# Groupoids and local cartesian closure

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**Abstract**

The 2-category of groupoids, functors and natural isomorphisms is shown to be locally cartesian closed in a weak sense of a pseudo-adjunction.

*Key words:* groupoids, 2-categories, local cartesian closure.

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## 1 Introduction

A *groupoid* is a category in which each morphism is invertible. The notion may be considered as a common generalisation of the notions of group and equivalence relation. A group is a one-object groupoid. Every equivalence relation on a set, viewed as a directed graph, is a groupoid. A recent independence proof in logic used groupoids. Hofmann and Streicher [8] showed that groupoids arise from a construction in Martin-Löf type theory. This is the so-called *identity type construction*, which applied to a type gives a groupoid turning the type into a projective object, in the category of types with equivalence relations. As a consequence there are plenty of choice objects in this category: every object has a projective cover [12] — a property which seems essential for doing constructive mathematics according to Bishop [1] internally to the category. For some time the groupoids associated with types were believed to be discrete. However, in [8] a model of type theory was given using fibrations over groupoids, showing that this need not be the case.

The purpose of this article is to draw some further conclusions for groupoids from the idea of Hofmann and Streicher. Since type theory has a Π-construction, one could expect that some kind of (weak) local cartesian closure should hold for groupoids. The small groupoids can be organised into a category **Gpd** by taking the morphisms to be functors. This category has many useful properties: it is complete and cocomplete, and unlike the

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category of groups, it is also cartesian closed [4]. However, it is *not* locally cartesian closed (see Section 2). The main goal of the present paper is to show that, from a 2-categorical perspective, the groupoids nevertheless satisfy a version of local cartesian closedness. The category **Gpd** may be reorganised as a 2-category $\mathbb{G}$ by letting the natural transformations between functors be the 2-cells, which are then necessarily natural isomorphisms. After some preliminaries on 2-categorical constructions (Section 3 and 4) the main result (Theorem 5.6) is presented. This involves a construction of dependent products of groupoids which is fairly natural, though the verification of its 2-universal property is quite involved. We should point out that the solution is not completely satisfactory, since the notion of "semi-strict pseudo-adjunction" is probably not general enough.

The proofs herein are constructive and choice free, and should be possible to formalise in a topos, or even in predicative versions of toposes.

## 2 The category of groupoids

In this section we see that the category **Gpd** of small groupoids and functors is not locally cartesian closed. This motivates the laborious 2-categorical constructions in Section 3 and onwards. For a general overview of groupoids we recommend Brown [4]. Higgins [7] contains the most basic and central constructions. The pullback construction will be of special interest, so we state it explicitly here. Let $f : A \longrightarrow Z$ and $g : B \longrightarrow Z$ be functors in **Gpd**. The pullback of these is constructed analogously to that of sets. Let $P$ be the groupoid where the objects are pairs $(x, y)$, where $x$ is an object of $A$ and $y$ is an object of $B$ such that $f(x) = g(y)$. A morphism from $(x, y)$ to $(x', y')$ is a pair of morphisms $(\varphi, \psi)$ where $\varphi : x \longrightarrow x'$ and $\psi : y \longrightarrow y'$ are morphisms satisfying $f(\varphi) = g(\psi)$. It follows by the functoriality of $f$ and $g$ that this defines a groupoid. Together with the first and second projection functor $\pi_1 : P \longrightarrow A$ and $\pi_2 : P \longrightarrow B$, this is a pullback of $f$ and $g$.

The coproduct of a set of groupoids can be formed by simply taking the disjoint union of objects and arrows. For the construction of coequalizers, see [7].

It is known that the category **Cat** of small categories is not locally cartesian closed (Conduché [5]). Johnstone [9, p. 48] gives a simple argument for this fact, which is easy to adapt to the category **Gpd**. Let $E_n$ be the groupoid given by the coarsest equivalence relation on the set $\mathbb{N}_n = \{0, 1, \ldots, n-1\}$, that is, for every pair of objects $i, j \in E_n$, there is a unique arrow $e_{ij} : i \longrightarrow j$. In the terminology of [7] this is the *simplicial groupoid* on $\mathbb{N}_n$.

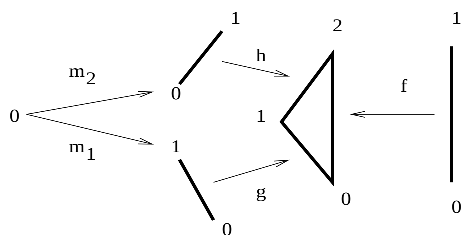


Figure 1

Let $g, h : E_2 \longrightarrow E_3$ be the unique functors such that $g(e_{01}) = e_{01}$ and $h(e_{01}) = e_{12}$. Then the canonical map $[g, h] : E_2 + E_2 \longrightarrow E_3$ is a regular epi. To wit, it is the coequalizer of the arrows $m_1, m_2 : E_1 \longrightarrow E_2 + E_2$ where $m_1$ maps 0 to 1 of the first summand and $m_2$ maps 0 to 0 of the second summand. Now let $f : E_2 \longrightarrow E_3$ be the unique functor with $f(e_{01}) = e_{02}$. (See Figure 1.) The pullback of the arrow $[g, h]$ along $f$ is then isomorphic to $k = [k_0, k_1] : E_1 + E_1 \longrightarrow E_2$ where $k_j$ is the unique functor mapping the object 0 to $j$. This $k$ is clearly not epi.

**Proposition 2.1** *The category* **Gpd** *is not locally cartesian closed.*

**Proof.** Suppose that **Gpd** is locally cartesian closed. Then since **Gpd** has coequalizers [7, Ch. 9] it is also regular [9, Lemma 1.5.13]. Consequently, the pullback of a regular epi should be a regular epi. But we have just presented a counter-example to this in **Gpd**. This is a contradiction. ∎

## 3 Some 2-categorical notions and constructions

Let $\mathcal{C}$ be a 2-category. We employ the notation of Borceux [3]. The composition of 1-cells is denoted $\circ$, and for a 1-cell $f : A \longrightarrow B$, the left and right identities are $1_A$ and $1_B$ respectively. A 2-cell $\alpha$ from $f : A \longrightarrow B$ to $g : A \longrightarrow B$ is written $\alpha : f \Rightarrow g$. The vertical composition of 2-cells is denoted $\odot$. The left and right vertical identities of $\alpha$ are $i_f$ and $i_g$. The operation $*$ stands for horizontal composition of 2-cells. The left and right horizontal identities for $\alpha$ from $f : A \longrightarrow B$ to $g : A \longrightarrow B$ are $i_{1_A}$ and $i_{1_B}$, but we will denote these by $i_A$ and $i_B$ respectively. Apart from the usual identity and associativity laws these operations satisfy a further identity law and the exchange law

$$\begin{aligned} i_g * i_f &= i_{g \circ f}, \\ (\gamma \odot \delta) * (\alpha \odot \beta) &= (\gamma * \alpha) \odot (\delta * \beta). \end{aligned}$$

for $f_1 \stackrel{\beta}{\Longrightarrow} f_2 \stackrel{\alpha}{\Longrightarrow} f_3$ and $g_1 \stackrel{\delta}{\Longrightarrow} g_2 \stackrel{\gamma}{\Longrightarrow} g_3$, where $f_1, f_2, f_3 : A \longrightarrow B$, $g_1, g_2, g_3 : B \longrightarrow C$. We have distributivity of vertical identities

$$\begin{aligned} i_g * (\alpha \odot \beta) &= (i_g * \alpha) \odot (i_g * \beta) \\ (\gamma \odot \delta) * i_f &= (\gamma * i_f) \odot (\delta * i_f), \end{aligned}$$

where $f : A \longrightarrow B$ and $g : B \longrightarrow C$. Inverses can be computed as

$$(\alpha * \beta)^{-1} = \alpha^{-1} * \beta^{-1},$$

for invertible $\alpha$ and $\beta$. Note that for 2-cells $\alpha : h \Rightarrow k$ and $\beta : f \Rightarrow g$, where $f, g : A \longrightarrow B$ and $h, k : B \longrightarrow C$, the interchange law gives the decomposition laws

$$\begin{aligned} \alpha * \beta &= (\alpha * i_g) \odot (i_h * \beta), \\ \alpha * \beta &= (i_k * \beta) \odot (\alpha * i_f). \end{aligned}$$

In the following we shall of use the syntactic convention that the operation $*$ binds stronger than $\odot$.

The 2-category of groupoids to be studied in Sections 4 and 5 is the following.

**Example 3.1** The 2-category $\mathbb{G}$ has small groupoids as objects, functors as 1-cells and natural transformations as 2-cells. The vertical composition $\odot$ is the usual vertical composition $\cdot$ of natural transformations. The horizontal composition of $\alpha * \beta$ (with $f, g, h, k$ as above) is given by

$$(\alpha * \beta)_a = \alpha_{g(a)} h(\beta_a) = k(\beta_a) \alpha_{f(a)}.$$

The identity $i_f$ is the natural transformation $i_f : f \Rightarrow f$ defined by $(i_f)_a = 1_{f(a)}$.

## 3.1 2-slices

In an ordinary slice category $\mathcal{C}/X$ a morphism from $\alpha : A \longrightarrow X$ to $\beta : B \longrightarrow X$ is a morphism $f : A \longrightarrow B$ in $\mathcal{C}$ satisfying the equality $\alpha = \beta \circ f$. For morphisms between the slices of a 2-category (to be defined below), this equality is replaced by a 2-cell. The specific information given by the 2-cell seems necessary to obtain the pseudo-adjoints of Section 4 and 5.

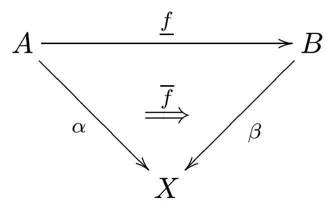
**Definition 3.2** Let $\mathcal{C}$ be a 2-category. For every object $X$ of $\mathcal{C}$ the *2-slice category* $\mathcal{C}//X$ of $\mathcal{C}$ by $X$ is the 2-category given by the following data. (When the intended category is obvious from the context we simply write $\bar{\bar{X}}$ for $\mathcal{C}//X$.)

- The objects of $\mathcal{C}//X$ are arrows (1-cells) of the form $\alpha : A \longrightarrow X$ of $\mathcal{C}$. We also write $(\alpha, A)$ or even $\alpha$.

- The arrows (1-cells) from $(\alpha, A)$ to $(\beta, B)$ in $\mathcal{C}//X$ are pairs

$$f = (\underline{f}, \overline{f})$$

such that $\underline{f} : A \longrightarrow B$ is an arrow of $\mathcal{C}$ and $\overline{f} : \alpha \Rightarrow \beta \circ \underline{f}$ is a 2-cell of $\mathcal{C}$.




  - Suppose that $f : (\alpha, A) \longrightarrow (\beta, B)$ and $g : (\beta, B) \longrightarrow (\gamma, C)$ are 1-cells. Then their composition is given by

$$g \circ f = (\underline{g} \circ \underline{f}, (\overline{g} * i_{\underline{f}}) \odot \overline{f}). \qquad (1)$$

  - The identity $1_{(\alpha,A)}$ on $(\alpha, A)$ is $(1_A, i_\alpha)$.

- Suppose $f, g : (\alpha, A) \longrightarrow (\beta, B)$ are 1-cells. A 2-cell $\tau$ from $f$ to $g$ is by definition a 2-cell $\tau : \underline{f} \Rightarrow \underline{g}$ of $\mathcal{C}$ such that the *pasting condition*

$$(i_\beta * \tau) \odot \overline{f} = \overline{g} \qquad (2)$$

is satisfied.

  - Suppose that $\sigma : f \Rightarrow g$ and $\tau : g \Rightarrow h$ are 2-cells, where $f, g, h : (\alpha, A) \longrightarrow (\beta, B)$. The vertical composition $\tau \odot \sigma$ is then just the vertical composition $\tau \odot \sigma$ in $\mathcal{C}$.
  - Suppose that $\sigma : f \Rightarrow g$ and $\tau : h \Rightarrow k$ are 2-cells, where $f, g : (\alpha, A) \longrightarrow (\beta, B)$ and $h, k : (\beta, B) \longrightarrow (\gamma, C)$. The horizontal composition $\tau * \sigma : h \circ f \Rightarrow k \circ g$ is then the horizontal composition $\tau * \sigma$ in $\mathcal{C}$.
  - The identity 2-cell $i_f : f \Rightarrow f$ is $i_{\underline{f}}$.

**Remark 3.3** Note that in $\mathbb{G}//X$ the composition (1) becomes $(\underline{g} \circ \underline{f}, \overline{g}\underline{f} \cdot \overline{f})$ and the pasting condition (2) becomes $\beta\tau \cdot \overline{f} = \overline{g}$, using standard notation for composing functors and natural transformations (Mac Lane [11]).

**Proposition 3.4** *If $\mathcal{C}$ is a 2-category and $X$ is an object of $\mathcal{C}$, then $\mathcal{C}//X$ is a 2-category.*

**Proof.** The verification is straightforward, except possibly for the fact that horizontal composition of 2-cells is well-defined. Let $f_1, f_2 : (\alpha, A) \longrightarrow (\beta, B)$ and $g_1, g_2 : (\beta, B) \longrightarrow (\gamma, C)$. Suppose that $\sigma : f_1 \Rightarrow f_2$ and $\tau : g_1 \Rightarrow g_2$. Their horizontal composition is $\tau * \sigma : \underline{g_1} \circ \underline{f_1} \Rightarrow \underline{g_2} \circ \underline{f_2}$. We show that it satisfies the pasting condition.

$$\begin{aligned}
(i_\gamma * \tau * \sigma) \odot \overline{g_1 \circ f_1} &= (i_\gamma * \tau * \sigma) \odot (\overline{g_1} * i_{\underline{f_1}}) \odot \overline{f_1} \\
&= (((i_\gamma * \tau) \odot \overline{g_1}) * (\sigma \odot i_{\underline{f_1}})) \odot \overline{f_1} \\
&= ((\overline{g_2} * (\sigma \odot i_{\underline{f_1}})) \odot \overline{f_1} \\
&= (\overline{g_2} * \sigma) \odot \overline{f_1} \\
&= (\overline{g_2} * i_{\underline{f_2}}) \odot (i_\beta * \sigma) \odot \overline{f_1} \\
&= (\overline{g_2} * i_{\underline{f_2}}) \odot \overline{f_2} = \overline{g_2 \circ f_2}
\end{aligned}$$

The equations are obtained by applying (in order): the interchange law, the pasting condition of $\tau$, identity law, the decomposition law and the pasting condition of $\sigma$. ∎

**Remark 3.5** Note that if $\tau : f \Rightarrow g$ is a 2-cell in $\mathcal{C}//X$ and $\tau$ is invertible in $\mathcal{C}$, then $\tau^{-1} : g \Rightarrow f$ is inverse to $\tau$ in $\mathcal{C}//X$. In particular, this means that the 2-cells of $\mathbb{G}//X$ are all invertible.

### 3.2 2-dimensional functors

**Definition 3.6** Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. A *2-functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ consists of an object $F(A)$ of $\mathcal{D}$ for each object $A$ of $\mathcal{C}$, and for each pair of objects $A, B \in \mathcal{C}$ a functor

$$F_{A,B} : \mathcal{C}(A, B) \longrightarrow \mathcal{D}(FA, FB)$$

such that

$$F_{A,A}(1_A) = 1_{FA} \quad F_{A,C}(g \circ f) = F_{B,C}(g) \circ F_{A,B}(f), \tag{3}$$

for 1-cells $f : A \longrightarrow B$, $g : B \longrightarrow C$, and moreover

$$F_{A,A}(i_A) = i_{FA} \quad F_{A,C}(\beta * \alpha) = F_{B,C}(\beta) * F_{A,B}(\alpha), \tag{4}$$

for 2-cells $\alpha : f_1 \Rightarrow f_2$ and $\beta : g_1 \Rightarrow g_2$, where $f_1, f_2 : A \longrightarrow B$ and $g_1, g_2 : B \longrightarrow C$.

The *composition* $GF : \mathcal{C} \longrightarrow \mathcal{E}$ with another 2-functor $G : \mathcal{D} \longrightarrow \mathcal{E}$ is given by

$$(G \circ F)(A) = G(F(A)) \qquad (G \circ F)_{A,B} = G_{FA,FB} \circ F_{A,B}.$$

**Remark 3.7** Note that the first equality of (4) actually follows from (3) and the fact that $F_{A,A}$ is a functor.

**Example 3.8** Let $\mathcal{C}$ be a 2-category. The identity 2-functor $I_{\mathcal{C}}$ on $\mathcal{C}$ is defined by $I_{\mathcal{C}}(A) = A$, and $(I_{\mathcal{C}})_{A,B}$ is the identity functor on $\mathcal{C}(A, B)$.

**Example 3.9** Let $\mathcal{C}$ be a 2-category and let $h : X \longrightarrow Y$ be an arrow (1-cell) in this category. Then we define the post-composition 2-functor

$$\Sigma_h : \mathcal{C}//X \longrightarrow \mathcal{C}//Y$$

as follows on 0-cells, 1-cells and 2-cells respectively:

$$\begin{aligned} \Sigma_h(\alpha, A) &= (h \circ \alpha, A), \\ (\Sigma_h)_{(\alpha,A),(\beta,B)}(f) &= (\underline{f}, i_h * \overline{f}), \\ (\Sigma_h)_{(\alpha,A),(\beta,B)}(\tau) &= \tau. \end{aligned}$$

## 3.3 2-dimensional transformations and adjunctions

We recall the definition of a 2-natural transformation [3].

**Definition 3.10** Suppose that $F, G : \mathcal{A} \longrightarrow \mathcal{B}$ are 2-functors. A *2-natural transformation* $\theta : F \stackrel{\cdot}{\longrightarrow} G$ is a collection of arrows of $\mathcal{B}$, $\theta_A : F(A) \longrightarrow G(A)$, where $A$ ranges over objects of $\mathcal{A}$. They satisfy the following naturality condition

$$\mathcal{B}(FA, \theta_{A'}) \circ F_{A,A'} = \mathcal{B}(\theta_A, GA') \circ G_{A,A'} \tag{5}$$

for objects $A$ and $A'$ of $\mathcal{A}$. This means that for 1-cells $f \in \mathcal{A}(A, A')$

$$\theta_{A'} \circ F_{A,A'}(f) = G_{A,A'}(f) \circ \theta_A \tag{6}$$

and for 2-cells $\alpha : f \Rightarrow f'$ in $\mathcal{A}(A, A')$

$$i_{\theta_{A'}} * F_{A,A'}(\alpha) = G_{A,A'}(\alpha) * i_{\theta_A}. \tag{7}$$

**Example 3.11** Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a 2-functor. The identity 2-natural transformation on $F$ is $I_F : F \stackrel{\cdot}{\longrightarrow} F$ given by $(I_F)_A = 1_{FA}$.

**Example 3.12** Specialising to $\mathcal{A} = \mathbb{G}//X$, $\mathcal{B} = \mathbb{G}//Y$, and $\theta_A = (\underline{\theta}_A, \overline{\theta}_A)$ then (7) reads

$$\underline{\theta}_{A'} F_{A,A'}(\alpha) = G_{A,A'}(\alpha) \underline{\theta}_A. \tag{8}$$

The definition of composition of 2-functors and 2-natural transformation are formally identical to the 1-dimensional case. Let $H : \mathcal{B} \longrightarrow \mathcal{C}$, $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ and $K : \mathcal{D} \longrightarrow \mathcal{E}$ be 2-functors, and suppose that $\varepsilon : F \stackrel{\cdot}{\longrightarrow} G$ is a 2-natural transformation. Then define 2-natural transformations

$$\varepsilon H : FH \stackrel{\cdot}{\longrightarrow} GH, \qquad K\varepsilon : KF \stackrel{\cdot}{\longrightarrow} KG$$

by

$$(\varepsilon H)_A = \varepsilon_{HA} : FHA \longrightarrow GHA$$

and

$$(K\varepsilon)_A = K_{FA,GA}(\varepsilon_A) : KFA \longrightarrow KGA.$$

To formulate the desired adjunction property we need a more liberal version of 2-natural transformations. The notion of *lax 2-natural transformation* (Borceux [3]) can be applied to (strict) 2-functors as well. We specialise the notion to this case. (Note the direction of $\tau_{A,B}$.)

**Definition 3.13** Let $F, G : \mathcal{A} \longrightarrow \mathcal{B}$ be 2-functors. A *lax natural transformation* from $F$ to $G$ is a pair $(\alpha, \tau)$ (notation: $(\alpha, \tau) : F \stackrel{\cdot}{\longrightarrow} G$) consisting of arrows $\alpha_A : FA \longrightarrow GA$, for $A \in \mathcal{A}$, and for each pair of objects $A, B \in \mathcal{A}$ a natural transformation

$$\tau_{A,B} : \mathcal{B}(\alpha_A, GB) \circ G_{A,B} \longrightarrow \mathcal{B}(FA, \alpha_B) \circ F_{A,B}. \tag{9}$$

These should satisfy the following functoriality conditions:

$$(\tau_{A,A})_{1_A} = i_{\alpha_A} \tag{10}$$

and

$$(\tau_{A,C})_{g \circ f} = (\tau_{B,C})_g * i_{F_{A,B}(f)} \odot i_{G_{B,C}(g)} * (\tau_{A,B})_f \tag{11}$$

for $f : A \longrightarrow B$ and $g : B \longrightarrow C$. In case each $\tau_{A,B}$ is a natural isomorphism, $(\alpha, \tau)$ is called a *pseudo-natural transformation.*

**Remark 3.14** Condition (9) means, explicitly, that for $f, g : A \longrightarrow B$ and $\sigma : f \Rightarrow g$ the following diagram of 2-cells commutes:

$$\begin{array}{ccc} G_{A,B}(f) \circ \alpha_A & \xRightarrow{(\tau_{A,B})_f} & \alpha_B \circ F_{A,B}(f) \\ {\scriptstyle G_{A,B}(\sigma) * i_{\alpha_A}} \Big\Downarrow & & \Big\Downarrow {\scriptstyle i_{\alpha_B} * F_{A,B}(\sigma)} \\ G_{A,B}(g) \circ \alpha_A & \xRightarrow[(\tau_{A,B})_g]{} & \alpha_B \circ F_{A,B}(g) \end{array} \tag{12}$$

**Example 3.15** Given a 2-natural transformation $\alpha : F \stackrel{\cdot}{\longrightarrow} G$, then $G_{A,B}(f) \circ \alpha_A = \alpha_B \circ F_{A,B}(f)$. We may define

$$(\tau_{A,B})_f = i_{G_{A,B}(f)\circ\alpha_A} = i_{\alpha_B \circ F_{A,B}(f)}.$$

It is straightforward to see that $(\alpha, \tau)$ defines a pseudo-natural transformation $F \stackrel{\cdot}{\longrightarrow} G$, the *lax version* of $\alpha$. Conversely, any lax $(\alpha, \tau)$ where $(\tau_{A,B})_f$ is the identity for each $f$, is called *2-natural*. In case $\alpha = I_F : F \stackrel{\cdot}{\longrightarrow} F$ is the identity 2-natural transformation on F, then $(\tau_{A,B})_f = i_{F_{A,B}(f)}$.

The compositions given just before Definition 3.13 are now generalised to lax natural transformations.

**Definition 3.16** Let $(\alpha, \tau) : F \stackrel{\cdot}{\longrightarrow} G$ be a lax natural transformation between the 2-functors $F, G : \mathcal{A} \longrightarrow \mathcal{B}$. We define the left and right compositions with the 2-functors $H : \mathcal{Z} \longrightarrow \mathcal{A}$ and $K : \mathcal{B} \longrightarrow \mathcal{C}$. Define

$$(\alpha, \tau)H =_{\text{def}} (\alpha H, \tau H) : FH \stackrel{\cdot}{\longrightarrow} GH$$

by letting

$$(\alpha H)_U = \alpha_{HU}$$

and

$$(\tau H)_{U,V} = \tau_{HU,HV} H_{U,V},$$

where the right hand side is an ordinary composition of a natural transformation with a functor. Define

$$K(\alpha, \tau) =_{\text{def}} (K\alpha, K\tau) : KF \stackrel{\cdot}{\longrightarrow} KG$$

by assigning

$$(K\alpha)_A = K_{FA,GA}(\alpha_A)$$

and

$$(K\tau)_{A,B} = K_{FA,GB}\tau_{A,B}.$$

Here the right hand side is the composition of a functor with a natural transformation.

**Proposition 3.17** *With assumptions as in Definition 3.16:*

*(i)* $(\alpha, \tau)H$ *is lax natural, and it is pseudo-natural whenever* $(\alpha, \tau)$ *is,*

*(ii)* $K(\alpha, \tau)$ *is lax natural, and it is pseudo-natural whenever* $(\alpha, \tau)$ *is.*

**Remark 3.18** These compositions generalise those for 2-natural transformation. (Cf. Example 3.15.) So, e.g., that if $(\alpha, \tau)$ is 2-natural, then $(\alpha, \tau)H$ is the lax version of $\alpha H$.

**Definition 3.19** Let $F, G, H : \mathcal{A} \overset{\cdot}{\longrightarrow} \mathcal{B}$ be 2-functors. Let $(\alpha, \sigma) : F \overset{\cdot}{\longrightarrow} G$ and $(\beta, \tau) : G \overset{\cdot}{\longrightarrow} H$ be lax natural transformations. Define their *vertical composition*

$$(\beta, \tau) \odot (\alpha, \sigma) =_{\text{def}} (\beta \odot \alpha, \tau \odot \sigma) : F \overset{\cdot}{\longrightarrow} H$$

as follows

$$(\beta \odot \alpha)_A = \beta_A \circ \alpha_A$$

and, for $f : A \longrightarrow B$,

$$((\tau \odot \sigma)_{A,B})_f = i_{\beta_B} * (\sigma_{A,B})_f \odot (\tau_{A,B})_f * i_{\alpha_A}. \tag{13}$$

**Remark 3.20** If $(\alpha, \sigma)$ is 2-natural, the right hand side of (13) becomes $(\tau_{A,B})_f * i_{\alpha_A}$, and in case $(\beta, \tau)$ is 2-natural, it becomes $i_{\beta_B} * (\sigma_{A,B})_f$.

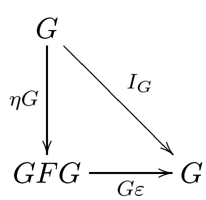
**Proposition 3.21** *With assumptions as in Definition 3.19: $(\beta, \tau) \odot (\alpha, \sigma)$ is a lax natural transformation, and it is pseudo-natural if both $(\beta, \tau)$ and $(\alpha, \sigma)$ are. Moreover, in case $(\beta, \tau)$ and $(\alpha, \sigma)$ are both 2-natural transformations (considered as lax natural transformation), then $(\beta, \tau) \odot (\alpha, \sigma)$ is again a 2-natural transformation.*

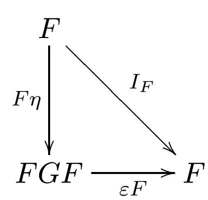
We now define the notion of adjunction to be employed.

**Definition 3.22** A *semi-strict pseudo-adjunction* $\langle F, G, \varepsilon, \eta \rangle : \mathcal{C} \longrightarrow \mathcal{D}$ consists of the 2-functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ and pseudo-natural transformations

$$\varepsilon : FG \overset{\cdot}{\longrightarrow} I_{\mathcal{D}} \qquad \eta : I_{\mathcal{C}} \overset{\cdot}{\longrightarrow} GF$$

such that both




commute. These are called the *triangular identities* for $G$ and $F$, respectively. Here $I_H$ denotes the lax version of the identity 2-natural transformation on the 2-functor $H$. (See Example 3.15.)

Given an adjunction as in Definition 3.22, we define functors

$$P_{A,B} : \mathcal{C}(A, GB) \longrightarrow \mathcal{D}(FA, B)$$

and

$$Q_{A,B} : \mathcal{D}(FA, B) \longrightarrow \mathcal{C}(A, GB)$$

as follows for arrows $f$ and 2-cells $\alpha$:

$$P_{A,B}(f) = \varepsilon^1_B \circ F_{A,GB}(f), \qquad P_{A,B}(\alpha) = i_{\varepsilon^1_B} * F_{A,GB}(\alpha) \tag{14}$$

and

$$Q_{A,B}(f) = G_{FA,B}(f) \circ \eta^1_A, \qquad Q_{A,B}(\alpha) = G_{FA,B}(\alpha) * i_{\eta^1_A}, \tag{15}$$

where $\eta = (\eta^1, \eta^2)$ and $\varepsilon = (\varepsilon^1, \varepsilon^2)$.

**Theorem 3.23** *Let $\langle F, G, \varepsilon, \eta\rangle : \mathcal{C} \longrightarrow \mathcal{D}$ be a semi-strict pseudo-adjunction. With $P$ and $Q$ defined as in (14) and (15):*

*(i) $Q_{A,B} \circ P_{A,B} \simeq I_{\mathcal{C}(A,GB)}$ and $Q_{A,B} \circ P_{A,B} = I_{\mathcal{C}(A,GB)}$, whenever $\eta$ is 2-natural,*

*(ii) $P_{A,B} \circ Q_{A,B} \simeq I_{\mathcal{D}(FA,B)}$, and $P_{A,B} \circ Q_{A,B} = I_{\mathcal{D}(FA,B)}$, whenever $\varepsilon$ is 2-natural.*

**Proof.** (i): Write $\eta = (\eta^1, \eta^2)$ and $\varepsilon = (\varepsilon^1, \varepsilon^2)$.

For arrows $g : A \longrightarrow GB$ we have

$$\begin{aligned} Q_{A,B}(P_{A,B}(g)) &= G_{FA,B}(\varepsilon^1_B \circ F_{A,GB}(g)) \circ \eta^1_A \\ &= G_{FGB,B}(\varepsilon^1_B) \circ G_{FA,FGB}(F_{A,GB}(g)) \circ \eta^1_A \\ &= G_{FGB,B}(\varepsilon^1_B) \circ (GF)_{A,GB}(g) \circ \eta^1_A \end{aligned}$$

Let $\tau_g = i_{G_{FGB,B}(\varepsilon^1_B)} * (\eta^2_{A,GB})_g$, for $g : A \longrightarrow GB$. Thus

$$\tau_g : G_{FGB}(\varepsilon^1_B) \circ (GF)_{A,GB}(g) \circ \eta^1_A \Longrightarrow G_{FGB,B}(\varepsilon^1_B) \circ \eta^1_{GB} \circ (I_{\mathcal{C}})_{A,GB}(g),$$

by the fact that $\eta$ is pseudo-natural. Moreover, by the triangular identity for $G$:

$$G_{FGB,B}(\varepsilon^1_B) \circ \eta^1_{GB} \circ (I_{\mathcal{C}})_{A,GB}(g) = 1_{GB} \circ g = g$$

By the above equations, it follows that $\tau_g : Q_{A,B}(P_{A,B}(g)) \Rightarrow g$ is a well-defined isomorphism. To demonstrate that $\tau$ gives the desired equivalence it suffices to show that $\tau$ is natural. Suppose $\alpha : f \Rightarrow g$. Then:

$$\begin{aligned} \tau_g \odot Q_{A,B}(P_{A,B}(\alpha)) &= \tau_g \odot G_{FA,B}(i_{\varepsilon^1_B} * F_{A,GB}(\alpha)) * i_{\eta^1_A} \\ &= \tau_g \odot G_{FGB,B}(i_{\varepsilon^1_B}) * G_{FA,FGB}(F_{A,GB}(\alpha)) * i_{\eta^1_A} \\ &= \tau_g \odot i_{G_{FGB,B}(\varepsilon^1_B)} * G_{FA,FGB}(F_{A,GB}(\alpha)) * i_{\eta^1_A} \\ &= i_{G_{FGB,B}(\varepsilon^1_B)} * ((\eta^2_{A,GB})_g \odot G_{FA,FGB}(F_{A,GB}(\alpha)) * i_{\eta^1_A}) \\ &= i_{G_{FGB,B}(\varepsilon^1_B)} * ((\eta^2_{A,GB})_g \odot (GF)_{A,GB}(\alpha) * i_{\eta^1_A}) \\ &= i_{G_{FGB,B}(\varepsilon^1_B)} * (i_{\eta^1_{GB}} * \alpha \odot (\eta^2_{A,GB})_f) \qquad \text{(nat.)} \\ &= i_{G_{FGB,B}(\varepsilon^1_B) \circ \eta^1_{GB}} * \alpha \odot \tau_f \qquad \text{(def. of } \tau_f) \\ &= i_{1_{GB}} * \alpha \odot \tau_f = \alpha \odot \tau_f. \qquad \text{(triang.)} \end{aligned}$$

In this calculation we used distributivity of vertical identities, that $\eta^2_{A,GB}$ is natural and the triangular equality for $G$. This shows that $\tau$ is an equivalence. Suppose now that $\eta$ is a 2-natural transformation. Then $(\eta^2_{A,GB})_g$ is the identity, so $\tau_g$ is the identity for each $g$. Hence, in this case, $Q_{A,B} \circ P_{A,B} = I_{\mathcal{C}(A,GB)}$.

The proof of (ii) is analogous. ∎

**Remark 3.24** A good notion of "pseudo-adjunction" should probably be flexible enough to allow the functors to be pseudo-functors rather than strict, and the triangular identities to be "iso-modifications". (See also Remark 5.2 in [2].) We have not investigated this possibility fully.

# 4 Groupoids and weak pullbacks

We consider here the 2-category $\mathbb{G}$ of small groupoids. For arrows $f : A \longrightarrow C$ and $g : B \longrightarrow C$ in $\mathbb{G}$ the *comma groupoid* $f \downarrow g$ consists of triples $(a, b, \varphi)$ such that $\varphi : f(a) \longrightarrow g(b)$ as objects. A morphism $(a, b, \varphi) \longrightarrow (a', b', \varphi')$ is a pair $(\alpha, \beta)$ of arrows $\alpha : a \longrightarrow a'$, $\beta : b \longrightarrow b'$ such that

$$\begin{array}{ccc} f(a) & \xrightarrow{\varphi} & g(b) \\ {\scriptstyle f(\alpha)}\downarrow & & \downarrow{\scriptstyle g(\beta)} \\ f(a') & \xrightarrow[\varphi']{} & g(b') \end{array} \tag{16}$$

commutes. The identity morphism on $(a, b, \varphi)$ is $(1_a, 1_b)$. It is immediate that all arrows of the category $f \downarrow g$ are invertible, so that it constitutes a groupoid. Denote by $\mathbf{p}_1 : (f \downarrow g) \longrightarrow A$ the functor given by the first projections $\mathbf{p}_1(a, b, \varphi) = a$ and $\mathbf{p}_1(\alpha, \beta) = \alpha$. Similarly, there is a functor $\mathbf{p}_2 : (f \downarrow g) \longrightarrow B$ given by second projections. We do not in general have $f\mathbf{p}_1 = g\mathbf{p}_2$. However, there is a natural isomorphism

$$\sigma : f\mathbf{p}_1 \Rightarrow g\mathbf{p}_2$$

given by $\sigma_{(a,b,\varphi)} = \varphi$. Writing $\mathbf{p}_1^{f,g} = \mathbf{p}_1$, $\mathbf{p}_2^{f,g} = \mathbf{p}_2$ and $\sigma^{f,g} = \sigma$, we depict this as:

$$\begin{array}{ccc} (f \downarrow g) & \xrightarrow{\mathbf{p}_2^{f,g}} & B \\ {\scriptstyle \mathbf{p}_1^{f,g}}\downarrow & \overset{\sigma^{f,g}}{\Rightarrow} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array} \tag{17}$$

Now given $h : D \longrightarrow A$ and $k : D \longrightarrow B$ and a natural isomorphism

$$\rho : fh \Rightarrow gk.$$

Define a functor $t : D \longrightarrow (f \downarrow g)$ by $t(d) = (h(d), k(d), \rho_d)$ and $t(\theta) = (h(\theta), k(\theta))$. Then $\mathbf{p}_1 t = h$ and $\mathbf{p}_2 f = k$.

### 4.1 2-pullback functor

Next we introduce what can be considered as a 2-analogue of the pullback functor $k^*$ along a morphism $k$. We use the notation $\bar{\bar{X}}$ for the 2-slice $\mathbb{G}//X$. For each $f$ the construction of a comma groupoid $(f \downarrow g)$ can be considered as a 2-functor, which is a right pseudo-adjoint of $\Sigma_f$ (cf. Theorem 4.4 below). Let $f : X \longrightarrow Y$ be a functor in $\mathbb{G}$. Define a 2-functor $f^+ : \bar{\bar{Y}} \longrightarrow \bar{\bar{X}}$ that constructs comma groupoids along $f$. For $\alpha : A \longrightarrow Y$ let

$$f^+(\alpha, A) = (\mathbf{p}_1^{f,\alpha}, f \downarrow \alpha)$$

Write $\mathbf{A} = (\alpha, A)$, $\mathbf{B} = (\beta, B)$ etc. for objects in 2-slice categories. Let $\mathbf{A}$ and $\mathbf{B}$ be objects in $\bar{\bar{Y}}$, and suppose that $h : \mathbf{A} \longrightarrow \mathbf{B}$ is an arrow. Define $f^+_{\mathbf{A},\mathbf{B}}(h) : f^+(\mathbf{A}) \longrightarrow f^+(\mathbf{B})$ by

$$f^+_{\mathbf{A},\mathbf{B}}(h) = (\underline{F}(h), \overline{F}(h)) \tag{18}$$

where

$$\begin{aligned} \underline{F}(h)(x, a, \varphi) &= (x, \underline{h}(a), \overline{h}_a \circ \varphi), \\ \underline{F}(h)(\theta_1, \theta_2) &= (\theta_1, \underline{h}(\theta_2)), \\ \overline{F}(h)_{(x,a,\varphi)} &= 1_x. \end{aligned}$$

Let $h, p : \mathbf{A} \longrightarrow \mathbf{B}$ be two arrows, and let $\tau : h \Rightarrow p$ be a 2-cell between these arrows in $\bar{\bar{Y}}$. Then define $f^+_{\mathbf{A},\mathbf{B}}(\tau) : f^+_{\mathbf{A},\mathbf{B}}(h) \Rightarrow f^+_{\mathbf{A},\mathbf{B}}(p)$ by

$$(f^+_{\mathbf{A},\mathbf{B}}(\tau))_{(x,a,\varphi)} = (1_x, \tau_a).$$

**Lemma 4.1** *For any functor $f : X \longrightarrow Y$ in $\mathbb{G}$, $f^+ : \bar{\bar{Y}} \longrightarrow \bar{\bar{X}}$ is a 2-functor.*

Let $f : X \longrightarrow Y$ be a functor in $\mathbb{G}$. Consider the composition of 2-functors

$$\Sigma_f \circ f^+ : \bar{\bar{Y}} \longrightarrow \bar{\bar{Y}}.$$

For each $\mathbf{A} \in \bar{\bar{Y}}$ define

$$\varepsilon_{\mathbf{A}} = (\mathbf{p}_2^{f,\alpha}, \sigma^{f,\alpha}),$$

where $\mathbf{p}_1^{f,\alpha}, \mathbf{p}_2^{f,\alpha}$ and $\sigma^{f,\alpha}$ are the projection and 2-cell associated with the comma category $f \downarrow \alpha$, see diagram (17). Let $I = I_{\bar{\bar{Y}}} : \bar{\bar{Y}} \stackrel{\cdot}{\longrightarrow} \bar{\bar{Y}}$ be the identity 2-natural transformation. Note that $\Sigma_f \circ f^+(\mathbf{A}) = (f \circ \mathbf{p}_1^{f,\alpha}, f \downarrow \alpha)$, so $\varepsilon_{\mathbf{A}} : \Sigma_f \circ f^+(\mathbf{A}) \Rightarrow \mathbf{A}$. In fact, we have

**Lemma 4.2** *$\varepsilon : \Sigma_f \circ f^+ \overset{\cdot}{\longrightarrow} I$ is a 2-natural transformation.*

The counit of the adjunction will merely be a pseudo-natural transformation. Let $f : X \longrightarrow Y$ be a functor in $\mathbb{G}$, and consider the composition

$$f^+ \circ \Sigma_f : \bar{\bar{\bar{X}}} \longrightarrow \bar{\bar{\bar{X}}}.$$

For $\mathbf{B} = (\beta, B)$ we have $f^+ \circ \Sigma_f(\mathbf{B}) = (\mathbf{p}_1^{f, f\circ\beta}, f \downarrow f \circ \beta)$. Define a pseudo-natural transformation

$$(\eta, \rho) : I \overset{\cdot}{\longrightarrow} f^+ \circ \Sigma_f$$

as follows: $\eta_{\mathbf{B}} = (\underline{\eta}_{\mathbf{B}}, \overline{\eta}_{\mathbf{B}})$ where $\underline{\eta}_{\mathbf{B}} : B \longrightarrow f \downarrow (f \circ \beta)$ is the functor defined by

$$\underline{\eta}_{\mathbf{B}}(b) = (\beta(b), b, 1_{f(\beta(b))})$$

for $b \in B$ and $\underline{\eta}_{\mathbf{B}}(\theta) = (\beta(\theta), \theta)$ for $\theta : b \longrightarrow b'$. Moreover, $\overline{\eta}_{\mathbf{B}} = i_\beta$. For $\mathbf{A}$ and $\mathbf{B}$ in $\bar{\bar{\bar{X}}}$, define $\rho_{\mathbf{A},\mathbf{B}}$ as

$$((\rho_{\mathbf{A},\mathbf{B}})_h)_a = (\overline{h}_a, 1_{\underline{h}(a)})$$

for arrows $h : \mathbf{A} \longrightarrow \mathbf{B}$ and $a \in A$.

**Lemma 4.3** *$(\eta, \rho) : I \overset{\cdot}{\longrightarrow} f^+ \circ \Sigma_f$ is a pseudo-natural transformation.*

**Theorem 4.4** *For any functor $f : X \longrightarrow Y$ in $\mathbb{G}$,*

$$\langle \Sigma_f, f^+, \varepsilon, (\eta, \rho) \rangle : \bar{\bar{\bar{X}}} \longrightarrow \bar{\bar{\bar{Y}}}$$

*is a semi-strict pseudo-adjunction, where $\varepsilon$ is 2-natural.*

**Remark 4.5** The above adjunction seems to be known, in some form, among category theorists, but I have not been able to locate a reference. It might implicitly be contained in the theory of 2-monad [2], but then again we have not be able figure this out.

## 5 Dependent products of groupoids

First we introduce some auxiliary notions. Let $f : X \longrightarrow Y$ be a functor in $\mathbb{G}$ and let $y$ be an object of $Y$. Define $f^{-1}(y)$ be the groupoid consisting of objects $(x, \varphi)$ where $\varphi : f(x) \longrightarrow y$ is an arrow in $Y$. A morphism $\psi : (x, \varphi) \longrightarrow (x', \varphi')$ in $f^{-1}(y)$ is an arrow $\psi : x \longrightarrow x'$ in $X$ such that $\varphi = \varphi' \circ f(\psi)$. In fact, this construction can be considered as a special comma category: $f \downarrow y$. For $\theta : y \longrightarrow y'$ in $Y$ we have a functor $f^{-1}(\theta) : f^{-1}(y) \longrightarrow f^{-1}(y')$ defined by $f^{-1}(\theta)(x, \varphi) = (x, \theta\varphi)$ and $f^{-1}(\theta)(\psi) = \psi$. Also, $f^{-1}(\theta'\theta) = f^{-1}(\theta')f^{-1}(\theta)$, if $\theta' : y' \longrightarrow y''$, and $f^{-1}(1_y) = 1_{f^{-1}(y)}$.

For each object $y$ of $Y$ there is a projection functor $p_y : f^{-1}(y) \longrightarrow X$ given by $p_y(x, \varphi) = x$ and $p_y(\psi) = \psi$. For $\theta : y \longrightarrow y'$ we have $p_{y'} \circ f^{-1}(\theta) = p_y$ so

$$\mathbf{f}^{-1}(\theta) =_{\mathrm{def}} (f^{-1}(\theta), 1_{p_y}) : p_y \longrightarrow p_{y'}$$

is a 1-cell in $\bar{\bar{\bar{X}}}$.

**Definition 5.1** Let $\gamma : C \longrightarrow X$ and $f : X \longrightarrow Y$ be functors in $\mathbb{G}$. The groupoid $\Pi_f(\gamma)$ is constructed as follows.

- Objects are pairs $(y, h)$ where $y \in Y$, and $h = (\underline{h}, \overline{h}) : p_y \longrightarrow \gamma$ is a 1-cell in $\bar{\bar{\bar{X}}}$. More explicitly

  $$\underline{h} : f^{-1}(y) \longrightarrow C$$

  and

  $$\overline{h} : p_y \Rightarrow \gamma\underline{h}.$$

  In a diagram:

  $$\begin{array}{ccc} f^{-1}(y) & \xrightarrow{\quad \underline{h} \quad} & C \\ & \overset{\overline{h}}{\Longrightarrow} & \\ {\scriptstyle p_y} \searrow & & \swarrow {\scriptstyle \gamma} \\ & X & \end{array}$$

- Morphisms are pairs $(\theta, \eta) : (y, h) \longrightarrow (y', h')$ where

  $$\theta : y \longrightarrow y'$$

  and

  $$\eta : h \Longrightarrow h' \circ \mathbf{f}^{-1}(\theta)$$

  is a 2-cell in $\bar{\bar{\bar{X}}}$. Explicitly, this means that the 2-cell in $\mathbb{G}$

  $$\eta : \underline{h} \Rightarrow \underline{h'} \circ f^{-1}(\theta)$$

  is such that the diagram of natural transformations

  $$\begin{array}{ccc} p_y & \xRightarrow{\quad \overline{h} \quad} & \gamma\underline{h} \\ {\scriptstyle i_{p_y}} \Big\Downarrow & & \Big\Downarrow {\scriptstyle i_\gamma * \eta} \\ p_{y'} f^{-1}(\theta) & \xRightarrow[\overline{h'} * i_{f^{-1}(\theta)}]{} & \gamma\underline{h'} f^{-1}(\theta) \end{array} \qquad (19)$$

  commutes, i.e. $\overline{h'} * i_{f^{-1}(\theta)} = (i_\gamma * \eta) \odot \overline{h}$. This is the *pasting condition* for $\eta$.

- The identity morphism on $(y,h)$ is $(1_y, i_{\underline{h}})$.
- Composition of $(\theta,\eta) : (y,h) \longrightarrow (y',h')$ and $(\theta',\eta') : (y',h') \longrightarrow (y'',h'')$ is given by

$$(\theta',\eta') \circ (\theta,\eta) = (\theta' \circ \theta, \eta' * i_{f^{-1}(\theta)} \odot \eta).$$

- The inverse of $(\theta,\eta) : (y,h) \longrightarrow (y',h')$ is $(\theta^{-1}, \eta^{-1} * i_{f^{-1}(\theta^{-1})})$.

The following is now easily verified.

**Lemma 5.2** *$\Pi_f(\gamma)$ is a groupoid.*

The first projection of this groupoid $\pi_f(\gamma) = \pi_{f,\gamma} : \Pi_f(\gamma) \longrightarrow Y$ defined by $\pi_f(\gamma)(y,h) = y$ and $\pi_f(\gamma)(\theta,\eta) = \theta$ is clearly a functor. We now extend the $\Pi$-construction to a 2-functor

$$\mathbf{\Pi}_f : \bar{\bar{X}} \longrightarrow \bar{\bar{Y}},$$

for each $f : X \longrightarrow Y$. For objects $\mathbf{C} = (\gamma, C)$ of $\bar{\bar{X}}$, let

$$\mathbf{\Pi}_f(\mathbf{C}) = (\pi_f(\gamma), \Pi_f(\gamma)).$$

For each pair $\mathbf{C}, \mathbf{D}$ of objects in $\bar{\bar{X}}$ define the functor

$$(\mathbf{\Pi}_f)_{\mathbf{C},\mathbf{D}} : \bar{\bar{X}}(\mathbf{C},\mathbf{D}) \longrightarrow \bar{\bar{Y}}(\mathbf{\Pi}_f(\mathbf{C}), \mathbf{\Pi}_f(\mathbf{D}))$$

as follows.

- For a 1-cell $q = (\underline{q}, \overline{q})$ in $\bar{\bar{X}}$, let $(\mathbf{\Pi}_f)_{\mathbf{C},\mathbf{D}}(q) = (\underline{P}(q), \overline{P}(q))$ where

$$\begin{aligned} \underline{P}(q)(y,h) &=_{\text{def}} (y, q \circ h) = (y, (\underline{q}\underline{h}, \overline{q} * i_{\underline{h}} \odot \overline{h})) \\ \underline{P}(q)(\theta,\eta) &= (\theta, i_{\underline{q}} * \eta) \end{aligned}$$

and $\overline{P}(q) : \pi_f(\gamma) \Rightarrow \pi_f(\delta) \circ \underline{P}(q)$ is given by

$$\overline{P}(q)_{(y,h)} = 1_y.$$

- For a 2-cell $\beta : q \Rightarrow r$ in $\bar{\bar{X}}$ define the 2-cell $(\mathbf{\Pi}_f)_{\mathbf{C},\mathbf{D}}(\beta)$ in $\bar{\bar{Y}}$ by

$$(\mathbf{\Pi}_f)_{\mathbf{C},\mathbf{D}}(\beta)_{(y,h)} = (1_y, \beta * i_{\underline{h}}).$$

**Theorem 5.3** *For every functor $f : X \longrightarrow Y$ in $\mathbb{G}$,*

$$\mathbf{\Pi}_f : \bar{\bar{X}} \longrightarrow \bar{\bar{Y}}$$

*is a 2-functor.*

**Proof.** A tedious but straightforward verification. ∎

Next define the unit, or the *evaluation operator:* $\varepsilon : f^+ \circ \mathbf{\Pi}_f \overset{\cdot}{\longrightarrow} I_{\bar{\bar{X}}}$. Construct the composition of 2-functors

$$\Phi = f^+ \circ \mathbf{\Pi}_f : \bar{\bar{X}} \longrightarrow \bar{\bar{X}}.$$

For $\mathbf{C} = (\gamma, C)$ of $\bar{\bar{X}}$, observe that

$$\Phi(\mathbf{C}) = (\mathbf{p}_1^{f,\pi_f(\gamma)}, (f \downarrow \pi_f(\gamma)))$$

Explicitly, the objects of $(f \downarrow \pi_f(\gamma))$ will be tuples $(x, (y, h), \nu)$ where $\nu : f(x) \longrightarrow y$. A morphism

$$(\beta, (\theta, \eta)) : (x, (y, h), \nu) \longrightarrow (x', (y', h'), \nu') \tag{20}$$

consists of $\beta : x \longrightarrow x'$ and $(\theta, \eta) : (y, h) \longrightarrow (y', h')$ in $\Pi_f(\gamma)$ such that $\theta\nu = \nu' f(\beta)$. For $\mathbf{C} \in \bar{\bar{X}}$ define a 1-cell

$$\varepsilon_{\mathbf{C}} = (\underline{\varepsilon}_{\mathbf{C}}, \overline{\varepsilon}_{\mathbf{C}}) : \Phi(\mathbf{C}) \longrightarrow \mathbf{C}$$

as follows: Let

$$\underline{\varepsilon}_{\mathbf{C}}(x, (y, h), \nu) = \underline{h}(x, \nu)$$

on objects, and for a morphism $(\beta, (\theta, \eta))$ as in (20), let

$$\underline{\varepsilon}_{\mathbf{C}}(\beta, (\theta, \eta)) = \underline{h'}(\beta) \circ \eta_{(x,\nu)}. \tag{21}$$

It can readily be checked that $\underline{\varepsilon}_{\mathbf{C}}$ is a functor $(f \downarrow \pi_f(\gamma)) \longrightarrow C$. There is a natural transformation

$$\overline{\varepsilon}_{\mathbf{C}} : \mathbf{p}_1^{f,\gamma} \Rightarrow \gamma \circ \underline{\varepsilon}_{\mathbf{C}}$$

given by

$$(\overline{\varepsilon}_{\mathbf{C}})_{(x,(y,h),\nu)} = \overline{h}_{(x,\nu)}.$$

A calculation shows

**Lemma 5.4** *$\varepsilon : f^+ \circ \mathbf{\Pi}_f \overset{\cdot}{\longrightarrow} I_{\bar{\bar{X}}}$ is a 2-natural transformation.*

Next we define a counit $I_{\bar{\bar{Y}}} \overset{\cdot}{\longrightarrow} \mathbf{\Pi}_f \circ f^+$, which will merely be a pseudo-natural transformation. For $\mathbf{C} = (\gamma, C)$ note that

$$(\mathbf{\Pi}_f \circ f^+)(\mathbf{C}) = (\pi_f(\mathbf{p}_1^{f,\gamma}), \Pi_f(\mathbf{p}_1^{f,\gamma})).$$

Define a 1-cell in $\bar{\bar{Y}}$: $\psi_{\mathbf{C}} = (\underline{\psi}_{\mathbf{C}}, \overline{\psi}_{\mathbf{C}}) : \mathbf{C} \longrightarrow (\mathbf{\Pi}_f \circ f^+)(\mathbf{C})$ by

$$\underline{\psi}_{\mathbf{C}}(c) = (\gamma(c), (k^c, \tau^c)) \qquad (c \in C)$$

where

- $k^c : f^{-1}(\gamma(c)) \longrightarrow (f \downarrow \gamma)$ is given by
  - $k^c(x, \varphi) = (x, c, \varphi)$ for $(x, \varphi) \in f^{-1}(\gamma(c))$ and
  - $k^c(\alpha) = (\alpha, 1_c)$ for $\alpha : (x, \varphi) \longrightarrow (x', \varphi')$ in $f^{-1}(\gamma(c))$
- $\tau^c : p_{\gamma(c)} \Rightarrow \mathbf{p}_1^{f,\gamma} k^c$ is defined by $\tau^c_{(x,\varphi)} = 1_x$.

Furthermore for $\beta : c \longrightarrow c'$ let $\underline{\psi}_{\mathbf{C}}(\beta) = (\gamma(\beta), \eta^\beta)$ where

$$\eta^\beta_{(x,\varphi)} = (1_x, \beta).$$

Let

$$\overline{\psi}_{\mathbf{C}} = i_\gamma : \gamma \Rightarrow \pi_f(\mathbf{p}_1^{f,\gamma}) \circ \underline{\psi}_{\mathbf{C}}.$$

This defines the first component $\psi$ of the pseudo-natural transformation. The second component $\rho$ is a family of 2-cells

$$\mathbf{C} \underset{\psi_{\mathbf{D}} \circ q}{\overset{(\mathbf{\Pi}_f \circ f^+)_{\mathbf{C},\mathbf{D}}(q) \circ \psi_{\mathbf{C}}}{\rightrightarrows}} (\mathbf{\Pi}_f \circ f^+)(\mathbf{D}) \qquad \Downarrow (\rho_{\mathbf{C},\mathbf{D}})_q$$

(for $q : \mathbf{C} \longrightarrow \mathbf{D}$ in $\bar{\bar{Y}}$) given by

$$((\rho_{\mathbf{C},\mathbf{D}})_q)_c = (\overline{q}_c, i_{\underline{F}(q)k^c}) \qquad (c \in \mathbf{C}). \tag{22}$$

Here $F$ is defined as in (18).

**Lemma 5.5** $(\psi, \rho) : I_{\bar{\bar{Y}}} \overset{\cdot}{\longrightarrow} \mathbf{\Pi}_f \circ f^+$ *is a pseudo-natural transformation.*

**Proof.** We consider first the obstacle for $\psi$ being a strict 2-natural transformation. Write $\mathbf{C} = (\gamma, C)$ and $\mathbf{D} = (\delta, D)$. For a 1-cell $q : \mathbf{C} \longrightarrow \mathbf{D}$ the first component of $(\mathbf{\Pi}_f \circ f^+)(q) \circ \psi_{\mathbf{C}}$ at the object $c \in C$ is

$$\underline{P}(F(q))(\psi_{\mathbf{C}}(c)) = (\gamma(c), F(q) \circ (\kappa^c, \tau^c)) = (\gamma(c), (\underline{F}(q)\kappa^c, \overline{F}(q) * i_{\kappa^c} \odot \tau^c)) \tag{23}$$

whereas the first component of $\psi_{\mathbf{D}} \circ q$ computed at $c$ is

$$\underline{\psi}_{\mathbf{D}}(\underline{q}(c)) = (\delta(\underline{q}(c)), (\kappa^{\underline{q}(c)}, \tau^{\underline{q}(c)})). \tag{24}$$

These are clearly not equal. However, it can be checked that $(\overline{q}_c, i_{\underline{F}(q)\kappa^c})$ as in (22) above is an isomorphism from (23) to (24). We next check that $\rho_{\mathbf{C},\mathbf{D}}$ is natural: we need to show that for $q, r : \mathbf{C} \longrightarrow \mathbf{D}$ and $\alpha : q \Rightarrow r$

$$i_{\psi_{\mathbf{D}}} * \alpha \odot (\rho_{\mathbf{C},\mathbf{D}})_q = (\rho_{\mathbf{C},\mathbf{D}})_r \odot (\mathbf{\Pi}_f \circ f^+)_{\mathbf{C},\mathbf{D}}(\alpha) * i_{\psi_{\mathbf{C}}}. \tag{25}$$

Computing the left hand side of (25) at $c$ yields

$$\begin{array}{rcl} (i_{\underline{\psi}_{\mathbf{D}}} * \alpha \odot (\rho_{\mathbf{C},\mathbf{D}})_q)_c & = & (i_{\underline{\psi}_{\mathbf{D}}} * \alpha)_c \circ ((\rho_{\mathbf{C},\mathbf{D}})_q)_c \\ & = & \underline{\psi}_{\mathbf{D}}(\alpha_c) \circ (\overline{q}_c, i_{\underline{F}(q)\kappa^c}) \\ & = & (\delta(\alpha_c) \circ \overline{q}_c, \eta^{\alpha_c} * i_{f^{-1}(\overline{q}_c)} \odot i_{\underline{F}(q)\kappa^c}). \end{array}$$

The right hand side of (25) at $c$ is

$$\begin{array}{rcl} ((\rho_{\mathbf{C},\mathbf{D}})_r \odot (\mathbf{\Pi}_f \circ f^+)_{\mathbf{C},\mathbf{D}}(\alpha) * i_{\psi_{\mathbf{C}}})_c & = & (\overline{r}_c, i_{\underline{F}(r)\kappa^c}) \circ (\mathbf{\Pi}_f \circ f^+)_{\mathbf{C},\mathbf{D}}(\alpha)_{\underline{\psi}_{\mathbf{C}}(c)} \\ & = & (\overline{r}_c, i_{\underline{F}(r)\kappa^c}) \circ (\mathbf{\Pi}_f)_{f^+\mathbf{C},f^+\mathbf{D}}(f^+_{\mathbf{C},\mathbf{D}}(\alpha))_{(\gamma(c),(\kappa^c,\tau^c))} \\ & = & (\overline{r}_c, i_{\underline{F}(r)\kappa^c}) \circ (1_{\gamma(c)}, f^+_{\mathbf{C},\mathbf{D}}(\alpha) * i_{\kappa^c}) \\ & = & (\overline{r}_c, i_{\underline{F}(r)\kappa^c} * i_{f^{-1}(1_{\gamma(c)})} \odot f^+_{\mathbf{C},\mathbf{D}}(\alpha) * i_{\kappa^c}) \\ & = & (\overline{r}_c, f^+_{\mathbf{C},\mathbf{D}}(\alpha) * i_{\kappa^c}) \end{array}$$

From $\alpha : q \Rightarrow r$ follows, by the pasting condition, $\delta(\alpha_c) \circ \overline{q}_c = \overline{r}_c$. This shows that the first components of (25) are the same. As for the second components evaluate at an arbitrary $(x, \varphi) \in f^{-1}(\gamma)$:

$$\begin{array}{rcl} (\eta^{\alpha_c} * i_{f^{-1}(\overline{q}_c)})_{(x,\varphi)} & = & \eta^{\alpha_c}_{f^{-1}(\overline{q}_c)(x,\varphi)} \\ & = & \eta^{\alpha_c}_{(x,\overline{q}_c,\varphi)} \\ & = & (1_x, \alpha_c) \\ & = & f^+_{\mathbf{C},\mathbf{D}}(\alpha)_{(x,c,\varphi)} \\ & = & f^+_{\mathbf{C},\mathbf{D}}(\alpha)_{\kappa^c(x,\varphi)} \\ & = & (f^+_{\mathbf{C},\mathbf{D}}(\alpha) * i_{\kappa^c})_{(x,\varphi)}. \end{array}$$

showing that they are identical as well. The functoriality conditions (10) and (11) on $\rho$ are straightforward to check. ∎

**Theorem 5.6** *For each $f : X \longrightarrow Y$ in $\mathbb{G}$,*

$$\langle f^+, \mathbf{\Pi}_f, \varepsilon, (\psi, \rho) \rangle : \bar{\bar{Y}} \longrightarrow \bar{\bar{X}}$$

*is a semi-strict pseudo-adjunction, where $\varepsilon$ is 2-natural.*

**Proof.** We compute the composition of pseudo-natural transformations

$$(\theta, \sigma) = \varepsilon f^+ \odot f^+(\psi, \rho) : f^+ \longrightarrow (f^+ \circ \Pi_f \circ f^+) \longrightarrow f^+.$$

For $\mathbf{A} \in \bar{\bar{Y}}$ we have

$$\theta_{\mathbf{A}} = (\underline{\varepsilon}_{\mathbf{B}} \circ \underline{F}(\psi_{\mathbf{A}}), \overline{\varepsilon}_{\mathbf{B}} * i_{\underline{F}(\psi_{\mathbf{A}})} \odot \overline{F}(\psi_{\mathbf{A}}) \quad (26)$$

where $\mathbf{B} = f^+\mathbf{A}$. Now evaluating the first component of (26) at an arbitrary object $(x, a, \varphi) \in (f \downarrow \alpha)$ we get

$$\underline{\varepsilon}_{\mathbf{B}}(\underline{F}(\psi_{\mathbf{A}})(x, a, \varphi)) = \underline{\varepsilon}_{\mathbf{B}}(x, (\alpha(a), (\kappa^a, \tau^a)), 1_{\alpha(a)} \circ \varphi) = \kappa^a_{(x,\varphi)} = (x, a, \varphi).$$

For an arbitrary morphism $(\theta_1, \theta_2) : (x, a, \varphi) \longrightarrow (x', a', x')$ we have

$$\begin{aligned}
\underline{\varepsilon}_{\mathbf{B}}(\underline{F}(\psi_{\mathbf{A}})(\theta_1, \theta_2)) &= \underline{\varepsilon}_{\mathbf{B}}(\theta_1, (\alpha(\theta_2), \eta^{\theta_2})) \\
&= \kappa^{a'}(\theta_1) \circ \eta^{\theta_2}_{(x,\varphi)} \\
&= (\theta_1, 1_{a'}) \circ (1_x, \theta_2) = (\theta_1, \theta_2).
\end{aligned}$$

This shows that $\underline{\varepsilon}_{\mathbf{B}} \circ \underline{F}(\psi_{\mathbf{A}}) = 1_{f\downarrow\alpha}$. The second component in (26) is

$$\begin{aligned}
(\underline{\varepsilon}_{\mathbf{B}} * i_{\underline{F}(\psi_{\mathbf{A}})} \odot \overline{F}(\psi_{\mathbf{A}}))_{(x,a,\varphi)} &= (\underline{\varepsilon}_{\mathbf{B}})_{\underline{F}(\psi_{\mathbf{A}})(x,a,\varphi)} \overline{F}(\psi_{\mathbf{A}})_{(x,a,\varphi)} \\
&= (\underline{\varepsilon}_{\mathbf{B}})_{(x, \underline{\psi}_{\mathbf{A}}(a), (\overline{\psi}_{\mathbf{A}})_a \circ \varphi)} \circ 1_x \\
&= (\underline{\varepsilon}_{\mathbf{B}})_{(x, (\alpha(a), (\kappa^a, \tau^a)), 1_{\alpha(a)} \circ \varphi)} \\
&= \tau^a(x, \varphi) = 1_x.
\end{aligned}$$

But $1_x = 1_{\mathbf{p}_1^{f,\alpha}}(x, a, \varphi) = (i_{\mathbf{p}_1^{f,\alpha}})_{(x,a,\varphi)}$. Thus

$$\theta_{\mathbf{A}} = (1_{f\downarrow\alpha}, i_{\mathbf{p}_1^{f,\alpha}}) = 1_{\mathbf{A}}.$$

Next consider $\sigma$. Let $\mathbf{A}$ and $\mathbf{B}$ be arbitrary and let $h : \mathbf{A} \longrightarrow \mathbf{B}$. Then since $\varepsilon f^+$ is 2-natural we have by Remark 3.20

$$(\sigma_{\mathbf{A},\mathbf{B}})_h = i_{(\varepsilon f^+)_{\mathbf{B}}} * ((f^+\rho)_{\mathbf{A},\mathbf{B}})_h. \tag{27}$$

and

$$(\sigma_{\mathbf{A},\mathbf{B}})_h : (f^+)_{\mathbf{A},\mathbf{B}}(h) \circ \theta_{\mathbf{A}} \Longrightarrow \theta_{\mathbf{B}} \circ (f^+)_{\mathbf{A},\mathbf{B}}(h).$$

Since $\theta_{\mathbf{C}} = 1_{\mathbf{C}}$ for all $\mathbf{C}$ we have

$$(\sigma_{\mathbf{A},\mathbf{B}})_h : (f^+)_{\mathbf{A},\mathbf{B}}(h) \Longrightarrow (f^+)_{\mathbf{A},\mathbf{B}}(h).$$

Consider an arbitrary $(x, a, \varphi)$. Then by (27)

$$((\sigma_{\mathbf{A},\mathbf{B}})_h)_{(x,a,\varphi)} = (\varepsilon f^+)_{\mathbf{B}}(\omega_{(x,a,\varphi)})$$

where

$$\omega = ((f^+\rho)_{\mathbf{A},\mathbf{B}})_h = f^+_{\mathbf{A},(\mathbf{\Pi}_f \circ f^+)(\mathbf{B})}((\rho_{\mathbf{A},\mathbf{B}})_h)$$

and

$$(\rho_{\mathbf{A},\mathbf{B}})_h : (\mathbf{\Pi}_f \circ f^+)_{\mathbf{A},\mathbf{B}} \circ \psi_{\mathbf{A}} \Longrightarrow \psi_{\mathbf{B}} \circ h. \tag{28}$$

Denote the domain and codomain in (28) by $q$ and $r$ respectively. We need to find the data for (21) so we compute domain and codomains for

$$\omega_{(x,a,\varphi)} = (1_x, (\overline{h}_a, i_{\underline{F}(h)\kappa^a})) : (x, \underline{q}(a), \overline{q}_a \circ \varphi) \longrightarrow (x, \underline{r}(a), \overline{r}_a \circ \varphi).$$

Here we have

$$\underline{q}_a = \overline{P}(\underline{F}(h), \overline{F}(h))_{(\alpha(a),\kappa^a,\tau^a)} = 1_{\alpha(a)}$$

and $\underline{r}(a) = (\beta(\underline{h}(a)), (\kappa^{\underline{h}(a)}, \tau^{\underline{h}(a)}))$. Thus

$$\begin{aligned}
(\varepsilon f^+)_{\mathbf{B}}(\omega_{(x,a,\varphi)}) &= \underline{\varepsilon}_{f^+\mathbf{B}}((1_x, (\overline{h}_a, i_{\underline{F}(h)\kappa^a}))) \\
&= \kappa^{\underline{h}(a)}(1_x) \circ (i_{\underline{F}\kappa^a})_{(x,\varphi)} \\
&= (1_x, 1_{\underline{h}(a)}) \circ 1_{\underline{F}(h)\kappa^a(x,\varphi)} \\
&= 1_{\underline{F}(h)(x,a,\varphi)} \\
&= (i_{\underline{F}(h)})_{(x,a,\varphi)}.
\end{aligned}$$

This shows $(\sigma_{\mathbf{A},\mathbf{B}})_h = i_{\underline{F}(h)}$, so $(\theta, \sigma)$ is indeed the lax version of the identity 2-natural transformation.

We now prove the second triangular equality, by computing the composition

$$(\xi, \zeta) = (\mathbf{\Pi}_f)\varepsilon \odot (\psi, \rho)\mathbf{\Pi}_f : \mathbf{\Pi}_f \longrightarrow (\mathbf{\Pi}_f \circ f^+ \circ \mathbf{\Pi}_f) \longrightarrow \mathbf{\Pi}_f. \tag{29}$$

Then for a given $\mathbf{A}$

$$\xi_{\mathbf{A}} = (\underline{P}(\varepsilon_{\mathbf{A}}) \circ \underline{\psi}_{\mathbf{C}}, \ \overline{P}(\psi_{\mathbf{A}}) * i_{\underline{\psi}_{\mathbf{C}}} \odot \overline{\psi}_{\mathbf{C}}), \tag{30}$$

where $\mathbf{C} = \mathbf{\Pi}_f(\mathbf{A})$. We examine the first component, a functor. For an object $(y, h) \in \Pi_f(\alpha)$

$$\begin{aligned}
\underline{P}(\varepsilon_{\mathbf{A}})(\underline{\psi}_{\mathbf{C}}(y,h)) &= \underline{P}(\varepsilon_{\mathbf{A}})(\pi_f(\alpha)(y,h), (\kappa^{(y,h)}, \tau^{(y,h)})) \\
&= (y, (\underline{\varepsilon}_{\mathbf{A}} \circ \kappa^{(y,h)}, \overline{\varepsilon}_{\mathbf{A}} * i_{\kappa^{(y,h)}} \odot \tau^{(y,h)})) \\
&= (y, (\underline{h}, \overline{h})) = (y, h).
\end{aligned}$$

The third step follows because of the equations (31, 32, 33)

$$\underline{\varepsilon}_{\mathbf{A}}(\kappa^{(y,h)}(x,\varphi)) = \underline{\varepsilon}_{\mathbf{A}}((x,(y,h),\varphi)) = \underline{h}(x,\varphi) \tag{31}$$

for $(x, \varphi) \in f^{-1}(y)$. For $\beta : (x,\varphi) \longrightarrow (x', \varphi')$ in $f^{-1}(y)$, and so $\kappa^{(y,h)}(\beta) : (x,(y,h),\varphi) \longrightarrow (x',(y,h),\varphi')$, we have

$$\underline{\varepsilon}_{\mathbf{A}}(\kappa^{(y,h)}(\beta)) = \underline{\varepsilon}_{\mathbf{A}}((\beta, 1_{(y,h)})) = \underline{\varepsilon}_{\mathbf{A}}((\beta, (1_y, i_{\underline{h}}))) = \underline{h}(\beta) \circ (i_{\underline{h}})_{(x,\varphi)} = \underline{h}(\beta). \tag{32}$$

For $(x, \varphi) \in f^{-1}(y)$

$$(\overline{\varepsilon}_{\mathbf{A}} * i_{\kappa^{(y,h)}} \odot \tau^{(y,h)})_{(x,\varphi)} = (\overline{\varepsilon}_{\mathbf{A}})_{\kappa^{(y,h)}(x,\varphi)} \tau^{(y,h)}_{(x,\varphi)} = (\overline{\varepsilon}_{\mathbf{A}})_{(x,(y,h),\varphi)} = \overline{h}_{(x,\varphi)}. \tag{33}$$

This proves that the first component is the identity on objects. As for morphisms consider $(\theta, \delta) : (y,h) \longrightarrow (y', h')$

$$\underline{P}(\varepsilon_{\mathbf{A}})(\underline{\psi}_{\mathbf{C}}(\theta,\delta)) = \underline{P}(\varepsilon_{\mathbf{A}})(\theta, \eta^{(\theta,\delta)}) = (\theta, i_{\underline{\varepsilon}_{\mathbf{A}}} * \eta^{(\theta,\delta)}) \tag{34}$$

Now $\eta^{(\theta,\delta)}_{(x,\varphi)} = (1_x, (\theta,\delta)) : (x,(y,h),\varphi) \longrightarrow (x,(y',h'),\varphi)$ so

$$(i_{\underline{\varepsilon}_{\mathbf{A}}} * \eta^{(\theta,\delta)})_{(x,\varphi)} = \underline{\varepsilon}_{\mathbf{A}}((1_x,(\theta,\delta))) = \underline{h}'(1_x) \circ \delta_{(x,\varphi)} = \delta_{(x,\varphi)}.$$

Thus (34) is $(\theta,\delta)$ and consequently

$$\underline{P}(\varepsilon_{\mathbf{A}}) \circ \underline{\psi}_{\mathbf{C}} = 1_{\Pi_f(\alpha)}.$$

We now consider the second component in (30):

$$\begin{aligned}
(\overline{P}(\psi_{\mathbf{A}}) * i_{\underline{\psi}_{\mathbf{C}}} \odot \overline{\psi}_{\mathbf{C}}))_(y,h) &= \overline{P}(\psi_{\mathbf{A}})_{\underline{\psi}_{\mathbf{C}}(y,h)}(\overline{\psi}_{\mathbf{C}})_{(y,h)} \\
&= \overline{P}(\psi_{\mathbf{A}})_(y,(\kappa^{(y,h)},\tau^{(y,h)})) \circ 1_{\pi_f(\alpha)(y,h)} \\
&= 1_y \circ 1_{\pi_f(\alpha)(y,h)} \\
&= 1_{\pi_f(\alpha)(y,h)} = (i_{\pi_f(\alpha)})_{(y,h)}.
\end{aligned}$$

Thus

$$\xi_{\mathbf{A}} = (1_{\Pi_f(\alpha)}, i_{\pi_f(\alpha)}) = 1_{\mathbf{\Pi}_f(\mathbf{A})}.$$

Finally, we compute $\zeta$. For $q : \mathbf{A} \longrightarrow \mathbf{B}$ we have

$$\zeta' = (\zeta_{\mathbf{A},\mathbf{B}})_q : H(q) \circ \xi_{\mathbf{A}} \Longrightarrow \xi_{\mathbf{B}} \circ H(q),$$

where $H(q) = (\mathbf{\Pi}_f)_{\mathbf{A},\mathbf{B}}$. Noting that $\xi_{\mathbf{A}} = 1_{\mathbf{A}}$ and $\xi_{\mathbf{B}} = 1_{\mathbf{B}}$ we get $\zeta' : H(q) \Longrightarrow H(q)$ and so

$$\zeta' : \underline{P}(q) \Longrightarrow \underline{P}(q),$$

Since $\mathbf{\Pi}_f\varepsilon$ is strict 2-natural we may, according to Remark 3.20, compute

$$\zeta' = i_{(\mathbf{\Pi}_f\varepsilon)_{\mathbf{B}}} * ((\rho\mathbf{\Pi}_f)_{\mathbf{A},\mathbf{B}})_q = i_{\underline{P}(\varepsilon_{\mathbf{B}})} * (\rho_{\mathbf{\Pi}_f(\mathbf{A}),\mathbf{\Pi}_f(\mathbf{B})})_r$$

where $r = H(q)$. Thus

$$\begin{aligned}
(\zeta')_{(y,h)} &= \underline{P}(\varepsilon_{\mathbf{B}})(((\rho_{\mathbf{\Pi}_f(\mathbf{A}),\mathbf{\Pi}_f(\mathbf{B})})_r)_{(y,h)}) \\
&= \underline{P}(\varepsilon_{\mathbf{B}})(\overline{P}(r)_{(y,h)}, i_{\underline{F}(r)\kappa^{(y,h)}}) \\
&= (\overline{P}(r)_{(y,h)}, i_{\underline{\varepsilon}_{\mathbf{B}}} * i_{\underline{F}(r)\kappa^{(y,h)}}) \\
&= (1_y, i_{\underline{\varepsilon}_{\mathbf{B}} \circ \underline{F}(r)\kappa^{(y,h)}})
\end{aligned}$$

The last component can be simplified as follows: consider an arbitrary $(x,\varphi)$

$$\begin{aligned}
\underline{\varepsilon}_{\mathbf{B}}(\underline{F}(r)(\kappa^{(y,h)}(x,\varphi))) &= \underline{\varepsilon}_{\mathbf{B}}(\underline{F}(r)(x,(y,h),\varphi)) \\
&= \underline{\varepsilon}_{\mathbf{B}}(x, \underline{r}(y,h), \overline{r}_{(y,h)} \circ \varphi)) \\
&= \underline{\varepsilon}_{\mathbf{B}}(x,(y,(q\underline{h}, \overline{q} * i_{\underline{h}} \odot \overline{h})), 1_y \circ \varphi)) \\
&= \underline{qh}(x,\varphi)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(i_{H(q)})_{(y,h)} &= (i_{\underline{P}(q)})_{(y,h)} \\
&= 1_{\underline{P}(q)(y,h)} \\
&= 1_{(y,(q\underline{h},\overline{q}*i_{\underline{h}}\odot\overline{h}))} \\
&= (1_y, i_{q\underline{h}}) \\
&= \zeta'_{(y,h)}
\end{aligned}$$

Hence

$$\zeta' = i_{H(q)}$$

which shows that $(\xi, \zeta)$ a lax version of the identity 2-functor as well. ■

# 6 Further work

There are several further type constructions [8] whose counterpart in $\mathbb{G}$ might be worth investigating, if they exist. For instance, general inductive and recursive types, in particular the type of well-founded trees with infinite branching, and universes closed under type construction. Whether there exists some counterpart to the subobject classifier seems also to be an interesting question.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