\mathfrak{m}} = [\theta_{\mathfrak{h}}, \theta_{\mathfrak{m}}] \ .$$

It follows that a multilinear form F on $\mathfrak{m}$ is $\mathfrak{h}$-invariant if and only if it is parallel with respect to the canonical connection.

Now skew-symmetrization of $\tilde{\nabla}F = 0$ leads to,

$$0 = dF - \mathrm{Alt}(S(F)) = -\mathrm{Alt}(S(F)) \ . \tag{6.1.3}$$

Since the metric is invariant, the Hodge star of F must also be invariant, whence skew-symmetrization of the equation $\tilde{\nabla} * F = 0$ leads to

$$0 = d * F - \mathrm{Alt}(S(*F)) = \lambda F \wedge F - \mathrm{Alt}(S(*F)) \ . \tag{6.1.4}$$

If $S = 0$, so that the reductive split is symmetric and the canonical connection coincides with the Levi-Civitá connection, then any invariant form is parallel and hence both closed and co-closed. So the equations (6.1.1) reduce to the algebraic condition

$$\lambda F \wedge F = 0 \ . \tag{6.1.5}$$

If S is of type $\mathcal{T}_1$ then it is of the form

$$S(X, Y) = g(X, Y)\xi - g(X, \xi)Y \ ,$$

for some vector field ξ. There are two cases to consider, when ξ is null and when it is not null.

When ξ is not null, then Gadea and Oubiñia [41] showed that (M, g) must in fact be locally isometric to anti de-Sitter space (which is locally symmetric.)

When ξ is null, then Montesinos Amilibia [42] showed that (M, g) must be a singular plane wave. In this case

$$S_X(F) = X^\flat \otimes \iota_\xi F - \iota_X F \otimes \xi^\flat \ . \tag{6.1.6}$$

Consequently $\mathrm{Alt}(S(F)) = -\xi^\flat \wedge F$, and if we apply equation (6.1.3) we find F must be of the form

$$F = \xi^\flat \wedge \omega \ , \tag{6.1.7}$$

where ω is a 3-form independent of $\xi^\flat$. Taking the Hodge dual of this we have $*F = \xi^\flat \wedge \nu$ where ν is a 6-form. This clearly satisfies the condition

$$\lambda F \wedge F - \mathrm{Alt}(S(*F)) = 0 \ .$$

If we write the plane-wave in Brinkmann coordinates

$$g = 2dx^+dx^- + H(x^+, \boldsymbol{x}, \boldsymbol{x})(dx^+)^2 + |d\boldsymbol{x}|^2 \ , \tag{6.1.8}$$

then ω is of the form $\omega_{ijk}dx^i \wedge dx^j \wedge dx^k$ for $i, j, k \in (1, \ldots, n-2)$, and $F = dx^+ \wedge \omega$. Of course, we have not checked when F is actually invariant. We will do this in the next section when we consider supergravity on all homogeneous plane-wave backgrounds, and not just those that admit a $\mathcal{T}_1$-structure.

The other classes of homogeneous structures don't immediately say much about the forms F. However, as already mentioned in section 2.3, a homogeneous structure of type $\mathcal{T}_1 \oplus \mathcal{T}_3$ is either isometric to AdS_n and therefore symmetric, or a singular homogeneous plane-wave.

On a reductive homogeneous background all of the objects in the Einstein equation are of course invariant, and the equation may be evaluated at a point o using equations (2.4.9) and (2.4.10). Again, these equations will simplify for some of the different types of homogeneous structures. For example, for a symmetric space the Ricci and scalar curvatures become

$$\mathrm{Ric}_{ij} = -\frac{1}{2}\sum_k \langle [E_i, [E_j, E_k]_{\mathfrak{h}}]_{\mathfrak{m}} + [E_j, [E_i, E_k]_{\mathfrak{h}}]_{\mathfrak{m}}, E_k\rangle \tag{6.1.9}$$

$$s = -\sum_{j,k} \langle [E_k, [E_k, E_j]_{\mathfrak{h}}]_{\mathfrak{m}}, E_j\rangle \ , \tag{6.1.10}$$

which may be put into an Einstein equation such as (1.1.2).

We can use theorem 5.4.1 to solve for supersymmetries on a homogeneous 11-dimensional supergravity background. We saw that every Killing bispinor on a supergravity background is generated by Killing spinors, so a solution to the equations (5.4.13) will determine a supersymmetry. Since a Killing bispinor is completely determined by its value at a point, we may restrict to the origin o of M and solve the equations (5.4.13) on $\mathfrak{m}$. We shall see an example of this in the next section when we consider plane-wave backgrounds.

Some of the examples of reductive spaces that we considered at the end of chapter 4 are solutions to supergravity theories. The five dimensional Gödel universe is a supergravity background and its riemannian product with flat space is an eleven-dimensional solution. The Kaigorodov space is a purely gravitational solution to Einstein's gravity with a cosmological constant and its riemannian product with the sphere $K_{n+3} \times S^{8-n}$ can be seen to be an eleven-dimensional supergravity solution [31]. However, there is no differential form F for which the Kaplan space is a supergravity solution in six-dimensions.

6.2 Plane-wave backgrounds

Using the algebra (3.1.7) it is simple to calculate the $\mathfrak{h}$-invariant multilinear forms F on $\mathfrak{m}$ for a plane-wave. We find that these have the same form as derived above for the $\mathcal{T}_1$ case in equation (6.1.7), that is the wedge product between the dual of the parallel vector and a transversal 3-form. Using the Maurer-Cartan equations

$$d\theta^i = C^i_{jk}\theta^j \wedge \theta^k \ ,$$

where C^i_{jk} are the structure constants of the Lie algebra $\mathfrak{g}$, it is easy to check that $F = \xi^\flat \wedge \omega$ is closed and co-closed. If we use the Maurer-Cartan one-form θ to recover the full form of F then we find that ω_{ijk} is constant for the regular-waves or a constant multiple of $(x^+)^{-1}$ for the singular waves.

We will consider plane-waves in 11-dimensional supergravity with the Maxwell and Einstein type equations given by (1.1.1) and (1.1.2), although a suitable generalisation of the following discussion will hold for lower dimensional supergravity theories. For the plane-wave, the 4-form F is null so $|F|^2$ and s are zero. The Ricci tensor of a homogeneous plane-wave (3.1.2) can easily be calculated as it was in [28], and we find that the Einstein equation has only one non-zero component:

$$-\operatorname{tr} H(x^+) = \mathrm{Ric}_{++} = \frac{1}{12}|\omega|^2 \ . \tag{6.2.1}$$

The left-hand side can be calculated for both the regular and singular homogeneous plane-waves:

$$\begin{aligned} \text{Regular:}\quad \mathrm{Ric}_{++} &= -\operatorname{tr} H_0 \\ \text{Singular:}\quad \mathrm{Ric}_{++} &= -(x^+)^{-2}\operatorname{tr} H_0 \ . \end{aligned}$$

Therefore it is clear that an appropriate choice of constant 3-form ω on $\mathfrak{m}$ will solve the Einstein equation for any given homogeneous plane-wave.

It has been shown in [5] that every 11-dimensional plane-wave background has at least 16 linearly independent Killing spinors ε characterised by the projection $\partial_-^\flat \cdot \varepsilon = 0$. We may see this as a solution to the equations (5.4.13). We saw in section 5.3.1 that $\partial_-^\flat \cdot \psi = \partial_-^\flat \cdot \phi = 0$ implies the Killing vector $\xi[\psi, \phi]$ is proportional to the parallel vector ∂_-. A similar argument for B and C shows that

$$B[\psi, \phi] = dx^+ \wedge B_i\, dx^i \quad \text{and} \quad C[\psi, \phi] = dx^+ \wedge C_{ijkl}\, dx^i \wedge dx^j \wedge dx^k \wedge dx^l \ , \tag{6.2.2}$$

where B_i and C_{ijkl} are constant for the regular wave and constant multiples of $(x^+)^{-1}$ for the singular wave. It is easy to check that these do indeed solve equations (5.4.13) on $\mathfrak{m}$ for all plane-wave backgrounds and thus define a Killing bispinor. Counting the number of such solutions, there are 126 linearly independent four-forms and 9 linearly independent one-forms in the nine transversal directions. Add in the 1 constant factor $\xi[\psi, \phi] = k\partial_-$ and we have 136 Killing bispinors, which agrees with the symmetric square of 16 Killing spinors.

Any extra Killing spinor further than these 16 must satisfy $\partial_-^\flat \cdot \varepsilon \neq 0$. As discussed in section 3.1, any plane-wave is cohomogeneity one and is homogeneous if there is an extra Killing vector in the ∂_+ direction. It was shown in [17] that any extra Killing spinor squares to a Killing vector with a component in this required direction. Indeed, the component of such a Killing vector in the ∂_+ direction is

$$(\varepsilon, \partial_+^\flat \cdot \varepsilon) = \frac{1}{\sqrt{2}}\varepsilon^T\Gamma^-\Gamma^+\varepsilon = \frac{1}{\sqrt{2}}(\Gamma^-\varepsilon)^T(\Gamma^-\varepsilon) .$$

The projection condition $\partial_-^\flat \cdot \varepsilon = \Gamma^-\varepsilon \neq 0$ ensures that this does not vanish. Notice that this result is different from our general result in section 5.5. Let ψ and ϕ each be one of these *extra* Killing spinors which in addition have a component of $\xi[\psi, \phi]|_o$ in the X direction. By calculating $\nabla(X + \alpha^i Y_i + \beta Z)$ for $X, Y_i, Z \in \mathfrak{m}$ satisfying the algebra (3.1.7) and using the algebraic identities (5.4.3), it is not difficult to show that if

$$B[\psi, \phi]|_o(Z, Y_k) = B^k \quad \text{then} \quad f_{ij} = -\frac{1}{3}B^k\omega_{ijk} . \tag{6.2.3}$$

Now, suppose that an 11-dimensional supergravity plane-wave background has greater than 16 linearly independent Killing spinors. The algebraic formulae (4.2.8) for the plane-wave limit give the constant c in terms of the homogeneous structure $-S(U, U, V)$, which can be calculated using the definition (2.3.2) and theorem 5.4.1:

$$-S(\partial_+, \partial_+, \partial_-) = g(\nabla_{\partial_+}\partial_+, \partial_-) = -\frac{1}{3}\iota_{B_+}\iota_{\partial_-}\iota_{\partial_+}F = 0 .$$

Therefore the plane-wave must be regular.

It follows that F is of the form $F = dx^+ \wedge \omega$ with ω a constant transversal 3-form. Obviously $F \wedge F = 0$, and it is not difficult to see that $\nabla_{\partial_i}F = 0$. These two facts considerably simplify formula (5.4.21), which we may apply to find

$$\begin{aligned}(H_0)_{ij} &= -R(X, Y_i, X, Y_j) \\ &= -\frac{1}{9}\langle\iota_{Y_i}\omega, \iota_{Y_j}\omega\rangle + \frac{1}{36}\langle Y_i^\flat \wedge \omega, Y_j^\flat \wedge \omega\rangle \\ &\quad - \frac{1}{9}C[\psi, \phi](Z^\flat \wedge \iota_{Y_i}\omega \wedge \iota_{Y_j}\omega) \\ &\quad - \frac{1}{36}C[\psi, \phi](Z^\flat \wedge ((Y_i^\flat \wedge \omega) *_1 \iota_{Y_j}\omega) + Z^\flat \wedge ((Y_j^\flat \wedge \omega) *_1 \iota_{Y_i}\omega)).\end{aligned} \tag{6.2.4}$$

It follows, as was already noted in [83] by analyzing the holonomy representation, that a plane-wave background with greater than 16 supersymmetries and $F = 0$ must be flat. We can also see that if F is simple then a plane-wave background with greater than 16 supersymmetries must be symmetric. Indeed, we can suppose without loss of generality that the only non-zero component of F is $dx^+ \wedge w_{123} dx^1 \wedge dx^2 \wedge dx^3$. Then from equation (6.2.3) we see that the only non-zero components of f_{ij} are f_{12}, f_{13} and f_{23}. Equation (6.2.4) implies that H_0 has the form

$$\begin{pmatrix} -\frac{|\omega|^2}{9}\mathbb{1}_{3\times3} & \\ & \frac{|\omega|^2}{36}\mathbb{1}_{6\times6} \end{pmatrix} , \tag{6.2.5}$$

which commutes with f and therefore the plane-wave is symmetric.

Equation (6.2.3) makes it clear that if we can find a Killing bispinor $\psi \odot \phi$ whose square at o has an X component and also has $B^k = 0$ for all k, then $f_{ij} = 0$ and the plane-wave must be symmetric. Let P be the subspace which is the direct sum of the one-dimensional space spanned by X and the subspace of 2-forms of the form $B^k X^\flat \wedge Y_k^\flat$. If there are "enough" of the *extra* Killing spinors so that under the map $\xi \oplus B : \mathcal{S} \odot \mathcal{S} \longrightarrow \wedge^1 \oplus \wedge^2$ we can surjectively hit P, then we can guarantee that there is at least one Killing spinor ε whose square contains an X component and $B^k = 0$. We claim that " enough" is greater than 8 linearly independent *extra* Killing spinors above the 16 that every plane-wave background has, making a total of greater than 24. Our proof of this claim, which follows immediately, uses the same arguments that we exhibited in section 5.5 for the proof that greater than 24 supersymmetries implies local homogeneity.

Let $\mathcal{S}$ be the 32-dimensional spinor bundle restricted to the point o. The space of Killing spinors at o which satisfy $Z^\flat \cdot \varepsilon = 0$ is the 16-dimensional subspace $\ker(Z^\flat \cdot) \subset \mathcal{S}$. We saw in section 5.5 that $\ker(Z^\flat \cdot) = \mathrm{Im}\,(Z^\flat \cdot)$ and $\ker(X^\flat \cdot) = \mathrm{Im}\,(X^\flat \cdot)$ are complementary lagrangian subspaces which partition $\mathcal{S}$. Let W denote the space of all Killing spinors and let V be a complement to $\ker(Z^\flat \cdot)$ so that $W = \ker(Z^\flat \cdot) \oplus V$. Then V comprises of *extra* Killing spinors ε: those Killing spinors that satisfy $Z^\flat \cdot \varepsilon \neq 0$. We may assume that $\dim V > 4$ so that $\dim W > 20$, since there are known examples of non-symmetric plane-waves with 20-supersymmetries (see section 1.1), but this will not actually be necessary.

Recall that the map $\xi \oplus B : W \odot W \longrightarrow T_oM \oplus \wedge^2(T_oM)$ is defined by

$$\langle \xi[\psi, \phi], v \rangle = (\psi, v^\flat \cdot \phi) \quad \text{and} \quad \langle B[\psi, \phi], v \wedge w \rangle = (\psi, v^\flat \wedge w^\flat \cdot \phi) , \tag{6.2.6}$$

where $v, w \in T_oM$. The map $\xi \oplus B$ is surjective onto P if and only if

$$(\psi, (kZ^\flat + v^\flat \wedge Z^\flat) \cdot \phi) = \langle (\xi + B)[\psi, \phi], kZ^\flat + v^\flat \wedge Z^\flat \rangle = 0 \tag{6.2.7}$$

for all $\psi, \phi \in W$ implies that $kZ^\flat + v^\flat \wedge Z^\flat = 0$, that is $k = 0$ and $v \propto Z$. The existence of $kZ^\flat + v^\flat \wedge Z^\flat$ which satisfies (6.2.7) is equivalent to saying that as a Clifford endomorphism

$$(kZ^\flat + v^\flat \wedge Z^\flat)\cdot : W \longrightarrow W^\perp \ . \tag{6.2.8}$$

Since $\dim W > 16$, the endomorphism $kZ^\flat + v^\flat \wedge Z^\flat\cdot$ must have kernel. Writing $v = v^i Y_i + v^X X$, it is not difficult to see that

$$(kZ^\flat + v^\flat \wedge Z^\flat\cdot)^2 = -(v^X)^2 X^\flat \cdot Z^\flat \cdot \ . \tag{6.2.9}$$

Let $\varepsilon \in \ker(kZ^\flat + v^\flat \wedge Z^\flat\cdot)$ and decompose it $\varepsilon = \varepsilon_Z + \varepsilon_V$ where $\varepsilon_Z \in \ker(Z^\flat\cdot)$ and $\varepsilon_V \in V$. If $\varepsilon_V \neq 0$ then applying equation (6.2.9) to ε implies that $v^X = 0$, because $Z^\flat \cdot \varepsilon \subset \mathrm{Im}\,(Z^\flat\cdot)$ and $\ker(X^\flat\cdot)$ is complementary to $\mathrm{Im}\,(Z^\flat\cdot)$. On the other hand, if $\varepsilon_V = 0$, then using equation (5.1.3) we find

$$0 = (kZ^\flat + v^\flat \wedge Z^\flat) \cdot \varepsilon = v^i Y_i^\flat \cdot Z^\flat \cdot \varepsilon + v^X X^\flat \cdot Z^\flat \cdot \varepsilon + v^X \varepsilon = v^X \varepsilon \ , \tag{6.2.10}$$

and therefore $v^X = 0$ in this case too. Hence v must be transversal; that is, in the span of (Y_i).

The Clifford endomorphism $(kZ^\flat + v^\flat \wedge Z^\flat)\cdot$ satisfies $|kZ^\flat + v^\flat \wedge Z^\flat|^2 = 0$, and $kX^\flat + v^\flat \wedge X^\flat$ is complementary in that it satisfies

$$(kZ^\flat + v^\flat \wedge Z^\flat) \cdot (kX^\flat + v^\flat \wedge X^\flat) + (kX^\flat + v^\flat \wedge X^\flat) \cdot (kZ^\flat + v^\flat \wedge Z^\flat) = 2(k^2 + |v|^2)\mathbb{1} \ . \tag{6.2.11}$$

Therefore, we can conclude from section 5.5 that $\ker(kZ^\flat + v^\flat \wedge Z^\flat\cdot)$ and $\ker(kX^\flat + v^\flat \wedge X^\flat\cdot)$ are complementary 16-dimensional lagrangian subspaces. In fact, because v is spacelike, it is not difficult to see that $\ker(kZ^\flat + v^\flat \wedge Z^\flat\cdot) = \ker(Z^\flat\cdot)$ and $\ker(kX^\flat + v^\flat \wedge X^\flat) = \ker(X^\flat\cdot)$.

Therefore the symmetric bilinear form

$$\beta(\psi, \phi) = (\psi, (kZ^\flat + v^\flat \wedge Z^\flat) \cdot \phi) \tag{6.2.12}$$

has rank 16. Now we can follow the same argument given in section 5.5 to estimate the maximum possible rank of the matrix for β, and conclude that the dimension of a complementary subspace to W must be at least 8 dimensional. Whence, if the dimension of W is greater than 24 then $kZ^\flat + v^\flat \wedge Z^\flat$ must be zero and the result follows.

The only thing left to do is to exhibit a homogeneous plane-wave background that admits 24-supersymmetries and is **not** symmetric. Using a similar strategy to that used for the 24+ conjecture in section 5.6, we could exhibit a subspace of Killing spinors which under the map B would always have a $B^k X^\flat \wedge Y_k$ component and therefore, if F were chosen wisely, would lead to a non-zero f_{ij}. However, actually finding a supergravity plane-wave background which satisfies this is a non-trivial task which we have not been able to do.

6.3 Five and six dimensional supergravity

In [87] Agricola considered naturally reductive homogeneous models of string theory in which the string theory 3-form occurs as the homogeneous structure. In 5 and in 6-dimensions, we will show below that the Maxwell type form F of a homogeneous supergravity theory can be constructed naturally from the skew-symmetrization of any homogeneous structure and show what this means for the Einstein equation. First, we will look at 5-dimensional supergravity.

A 5-dimensional supergravity bosonic background consists of a triple (M, g, F) where (M, g) is a 5-dimensional lorentzian spin manifold with metric g and F is a closed two form such that the Maxwell and Einstein type equations are satisfied:

$$\begin{aligned} d * F &= F \wedge F \ , \\ \mathrm{Ric}_{ij} &= F_{ik}F_j^k + |F|^2 g_{ij} \ , \end{aligned}$$

where $|F|^2 = F_{ij}F^{ij}$, the $*$ is the Hodge star and we are using the Einstein summation convention.

Suppose that (M, g) is a reductive homogeneous solution to these equations with $F = -*\mathrm{Alt}(S)$ defined by the homogeneous structure S, where $\mathrm{Alt}(-)$ means the skew symmetric part. This 2-form is invariant since S and the Hodge star are both invariant, therefore F is parallel with respect to the canonical connection $\tilde{\nabla}$. By taking the skew-symmetric part of $\tilde{\nabla} F = 0$ we have the equation,

$$d * F = \mathrm{Alt}(S(*F)) = -\mathrm{Alt}(\mathrm{Alt}(S)(\mathrm{Alt}(S))) \ . \tag{6.3.1}$$

Let σ^i be a pseudo-orthonormal coframe for T^*M with $|\sigma^1|^2 = -1$. For a 3-form T in 5-dimensions we have

$$\begin{aligned} \mathrm{Alt}(T(T)) &= \mathrm{Alt}((T^i_{jk}(T_{imn}\sigma^j \wedge \sigma^k \otimes \sigma^m \wedge \sigma^n) \\ &= \mathrm{Alt}(T^i_{jk}T_{imn})\sigma^j \wedge \sigma^k \wedge \sigma^m \wedge \sigma^n \\ &= -\mathrm{Alt}(T^{ijk}T^{imn})\sigma^j \wedge \sigma^k \wedge \sigma^m \wedge \sigma^n \\ &= -*T \wedge *T \ , \end{aligned}$$

where in the third equality we have used that i, j, k, m, n must be distinct for it to be non-zero. It follows that $d * F = F \wedge F$. Similarly,

$$dF = \mathrm{Alt}(S(F)) = -\mathrm{Alt}(\mathrm{Alt}(S)(*\mathrm{Alt}(S))) \ ,$$

and for any 3-form T we have

$$\begin{aligned} \mathrm{Alt}(T(*T)) &= \mathrm{Alt}(T^i_{jk}(T^{mjk}\sigma^j \wedge \sigma^k \otimes \sigma^n) \\ &= \mathrm{Alt}(T^i_{jk}T^{mjk})\sigma^j \wedge \sigma^k \wedge \sigma^n = 0 \ . \end{aligned}$$

Therefore F is closed. Now, for the Einstein equation we have

$$F_{ik}F_j^k = *\mathrm{Alt}(S)_{ik} * \mathrm{Alt}(S)_j^k = \epsilon_{ikjmn}\epsilon_{jkimn}|\sigma^k|^2\mathrm{Alt}(S)^{jmn}\mathrm{Alt}(S)^{imn} \ , \qquad (6.3.2)$$

where ϵ_{ijklm} is the sign of the permutation $ijklm$ and the $|\sigma^k|^2$ takes account of the sign when raising the k index. The two epsilon symbols are of opposite sign, combining this with the fact that $ijklm$ are all the indices in 5-dimensions we find that equation (6.3.2) is equal to $\mathrm{Alt}(S)_j^{mn}\mathrm{Alt}(S)_{imn}$. Therefore the Einstein condition becomes

$$\mathrm{Ric}_{ij} = \mathrm{Alt}(S)_{imn}\mathrm{Alt}(S)_j^{mn} + |\mathrm{Alt}(S)|^2 g_{ij} \ . \qquad (6.3.3)$$

If we take the trace of equation (2.2.15) to obtain the Ricci tensor of the canonical connection, one finds the following expression:

$$\tilde{\mathrm{Ric}}_{ij} = \mathrm{Ric}_{ij} - S_{imn}S_j^{mn} \ . \qquad (6.3.4)$$

For a naturally reductive space we have $S = \mathrm{Alt}(S)$, thus the Einstein condition in this case may be rewritten as

$$\tilde{\mathrm{Ric}}_{ij} = |S|^2 g_{ij} \ , \qquad (6.3.5)$$

which is the Einstein condition with vanishing field strength for the canonical connection.

An example of a supergravity background of this type is the 5-dimensional Gödel universe considered in section 4.4.1.1. Recall that this has a one-parameter family of homogeneous structures S_α labelled by α generically of type $\mathcal{T}_2 \oplus \mathcal{T}_3$, but of type $\mathcal{T}_3$ for $\alpha = -1$ and of type $\mathcal{T}_2$ for $\alpha = 1$. The skew symmetric part of these homogeneous structures is the naturally reductive structure at $\alpha = -1$, that is $\mathrm{Alt}(S_\alpha) = S_{-1}$. From the explicit form of S_{-1} given in equation (4.4.5) we find F to be

$$F = - * \mathrm{Alt}(S_\alpha) = - * S_{-1} = \frac{1}{2}\Omega_{ij}dx^1 \wedge dx^2 \ , \qquad (6.3.6)$$

which, up to a factor of two, agrees with F given in section 4.4.1.1.

The homogeneous plane-waves also provide examples of these backgrounds. For the regular waves we have

$$F = - * S = -\frac{1}{2}\sum_{i,j,k} \epsilon_{ijk} f_{ij} \ dx^+ \wedge dx^k \ ,$$

and for the singular waves

$$F = - * \mathrm{Alt}(S) = -\frac{1}{2x^+}\sum_{i,j,k} \epsilon_{ijk} f_{ij} \ dx^+ \wedge dx^k \ .$$

Then the Einstein equation in both cases becomes

$$\operatorname{tr} H_0 = \frac{1}{48} \operatorname{tr} f^2 . \tag{6.3.7}$$

If we pick H_0 and f to satisfy this equation then the resultant plane-wave together with F is a supergravity background.

The data for a 6-dimensional supergravity bosonic background is a lorentzian manifold (M, g) together with a closed and co-closed 3-form F such that

$$\mathrm{Ric}_{ij} = F_{ikl} F_j^{kl} + |F|^2 g_{ij} . \tag{6.3.8}$$

Again, let us consider reductive homogeneous solutions to these equations where the form F is defined by the homogeneous structure: this time by $F = \mathrm{Alt}(S)$. The same calculation as in the 5-dimensional case shows that F is automatically co-closed, and is closed if and only if $\mathrm{Alt}(\mathrm{Alt}(S)_{ij}^k \mathrm{Alt}(S)_{kmn}) = 0$.

The Einstein equation again reduces to (6.3.3), and in the naturally reductive case to (6.3.5).

For 6-dimensional supergravity the homogeneous plane-waves again provide examples. Again the Einstein equation becomes (6.3.7) and we can choose H_0 and f to satisfy this.

Chapter 7

Conclusions

The work in this thesis can be broadly separated into two not unrelated threads: an investigation of some hereditary properties of plane-wave limits, in particular that of homogeneity, and a study of the relationship between supersymmetry and homogeneity for 11-dimensional supergravity backgrounds.

For the first thread, we saw that the plane-wave limit preserves some natural geometric properties such as Einstein's equation, submanifold geometries, and the number of linearly independent Killing vectors and spinors. We showed that a sufficient condition for the plane-wave limit to be homogeneous is that the null geodesic is homogeneous and have given concrete algebraic formulae for the plane-wave limit of a reductive homogeneous space along a homogeneous geodesic. We have noted that this however is not a necessary condition, and have given a method for deciding when the plane-wave limit of a reductive space is homogeneous. This method allows one to calculate the limit when it is homogeneous. We have applied these methods to several interesting homogeneous examples.

For the second thread, we showed how supersymmetries generate the Killing superalgebra of an 11-dimensional supergravity background. We have formulated the Killing bispinor equation on the bundle $\wedge^1 \oplus \wedge^2 \oplus \wedge^5$ of differential forms, and have shown that the integrability condition implies that these must in fact correspond to symmetric products of Killing spinors. We have proven that if a background preserves strictly greater than 24 supersymmetries then the ideal of the Killing superalgebra generated by these supersymmetries acts locally transitively on the background. In particular, these 24+ backgrounds are locally homogeneous. We have also provided evidence towards the conjecture that this bound is sharp, and there exists a non-homogeneous background which preserves 24 supersymmetries; although to prove this conjecture we would need to exhibit such a supergravity background. Finally, we also considered plane-wave backgrounds and showed that a plane-wave which preserves greater than 16 supersymmetries is necessarily naturally reductive. By a similar method to that used for the local homogeneity of 24+ backgrounds, we showed that plane-waves which preserve greater than 24 supersymmetries are symmetric.

These results do raise some interesting questions and potential for further study. First, although the task of finding a non-homogeneous supergravity background with 24 supersymmetries is a difficult one, the task of finding a non-symmetric plane-wave background with 24 supersymmetries is more tractable. To specify a regular homogeneous wave background we need to specify two matrices f and H_0 and a constant 3-form ω in the 9 transversal directions, giving 165 degrees of freedom, modulo the relations given by the Einstein equation, $[f, H_0] \neq 0$ and that ω is not simple. Of course, this is still a large number to systematically check, but it is at least approachable.

In order to make further progress algebraically with the 24+ backgrounds we need them to be reductive. However, it is not clear whether they *are* necessarily reductive; that is whether the Killing superalgebra generated by the supersymmetries necessarily defines a reductive transitive subalgebra. One attempt to understand under what circumstances they are reductive is to lift the reductivity condition to the symmetric square of the spinor bundle and hope to find a natural solution there. Indeed, one could use the formulae (5.4.13) to formulate the Cartan-Killing form K in terms of the Killing spinors, and perhaps derive a condition for reductivity based on non-degeneracy of the restriction of K to the isotropy subalgebra. However the lack of a natural Lie bracket on $\mathbb{S} \odot \mathbb{S}$ makes this difficult. Given that in four dimensions all lorentzian homogeneous spaces admit a reductive transitive subalgebra, it may not be unreasonable to assume that the backgrounds are reductive.

Preservation of supersymmetries under the plane-wave limit implies that the limit of an 11-dimensional supergravity background which preserves greater than 16 supersymmetries must be a regular homogeneous wave, and thus all null homogeneous geodesics of such a background must be absolutely homogeneous. However, examples such as the Kaigorodov space show that not all the geodesics need be homogeneous. Similarly, a background which preserves greater than 24 supersymmetries must have a symmetric plane-wave limit with $f = 0$, and thus if the background is reductive then $S(U, Y_i, Y_j) = 0$ for all null geodetic U. Unfortunately we can not say any more, but this last condition must be quite strong. For example, on a naturally reductive space this implies that the only non-zero component of the homogeneous structure is S_{uvi}. A further study of the implications for the 24+ solutions could be interesting.

Finally, the reduction of the supergravity equations of motion to algebraic equations could in principle lead to a classification of homogeneous solutions. This problem is certainly tractable for a restricted class of homogeneous spaces such as symmetric spaces, where all lorentzian symmetric spaces have been classified [23]. However one problem, which Komrakov's classification illustrates, is the scale; even in four dimensions there are 211 families of solutions to the Einstein-Maxwell equations solved by Komrakov.

Appendix A

Geometric Killing spinors

It is interesting to repeat some of the same analysis of section 5.4 for **geometric Killing spinors**: spinors $\psi \in S$ which satisfy

$$\nabla_X \psi = \lambda X \cdot \psi \ , \tag{A.0.1}$$

for all $X \in TM$, where $\lambda \in \mathbb{R}$ is called the **Killing constant**. We can always take the Killing constant to be either $\pm\frac{1}{2}$ or 0. Geometric Killing spinors[1] like supergravity Killing spinors have the fundamental property that if ψ and ϕ are both Killing then $\xi[\psi, \phi]$ is a Killing vector. We shall see that the some of the issues simplify significantly for the geometric Killing spinors, so they are a good toy model for the supergravity case.

Naturally, we call a bispinor $\psi \odot \phi$ Killing if it satisfies $\nabla_X(\psi \odot \phi) = \lambda X \cdot (\psi \odot \phi)$. The argument given before equation (5.4.7) shows that $\psi \odot \phi$ is Killing if and only if

$$\nabla_X \psi = (\lambda X + \mu(X)) \cdot \psi \quad \text{and} \quad \nabla_X \phi = (\lambda X - \mu(X)) \cdot \phi \ , \tag{A.0.2}$$

and if the 1-form μ is closed then we may change gauge to make both ψ and ϕ Killing. However, the integrability condition does not necessarily imply that μ is closed unless we impose extra conditions such as the Einstein equation, and at the end of this section we shall exhibit a space which admits a Killing bispinor which does not originate from Killing spinors.

Again we consider the isomorphism (5.4.2) between the symmetric square of the spinor bundle $\mathbb{S} \odot \mathbb{S}$ in 11-dimensions and the bundle $\wedge^1 \oplus \wedge^2 \oplus \wedge^5$. Similar calculations to those for the supergravity case lead to the following result:

Theorem A.0.1. *A triple (ξ, B, C) is a Killing bispinor if and only if*

$$\nabla_X(\xi, A, C) = 2\lambda(A(X), \ X^\flat \wedge \xi^\flat \ , \ \iota_X(*C)) \ . \tag{A.0.3}$$

[1] For the rest of this chapter we shall refer to geometric Killing spinors simply as *Killing spinors*, making the distinction with supergravity Killing spinors when necessary.

If we apply Killings identity (2.1.2) to this equation we find

$$R(\xi, X)(Y, Z) = X^\flat \wedge \xi^\flat(Y, Z) = g(X, Y)g(\xi, Z) - g(X, Z)g(\xi, Y) \ ,$$

from which it follows that the plane spanned by ξ and X has sectional curvature equal to 1. Recall the result of section 5.5, that if the dimension of the bundle of Killing spinors is greater than 24 then the space is locally homogeneous. The proof of this result is purely linear algebra and the result still holds if we replace the supergravity with geometric Killing spinors. Thus an 11-dimensional spin manifold whose bundle of Killing spinors is greater than 24 dimensional must have constant sectional curvature, which can also be easily seen as a consequence of the classification of lorentzian spaces admitting real Killing spinors in [88].

If (M, g) were riemannian, equation (A.0.3) would define a sasakian structure on M with sasakian vector field ξ. The existence of a sasakian structure is equivalent to the existence of a Kähler form on the metric cone $C(M)$ (see for example [89]), so that the holonomy of the cone over an n-dimensional M is contained in $\mathrm{U}(n+1)$. The cone $C(M)$ is Kähler if and only if there exists a Spin^c-structure on $C(M)$ and a parallel spinor^c [90], so here the existence of a Killing bispinor is equivalent to the existence of a parallel spinor^c. Of course, we are interested in the case where (M, g) is lorentzian and the notion of Kähler does not exist there. Nevertheless, we shall use Spin^c-structures to construct lorentzian spaces which admit a Killing bispinor which does not originate from Killing spinors.

The Spin^c-bundle S^c is *locally* the tensor product between the spinor bundle and a square root of the canonical line bundle

$$S^c = S \otimes K^{-\frac{1}{2}} \ . \tag{A.0.4}$$

If M is spin, then a connection ω on the $U(1)$-bundle associated to $K^{-\frac{1}{2}}$ together with the Levi-Cività connection induces a covariant derivative ∇^ω on the spinor^c bundle S^c. If ω is flat then ∇ and ∇^ω coincide.

We say that a spinor^c is a real (geometric) **Killing** spinor^c if it satisfies the equation

$$\nabla^\omega_X \psi = \lambda X \cdot \psi \ ,$$

for all $X \in TM$ with $\lambda \in \mathbb{R}$. In [90], it was proven that a simply connected riemannian Spin^c manifold M carries a parallel spinor^c if and only if it is isometric to the product $M_1 \times M_2$ between a simply connected Kähler manifold and a simply connected spin manifold carrying a parallel spinor. It was also proved that a simply connected riemannian Spin^c-manifold admitting a real Killing spinor^c then either the $U(1)$ connection ω is flat and M admits a Killing spinor on the Spin-bundle, or M is Sasakian; that is, M admits a Killing vector ξ of unit length such that the tensor $A = -\nabla\xi$ satisfies

$$(\nabla_X A)Y = g(X,Y)\xi - \xi^\flat(Y)X \ .$$

The spinor^c bundle S^c inherits an inner product $(-,-)$ from the usual spinor inner product on S and hermitian inner product

$$\langle\alpha,\beta\rangle = \int_M \alpha \wedge *\bar{\beta} \ ,$$

(assuming some form of local compactness for M.) The Spin^c connection preserves this metric. This inner product gives each 11-dimensional Spin^c-manifold a pairing which is an extension of the isomorphism (5.4.2) to a map $\mathcal{B} : S^c \odot S^c \to \mathcal{E}_{\mathbb{C}}$ where $\mathcal{E}_{\mathbb{C}}$ is the complexification of $\mathcal{E}$, given by

$$\psi \odot \phi \mapsto (\xi_{\mathbb{C}}[\psi \odot \phi], B_{\mathbb{C}}[\psi \odot \phi], C_{\mathbb{C}}[\psi \odot \phi]) \ , \tag{A.0.5}$$

where the forms $\xi_{\mathbb{C}}, B_{\mathbb{C}}$ and $C_{\mathbb{C}}$ are defined using the Spin^c inner product in equation (5.4.2).

Suppose that M is an 11-dimensional Spin^c-manifold that also admits a Spin-structure. Suppose also that it admits two real Killing spinor^{c}s ψ and ϕ both with Killing constant λ. We may use the isomorphism (A.0.5) to square the spinor^{c}s to obtain a complex bispinor (ξ, B, C). In particular,

$$\mathcal{B}(\psi \odot \phi) = (\xi, B, C)$$

As the spinor^c inner product is preserved by the Spin^c connection we have

$$\begin{aligned}\nabla_X(\psi, Y\cdot\phi) &= (\nabla^\omega_X\psi, Y\cdot\phi) + (\psi, Y\cdot\nabla^\omega_X\psi)\\ &= 2\lambda(\psi, X\wedge Y\cdot\phi) \ ,\end{aligned}$$

where we have made use of (5.1.3) and (5.1.8). Therefore $\nabla_X\xi = 2\lambda B(X)$. Similarly $\nabla_X B = 2\lambda X^\flat \wedge \xi^\flat$ and $\nabla_X C = 2\lambda\iota_X(*C)$. Taking the real part of the complex forms, we find that (ξ, B, C) defines (up to a rescaling) a Killing bispinor.

This bispinor does not necessarily originate from Killing spinors since M may not admit Killing spinors. For example, consider the product space $M = N^{2k} \times \mathbb{E}^{1,10-2k}$ where (N, g_N) is a Kähler manifold of dimension $2k$ that is *not* Ricci flat (and hence does not admit a parallel spinor) and $\mathbb{E}^{1,10-2k}$ is the $11-2k$-dimensional Minkowski space. Minkowski space $\mathbb{E}^{1,10-2k}$ has 2^{10-2k} linearly independent parallel spinors ϕ_i, thus we can construct 2^{10-2k} Spin^c-parallel spinors on the Spin^c-bundle of $N \times T$:

$$\psi_i = \chi \otimes \phi_i \ ,$$

where χ is the spinor^c on N. The associated real bispinor $\Re e(\mathcal{B}(\psi \odot \phi))$ is parallel but M does not admit parallel spinors. If one imposes the Einstein condition on M, then N must be Ricci flat and it follows that all parallel spinor^{c}s are in fact parallel spinors.

Appendix B

Komrakov's lorentzian List

In this appendix we give the list of all 4-dimensional lorentzian homogeneous metrics calculated from Komrakov's classification [48], as promised in section 2.6. For each metric we have used the GRTensor package for Maple to calculate whether the metric is Einstein, Ricci flat, flat or locally symmetric. We have made no attempt to take out isometric metrics; for example, the many flat metrics which appear.

K	g	Properties
$1.1^1.1$ $\lambda = 0$ $b_{13} = 1$	$2e^y dudv + b_{22}(ve^y du + e^y dx)^2$ $+2b_{24}(ve^y du + e^y dx)dy + b_{44}dy^2$	$\det B = 1$ $\downarrow$ Einstein Symmetric
$1.1^1.2$ $\lambda = 0$ $b_{13} = 1$	$2e^y dudv + b_{22}e^{2py}dx^2$ $+2b_{24}e^{py}dxdy + b_{44}dy^2$	$p = \frac{1}{2}$ Einstein $p = 0, \frac{1}{2}$ Symmetric
$1.1^1.3$ $\lambda = 0$ $b_{13} = 1$	$2du(dv - v^2du/2) + b_{22}(vdu + dx)^2$ $+2b_{24}(vdu + dx)dy + b_{44}dy^2$	$b_{22} = 1$ $\downarrow$ Symmetric
$1.1^1.4$ $\lambda = 0$ $b_{13} = 1$	$2dudv + b_{22}(vdu + dx)^2$ $+2b_{24}(vdu + dx)dy + b_{44}dy^2$	$b_{22} = 0$ $\downarrow$ Symmetric
$1.1^1.5$ $\lambda = 0$ $b_{13} = 1$	$2du(dv - v^2du/2) + b_{22}e^{2y}dx^2$ $+2b_{24}e^y dxdy + b_{44}dy^2$	Symmetric $\det B = -b_{22}$ Einstein

K	g	Properties
$1.1^1.6$ $\lambda=0$ $b_{13}=1$	$2dudv+b_{22}e^{2y}dx^2$ $+2b_{24}e^{y}dxdy+b_{44}dy^2$	$b_{22}=0$ $\downarrow$ Flat
$1.1^1.7$ $\lambda=0$ $b_{13}=1$	$2du(dv-v^2du/2)+b_{22}dx^2$ $+2b_{24}dxdy+b_{44}dy^2$	Symmetric
$1.1^1.10$ $\lambda=0$ $b_{13}=1$	Flat	Flat
$1.1^2.1$ $\lambda=0$ $b_{11}=1$	$e^{2y}du^2+b_{22}e^{4y}(dx-vdu)^2$ $+e^{2y}dv^2+b_{44}dy^2$ $+2b_{13}e^{2y}(dx-vdu)dy$	$\det B=4$ Symmetric
$1.1^2.2$ $\lambda=0$ $b_{11}=1$	$e^{2y}du^2+b_{22}e^{2py}dx^2$ $+e^{2y}dv^2+2b_{13}e^{py}dxdy+b_{44}dy^2$	$p=1, b_{22}=1$ Einstein $p=0,1$ Symmetric
$1.1^2.3$ $\lambda=0$ $b_{11}=1$	$b_{22}(dx+\sin(v)du)^2$ $+\cos^2(v)du^2+dv^2+b_{44}dy^2$ $+2b_{13}(dx+\sin(v)du)dy$	$\left.\begin{array}{l} b_{13}^2=b_{44} \\ b_{22}=1 \end{array}\right\}$ Einstein $b_{22}=1$ Symmetric
$1.1^2.4$ $\lambda=0$ $b_{11}=1$	$b_{22}(dx-\sinh(v)du)^2$ $+\cosh^2(v)du^2+dv^2$ $+2b_{13}(dx-\sinh(v)du)dy+b_{44}dy^2$	$\left.\begin{array}{l} b_{13}^2=b_{44} \\ b_{22}=-1 \end{array}\right\}$ Einstein $b_{22}=-1$ Symmetric
$1.1^2.5$ $\lambda=0$ $b_{11}=1$	$du^2+b_{22}(dx+vdu)^2$ $+dv^2+2b_{13}(dx+vdu)dy+b_{44}dy^2$	$b_{22}=0$ Symmetric
$1.1^2.6$ $\lambda=0$ $b_{11}=1$	$\cos^2(v)du^2+b_{22}e^{2y}dx^2$ $+dv^2+2b_{13}e^{y}dxdy+b_{44}dy^2$	$\det B=-b_{22}$ Einstein Symmetric
$1.1^2.7$ $\lambda=0$ $b_{11}=1$	$\cosh^2(v)du^2+b_{22}e^{2y}dx^2$ $+dv^2+2b_{13}e^{y}dxdy+b_{44}dy^2$	$\det B=b_{22}$ Einstein Symmetric
$1.1^2.8$ $\lambda=0$ $b_{11}=1$	$du^2+b_{22}e^{2y}dx^2$ $+dv^2+2b_{13}e^{y}dxdy+b_{44}dy^2$	$b_{22}=0$ Flat Symmetric

K	g	Properties
$1.1^2.9$ $\lambda=0$ $b_{11}=1$	$\cos^2(v)du^2+b_{22}dx^2$ $+dv^2+2b_{13}dxdy+b_{44}dy^2$	Symmetric
$1.1^2.10$ $\lambda=0$ $b_{11}=1$	$\cosh^2(v)du^2+b_{22}dx^2$ $+dv^2+2b_{13}dxdy+b_{44}dy^2$	Symmetric
1.1^3	Flat	Flat
1.1^4	Flat	Flat
$1.4^1.1$ $b_{13}=1$ $b_{44}>0$	$dx^2-2e^x dudv+b_{44}dy^2$ $+b_{33}e^{-2y}(v^2e^x du/2+vdx+dv)^2$ $+2b_{34}e^{-y}(v^2e^x du/2+vdx+dv)dy$	–
$1.4^1.2$ $b_{13}=1$ $b_{44}>0$	$e^{-2y(1-p)}dx^2-2e^{2y(p-1)}dudv$ $+b_{33}e^{-2y(2-p)}dv^2$ $+2b_{34}e^{-y(2-p)}dvdy+b_{44}dy^2$	$p=\frac{5}{3}$ Einstein
$1.4^1.3$ $b_{13}=1$ $b_{44}>0$	$\cosh^2(v)e^{2y}dx^2-2e^{2y}dudv$ $+b_{33}dv^2+2b_{34}dvdy+b_{44}dy^2$	$b_{33}=b_{44}$ $\downarrow$ Einstein Symmetric
$1.4^1.4$ $b_{13}=1$ $b_{44}>0$	$\cos^2(v)e^{2y}dx^2-2e^{2y}dudv$ $+b_{33}dv^2+2b_{34}dvdy+b_{44}dy^2$	$b_{33}=-b_{44}$ $\downarrow$ Einstein Symmetric
$1.4^1.5$ $b_{13}=1$ $b_{44}>0$	$-2e^{2y}dudv+2e^{2x}v^2du^2$ $+b_{33}(-e^x v^2du/2+vdx+dv)^2$ $+2b_{34}(-e^x v^2du/2+vdx+dv)dy$ $+dx^2+b_{44}dy^2$	–
$1.4^1.6$ $b_{13}=1$ $b_{44}>0$	$e^{2y}dx^2-2e^{2y}(du+ydv)dv$ $+b_{33}e^{2y}dv^2+2b_{34}e^y dvdy+b_{44}dy^2$	–
$1.4^1.7$ $b_{13}=1$ $b_{44}>0$	$e^{2y}dx^2-2e^{2y}(du-ydv)dv$ $+b_{33}e^{2y}dv^2+2b_{34}e^y dvdy+b_{44}dy^2$	–
$1.4^1.8$ $b_{13}=1$ $b_{44}>0$	$e^{2y}dx^2-2e^{2y}dudv$ $+b_{33}e^{2y}dv^2+2b_{34}e^y dvdy+b_{44}dy^2$	$-3b_{33}=b_{44}$ $\downarrow$ Einstein

K	g	Properties
$1.4^1.9$ $b_{13}=1$ $b_{44}>0$	$-2(du - x^2rdv/2 - udv)dv$ $+(dx - xdv)^2 + b_{33}dv^2$ $+2b_{34}(e^{-pv}dy - xdv)dv$ $+b_{44}(e^{-pv}dy - xdv)^2$	$r + b_{44}/2$ $+p + p^2 = 0$ $\downarrow$ Ricci Flat
$1.4^1.10$ $b_{13}=1$ $b_{44}>0$	$-2(du - x^2rdv/2 - udv)dv$ $+(dx - xdv)^2 + 2b_{34}e^{-pv}dvdy$ $+b_{33}dv^2 + b_{44}e^{-2pv}dy^2$	$r + p + p^2 = 0$ $\downarrow$ Ricci Flat
$1.4^1.11$ $b_{13}=1$ $b_{44}>0$	$-2(du - x^2rdv/2 - udv - ve^vdy)dv$ $+(dx - xdv)^2 + 2b_{34}(e^vdy - xdv)dv$ $+b_{33}dv^2 + b_{44}(e^vdy - xdv)^2$	$r + b_{44}/2$ $+p + p^2 = 0$ $\downarrow$ Ricci Flat
$1.4^1.12$ $b_{13}=1$ $b_{44}>0$	$-2(du - x^2rdv/2 - udv - ve^vdy)dv$ $+(dx - xdv)^2 + 2b_{34}e^vdvdy$ $+b_{33}dv^2 + b_{44}e^{2v}dy^2$	$r + p + p^2 = 0$ $\downarrow$ Ricci Flat
$1.4^1.13$ $b_{13}=1$ $b_{44}>0$	$-2(du - x^2rdv/2)dv$ $+dx^2 + 2b_{34}(e^vdy - xdv)dv$ $+b_{33}dv^2 + b_{44}(e^vdy - xdv)^2$	$r + b_{44}/2$ $+p^2 = 0$ $\downarrow$ Ricci Flat
$1.4^1.14$ $b_{13}=1$ $b_{44}>0$	$-2dudv - ve^vdy)dv$ $+\cosh^2(\sqrt{r}v)dx^2$ $+2b_{34}(dy + ydv)dv$ $+b_{33}dv^2 + b_{44}(dy + ydv)^2$	$r = -1$ $\downarrow$ Ricci Flat
$1.4^1.15$ $b_{13}=1$ $b_{44}>0$	$-2dv(du + (1 - \cosh(v))dx + ydv)$ $+\cosh^2(v)dx^2 + b_{33}dv^2$ $+2b_{34}(dy + \sinh(v)dx)dv$ $+b_{44}(dy + \sinh(v)dx)^2$	$b_{44} = -2$, not $(1,3)$ $\downarrow$ Ricci Flat
$1.4^1.16$ $b_{13}=1$ $b_{44}>0$	$-2dudv + \cos^2(v)dx^2 + b_{33}dv^2$ $-2dv((-1 + \cos(v))dx + ydv)$ $+2b_{34}(dy + \sin(v)dx)dv$ $+b_{44}(dy + \sin(v)dx)^2$	$b_{44} = 2$ $\downarrow$ Ricci Flat
$1.4^1.17$ $b_{13}=1$ $b_{44}>0$	$-2(du - v^2dx/2 + ydv)dv$ $+dx^2 + 2b_{34}(dy + vdx)dv$ $+b_{33}dv^2 + b_{44}(dy + vdx)^2$	$b_{44} = 0$ $\downarrow$ Ricci Flat
$1.4^1.18$ $b_{13}=1$ $b_{44}>0$	$-2dudv + \cosh^2(v)dx^2$ $+2b_{34}(dy + \sinh(v)dx)dv$ $+b_{33}dv^2 + b_{44}(dy + \sinh(v)dx)^2$	$b_{44} = -2$ not $(1,3)$ $\downarrow$ Ricci Flat