Configuration spaces and homological stability

Martin Palmer

Merton College

University of Oxford

A thesis submitted for the degree of

Doctor of Philosophy

Michaelmas 2012

To my parents,

for their love, support and encouragement.

I don’t believe it. Prove it to me and I still won’t believe it.

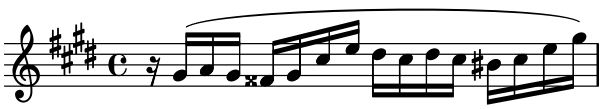
Ford Prefect on proof
Douglas Adams
Life, the Universe and Everything, 1982

La géométrie est une magie qui réussit. …toute magie, dans la mesure où elle réussit, n’est-elle pas nécessairement une géométrie?

René Thom
Stabilité Structurelle et Morphogénèse, 1972

Die Topologie hat das Eigentümliche an sich, daß die zu ihr gehörigen Fragen unter Umständen sicher entscheidbar sind selbst dann, wenn die Kontinua, an welche sie gestellt werden, nicht exakt, sondern nur vage gegeben sind, wie es in der Wirklichkeit stets der Fall ist.

Hermann Weyl
Philosophie der Mathematik und Naturwissenschaft, 1927



Frédéric Chopin
Fantaisie Impromptu in $C^\sharp$ minor, 1834

Abstract

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In this thesis we study the homological behaviour of configuration spaces as the number of objects in the configuration goes to infinity. For unordered configurations of distinct points (possibly equipped with some internal parameters) in a connected, open manifold it is a well-known result, going back to G. Segal and D. McDuff in the 1970s, that these spaces enjoy the property of *homological stability*.

In Chapter 2 we prove that this property also holds for so-called *oriented* configuration spaces, in which the points of a configuration are equipped with an ordering up to even permutations. There are two important differences from the unordered setting: the rate (or slope) of stabilisation is strictly slower, and the stabilisation maps are not in general split-injective on homology. This can be seen by some explicit calculations of Guest-Kozłowski-Yamaguchi in the case of surfaces. In Chapter 3 we refine their calculations to show that, for an odd prime p, the *difference* between the mod-p homology of the oriented and the unordered configuration spaces on a surface is zero in a stable range whose slope converges to 1 as $p \to \infty$.

In Chapter 4 we prove that unordered configuration spaces satisfy homological stability with respect to finite-degree *twisted coefficient systems*, generalising the corresponding result of S. Betley for the symmetric groups. We deduce this from a general "twisted stability from untwisted stability" principle, which also applies to the configuration spaces studied in the next chapter.

In Chapter 5 we study configuration spaces of *submanifolds* of a background manifold M. Roughly, these are spaces of pairwise unlinked, mutually isotopic copies of a fixed closed, connected manifold P in M. We prove that if the dimension of P is at most $\frac{1}{2}(\dim(M)-3)$ then these configuration spaces satisfy homological stability w.r.t. the number of copies of P in the configuration. If P is a sphere this upper bound on its dimension can be increased to $\dim(M)-3$.

Acknowledgements

First, I would like to thank Ulrike Tillmann, for her invaluable advice and guidance throughout my Ph.D., and for being an inspiring supervisor. I would also like to thank Oscar Randal-Williams for many enlightening discussions; I have learnt a lot of what I know from him, and the content of this thesis has been considerably influenced by his work. Many other people have helped me along the way, variously teaching me new mathematics, answering my questions, discussing ideas with me, and in general making me feel like part of the mathematical community: Jeffrey Giansiracusa, Richard Hepworth, Christopher Douglas, Federico Cantero and Jeremy Miller, to name but a few.

[Two more specific acknowledgements: I am indebted to certain discussions with Oscar for the idea of the "factorisation lemma" (see §2.A of Chapter 2), and to a conversation with Federico for the idea of the "second resolution" (see §5.6 of Chapter 5).]

I would like to thank my family for their support and encouragement of every kind. Thanks go to my office (and inter-office) mates for keeping me sane, providing distractions and suggesting trips to the pub with an appropriate degree of regularity. Also thanks to David and Rok for being great housemates, and improving the culinary aspect of my life at Oxford. On the subject of food, the Monday afternoon pre-topology-seminar cake club has been an important source of harmony and refined sugar for the last few years.

Finally, thanks (and mild surprise) to anyone who actually reads my thesis.

Contents

CHAPTER 1

Introduction

Beginning is easy; continuing, hard.

Japanese proverb

This thesis is centred around the phenomenon of *homological stability* for a sequence of spaces $\{X_n\}$. By this we mean that in any fixed degree $*$ the homology $H_*(X_n)$ is eventually independent of the parameter n. When true, this property of $\{X_n\}$ gives a powerful link between the topology of the individual spaces X_n and the *limiting space* $X_\infty = \lim_{n\to\infty} X_n$.[1] Often the limiting space has more structure than the individual spaces, and can be identified in terms of more accessible spaces, making its homology more amenable to calculation. Thus homological stability often affords a calculation of $H_*(X_n)$ in the *stable range* — the range in which this is independent of n.

Many examples of interest are sequences of classifying spaces of discrete groups. One very important example is the family of mapping class groups $\mathrm{MCG}(\Sigma_g^r) = \pi_0\mathrm{Diff}(\Sigma_g^r; \partial\Sigma_g^r)$ of compact, connected, orientable surfaces of genus g and with r boundary-components. In this case the family is parametrised by two numbers, and it was proved by Harer [Har85] that for any fixed degree $*$, the homology $H_*(\mathrm{MCG}(\Sigma_g^r))$ is independent of both g and r once g is sufficiently large. This homological stability property was used by Tillmann [Til97] to prove that the classifying space of the *stable mapping class group* $\lim_{g\to\infty}\mathrm{MCG}(\Sigma_{g,1})$, after applying the Quillen plus-construction, is an infinite loop space. A recent breakthrough was an explicit identification of this infinite loop space in terms of Thom spaces of bundles over oriented Grassmannians, which was conjectured by Madsen and Tillmann [MT01] and later proved by Madsen and Weiss [MW07], whose proof also used homological stability for mapping class groups. This identified the cohomology ring of the stable mapping class group, in particular proving Mumford's conjecture [Mum83], and therefore also, via homological stability again, identified the cohomology of the mapping class groups in the stable range. There is a recent survey [Coh09] of homological stability phenomena which discusses many other examples, and puts this into a broader context.

In particular we will be concerned with homological stability for sequences of *configuration spaces*. In the simplest case this means the space (suitably topologised) of cardinality-n subsets of M, for some fixed integer n and background space M (usually a manifold). There are more sophisticated versions of this notion, in which for example the n "particles" in M are equipped with a parameter taking values in a fixed space, or more generally in a bundle over M. One can also give the particles an ordering, or alternatively just a "shadow" of an ordering by taking the quotient w.r.t. the action of a subgroup of the symmetric group. In another direction, instead of considering configurations of points, i.e. 0-dimensional connected submanifolds of M, one could instead consider configurations of *higher-dimensional* connected submanifolds of M. A brief history of some homological stability results for

[1] We assume that we also have maps $X_n \to X_{n+1}$, and the abstract isomorphisms $H_*(X_n) \cong H_*(X_{n+1})$ for $n \gg *$ are induced by these maps.

configuration spaces is given in §2.1.1 below.

The importance of configuration spaces lies in their ubiquity: they appear in various guises in many different settings in, for example, homotopy theory, knot theory and algebraic geometry. In homotopy theory, for instance, configuration spaces give tractable models for *mapping spaces*, and more generally *section spaces* of fibre bundles (see for example [Böd87]). Another fairly direct link to configuration spaces is with the classical *braid groups*, whose classifying spaces can be modelled as configuration spaces of particles on the plane $\mathbb{R}^2$. Studying the homology of configuration spaces therefore includes the study of the homology of (generalised) braid groups.

Motivated by this, the purpose of this thesis is to establish homological stability results for configuration spaces in several different contexts (i.e. the precise meanings of the words "homological stability" and "configuration spaces"). We will introduce and outline each chapter in the next section, and then finish this introduction with some remarks on possible further directions in which this work may be taken.

1.1 Outline of the thesis

As remarked above, there are various ways of making the basic configuration spaces of unordered points more sophisticated. We will always allow points to possess parameters with values in a fixed parameter-space. In Chapters 2 and 3 we are concerned with configurations which additionally have an "orientation" (or "alternating") structure, meaning an ordering up to even permutations. In Chapter 4 we return to unordered configuration spaces, but consider their homology with coefficients in a "twisted coefficient system". Finally, in Chapter 5 we consider configurations of submanifolds.

The chapters are designed to be essentially self-contained, so there is a small amount of repetition in their introductory sections.

Some notation

Let M be a connected manifold which is the interior of a manifold $\bar{M}$ with non-empty boundary. Let X be a space. We then define the *ordered configuration space* on M with labels in X to be

$$\widetilde{C}_n(M,X) \coloneqq \big\{(p_1,\ldots,p_n) \in M^n \mid p_i \neq p_j \text{ for } i \neq j\big\} \times X^n.$$

The symmetric group Σ_n acts diagonally on this space, by permuting the coordinates of both M^n and X^n, and we define the *unordered* and *oriented* configuration spaces to be the quotients

$$C_n(M,X) \coloneqq \widetilde{C}_n(M,X)/\Sigma_n$$
$$C_n^+(M,X) \coloneqq \widetilde{C}_n(M,X)/A_n$$

respectively. Choosing a boundary-component B of $\bar{M}$ and a basepoint $x_0 \in X$, there is a map

$$s_n \colon \widetilde{C}_n(M,X) \longrightarrow \widetilde{C}_{n+1}(M,X)$$

which "pushes" a new configuration point into the manifold from B and labels it by x_0 (this is defined rigorously in §2.2.2 below). Since it is equivariant with respect to the action of Σ_n it descends to maps of unordered and oriented configuration spaces (also called s_n). It is called the "stabilisation map" since it turns out to induce the isomorphisms $H_*(C_n(M,X)) \cong H_*(C_{n+1}(M,X))$ in the statement of homological stability for $\{C_n(M,X)\}$, and similarly for $\{C_n^+(M,X)\}$.

The *limiting space* $C_\infty(M,X)$ is defined to be the (homotopy) colimit of the sequence

$$\cdots \xrightarrow{s_{n-1}} C_n(M,X) \xrightarrow{s_n} C_{n+1}(M,X) \xrightarrow{s_{n+1}} \cdots$$

with respect to these maps; $C_\infty^+(M,X)$ is defined analogously.

In Chapter 5 we consider a more general notion of *configuration space of submanifolds*. We alert the reader that there is also a change of notation: the 'C' is replaced by a 'Σ', by analogy with the symmetric groups ($C_n(\mathbb{R}^\infty) \simeq B\Sigma_n$ in the old notation). Let M and X be as before, and now also choose a closed, connected manifold P and a subgroup $G \leq \mathrm{Diff}(P)$ of its diffeomorphism group. In §5.1.2 we define rigorously the space

$$\Sigma_n^P(M,X|G),$$

which can be thought of as the space of n unordered, mutually isotopic copies of P embedded in M, equipped with labels in X and parametrised up to G (in other words the embedding into M is only remembered up to the action of G). There are a few technical assumptions to be made here, including that the copies of P must all be isotopic to an embedding into the boundary $\partial\bar{M}$, and that they are "pairwise unlinked". See §5.1.2 for more details.

Oriented configuration spaces

Homological stability is well-known to hold for unordered configuration spaces [Seg73, McD75, Seg79, RW11]. The main result of Chapter 2 is a quantitative homological stability result for *oriented* configuration spaces: if $\dim(M) \geq 2$ and X is path-connected then the stabilisation map

$$(s_n)_* \colon H_*\big(C_n^+(M,X)\big) \longrightarrow H_*\big(C_{n+1}^+(M,X)\big)$$

is an isomorphism in the range $* \leq \frac{n-5}{3}$ and surjective in the range $* \leq \frac{n-2}{3}$. To prove this result we adopt the strategy of "taking resolutions" which was introduced in [RW11].

Part of the interest of this result is that it cannot be proved by the "scanning" techniques of McDuff [McD75]. Firstly, the *scanning map* used in [McD75] (which was introduced in [Seg73] and first used under the name "scanning" in [Seg79]) can only see local data, such as parameters attached to the configuration points, whereas an ordering up to even permutations is an example of *non-local* data. So there is no natural geometric analogue of the scanning map for $C_n^+(M,X)$.[2] Secondly, the method of [McD75] relies on the fact that the maps $(s_n)_* \colon H_*(C_n(M,X)) \to H_*(C_{n+1}(M,X))$ are always split-injective; however, this is *false* for $(s_n)_* \colon H_*(C_n^+(M,X)) \to H_*(C_{n+1}^+(M,X))$. Counterexamples can be found by the calculations in [GKY96], which compute the difference between $H_*(C_n^+(S,pt);\mathbb{F}_p)$ and $H_*(C_n(S,pt);\mathbb{F}_p)$ for certain surfaces S and odd primes p.

A noticeable fact about our homological stability result for $C_n^+(M,X)$ is that it holds in a range with a *stability slope* of $\frac{1}{3}$, rather than the larger stability slope of $\frac{1}{2}$ for $C_n(M,X)$. Again looking at the calculations of [GKY96], one can find counterexamples to improving the slope beyond $\frac{1}{3}$ for oriented configuration spaces. On the other hand these counterexamples only appear at the prime 3, and indeed if we take coefficients in $\mathbb{Z}[\frac{1}{3}]$ the calculations imply that homological stability with a stability slope of $\frac{1}{2}$ *does* hold for $C_n^+(S,pt)$ — at least for the surfaces S considered in [GKY96].

Chapter 2 has appeared, essentially as it is now (but without §2.ℵ), as the preprint:

Martin Palmer, *Homological stability for oriented configuration spaces*, arXiv:1106.4540, 2011.

It is also shortly to appear in the Transactions of the American Mathematical Society. Since this chapter was written less recently than the others, we have added an Addendum (§2.ℵ) which mentions some simpler models for certain maps between configuration spaces used in the proof, and also discusses configuration spaces on *closed* manifolds. Homological stability does *not* hold in general for unordered configuration spaces on closed manifolds,[3] but it does hold for certain coefficients or dimensions, by [RW11, Theorem C] and [Chu12]. We explain how the proof of [RW11] for unordered configuration spaces on closed manifolds works, and why it does *not* carry over to oriented configuration spaces.

In Chapter 3 we refine the calculations of [GKY96] to show that, for odd primes p, the twisted homology group $H_*\big(C_n(S,pt);\mathbb{F}_p^{(-1)}\big)$ — which measures the difference between the $\mathbb{F}_p$-homology of $C_n^+(S,pt)$ and of $C_n(S,pt)$ — vanishes in the range

$$* \leq \Big(\tfrac{p-2}{p}\Big)n - 1$$

for any connected surface S. So looking through a mod-p filter, the rate at which $C_n(S,pt)$

[2] See §1.2 below, however.

[3] One can see from a presentation [Bir74, Theorem 1.11] of $\pi_1 C_n(S^2)$ that $H_1(C_n(S^2)) \cong \mathbb{Z}/(2n-2)$, for example.

and $C_n^+(S,pt)$ become "the same" increases from $\frac{1}{3}$ to 1 as $p \to \infty$. From this and homological stability for unordered configuration spaces it follows that we have homological stability

$$H_*\big(C_n^+(S,pt);\mathbb{Z}[\tfrac{1}{3}]\big) \quad\cong\quad H_*\big(C_{n+1}^+(S,pt);\mathbb{Z}[\tfrac{1}{3}]\big)$$

in the range $* \leq \frac{n-1}{2}$, for any connected, open surface S.

To illustrate the calculation we also give some tables of $\dim\big(H_*\big(C_n(S,pt);\mathbb{F}_p^{(-1)}\big)\big)$ for small values of $*$, n and p, and for the surface S equal to the plane, sphere, torus and once-punctured torus.

Twisted homological stability

In Chapter 4 we extend stability of the sequence $H_*(C_n(M,X);\mathbb{Z})$ to stability for $H_*(C_n(M,X);T_n)$, where the sequence of Σ_n-modules T_n comes from a "finite-degree coefficient system".

Similar twisted homological stability results are known for general linear groups [Dwy80], mapping class groups of surfaces [Iva93, CM09, Bol12] and symmetric groups [Bet02]; our method is a generalisation of that of [Bet02]. In each case one must carefully define exactly what is meant by a "twisted coefficient system" for $\{Y_n\}$. It is *not* enough to simply have a $\pi_1 Y_n$-module for each n, with no relations between them; such a sequence of modules is a functor $\pi_1(\{Y_n\}) \to \mathsf{Ab}$, where $\pi_1(\{Y_n\})$ has objects $\{1,2,3\ldots\}$, automorphisms $\mathrm{Aut}(n) = \pi_1 Y_n$ and no other morphisms. A *twisted coefficient system* is an abelian-group-valued functor from a certain larger category, with some non-endomorphisms added, whose precise definition depends on the sequence $\{Y_n\}$ one is considering. The *degree* of such a functor is then defined in terms of these new morphisms. In our case the correct category is the *partial braid category* $\mathcal{B}(M,X)$, which is built out of "partial braids" on M with strands decorated by paths in X.

The precise statement of our twisted homological stability result is as follows: for $\dim(M) \geq 2$, X path-connected and T a twisted coefficient system of finite degree d, there is an isomorphism

$$H_*\big(C_n(M,X);T_n\big) \quad\cong\quad H_*\big(C_{n+1}(M,X);T_{n+1}\big)$$

in the range $* \leq \frac{n-d}{2}$. For example we may take T_n to be the Σ_n-module $H_q(Z^n;F)$, for a fixed based space Z, field F and degree q, or $\mathbb{Z}[\Sigma_n/\Sigma_{n-k}]$ for fixed k.

As a curious consequence of (the rational version of) the latter example, one can prove (see §4.10) that *if* the sequence

$$\dim\big(V(\lambda)^* \otimes_{\Sigma_n} H^*(\widetilde{C}_n(M,X);\mathbb{Q})\big) \tag{1.1.1}$$

is non-decreasing for all $*$ and λ, then it is eventually constant for all $*$ and λ. Here λ denotes a stable Young diagram[4] and $V(\lambda)$ denotes the corresponding irreducible Σ_n-representation. This last statement is *multiplicity stability* for the cohomology of the ordered configuration spaces $\widetilde{C}_n(M,X)$, in the sense of *representation stability* (see [CF10]). We emphasise that we only know how to deduce this from $\mathbb{Q}[\Sigma_n/\Sigma_{n-k}]$-twisted homological stability for $C_n(M,X)$ if we already assume that (1.1.1) is non-decreasing.

Finally, we note that homological stability for oriented configuration spaces (Chapter 2) is equivalent to homological stability for unordered configuration spaces, with twisted coefficients in the $\pi_1 C_n(M,X)$-module $V \cong \mathbb{Z}^2$, with the action

$$\pi_1 C_n(M,X) \twoheadrightarrow \pi_1 C_n(M,X)/\pi_1 C_n^+(M,X) = \mathbb{Z}/2 \curvearrowright \mathbb{Z}^2$$

given by $(x,y) \mapsto (y,x)$. However, this result is *disjoint* from the twisted homological stability result of this chapter, since one can easily show that this sequence of $\pi_1 C_n(M,X)$-modules $\{V\}$ cannot be part of a twisted coefficient system for $\{C_n(M,X)\}$. Alternatively, by [GKY96] or Chapter 3 we know that homological stability with a stability slope of $\frac{1}{2}$ *cannot* hold integrally for oriented configuration spaces, so the result of this chapter cannot apply to the sequence of coefficients $\{V\}$.

Configuration spaces of submanifolds

In Chapter 5 we generalise homological stability for unordered configuration spaces in a different direction: to homological stability for unordered configuration spaces *of submanifolds of M*, in a suitable sense.

Recall from earlier in this section the description of the space $\Sigma_n^P(M,X|G)$ of "configurations of copies of P in M". When the subgroup G of $\mathrm{Diff}(P)$ is the trivial group this is denoted $\widehat{\Sigma}_n^P(M,X)$, and consists of n *parametrised* copies of P in M, labelled by X (and which must all be isotopic to a chosen embedding $\iota\colon P \hookrightarrow \partial\bar{M}$, and be "pairwise unlinked"). In this case the main theorem of Chapter 5 can be stated as follows.

(a) If X is path-connected and $\dim(M) \geq 2\dim(P)+3$ then the *stabilisation map*

$$\widehat{\Sigma}_n^P(M,X) \longrightarrow \widehat{\Sigma}_{n+1}^P(M,X)$$

is an isomorphism on homology up to degree $\frac{n-2}{2}$, and a surjection up to degree $\frac{n}{2}$.

(b) If P is a point or k-sphere (with the chosen embedding $\iota\colon S^k \hookrightarrow \partial\bar{M}$ a *standard* embedding), then the dimension condition above may be relaxed to $\dim(M) \geq \dim(P)+3$.

More generally, we have the same result for $\Sigma_n^P(M,X|G)$, subject to certain conditions on $G \leq \mathrm{Diff}(P)$. In both cases it must be either finite or open in $\mathrm{Diff}(P)$, and in the extension (b) for points and spheres it must also satisfy an extra technical condition; see §5.1 for the full details and a discussion of the hypotheses.

[4] A Young diagram with any number of boxes, up to the equivalence relation generated by identifying λ with λ^+, which is λ with one box added to the top row.

We can also combine the ideas of Chapters 4 and 5 to obtain *twisted* homological stability for configuration spaces of submanifolds: under the same conditions, there is an isomorphism

$$H_*\big(\Sigma_n^P(M,X|G);T_n\big) \quad\cong\quad H_*\big(\Sigma_{n+1}^P(M,X|G);T_{n+1}\big)$$

in the range $* \leq \frac{n-d-2}{2}$, for any degree-d twisted coefficient system T.

There seems to be no essential obstruction to combining the ideas of Chapters 2 and 5 to obtain homological stability for *oriented* (or, perhaps better terminology in this context, *alternating*) configuration spaces of submanifolds: there is an isomorphism

$$H_*\big(A_n^P(M,X|G)\big) \quad\cong\quad H_*\big(A_{n+1}^P(M,X|G)\big)$$

in the stable range $* \leq \frac{n-5}{3}$ under the same conditions as in the main theorem, where $A_n^P(M,X|G)$ denotes the alternating version of $\Sigma_n^P(M,X|G)$. However, we do not claim to have checked this explicitly.

The dimension assumption $\dim(M) \geq 2\dim(P)+3$ in part (a) of the main theorem arises due to the need for a transversality argument at a certain point in the proof (and we need *transverse* to imply *disjoint*). The weakening of this dimension assumption for points and spheres is possible since this step can be done "by hand" in this special case. However, we are still left with a codimension (at least) 3 requirement, which arises since we need M to remain connected after cutting out various embedded copies of $P \times [0,1]$.

We conjecture that, in case (b) of the main theorem, the codimension requirement may in fact be reduced to 2. This seems plausible as (i) homological stability *does* hold in codimension 2 when P is a point, and (ii) homological stability holds for the sequence of fundamental groups $\pi_1\Sigma_n^{S^1}(\mathbb{R}^3)$, where implicitly we take $X = pt$ and $G = \mathrm{Diff}(S^1)$. This is because $\pi_1\Sigma_n^{S^1}(\mathbb{R}^3) \cong \Sigma\mathrm{Aut}(F_n)$, the symmetric automorphism group of the free group F_n, and homological stability for the latter was proved in [HW10, Corollary 1.2]. This second fact is just evidence by analogy though, since $\Sigma_n^{S^1}(\mathbb{R}^3)$ is *not* aspherical, so this is not a special case of the conjecture. See §5.1.4 for more details.

1.2 Further directions

One natural next question to think about is whether there are "scanning" theorems complementing the homological stability theorems of Chapters 2 and 5. In other words: is it possible, up to homology, to identify the limiting space in terms of more accessible spaces, in the setting of *oriented* configurations, or configurations of *submanifolds*? The author has an ongoing joint project (see the section below on "scanning for oriented configurations") which is closely related to the first question, and intends to think about the second question in the near future.

Another objective is to extend the result in Chapter 5 for configuration spaces of spheres in M to the codimension-2 case, as conjectured in §5.1.4.

One may be able to find some applications of homological stability for oriented configuration spaces in proving *homotopical* stability results for certain other types of configuration spaces. An application along these lines appears in an unpublished preprint of Guest, Kozłowski and Yamaguchi [GKY], for certain spaces of "positive and negative particles" (related to those of McDuff [McD75]). The sequence of configuration spaces in question is just $\mathbb{Z}/2 \to \mathbb{Z}/2 \to \cdots$ on π_1, so homotopical stability is equivalent to homological stability of the universal cover of this sequence, and being a *double* cover it turns out to be sufficiently closely related to oriented configuration spaces that homological stability carries over.

In relation to Chapter 4, it would be good to have some interesting examples of twisted coefficient systems for configuration spaces $\{C_n(M,X)\}$ which are *not* pulled back from a twisted coefficient system for the symmetric groups $\{\Sigma_n\}$; this may be possible by a more geometric construction than that of the examples given in §4.5. See the end of §4.5 for a discussion of this. Finally, it would be very interesting to investigate further the apparent link, mentioned in §4.10, between twisted homological stability for unordered configuration spaces (for a certain twisted coefficient system) and representation stability for the cohomology of ordered configuration spaces.

Scanning for oriented configurations

The author is currently working on a joint project, together with Jeremy Miller,[5] which in particular involves "scanning" questions for oriented configuration spaces. Let M be a connected, open manifold, and denote the unordered and oriented configuration spaces on M by $\Sigma_n(M)$ and $A_n(M)$ respectively (as in Chapter 5).[6] This is to avoid conflict of notation with the Quillen plus-construction. Let $T^cM \to M$ be the fibrewise one-point compactified tangent bundle of M, and denote by $\Gamma(T^cM)$ the compactly-supported sections of this bundle. There is a *scanning map* $s\colon \mathbb{Z}\times\Sigma_\infty(M) \to \Gamma(T^cM)$, which in [McD75, Theorem 1.2] was proved to be a homology-equivalence. The main aim of our project is to strengthen this to the statement that the scanning map is *acyclic*, i.e. induces an isomorphism

$$H_*\bigl(\mathbb{Z}\times\Sigma_\infty(M);s^*A\bigr) \quad\cong\quad H_*\bigl(\Gamma(T^cM);A\bigr) \tag{1.2.1}$$

for any local coefficient system A on $\Gamma(T^cM)$.

Now, the double cover $\mathbb{Z}\times A_\infty(M)$ of $\mathbb{Z}\times\Sigma_\infty(M)$ is classified by a cohomology class in $H^1(\mathbb{Z}\times\Sigma_\infty(M);\mathbb{Z}/2)$, so since s is an isomorphism on $H^1(-;\mathbb{Z}/2)$ we conclude that $\mathbb{Z}\times A_\infty(M)$ is the pullback of some double cover of $\Gamma(T^cM)$, which we denote by $\Gamma^A(T^cM)$. Acyclicity is a property of maps which lifts to covering spaces,[7] so (1.2.1) implies that the

[5] `jmiller@gc.cuny.edu`

[6] We will just speak of *unlabelled* configuration spaces in this section, for simplicity.

[7] Since an equivalent characterisation of acyclicity for a map $X \to Y$ is that its lift to $X\times_Y\widetilde{Y}\to\widetilde{Y}$ is a homology-equivalence, where $\widetilde{Y}$ is the universal cover of Y.

lifted map

$$\mathbb{Z} \times A_\infty(M) \to \Gamma^A(T^c M)$$

is acyclic, and in particular a homology-equivalence. This could be thought of as the corresponding scanning result for oriented configuration spaces.

A brief outline of our strategy is as follows. The proof proceeds in two steps, first proving (1.2.1) for Euclidean spaces M (in fact, for any manifold of the form $\mathbb{R}^2 \times N$), and then deducing from this the general case. We add a disclaimer that the following sketch has not yet been written up in detail, and so should be treated with a healthy dose of either scepticism or optimism.

When $M = \mathbb{R}^2 \times N$ we have that $\coprod_n \Sigma_n(M)$ and $\Gamma(T^c M)$ are each homotopy-equivalent to homotopy-commutative topological monoids, by a construction similar to that of C'_n on the first page of [Seg73]. Hence the plus-constructed scanning map

$$s^+ \colon \mathbb{Z} \times \Sigma_\infty(M)^+ \to \Gamma(T^c M)$$

is a homology-equivalence (by [McD75]) with target a *simple* space (π_1 acting trivially on π_n for all $n \geq 1$). If we can show that $\mathbb{Z} \times \Sigma_\infty(M)^+$ is also a simple space then s^+ must be a weak equivalence and hence s must be acyclic. Now, a general condition for BG^+ to be an H-space, and therefore simple, is given in [Wag72, Proposition 1.2], and this covers the special cases of the infinite braid group $\Sigma_\infty(\mathbb{R}^2)^+ \simeq B\beta_\infty^+$ and the infinite symmetric group $\Sigma_\infty(\mathbb{R}^\infty)^+ \simeq B\Sigma_\infty^+$. However, the method crucially uses properties of group homology, and so does not suffice in general.

In the general case we need a *twisted* version of the group-completion theorem, implicit in [MS76] and recently written up in detail in [RW], as follows. Let $\mathcal{M}$ be a homotopy-commutative topological monoid, with $\pi_0\mathcal{M} = \mathbb{N}$,[8] and let $\mathcal{M}_\infty$ be the homotopy colimit of the sequence $\mathcal{M} \to \mathcal{M} \to \cdots$ given by multiplication by a fixed element of the 1-component; so $\mathcal{M}$ acts on $\mathcal{M}_\infty$ and we have a map $\pi \colon E\mathcal{M} \times_{\mathcal{M}} \mathcal{M}_\infty \to B\mathcal{M}$. Then the canonical map $\mathcal{M}_\infty \to \mathrm{hofib}(\pi) \simeq \Omega B\mathcal{M}$ is *acyclic*. Hence $\mathcal{M}_\infty^+$ is weakly equivalent to $\Omega B\mathcal{M}$, so in particular it is simple. Applying this to $\mathcal{M} \simeq \coprod_n \Sigma_n(M)$ we therefore conclude that $\mathcal{M}_\infty^+ \simeq \mathbb{Z} \times \Sigma_\infty(M)^+$ is simple, as required.

For the second step we have M any connected manifold which is the interior of a manifold $\bar{M}$ with non-empty boundary. Choose a disc D in $\bar{M}$ which intersects $\partial\bar{M}$ (far away from

[8] This is not necessary, but makes the hypotheses simpler to state.

where the stabilisation map acts). We then have a square of maps

$$\begin{tikzcd} \Sigma(M) \arrow[r, "s"] \arrow[d, "\pi"'] & \Gamma(T^c M) \arrow[d, "\pi'"] \\ \Sigma(M,D) \arrow[r, "s"'] & \Gamma(T^c M, D) \end{tikzcd} \tag{1.2.2}$$

where $\Sigma(M) = \coprod_n \Sigma_n(M)$ and the bottom map is the stabilisation map for *relative* configuration and section spaces, which was shown to be a weak equivalence (as long as $\pi_0(D) \to \pi_0(M)$ is surjective, which is true in our case) by [Böd87, Proposition 2]. Taking the homotopy colimit of (1.2.2) under stabilisation maps for the top two spaces and the identities on the bottom two spaces, we obtain the square

$$\begin{tikzcd} \mathbb{Z} \times \Sigma_\infty(M) \arrow[r, "s"] \arrow[d, "\pi_\infty"'] & \Gamma(T^c M) \arrow[d, "\pi'"] \\ \Sigma(M,D) \arrow[r, "s"'] & \Gamma(T^c M, D) \end{tikzcd} \tag{1.2.3}$$

for which the map of (point-set) fibres is $s\colon \mathbb{Z} \times \Sigma_\infty(D) \to \Gamma(T^c D)$, which is therefore acyclic by the first half of the proof. Since π' is a fibration and π_∞ is a "strong homology fibration" (explained in a moment), this implies that the map of *homotopy* fibres is acyclic. But the map of base spaces is a *weak equivalence*, so the map of total spaces is also acyclic.

A *strong homology fibration* is a map f such that the inclusion of its point-set fibres $f^{-1}(b)$ into its homotopy fibre $\mathrm{hofib}(f)$ are *strong homology-equivalences*: they induce isomorphisms on homology for any *abelian* local coefficient system pulled back from $\mathrm{hofib}(f)$. To show that π_∞ is a strong homology fibration one has to prove a strong homology fibration criterion (similar to the homology fibration criterion of [McD75, Proposition 5.1] and the quasifibration criterion of [DT58]), and then verify this criterion for π_∞.

If we wanted to avoid *strong* homology fibrations we could still at least deduce the corollary that $\mathbb{Z} \times A_\infty(M) \to \Gamma^A(T^c M)$ is a homology-equivalence by showing that the map $\mathbb{Z} \times A_\infty(M) \to \Sigma(M,D)$ is an (ordinary) homology fibration.

CHAPTER 2

Homological stability for oriented configuration spaces

2.1 Introduction

Recall from §1.1 that for a manifold M and space X, we define the n-point unordered configuration space to be

$$C_n(M,X) \coloneqq \left\{(p_1,\ldots,p_n)\in M^n \mid p_i\neq p_j \text{ for } i\neq j\right\} \times_{\Sigma_n} X^n.$$

This is the space of configurations of n distinct points (or 'particles') in M, each carrying a label (or 'parameter') in X. When the label-space X is just a point we call $C_n(M,pt) = \{(p_1,\ldots,p_n)\in M^n \mid p_i\neq p_j \text{ for } i\neq j\}/\Sigma_n$ an *unlabelled* configuration space. The *oriented* configuration space is defined to be the double cover

$$C_n^+(M,X) \coloneqq \left\{(p_1,\ldots,p_n)\in M^n \mid p_i\neq p_j \text{ for } i\neq j\right\} \times_{A_n} X^n.$$

of this space, so oriented configurations have an additional global parameter: an ordering of the n points up to even permutations. If M "admits a boundary" there is a natural map s which adds a new point to the configuration near this boundary (see §2.2.2 for precise definitions).

Main Theorem *If M is the interior of a connected manifold-with-boundary of dimension at least 2, and X is any path-connected space, then*

$$s\colon C_n^+(M,X) \longrightarrow C_{n+1}^+(M,X)$$

is an isomorphism on homology up to degree $\frac{n-5}{3}$, and a surjection up to degree $\frac{n-2}{3}$.

Remark 2.1.1 If either M or X is not path-connected, then the number of path-components of $C_n^+(M,X)$ grows unboundedly as $n\to\infty$, so homological stability fails even in degree zero. We also exclude the case of 1-dimensional manifolds, where homological stability also fails in general: the space $C_n^+(\mathbb{R},X)$ deformation retracts onto $X^n \sqcup X^n$.

When such a statement holds for a range of degrees $*\leq \alpha n + c$, we say that the *stability slope* is α; so in this case we have homological stability for oriented configuration spaces with a stability slope of $\frac{1}{3}$.

The underlying method we use for the proof is that of taking "resolutions of moduli spaces", as introduced and studied by Randal-Williams in [RW10]. This method involves considering a semi-simplicial space augmented by the space of interest, where in the 'standard' strategy for proving homological stability one would consider a simplicial complex acted on by the group of interest. The method was applied in [RW11] to prove the analogous theorem for unordered configuration spaces, which has a stability slope of $\frac{1}{2}$. Our method is a modified version of that of [RW11]; however, some important complications arise in going from the unordered to the oriented case, which are outlined in §2.3 below.

In particular, §2.3.2 explains why the stability slope goes from $\frac{1}{2}$ to $\frac{1}{3}$ when we apply the techniques of [RW11] to the oriented case.

Remark 2.1.2 We note that the stability slope of $\frac{1}{3}$ is the best possible for oriented configuration spaces (for $\mathbb{Z}$-coefficients), as can be seen by the calculations of [Hau78] or [GKY96] (see also §2.8.3 and Chapter 3).

2.1.1 Background

A brief history of homology-stability theorems for unordered and oriented configuration spaces is as follows.

Unordered configuration spaces. Two special cases which were proved early on are homology-stability for the sequences of symmetric and braid groups, corresponding to $M = \mathbb{R}^\infty, \mathbb{R}^2$ respectively (and $X = pt$, i.e. unlabelled). The result for the symmetric groups is due to Nakaoka [Nak60], and the result for the braid groups was proved later by Arnol'd [Arn70b]. The stability slope obtained in each case was $\frac{1}{2}$. Using more indirect methods, Segal [Seg73] proved homology-stability for all Euclidean spaces $M = \mathbb{R}^d$ and arbitrary path-connected label-spaces X, but this time without an explicit range of stability (see also [LS01, §3]). Generalising in a different direction, in [McD75] McDuff proved homology-stability for arbitrary manifolds M (assuming connectivity and that M admits a boundary) but without labels ($X = pt$) and also without an explicit stability range. (She remarked, however, that her methods would generalise to labelled configuration spaces.) Later, Segal [Seg79] showed by a different method that in this case we do in fact have a stability slope of $\frac{1}{2}$, as with the symmetric and braid groups.

The most general result for unordered configuration spaces is due to Randal-Williams [RW11, Theorem A],[1] which allows arbitrary manifolds *and* label-spaces. Specifically, he proves homology-stability for $C_n(M,X)$, with a slope of $\frac{1}{2}$, under the same assumptions on M and X as stated in the Main Theorem above.

A recent result of Church [Chu12] concerning *representation stability* shows, as a corollary of his main theorem, that rational homology-stability holds (with slope 1) for unordered configuration spaces on a manifold M which may be *closed*. In this case M does not admit a boundary, and there is no natural map s adding a point to the configuration, but nevertheless stability still holds rationally. The isomorphism in this case is induced by a transfer map which *removes* a point from the configuration. This result is also proved directly in Theorems B and C of [RW11] (although here the increased stability slope of 1 is only obtained when the manifold has dimension at least 3).

Oriented configuration spaces. Homology-stability for oriented configuration spaces $C_n^+(M,X)$ has been proved in two special cases. For the alternating groups ($M = \mathbb{R}^\infty$,

[1] This is also recalled as Theorem 2.3.1 below.

$X = pt$) it can be quickly deduced from a result of Hausmann [Hau78, page 130], with a stability slope of $\frac{1}{3}$, which can be improved to $\frac{1}{2}$ by taking $\mathbb{Z}[\frac{1}{3}]$-coefficients. For M a compact connected Riemann surface minus a non-empty finite set of points (and $X = pt$), Guest-Kozlowsky-Yamaguchi [GKY96] proved homology-stability with a slope of $\frac{1}{3}$, which again can be improved to $\frac{1}{2}$ by taking $\mathbb{Z}[\frac{1}{3}]$-coefficients.[2] The proofs of [Hau78] and [GKY96] involve explicit calculations, using methods which are specific to their respective cases, so do not generalise naturally to all manifolds. The main result of this chapter answers a question in [GKY96], which asks whether their result generalises to arbitrary open manifolds.

In general, *rational* homology-stability for oriented configuration spaces follows from the result of Church mentioned above. It corresponds to stability for the multiplicities of the trivial and alternating representations of Σ_n in the rational cohomology of the ordered configuration space $\widetilde{C}_n(M,X)$. Representation stability for $\widetilde{C}_n(M,X)$ [Chu12, Theorem 1] includes multiplicity stability for the trivial representation, and indirectly shows that the multiplicity of the alternating representation is eventually zero (cf. the discussion after the statement of Theorem 1 in [Chu12]).

2.1.2 Remarks

Remark 2.1.3 The Serre spectral sequence for the fibration $\mathbb{Z}_2 \to C_n^+(M,X) \to C_n(M,X)$ implies that

$$H_*(C_n^+(M,X);\mathbb{Z}) \;\cong\; H_*(C_n(M,X);\underline{\mathbb{Z}\oplus\mathbb{Z}}),$$

where the $\underline{\mathbb{Z}\oplus\mathbb{Z}}$-coefficients on the right are twisted by the action of $\pi_1 C_n(M,X)$ on $\mathbb{Z}\oplus\mathbb{Z}$ by first projecting to $\mathbb{Z}_2$ (corresponding to the index-2 subgroup $\pi_1 C_n^+(M,X)$) and then letting the generator of $\mathbb{Z}_2$ act by swapping the two $\mathbb{Z}$-summands. So the Main Theorem above is also twisted homological stability for unordered configuration spaces with this sequence of $\pi_1 C_n(M,X)$-modules. We note that in the $M = \mathbb{R}^\infty$, $X = pt$ case this sequence of Σ_n-modules does *not* extend to a (functorial) coefficient system in the sense of [Bet02]. See Chapter 4 for more on twisted homological stability for unordered configuration spaces.

Remark 2.1.4 The orientation of a configuration in $C_n^+(M,X)$ is an example of a *global* parameter on configuration spaces (the labels in X are local parameters); in a sense it is the simplest possible one. It is interesting that homological stability still holds for these spaces, since the 'scanning' method of Segal and McDuff does not work in this case. In this method one first uses a 'transfer-type' argument to show that, on homology of any degree, the adding-a-point maps s are inclusions of direct summands. Then one shows that the colimit of this sequence of maps is finitely generated (cf. the proof of Theorem 4.5 in [McD75]). However, for oriented configuration spaces the maps s are *not* always injective on homology (see §2.8.3 for counterexamples). Arguably, it is the existence of global data in $C_n^+(M,X)$ which causes this injectivity-on-homology to fail.

[2]Their calculations actually work for any connected, open surface; see Chapter 3.

Remark 2.1.5 A nice orientability property of oriented configuration spaces is the following: if M and X are both orientable manifolds, then $C_n^+(M,X)$ is again an orientable manifold. This is simpler than in the unordered case, where $C_n(M,X)$ is *non-orientable* (exactly) if either

- $\dim(M) \geq 2$ and $\dim(M)+\dim(X)$ is odd, or
- $M = S^1$ and $\dim(X)$ and n are even

(cf. the remark following Proposition A.1 in [Seg79]).

2.1.3 Corollaries

The Main Theorem has corollaries for homological stability of certain sequences of groups:

Corollary A *If G is any discrete group and S is the interior of a connected surface-with-boundary $\overline{S}$, then the natural maps*

$$G \wr A\beta_n^S \longrightarrow G \wr A\beta_{n+1}^S \quad \text{and} \quad G \wr A_n \longrightarrow G \wr A_{n+1}$$

are isomorphisms on homology up to degree $\frac{n-5}{3}$ and surjections up to degree $\frac{n-2}{3}$.

Here β_n^S is the braid group on n strands on the surface S, and $A\beta_n^S$ is its alternating subgroup, consisting of those braids whose induced permutation is even. Of course, these corollaries exactly parallel those of the unordered version of the Main Theorem, which concern $G \wr \Sigma_n$ and $G \wr \beta_n^S$. Homological stability for A_n and for $A\beta_n^S$ with $\overline{S}$ compact and orientable were known previously by [Hau78, Proposition A] (via the relative Hurewicz theorem) and [GKY96] respectively. The above corollaries are new (as far as the author is aware) for G non-trivial or for $\overline{S}$ non-orientable or non-compact.

Via the Universal Coefficient Theorem and the Atiyah-Hirzebruch spectral sequence, homological stability for (trivial) $\mathbb{Z}$-coefficients implies homological stability for any connective homology theory:

Corollary B *Under the hypotheses of the Main Theorem, if h_* is a connective homology theory with connectivity c, the map*

$$s\colon C_n^+(M,X) \longrightarrow C_{n+1}^+(M,X)$$

is an isomorphism on h_* for $* \leq \frac{n-5}{3} + c$ and surjective on h_* for $* \leq \frac{n-2}{3} + c$.

Organisation of the chapter

In §2.2 we define all the spaces, semi-simplicial spaces, and maps which will be used later. Section 2.3 contains an outline of the proof, and explains the differences between the methods in the unordered and the oriented cases. The proof itself is contained in §§2.4, 2.5 and 2.6. Section 2.4 produces some spectral sequences and establishes some facts about them, §2.5 uses excision to relate the connectivity of two different maps between configuration spaces, and §2.6 brings this together to prove the Main Theorem. Section 2.7 establishes the corollaries stated above, and §2.8 contains a note on the (failure of) injectivity of stabilisation maps on homology.

Some technical constructions have been deferred to the appendices, to avoid lengthy digressions during the proof of the Main Theorem. Appendix 2.A constructs a factorisation on homology for maps between mapping cones, under fairly general conditions, and Appendix 2.B recalls the details of the construction of various spectral sequences arising from semi-simplicial spaces.

2.2 Definitions and set-up

First we mention two general notational conventions: A connected manifold M with k points removed will be denoted by M_k (since it is connected, its homeomorphism type is independent of *which* k points are removed). The symbol $\rightarrowtail$ will be reserved for the canonical inclusion of the codomain of a map into its mapping cone:

$$Y \xrightarrow{f} Z \rightarrowtail Cf.$$

2.2.1 Configuration spaces

Definition 2.2.1 For a manifold M and space X, we define the *ordered configuration space* to be

$$\widetilde{C}_n(M,X) \coloneqq \mathrm{Emb}([n],M) \times X^n,$$

where $[n]$ is the n-point discrete space. This is the space of ordered, distinct points ('particles') in M, each carrying a label (or parameter) in X. The symmetric group acts diagonally on this space, permuting the points along with their labels, and we define the *unordered configuration space* to be the quotient

$$C_n(M,X) \coloneqq \widetilde{C}_n(M,X)/\Sigma_n.$$

If instead we just quotient out by the action of the alternating group, we obtain the *oriented configuration space*

$$C_n^+(M,X) \coloneqq \widetilde{C}_n(M,X)/A_n.$$

Notation 2.2.2 We will denote elements of ordered, oriented, unordered configuration spaces respectively by $\left(\begin{smallmatrix} p_1 & \cdots & p_n \\ x_1 & \cdots & x_n \end{smallmatrix}\right)$, $\left[\begin{smallmatrix} p_1 & \cdots & p_n \\ x_1 & \cdots & x_n \end{smallmatrix}\right]$, $\left\{\begin{smallmatrix} p_1 & \cdots & p_n \\ x_1 & \cdots & x_n \end{smallmatrix}\right\}$, where $p_i \in M$ and $x_i \in X$. So square brackets denote the equivalence class under even permutations of the columns. The orientation-reversing automorphism

$$\left[\begin{smallmatrix} p_1 & \cdots & p_n \\ x_1 & & x_n \end{smallmatrix}\right] \quad\mapsto\quad \left[\begin{smallmatrix} p_1 & \cdots & p_{n-2} & p_n & p_{n-1} \\ x_1 & & x_{n-2} & x_n & x_{n-1} \end{smallmatrix}\right]$$

of $C_n^+(M,X)$ will be denoted by ν. We will often abbreviate these spaces to $\widetilde{C}_n(M)$, $C_n^+(M)$, and $C_n(M)$ when the space of labels is clear, to avoid cluttering our notation.

2.2.2 Adding a point to a configuration space

To add a point to a configuration on M, there needs to be somewhere "at infinity" from which to push in this new configuration point. An appropriate condition is to "admit a boundary":

Definition 2.2.3 We say that M *admits a boundary* if it is the interior of some manifold-with-boundary $\bar{M}$. Note that we do not require $\bar{M}$ to be compact.

When M admits a boundary, there is a natural adding-a-point map, as follows:

Definition 2.2.4 Suppose that $M = \mathrm{int}(\bar{M})$, where $\bar{M}$ is a manifold-with-boundary, and choose a point $b_0 \in \partial\bar{M}$. Let $B_0 = \partial_0\bar{M}$ be the boundary-component containing b_0. Also choose a basepoint $x_0 \in X$. We initially define the adding-a-point map at the level of *ordered* configuration spaces to be

$$\left(\begin{smallmatrix} p_1 & \cdots & p_n \\ x_1 & & x_n \end{smallmatrix}\right) \quad\mapsto\quad \left(\begin{smallmatrix} p_1 & \cdots & p_n & b_0 \\ x_1 & & x_n & x_0 \end{smallmatrix}\right).$$

This is a map $\widetilde{C}_n(M,X) \to \widetilde{C}_{n+1}(M',X)$, where M' is $\bar{M}$ with an open collar attached at B_0:

$$M' = \bar{M} \cup_{B_0} \big(B_0 \times [0,1)\big).$$

Choosing a canonical homeomorphism $\phi\colon M' \cong M$ (with support contained in a small neighbourhood of B_0), which pushes this collar back into M, we obtain a map

$$s\colon \widetilde{C}_n(M,X) \longrightarrow \widetilde{C}_{n+1}(M,X).$$

This process is illustrated in Figure 2.2.1. The map s is equivariant w.r.t. the standard inclusion $\Sigma_n \hookrightarrow \Sigma_{n+1}$ (and hence also w.r.t. $A_n \hookrightarrow A_{n+1}$), so it descends to maps

$$s\colon C_n(M,X) \longrightarrow C_{n+1}(M,X)$$

and

$$s\colon C_n^+(M,X) \longrightarrow C_{n+1}^+(M,X).$$

Notation 2.2.5 We will generally refer to the adding-a-point maps s as *stabilisation maps*, since these are the maps with respect to which the unordered and oriented configuration spaces stabilise. When it is necessary to keep track of the number of points in a configuration, we write $s = s_n$ for the map which adds the $(n+1)$st point to a configuration.

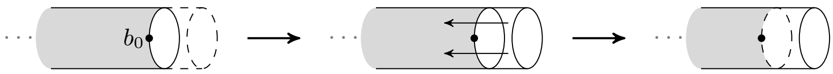


Figure 2.2.1: The adding-a-point map s. The original configuration is contained in the interior of the shaded region in each picture.

In the oriented case, we define $-s := \nu \circ s$ (and $+s := s$). So $-s$ just takes the opposite orientation convention in its definition, sending $\left[\begin{smallmatrix} p_1 & \cdots & p_n \\ x_1 & \cdots & x_n \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} p_1 & \cdots & p_{n-1} & b_0 & p_n \\ x_1 & \cdots & x_{n-1} & x_0 & x_n \end{smallmatrix}\right]$ instead of $\left[\begin{smallmatrix} p_1 & \cdots & p_n & b_0 \\ x_1 & \cdots & x_n & x_0 \end{smallmatrix}\right]$.

Remark 2.2.6 Up to homotopy, the stabilisation map s depends only on the choice of boundary-component B_0, and the choice of path-component of X containing x_0. Later we will only consider the case when X is path-connected, so s will only depend on which 'end' of the manifold the new configuration point is pushed in from.

Remark 2.2.7 Since $\pm s$ only differ by an automorphism of their common codomain, they have exactly the same properties w.r.t. injectivity- and surjectivity-on-homology, so they are interchangeable for the purposes of homology-stability.

2.2.3 Semi-simplicial spaces

A semi-simplicial space (which in this chapter we will call a Δ-space) is a diagram of the form

$$\cdots \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; Y_1 \;\rightrightarrows\; Y_0$$

where the "face maps" $d_i \colon Y_k \to Y_{k-1}$ $(1 \leq i \leq k+1)$ satisfy the simplicial identities $d_i d_j = d_{j-1} d_i$ whenever $i < j$. The Δ-space as a whole is denoted by $Y_\bullet$. An *augmented* Δ-space is a diagram of the form

$$\cdots \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; Y_1 \;\rightrightarrows\; Y_0 \longrightarrow Y_{-1}$$

where again the face maps satisfy the simplicial identities. In other words this is a Δ-space together with an "augmentation map" $Y_0 \to Y_{-1}$ which equalises the two face maps $d_1, d_2 \colon Y_1 \rightrightarrows Y_0$. A map of (augmented) Δ-spaces is a collection of maps, one for each level k, which commutes with d_i for each i.

The (thick) geometric realisation of a Δ-space $Y_\bullet$ is $\|Y_\bullet\| = \left(\coprod_{k \geq 0} Y_k \times \Delta^k\right) / \sim$, where $\sim$ is the equivalence relation generated by the face relations $(d_i(y), z) \sim (y, \delta_i(z))$, where δ_i is the inclusion of the ith face of Δ^{k+1}. If $Y_\bullet$ is an augmented Δ-space, there is a unique composition of face maps $Y_k \to Y_{-1}$ for each k. These fit together to give an induced map $\|Y_\bullet\| \to Y_{-1}$, where $\|Y_\bullet\|$ is the geometric realisation of the *non-augmented* part of $Y_\bullet$.

Definition 2.2.8 A Δ-space $Y_\bullet$ with an augmentation to Y_{-1} such that the induced map $\|Y_\bullet\| \to Y_{-1}$ is n-connected is called an *n-resolution* of Y_{-1} in the terminology of [RW10] and [RW11].

2.2.3.1 A configuration Δ-space.

We now extend the oriented configuration space $C_n^+(M,X)$ so that it is the (-1)st level of an augmented Δ-space.

Definition 2.2.9 The augmented Δ-space $C_n^+(M,X)^\bullet$ is defined as follows: The elements of the space of i-simplices $C_n^+(M,X)^i$ are configurations $[\begin{smallmatrix} p_1 \\ x_1 \end{smallmatrix} \cdots \begin{smallmatrix} p_n \\ x_n \end{smallmatrix}]$, together with an *ordering* of $i+1$ of the pairs $(\begin{smallmatrix} p_i \\ x_i \end{smallmatrix})$. In particular, $C_n^+(M,X)^{-1}$ is just $C_n^+(M,X)$, and $C_n^+(M,X)^0$ consists of (oriented, labelled) configurations with one of the points marked out as 'special'. The face map d_j is given by forgetting the jth element of the $(i+1)$-ordering; in particular, the augmentation map is the map $C_n^+(M,X)^0 \to C_n^+(M,X)$ which forgets which point is 'special'.

Remark 2.2.10 We will show later (see Corollary 2.4.8) that $C_n^+(M,X)^\bullet$ is an $(n-1)$-resolution of $C_n^+(M,X)$.

Note The definition of $\pm s$ clearly extends to each level $C_n^+(M,X)^i$ and commutes with the face maps, so we have maps of augmented Δ-spaces:

$$C_n^+(M,X)^\bullet \xrightarrow{\pm s^\bullet} C_{n+1}^+(M,X)^\bullet.$$

As before, we will often abbreviate the augmented Δ-space $C_n^+(M,X)^\bullet$ as $C_n^+(M)^\bullet$.

2.2.4 Maps between configuration spaces

We will make use of the following maps between configuration spaces in the proof of the Main Theorem.

2.2.4.1 ε_n and a_n.

These automatically come from the structure of the augmented Δ-space $C_n^+(M,X)^\bullet$: a_n denotes the augmentation map

$$a_n \colon C_n^+(M,X)^0 \longrightarrow C_n^+(M,X),$$

which forgets which point is the 'special' point, and ε_n is the induced map

$$\varepsilon_n \colon \|C_n^+(M,X)^\bullet\| \longrightarrow C_n^+(M,X)$$

from the geometric realisation of the unaugmented part of the Δ-space to the augmentation.

Aside (*Puncturing M*) Recall from the beginning of the section that M_1, M_k denote the connected manifold M with any point, or more generally any k points, removed. Since M is connected, the manifolds resulting from removing different choices of a set of k points can all be (non-canonically) identified. So where necessary we may assume that M_1 means M

with a point *near* B_0 (in the sense of the definition of the stabilisation map) removed. It is also implicitly assumed that we remember the inclusion $M_k \hookrightarrow M$, as well as the abstract manifold M_k.

2.2.4.2 p_n and u_n.

The maps
$$C_n^+(M) \xrightarrow{p_n} C_n^+(M_1) \xrightarrow{u_n} C_n^+(M)$$
'puncture' and 'unpuncture' the manifold M respectively. The second map is easiest to describe: it is just induced by the inclusion $M_1 \hookrightarrow M$. The puncturing map p_n is defined similarly to the stabilisation map. Let $M^{(1)}$ be $\bar{M}$ with an open collar attached at B_0, and then punctured at b_0:
$$M^{(1)} = \bar{M} \cup_{B_0} \big(B_0 \times [0,1)\big) \smallsetminus \{b_0\}.$$
Then the map p_n is induced by the inclusion $M \hookrightarrow M^{(1)}$ and the canonical homeomorphism $\phi|_{M^{(1)}} \colon M^{(1)} \cong M_1$ (from the definition of the stabilisation map) which pushes the collar back into M.

Remark 2.2.11 The composition $u_n \circ p_n$ is homotopic to the identity, since it just pushes the configuration away from B_0 slightly.

2.2.4.3 $\pi_{n,i}$ and $j_{n,i}$.

The projection
$$\pi_{n,i} \colon C_n^+(M)^i \longrightarrow \widetilde{C}_{i+1}(M)$$
forgets all but the $(i+1)$-ordered points of the configuration in $C_n^+(M)^i$. It clearly commutes with the stabilisation map: $\pi_{n+1,i} \circ s_n = \pi_{n,i}$.

This is a fibre bundle, with fibre homeomorphic to $C_{n-i-1}^+(M_{i+1})$ when $i \leq n-3$. It is closely analogous to the fibre bundle constructed by Fadell-Neuwirth in [FN62a, Theorem 3], and the fact that *this* is a fibre bundle is proved in detail as Lemma 1.26 in [KT08, page 26], so we refer to this for a detailed exposition. Alternatively, see Lemma 5.5.7 of Chapter 5, where a much more general version of this is proved (although in the unordered case). To find a trivialising neighbourhood for $\left(\begin{smallmatrix} p_1 & \cdots & p_{i+1} \\ x_1 & & x_{i+1}\end{smallmatrix}\right) \in \widetilde{C}_{i+1}(M)$ one just needs to choose pairwise disjoint open neighbourhoods for the points $p_1, \ldots, p_{i+1} \in M$. The condition $i \leq n-3$ is to ensure that the fibre is path-connected; in the cases $i = n-2$ and $i = n-1$, the fibre is $M_{n-1} \times X \times [2]$ and $[2]$ respectively (where $[2]$ is the two-point discrete space).

Pick a basepoint $\left(\begin{smallmatrix} m_1 & \cdots & m_{i+1} \\ x_0 & & x_0\end{smallmatrix}\right) \in \widetilde{C}_{i+1}(M)$ and define $j_{n,i}$ to be the inclusion of the fibre
$$j_{n,i} \colon C_{n-i-1}^+(M_{i+1}) = C_{n-i-1}^+(M \smallsetminus \{m_1, \ldots, m_{i+1}\}) \hookrightarrow C_n^+(M)^i.$$
In identifying the fibre abstractly with $C_{n-i-1}^+(M_{i+1})$, we have implicitly chosen a convention for combining the orientation of $\left[\begin{smallmatrix} p_1 & \cdots & p_{n-i-1} \\ x_1 & & x_{n-i-1}\end{smallmatrix}\right] \in C_{n-i-1}^+(M_{i+1})$ with the ordering

$\left(\begin{smallmatrix} m_1 \\ x_0 \end{smallmatrix} \cdots \begin{smallmatrix} m_{i+1} \\ x_0 \end{smallmatrix}\right)$ to induce an orientation of all n points. We declare this convention to be $\left[\begin{smallmatrix} m_1 \\ x_0 \end{smallmatrix} \cdots \begin{smallmatrix} m_{i+1} \\ x_0 \end{smallmatrix} \begin{smallmatrix} p_1 \\ x_1 \end{smallmatrix} \cdots \begin{smallmatrix} p_{n-i-1} \\ x_{n-i-1} \end{smallmatrix}\right]$, which completes the definition of $j_{n,i}$.

So abstractly $j_{n,i}$ is a map which replaces $i+1$ punctures with $i+1$ new configuration points, which are additionally given an $(i+1)$-ordering. The new points are all labelled by $x_0 \in X$, and the orientation of the new, larger configuration is given by the convention stated above.

Remark 2.2.12 Due to our choice of orientation convention for $j_{n,i}$, these maps commute with stabilisation maps, and we have a map of fibre bundles

$$\begin{tikzcd} C^+_{n-i}(M_{i+1}) \arrow[r, hook, "j_{n+1,i}"] & C^+_{n+1}(M)^i \arrow[dr, two heads, "\pi_{n+1,i}"] & \\ & & \widetilde{C}_{i+1}(M) \\ C^+_{n-i-1}(M_{i+1}) \arrow[uu, "s"] \arrow[r, hook, "j_{n,i}"'] & C^+_n(M)^i \arrow[uu, "s^i"'] \arrow[ur, two heads, "\pi_{n,i}"'] & \end{tikzcd} \tag{2.2.1}$$

Remark 2.2.13 The composition

$$C^+_n(M) \xrightarrow{p_n} C^+_n(M_1) \xrightarrow{j_{n+1,0}} C^+_{n+1}(M)^0 \xrightarrow{a_{n+1}} C^+_{n+1}(M)$$

sends $\left[\begin{smallmatrix} p_1 \\ x_1 \end{smallmatrix} \cdots \begin{smallmatrix} p_n \\ x_n \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} b_0 \\ x_0 \end{smallmatrix} \begin{smallmatrix} \bar{p}_1 \\ x_1 \end{smallmatrix} \cdots \begin{smallmatrix} \bar{p}_n \\ x_n \end{smallmatrix}\right]$, where $\bar{p}_i = \phi(p_i)$ is p_i pushed slightly away from B_0 if it is near B_0. Hence this is a factorisation of $(-1)^n s_n$.

This factorisation will be key to the proof of the Main Theorem, and the appearance of $(-1)^n$ here is in a sense where the extra complication (compared to the unordered case) comes from — and why we only obtain a stability slope of $\frac{1}{3}$.

2.2.5 Relative configuration spaces

Definition 2.2.14 We define the *relative configuration space* to be the mapping cone of the (positive) stabilisation map:

$$R^+_n(M,X) := \operatorname{hocofib}\big(C^+_n(M,X) \xrightarrow{s_n} C^+_{n+1}(M,X)\big).$$

Similarly, $R^+_n(M,X)^i$ is defined to be the mapping cone of the stabilisation map s^i_n between the ith levels of the corresponding Δ-spaces. Since the face maps commute exactly with the stabilisation maps, they induce relative face maps which give $\{R^+_n(M,X)^i\}_{i\geq -1}$ the structure of an augmented Δ-space $R^+_n(M,X)^\bullet$. Again, we will usually abbreviate the notation to $R^+_n(M)$ and $R^+_n(M)^\bullet$ when X is understood.

2.2.6 Maps between relative configuration spaces

All our maps $\widetilde{f}\colon R^+_n(M)^i \longrightarrow R^+_{n'}(M')^{i'}$ between relative configuration spaces will be induced by maps defined on the non-relative configuration spaces

$$\begin{tikzcd}
C_n^+(M)^i \arrow[r, "s"] \arrow[d, "f"'] \arrow[dr, phantom, "H_f"] & C_{n+1}^+(M)^i \arrow[r] \arrow[d, "f"] & R_n^+(M)^i \arrow[d, "\widetilde{f}"] \\
C_{n'}^+(M')^{i'} \arrow[r, "s"'] & C_{n'+1}^+(M')^{i'} \arrow[r] & R_{n'}^+(M')^{i'}
\end{tikzcd}$$

Note that $\widetilde{f}$ (even up to homotopy) depends on the non-relative maps f, *and* the homotopy H_f chosen to fill the square.

2.2.6.1 $\widetilde{\jmath}_{n,i}$, $\widetilde{a}_n$, and $\widetilde{u}_n$.

We define relative versions of the inclusion-of-the-fibre, augmentation, and unpuncturing maps as follows:

Definition 2.2.15 The maps $j_{n,i}$, a_n, and u_n commute exactly with stabilisation maps, so we may define

$$\begin{aligned}
\widetilde{\jmath}_{n,i} \colon R_{n-i-1}^+(M_{i+1}) &\longrightarrow R_n^+(M)^i, \\
\widetilde{a}_n \colon R_n^+(M)^0 &\longrightarrow R_n^+(M), \\
\widetilde{u}_n \colon R_n^+(M_1) &\longrightarrow R_n^+(M)
\end{aligned}$$

as explained above, taking the homotopy H_f to be the *constant* homotopy in each case.

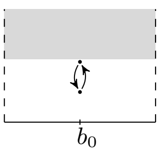
2.2.6.2 $\widetilde{p}_n$ and relative stabilisation maps.

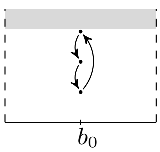
We will now define relative versions of the puncturing map p_n, the stabilisation map $s\colon C_n(M) \to C_{n+1}(M)$ and the negative iterated stabilisation map $-s^2 = \nu \circ s \circ s \colon C_n^+(M) \to C_{n+2}^+(M)$.

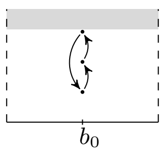
Notation 2.2.16 To differentiate clearly between the unordered and oriented cases, we will temporarily (for Definition 2.2.17 and Remark 2.2.18 below) use the following notation when we want to emphasise which case we are dealing with: $\breve{s}$ denotes the stabilisation map between *u*nordered configuration spaces, and $\mathring{s}$ denotes the stabilisation map between *o*riented configuration spaces. So we want to define relative versions of p_n, $\breve{s}$, and $-\mathring{s}^2$.

Definition 2.2.17 Embed $\mathbb{H} = \{(x,y) \in \mathbb{R}^2 | y \geq 0\}$ in $\bar{M}$, in a neighbourhood of the boundary-component B_0, so that b_0 is identified with $(0,0)$, and such that the homeomorphism $\phi\colon M' \cong M$ from the definition of the stabilisation map restricts to $(x,y) \mapsto (x, y+1)$ on $\mathbb{H}$. So on $\mathbb{H}$, the stabilisation map pushes points up by 1 and adds a new point at $(0,1)$.

Define the self-homotopies $(12)\colon \breve{s}^2 \simeq \breve{s}^2$ and $(123), (132)\colon -\mathring{s}^3 \simeq -\mathring{s}^3$ to fix the original configuration, and move the new configuration points around on $\mathbb{H}$ as illustrated below:

(12): 

(123): 

(132): 

(The original configuration is contained in the interior of the shaded region in each case.) Now, the left square below

$$
\begin{tikzcd}
\cdot \arrow[r, "\breve{s}"] \arrow[d, "\breve{s}"'] \arrow[dr, phantom, "\swarrow"] & \cdot \arrow[d, "\breve{s}"] \\
\cdot \arrow[r, "\breve{s}"'] & \cdot
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\cdot \arrow[r, "\mathring{s}"] \arrow[d, "-\mathring{s}^2"'] \arrow[dr, phantom, "\swarrow"] & \cdot \arrow[d, "-\mathring{s}^2"] \\
\cdot \arrow[r, "\mathring{s}"'] & \cdot
\end{tikzcd}
\tag{2.2.2}
$$

admits the identity homotopy 1 and the homotopy (12). These induce *relative stabilisation maps*

$$\widetilde{s}_1, \widetilde{s}_{(12)} \colon R_n(M) \longrightarrow R_{n+1}(M)$$

on relative unordered configuration spaces. Similarly the right square admits 1, (123) and (132), which induce *relative double stabilisation maps*

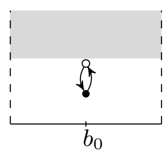
$$\widetilde{s}^2_1, \widetilde{s}^2_{(123)}, \widetilde{s}^2_{(132)} \colon R^+_n(M) \longrightarrow R^+_{n+2}(M)$$

on relative oriented configuration spaces.

Remark 2.2.18 The natural self-homotopies $\breve{s}^2 \simeq \breve{s}^2$ come from the different ways of moving the two new configuration points around in the collar neighbourhood $B_0 \times [0,1)$, so they are parametrised by $\pi_1 C_2(B_0 \times [0,1))$. We are only considering those which are supported in a coordinate neighbourhood near b_0, which are parametrised by $\pi_1 C_2(\mathbb{R}^d)$. This is either Σ_2 ($d \geq 3$) or β_2 ($d = 2$); the homotopy (12) defined above corresponds respectively to the only non-trivial element or a generator.

The analogous statement holds for self-homotopies $-\mathring{s}^3 \simeq -\mathring{s}^3$, replacing '$C_2$' by '$C_3^+$'. In this case $\pi_1 C_3^+(\mathbb{R}^d)$ is either A_3 ($d \geq 3$) or $A\beta_3$ ($d = 2$). This time the homotopies (123), (132) defined above correspond respectively to either the only non-trivial elements or a generating pair.

Definition 2.2.19 To define the relative puncturing map $\widetilde{p}_n \colon R^+_n(M) \longrightarrow R^+_n(M_1)$, we need to choose a homotopy $sp_n \simeq p_n s$. Similarly to the definition of the relative stabilisation maps, we define this to fix the original configuration, and swap the puncture $\circ$ and the new configuration point $\bullet$ on $\mathbb{H}$ as illustrated below:



This homotopy will be called $(12)_p$, to distinguish it from the homotopy (12) fitting in to the left square of (2.2.2).

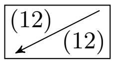
Remark 2.2.20 By Remark 2.2.13, and the definitions above, the composition

$$\widetilde{a}_{n+2} \circ \widetilde{\jmath}_{n+2,0} \circ \widetilde{p}_{n+1} \circ \widetilde{a}_{n+1} \circ \widetilde{\jmath}_{n+1,0} \circ \widetilde{p}_n \colon R_n^+(M) \longrightarrow R_{n+2}^+(M)$$

is a factorisation of $\widetilde{s}_H^2$, where H is the composite homotopy

$$\boxed{\begin{array}{c}(12)\\ \hline (12)\end{array}}$$

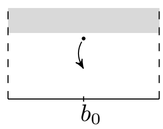
We are reading this in the direction $\swarrow$, so this is (132). We note that the homotopy (123) *also* factorises into two copies of (12), but pasted together differently:



The diagonal map here is the (positive) stabilisation map $s\colon C_{n+1}^+(M) \longrightarrow C_{n+2}^+(M)$.

Remark 2.2.21 We note that the composition $\widetilde{u}_n \circ \widetilde{p}_n$ is homotopic to the identity (cf. Remark 2.2.11). Indeed, composing the diagrams defining $\widetilde{u}_n$ and $\widetilde{p}_n$ results in

$$\begin{tikzcd}
\cdot \arrow[r, "s"] \arrow[d, "u_n \circ p_n"'] \arrow[dr, phantom, "\swarrow_H"] & \cdot \arrow[r] \arrow[d, "u_{n+1} \circ p_{n+1}"] & \cdot \arrow[d, "\widetilde{u}_n \circ \widetilde{p}_n"] \\
\cdot \arrow[r, "s"'] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

$H =$ 

and a little thought shows that the maps $u_n \circ p_n$, $u_{n+1} \circ p_{n+1}$ and the homotopy H can be simultaneously homotoped to identities, which induces a homotopy from $\widetilde{u}_n \circ \widetilde{p}_n$ to the identity.

2.3 Sketch of the proof

The aim of this section is to explain some of the ideas in the proof of the Main Theorem, and especially how the proof differs from the proof of the *unordered* version of this theorem. The proof itself is contained in §§2.4, 2.5, and 2.6 below, and does not depend on the contents of the present section, which is purely an overview.

2.3.1 The unordered case

We first outline the proof of the *unordered* version of the Main Theorem, due to Randal-Williams:

Theorem 2.3.1 ([RW11]) *If M is the interior of a connected manifold-with-boundary of dimension at least 2, and X is any path-connected space, then the stabilisation map*

$$C_n(M,X) \xrightarrow{s} C_{n+1}(M,X)$$

is an isomorphism on homology up to degree $\frac{n-2}{2}$ *and a surjection up to degree* $\frac{n}{2}$*. Equivalently,*

$$\widetilde{H}_* R_n(M, X) = 0 \textit{ for } * \leq \tfrac{n}{2}. \tag{2.3.1}$$

Sketch of proof. Since M and X are path-connected and $\dim(M) \geq 2$, all the configuration and relative configuration spaces are also path-connected, so $\widetilde{H}_0 R_n(M, X) = 0$ for all n. This proves the $n = 0, 1$ cases of (2.3.1); the general result is proved by induction on n.

The strategy is to construct some map with target $R_n(M, X) = R_n(M)$, and then prove that it is both zero and surjective on homology up to degree $\frac{n}{2}$. Two possible maps into $R_n(M)$ are the relative stabilisation maps $\widetilde{s}_1$ and $\widetilde{s}_{(12)}$ defined in §2.2.6.2, which are induced by putting the homotopies 1, (12) into the left-hand square of (2.2.2). By the unordered version of Remark 2.2.13, the vertical maps s in this square factorise into $a \circ j \circ p$ (corresponding to puncturing the manifold, then replacing the puncture by a new configuration point which is marked as special, and then forgetting which point is special). Now, the unordered versions of the maps $\widetilde{p}$, $\widetilde{\jmath}$, $\widetilde{a}$ on relative configuration spaces are defined similarly to the oriented ones: $\widetilde{p}$ is induced by a square containing the homotopy $(12)_p$ and $\widetilde{\jmath}$, $\widetilde{a}$ are induced by squares containing the identity homotopy. Hence the homotopy (12) respects the factorisation $s = a \circ j \circ p$ (i.e. it factorises into $\boxed{\rightrightarrows}$), whereas the identity homotopy 1 does not. So $\widetilde{s}_{(12)}$ has an induced factorisation $\widetilde{s}_{(12)} = \widetilde{a} \circ \widetilde{\jmath} \circ \widetilde{p}$. On the other hand 1 trivially factorises into triangles $\boxed{\swarrow}$ (here the diagonal map and both homotopies are just identities), but (12) does not.

Now, by some intricate arguments (this is where the bulk of the proof lies, and is contained in §§2.4, 2.5, 2.6 for the oriented case) the induced factorisation of $\widetilde{s}_{(12)}$ into $\widetilde{a} \circ \widetilde{\jmath} \circ \widetilde{p}$ allows us, using the inductive hypothesis, to prove that it is *surjective* on homology up to the required degree. On the other hand a factorisation into triangles $\boxed{\swarrow}$ automatically gives a nullhomotopy of the induced map on mapping cones; hence $\widetilde{s}_1$ is *zero* on homology (in all degrees). But neither map factorises both ways, so this doesn't yet finish the inductive step. Instead, in the unordered case, the following trick suffices to complete it:

We have a map of long exact sequences

$$\begin{tikzcd}
\arrow[r] & H_* C_n(M) \arrow[r] \arrow[d] \arrow[dr, bend right, "0"'] & H_* R_{n-1}(M) \arrow[r] \arrow[d, "(\widetilde{s}_{(12)})_*"] & H_{*-1} C_{n-1}(M) \arrow[r, "s_*"] \arrow[d] & H_{*-1} C_n(M) \arrow[r] \arrow[d] & {} \\
\arrow[r] & H_* C_{n+1}(M) \arrow[r] & H_* R_n(M) \arrow[r] & H_{*-1} C_n(M) \arrow[r] & H_{*-1} C_{n+1}(M) \arrow[r] & {}
\end{tikzcd}$$

where the indicated composition is zero since it is induced by a cofibration sequence. In the range of degrees under consideration we know that $(\widetilde{s}_{(12)})_*$ is surjective, so it is sufficient to prove surjectivity of the map $H_* C_n(M) \longrightarrow H_* R_{n-1}(M)$. By exactness, this is equivalent to injectivity of $s_* \colon H_{*-1} C_{n-1}(M) \longrightarrow H_{*-1} C_n(M)$. The inductive hypothesis only gives us this in the range $* \leq \frac{n-1}{2}$, which is not quite enough. However, in the unordered case one can show, by a completely different argument, that s_* is split-injective in *every* degree (see §2.8.1), so this completes the proof.

2.3.2 The oriented case

The oriented version of this theorem is the Main Theorem of this chapter:

Main Theorem *If M is the interior of a connected manifold-with-boundary of dimension at least 2, and X is any path-connected space, then the stabilisation map*

$$C_n^+(M,X) \xrightarrow{s} C_{n+1}^+(M,X)$$

is an isomorphism on homology up to degree $\frac{n-5}{3}$ and a surjection up to degree $\frac{n-2}{3}$. Equivalently,

$$\widetilde{H}_* R_n^+(M,X) = 0 \text{ for } * \leq \tfrac{n-2}{3}. \tag{2.3.2}$$

Sketch of proof. The basic strategy for the inductive step in the oriented case is the same: find a map with target $R_n^+(M,X) = R_n^+(M)$ which is both zero and surjective on homology up to degree $\frac{n-2}{3}$. By analogy with the unordered case, the first thing one might try is the relative stabilisation maps induced by

$$\begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d, "\pm s"'] \arrow[dr, phantom, "H"] & \cdot \arrow[d, "\pm s"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd} \qquad \text{with } H = \left\{ \begin{array}{ll} 1 & \text{if the vertical maps have the same sign} \\ (12) & \text{if the vertical maps have opposite signs} \end{array} \right\}.$$

Similarly to before, we would like the homotopy H to factorise like , so we need to choose the case where the vertical maps have opposite signs and $H = (12)$. This gives an induced factorisation of the relative stabilisation map into $\widetilde{a} \circ \widetilde{\jmath} \circ \widetilde{p}$, which allows us to prove that it is surjective on homology, by the same kind of arguments as in the unordered case.

However, (12) does *not* factorise into triangles , so we cannot deduce that it is also zero on homology. So far this is just as in the unordered case, but this time the "ladder trick" which completed the inductive step in the unordered case does *not* work: It depends on knowing injectivity of s_* in all degrees, in advance, by a separate argument, but in the oriented case s_* is *not* always injective (see §2.8).

So to solve this we will instead construct a *different* factorisation of the relative stabilisation map on homology, and then use this factorisation (and naturality of the factorisation w.r.t. stabilisation maps) to show that it factors through the zero map in the required range of degrees. This new factorisation is actually just a general construction for homotopy-commutative squares: the map on mapping cones induced by choosing any particular homotopy to fill the square has a certain factorisation on homology — as long as the square admits *some* homotopy which factorises into triangles . However, we do not currently have such a split homotopy. To remedy this, we can stack two copies of our square on top of each other; this produces the right-hand square of diagram (2.2.2), filled by the homotopy (132). So we have extended our map into $R_n^+(M)$ further back, to $\widetilde{s}^2_{(132)} \colon R^+_{n-2}(M) \to R^+_n(M)$.

Now we also have the homotopy (123) filling the same square, and as noted in Remark 2.2.20, this factorises into triangles (as does the identity homotopy, in fact). This allows us to construct the aforementioned factorisation of $\widetilde{s}^2_{(132)}$ on homology, which is

$$R^+_{n-2}(M) \longrightarrow \Sigma C^+_{n-2}(M) \dashrightarrow C^+_{n+1}(M) \longrightarrow R^+_n(M),$$

where a dotted arrow indicates a map defined only on homology.

In this case one can also check that the middle part of the factorisation commutes with stabilisation maps in the following way:

$$\begin{tikzcd}
R^+_{n-2}(M) \arrow[r] & \Sigma C^+_{n-2}(M) \arrow[r, dashed] \arrow[dr, phantom, "\circlearrowleft"] & C^+_{n+1}(M) \arrow[r] & R^+_n(M) \\
& \Sigma C^+_{n-3}(M) \arrow[r, dashed] \arrow[u, "\Sigma(-s_{n-3})"] & C^+_n(M) \arrow[u, "s_n"'] &
\end{tikzcd}$$

Now we can show that the top row ($\widetilde{s}^2_{(132)}$ on homology) is zero in the desired range.

The inductive hypothesis implies that $\Sigma(-s_{n-3})$ is surjective on homology in this range, so we can factor the top row along the bottom of the diagram like . In particular, it factors through $C^+_n(M) \xrightarrow{s_n} C^+_{n+1}(M) \to R^+_n(M)$, which is zero on homology since it is induced by a cofibration sequence.

This completes the inductive step, since surjectivity-on-homology can be proved as before, using the factorisation $\widetilde{s}^2_{(132)} = \widetilde{a} \circ \widetilde{\jmath} \circ \widetilde{p} \circ \widetilde{a} \circ \widetilde{\jmath} \circ \widetilde{p}$. However, note that we are now using the inductive hypothesis from further back (to prove surjectivity for the 'older' copies of $\widetilde{a}_*$, $\widetilde{\jmath}_*$, $\widetilde{p}_*$), which results in a smaller improvement in the range of stability during each inductive step — and hence the slower rate of stabilisation in the oriented case.

Remark 2.3.2 This narrative outlines a fairly direct link from the existence of a global parameter on configuration spaces to the reduced stability slope: Firstly it means that injectivity of s_* fails (see §2.8 for more on this), cutting off one line of attack, and secondly it makes the other line of attack weaker: The global parameter is an obstruction to the existence of certain self-homotopies of iterated stabilisation maps, which are needed to do the *zero*-on-homology half of the proof in this line of attack. Hence we need to extend our map into $R^+_n(M)$ further back to obtain such self-homotopies. This means we need to use the inductive hypothesis from further back to prove *surjectivity*-on-homology for the 'older' parts of this map, and so this only goes through for a smaller range of degrees. Hence we get a smaller increase in the stability range with each inductive step, and so the rate of stabilisation is slower.

2.4 Two spectral sequences

In this section we first establish the two spectral sequences to be used in the proof of the Main Theorem, and then show that (as mentioned in Remark 2.2.10) the augmented Δ-space $C_n^+(M,X)^\bullet$ is an $(n-1)$-resolution, implying that one of our spectral sequences converges to zero in a range of degrees.

2.4.1 General constructions

The first spectral sequence we will make use of is a relative version of the Serre spectral sequence. We denote the mapping cone of a map g by Cg.

Proposition 2.4.1 *Suppose f is a map of fibrations over a path-connected space B*

$$\begin{tikzcd} E_0 \arrow[rr, "f"] \arrow[dr] & & E_1 \arrow[dl] \\ & B & \end{tikzcd} \tag{2.4.1}$$

Let F_0, F_1 be the fibres over a point $b \in B$, and denote the restriction of f to $F_0 \to F_1$ by f_b. Then there is a first quadrant spectral sequence

$$E^2_{s,t} \;\cong\; H_s(B; \widetilde{H}_t(Cf_b)) \quad\Rightarrow\quad \widetilde{H}_*(Cf)$$

in which the rth differential has bidegree $(-r, r-1)$. The edge homomorphism

$$\widetilde{H}_t(Cf_b) \cong E^2_{0,t} \twoheadrightarrow E^\infty_{0,t} \hookrightarrow \widetilde{H}_t(Cf)$$

is the map on $\widetilde{H}_t$ induced by the inclusion $Cf_b \hookrightarrow Cf$.

This is mentioned as Remark 2 on p. 351 of [Swi75] and as Exercise 5.6 of [McC01]. (In these two places it is assumed that f is an inclusion, but this can be ensured by replacing (2.4.1) by a homotopy-equivalent diagram.) We will show how to derive this from the usual (absolute) Serre spectral sequence:

Proof. Let $C^{\mathrm{fib}}E_0$ be the *fibrewise* cone on E_0, i.e. $E_0 \times [0,1]$ with $F_b \times \{1\}$ collapsed to a point separately for each fibre F_b, and let

$$C^{\mathrm{fib}} f = E_1 \cup_f C^{\mathrm{fib}} E_0$$

(compare $Cf = E_1 \cup_f CE_0$). There is an induced fibration $p\colon C^{\mathrm{fib}} f \to B$, whose fibre is Cf_b, and which has a section $s\colon B \to C^{\mathrm{fib}} f$ taking $b' \in B$ to the tip of the cone in the fibre over b'. Collapsing this section gives a map $c\colon C^{\mathrm{fib}} f \to Cf$. These maps fit into the diagram

$$\begin{tikzcd}
\{b\} \arrow[r] \arrow[d] & Cf_b \arrow[r, "1"] \arrow[d, hook] & Cf_b \arrow[d, hook] \\
B \arrow[r, "s"] \arrow[d, "1"'] & C^{\mathrm{fib}}f \arrow[r, "c"] \arrow[d, "p"] & Cf \\
B \arrow[r, "1"'] & B &
\end{tikzcd}$$

where each *vertical* sequence is a fibration sequence and each *horizontal* sequence is a split cofibration sequence.

The required spectral sequence will be a direct summand of the Serre spectral sequence associated to the middle fibration. This can be seen as follows: The map of fibrations in the diagram above induces a map of Serre spectral sequences, and the fact that the horizontal sequences are split cofibrations means that we can identify this map of spectral sequences, on each page E^r and in the limit, as an inclusion of a direct summand. In particular,

$$\begin{array}{lrcl}
\text{on the } E^2 \text{ page:} & H_s(B; H_t(pt)) & \hookrightarrow & H_s(B; H_t(pt)) \oplus H_s(B; \widetilde{H}_t(Cf_b)); \\
\text{in the limit:} & H_*(B) & \hookrightarrow & H_*(B) \oplus \widetilde{H}_*(Cf).
\end{array}$$

Passing to the *other* direct summand now gives the required spectral sequence.

The claim about edge homomorphisms follows from the analogous fact about edge homomorphisms for the Serre spectral sequence associated to $Cf_b \hookrightarrow C^{\mathrm{fib}}f \twoheadrightarrow B$, of which our spectral sequence is a direct summand. $\square$

To state the next construction of a spectral sequence, we first define the notion of a "double mapping cone":

Definition 2.4.2 Given a square of maps which commutes up to homotopy, and a chosen homotopy to fill this square, one can apply the mapping cone construction either vertically then horizontally, or horizontally then vertically. The resulting 2-by-2 grid $\boxplus$ of spaces and maps is the same up to homeomorphism whichever way around this is done. In particular, the mapping cone (taken horizontally) of the induced map-on-mapping cones (taken vertically) is homeomorphic to the mapping cone (taken vertically) of the induced map-on-mapping-cones (taken horizontally). We call this the *double mapping cone* of the original square-with-homotopy.

The second spectral sequence we will need is constructed from a map of augmented Δ-spaces. There are versions of this construction for Δ-spaces and for augmented Δ-spaces, which can be either basepointed or non-basepointed, and maps of any of the above. The version we will use is:

Proposition 2.4.3 *Given a map of augmented Δ-spaces $Y_\bullet \to Z_\bullet$, there is an induced* *square of maps*

$$\begin{tikzcd}
\|Y_\bullet\| \arrow[r] \arrow[d] & \|Z_\bullet\| \arrow[d] \\
Y_{-1} \arrow[r] & Z_{-1}
\end{tikzcd} \tag{2.4.2}$$

Denote the double mapping cone of this square by $C^2(Y_\bullet \to Z_\bullet)$*, and as before denote the mapping cone of* $Y_s \to Z_s$ *by* $C(Y_s \to Z_s)$*. Then there is a spectral sequence contained in* $\{s \geq -1, t \geq 0\}$*:*

$$E^1_{s,t} \;\cong\; \widetilde{H}_t(C(Y_s \to Z_s)) \quad\Rightarrow\quad \widetilde{H}_{*+1}(C^2(Y_\bullet \to Z_\bullet)),$$

where the first differential is the alternating sum of the maps on homology induced by the relative face maps. In particular, $E^1_{-1,t} \longleftarrow E^1_{0,t}$ *is* $\widetilde{H}_t$ *of the relative augmentation map.*

Proof. The construction is given in Appendix 2.B. □

2.4.2 The spectral sequences to be used in the proof of the Main Theorem

Proposition 2.4.4 *We have the following spectral sequences:*

$$E^2_{s,t} \;\cong\; H_s\big(\widetilde{C}_{i+1}(M); \widetilde{H}_t(R^+_{n-i-1}(M_{i+1}))\big) \quad\Rightarrow\quad \widetilde{H}_*(R^+_n(M)^i) \tag{RSSS$_i$}$$

$$E^1_{s,t} \;\cong\; \widetilde{H}_t(R^+_n(M)^s) \quad\Rightarrow\quad \widetilde{H}_{*+1}C\widetilde{\varepsilon}_n \tag{ΔSS}$$

for $0 \leq i \leq n-3$*, where* $C\widetilde{\varepsilon}_n$ *is as follows:*

$$\begin{tikzcd}
\|C^+_n(M)^\bullet\| \arrow[r] \arrow[d, "\varepsilon_n"'] & \|C^+_{n+1}(M)^\bullet\| \arrow[r, tail] \arrow[d, "\varepsilon_{n+1}"] & \|R^+_n(M)^\bullet\| \arrow[d, "\widetilde{\varepsilon}_n"] \\
C^+_n(M) \arrow[r] \arrow[d, tail] & C^+_{n+1}(M) \arrow[r, tail] \arrow[d, tail] & R^+_n(M) \arrow[d, tail] \\
C\varepsilon_n \arrow[r] & C\varepsilon_{n+1} \arrow[r, tail] & C\widetilde{\varepsilon}_n
\end{tikzcd} \tag{2.4.3}$$

The edge homomorphisms on the vertical axis of (RSSS$_i$) *are the maps on* $\widetilde{H}_t$ *induced by* $\widetilde{\jmath}_{n,i}$*, and the leftmost* d^1*-differentials of* (ΔSS) *are the maps on* $\widetilde{H}_t$ *induced by* $\widetilde{a}_n$*.*

Proof. This follows immediately by applying Proposition 2.4.1 to the map of fibre bundles (2.2.1), and applying Proposition 2.4.3 to the map of augmented Δ-spaces

$$s^\bullet_n \colon C^+_n(M, X)^\bullet \longrightarrow C^+_{n+1}(M, X)^\bullet.$$ □

2.4.3 $C_n^+(M,X)^\bullet$ is an $(n-1)$-resolution

In the remainder of this section, we prove that the spectral sequence (ΔSS) converges to zero up to total degree $n-1$, which will follow from the fact that $C_n^+(M,X)^\bullet$ is an $(n-1)$-resolution. First we define a certain semi-simplicial set:

Definition 2.4.5 Let $\mathrm{inj}([i+1],[n])$ be the discrete space of all injections $[i+1] \hookrightarrow [n]$. These combine to form a Δ-space $\mathrm{inj}([\bullet+1],[n])$, with face maps induced by all strictly increasing functions $[i] \to [i+1]$.

This appears as the fibre of the map ε_n, which we next prove is a fibre bundle.[3]

Lemma 2.4.6 *The map $\varepsilon_n \colon \|C_n^+(M)^\bullet\| \to C_n^+(M)$ is a fibre bundle, with fibre homeomorphic to $\|\mathrm{inj}([\bullet+1],[n])\|$.*

Proof. For each level $i \geq 0$, the (unique) composition of face maps $f_i \colon C_n^+(M)^i \to C_n^+(M)$ is a finite-sheeted covering map, so in particular it is a fibre bundle. Moreover, this collection can be *simultaneously* locally trivialised: each point $c \in C_n^+(M)$ has an open neighbourhood U_c over which f_i is a trivial bundle for all i. Explicitly, we may take U_c to be the following. Choose pairwise disjoint open balls around the n points of c, and associate to these open balls the orientation inherited from c. Then let U_c be all configurations in $C_n^+(M)$ which have one point in each open ball, and whose orientation matches that of the open balls.

Over U_c, the trivialisation itself can be described as follows. Choose an arbitrary, fixed ordering of the n open balls: $(B_1,...,B_n)$. Given $a \in f_i^{-1}(U_c)$, the $(i+1)$-ordering of a induces an injection $[i+1] \to \{B_1,...,B_n\}$, and hence an element $\mathrm{ord}(a) \in \mathrm{inj}([i+1],[n])$. Define the trivialisation $f_i^{-1}(U_c) \cong U_c \times \mathrm{inj}([i+1],[n])$ to be $a \mapsto (f_i(a), \mathrm{ord}(a))$.

Since we have a simultaneous local trivialisation for $\{f_i\}$, we get a local trivialisation for the map $\coprod_i C_n^+(M)^i \times \Delta^i \to C_n^+(M)$, which identifies the preimage of U_c with $U_c \times \big(\coprod_i \mathrm{inj}([i+1],[n]) \times \Delta^i\big)$. Under this identification, the face relations for $C_n^+(M)^\bullet$ correspond exactly to the face relations for $\mathrm{inj}([\bullet+1],[n])$, since the squares

$$\begin{tikzcd}
U_c \times \mathrm{inj}([i+1],[n]) \arrow[r, phantom, "\cong"] \arrow[d, "1\times d_j"'] & f_i^{-1}(U_c) \arrow[d, "d_j"] \\
U_c \times \mathrm{inj}([i],[n]) \arrow[r, phantom, "\cong"] & f_{i-1}^{-1}(U_c)
\end{tikzcd}$$

all commute. Hence we have an induced local trivialisation of the quotient map

$$\varepsilon_n \colon \|C_n^+(M)^\bullet\| = \Big(\coprod_i C_n^+(M)^i \times \Delta^i\Big) \Big/ \sim \longrightarrow C_n^+(M),$$

which identifies the preimage of U_c with $U_c \times \|\mathrm{inj}([\bullet+1],[n])\|$. In particular, the fibre over a point is identified with $\|\mathrm{inj}([\bullet+1],[n])\|$. $\square$

[3]See also Lemma 5.5.5 of Chapter 5.

The homotopy type of $\|\mathrm{inj}([\bullet+1],[n])\|$ was identified by Randal-Williams in [RW11]:[4]

Proposition 2.4.7 (Proposition 3.2 of [RW11]) *The geometric realisation of the Δ-space $\mathrm{inj}([\bullet+1],[n])$ is a wedge of $(n-1)$-spheres:*

$$\|\mathrm{inj}([\bullet+1],[n])\| \simeq \bigvee S^{n-1}.$$

Putting this together, we immediately get:

Corollary 2.4.8 *The map $\varepsilon_n \colon \|C_n^+(M)^\bullet\| \to C_n^+(M)$ is $(n-1)$-connected. In other words $C_n^+(M)^\bullet$ is an $(n-1)$-resolution of $C_n^+(M)$.*

By the relative Hurewicz theorem and a diagram chase in (2.4.3), this in turn immediately implies that $\widetilde{H}_* C\widetilde{\varepsilon}_n = 0$ for $* \leq n$, and hence

Corollary 2.4.9 *The spectral sequence (ΔSS) converges to zero in total degree $\leq n-1$.*

2.5 The connectivity of the unpuncturing map

In this section we relate the homology-connectivity of the relative unpuncturing map

$$\widetilde{u}_n \colon R_n^+(M_1) \longrightarrow R_n^+(M)$$

(which was defined in §2.2.6.1) to the homology-connectivity of the stabilisation map

$$s_{n-1} \colon C_{n-1}^+(M) \longrightarrow C_n^+(M).$$

First, we define precisely what we mean by "homology-connectivity":

Definition 2.5.1 For a map $f \colon Y \to Z$, the *homology-connectivity* of f is

$$hconn(f) := \max\left\{ * \;\middle|\; \begin{array}{c} f \text{ is surjective on homology up to degree } * \\ f \text{ is injective on homology up to degree } *-1 \end{array} \right\}.$$

Equivalently, this is the degree up to which the reduced homology of the mapping cone Cf is zero.

Proposition 2.5.2 *For $n \geq 3$, $hconn(\widetilde{u}_n) \geq hconn(s_{n-1}) + \dim(M)$.*

To prove this we will first construct an excisive square. Let $d = \dim(M)$, and let $D \subset M$ be an open, d-dimensional disc embedded in the interior of M, far away from the boundary-component B_0 of $\bar{M}$. We identify D with the standard d-dimensional disc with its metric. Let $U_n^+(M) \subseteq C_n^+(M)$ be the subspace of configurations which have a unique closest point

[4] As noted in [RW11], this fact has been proved before in the literature, where $\mathrm{inj}([\bullet+1],[n])$ is known as the "complex of injective words".

in D to $0 \in D$. (In particular, configurations in $U_n^+(M)$ are required to *have* a point in D.) The pair $\{U_n^+(M), C_n^+(M \smallsetminus 0)\}$ is an open cover of $C_n^+(M)$, so the square

$$\begin{tikzcd}
C_n^+(M \smallsetminus 0) \arrow[rr, hook, "u_n"] & & C_n^+(M) & \\
& U_n^+(M \smallsetminus 0) \arrow[ul, hook] \arrow[rr, hook] & & U_n^+(M) \arrow[ul, hook]
\end{tikzcd} \tag{2.5.1}$$

is excisive.

Now, $U_n^+(M)$ may be decomposed as follows:

Lemma 2.5.3 *For $n \geq 3$, $U_n^+(M) \cong C_{n-1}^+(M \smallsetminus 0) \times D \times X$.*

Proof. First, choose a family of homeomorphisms $\psi_r \colon M \smallsetminus \bar{B}_r(0) \cong M \smallsetminus 0$, with support contained in D, depending continuously on the parameter $r \in [0,1)$. Here, $\bar{B}_r(0)$ means the closed ball in D, of radius r centred at $0 \in D$.

Given $\left[\begin{smallmatrix} p_1 & \cdots & p_n \\ x_1 & \cdots & x_n \end{smallmatrix}\right] \in U_n^+(M)$, we may assume by applying an even permutation (since $n \geq 3$) that the unique closest point in D to 0 for this configuration is p_n. Sending this to

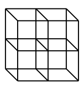
$$\left(\left[\begin{smallmatrix} \psi_{|p_n|}(p_1) & \ldots & \psi_{|p_n|}(p_{n-1}) \\ x_1 & & x_{n-1} \end{smallmatrix} \right], p_n, x_n \right) \in C_n^+(M \smallsetminus 0) \times D \times X$$

defines the required homeomorphism. $\square$

This identification restricts to $U_n^+(M \smallsetminus 0) \cong C_{n-1}^+(M \smallsetminus 0) \times (D \smallsetminus 0) \times X$, and under the identification,

- the inclusion at the bottom of (2.5.1) is the identity on the first and third factors, and the inclusion $D \smallsetminus 0 \hookrightarrow D$ on the middle factor;
- restricting the stabilisation map $s_n \colon C_n^+(M) \to C_{n+1}^+(M)$ to $U_n^+(M) \to U_{n+1}^+(M)$ yields
$$s_{n-1} \times 1 \times 1 \colon C_{n-1}^+(M \smallsetminus 0) \times D \times X \longrightarrow C_n^+(M \smallsetminus 0) \times D \times X,$$
and similarly for $s_n \colon C_n^+(M \smallsetminus 0) \to C_{n+1}^+(M \smallsetminus 0)$. In other words the identification commutes with stabilisation maps; this is because we embedded D far away from the boundary-component B_0. (More precisely, it is ensured by embedding D sufficiently far away from B_0 so that the homeomorphism $\phi \colon M' \cong M$ from the definition of the stabilisation map has support disjoint from D.)

Having done this set-up, we can now prove the main result of this section:

Proof of Proposition 2.5.2. Apply stabilisation maps vertically to the square (2.5.1), to get a commuting cube of maps, and then take mapping cones horizontally and vertically, to produce a commutative lattice of maps of the form . The back face of this can be identified as

$$
\begin{tikzcd}
R_n^+(M \smallsetminus 0) \arrow[r, "\widetilde{u}_n"] & R_n^+(M) \arrow[r, tail] & C\widetilde{u}_n \\
C_{n+1}^+(M \smallsetminus 0) \arrow[r, "u_{n+1}"] \arrow[u, tail] & C_{n+1}^+(M) \arrow[r, tail] \arrow[u, tail] & Cu_{n+1} \arrow[u, tail] \\
C_n^+(M \smallsetminus 0) \arrow[r, "u_n"'] \arrow[u, "s_n"] & C_n^+(M) \arrow[r, tail] \arrow[u, "s_n"'] & Cu_n \arrow[u]
\end{tikzcd}
$$

Using Lemma 2.5.3 and the fact that the mapping cone of $A \times Y \xrightarrow{1\times f} A \times Z$ is $C(1\times f) \cong (A_+) \wedge Cf$, the front face can be identified as

$$
\begin{tikzcd}
C(s_{n-1} \times 1 \times 1) \arrow[r] & C(s_{n-1} \times 1 \times 1) \arrow[r, tail] & C\Sigma^d(s_{n-1} \times 1)_+ \\
C_n^+(M \smallsetminus 0) \times (D \smallsetminus 0) \times X \arrow[r, hook] \arrow[u, tail] & C_n^+(M \smallsetminus 0) \times D \times X \arrow[r, tail] \arrow[u, tail] & \Sigma^d(C_n^+(M \smallsetminus 0) \times X)_+ \arrow[u, tail] \\
C_{n-1}^+(M \smallsetminus 0) \times (D \smallsetminus 0) \times X \arrow[r, hook] \arrow[u, "s_{n-1} \times 1 \times 1"] & C_{n-1}^+(M \smallsetminus 0) \times D \times X \arrow[r, tail] \arrow[u, "s_{n-1} \times 1 \times 1"'] & \Sigma^d(C_{n-1}^+(M \smallsetminus 0) \times X)_+ \arrow[u, "\Sigma^d(s_{n-1} \times 1)_+"']
\end{tikzcd}
$$

Now, one way of stating the excision theorem is that the map-on-mapping-cones induced by an excisive square is a homology-equivalence. Hence the homology of the right-hand columns of the two diagrams above is the same; in particular, $\widetilde{H}_* C\widetilde{u}_n \cong \widetilde{H}_* C\Sigma^d(s_{n-1} \times 1)_+$. So:

$$
\begin{aligned}
hconn(\widetilde{u}_n) &= hconn\big(\Sigma^d(s_{n-1} \times 1)_+\big) \\
&= d + hconn(s_{n-1} \times 1)_+ && \text{by the suspension isomorphism} \\
&= d + hconn(s_{n-1} \times 1) \\
&\geq d + hconn(s_{n-1}) && \text{by the Künneth theorem.} \qquad \square
\end{aligned}
$$

2.6 Proof of the Main Theorem

We now apply the constructions and results of the previous two sections to prove the Main Theorem. This can be rephrased in terms of relative configuration spaces (as defined in §2.2.5):

Main Theorem *If M is the interior of a connected manifold-with-boundary of dimension at least 2, and X is a path-connected space, then*

$$\widetilde{H}_* R_n^+(M, X) = 0 \quad \textit{for} \quad * \leq \tfrac{n-2}{3}. \tag{2.6.1}$$

2.6.1 Strategy of the proof

We defined in §2.2.6.2 the "relative double stabilisation map"

$$\widetilde{s}^2_{(132)} \colon R^+_{n-2}(M,X) \longrightarrow R^+_n(M,X).$$

The proof will be by induction on n, and the idea is to show, using the inductive hypothesis, that this map is both *surjective* and the *zero-map* on homology, up to the required degree. We will use completely different factorisations of $\widetilde{s}^2_{(132)}$ for each of these. The first will allow us to prove surjectivity-on-homology *piece by piece*, using different methods for the different pieces of the factorisation, and the second (which only exists on homology) will turn out to factor through the zero map in the required range of degrees.

Proof of the Main Theorem, by induction on n. Since M and X are path-connected and $\dim(M) \geq 2$, $C^+_n(M,X)$ is path-connected for all n, and hence so is $R^+_n(M,X)$. So the theorem is true for $n \leq 4$ — this is the base case.

Now assume $n \geq 5$. By Lemmas 2.6.1 and 2.6.5 below, the map

$$(\widetilde{s}^2_{(132)})_* \colon \widetilde{H}_* R^+_{n-2}(M,X) \longrightarrow \widetilde{H}_* R^+_n(M,X)$$

is surjective and zero for $* \leq \frac{n-2}{3}$. Hence $\widetilde{H}_* R^+_n(M,X) = 0$ in this range. □

Of course the main content of the proof is contained in the proofs of Lemmas 2.6.1 and 2.6.5 below. We begin with the one asserting *surjectivity* of $(\widetilde{s}^2_{(132)})_*$ for $* \leq \frac{n-2}{3}$.

2.6.2 Surjectivity on homology

As noted in Remark 2.2.20, $\widetilde{s}^2_{(132)}$ factorises into

$$\begin{tikzcd}
R^+_{n-2}(M) \arrow[r, "\widetilde{p}_{n-2}"] & R^+_{n-2}(M_1) \arrow[r, "\widetilde{\jmath}_{n-1,0}"] & R^+_{n-1}(M)^0 \arrow[r, "\widetilde{a}_{n-1}"] & R^+_{n-1}(M) \arrow[dlll, "="] \\
R^+_{n-1}(M) \arrow[r, "\widetilde{p}_{n-1}"'] & R^+_{n-1}(M_1) \arrow[r, "\widetilde{\jmath}_{n,0}"'] & R^+_n(M)^0 \arrow[r, "\widetilde{a}_n"'] & R^+_n(M)
\end{tikzcd} \tag{2.6.2}$$

which is the mapping cone construction applied to

$$\begin{tikzcd}
\cdot \arrow[r, "p_{n-1}"] & \cdot \arrow[r, "j_{n,0}"] & \cdot \arrow[r, "a_n"] & \cdot \arrow[r, "p_n"] & \cdot \arrow[r, "j_{n+1,0}"] & \cdot \arrow[r, "a_{n+1}"] & \cdot \\
\cdot \arrow[u] \arrow[r, "p_{n-2}"'] \arrow[ur, phantom, "(12)_p"] & \cdot \arrow[u] \arrow[r, "j_{n-1,0}"'] \arrow[ur, phantom, "1"] & \cdot \arrow[u] \arrow[r, "a_{n-1}"'] \arrow[ur, phantom, "1"] & \cdot \arrow[u] \arrow[r, "p_{n-1}"'] \arrow[ur, phantom, "(12)_p"] & \cdot \arrow[u] \arrow[r, "j_{n,0}"'] \arrow[ur, phantom, "1"] & \cdot \arrow[u] \arrow[r, "a_n"'] \arrow[ur, phantom, "1"] & \cdot \arrow[u]
\end{tikzcd} \tag{2.6.3}$$

where vertical maps are stabilisation maps. Recall that p punctures the manifold, j replaces the puncture by a new configuration point which is marked as special, and a forgets which point is special. This is the factorisation we will use to show surjectivity-on-homology.

Lemma 2.6.1 *Let $n \geq 5$, and assume as an inductive hypothesis that* (2.6.1) *holds for smaller values of n. Then $\widetilde{s}^2_{(132)}$ is surjective on homology up to degree $\frac{n-2}{3}$.*

Proof. We will show that the six maps in (2.6.2) are each surjective on homology up to this degree.

The relative puncturing maps $\widetilde{p}_{n-1}$ and $\widetilde{p}_{n-2}$. Recall from §2.5 that

$$\mathit{h}\mathrm{conn}(f) \coloneqq \max\left\{ * \;\middle|\; \begin{array}{c} f \text{ is surjective on homology up to degree } * \\ f \text{ is injective on homology up to degree } * - 1 \end{array} \right\}.$$

In this notation the inductive hypothesis is

$$\mathit{h}\mathrm{conn}(s_{n'}) \geq \lfloor \tfrac{n'-2}{3} \rfloor, \; \forall n' < n.$$

As noted in Remark 2.2.21, $\widetilde{u}_r \circ \widetilde{p}_r$ is homotopic to the identity, so $(\widetilde{u}_r)_* \circ (\widetilde{p}_r)_* = \mathrm{id}$. Hence $(\widetilde{u}_r)_*$ is injective up to the same degree to which $(\widetilde{p}_r)_*$ is surjective, so $\mathit{h}\mathrm{conn}(\widetilde{p}_r) = \mathit{h}\mathrm{conn}(\widetilde{u}_r) - 1$. Combining this with Proposition 2.5.2 we have

$$\mathit{h}\mathrm{conn}(\widetilde{p}_r) \geq \mathit{h}\mathrm{conn}(s_{r-1}) + \dim(M) - 1,$$

for $r \geq 3$. Using the inductive hypothesis and the fact that $\dim(M) \geq 2$ we obtain

$$\mathit{h}\mathrm{conn}(\widetilde{p}_{n-1}) \geq \lfloor \tfrac{n-1}{3} \rfloor \qquad \text{and} \qquad \mathit{h}\mathrm{conn}(\widetilde{p}_{n-2}) \geq \lfloor \tfrac{n-2}{3} \rfloor.$$

The relative inclusion-of-the-fibre maps $\widetilde{\jmath}_{n,0}$ and $\widetilde{\jmath}_{n-1,0}$. Recall the spectral sequence

$$E^2_{s,t} \;\cong\; H_s\big(\widetilde{C}_{i+1}(M); \widetilde{H}_t(R^+_{n-i-1}(M_{i+1}))\big) \quad \Rightarrow \quad \widetilde{H}_*(R^+_n(M)^i) \tag{RSSS$_i$}$$

from Proposition 2.4.4. The edge homomorphism

$$\widetilde{H}_t(R^+_{n-i-1}(M_{i+1})) \cong E^2_{0,t} \twoheadrightarrow E^\infty_{0,t} \hookrightarrow \widetilde{H}_t(R^+_n(M)^i)$$

is the map on $\widetilde{H}_t$ induced by $\widetilde{\jmath}_{n,i}$.

Now, the inductive hypothesis implies that $E^2_{s,t} = 0$ for $t \leq \frac{n-i-3}{3}$, so the E^2 page is as illustrated in Figure 2.6.1(a). Hence in degrees $t \leq \frac{n-i-3}{3}$ the map $\widetilde{\jmath}_{n,i}$ induces $0 \to 0$ on $\widetilde{H}_t$, which is trivially surjective. Moreover in the larger range $t \leq \frac{n-i}{3}$ we can see from Figure 2.6.1(a) that the inclusion $E^\infty_{0,t} \hookrightarrow \widetilde{H}_t(R^+_n(M)^i)$ is an isomorphism, so $\widetilde{\jmath}_{n,i}$ still induces a surjection on $\widetilde{H}_t$. Setting $i = 0$, this proves that $\widetilde{\jmath}_{n,0}$ is surjective on homology up to degree $\frac{n}{3}$. The argument goes through identically when n is replaced by $n-1$, and proves that $\widetilde{\jmath}_{n-1,0}$ is surjective on homology up to degree $\frac{n-1}{3}$.

The relative augmentation maps $\widetilde{a}_n$ and $\widetilde{a}_{n-1}$. Recall the spectral sequence

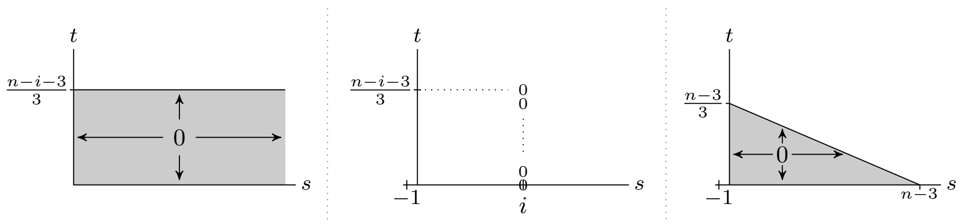
$$E^1_{s,t} \;\cong\; \widetilde{H}_t(R_n^+(M)^s) \quad\Rightarrow\quad \widetilde{H}_{*+1}C\widetilde{\varepsilon}_n \tag{ΔSS}$$

from Proposition 2.4.4. The differential $E^1_{-1,t} \longleftarrow E^1_{0,t}$ is the map on $\widetilde{H}_t$ induced by $\widetilde{a}_n$.

Now, as noted above, the spectral sequence (RSSS$_i$) has E^2 page as illustrated in Figure 2.6.1(a) — hence it converges to zero in total degree up to $\frac{n-i-3}{3}$. The limit of (RSSS$_i$) is the ith column of the E^1 page of (ΔSS), so we have a column of zeros on the E^1 page of (ΔSS) as shown in Figure 2.6.1(b). There is a spectral sequence (RSSS$_i$) for each $0 \leq i \leq n-3$, so there is a triangle of zeros on the E^1 page of (ΔSS) as shown in Figure 2.6.1(c).

Now assume that $t-1 \leq \frac{n-4}{3}$. Looking at Figure 2.6.1(c) we see that the *first* differential is the only possible non-trivial differential hitting $E^1_{-1,t}$. Also, by Corollary 2.4.9, the spectral sequence (ΔSS) converges to zero in total degree $\leq \frac{n-4}{3} \leq n-1$, so we have $E^\infty_{-1,t} = 0$. Hence the first differential $E^1_{-1,t} \longleftarrow E^1_{0,t}$ must be surjective.

So $\widetilde{a}_n$ induces surjections on $\widetilde{H}_t$ for $t-1 \leq \frac{n-4}{3}$, i.e. for $t \leq \frac{n-1}{3}$. The argument goes through identically when n is replaced by $n-1$, and proves that $\widetilde{a}_{n-1}$ induces surjections on $\widetilde{H}_t$ for $t \leq \frac{n-2}{3}$. $\square$



(a) (b) (c)

Figure 2.6.1: The two spectral sequences from the proof of surjectivity: (a) is the E^2 page of (RSSS$_i$); (b) and (c) are the E^1 page of (ΔSS).

2.6.3 Zero on homology

The factorisation of $\widetilde{s}^2_{(132)}$ (on homology) we will use for this part comes from a more general factorisation lemma, so we begin by stating this.

2.6.3.1 A general factorisation lemma.

As before, we let Cf denote the mapping cone of a map f. Suppose we have a homotopy-commutative square of maps

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "j"'] & Y \end{tikzcd} \tag{$\mathcal{S}$}$$

Choosing any particular homotopy $H\colon g \circ i \simeq j \circ f$ to fill this square induces a map $CH\colon Ci \to Cj$, and completes an exact 'ladder' on homology

$$\begin{tikzcd} \cdots \arrow[r] & \widetilde{H}_*X \arrow[r] \arrow[d, "g_*"] & \widetilde{H}_*Ci \arrow[r] \arrow[d, "CH_*"] & \widetilde{H}_{*-1}A \arrow[r] \arrow[d, "f_*"] & \cdots \\ \cdots \arrow[r] & \widetilde{H}_*Y \arrow[r] & \widetilde{H}_*Cj \arrow[r] & \widetilde{H}_{*-1}B \arrow[r] & \cdots \end{tikzcd} \tag{2.6.4}$$

We say that ($\mathcal{S}$) *splits into triangles* if there exists a map $d\colon X \to B$, together with homotopies $F_1\colon d \circ i \simeq f$, $F_2\colon g \simeq j \circ d$. In other words the square can be filled in as

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "\simeq" near start] & X \arrow[d, "g"] \arrow[dl] \arrow[dl, phantom, "\simeq" near end, shift right=4] \\ B \arrow[r, "j"'] & Y \end{tikzcd} \tag{2.6.5}$$

Lemma 2.6.2 ("Factorisation lemma") *If the square* ($\mathcal{S}$) *splits into triangles, and H is any homotopy filling this square, then CH_* factors through a map $z_H\colon \widetilde{H}_{*-1}A \to \widetilde{H}_*Y$ in diagram* (2.6.4). *Hence, in particular, the composition $\widetilde{H}_*X \to \widetilde{H}_*Cj$ in* (2.6.4) *is zero.*

Moreover, z_H itself factorises as follows:

$$\widetilde{H}_{*-1}A \hookrightarrow \widetilde{H}_*(S^1 \times A) \xrightarrow{\gamma_*} \widetilde{H}_*Y,$$

where the first map is the inclusion of a direct summand in the Künneth decomposition $\widetilde{H}_*(S^1 \times A) \cong \widetilde{H}_{*-1}(A) \oplus H_{*-1}(pt) \oplus \widetilde{H}_*(A)$, and the second map is induced by the self-homotopy $\gamma\colon S^1 \times A \to Y$ built out of H and the two homotopies F_1 and F_2 occurring in (2.6.5).

Proof. See Appendix 2.A.

2.6.3.2 Applying the factorisation lemma.

In particular we may take ($\mathcal{S}$) to be the square

$$\begin{tikzcd} C^+_{n-2}(M) \arrow[r, "s"] \arrow[d, "-s^2"'] & C^+_{n-1}(M) \arrow[d, "-s^2"] \\ C^+_n(M) \arrow[r, "s"'] & C^+_{n+1}(M) \end{tikzcd}$$

(for $n \geq 3$). This is the right-hand square from (2.2.2). It splits into triangles, since we may for example take the diagonal map to be $-s\colon C^+_{n-1}(M) \to C^+_n(M)$, and the two homotopies to be constant. (See also Remark 2.2.20.) Taking H to be the homotopy (132), as defined in §2.2.6.2, Lemma 2.6.2 implies the following factorisation of $(\widetilde{s}^2_{(132)})_*$:

Corollary 2.6.3 *The map* $(\widetilde{s}^2_{(132)})_*\colon \widetilde{H}_*R^+_{n-2}(M) \to \widetilde{H}_*R^+_n(M)$ *factorises as follows*:

$$\widetilde{H}_*R^+_{n-2}(M) \to \widetilde{H}_{*-1}C^+_{n-2}(M) \hookrightarrow \widetilde{H}_*(S^1 \times C^+_{n-2}(M)) \xrightarrow{\gamma_*} \widetilde{H}_*C^+_{n+1}(M) \to \widetilde{H}_*R^+_n(M).$$

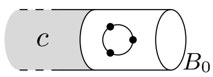
The first and last maps come from the long exact sequences for $C^+_{n-2}(M) \to C^+_{n-1}(M)$ *and* $C^+_n(M) \to C^+_{n+1}(M)$ *respectively, the second map comes from the Künneth decomposition of* $\widetilde{H}_*\big(S^1 \times C^+_{n-2}(M)\big)$, *and*

$$\gamma\colon S^1 \times C^+_{n-2}(M) \to C^+_{n+1}(M)$$

is the self-homotopy (132).

Proof. This is immediate from Lemma 2.6.2, once we note that in this case we can take the split homotopy (2.6.5) to be the *constant* homotopy, so that γ is just $H = (132)$. □

Rephrasing the definition of the homotopy (132) in §2.2.6.2, we may describe γ, as a map $S^1 \times C^+_{n-2}(M) \to C^+_{n+1}(M)$, concretely as follows:

$(t, c) \quad \mapsto$ 

The configuration c is pushed away from the chosen boundary-component B_0, and three new points are added on a small embedded circle near B_0, at the positions $\{t^{1/3}, \omega t^{1/3}, \omega^2 t^{1/3}\}$ where $\omega = \exp(\frac{2}{3}\pi i)$. Fix an orientation of the circle: this gives the three new points a cyclic ordering $[p_1, p_2, p_3]$, and we use the orientation convention $[c, p_1, p_2, p_3]$.

We can use this description to check that γ is natural w.r.t. stabilisation maps:

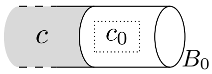
Lemma 2.6.4 *The following square is commutative up to homotopy*:

$$\begin{tikzcd} S^1 \times C^+_{n-2}(M) \arrow[r, "\gamma"] & C^+_{n+1}(M) \\ S^1 \times C^+_{n-3}(M) \arrow[r, "\gamma"'] \arrow[u, "1 \times (-s)"] & C^+_n(M) \arrow[u, "s"'] \end{tikzcd}$$

Proof. The two ways around this square are both of the form

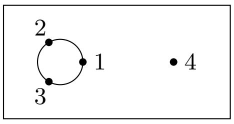
$$S^1 \times C^+_{n-3}(M) \xrightarrow{?\times 1} C^+_4(\mathbb{R}^d) \times C^+_{n-3}(M) \longrightarrow C^+_{n+1}(M),$$

where the second map is

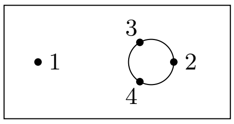
$(c_0, c) \quad \mapsto$ 

Here, the configuration c is pushed away from B_0, and the configuration c_0 is inserted into a coordinate neighbourhood near B_0 (and we use the orientation convention $[c, c_0]$).

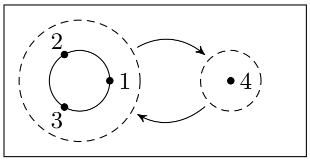
The map '?': $S^1 \longrightarrow C^+_4(\mathbb{R}^d)$ is either



or $-$



(the numberings represent orientations of the configurations; the '$-$' in the right diagram indicates that the orientation should in fact be the *opposite* of that which is illustrated). So it is enough to find a homotopy $h\colon S^1 \times I \longrightarrow C^+_4(\mathbb{R}^d)$ connecting these two maps. Such a homotopy clearly does exist: for example define $h(t, u)$ to be



where $t \in S^1$ determines the positions of the 3 points on the circle, and $u \in I$ determines how far along the arrows to move the dotted regions. $\square$

2.6.3.3 Zero on homology.

Finally, we may apply our new factorisation of $(\widetilde{s}^2_{(132)})_*$ to deduce that it is zero in the required range:

Lemma 2.6.5 *Let $n \geq 4$, and assume as an inductive hypothesis that* (2.6.1) *holds for smaller values of n. Then $\widetilde{s}^2_{(132)}$ is the zero map on (reduced) homology up to degree $\frac{n-2}{3}$.*

Proof. By Corollary 2.6.3, Lemma 2.6.4 and the naturality of the Künneth decomposition we have a commutative diagram

$$\begin{tikzcd}
\widetilde{H}_* R^+_{n-2}(M) \arrow[r] & \widetilde{H}_{*-1} C^+_{n-2}(M) \arrow[r] & \widetilde{H}_* C^+_{n+1}(M) \arrow[r] & \widetilde{H}_* R^+_n(M) \\
 & \widetilde{H}_{*-1} C^+_{n-3}(M) \arrow[u, "(-s)_*"] \arrow[r] & \widetilde{H}_* C^+_n(M) \arrow[u, "s_*"'] \arrow[ur, "0"'] &
\end{tikzcd}$$

where the composition along the top row is $(\widetilde{s}^2_{(132)})_*$. The composition on the right is zero since it is induced by a cofibration sequence. By definition, the maps $\pm s$ differ only by an automorphism of their common codomain, so (as noted in Remark 2.2.7) they have the same surjectivity-on-homology properties. Hence by the inductive hypothesis $(-s)_*$ is surjective for $* - 1 \leq \frac{n-5}{3}$, i.e. for $* \leq \frac{n-2}{3}$. So in this range $(\widetilde{s}^2_{(132)})_*$ factors through the zero map, and hence is itself zero. $\square$

2.7 Corollaries

2.7.1 Stability for generalised homology theories

First we will prove Corollary B (stated in §2.1.3). This follows directly from the Main Theorem and the following lemma:

Lemma 2.7.1 *If h_* is a connective generalised homology theory with connectivity c (i.e. its associated spectrum has connectivity c), and if the map $f\colon X \to Y$ is an isomorphism on $H_*(-;\mathbb{Z})$ up to degree $k-1$ and surjective up to degree k, then f is an isomorphism on h_* up to degree $k-1+c$ and surjective on h_* up to degree $k+c$.*

Proof. By the long exact sequence for cofibration sequences, this is equivalent to the claim that

$$\widetilde{H}_*(Cf;\mathbb{Z}) = 0 \ \forall * \leq k \quad \Rightarrow \quad \widetilde{h}_*(Cf) = 0 \ \forall * \leq k+c.$$

If E is the spectrum associated to h_*, then we have the Atiyah-Hirzebruch spectral sequence (see [McC01, Theorem 11.16])

$$E^2_{s,t} \cong H_s(Cf;\pi_t(E)) \quad \Rightarrow \quad h_*(Cf).$$

Removing an $H_s(pt;\pi_t(E))$ summand from the E^2 page, and correspondingly an $h_*(pt) = \pi_*(E)$ summand from the limit, gives the reduced version

$$E^2_{s,t} \cong \widetilde{H}_s(Cf;\pi_t(E)) \quad \Rightarrow \quad \widetilde{h}_*(Cf).$$

By the Universal Coefficient Theorem, and since E is c-connected, the E^2 page is zero for $s \leq k$ or $t \leq c$. Therefore the limit is zero for total degrees $* \leq k+c$. $\square$

Remark 2.7.2 Alternatively, one could consider the map of (non-reduced) Atiyah-Hirzebruch spectral sequences induced by f, and apply the Zeeman comparison theorem [Zee57].

We now revert to talking only about ordinary homology again, but of course the corollaries for sequences of groups below also have similar generalised homology versions.

2.7.2 Wreath products with alternating braid groups

Let S be the interior of a connected surface-with-boundary $\overline{S}$, and let G be any discrete group.

Definition 2.7.3 The *braid group on n strands on S* is $\beta_n^S := \pi_1 C_n(S, pt)$. When $S = \mathbb{R}^2$ this recovers the definition of the Artin braid group β_n (by [FN62b]). A based loop in $C_n(S, pt)$ induces a permutation of the basepoint configuration, so there is a natural projection $\beta_n^S \twoheadrightarrow \Sigma_n$. The *alternating braid group on n strands on S*, $A\beta_n^S$, is defined to be the index-2 subgroup of braids whose induced permutation is even. A loop in $C_n(S, pt)$ induces an even permutation iff it lifts to a loop in $C_n^+(S, pt)$, so this is equivalent to defining $A\beta_n^S := \pi_1 C_n^+(S, pt)$.

The wreath product $G \wr A\beta_n^S$ is defined to be the semi-direct product

$$1 \to G^n \hookrightarrow G^n \rtimes A\beta_n^S \twoheadrightarrow A\beta_n^S \to 1 \tag{2.7.1}$$

where $A\beta_n^S$ acts on G^n by permuting the n factors through its projection to $A_n \leq \Sigma_n$.

The first half of Corollary A (see §2.1.3) follows directly from the Main Theorem and the following lemma:

Lemma 2.7.4 *Pick a model for the classifying space BG. Then $C_n^+(S, BG)$ is a model for the classifying space $B(G \wr A\beta_n^S)$.*

Proof. First we show that $C_n^+(S, BG)$ is aspherical. In the case where $\overline{S}$ is compact, using the classification of compact connected surfaces-with-boundary we can draw an explicit deformation retraction from $\overline{S}$ onto a wedge of circles, so it is aspherical. In general, any map of a sphere into $\overline{S}$ will have its image contained in a *compact* connected subsurface-with-boundary, so $\overline{S}$ is also aspherical without the compactness assumption. Hence S is aspherical. Moreover, $S \smallsetminus \{\text{finitely many points}\}$ is again the interior of a connected surface-with-boundary, and so is also aspherical by the previous argument.

Via the fibration sequences

$$S_{n-1} \times BG \longrightarrow \widetilde{C}_n(S, BG) \longrightarrow \widetilde{C}_{n-1}(S, BG)$$

and induction on n, this implies that $\widetilde{C}_n(S, BG)$ is aspherical for all n. This is a covering space of $C_n^+(S, BG)$, so $C_n^+(S, BG)$ is also aspherical for all n.

Now we check that $\pi_1 C_n^+(S, BG) \cong G \wr A\beta_n^S$. Forgetting the labels gives a fibration

$$(BG)^n \hookrightarrow C_n^+(S, BG) \xrightarrow{\textit{forget}} C_n^+(S, pt),$$

which admits a section. So on π_1 this induces a split short exact sequence

$$1 \to G^n \hookrightarrow G^n \rtimes A\beta_n^S \twoheadrightarrow A\beta_n^S \to 1.$$

It remains to show that this is the *same* semi-direct product as $G \wr A\beta_n^S$, (2.7.1). This can be seen most easily by just thinking about what concatenation of based loops in $C_n^+(S, BG)$ does under this identification: it concatenates the corresponding braids, and multiplies the elements of G in pairs, according to which strands have been glued together. So the multiplication in the G^n component is twisted by the induced permutation coming from the $A\beta_n^S$ component. $\square$

Example 2.7.5 A special case of the first half of Corollary A, taking $S = \mathbb{R}^2$ and $G = *$, is homological stability for the *alternating Artin braid groups*, an index-2 subfamily of the sequence of Artin braid groups. Another special case, taking $S = \mathbb{R}^2$ and $G = \mathbb{Z}$, is homological stability for the sequence of *alternating ribbon braid groups*.

Remark 2.7.6 The elements of $G \wr A\beta_n^S$ can be thought of as braids embedded in $S \times I$, with an element of G 'attached' to each strand. In this description the "natural map" $G \wr A\beta_n^S \to G \wr A\beta_{n+1}^S$ referred to in the statement of Corollary A is given by adding a new strand (with the identity of G attached) near a chosen boundary-component of $\overline{S}$.

2.7.3 Wreath products with alternating groups

We now want to take configurations in the 'manifold' $M = \mathbb{R}^\infty$:

Corollary 2.7.7 *For any path-connected space X, the map*

$$s\colon C_n^+(\mathbb{R}^\infty, X) \longrightarrow C_{n+1}^+(\mathbb{R}^\infty, X)$$

is an isomorphism on homology up to degree $\frac{n-5}{3}$ and surjective up to degree $\frac{n-2}{3}$.

Proof. By the Main Theorem, the analogous statement is true for

$$C_n^+(\mathbb{R}^N, X) \longrightarrow C_{n+1}^+(\mathbb{R}^N, X)$$

for all N. These fit into a commutative ladder of maps, where the vertical maps are induced by the standard inclusions $\mathbb{R}^N \hookrightarrow \mathbb{R}^{N+1}$, and the map we are interested in is the vertical colimit of this ladder. Injectivity- and surjectivity-on-H_* properties of the horizontal maps are preserved under taking this colimit, so the result follows. $\square$

Remark 2.7.8 This corollary depends on having an explicit range for homological stability which is *independent* of the manifold M. If we only knew qualitatively that homological stability held for *some* (unknown) range, then we would not have been able to take a direct limit and keep homological stability, as we did in the proof above. (A priori, the stability slope could $\to 0$ as the dimension of $M \to \infty$, for example.)

Remark 2.7.9 We note that $\mathrm{inj}([n], \mathbb{R}^\infty)$ is contractible, and the action of A_n on it is free, so it is a model for EA_n. This means that the oriented configuration space on $\mathbb{R}^\infty$ with

X-labels is a model for the homotopy quotient, or Borel construction

$$C_n^+(\mathbb{R}^\infty, X) \;=\; \mathrm{inj}([n], \mathbb{R}^\infty) \times_{A_n} X^n \;\simeq\; EA_n \times_{A_n} X^n \;=\; X^n /\!\!/ A_n.$$

So by Corollary 2.7.7 we have homological stability for the sequence

$$\cdots \to X^n /\!\!/ A_n \to X^{n+1} /\!\!/ A_{n+1} \to \cdots .$$

In the special case $X = BG$, we have the following:

Corollary 2.7.10 (Second half of Corollary A) *For any discrete group G, the natural map $G \wr A_n \to G \wr A_{n+1}$ is an isomorphism on homology up to degree $\frac{n-5}{3}$ and surjective up to degree $\frac{n-2}{3}$.*

Here the wreath product $G \wr A_n$ is the semi-direct product

$$1 \to G^n \hookrightarrow G^n \rtimes A_n \twoheadrightarrow A_n \to 1 \tag{2.7.2}$$

where A_n acts by permuting the n factors of G^n.

Proof. By Corollary 2.7.7 we just need to show that $C_n^+(\mathbb{R}^\infty, BG)$ is a model for the classifying space $B(G \wr A_n)$. Now, $\mathbb{R}^\infty \smallsetminus \{\text{finitely many points}\}$ is contractible, so by considering the fibration sequences

$$(\mathbb{R}^\infty \smallsetminus \{n-1 \text{ points}\}) \times BG \hookrightarrow \widetilde{C}_n(\mathbb{R}^\infty, BG) \twoheadrightarrow \widetilde{C}_{n-1}(\mathbb{R}^\infty, BG)$$

we can inductively show that $\widetilde{C}_n(\mathbb{R}^\infty, BG)$, and hence also $C_n^+(\mathbb{R}^\infty, BG)$, is aspherical for all n.

To show that $\pi_1 C_n^+(\mathbb{R}^\infty, BG) \cong G \wr A_n$, we first consider $\pi_1 C_n^+(\mathbb{R}^\infty, pt)$. A based loop (up to $\simeq$) in $C_n^+(\mathbb{R}^\infty, pt)$ is an n-strand braid on $\mathbb{R}^\infty$. Any braid in $\mathbb{R}^\infty$ can be 'untangled', so it is just a permutation of the basepoint configuration, which in this case must be *even* to preserve the orientation. So $\pi_1 C_n^+(\mathbb{R}^\infty, pt) \cong A_n$. As in the proof of Lemma 2.7.4 we have a fibration

$$(BG)^n \hookrightarrow C_n^+(\mathbb{R}^\infty, BG) \xrightarrow{\textit{forget}} C_n^+(\mathbb{R}^\infty, pt),$$

which admits a section, so on π_1 we have a split short exact sequence

$$1 \to G^n \hookrightarrow G^n \rtimes A_n \twoheadrightarrow A_n \to 1.$$

By considering what concatenation of based loops in $C_n^+(\mathbb{R}^\infty, BG)$ does under this identification, we can see that the action of A_n on G^n in this semi-direct product is just permutation of the n factors, as in (2.7.2). Hence $\pi_1 C_n^+(\mathbb{R}^\infty, BG) \cong G \wr A_n$. □

2.7.4 Aside: the limiting spaces for A_n and $A\beta_n$.[5]

When $S=\mathbb{R}^2$ we denote the alternating braid group $A\beta_n^{\mathbb{R}^2}$ by just $A\beta_n$.

Corollary A relates the homology of the families of groups (A_n) and $(A\beta_n)$, in the stable range, to the homology of the *limiting spaces* BA_∞^+ and $BA\beta_\infty^+$, where $G_\infty = \lim_n G_n$ and $(\cdot)^+$ is the Quillen plus-construction. In these two cases we can identify the limiting spaces explicitly: The 'scanning' argument of Segal and McDuff implies [McD75, Theorem 4.5] that

$$B\Sigma_\infty^+ \simeq \Omega_0^\infty S^\infty = \mathcal{Q}_0 S^0 \qquad \text{and} \qquad B\beta_\infty^+ \simeq \Omega_0^2 S^2 \simeq \Omega^2 S^3. \tag{2.7.3}$$

(The first of these is the Barratt-Priddy-Quillen theorem [BP72].) Plus-constructing preserves double-covering maps (see for example [Ber82, Theorem 6.4]), so

$$BA_\infty^+ \simeq \widetilde{\mathcal{Q}_0 S^0} \qquad \text{and} \qquad BA\beta_\infty^+ \simeq \widetilde{\Omega^2 S^3}, \tag{2.7.4}$$

the universal cover of $\mathcal{Q}_0 S^0$ and the unique connected double cover of $\Omega^2 S^3$. Let Cob_n denote the category of $(n-1)$-dimensional manifolds and n-dimensional cobordisms between them (embedded in $\mathbb{R}^\infty$), as defined and studied in [GMTW09], and let $\mathrm{Cob}_n(\mathbb{R}^2)$ denote the version with embeddings into $\mathbb{R}^2$. In this language (2.7.3) can be reinterpreted (by the group-completion theorem) as

$$\Omega B\mathrm{Cob}_0 \simeq \mathcal{Q}S^0 \qquad \text{and} \qquad \Omega B\mathrm{Cob}_0(\mathbb{R}^2) \simeq \Omega^2 S^2. \tag{2.7.5}$$

Now if Cob_0^+, $\mathrm{Cob}_0^+(\mathbb{R}^2)$ denote the corresponding (embedded) cobordism categories where 0-manifolds have an ordering-up-to-even-permutations (this is a non-tangential, i.e. 'global', structure), then by the group-completion theorem (2.7.4) becomes

$$\Omega B\mathrm{Cob}_0^+ \simeq \widetilde{\mathcal{Q}S^0} \qquad \text{and} \qquad \Omega B\mathrm{Cob}_0^+(\mathbb{R}^2) \simeq \widetilde{\Omega^2 S^2}, \tag{2.7.6}$$

where we are taking double covers componentwise.

So in a very special case, and up to delooping once, this identifies the homotopy type of a cobordism category of manifolds with some kind of *non-local* structure.

2.8 Failure of injectivity

In this section we elaborate on one way in which the oriented case is harder to deal with than the unordered case: the failure of the stabilisation maps to be injective on homology in general. In §2.8.1 we recall how injectivity-on-homology can be proved in the unordered case, and in §2.8.2 explain why the analogous argument breaks down in the oriented case. Then in

[5] See also §1.2 of the Introduction.

§2.8.3 we give some explicit examples demonstrating non-injectivity of $s_* \colon H_* C_n^+(M,X) \longrightarrow H_* C_{n+1}^+(M,X)$.

2.8.1 Injectivity in the unordered case

The stabilisation maps s are split-injective on homology in all degrees in the case of unordered configuration spaces. This can be shown with the help of the following lemma proved by Dold (and used earlier by Nakaoka in [Nak60]):

Lemma 2.8.1 (Lemma 2 of [Dol62]) *Given a sequence of abelian groups and homomorphisms* $0 \to A_1 \xrightarrow{s_1} A_2 \xrightarrow{s_2} \cdots$, *if there are 'transfer' maps* $\tau_{k,n} \colon A_n \to A_k$ $(1 \leq k \leq n)$ *satisfying*

$$\tau_{n,n} = \mathrm{id} \qquad \textit{and} \qquad \tau_{k,n} = \tau_{k,n+1} \circ s_n \mod \mathrm{im}(s_{k-1}),$$

then every s_n *is split-injective.*

If the abelian groups are in fact $\mathbb{Q}$-vector spaces, then it suffices to find transfer maps going back just one step:

Corollary 2.8.2 *Given a sequence of* $\mathbb{Q}$*-vector spaces* $0 \to A_1 \xrightarrow{s_1} A_2 \xrightarrow{s_2} \cdots$, *if there are 'transfer' maps* $t_n \colon A_n \to A_{n-1}$ $(n \geq 1)$ *satisfying*

$$t_{n+1} \circ s_n = \mathrm{id} + s_{n-1} \circ t_n,$$

then every s_n *is split-injective.*

Proof. Define $\tau_{k,n} := \frac{1}{(n-k)!} t_{k+1} \circ \cdots \circ t_n$ for $1 \leq k < n$, so that $\tau_{k,n+1} \circ s_n = \tau_{k,n} + s_{k-1} \circ \tau_{k-1,n}$, and apply Lemma 2.8.1. □

Lemma 2.8.1 can be applied to prove injectivity of s_* in the unordered case by defining

$$\tau_{k,n} \colon SP^\infty C_n(M,X) \longrightarrow SP^\infty C_k(M,X)$$

to take an n-point configuration in M to the formal sum of its $\binom{n}{k}$ different k-point subsets (cf. the proof of Theorem 4.5 in [McD75]). This uses the Dold-Thom theorem: $\pi_* SP^\infty \cong H_*$ for $* \geq 1$ [DT58].

2.8.2 Failure of injectivity in the oriented case

This trick doesn't work for *oriented* configuration spaces, however, since there is no way for an oriented n-point configuration to induce an orientation on a k-point subset unless $k = n-1$. If we instead define $\tau_{k,n}$ to take an oriented n-point configuration to the sum of all its oriented k-point subsets — with *either* orientation — then $\tau_{n,n} = \mathrm{id} + \nu$, so the first hypothesis of Lemma 2.8.1 is not satisfied.

Alternatively, we could try to just prove injectivity on rational homology using Corollary 2.8.2, since this only requires maps removing a *single* configuration point, and in this case there is an induced orientation on the subconfiguration. However, defining

$$t_n \colon SP^\infty C_n^+(M,X) \longrightarrow SP^\infty C_{n-1}^+(M,X)$$

to take an oriented n-point configuration to the sum of its n different $(n-1)$-point subsets (with their induced orientations) results in equations $t_{n+1} \circ s_n = \mathrm{id} + \nu \circ s_{n-1} \circ t_n$, which are *not* the correct equations for the hypothesis of Corollary 2.8.2 to hold (note the appearance of the orientation-reversing automorphism ν).

2.8.3 Counterexamples

As mentioned in Remark 2.1.4 in the Introduction, the calculations in [GKY96] provide counterexamples to injectivity of the maps s_* in the case of oriented configuration spaces. The same examples also serve to show that a stability slope of $\frac{1}{3}$ is the best possible in the oriented case.

First, though, we mention a much simpler counterexample:

Counterexample 2.8.3 The simplest counterexample to injectivity of s_* is the map $H_1(C_4^+(\mathbb{R}^\infty, pt)) \to H_1(C_5^+(\mathbb{R}^\infty, pt))$, which is $H_1 A_4 \to H_1 A_5$, which is $\mathbb{Z}/3 \to 0$. This is the colimit of the maps $s_* \colon H_1(C_4^+(\mathbb{R}^k, pt)) \to H_1(C_5^+(\mathbb{R}^k, pt))$ as $k \to \infty$, and injectivity is *preserved* by taking such a colimit, so this provides counterexamples: s_* must be non-injective for infinitely many values of k.

The [GKY96] calculations.

For an odd prime p, there is a splitting

$$H_q(C_n^+(M,X); \mathbb{F}_p) \cong H_q(C_n(M,X); \mathbb{F}_p) \oplus H_q(C_n(M,X); \mathbb{F}_p^{(-1)}),$$

where on the right summand $\pi_1 C_n^+(M,X) \leq \pi_1 C_n(M,X)$ acts on $\mathbb{F}_p$ by the identity, and its complement acts by multiplication by -1. Correspondingly, the stabilisation map s_* splits into two summands. One is the stabilisation map from the unordered case, which *is* split-injective by §2.8.1 above, and the other is the map induced by the stabilisation map (from the unordered case) on *twisted* homology:

$$H_q(C_n(M,X); \mathbb{F}_p^{(-1)}) \longrightarrow H_q(C_{n+1}(M,X); \mathbb{F}_p^{(-1)}). \tag{2.8.1}$$

The calculations in [GKY96] use a result of Bödigheimer-Cohen-Milgram-Taylor [BCT89, BCM93, Corollary 8.4] to write this (under some conditions) in terms of the homology of iterated loopspaces of spheres, and then apply the Snaith splitting theorem [Sna74] and knowledge of the structure of $H_*\Omega^2 S^3$ to analyse the result. See Chapter 3 for more details.

Going through their calculations one can see that the map (2.8.1) is the map $\mathbb{F}_p \to 0$ for M any connected open surface, $X = pt$, and

$$(n, q) = (\lambda p + 1, \lambda(p-2)) \quad \text{for any } \lambda \geq 1$$

(although they state their result in slightly less generality).

This provides an infinite family of counterexamples to injectivity at each odd prime, and taking $p = 3$ also provides counterexamples to demonstrate that $\frac{1}{3}$ is the best possible stability slope for oriented configuration spaces, as mentioned in Remark 2.1.2 in the Introduction.

2.A Appendix: Proof of the factorisation lemma

In this appendix we prove the general factorisation lemma which is used in the proof of the Main Theorem in §2.6.

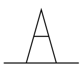
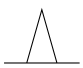
Proof of Lemma 2.6.2 (page 40). We have a square with a given homotopy filling it

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "H"] & X \arrow[d, "g"] \\ B \arrow[r, "j"'] & Y \end{tikzcd}$$

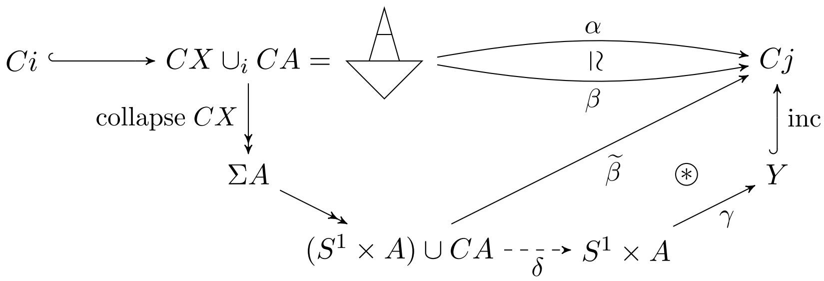
and also know that there exists a *split* homotopy filling the same square

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "F_1" near start] & X \arrow[d, "g"] \arrow[dl, "d"'] \arrow[dl, phantom, "F_2" near end, shift right=4] \\ B \arrow[r, "j"'] & Y \end{tikzcd}$$

We want to find a factorisation of $CH_* \colon \widetilde{H}_* Ci \to \widetilde{H}_* Cj$, so we begin by factorising the map $CH \colon Ci \to Cj$ itself. Schematically, CH looks like

$Ci = X \cup_i CA =$  $\longrightarrow$  $= Y \cup_j CB = Cj$

where the top part of CA is mapped to (all of) CB by f, levelwise, the middle section $A \times I$ is mapped to Y by the homotopy H, and X is mapped to Y by g. We will factorise this as follows:



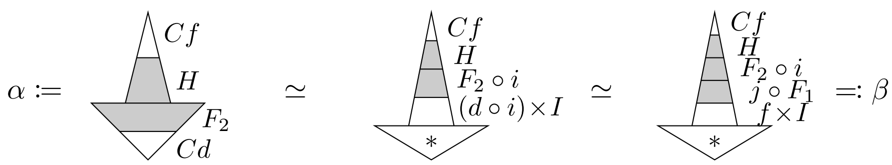
(2.A.1)

This requires some explanation: we will define α so that the map across the top is CH (so α is an extension of CH). Then we will homotope α to a map β which descends to $\widetilde{\beta}\colon (S^1 \times A) \cup CA \longrightarrow Cj$ when you collapse CX and then glue a small cone at the top of ΣA to a small cone at the bottom.[6] Then we will show that $\widetilde{\beta}$ factors through the square ⊛ as indicated (a dotted arrow denotes a map which is only defined on homology).

The composition $Ci \longrightarrow \Sigma A$ is the map in the Puppe sequence inducing the boundary map in (2.6.4), so this will prove the first half of the lemma, with z_H induced by the composition

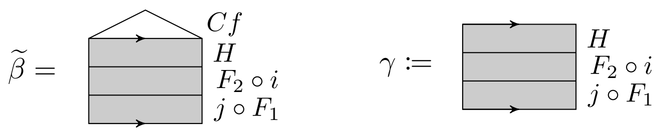
$$\Sigma A \twoheadrightarrow (S^1 \times A) \cup CA \dashrightarrow S^1 \times A \xrightarrow{\gamma} Y.$$

First, we define α and β as follows: Each region is mapped to a part of $Cj = Y \cup_j CB$ by the map or homotopy indicated; $*$ means it is sent to the tip of the cone CB; shaded regions have target Y, whereas unshaded regions are mapped (levelwise) to CB. By temporary abuse of notation, Cf in this diagram means the map $CA \to CB$ which is levelwise f; similarly for Cd.



Intuitively: the left homotopy "pulls α upwards" to obtain the map pictured in the middle, then the right homotopy gradually morphs the levelwise-$(d \circ i)$ part of this map into the homotopy F_1, and then "stretches" one end of it into levelwise-f.

It is clear from its definition that β descends to a map $\widetilde{\beta}$ as described above; we define γ to be the restriction of $\widetilde{\beta}$ to $S^1 \times A$.



Now we need to construct the map δ: This comes from the split cofibration sequence

[6] Here ΣA means the *unreduced* suspension.

$$\begin{tikzcd} A \arrow[r, hook] & S^1 \times A \arrow[l, bend right] \arrow[r, hook, "\varepsilon"'] & (S^1 \times A) \cup CA. \arrow[l, dashed, bend right, "\delta"'] \end{tikzcd}$$

We have an *actual* splitting $A \leftarrow S^1 \times A$, which induces a splitting on homology, which implies the existence of a splitting $S^1 \times A \dashleftarrow (S^1 \times A) \cup CA$ on homology.

Since we defined γ to be the restriction of $\widetilde{\beta}$, we have $(\mathrm{inc}) \circ \gamma = \widetilde{\beta} \circ \varepsilon$. But δ is a splitting on homology, so $\varepsilon_* \circ \delta = \mathrm{id}$. Hence

$$\begin{aligned} (\mathrm{inc})_* \circ \gamma_* \circ \delta &= \widetilde{\beta}_* \circ \varepsilon_* \circ \delta \\ &= \widetilde{\beta}_*, \end{aligned}$$

so the square $\circledast$ commutes on homology, as required. This completes the proof of the first half of the lemma.

Now, the map z_H was constructed as the composition

$$\widetilde{H}_* \Sigma A = \widetilde{H}_{*-1} A \longrightarrow \widetilde{H}_*\big((S^1 \times A) \cup CA\big) \xrightarrow{\delta} \widetilde{H}_*(S^1 \times A) \xrightarrow{\gamma_*} \widetilde{H}_* Y$$

induced by the three maps along the bottom of diagram (2.A.1). As defined above, γ is the composition of the homotopy H and the split homotopy $(j \circ F_1) * (F_2 \circ i)$. Hence to prove the second half of the lemma, it just remains to show that the composition of the first two maps is the inclusion coming from the Künneth decomposition for $\widetilde{H}_*(S^1 \times A)$.

This can be seen as follows. Using the homotopy equivalence $(S^1 \times A) \cup CA \simeq \Sigma A \vee S^1$, the Künneth decomposition and the suspension isomorphism we identify

$$\begin{aligned} \widetilde{H}_* \Sigma A &= \widetilde{H}_{*-1} A, \\ \widetilde{H}_*\big((S^1 \times A) \cup CA\big) &= \widetilde{H}_{*-1} A \oplus \widetilde{H}_* S^1, \\ \widetilde{H}_*(S^1 \times A) &= \widetilde{H}_{*-1} A \oplus H_{*-1}(pt) \oplus \widetilde{H}_* A. \end{aligned}$$

Carefully analysing the map on homology induced by ε, we see that under this identification it sends $\widetilde{H}_{*-1} A$ to itself by the identity, $H_{*-1}(pt)$ isomorphically to $\widetilde{H}_* S^1$, and $\widetilde{H}_* A$ to 0. Hence its right-inverse δ must send $\widetilde{H}_{*-1} A$ to itself by the identity, and $\widetilde{H}_* S^1$ isomorphically to $H_{*-1}(pt)$.

Under the identification $(S^1 \times A) \cup CA \simeq \Sigma A \vee S^1$, the map $\Sigma A \twoheadrightarrow (S^1 \times A) \cup CA$ becomes the inclusion $\Sigma A \hookrightarrow \Sigma A \vee S^1$, so on homology it induces the inclusion of the direct summand $\widetilde{H}_{*-1} A \hookrightarrow \widetilde{H}_{*-1} A \oplus \widetilde{H}_* S^1$.

Hence overall the composition $\widetilde{H}_{*-1} A \longrightarrow \widetilde{H}_*\big((S^1 \times A) \cup CA\big) \longrightarrow \widetilde{H}_*(S^1 \times A)$ is the inclusion of the direct summand

$$\widetilde{H}_{*-1} A \quad \hookrightarrow \quad \widetilde{H}_{*-1} A \oplus H_{*-1}(pt) \oplus \widetilde{H}_* A$$

into the Künneth decomposition for $\widetilde{H}_*(S^1 \times A)$. $\square$

2.B Appendix: Spectral sequences from semi-simplicial spaces

The aim of this appendix is to prove Proposition 2.4.3 — the construction of a spectral sequence associated to a map of augmented Δ-spaces. We will work up to this gradually, starting with the spectral sequence associated to a Δ-space, and will use the general construction recalled below.

2.B.1 General construction

Recall the following construction (see for example [MT68, chapter 7]): given a filtration

$$\varnothing = X_{-1} \subseteq X_0 \subseteq \cdots \quad \cdots \subseteq X_n \subseteq \cdots \quad \subseteq X$$

of a space X such that

$$\bigcup_{n\geq 0} X_n = X \qquad \text{and} \qquad H_*(X_n, X_{n-1}) = 0 \text{ for } * < n, \tag{2.B.1}$$

the filtered chain complex $C_*(X)$ induces a first quadrant spectral sequence

$$E^1_{s,t} \;\cong\; H_{s+t}(X_s, X_{s-1}) \quad \Rightarrow \quad H_*(X).$$

The first differential in this spectral sequence is the boundary map for the pair (X_s, X_{s-1}) composed with the quotient map for the pair (X_{s-1}, X_{s-2}).

2.B.2 Δ-spaces

We first describe the construction of the spectral sequence associated to a Δ-space $Y_\bullet$. Filter $X = \|Y_\bullet\|$ by its skeleta,

$$X_n = \|Y_\bullet\|^n \;=\; \coprod_{n\geq k\geq 0} Y_k \times \Delta^k \big/ \sim .$$

The filtration quotients are $X_n/X_{n-1} \cong (Y_n)_+ \wedge S^n$, and the inclusions $X_{n-1} \hookrightarrow X_n$ are cofibrations, so

$$\begin{aligned} H_{s+t}(X_s, X_{s-1}) \;&\cong\; \widetilde{H}_{s+t}((Y_s)_+ \wedge S^s) \\ &\cong\; \widetilde{H}_t((Y_s)_+) \;=\; H_t(Y_s). \end{aligned} \tag{2.B.2}$$

This is zero for $t < 0$, so (2.B.1) is satisfied and we get the spectral sequence

$$E^1_{s,t} \;\cong\; H_t(Y_s) \quad \Rightarrow \quad H_*(\|Y_\bullet\|).$$

The formula for the boundary map of the pair (X_s, X_{s-1}), under the identification (2.B.2), gives the first differential as the alternating sum of H_t of the face maps $Y_s \to Y_{s-1}$.

2.B.3 Augmented Δ-spaces

For an augmented Δ-space $Y_\bullet$, we filter the mapping cone $X = C(\|Y_\bullet\| \to Y_{-1})$ by

$$X_n = C(\|Y_\bullet\|^{n-1} \to Y_{-1})$$

for $n \geq 1$ and $X_0 = Y_{-1} \sqcup \{\text{tip of cone}\}$. The filtration quotients are now $X_n/X_{n-1} \cong (Y_{n-1})_+ \wedge S^n$ for $n \geq 1$, so similarly to before we have

$$H_{s+t}(X_s, X_{s-1}) \;\cong\; H_t(Y_{s-1}),$$

except with an extra $\mathbb{Z}$-summand when $s = t = 0$. Again this satisfies (2.B.1), so we have a spectral sequence converging from this E^1 page to $H_*(C(\|Y_\bullet\| \to Y_{-1}))$. Removing the extra $\mathbb{Z}$-summand from the E^1 page turns the limit into the *reduced* homology, so if we also regrade $s \mapsto s+1$ we obtain the spectral sequence

$$E^1_{s,t} \;\cong\; H_t(Y_s) \quad\Rightarrow\quad \widetilde{H}_{*+1}(C(\|Y_\bullet\| \to Y_{-1})),$$

which lives in $\{s \geq -1, t \geq 0\}$. Again, d^1 is the alternating sum of the maps on H_t induced by the face maps; in particular, the differential $E^1_{0,t} \to E^1_{-1,t}$ is H_t of the augmentation map.

2.B.4 Basepoints

Now we will introduce basepoints. Let $Y_\bullet$ be an augmented Δ-object in the category of pointed spaces. The pointed geometric realisation $\|Y_\bullet\|_\star$ is $\coprod_{n\geq 0} Y_n \times \Delta^n$ quotiented out by $\coprod_{n\geq 0} * \times \Delta^n$ and then by the face relations, and again there is an induced map $\|Y_\bullet\|_\star \to Y_{-1}$.

Filter $X = C(\|Y_\bullet\|_\star \to Y_{-1})$ by $X_n = C(\|Y_\bullet\|_\star^{n-1} \to Y_{-1})$ for $n \geq 1$ and $X_0 = Y_{-1}$. The filtration quotients are $X_n/X_{n-1} \cong Y_{n-1} \wedge S^n$ for $n \geq 1$, so

$$H_{s+t}(X_s, X_{s-1}) \;\cong\; \widetilde{H}_t(Y_{s-1}),$$

except again with an extra $\mathbb{Z}$-summand when $s = t = 0$. This satisfies (2.B.1), so removing the extra $\mathbb{Z}$-summand and regrading as before we get a spectral sequence

$$E^1_{s,t} \;\cong\; \widetilde{H}_t(Y_s) \quad\Rightarrow\quad \widetilde{H}_{*+1}(C(\|Y_\bullet\|_\star \to Y_{-1}))$$

in $\{s \geq -1, t \geq 0\}$.

2.B.5 Maps of augmented Δ-spaces

We can now deduce Proposition 2.4.3 from the last construction above.

Proof of Proposition 2.4.3. We are given a map of augmented Δ-spaces $Y_\bullet \to Z_\bullet$. Since homotopy colimits commute,

$$\mathrm{hocofib}(\|Y_\bullet\| \to \|Z_\bullet\|) \;\simeq\; \|\mathrm{hocofib}(Y_\bullet \to Z_\bullet)\|_\star,$$

i.e. $C(\|Y_\bullet\| \to \|Z_\bullet\|) \;\simeq\; \|C(Y_\bullet \to Z_\bullet)\|_\star$, where the *pointed* realisation appears on the right since mapping cones are naturally pointed spaces. The face and augmentation maps of $Y_\bullet$ and $Z_\bullet$ give $C(Y_\bullet \to Z_\bullet)$ the structure of an augmented Δ-object in the category of pointed spaces, so we may apply the construction of 2.B.4 above. This yields a spectral sequence in $\{s \geq -1, t \geq 0\}$ with

$$E^1_{s,t} \;\cong\; \widetilde{H}_t(C(Y_s \to Z_s)),$$

and converging to $\widetilde{H}_{*+1}$ of the mapping cone of

$$C(\|Y_\bullet\| \to \|Z_\bullet\|) \simeq \|C(Y_\bullet \to Z_\bullet)\|_\star \longrightarrow C(Y_{-1} \to Z_{-1}),$$

which is the double mapping cone $C^2(Y_\bullet \to Z_\bullet)$ of the square (2.4.2). The first differential can be identified as in the other constructions above. □

2.ℵ Addendum

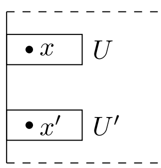
In this Addendum we mention in §2.ℵ.1 an alternative, slightly simpler model for the various maps between configuration spaces constructed in §2.2.4, and in §§2.ℵ.2 and 2.ℵ.3 we discuss the question of homological stability for *closed* manifolds.

Homological stability for *unordered* configuration spaces on closed manifolds was proved in [RW11, §9] in the three cases (i) $\dim(M)$ is odd; (ii) taking homology with $\mathbb{F}_2$ coefficients; (iii) taking homology with $\mathbb{Q}$ coefficients (this case also follows from the main result of [Chu12]). We will explain how each of these three cases is deduced from homological stability for open manifolds, to point out exactly why the methods do *not* carry over to the oriented case. As such (and as a disclaimer), we emphasise that §§2.ℵ.2 and 2.ℵ.3 are not original; they are just intended as an account of [RW11, §9], written in a slightly different style, in order to explain the difficulty with the oriented case. Also, we remark that although these methods do not carry over, case (iii) *is* nevertheless true in the oriented case: it simply follows from the main result of [Chu12], as in the unordered case.

2.ℵ.1 Maps between configuration spaces

The following alternative explicit models for the maps between configuration spaces defined in §2.2.4 do not essentially change the proofs of this chapter, but we mention them here as they may be more convenient to think about as more of the relevant diagrams commute on the nose. They are also more in line with the constructions in §5.2 of Chapter 5.

Choose a coordinate neighbourhood of part of the boundary of $\bar{M}$, and choose two disjoint subneighbourhoods U and U' (again homeomorphic to $\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 \geq 0\}$) and a point in each of them, as below.



Choose a self-embedding $e\colon M \hookrightarrow M$ whose support lies inside U and whose image misses x, and choose $e'\colon M \hookrightarrow M$ similarly. Take M_1 to be $M \smallsetminus \{x'\}$.

We can then define the following maps:

$p\colon C_n^+(M) \to C_n^+(M_1)$	apply e' to the configuration
$j\colon C_n^+(M_1) \to C_{n+1}^+(M)^0$	add a point to the configuration at x', and mark it
$s\colon C_n^+(M) \to C_{n+1}^+(M)$	apply e to the configuration, then add a point at x
$s'\colon C_n^+(M) \to C_{n+1}^+(M)$	apply e' to the configuration, then add a point at x'

and the augmentation map $a\colon C_n^+(M)^0 \to C_n^+(M)$ forgets the marked configuration point, as before. We have implicitly fixed a convention for choosing a new orientation for a configuration when we add a point to it. Denote the orientation-reversing automorphism by ν; the two conventions then differ by post-composition by ν.

We can also define versions of the 'j' map which fill in more than one point. Choose a sequence x'_k of distinct points in U' such the the image of $(e')^k \colon M \hookrightarrow M$ misses $\{x'_1, \ldots, x'_k\}$, and let $M_k = M \smallsetminus \{x'_1, \ldots, x'_k\}$. Then the map $j \colon C^+_{n-i-1}(M_{i+1}) \to C^+_n(M)^i$ adds $i{+}1$ points to the configuration at $x'_1, \ldots, x'_{i+1}$, with an appropriate convention for the orientation of the new configuration, and the obvious $(i+1)$-ordering. This is the inclusion of a fibre for the fibre bundle $\pi \colon C^+_n(M)^i \to \widetilde{C}_{i+1}(M)$, as before.

With these definitions, the diagram (2.6.3) in the proof of the Main Theorem becomes the following (or rather two copies of it, for n and $n-1$, side-by-side):

$$\begin{tikzcd}
C^+_n(M) \arrow[r, "p"] & C^+_n(M_1) \arrow[r, "\nu \circ j"] & C^+_{n+1}(M)^0 \arrow[r, "a"] & C^+_{n+1}(M) \\
C^+_{n-1}(M) \arrow[r, "p"'] \arrow[u, "s"] \arrow[ur, phantom, "\circlearrowleft"] & C^+_{n-1}(M_1) \arrow[r, "j"'] \arrow[u, "s"] \arrow[ur, phantom, "\circlearrowleft"] & C^+_n(M)^0 \arrow[r, "a"'] \arrow[u, "s"'] \arrow[ur, phantom, "\circlearrowleft"] & C^+_n(M) \arrow[u, "s"']
\end{tikzcd} \tag{2.ℵ.1}$$

The squares commute on the nose and the three horizontal maps compose to give $\nu \circ s'$ and s' on the top and bottom respectively. The arguments of §2.6.2 can then be run on this diagram to show that the map of mapping cones (with mapping cones taken w.r.t. the vertical maps) is surjective on homology in the desired stable range. Gluing two copies of (2.ℵ.1) together side-by-side, we can factorise the outer rectangle of maps up to homotopy as follows:

$$\begin{tikzcd}
C^+_{n-1}(M) \arrow[r, "(s')^2"] \arrow[dr, "s"] & C^+_{n+1}(M) \\
C^+_{n-2}(M) \arrow[r, "(s')^2"'] \arrow[u, "s"] & C^+_n(M) \arrow[u, "s"']
\end{tikzcd} \tag{2.ℵ.2}$$

and then run the arguments of §2.6.3 to show that the map of mapping cones is also trivial on homology in the desired stable range. The homotopies $\simeq$ in the triangles above are slightly different to the ones used in §2.6.3, due to our different choice of explicit models for the various maps between configuration spaces, but they still satisfy the property needed for the proof: Lemma 2.6.4 is still true.

2.ℵ.2 Closed manifolds and $\mathbb{F}_2$ coefficients or odd dimension

We will first describe the proof of [RW11, §9] in these cases, and then explain at the end what goes wrong if one tries to apply the same method to oriented configuration spaces. For simplicity we will just consider *unlabelled* configuration spaces, but the same argument works for labelled configuration spaces with essentially no more difficulty.

Let M be a closed, connected manifold and let M' be M with a point removed. There is an inclusion map $C_n(M') \hookrightarrow C_n(M)$ whose homotopy cofibre, up to homology, can be identified with $\Sigma^d C_{n-1}(M')_+$. (See §2.5 for the corresponding fact for oriented configuration spaces.) Hence, up to homology, a segment of the Puppe sequences for $C_n(M') \hookrightarrow C_n(M)$ and for $C_{n+1}(M') \hookrightarrow C_{n+1}(M)$ looks as follows:

$$\begin{tikzcd}
\Sigma^d C_{n-1}(M')_+ \arrow[r, "\Delta_n"] \arrow[d, "\Sigma^d s_{n-1}"'] \arrow[dr, phantom, "(*)"] & \Sigma C_n(M')_+ \arrow[r] \arrow[d, "\Sigma s_n"] & \Sigma C_n(M)_+ \arrow[r] & \Sigma^{d+1} C_{n-1}(M')_+ \arrow[r, "\Sigma\Delta_n"] \arrow[d, "\Sigma^{d+1} s_{n-1}"'] & \Sigma^2 C_n(M')_+ \arrow[d, "\Sigma^2 s_n"] \\
\Sigma^d C_n(M')_+ \arrow[r, "\Delta_{n+1}"'] & \Sigma C_{n+1}(M')_+ \arrow[r] & \Sigma C_{n+1}(M)_+ \arrow[r] & \Sigma^{d+1} C_n(M')_+ \arrow[r, "\Sigma\Delta_{n+1}"'] & \Sigma^2 C_{n+1}(M')_+
\end{tikzcd}$$

We have indicated the stabilisation maps connecting the two Puppe sequences, but we do *not* claim commutativity of this diagram; in fact it does not commute in general, as we show below. Taking homology with coefficients in a field $\mathbb{F}$, we get a long exact sequence from the top row and deduce that

$$\begin{aligned}
\dim H_* C_n(M) = {} & \dim H_* C_n(M') + \dim H_{*-d} C_{n-1}(M') \\
& - \mathrm{rank}\big((\Delta_n)_* \colon H_{*-d+1} C_{n-1}(M') \to H_* C_n(M')\big) \\
& - \mathrm{rank}\big((\Delta_n)_* \colon H_{*-d} C_{n-1}(M') \to H_{*-1} C_n(M')\big).
\end{aligned} \tag{2.ℵ.3}$$

Since M' is open we know that the first two terms of (2.ℵ.3) are stable: they are independent of n for $n \gg *$. So it is sufficient to prove that $\mathrm{rank}(\Delta_n)_*$ is also stable. Note that *if* the square $(*)$ in the map of Puppe sequences above commutes, then $(\Sigma s_n, \Sigma^d s_{n-1})$ provides an isomorphism between the maps $(\Delta_n)_*$ and $(\Delta_{n+1})_*$ in a stable range. So the aim is to show that when $\dim(M)$ is odd or $\mathbb{F} = \mathbb{F}_2$, this square does in fact commute.

There is a map $g \colon C_2(S^{d-1} \times [0,1]) \times C_{n-1}(M') \to C_{n+1}(M')$ given by gluing $S^{d-1} \times [0,1]$ to the puncture of M' as a collar neighbourhood. Consider the square

$$\begin{tikzcd}
S^{d-1} \times C_0(S^{d-1} \times [0,1]) \arrow[r, "r"] \arrow[d, "\mathrm{id} \times s"'] & C_1(S^{d-1} \times [0,1]) \arrow[d, "s"] \\
S^{d-1} \times C_1(S^{d-1} \times [0,1]) \arrow[r, "r"'] & C_2(S^{d-1} \times [0,1])
\end{tikzcd} \tag{2.ℵ.4}$$

where s and r are as follows. Fix a basepoint $y_0 \in S^{d-1}$. Given a configuration c on the cylinder $S^{d-1} \times [0,1]$, the stabilisation map s pushes this configuration inwards from the 0 end of the cylinder and then adds a new point to the configuration at $(y_0, \frac{1}{2})$. Given a configuration c and a parameter $y \in S^{d-1}$, the map r pushes the configuration inwards from the 0 end of the cylinder and then adds a new point to the configuration at $(y, \frac{1}{2})$.

On homology the route $\rightarrow\!\downarrow$ around the square $(*)$ is induced by $g \circ ((s \circ r) \times \mathrm{id})$, and the route $\downarrow\!\rightarrow$ around $(*)$ is induced by $g \circ ((r \circ (\mathrm{id} \times s)) \times \mathrm{id})$, so it is enough to check that (2.ℵ.4) commutes on $\mathbb{F}$-homology, and of course we just have to check this for the fundamental class of $S^{d-1} = S^{d-1} \times C_0(S^{d-1} \times [0,1])$.

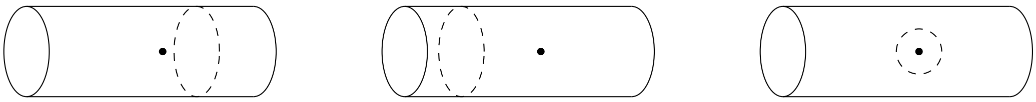
The image of the fundamental class under the composition $\rightarrow\!\downarrow$ of (2.ℵ.4) is represented by the $(d-1)$-cycle depicted in Figure 2.ℵ.1(a), and its image under the composition $\downarrow\!\rightarrow$ of (2.ℵ.4) is represented by the $(d-1)$-cycle in Figure 2.ℵ.1(b). Their difference is homologous to the $(d-1)$-cycle in Figure 2.ℵ.1(c), which is the image under

$$\mathbb{RP}^{d-1} \simeq C_2(\mathbb{R}^d) \longrightarrow C_2(S^{d-1} \times [0,1])$$

of the $(d-1)$-cycle τ in $\mathbb{RP}^{d-1}$ which is the image of the fundamental class of S^{d-1} under the quotient map $S^{d-1} \twoheadrightarrow S^{d-1}/(x \sim -x) = \mathbb{RP}^{d-1}$.

When d is odd, $H_{d-1}(\mathbb{RP}^{d-1};\mathbb{Z}) = 0$. When d is even, this $(d-1)$-cycle represents *twice* a generator of $H_{d-1}(\mathbb{RP}^{d-1};\mathbb{Z}) \cong \mathbb{Z}$, so it represents the trivial element of $H_{d-1}(\mathbb{RP}^{d-1};\mathbb{F}_2)$. Hence in these cases the square (2.ℵ.4) (and therefore also the square $(*)$) commutes on $\mathbb{F}$-homology.

The oriented case. Replacing $C_n(M)$ with $C_n^+(M)$ in the above argument, we instead end up with a $(d-1)$-cycle τ representing the fundamental class of S^{d-1}, which is of course never zero. Hence this argument cannot work in the oriented case.



(a) (b) (c)

Figure 2.ℵ.1: Some $(d-1)$-cycles in $C_2(S^{d-1} \times [0,1])$ (indicated by dots and dashes).

2.ℵ.3 Closed manifolds and $\mathbb{Q}$ coefficients

There are two proofs in [RW11, §9] of $\mathbb{Q}$-homological stability for unordered configuration spaces on closed manifolds. One is similar to the method above for $\mathbb{F}_2$ coefficients or odd dimension, and the other involves a spectral sequence argument and a different resolution of the configuration spaces. Both of them use Dold's Lemma (stated as Lemma 2.8.1 above), and therefore cannot work in the oriented case, since Dold's Lemma simply does not apply to oriented configuration spaces (see §2.8.2 for more details on this). We will explain both methods in detail below, to show exactly where this lemma is used, but first we will say a little more about the lemma itself.

Dold's Lemma. See §2.8 for the notation. Lemma 2.8.1 is proved by induction with the inductive hypothesis that

$$(\tau_{k,n})_{k=1}^{n} \colon A_n \longrightarrow \bigoplus_{k=1}^{n} \mathrm{coker}(s_{k-1}) \tag{2.ℵ.5}$$

is an isomorphism (where by abuse of notation we denote the composite $A_n \xrightarrow{\tau_{k,n}} A_k \twoheadrightarrow \mathrm{coker}(s_{k-1})$ also by $\tau_{k,n}$). The inductive step is as follows. In the square

$$\begin{tikzcd}
A_n \arrow[r, "s_n"] \arrow[d, "(\tau_{k,n})_{k=1}^{n}"'] & A_{n+1} \arrow[d, "(\tau_{k,n+1})_{k=1}^{n}"] \\
\bigoplus_{k=1}^{n} \mathrm{coker}(s_{k-1}) \arrow[r, phantom, "="] & \bigoplus_{k=1}^{n} \mathrm{coker}(s_{k-1})
\end{tikzcd}$$

the left-hand arrow is an isomorphism by the inductive hypothesis, so s_n has a left-inverse, and hence there is an isomorphism

$$\begin{tikzcd}
A_{n+1} \arrow[r, "\cong"] \arrow[d, phantom, "||"] & \mathrm{coker}(s_n) \oplus A_n \arrow[d, phantom, "\cong" rotate] \\
A_{n+1} \arrow[r, "(\tau_{k,n+1})_{k=1}^{n+1}"'] & \bigoplus_{k=1}^{n+1} \mathrm{coker}(s_{k-1})
\end{tikzcd}$$

Now, when we are in the category of $\mathbb{Q}$-vector spaces, and assume the hypotheses of Corollary 2.8.2, the maps $\tau_{k,n}$ also satisfy the relations $\tau_{k,n} \circ \tau_{n,n+1} = (n+1-k).\tau_{k,n+1}$, making the right-hand square below commute up to an automorphism of the bottom-right group:

$$\begin{tikzcd}
A_n \arrow[r, "s_n"] \arrow[d, "\cong"] & A_{n+1} \arrow[r, "\tau_{n,n+1}", "= t_{n+1}"'] \arrow[d, "\cong"] & A_n \arrow[d, "\cong"] \\
\bigoplus_{k=1}^{n} \mathrm{coker}(s_{k-1}) \arrow[r, hook] & \bigoplus_{k=1}^{n+1} \mathrm{coker}(s_{k-1}) \arrow[r, two heads] & \bigoplus_{k=1}^{n} \mathrm{coker}(s_{k-1})
\end{tikzcd}$$

The vertical maps are $(\tau_{k,n})_{k=1}^{n}$ and $(\tau_{k,n+1})_{k=1}^{n+1}$, and the left-hand square commutes on the nose. Hence the hypotheses of Corollary 2.8.2 also imply that:

$$t_{n+1} \circ s_n \text{ is an automorphism of } A_n. \tag{2.ℵ.6}$$

The first method. One of the proofs of $\mathbb{Q}$-homological stability for unordered configuration spaces on closed manifolds in [RW11, §9] goes as follows. The square $(*)$ of §2.ℵ.2 does not commute in this case, so we instead consider the square

$$
\begin{tikzcd}
\Sigma^d C_{n-1}(M')_+ \arrow[r, "\Delta_n"] & \Sigma C_n(M')_+ \\
\Sigma^d C_n(M')_+ \arrow[r, "\Delta_{n+1}"'] \arrow[u, dashed, "\Sigma^d t_n"] & \Sigma C_{n+1}(M')_+ \arrow[u, dashed, "\Sigma t_{n+1}"']
\end{tikzcd}
\tag{2.ℵ.7}
$$

involving the transfer maps mentioned in §2.8 which take a configuration c in $C_n(M')$ to the formal sum in $SP^\infty C_{n-1}(M')$ of the n configurations of $n-1$ points obtained by removing one point from c. The dashed arrows indicate that the map is only defined on homology (or SP^∞).

Since M' is an open manifold, the map $(s_n)_*\colon H_*(C_n(M');\mathbb{Q}) \to H_*(C_{n+1}(M');\mathbb{Q})$ is an isomorphism in the stable range. Since we are taking rational coefficients we can apply the discussion of **Dold's Lemma** above to see that $(t_{n+1})_* \circ (s_n)_*$ is an automorphism of $H_*(C_n(M');\mathbb{Q})$, and hence $(t_{n+1})_*$ is also an isomorphism in the stable range. So similarly to before, it is sufficient to show that (2.ℵ.7) commutes on reduced homology $\widetilde{H}_{*+1}$. Now, for the square

$$
\begin{tikzcd}
S^{d-1} \times C_{n-1}(M') \arrow[r, "r"] & C_n(M') \\
S^{d-1} \times C_n(M') \arrow[r, "r"'] \arrow[u, dashed, "\mathrm{id} \times t_n"] & C_{n+1}(M') \arrow[u, dashed, "t_{n+1}"']
\end{tikzcd}
\tag{2.ℵ.8}
$$

the commutator of the two maps ↱ and ⬏ on homology is the map induced by the projection $p\colon S^{d-1} \times C_n(M') \twoheadrightarrow C_n(M')$. The left-hand side of H_*(2.ℵ.8) has a Künneth decomposition, and one of these summands gives precisely $\widetilde{H}_{*+1}$(2.ℵ.7). But the projection p induces the trivial map on this summand, so $\widetilde{H}_{*+1}$(2.ℵ.7) commutes.

The second method. The second proof of $\mathbb{Q}$-homological stability for unordered configuration spaces on closed manifolds in [RW11, §9] uses an entirely different method, and shows that the transfer map $t_n\colon C_n(M) \to SP^\infty C_{n-1}(M)$, which removes a point from a configuration in all n possible ways, is an isomorphism on $\mathbb{Q}$-homology in the stable range. It goes as follows:

We begin by defining a new resolution of the oriented configuration space $C_n(M)$. For $i \geq -1$, let $D_n(M)^i$ be the space

$$\big\{(p_1,\ldots,p_{n+i+1}) \in M \;\big|\; p_i \neq p_j \text{ for } i \neq j\big\}/\Sigma_n$$

where Σ_n acts on the first n coordinates. These spaces form an augmented semi-simplicial space $D_n(M)^\bullet$ with $D_n(M)^{-1} = C_n(M)$ and with face maps $d_j\colon D_n(M)^i \to D_n(M)^{i-1}$

given by forgetting the $(n+j)$th point.

Similarly to Lemma 2.4.6, the map $\|D_n(M)^\bullet\| \to C_n(M)$ is a fibre bundle with fibre homeomorphic to $\|\mathrm{inj}([\bullet+1], M \smallsetminus n \text{ points})\|$. For any space Z the geometric realisation $\|\mathrm{inj}([\bullet+1], Z)\|$ is $(\mathrm{card}(Z)-2)$-connected (by Proposition 3.2 of [RW11] for example), so $\|D_n(M)^\bullet\| \to C_n(M)$ is a weak equivalence.

Let $C_{n,1}(M) = \big\{\{p_1,\ldots,p_n\} \in C_n(M), q \in M \mid q = p_i \text{ for some } i\big\}$ and form a similar semi-simplicial space $D_{n,1}(M)^\bullet$ with an augmentation to $C_{n,1}(M)$ such that $\|D_{n,1}(M)^\bullet\| \to C_{n,1}(M)$ is a weak equivalence. The spaces $D_n(M)^i$ and $D_{n,1}(M)^i$ have fibrations to $\widetilde{C}_{i+1}(M)$, with fibre $C_n(M \smallsetminus i+1 \text{ points})$ and $C_{n,1}(M \smallsetminus i+1 \text{ points})$ respectively.

The forgetful maps

$$D_{n,1}(M)^i \to D_{n-1}(M)^i \tag{2.ℵ.9}$$

are maps over $\widetilde{C}_{i+1}(M)$ w.r.t. these fibrations, and also commute with the face maps, so we have a map of augmented semi-simplicial spaces

$$D_{n,1}(M)^\bullet \to D_{n-1}(M)^\bullet. \tag{2.ℵ.10}$$

The other forgetful maps

$$D_{n,1}(M)^i \to D_n(M)^i \tag{2.ℵ.11}$$

are maps over $\widetilde{C}_{i+1}(M)$ w.r.t. these fibrations and are n-sheeted covering maps, so there are “fibrewise transfer” maps

$$D_n(M)^i \xrightarrow{\mathrm{trf}} SP^\infty_{\mathrm{fib}} D_{n,1}(M)^i \tag{2.ℵ.12}$$

which take a point to its preimage under (2.ℵ.11). (For a fibration $E \to B$ with basepoint $e \in E$, the fibrewise infinite symmetric product $SP^\infty_{\mathrm{fib}} E$ is the space of finite formal sums of points of E *in the same fibre*, with the relation that $e = 0$.) Note that the maps (2.ℵ.12) commute with the face maps, so we get a map of augmented semi-simplicial spaces, which we can compose with SP^∞_{fib}(2.ℵ.10) to get a map

$$D_n(M)^\bullet \xrightarrow{\mathrm{trf}} SP^\infty_{\mathrm{fib}} D_{n,1}(M)^\bullet \to SP^\infty_{\mathrm{fib}} D_{n-1}(M)^\bullet, \tag{2.ℵ.13}$$

which on level -1 is precisely the transfer map $t_n \colon C_n(M) \to SP^\infty C_{n-1}(M)$.

The map of augmented semi-simplicial spaces (2.ℵ.13) and the composite maps of fibrations SP^∞_{fib}(2.ℵ.9) $\circ$ (2.ℵ.12) give spectral sequences

$$\begin{aligned}
&\textit{in } \left\{\begin{smallmatrix} s\geq -1 \\ t\geq 0\end{smallmatrix}\right\}: && \widehat{E}^1_{s,t} = \widetilde{H}_t\big(\mathrm{cone}\big(D_n(M)^s \to SP^\infty_{\mathrm{fib}} D_{n-1}(M)^s\big);\mathbb{Q}\big) \Rightarrow 0 \\
&\textit{in } \left\{\begin{smallmatrix} p\geq 0 \\ q\geq 0\end{smallmatrix}\right\}: && {}^{(s)}E^2_{p,q} = H_p\Big(\widetilde{C}_{s+1}(M); \widetilde{H}_q\big(\mathrm{cone}\big(C_n(M_{s+1}) \to SP^\infty C_{n-1}(M_{s+1})\big);\mathbb{Q}\big)\Big) \\
&&& \qquad \Rightarrow \widetilde{H}_*\big(\mathrm{cone}\big(D_n(M)^s \to SP^\infty_{\mathrm{fib}} D_{n-1}(M)^s\big);\mathbb{Q}\big)
\end{aligned}$$

where M_{s+1} denotes $M \smallsetminus s{+}1$ points. The first one converges to zero since both $\|D_n(M)^\bullet\| \to C_n(M)$ and $\|D_{n,1}(M)^\bullet\| \to C_{n,1}(M)$ are weak equivalences. See §2.4.1 for a more detailed discussion of the properties of these spectral sequences.

By homological stability for unordered configuration spaces for *open* manifolds, when $s \geq 0$ the stabilisation map

$$C_{n-1}(M_{s+1}) \to C_n(M_{s+1})$$

is an isomorphism on integral homology up to degree $\frac{n-1}{2}$, and therefore by the discussion of **Dold's Lemma** earlier in this section, this implies that the transfer map

$$t_n \colon C_n(M_{s+1}) \to SP^\infty C_{n-1}(M_{s+1})$$

is an isomorphism on *rational* homology in the same range. Hence ${}^{(s)}E^2_{p,q} = 0$ for $s \geq 0$, $p \geq 0$ and $0 \leq q \leq \frac{n-1}{2}$, so for all $s \geq 0$ the spectral sequence ${}^{(s)}E$ converges to zero in total degree $* \leq \frac{n-1}{2}$. Hence $\widehat{E}^1_{s,t} = 0$ for $s \geq 0$ and $0 \leq t \leq \frac{n-1}{2}$. Since $\widehat{E}$ converges to zero in all degrees, we must therefore also have $\widehat{E}^1_{-1,t} = 0$ for $0 \leq t \leq \frac{n-1}{2}$. Hence

$$t_n \colon C_n(M) \to SP^\infty C_{n-1}(M)$$

is an isomorphism on $\mathbb{Q}$-homology up to degree $\frac{n-3}{2}$.

Note that it is always surjective on $\mathbb{Q}$-homology by the discussion of Dold's Lemma above. Also, this stable range may be improved, since the stable range for *open* manifolds is larger (it has slope 1) when taking $\mathbb{Q}$ coefficients.

CHAPTER 3

Some calculations of the homology of oriented configuration spaces

3.1 Introduction

Definition 3.1.1 Given a background manifold M, recall that the unordered, unlabelled configuration space $C_n(M)$ of n points in M is defined to be the quotient of

$$\{(p_1,\ldots,p_n)\in M^n \bigm| p_i \text{ are pairwise distinct}\}$$

by the natural action of the symmetric group Σ_n permuting the coordinates, and the *oriented* configuration space $C_n^+(M)$ is the double cover of $C_n(M)$ obtained by instead taking the quotient by the alternating group A_n.

An important property enjoyed by both the unordered and oriented configuration spaces is that they satisfy *homological stability*, meaning that for any fixed (connected, open) manifold M and degree q, the sequences of homology groups $H_q(C_n(M))$ and $H_q(C_n^+(M))$ are eventually independent of n, once n is sufficiently large. The first of these facts was proved most recently and in the most generality in [RW11] (see also [Seg73, McD75, Seg79]), and the second was proved in Chapter 2. "Sufficiently large" for unordered configuration spaces means that $n \geq 2q$, whereas for oriented configuration spaces it means $n \geq 3q+5$. Imagining this on a plane with n as the horizontal axis and q as the vertical axis, we say that $C_n(M)$ is homologically stable with a *stability slope* of $\frac{1}{2}$, whereas $C_n^+(M)$ has a stability slope of $\frac{1}{3}$.

When M is a surface S the stability slope for $C_n^+(S)$ can be improved, away from the prime 3, to:

Proposition A *If S is a connected open surface, then*

$$H_q\big(C_n^+(S);\mathbb{Z}[\tfrac{1}{3}]\big) \;\cong\; H_q\big(C_{n+1}^+(S);\mathbb{Z}[\tfrac{1}{3}]\big)$$

for $n \geq 2q+2$.

This was essentially stated in [GKY96] (at the beginning of §2), in the case where S is a compact Riemann surface with finitely many points removed. Their result follows from some detailed prime-by-prime calculations—the purpose of this short chapter is to point out that performing their calculations carefully yields the more precise Theorem B below.

Remark 3.1.2 Every element of $\pi_1(C_n(M))$ induces a permutation of the basepoint configuration, and the subgroup of elements which induce an *even* permutation is exactly the subgroup corresponding to the double cover $C_n^+(M)$. Given any ring R, let V_R be the $R[\pi_1(C_n(M))]$-module R^2, where the even elements of $\pi_1(C_n(M))$ act by the identity and the odd elements act by swapping the two coordinates. When $\mathrm{char}(R)\neq 2$ this is isomorphic to the direct sum $R\oplus R^{(-1)}$, where the action on R is trivial, and on $R^{(-1)}$ the odd elements act by multiplication by -1. So we have:

$$\begin{aligned} H_q(C_n^+(M);R) &\cong H_q(C_n(M);V_R) \\ &\cong H_q(C_n(M);R) \oplus H_q(C_n(M);R^{(-1)}). \end{aligned}$$

Proposition A follows from this decomposition, the universal coefficient theorem, homological stability for unordered configuration spaces, and the following two facts:

Theorem B *Let S be a connected surface and $q > 0$. Then for p an odd prime,*

$$\begin{aligned} H_q\big(C_n(S);\mathbb{F}_p^{(-1)}\big) &= 0 \quad \textit{for} \quad n \geq \big(\tfrac{p}{p-2}\big)(q+1); \\ H_q\big(C_n(S);\mathbb{Q}^{(-1)}\big) &= 0 \quad \textit{for} \quad n > q. \end{aligned}$$

Hence modulo p (for an odd prime p), and with a sign-twisting, the homological stability slope of $C_n(S)$ is $\frac{p-2}{p}$, which converges to 1 as $p \to \infty$.

Lemma 3.1.3 *For S a connected open surface, $H_q(C_n^+(S);\mathbb{F}_2) \cong H_q(C_{n+1}^+(S);\mathbb{F}_2)$ for $n \geq 2q+2$.*

Proof of Lemma 3.1.3. The rank-0 spherical fibration $S^0 \to C_n^+(S) \to C_n(S)$ gives a Gysin sequence for $H_*(-;\mathbb{F}_2)$. There is a map $s\colon C_n(S) \to C_{n+1}(S)$ defined by pushing the n-point configuration away from a chosen end of S and adding a new point to the configuration in the vacated region. This lifts to a map of spherical fibrations, and hence we get a map of Gysin sequences. The isomorphism $H_q(C_n(S)) \cong H_q(C_{n+1}(S))$, for $n \geq 2q$, in the statement of homological stability for unordered configuration spaces is induced by the map s, so the result follows by applying the 5-lemma to the map of Gysin sequences. □

The proof of Theorem B is based on calculations due to Bödigheimer, Cohen, Milgram and Taylor [BCT89, BCM93] which we recall in the next section, along with some other preliminaries.

3.2 Preliminaries

Definition 3.2.1 For a manifold M and space X, the *labelled* (unordered) configuration space $C_n(M,X)$ is defined to be the quotient of

$$\{(p_1,\ldots,p_n)\,|\,p_i \text{ are pairwise disjoint}\} \times X^n$$

by the diagonal action of Σ_n, permuting the coordinates of both M^n and X^n. The unlabelled configuration space $C_n(M)$ of Definition 3.1.1 is recovered by taking $X = pt$. If a basepoint $x_0 \in X$ is chosen, the space $D_n(M,X)$ is defined to be the quotient

$$C_n(M,X)/\left\{\text{subspace of configurations where at least one label is } x_0\right\}.$$

Proposition 3.2.2 ([BCM93, Corollary 8.4]) *For a smooth d-dimensional manifold M, where d is even, $0 < q < dn$, and for any $N \gg 0$, the sign-twisted homology of the unordered configuration space $C_n(M)$ with coefficients in a field $\mathbb{F}$ is*

$$H_q\big(C_n(M);\mathbb{F}^{(-1)}\big) \;\cong\; H_{q+(2N+1)n}\bigg(\prod_{i=0}^{d}(\Omega^{d-i}S^{d+2N+1})^{b_i(M)};\mathbb{F}\bigg),$$

where $b_i(M)$ is the ith Betti number of M.

Theorem 3.2.3 ("Snaith splitting", [Sna74]) *For any space X there is a stable splitting*

$$\Omega^k\Sigma^k X \simeq_s \bigvee_{i\geq 0} D_i(\mathbb{R}^k;X).$$

Proposition 3.2.4 ([CMM78]) *Let $C_i(\mathbb{R}^2,\mathbb{R}) \to C_i(\mathbb{R}^2)$ be the rank-i vector bundle given by forgetting the $\mathbb{R}$-labelling of a configuration. Then the direct sum of two copies of this bundle is trivial.*

3.3 The calculation

With this set up, we can now prove Theorem B, following the same strategy as that of [GKY96].

Proof of Theorem B. First assume that S is open or nonorientable closed, so that $b_2(S) = 0$. Let $b = b_1(S)$. Then by Proposition 3.2.2,

$$\begin{aligned} H_q\big(C_n(S);\mathbb{F}_p^{(-1)}\big) &\cong H_{q+(2N+1)n}\big(\Omega^2S^{2N+3}\times(\Omega S^{2N+3})^b;\mathbb{F}_p\big) \\ &\cong \bigoplus_{j\geq 0} H_j\big((\Omega S^{2N+3})^b;\mathbb{F}_p\big)\otimes H_{-j+q+(2N+1)n}\big(\Omega^2S^{2N+3};\mathbb{F}_p\big) \end{aligned}$$

for any $N \gg 0$.

By the Serre spectral sequence for the fibration sequence $\Omega S^{2N+3} \to PS^{2N+3} \to S^{2N+3}$, the product of loopspaces $(\Omega S^{2N+3})^b$ has homology concentrated in degrees a multiple of $2N+2$, and $H_{(2N+2)j}\big((\Omega S^{2N+3})^b;\mathbb{F}_p\big)$ has rank equal to

$$\begin{aligned} \alpha(j) &= \text{number of ordered } b\text{-tuples of nonnegative integers which sum to } j \\ &= \tbinom{j+b-1}{j} \quad \text{[with the conventions that } \tbinom{x-1}{x} = 0 \text{ for } x \geq 1 \text{ and } \tbinom{-1}{0} = 1]. \end{aligned}$$

(This last equality can be seen by induction on b.) By the Snaith splitting (Theorem 3.2.3), we have

$$\widetilde{H}_*\big(\Omega^2S^{2N+3};\mathbb{F}_p\big) \;\cong\; \bigoplus_{i\geq 0}\widetilde{H}_*\big(D_i(\mathbb{R}^2,S^{2N+1});\mathbb{F}_p\big).$$

Note that in the range we're interested in, $q < n$, so in particular $q \neq n$. Hence for $N \gg 0$,

$$\begin{aligned} & 2N+2 \nmid q-n, \\ \text{so} \quad & 2N+2 \nmid q+(2N+1)n, \\ \text{so} \quad & q+(2N+1)n-(2N+2)j \neq 0, \end{aligned}$$

and so $H_* = \widetilde{H}_*$ for $* = q+(2N+1)n-(2N+2)j$. Putting this all together we have:

$$H_q\big(C_n(S);\mathbb{F}_p^{(-1)}\big) \;\cong\; \bigoplus_{j\geq 0}\Big(\mathbb{F}_p^{\alpha(j)} \otimes \bigoplus_{i\geq 0} \widetilde{H}_{-(2N+2)j+q+(2N+1)n}\big(D_i(\mathbb{R}^2,S^{2N+1});\mathbb{F}_p\big)\Big). \quad (3.3.1)$$

So we need to understand the spaces $D_i(\mathbb{R}^2,S^{2N+1})$. By their definition,

$$\begin{aligned} D_i(\mathbb{R}^2,S^{2N+1}) \;&\cong\; \mathrm{Th}(C_i(\mathbb{R}^2,\mathbb{R}^{2N+1}) \to C_i(\mathbb{R}^2)) \\ &\cong\; \mathrm{Th}(\oplus_{2N+1}(C_i(\mathbb{R}^2,\mathbb{R}) \to C_i(\mathbb{R}^2))), \end{aligned}$$

the Thom space of the bundle given by forgetting the labelling of the configuration. By Proposition 3.2.4, denoting the rank-k trivial bundle by ε^k, this is

$$\begin{aligned} \mathrm{Th}(\varepsilon^{2Ni} \oplus (C_i(\mathbb{R}^2,\mathbb{R}) \to C_i(\mathbb{R}^2))) \;&\cong\; \Sigma^{2Ni}\mathrm{Th}(C_i(\mathbb{R}^2,\mathbb{R}) \to C_i(\mathbb{R}^2)) \\ &\cong\; \Sigma^{2Ni} D_i(\mathbb{R}^2,S^1). \end{aligned}$$

This simplifies (3.3.1) to

$$H_q\big(C_n(S);\mathbb{F}_p^{(-1)}\big) \;\cong\; \bigoplus_{i,j\geq 0}\Big(\widetilde{H}_{q+(2N+1)n-(2N+2)j-2Ni}\big(D_i(\mathbb{R}^2,S^1);\mathbb{F}_p\big)\Big)^{\alpha(j)}. \quad (3.3.2)$$

Note that the homological degree simplifies to $2N(n-i-j)+q+n-2j$ and that $D_i(\mathbb{R}^2,S^1)$ can be given the structure of a $3i$-dimensional cell complex, so its homology vanishes above degree $3i$.

Recall that N may be chosen arbitrarily large compared to n and q, i.e., $N \gg n,q$. Hence:

- there is no contribution to the sum if $(n-i-j)$ is negative;
- so in particular we may assume that $n \geq i,j$;
- so $N \gg n,q,i,j$;
- so since the dimension of $D_i(\mathbb{R}^2;S^1)$ is much smaller than N, there is also no contribution to the sum if $(n-i-j)$ is positive.

So we may assume that $j = n-i$, and simplify (3.3.2) to a finite sum:

$$H_q\big(C_n(S);\mathbb{F}_p^{(-1)}\big) \;\cong\; \bigoplus_{i=0}^{n} \widetilde{H}_{q+2i-n}\big(D_i(\mathbb{R}^2,S^1);\mathbb{F}_p\big)^{\alpha(n-i)} \quad (3.3.3)$$

Now, $\widetilde{H}_*(\Omega^2 S^3;\mathbb{F}_p) \cong \mathbb{F}_p[x_1,x_2,...] \otimes \Lambda[y_0,y_1,...]$ with $\dim(x_j) = 2p^j-2$ and $\dim(y_j) = 2p^j-1$. The Snaith splitting for $\Omega^2 S^3$ is

$$\Omega^2 S^3 \simeq_s \bigvee_{i\geq 0} D_i(\mathbb{R}^2;S^1)$$

and induces a weight filtration on the mod-p homology of $\Omega^2 S^3$ given by $\mathrm{wt}(x_j) = \mathrm{wt}(y_j) = p^j$. So $\widetilde{H}_*(D_i(\mathbb{R}^2;S^1);\mathbb{F}_p)$ is generated as an $\mathbb{F}_p$-vector space by monomials in $\mathbb{F}_p[x_1,x_2,...]\otimes \Lambda[y_0,y_1,...]$ with $\dim = *$ and $\mathrm{wt} = i$.

The set of such monomials is in bijection with the set of double sequences of integers

$$v = \left(\begin{smallmatrix} k_0 & k_1 & k_2 & \cdots \\ & \ell_1 & \ell_2 & \cdots \end{smallmatrix}\right) \qquad \text{with} \qquad \begin{smallmatrix} k_j=0 \text{ or } 1 \\ \ell_j \geq 0 \end{smallmatrix} \tag{3.3.4}$$

such that the dot products

$$v \cdot \left(\begin{smallmatrix} 1 & 2p-1 & 2p^2-1 & \cdots \\ & 2p-2 & 2p^2-2 & \cdots \end{smallmatrix}\right) = * \qquad \text{and} \qquad v \cdot \left(\begin{smallmatrix} 1 & p & p^2 & \cdots \\ & p & p^2 & \cdots \end{smallmatrix}\right) = i,$$

or equivalently such that

$$v \cdot \left(\begin{smallmatrix} 1 & 1 & 1 & 1 & \cdots \\ & 2 & 2 & 2 & \cdots \end{smallmatrix}\right) = 2i - * \qquad \text{and} \qquad v \cdot \left(\begin{smallmatrix} 1 & p & p^2 & \cdots \\ & p & p^2 & \cdots \end{smallmatrix}\right) = i.$$

So from (3.3.3) we have the combinatorial formula

$$\begin{aligned} \dim H_q\big(C_n(S);\mathbb{F}_p^{(-1)}\big) &= \sum_{i=0}^{n} \left(\begin{smallmatrix} n-i+b-1 \\ n-i \end{smallmatrix}\right) \dim \widetilde{H}_{q+2i-n}\big(D_i(\mathbb{R}^2,S^1);\mathbb{F}_p\big) \\ &= \sum_{i=0}^{n} \left(\begin{smallmatrix} n-i+b-1 \\ n-i \end{smallmatrix}\right) N_i(n-q), \end{aligned}$$

where $N_x(y)$ is the number of double sequences of integers (3.3.4) such that

$$v \cdot \left(\begin{smallmatrix} 1 & 1 & 1 & 1 & \cdots \\ & 2 & 2 & 2 & \cdots \end{smallmatrix}\right) = y \qquad \text{and} \qquad v \cdot \left(\begin{smallmatrix} 1 & p & p^2 & \cdots \\ & p & p^2 & \cdots \end{smallmatrix}\right) = x.$$

Hence in particular $\dim H_q\big(C_n(S);\mathbb{F}_p^{(-1)}\big)$ is given by a certain weighted sum of the values marked • in Figure 3.3.1(a). Note that

$$N_x(2\lambda) = \left\{\begin{smallmatrix} 1 & (x=p\lambda) \\ 0 & (x<p\lambda) \end{smallmatrix}\right\} \quad \text{and} \quad N_x(2\lambda+1) = \left\{\begin{smallmatrix} 1 & (x=p\lambda+1) \\ 0 & (x<p\lambda+1) \end{smallmatrix}\right\}.$$

The grid of values of $N_x(y)$ therefore looks like Figure 3.3.1(b). In particular, we have $N_x(y) = 0$ whenever $y \geq \frac{2}{p}x+1$, equivalently $x \leq \frac{p}{2}(y-1)$, and therefore $H_q\big(C_n(S);\mathbb{F}_p^{(-1)}\big)$ is zero whenever $n \leq \frac{p}{2}(n-q-1)$, equivalently $q \leq \left(\frac{p-2}{p}\right)n - 1$.

This completes the proof for $\mathbb{F}_p^{(-1)}$-coefficients when $b_2(S) = 0$. When S is orientable and closed so $b_2(S) = 1$, a very similar calculation to the above results in the combinatorial formula

$$\dim H_q\big(C_n(S);\mathbb{F}_p^{(-1)}\big) \;=\; \sum_{i=0}^{n}\binom{n-i+b-1}{n-i}N_i(n-q) \;+\; \sum_{i=0}^{n-1}\binom{n-i+b-2}{n-i-1}N_i(n+1-q),$$

and again the fact that we know that $N_x(y)$ vanishes in a certain range implies that this is zero for $q \leq \big(\frac{p-2}{p}\big)n - 1$.

For $\mathbb{Q}^{(-1)}$-coefficients (and $b_2(S) = 0$) we do the above calculation, for any $0 < q < n$, and obtain equation (3.3.2) with $\mathbb{F}_p^{(-1)}$ replaced by $\mathbb{Q}^{(-1)}$. Then note that $\widetilde{H}_*(\beta_n;\mathbb{Q}) = 0$, where β_n is the Artin braid group on n strands (this is proved in [Arn70b] for example). Since $\Omega^2S^3 \simeq B\beta_\infty^+$, the Quillen plus-construction of the classifying space of $\beta_\infty = \lim_n\beta_n$, we also have $\widetilde{H}_*(\Omega^2S^3;\mathbb{Q}) = 0$, and hence by the Snaith splitting $\widetilde{H}_*(D_i(\mathbb{R}^2,S^1);\mathbb{Q}) = 0$ for all i. Therefore by equation (3.3.2) for $\mathbb{Q}^{(-1)}$,

$$H_q\big(C_n(S);\mathbb{Q}^{(-1)}\big) = 0$$

for all $0 < q < n$.

The calculation can be modified for $b_2(S) = 1$, in the same way as for $\mathbb{F}_p^{(-1)}$-coefficients, to show that $H_q\big(C_n(S);\mathbb{Q}^{(-1)}\big) = 0$ for $0 < q < n$ also holds in this case. □

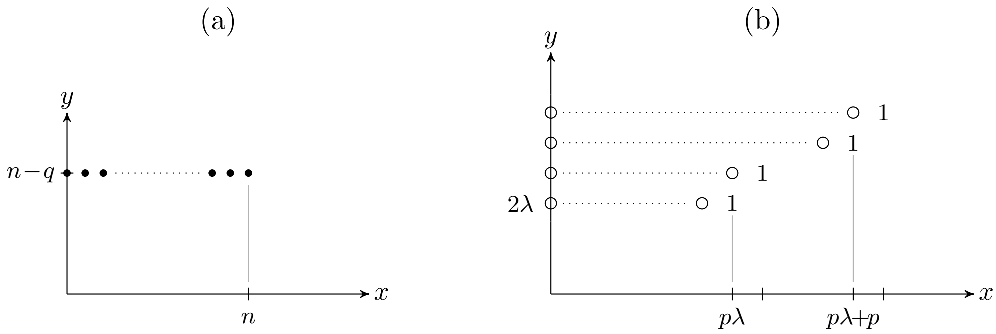


Figure 3.3.1: Schematic pictures of the values of $N_x(y)$.

Remark 3.3.1 The calculations also show that $\frac{p-2}{p}$ is the best possible slope for stability of the homology of $C_n(S)$ with coefficients in $\mathbb{F}_p$ with a sign-twisting. For example one can calculate that for all $\lambda \geq 1$, the stabilisation map

$$H_q\big(C_n(S);\mathbb{F}_p^{(-1)}\big) \longrightarrow H_q\big(C_{n+1}(S);\mathbb{F}_p^{(-1)}\big) \tag{3.3.5}$$

is $\mathbb{F}_p \to 0$ for $(n,q) = (p\lambda+1,(p-2)\lambda)$. So (3.3.5) fails to be an isomorphism on a line of slope $\frac{p-2}{p}$.

3.4 Tables

To illustrate the calculation, the tables on the following pages give the dimension of $H_q\big(C_n(S);\mathbb{F}_p^{(-1)}\big)$ for $p = 3, 5, 7$ and for the surface S equal to the plane $\mathbb{R}^2$, the sphere S^2, the torus T^2 and the once-punctured torus $T^2 \smallsetminus pt$. The number of particles n increases from left to right, and the homological degree q increases from top to bottom. One can observe the faster rate of homological stability for larger primes (namely $\frac{1}{3}$, $\frac{3}{5}$ and $\frac{5}{7}$ respectively for $p = 3$, 5 and 7), as well as some other interesting patterns.

For $S = \mathbb{R}^2$ (and $p = 3, 5, 7$ respectively).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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5	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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For $S = S^2$ (and $p = 3, 5, 7$ respectively).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
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2	1	1	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0

For $S = T^2 \setminus pt$ (and $p = 3, 5, 7$ respectively).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
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2	0	3	4	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	4	6	5	4	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	5	8	7	6	5	4	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	6	10	9	8	7	6	5	4	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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7	0	0	0	0	0	0	8	14	14	13	11	10	9	8	7	6	5	4	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	20	36	33	30	27	24	21	18	15	12	9	6	3	2	0	0	0	0	0	0	0	0

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
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2	0	3	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	5	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	6	6	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	7	8	3	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	8	10	5	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	9	12	7	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	10	14	9	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	11	16	11	6	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	12	18	13	8	3	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	13	20	15	10	5	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	14	22	17	12	7	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	15	24	19	14	9	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	16	26	21	16	11	6	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	17	28	23	18	13	8	3	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	18	30	25	20	15	10	5	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	19	32	27	22	17	12	7	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	20	34	29	24	19	14	9	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0

For $S = T^2$ (and $p = 3, 5, 7$ respectively).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	4	4	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	2	6	7	6	4	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	3	9	11	9	7	6	4	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	4	12	15	13	11	9	7	6	4	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	5	15	19	17	15	13	11	9	7	6	4	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	6	18	23	22	20	17	15	13	11	9	7	6	4	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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9	0	0	0	0	0	0	0	8	24	34	35	32	29	25	22	20	17	15	13	11	9	7	6	4	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	9	28	41	43	41	37	32	29	25	22	20	17	15	13	11	9	7	6	4	3	2	1	1	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	10	32	48	53	52	46	41	37	32	29	25	22	20	17	15	13	11	9	7	6	4	3	2	1	1	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	11	36	56	64	63	57	52	46	41	37	32	29	25	22	20	17	15	13	11	9	7	6	4	3	2	1	1	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	12	40	64	75	75	69	63	57	52	46	41	37	32	29	25	22	20	17	15	13	11	9	7	6	4	3	2	1	1
14	0	0	0	0	0	0	0	0	0	0	0	0	13	44	72	86	87	82	76	69	63	57	52	46	41	37	32	29	25	22	20	17	15	13	11	9	7	6	4	3

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
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2	1	4	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	2	6	4	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	3	8	6	3	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	4	10	9	6	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	5	13	13	9	4	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	6	16	17	12	6	3	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	7	19	21	15	9	6	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	8	22	25	19	13	9	4	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	9	25	29	23	17	12	6	3	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	10	28	33	27	21	15	9	6	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	11	31	37	31	25	19	13	9	4	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	12	34	41	35	29	23	17	12	6	3	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	13	37	45	39	33	27	21	15	9	6	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	4	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	2	6	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	3	8	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	4	10	6	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	5	12	8	3	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	6	14	11	6	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	7	17	15	9	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	8	20	19	12	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	9	23	23	15	6	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	10	26	27	18	8	3	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	11	29	31	21	11	6	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	12	32	35	25	15	9	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	13	35	39	29	19	12	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

CHAPTER 4

Twisted homological stability for configuration spaces

4.1 Introduction

Let M be a connected, open manifold of dimension at least 2, and let X be a path-connected space. Recall from §1.1 that:

Definition 4.1.1 The configuration space $C_n(M,X)$ of n unordered particles in M with labels in X is defined by

$$C_n(M,X) \coloneqq \left\{(p_1,\ldots,p_n) \in M^n \;\middle|\; p_i \text{ are pairwise distinct}\right\} \times X^n \big/ \Sigma_n,$$

where the symmetric group Σ_n acts diagonally by permuting the coordinates in M^n and X^n. We will write $C_n(M)$ if the label-space X is just a point; this is the space of *unlabelled* configurations.

The sequence $C_n(M,X)$ is known to be homologically stable as $n \to \infty$. In other words, we have an isomorphism

$$H_*(C_n(M,X)) \;\cong\; H_*(C_{n+1}(M,X))$$

whenever $n \gg *$ (in this case, the explicit range $n \geq 2*$ is known to be sufficient). The most recent proof of this is [RW11]; see [Seg73, McD75, Seg79] for earlier results using different methods.

Twisted coefficients. Several other families of groups or spaces which are homologically stable are also known to have homological stability for *twisted coefficients*. For example general linear groups [Dwy80], mapping class groups of surfaces [Iva93, CM09, Bol12] and the symmetric groups [Bet02] are known to satisfy this phenomenon.

In order for this not to be an empty statement one must say what one means by "twisted coefficients" for the sequence of spaces $\{Y_n\}$ one is considering. The minimum data needed for the question of homological stability to be defined at all is a functor $\pi_1(\{Y_n\}) \to \mathsf{Ab}$. By $\pi_1(\{Y_n\})$ we mean the category (groupoid) where the objects are the natural numbers, all morphisms are automorphisms, and $\mathrm{Aut}(n) = \pi_1(Y_n)$. In other words this is just a choice of $\pi_1(Y_n)$-module for each n. But there is no chance of stability with respect to such a general "twisted coefficient system", since the $\pi_1(Y_n)$-modules for various n need have no relation to each other. So to get a notion of twisted coefficient system that has a chance of stability one needs to add some (non-endo)morphisms to $\pi_1(\{Y_n\})$ and require that the functor from this new category to Ab satisfy some finiteness conditions defined in terms of the new morphisms. The correct way to do this depends on the specific context one is working in.

In §4.2 below we will carefully define what is meant by a *twisted coefficient system of degree d* for the configuration spaces $\{C_n(M,X)\}$. To state the main result of this chapter it is enough to note that it includes a $\pi_1(C_n(M,X))$-module T_n and a canonical map $\iota_n\colon T_n \to$

T_{n+1} for each n. Under the assumption that M is the interior of some manifold with non-empty boundary, there is a natural *stabilisation map* $s_n\colon C_n(M,X) \to C_{n+1}(M,X)$ (defined just below). The map ι_n is equivariant w.r.t. $\pi_1(s_n)$, so together these induce a map

$$(s_n;\iota_n)_*\colon H_*\bigl(C_n(M,X);T_n\bigr) \longrightarrow H_*\bigl(C_{n+1}(M,X);T_{n+1}\bigr). \tag{4.1.1}$$

Main Theorem *Let M be the interior of a connected manifold with non-empty boundary of dimension at least 2, let X be a path-connected space, and let T be any twisted coefficient system for $\{C_n(M,X)\}$ of degree d. Then the map (4.1.1) is split-injective for all values of $*$ and n, and surjective in the range $n \geq 2{*}+d$.*

This is a generalisation of the result of [Bet02], where twisted homological stability is proved for the symmetric groups $\{\Sigma_n\}$, which is the case $M = \mathbb{R}^\infty$ and $X = pt$.

Corollary 4.1.2 *In particular we have isomorphisms*

$$\begin{aligned} H_*\bigl(C_n(M,X);\mathbb{Z}\bigl[\Sigma_n/(\Sigma_k\times\Sigma_{n-k})\bigr]\bigr) &\cong H_*\bigl(C_{n+1}(M,X);\mathbb{Z}\bigl[\Sigma_{n+1}/(\Sigma_k\times\Sigma_{n+1-k})\bigr]\bigr),\\ H_*\bigl(C_n(M,X);\mathbb{Z}\bigl[\Sigma_n/\Sigma_{n-k}\bigr]\bigr) &\cong H_*\bigl(C_{n+1}(M,X);\mathbb{Z}\bigl[\Sigma_{n+1}/\Sigma_{n+1-k}\bigr]\bigr),\\ H_*\bigl(C_n(M,X);H_q(Z^n;F)\bigr) &\cong H_*\bigl(C_{n+1}(M,X);H_q(Z^{n+1};F)\bigr) \end{aligned}$$

for $n \geq 2{*}+k$ and $n \geq 2{*}+\lfloor\frac{q}{h+1}\rfloor$ respectively, where F is a field and Z is a based space with $\widetilde{H}_i(Z)=0$ for all $i\leq h$.

Proof. These follow from the Main Theorem and Examples 4.5.1 and 4.5.4 of twisted coefficient systems in §4.5. See that section for more details of these twisted coefficient systems. □

Remark 4.1.3 In the next chapter we prove that homological stability also holds for configuration spaces of *submanifolds*. The proof of the Main Theorem of this chapter actually follows from a general "twisted stability from untwisted stability" principle (§4.6), which applies equally well to the more general setting of configuration spaces of submanifolds; see §5.8 of Chapter 5.

Remark 4.1.4 There is a sequence of $\pi_1(C_n(M,X))$-modules which does not fit into the framework of this chapter (it doesn't even form a twisted coefficient system, let alone a finite-degree one), but which nevertheless does exhibit homological stability. Each element of $\pi_1(C_n(M,X))$ is either even or odd, depending on whether a loop representing it induces an even or odd permutation of the basepoint configuration. Let V be the $\pi_1(C_n(M,X))$-module $\mathbb{Z}^2$, where the even elements act by the identity and the odd elements act by swapping the two coordinates. The double cover of $C_n(M,X)$ corresponding to the subgroup of even elements is the space $C_n^+(M,X)$ of 'oriented' configurations, i.e. configurations equipped with an ordering of the n points which is only remembered up to even permutations. From the definition of twisted homology, or as a trivial application of the Serre spectral sequence for the fibration $\mathbb{Z}/2 \to C_n^+(M,X) \to C_n(M,X)$, we have

$$H_*\big(C_n^+(M,X);\mathbb{Z}\big) \;\;\cong\;\; H_*\big(C_n(M,X);V\big). \tag{4.1.2}$$

In Chapter 2 we proved that the sequence $C_n^+(M,X)$ has homological stability as $n \to \infty$ (in the range $n \geq 3{*}{+}5$), so via (4.1.2) this gives us twisted homological stability for the unordered configuration spaces $C_n(M,X)$ with coefficients in V.

The stabilisation map. Assume that the manifold M is the interior of $\bar{M}$, which has non-empty boundary. The following is one explicit model for the stabilisation map $C_n(M,X) \to C_{n+1}(M,X)$; up to homotopy it only depends on a choice of boundary component of $\bar{M}$ and path-component of X.

Definition 4.1.5 (Stabilisation map) Choose a boundary-component B of $\bar{M}$, a point $b \in B$, and a coordinate neighbourhood U of b, identified with the half-space $\mathbb{R}^d_+ = \{(x_1,\ldots,x_d) \,|\, x_1 \geq 0\}$ with b corresponding to 0. Also choose a basepoint $x_0 \in X$. Finally, choose a self-embedding $e\colon \bar{M} \hookrightarrow \bar{M}$ which is isotopic to the identity on $\bar{M}$, is *equal* to the identity outside U, and near $b = 0 \in \mathbb{R}^d_+$ is given by $x \mapsto x + (1,0,\ldots,0)$. The map $s\colon C_n(M,X) \to C_{n+1}(M,X)$ is then defined to be

$$\big\{(p_1,x_1),\ldots,(p_n,x_n)\big\} \quad\mapsto\quad \big\{(e(p_1),x_1),\ldots,(e(p_n),x_n),(e(b),x_0)\big\}.$$

A note on terminology. To keep our terminology from becoming ambiguous, we will always use "local coefficient system" and "twisted coefficient system" as follows. For a space Y, a *local coefficient system* for Y will have its usual meaning as a $\pi_1(Y)$-module,[1] whereas a *twisted coefficient system* for a sequence of spaces $\{Y_n\}$ is a local coefficient system for each Y_n, with some extra compatibility data and conditions (see Definition 4.2.12 for the precise definition).

Organisation of the chapter. We discuss twisted coefficient systems, for configuration spaces $\{C_n(M,X)\}$ and more abstractly, in §§4.2–4.5. We define them precisely in §4.2, and define the *degree* and *height* of a twisted coefficient system in §4.4. The definition of height depends on a certain decomposition result for twisted coefficient systems, which we establish in §4.3. Some examples of twisted coefficient systems are given in §4.5.

In §4.6 we state our general "twisted stability from untwisted stability" principle, and deduce from it the Main Theorem of the chapter (except the split-injectivity claim). The principle itself is proved in §4.8, after an interlude in §4.7 on a twisted version of the Serre spectral sequence which is needed in the proof. Finally, in §4.9 we prove the split-injectivity

[1] Alternatively: a functor $\pi(Y) \to \mathsf{Ab}$ from the fundamental groupoid of Y to the category of abelian groups (see §4.7), or a bundle of abelian groups over Y.

part of the Main Theorem and in §4.10 we briefly point out an interesting connection with representation stability for the cohomology of ordered configuration spaces.

4.2 Twisted coefficient systems

We first (in §4.2.1) define the notion of a twisted coefficient system for configuration spaces $\{C_n(M, X)\}$. In order to formulate a general "twisted stability from untwisted stability" principle in §4.6, we then (in §4.2.2) describe a more abstract framework in which one can define an analogous notion of "twisted coefficient system".

In later sections we will prove a decomposition theorem (§4.3) and define the *height* and the *degree* of a twisted coefficient system (§4.4) in this more abstract framework, and then give some examples of twisted coefficient systems for configuration spaces (§4.5).

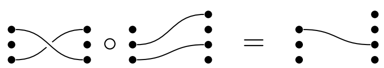
4.2.1 Twisted coefficient systems for configuration spaces

Let M be the interior of a connected manifold $\bar{M}$, with non-empty boundary and of dimension at least 2, and let X be a path-connected space. We keep the choices of Definition 4.1.5 above, and define a sequence of points $\{q_n\}$ in M by

$$q_1 = e(b), \quad q_n = e(q_{n-1}) \text{ for } n \geq 2.$$

We also choose an isotopy from the identity to $e\colon \bar{M} \hookrightarrow \bar{M}$, which gives a choice of path from q_n to q_{n+1}.

Definition 4.2.1 The category $\mathcal{B}(M, X)$ has objects tuples of elements of X, in symbols $\coprod_{n\geq 0} X^n$, and a morphism from $(x_1, \ldots, x_m)$ to $(y_1, \ldots, y_n)$ is a choice of $k \leq \min\{m, n\}$ and a path in $C_k(M, X)$ from a k-element subset of $\{(q_1, x_1), \ldots, (q_m, x_m)\}$ to a k-element subset of $\{(q_1, y_1), \ldots, (q_n, y_n)\}$, up to endpoint-preserving homotopy. Composition is given by concatenating paths and deleting configuration points for which the path is only defined half-way. For example (omitting the labels in X):



We call this the *category of partial braids on M with labels in X*. When the label-space X is just a point we will denote this category by $\mathcal{B}(M)$.

Definition 4.2.2 A *twisted coefficient system* for $\{C_n(M, X)\}$ is a functor $T\colon \mathcal{B}(M, X) \to$ Ab, where Ab denotes the category of abelian groups.

We will define the *degree* of such a twisted coefficient system in a more abstract setting in §4.4; here is the definition in the case of configuration spaces:

For every morphism ϕ of $\mathcal{B}(M,X)$ we have a commutative square of the form

$$\begin{tikzcd}
(x_1,\ldots,x_m) \arrow[r, "\iota_{\bar{x}}"] \arrow[d, "\phi"'] & (x_0,x_1,\ldots,x_m) \arrow[d, "\phi[1]"] \\
(y_1,\ldots,y_n) \arrow[r, "\iota_{\bar{y}}"'] & (x_0,y_1,\ldots,y_n)
\end{tikzcd} \tag{4.2.1}$$

The morphism $\phi[1]$ is obtained from ϕ by "pushing inwards and adding a vertical strand." More precisely, apply the embedding e to each particle in the path of configurations ϕ, then add a new particle at q_1 (labelled by x_0) which remains stationary. The morphism $\iota_{\bar{x}}\colon (x_1,\ldots,x_m) \to (x_0,x_1,\ldots,x_m)$ is the path from the configuration $\{(q_1,x_1),\ldots,(q_m,x_m)\}$ to the configuration $\{(q_2,x_1),\ldots,(q_{m+1},x_m)\}$ induced by our choice of isotopy from the identity to e. Putting this together we have an endofunctor $-[1]$ on $\mathcal{B}(M,X)$ and a natural transformation ι from the identity to $-[1]$.

Given any twisted coefficient system $T\colon \mathcal{B}(M,X) \to \mathsf{Ab}$, we get a natural transformation from T to $T\circ(-[1])$, which is a morphism in the abelian functor category $\mathsf{Ab}^{\mathcal{B}(M,X)}$. It has an obvious left-inverse, so it is split-injective. Denote its cokernel by $\Delta T\colon \mathcal{B}(M,X) \to \mathsf{Ab}$.

Definition 4.2.3 The zero functor has degree -1. Otherwise, T has degree d if ΔT has degree $d-1$.

Remark 4.2.4 Note that $\mathcal{B}(\mathbb{R}^\infty)$ is isomorphic to the category with objects $\{1,2,3,\ldots\}$ and morphisms partially-defined injections from $\{1,\ldots,m\}$ to $\{1,\ldots,n\}$ (this is called Σ in §4.2.2 below). There is a functor $\mathcal{B}(M,X) \to \mathcal{B}(\mathbb{R}^\infty)$ given by only remembering the partial injection induced by a partial braid (or equivalently by embedding M into $\mathbb{R}^\infty$), so any twisted coefficient system for $\{C_n(\mathbb{R}^\infty)\} = \{B\Sigma_n\}$ induces a twisted coefficient system for all $\{C_n(M,X)\}$ by composing with this functor.

4.2.2 A more abstract framework

We will now give a more general definition of "twisted coefficient system" which extracts the essential properties of the previous section.

Inputs for the definition.

Definition 4.2.5 Let $\underline{n}$ denote the set $\{1,2,\ldots,n\}$. Let Σ be the category with objects $\{\underline{n} \mid n \geq 1\}$ and morphisms partially-defined injective maps. In other words, a morphism in Σ from $\underline{m}$ to $\underline{n}$ is a choice of subset S of $\underline{m}$ and an injective map $S \hookrightarrow \underline{n}$. Let Σ' be the subcategory on the same objects, generated by the following two types of morphisms: (a) $i_n = (k \mapsto k+1)\colon \underline{n} \to \underline{n+1}$ for each n, and (b) morphisms which are the identity wherever they are defined.

Now let $\mathcal{C}$ be any category, and let $\mathcal{B}$ be another category with the same objects as Σ, equipped with a functor $\pi\colon \mathcal{B} \to \Sigma$ which is the identity on objects.

Definition 4.2.6 The *wreath product* category $\mathcal{C} \wr \mathcal{B}$ has objects $(c_1, \ldots, c_n)$ with $c_i \in \mathcal{C}$ and $n \geq 0$. Its morphisms from $(c_1, \ldots, c_m)$ to $(d_1, \ldots, d_n)$ consist of a morphism $\sigma \in \mathcal{B}(m, n)$ together with a morphism $c_i \to d_{\pi(\sigma)(i)}$ of $\mathcal{C}$ for each i in the domain of definition of $\pi(\sigma)$.

We also need two more technical definitions:

Definition 4.2.7 A *partial section* of the functor $\pi\colon \mathcal{B} \to \Sigma$ is a section of its restriction to $\mathcal{B}' = \pi^{-1}(\Sigma') \to \Sigma'$.

There is a "stabilising" endofunctor $-[1]\colon \Sigma \to \Sigma$ which sends $\underline{n}$ to $\underline{n+1}$ and a partial injection $j\colon \underline{m} \dashrightarrow \underline{n}$ to the partial injection $j[1]\colon \underline{m+1} \dashrightarrow \underline{n+1}$ which sends 1 to itself and k to $j(k-1)+1$ for $k \geq 2$. Furthermore, there is a natural transformation $\mathrm{id} \Rightarrow -[1]$ given by the morphisms $\{i_n\colon \underline{n} \to \underline{n+1}\}$.

Definition 4.2.8 The functor $\pi\colon \mathcal{B} \to \Sigma$ admits a *lift of the stabilising endofunctor* if there is an endofunctor $-[1]\colon \mathcal{B} \to \mathcal{B}$, equipped with a natural transformation $\mathrm{id} \Rightarrow -[1]$, such that $(-[1]) \circ \pi = \pi \circ (-[1])$. In particular this means that we have homomorphisms $\mathrm{Aut}_{\mathcal{B}}(\underline{n}) \to \mathrm{Aut}_{\mathcal{B}}(\underline{n+1})$ such that the following square commutes:

$$\begin{tikzcd}
\mathrm{Aut}_{\mathcal{B}}(\underline{n}) \arrow[r] \arrow[d, "\pi"'] & \mathrm{Aut}_{\mathcal{B}}(\underline{n+1}) \arrow[d, "\pi"] \\
\Sigma_n \arrow[r] & \Sigma_{n+1}
\end{tikzcd} \tag{4.2.2}$$

Note that the bottom horizontal map includes Σ_n as the stabiliser of 1 (not $n+1$) in Σ_{n+1}. The lift of the stabilising endofunctor is *compatible* with the partial section $s\colon \Sigma' \to \mathcal{B}'$ if the natural transformation $\mathrm{id} \Rightarrow -[1]$ for $\mathcal{B}$ is the lift $\{s(i_n)\}$ of the natural transformation $\{i_n\}\colon \mathrm{id} \Rightarrow -[1]$ for Σ_n. Hence we have

$$s(i_n) \circ \sigma = \sigma[1] \circ s(i_m) \tag{4.2.3}$$

for every morphism $\sigma\colon \underline{m} \to \underline{n}$ in $\mathcal{B}$.

Finally, fix an object c_0 of $\mathcal{C}$ and denote the n-tuple $(c_0, \ldots, c_0) \in \mathrm{ob}(\mathcal{C} \wr \mathcal{B})$ by $\overline{c}_0^n$.

Example 4.2.9 The canonical example for this is the category $\mathcal{B}(M)$ for a manifold M, as defined in §4.2.1. Forgetting everything apart from the partial injection of endpoints induced by a partial braid gives a functor $\pi\colon \mathcal{B} \to \Sigma$, as discussed in Remark 4.2.4. We chose an isotopy from $\mathrm{id}\colon M \to M$ to the self-embedding $e\colon M \hookrightarrow M$, which gives a canonical path from q_n to q_{n+1} for each n. Using these canonical paths, it is easy to construct a section of $\pi\colon \mathcal{B} \to \Sigma$ over the subcategory of all *order-preserving* morphisms of Σ. In particular this gives a partial section of π, which for example sends a morphism of type (b) in Definition 4.2.5 to the *constant* partial braid which covers it. There is also a lift of the stabilising endofunctor of Σ, which was defined in §4.2.1 and is compatible with this partial section.

If we also have a label-space X then the category $\mathcal{B}(M,X)$ decomposes as the wreath product $PX \wr \mathcal{B}(M)$, where PX denotes the path category of X.

Constructions. Suppose that we have $(\mathcal{C}, c_0)$ and $\pi\colon \mathcal{B} \to \Sigma$ as above, where π has a partial section and a compatible lift of the stabilising endomorphism of Σ. Denote a choice of partial section by $s\colon \Sigma' \to \mathcal{B}'$. We additionally assume that π is a *full* functor; since we are already assuming the existence of a partial section this is equivalent to asking for each homomorphism $\mathrm{Aut}_{\mathcal{B}}(\underline{n}) \to \Sigma_n$ to be surjective.

Definition 4.2.10 Let G_n denote the automorphism group $\mathrm{Aut}_{\mathcal{C}\wr\mathcal{B}}(\overline{c}_0^n) = \mathrm{Aut}_{\mathcal{C}}(c_0)\wr\mathrm{Aut}_{\mathcal{B}}(\underline{n})$. There is a canonical surjection

$$G_n \to \mathrm{Aut}_{\mathcal{B}}(\underline{n}) \xrightarrow{\pi} \Sigma_n;$$

denote the inverse image of $\Sigma_{n-k} \times \Sigma_k$ by $G_{k,n-k}$. From (4.2.2) we get a homomorphism $\gamma_n\colon G_n \to G_{n+1}$ which takes $G_{k,n-k}$ to $G_{k,n+1-k}$.

Now let T be any functor $\mathcal{C} \wr \mathcal{B} \to \mathsf{Ab}$ and abbreviate the G_n-module $T(\overline{c}_0^n)$ by T_n.

Definition 4.2.11 Recall that i_n is the inclusion $(k \mapsto k+1)\colon \underline{n} \to \underline{n+1}$. There is a canonical split-injective morphism $\overline{c}_0^n \to \overline{c}_0^{n+1}$ of $\mathcal{C} \wr \mathcal{B}$ given by $s(i_n)$ and n copies of the identity morphism $c_0 \to c_0$. Denote its image under T by $\iota_n\colon T_n \to T_{n+1}$.

Note that ι_n is γ_n-equivariant, essentially by (4.2.3).

The definition of a twisted coefficient system. Suppose are given a sequence of spaces

$$Y_1 \to Y_2 \to \cdots \to Y_n \xrightarrow{s_n} Y_{n+1} \to \cdots$$

Definition 4.2.12 A *twisted coefficient system for* $\{Y_n\}$ consists of the following data: a pointed category $(\mathcal{C}, c_0)$, a full functor $\pi\colon \mathcal{B} \to \Sigma$ which is the identity on objects, with a partial section and compatible lift of the stabilising endofunctor on Σ, a functor $T\colon \mathcal{C} \wr \mathcal{B} \to \mathsf{Ab}$ and an identification of $\pi_1(Y_n \xrightarrow{s_n} Y_{n+1})$ with $G_n \xrightarrow{\gamma_n} G_{n+1}$.

When the choice of the other data is clear, we refer to just the functors $T\colon \mathcal{C} \wr \mathcal{B} \to \mathsf{Ab}$ as "twisted coefficient systems". The other data just sets up the correct context in which these functors provide a "coherent" sequence of local coefficient systems for $\{Y_n\}$.

Example 4.2.13 For a sequence of configuration spaces $\{C_n(M,X)\}$ with a chosen basepoint $x_0 \in X$, we take $(\mathcal{C}, c_0)$ to be the path category (PX, x_0) and $\pi\colon \mathcal{B} \to \Sigma$ as explained in Example 4.2.9, so that $\mathcal{C} \wr \mathcal{B} = \mathcal{B}(M, X)$. Moreover there is a canonical identification

$$\begin{aligned} G_n &= \mathrm{Aut}_{PX}(x_0) \wr \mathrm{Aut}_{\mathcal{B}}(\underline{n}) \\ &= \pi_1(X, x_0) \wr \pi_1(C_n(M), \{q_1, \dots, q_n\}) \\ &= \pi_1(C_n(M, X), \{(q_1, x_0), \dots, (q_n, x_0)\}) \end{aligned}$$

under which γ_n corresponds to $(s_n)_*$. So a twisted coefficient system for $\{C_n(M, X)\}$ reduces to just a choice of functor $\mathcal{B}(M, X) \to \mathsf{Ab}$, as in Definition 4.2.2.

4.3 Decomposition of twisted coefficient systems

The following decomposition of T_n as a G_n-module will be central to the proof of twisted homological stability:

Proposition 4.3.1 *For $k = 0, \dots, n$ there is a direct summand (as abelian groups) $T_n^{(k)}$ of T_n, such that the action of $G_{k,n-k} \leq G_n$ on T_n preserves it—so it is a direct summand as a $G_{k,n-k}$-module. Moreover, there is a decomposition of T_n as a G_n-module:*

$$T_n \cong \bigoplus_{k=0}^{n} \left(\mathbb{Z}G_n \otimes_{G_{k,n-k}} T_n^{(k)} \right). \tag{4.3.1}$$

This identification is natural in the sense that $\iota_n \colon T_n \to T_{n+1}$ sends $T_n^{(k)}$ into $T_{n+1}^{(k)}$, and the map of the right-hand side induced by ι_n and γ_n corresponds under (4.3.1) to ι_n on the left-hand side.

This can be used to define the *height* of a twisted coefficient system (which will be related to its *degree* in §4.4 below).

Definition 4.3.2 A twisted coefficient system $T \colon \mathcal{C} \wr \mathcal{B} \to \mathsf{Ab}$ has height -1 if $T_n = 0$ for all n. Otherwise, T has height $h \geq 0$ if $T_n^{(k)} = 0$ for all n and all $k > h$, but $T_n^{(h)} \neq 0$ for some n. In other words, it is the height at which the decomposition (4.3.1) is truncated.

Remark 4.3.3 There is a general theory of cross-effects of a functor $\mathcal{C} \to \mathcal{A}$, where $\mathcal{C}$ is a pointed monoidal category and $\mathcal{A}$ is an abelian category, which includes a decomposition of the form (4.3.1). This goes back to [EML54] (see in particular §9); for a modern reference see for example [HPV12, §2]. Since our category $\mathcal{C} \wr \mathcal{B}$ is always pointed monoidal, the decomposition (4.3.1) is a special case of this theory (see [HPV12, Proposition 2.4]) and our functors $\mathcal{C} \wr \mathcal{B} \to \mathsf{Ab}$ of height h are "polynomial functors of degree at most h" in the language of cross-effects of functors.

However, the proof of the decomposition (4.3.1) is fairly elementary (although slightly involved), and it will be useful to have the details of the proof available to see the relation between height and degree, so in the rest of this section we will carefully establish this decomposition.[2] First we need to define some endomorphisms and subgroups of T_n.

Definition 4.3.4 For $S \subseteq \underline{n}$, consider the morphism $\sigma_S \colon \underline{n} \to \underline{n}$ of Σ which "forgets S", in other words $i \mapsto i$ for $i \in \underline{n} \smallsetminus S$ and undefined for $i \in S$. This is a morphism of type (b) in Definition 4.2.5, so it has a chosen lift to a morphism $s(\sigma_S) \colon \underline{n} \to \underline{n}$ of $\mathcal{B}$. Define $f_S \colon \bar{c}_0^n \to \bar{c}_0^n$ in $\mathcal{C} \wr \mathcal{B}$ to be the morphism $s(\sigma_S)$ together with the identity morphism $c_0 \to c_0$ for each $i \in \underline{n} \smallsetminus S = \mathrm{dom}(\sigma_S)$.

The induced maps $Tf_S \colon T_n \to T_n$ are group homomorphisms, but not in general G_n-module homomorphisms.

For $p \geq 0$ and $\{S_1, \ldots, S_p\}$ a partition of $S \subseteq \underline{n}$, define

$$T_n[S_1|\cdots|S_p] \;\coloneqq\; \mathrm{im}(Tf_{\underline{n} \smallsetminus S}) \cap \bigcap_{i=1}^{p} \ker(Tf_{S_i}).$$

These are subgroups, but not sub-G_n-modules, of T_n. Often we will write $T_n[S^\delta]$, where S^δ is understood to mean the *discrete* partition of S. In particular we define

$$T_n^{(k)} \;\coloneqq\; T_n[\{n-k+1, \ldots, n\}^\delta].$$

Aside on a pictorial notation. We briefly describe a pictorial notation for visualising morphisms of the wreath product category $\mathcal{C} \wr \mathcal{B}$.

Notation 4.3.5 Imagine a morphism of $\mathcal{C} \wr \mathcal{B}$ from $(c_1, \ldots, c_m)$ to $(d_1, \ldots, d_n)$ as a diagram of the form depicted in Fig.4.3.1(a), with sticks indicating the partial injection in Σ (the actual lift to a morphism of $\mathcal{B}$ is left implicit in this notation) and each stick decorated by a morphism of $\mathcal{C}$ between the objects at its endpoints. Composition is concatenation of two such diagrams, deleting any sticks that only make it half-way across the whole diagram, and composing the morphisms of $\mathcal{C}$ when two sticks are glued together. For morphisms from $\bar{c}_0^m$ to $\bar{c}_0^n$ we drop the labelling of nodes, and just indicate the m and n unless there is no ambiguity which objects the morphism is between—see Fig.4.3.1(b). A grey box (Fig.4.3.1(c)) is shorthand for a collection of parallel sticks, each decorated by the identity map $c_0 \to c_0$. (Note that in Figures 4.3.1 (c) and (d) the morphism of $\mathcal{B}$ is unambiguous; it is determined by the partial section s.) By abuse of notation, the image under T of this morphism, a map $T_m \to T_n$, is denoted by the same diagram. For example $\iota_n \colon T_n \to T_{n+1}$ is the map in Fig.4.3.1(d). We write function composition and application from left to right for these diagrams, so $x = \iota_n(y)$ is written as in Fig.4.3.1(e).

Remark 4.3.6 This pictorial notation is convenient to see for example that the map $\iota_n \colon T_n \to T_{n+1}$ sends $T_n^{(k)}$ into $T_{n+1}^{(k)}$ (as claimed in Proposition 4.3.1):

[2]The proof we give here was informed in part by reading [CDG11]; however, we note that the proof of their Lemme 1.5 (which corresponds to our decomposition (4.3.4) below) contains an error.

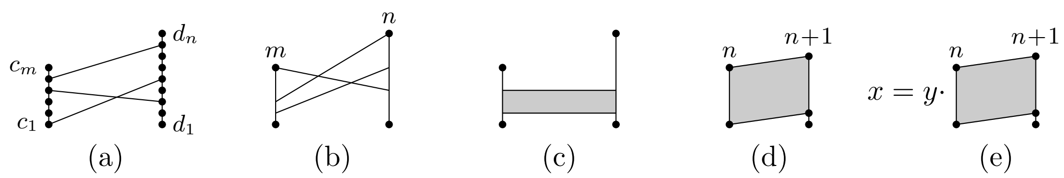


Figure 4.3.1: Some pictures of morphisms of $\mathcal{C} \wr \mathcal{B}$.

Suppose that $x = \iota_n(y)$ for $y \in T_n^{(k)}$. Then by definition $y = z \cdot$ for some $z \in T_n$ (the grey box has height k). Note that the composition is the identity, so the second map is split-surjective, and so $z = w \cdot$ for some $w \in T_{n+1}$. Hence

$$x = w \cdot \quad = \quad w \cdot$$

so $x \in \mathrm{im}(Tf_{\underline{n-k+1}})$. For any $n-k+2 \leq i \leq n+1$, we have

$$x \cdot \;{}^{i} \quad = \quad y \cdot \;{}^{i} \quad = \quad y \cdot \;{}^{i-1} \quad = \quad 0 \cdot \quad = \quad 0$$

since $y \in T_n^{(k)}$, so $x \in \ker(Tf_{\{i\}})$. Hence we have verified that $x \in T_{n+1}^{(k)}$.

Lemma 4.3.7 *For $k \leq m \leq n$, the map*

$$\iota_m^n = \iota_{n-1} \circ \cdots \circ \iota_m \colon T_m \to T_n$$

is split-injective and sends $T_m^{(k)}$ to $T_n^{(k)}$. Moreover, the restriction to $T_m^{(k)} \to T_n^{(k)}$ is a bijection.

Proof. The first sentence follows from the remark above. Given $x \in T_n^{(k)}$, we need to show that $x = \iota_m^n(z)$ for some $z \in T_m^{(k)}$. First note that $x = y \cdot$ for some $y \in T_n$ (the grey box has height k), and let $z := y \cdot \in T_m$. Then $\iota_m^n(z) = z \cdot \quad = \quad y \cdot \quad = \quad y \cdot$ $= x$, so we just need to check that $z \in T_m^{(k)}$. Firstly,

$$z = y \cdot \quad = \quad y \cdot$$

so $z \in \mathrm{im}(Tf_{\underline{m-k}})$. Secondly, for any $m-k+1 \leq i \leq m$, we have

$$z \cdot \; {}_{i} \quad = \quad z \cdot \;{}^{i+n-m} \quad = \quad x \cdot \;{}^{i+n-m} \quad = \quad 0$$

since $x \in T_n^{(k)}$. But ι_m^n is split-injective, so $z \in \ker(Tf_{\{i\}})$. $\square$

Remark 4.3.8 From now on, for typographical and space reasons, we will revert to using symbols rather than the pictorial notation. Any equations below which look a little too dense can be converted into cartoons as above, which are somewhat more enlightening.

Proof of the decomposition of T_n.

Observation 4.3.9 Some immediate observations from Definition 4.3.4 above are

- $\{Tf_S \mid S \subseteq \underline{n}\}$ is a set of idempotents on T_n.
- The composition of Tf_{S_1} and Tf_{S_2} is $Tf_{S_1 \sqcup S_2}$, so in particular $\{Tf_S \mid S \subseteq \underline{n}\}$ pairwise commute.
- $T_n[\,] = \mathrm{im}(Tf_{\underline{n}})$ and $T_n[\underline{n}] = \mathrm{im}(Tf_\varnothing) \cap \ker(Tf_{\underline{n}}) = \ker(Tf_{\underline{n}})$, since $f_\varnothing = \mathrm{id}$, so:

$$T_n = \mathrm{im}(Tf_{\underline{n}}) \oplus \ker(Tf_{\underline{n}}) = T_n[\,] \oplus T_n[\underline{n}]. \tag{4.3.2}$$

The decomposition (4.3.1) will follow by induction from the next lemma.

Lemma 4.3.10 *For all $\{S_1, \ldots, S_p\}$ partitioning $S \subseteq \underline{n}$, with $p \geq 2$, there is a split short exact sequence*

$$0 \to T_n[S_1|\cdots|S_p] \hookrightarrow T_n[S_1 \sqcup S_2|\cdots|S_p] \twoheadrightarrow T_n[S_1|S_3|\cdots|S_p] \oplus T_n[S_2|\cdots|S_p] \to 0.$$

The first map is inclusion, and a section of the second map is given by inclusion of each of the two factors, so in other words we have a decomposition

$$T_n[S_1 \sqcup S_2|\cdots|S_p] = T_n[S_1|\cdots|S_p] \oplus T_n[S_2|\cdots|S_p] \oplus T_n[S_1|S_3|\cdots|S_p].$$

Proof. One can check from the definitions that the following facts are true:[3]

1. Tf_{S_2} restricts to a map $T_n[S_1 \sqcup S_2|\cdots|S_p] \to T_n[S_1|S_3|\cdots|S_p]$, and similarly Tf_{S_1} restricts to a map $T_n[S_1 \sqcup S_2|\cdots|S_p] \to T_n[S_2|\cdots|S_p]$.
2. $T_n[S_1|S_3|\cdots|S_p]$ and $T_n[S_2|\cdots|S_p]$ are subgroups of $T_n[S_1 \sqcup S_2|\cdots|S_p]$.
3. For $i, j = 1, 2$, if $x \in T_n[S_i|S_3|\cdots|S_p]$, then $Tf_{S_j}(x)$ is x for $i \neq j$ and 0 for $i = j$.

These facts imply that the map (Tf_{S_2}, Tf_{S_1}) restricts to the required split surjection (with a section given by inclusion of each factor). The kernel of this is

$$\begin{aligned} &T_n[S_1 \sqcup S_2|S_3|\cdots|S_p] \cap \ker(Tf_{S_1}) \cap \ker(Tf_{S_2}) \\ &= \mathrm{im}(Tf_{\underline{n} \smallsetminus S}) \cap \bigcap_{i=3}^{p} \ker(Tf_{S_i}) \cap \ker(Tf_{S_1 \sqcup S_2}) \cap \ker(Tf_{S_1}) \cap \ker(Tf_{S_2}) \\ &= T_n[S_1|\cdots|S_p], \end{aligned}$$

[3]This can be seen using the pictorial notation above, or in symbols as follows:

1. Let $x \in T_n[S_1 \sqcup S_2|\cdots|S_p]$, so in other words $x = Tf_{\underline{n} \smallsetminus S}(y)$, $Tf_{S_1 \sqcup S_2}(x) = 0$ and $Tf_{S_i}(x) = 0$ for $i \geq 3$. Then $Tf_{S_2}(x) = Tf_{S_2}Tf_{\underline{n} \smallsetminus S}(y) = Tf_{\underline{n} \smallsetminus (S \smallsetminus S_2)}(y) \in \mathrm{im}(Tf_{\underline{n} \smallsetminus (S \smallsetminus S_2)})$, and we have $Tf_{S_i}Tf_{S_2}(x) = Tf_{S_2}Tf_{S_i}(x) = 0$ (for $i \geq 3$) and $Tf_{S_1}Tf_{S_2}(x) = Tf_{S_1 \sqcup S_2}(x) = 0$. Hence $Tf_{S_2} \in T_n[S_1|S_3|\cdots|S_p]$.
2. Let $y \in T_n[S_1|S_3|\cdots|S_p]$. Then $y = Tf_{\underline{n} \smallsetminus (S \smallsetminus S_2)}(z) = Tf_{\underline{n} \smallsetminus S}Tf_{S_2}(z) \in \mathrm{im}(Tf_{\underline{n} \smallsetminus S})$ and $Tf_{S_1 \sqcup S_2}(y) = Tf_{S_2}Tf_{S_1}(y) = 0$. Hence $y \in T_n[S_1 \sqcup S_2|\cdots|S_p]$.
3. Let $x \in T_n[S_1|S_3|\cdots|S_p]$. Then $Tf_{S_1}(x) = 0$ by definition. Also, $x = Tf_{\underline{n} \smallsetminus (S \smallsetminus S_2)}(y)$, so $Tf_{S_2}(x) = Tf_{S_2}Tf_{\underline{n} \smallsetminus (S \smallsetminus S_2)}(y) = Tf_{\underline{n} \smallsetminus (S \smallsetminus S_2)}(y) = x$.

since $\ker(Tf_{S_1}) \subseteq \ker(Tf_{S_1 \sqcup S_2})$. $\square$

Proposition 4.3.11 *For any $\varnothing \neq S \subseteq \underline{n}$ and $R \subseteq \underline{n} \smallsetminus S$ there is a decomposition*

$$T_n[S|R^\delta] = \bigoplus_{\varnothing \neq Q \subseteq S} T_n[(Q \sqcup R)^\delta]. \tag{4.3.3}$$

As before, Q^δ denotes the discrete partition of Q, so for example $T_n[\{1,2\}|\{3,4,5\}^\delta]$ means $T_n[\{1,2\}|\{3\}|\{4\}|\{5\}]$. Note that this decomposition is an equality of subgroups, not just an abstract isomorphism of groups.

Proof. The $|S| = 1$ case is obvious, so we assume that $|S| \geq 2$ and assume the theorem for smaller values of $|S|$ by induction. Pick an element $s \in S$. Then by Lemma 4.3.10,

$$T_n[S|R^\delta] \;=\; T_n[S \smallsetminus \{s\}|(R \sqcup \{s\})^\delta] \;\oplus\; T_n[S \smallsetminus \{s\}|R^\delta] \;\oplus\; T_n[\{s\}|R^\delta].$$

Apply the inductive hypothesis to the right-hand side. The proposition then follows from the observation that for $\varnothing \neq Q \subseteq S$, exactly one of the following holds: (i) $s \in Q$ but $Q \neq \{s\}$; (ii) $s \notin Q$; (iii) $Q = \{s\}$. $\square$

Finally, we can apply a special case of this proposition to obtain the desired decomposition (4.3.1) of T_n.

Proof of Proposition 4.3.1. Combining (4.3.3) with $R = \varnothing$ and $S = \underline{n}$ with (4.3.2) we obtain:

$$T_n = \bigoplus_{k=0}^{n} \bigoplus_{\substack{Q \subseteq \underline{n} \\ |Q| = k}} T_n[Q^\delta]. \tag{4.3.4}$$

The action of G_n on T_n permutes the summands via the projection $G_n \to \Sigma_n$ and the obvious action of Σ_n on subsets of $\underline{n}$. So:

- $T_n^{(k)} = T_n[\{n-k+1, \ldots, n\}^\delta]$ is preserved by the action of $G_{k,n-k} \leq G_n$ on T_n.
- The G_n-action on T_n preserves the outer direct sum.
- The inner direct sum is the induced module $\mathrm{Ind}_{G_{k,n-k}}^{G_n} T_n^{(k)} = \mathbb{Z}G_n \otimes_{G_{k,n-k}} T_n^{(k)}$.

This proves the decomposition of G_n-modules (4.3.1). We proved in Remark 4.3.6 above that $\iota_n \colon T_n \to T_{n+1}$ sends $T_n^{(k)}$ into $T_{n+1}^{(k)}$, and the naturality statement is clear. $\square$

4.4 Height and degree

Recall (Definition 4.3.2) that the *height* of a twisted coefficient system is the 'height' at which its decomposition (4.3.1) is truncated. We now define the *degree* of T as a direct generalisation of that given in §4.2.1.

Construction 4.4.1 The fact that $\pi\colon \mathcal{B} \to \Sigma$ admits a lift of the stabilising endofunctor of Σ means that we can also define a "stabilising endofunctor" on $\mathcal{C} \wr \mathcal{B}$. For each morphism $\phi\colon (c_1, \ldots, c_m) \to (d_1, \ldots, d_n)$ of $\mathcal{C} \wr \mathcal{B}$ we have a commutative square

$$\begin{tikzcd}
(c_1, \ldots, c_m) \arrow[r, "\iota_{\overline{c}}"] \arrow[d, "\phi"'] & (c_0, c_1, \ldots, c_m) \arrow[d, "\phi[1]"] \\
(d_1, \ldots, d_n) \arrow[r, "\iota_{\overline{d}}"'] & (c_0, d_1, \ldots, d_n)
\end{tikzcd} \tag{4.4.1}$$

where $\iota_{\overline{c}}$ is given by $s(i_m)$ and $\mathrm{id}_{c_1}, \ldots, \mathrm{id}_{c_m}$. If the morphism ϕ is given by $\sigma \in \mathcal{B}(\underline{m}, \underline{n})$ together with $c_k \to d_{\pi(\sigma)(k)}$ for each $k \in \mathrm{dom}(\pi(\sigma))$, then $\phi[1]$ is given by $\sigma[1]$ together with these morphisms plus id_{c_0}. Commutativity of the square is due to (4.2.3), i.e. *compatibility* of the lift of the stabilisation endofunctor with the partial section s.

This gives us an endofunctor $-[1]$ on $\mathcal{C} \wr \mathcal{B}$ and a natural transformation $\iota\colon \mathrm{id} \Rightarrow -[1]$. Given any functor $T\colon \mathcal{C} \wr \mathcal{B} \to \mathsf{Ab}$, it induces a natural transformation $T \Rightarrow T \circ (-[1])$, which is a morphism in the abelian category $\mathrm{Ab}^{\mathcal{C} \wr \mathcal{B}}$. Let $\Delta T\colon \mathcal{C} \wr \mathcal{B} \to \mathsf{Ab}$ be its cokernel.

In less fancy language: apply T to the diagram (4.4.1), take cokernels in the horizontal direction, and call this $\Delta T(\phi)\colon \Delta T(\overline{c}) \to \Delta T(\overline{d})$. The above says that this fits together into a new functor $\Delta T\colon \mathcal{C} \wr \mathcal{B} \to \mathsf{Ab}$.

Definition 4.4.2 The *degree* of a twisted coefficient system $T\colon \mathcal{C} \wr \mathcal{B} \to \mathsf{Ab}$ is defined inductively as follows. If $T_n = T(\overline{c}_0^n) = 0$ for all n then $\deg(T) = -1$. Otherwise, T has degree d if ΔT has degree $d - 1$.

Note that this specialises to the definition of degree given in §4.2.1, since in this case (4.4.1) = (4.2.1). The degree of T is related to its height as follows:

Lemma 4.4.3 *For any twisted coefficient system $T\colon \mathcal{C} \wr \mathcal{B} \to \mathsf{Ab}$,*

$$\mathrm{height}(T) \leq \deg(T).$$

Proof. We will use induction on d to prove the statement

$$\deg(T) \leq d \Rightarrow \mathrm{height}(T) \leq d \tag{IH$_d$}$$

for all $d \geq -1$, using the decomposition (4.3.4) above:

$$T_n \;=\; \bigoplus_{S \subseteq \underline{n}} T_n[S^\delta] \tag{4.4.2}$$

Recall that $\mathrm{height}(T) \leq d$ if and only if $T_n[S^\delta] = 0$ for all $|S| > d$ and all n.

When $d = -1$ the definitions of height and degree coincide. This deals with the base case, so let $d \geq 0$ and assume that (IH$_{d-1}$) holds. For all n we have a split short exact sequence $0 \to T_n \to T_{n+1} \to \Delta T_n \to 0$. Applying (4.4.2), this is

$$0 \to \bigoplus_{S \subseteq \underline{n}} T_n[S^\delta] \longrightarrow \bigoplus_{R \subseteq \underline{n+1}} T_{n+1}[R^\delta] \longrightarrow \bigoplus_{Q \subseteq \underline{n}} \Delta T_n[Q^\delta] \to 0.$$

Analysing the maps carefully we see that

(a) $T_n[S^\delta]$ is sent isomorphically onto $T_{n+1}[(S+1)^\delta]$ by the first map.
(b) $T_{n+1}[(Q \sqcup \{1\})^\delta]$ is sent isomorphically onto $\Delta T_n[(Q-1)^\delta]$ by the second map.

Suppose that $\deg(T) \leq d$. Then $\deg(\Delta T) \leq d-1$ by the definition of degree, and so by the inductive hypothesis (IH_{d-1}), $\mathrm{height}(\Delta T) \leq d-1$. By fact (b) above this implies that

$$T_{n+1}[R^\delta] = 0 \text{ whenever } |R| > d \text{ and } 1 \in R. \tag{4.4.3}$$

For any fixed k, the subgroups $\{T_{n+1}[R^\delta] \mid |R| = k\}$ are all abstractly isomorphic via the action of G_{n+1} on T_{n+1}. Also note that $d \geq 0$, so that $|R| > 0$, i.e. $R \neq \varnothing$. Hence:

$$T_{n+1}[R^\delta] = 0 \text{ for } \textit{all } |R| > d. \tag{4.4.4}$$

Therefore by (a), $T_n[S^\delta] = 0$ for all $|S| > d$; in other words, $\mathrm{height}(T) \leq d$. □

Remark 4.4.4 To prove that $\mathrm{height}(T) = \deg(T)$, one could try to reverse the argument above to get the other inequality. This goes wrong in one place though: Above we were able to deduce (4.4.4) from (4.4.3) because for every $|R| > d$, there is an R' of the same cardinality which contains 1. However, for the converse we would need to deduce (4.4.4) from:

$$T_{n+1}[R^\delta] = 0 \text{ whenever } |R| > d \text{ and } 1 \notin R. \tag{4.4.5}$$

Now there *is* a subset $R \subseteq \underline{n+1}$ for which there does not exist $R' \subseteq \underline{n+1}$ of the same cardinality and not containing 1; namely $\underline{n+1}$ itself.

This is the basic asymmetry which prevented us from proving an *equality* between height and degree. However, we do not know of any example for which the inequality $\mathrm{height}(T) \leq \deg(T)$ is strict.

Remark 4.4.5 The height of a twisted coefficient system is useful for the proof of the Main Theorem, but the degree is often easier to check in examples. We will prove the Main Theorem under the assumption that the height of T is at most d; in light of Lemma 4.4.3 this implies that it holds whenever the degree is at most d.

Remark 4.4.6 The notion of 'height' in this chapter is the same as the notion of degree in [Bet02] (for twisted coefficient systems for symmetric groups) and [Dwy80] (for general linear groups), whereas the notion of 'degree' in this chapter is in the same spirit as the notion of degree in [Iva93], [CM09] and [Bol12] (for mapping class groups of surfaces). Hence Lemma 4.4.3 provides a link between these two notions of degree.

Finally, we mention a couple of immediate facts about the degree of a twisted coefficient system.

Lemma 4.4.7 *For twisted coefficient systems $T, T' \colon \mathcal{C} \wr \mathcal{B} \to \mathsf{Ab}$ and a fixed abelian group A,*

(a) $\deg(T \oplus T') = \max\{\deg(T), \deg(T')\}$,
(b) $\deg(T \otimes A) \leq \deg(T)$,
and more generally, for $\deg(T)$ *and* $\deg(T')$ *non-negative,*
(c) $\deg(T \otimes T') \leq \deg(T) + \deg(T')$,

where $\oplus$ and $\otimes$ are defined objectwise.

Proof. Fact (a) follows by induction from the fact that $\Delta(T \oplus T') = \Delta T \oplus \Delta T'$. Fact (b) follows from the fact that $\Delta(T \otimes A) = \Delta T \otimes A$, which is true because tensoring a *split* short exact sequence with A preserves split-exactness. Fact (c) is proved by induction with base case (b), and inductive step using the fact that

$$\Delta(T \otimes T') \;=\; (T \otimes \Delta T') \oplus (\Delta T \otimes T') \oplus (\Delta T \otimes \Delta T'). \qquad \square$$

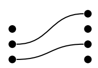
4.5 Examples of twisted coefficient systems

Our two main examples will be in the case where $\mathcal{C}$ is the trivial category (one object, one morphism) and π is the identity $\mathcal{B} \to \mathcal{B}$. So $\mathcal{C} \wr \mathcal{B} = \Sigma = \mathcal{B}(\mathbb{R}^\infty)$ and functors $\Sigma \to \mathsf{Ab}$ are twisted coefficient systems for $\{B\Sigma_n\} = \{C_n(\mathbb{R}^\infty)\}$. Recall that there is a canonical functor $\mathcal{C} \wr \mathcal{B} \to \Sigma$ for any other choice of $\mathcal{C}$ and $\pi \colon \mathcal{B} \to \Sigma$, so functors $\Sigma \to \mathsf{Ab}$ also give twisted coefficient systems for $\{C_n(M, X)\}$ in general.

Example 4.5.1 Fix a path-connected based space $(Z, *)$, an integer $q \geq 0$ and a field F. The functor $\hat{T}_Z \colon \Sigma \to \mathsf{Top}$ is defined on objects by $n \mapsto Z^n$, and on morphisms as follows: given a partially-defined injection $j \colon \{1, \ldots, m\} \dashrightarrow \{1, \ldots, n\}$ in Σ, define $\hat{T}_Z(j) \colon Z^m \to Z^n$ to be the map

$$(z_1, \ldots, z_m) \mapsto (z_{j^{-1}(1)}, \ldots, z_{j^{-1}(n)}),$$

where $z_\varnothing$ is taken to mean the basepoint $*$, for example



$$: \quad (z_1, z_2, z_3) \mapsto (*, z_1, *, z_2).$$

The functor $T_{Z,q,F} \colon \Sigma \to \mathsf{Ab}$ is then the composite functor $H_q(-; F) \circ \hat{T}_Z$.

Lemma 4.5.2 *The twisted coefficient system $T_{Z,q,F}$ has degree at most $\lfloor \frac{q}{h+1} \rfloor$, where for a path-connected space Z,*

$$h = h\mathrm{conn}_F(Z) \coloneqq \max\{k \geq 0 \mid \widetilde{H}_i(Z; F) = 0 \textit{ for all } i \leq k\} \geq 0.$$

Proof. First note that the Künneth theorem gives us natural split short exact sequences

$$0 \to H_q(Z^n; F) \longrightarrow H_q(Z^{n+1}; F) \longrightarrow \bigoplus_{i=1}^{q} H_{q-i}(Z^n; F) \otimes H_i(Z; F) \to 0,$$

which together with the fact that $H_i(Z; F) = 0$ for $1 \leq i \leq h$ implies that

$$\Delta T_{Z,q,F} = \bigoplus_{i=h+1}^{q} T_{Z,q-i,F} \otimes H_i(Z; F).$$

So by Lemma 4.4.7 above, $\deg(T_{Z,q,F}) \leq 1 + \max\{\deg(T_{Z,q-i,F}) \mid h+1 \leq i \leq q\}$. Abbreviating $\deg(T_{Z,q,F})$ to t_q, we have the recurrence inequality

$$t_q \;\leq\; 1 + \max\{t_0, \ldots, t_{q-h-1}\}. \tag{4.5.1}$$

Note that $H_0(Z^n; F) \to H_0(Z^{n+1}; F)$ is the identity map $F \to F$ for all n, so $\Delta T_{Z,0,F} = 0$, and hence $\deg(T_{Z,0,F}) = 0$. Also note that for $1 \leq q \leq h$, $h\mathrm{conn}_F(Z) \geq q$ implies that $h\mathrm{conn}_F(Z^n) \geq q$ for all n (by the Künneth theorem), so $T_{Z,q,F}(n) = H_q(Z^n; F) = 0$, and hence $\deg(T_{Z,q,F}) = -1 \leq 0$. So we also have the initial conditions

$$t_0, t_1, \ldots, t_h \leq 0. \tag{4.5.2}$$

It now remains to prove that the recurrence inequality (4.5.1) and the initial conditions (4.5.2) imply that $t_q \leq \lfloor \frac{q}{h+1} \rfloor$ for all $q \geq 0$. This will be done by induction on q. The base case is $0 \leq q \leq h$ which is covered by the initial conditions (4.5.2). Assume that $q \geq h+1$. Then:

$$\begin{aligned} t_q &\leq 1 + \max\{t_0, \ldots, t_{q-h-1}\} \\ &\leq 1 + \lfloor \tfrac{q-h-1}{h+1} \rfloor \\ &= \lfloor \tfrac{q}{h+1} \rfloor \end{aligned}$$

□

Remark 4.5.3 See also [Han09a, Proposition 12], where it is proved (in the terminology of this chapter) that the height of $T_{Z,q,F}$ is at most q.

Example 4.5.4 Let Γ^{op} be the category of finite sets and partially-defined functions. There is a free functor $\mathbb{Z}-\colon \Gamma^{\mathrm{op}} \to \mathsf{Ab}$ taking S to $\mathbb{Z}S$ and taking $j\colon S \dashrightarrow R$ to the homomorphism $\sum_{s\in S} n_s s \mapsto \sum_{s\in S} n_s j(s)$, where $j(s)$ means $0 \in \mathbb{Z}R$ if j is undefined on s. So any functor $\Sigma \to \Gamma^{\mathrm{op}}$ gives a twisted coefficient system for Σ by composing with $\mathbb{Z}-$.

For example one can just take $\Sigma \hookrightarrow \Gamma^{\mathrm{op}}$ to be the inclusion as a subcategory. More generally, for $0 \leq a \leq b$ one can take the functor $P_{a,b}\colon \Sigma \to \Gamma^{\mathrm{op}}$ which on objects is

$$\underline{n} \;\mapsto\; P_{a,b}(\underline{n}) = \{S \subseteq \underline{n} \mid a \leq |S| \leq b\}$$

and which takes $j\colon \{1,\dots,m\} \dashrightarrow \{1,\dots,n\}$ to

$$S \;\mapsto \begin{cases} j(S) & \text{if } a \leq |j(S)| \leq b \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{4.5.3}$$

Denote the composite functor $\Sigma \to \mathsf{Ab}$ by $\mathbb{Z}P_{a,b}$. Note that $\mathbb{Z}P_{b,b}(n) \cong \mathbb{Z}\big[\Sigma_n/(\Sigma_b \times \Sigma_{n-b})\big]$ as Σ_n-modules.

From the definitions one can check that $\Delta\mathbb{Z}P_{0,0} = 0$, $\Delta\mathbb{Z}P_{0,b} = \mathbb{Z}P_{0,b-1}$ for $b \geq 1$, and $\Delta\mathbb{Z}P_{a,b} = \mathbb{Z}P_{a-1,b-1}$ for $a \geq 1$. Hence by induction,

$$\deg(\mathbb{Z}P_{a,b}) = b.$$

(With a bit more work, one can check directly that the height of $\mathbb{Z}P_{a,b}$ is also exactly b.)

There is also an ordered version of this. The functor $\widetilde{P}_{a,b}\colon \Sigma \to \Gamma^{\mathrm{op}}$ takes $\underline{n}$ to the *ordered* subsets of $\underline{n}$ with cardinality between a and b, and it is defined on morphisms as above, where $j(S)$ inherits its ordering from S. Again, denote the composite functor $\Sigma \to \mathsf{Ab}$ by $\mathbb{Z}\widetilde{P}_{a,b}$. Note that $\mathbb{Z}\widetilde{P}_{b,b}(n) \cong \mathbb{Z}\big[\Sigma_n/\Sigma_{n-b}\big]$ as Σ_n-modules.

To find the degree of $\mathbb{Z}\widetilde{P}_{a,b}$ we need to consider something slightly more general. For $0 \leq a \leq b$ and a finite set R disjoint from $\{1,2,3,\dots\}$, let $\widetilde{P}^R_{a,b}$ be the functor $\Sigma \to \Gamma^{\mathrm{op}}$ which takes $\underline{n}$ to the set of subsets $S \subseteq \underline{n}$ of cardinality between a and b, equipped with an ordering of $S \sqcup R$. Then one can check from the definitions that $\Delta\mathbb{Z}\widetilde{P}^R_{0,0} = 0$, $\Delta\mathbb{Z}\widetilde{P}^R_{0,b} = \mathbb{Z}\widetilde{P}^{R^+}_{0,b-1}$ for $b \geq 1$, and $\Delta\mathbb{Z}\widetilde{P}^R_{a,b} = \mathbb{Z}\widetilde{P}^{R^+}_{a-1,b-1}$ for $a \geq 1$, where $R^+ = R \sqcup \{*\}$. Hence by induction on b,

$$\deg(\mathbb{Z}\widetilde{P}^R_{a,b}) = b.$$

Remark 4.5.5 For any $\mathcal{C}$ and $\pi\colon \mathcal{B} \to \Sigma$, denote the canonical map $\mathcal{C} \wr \mathcal{B} \to \Sigma$ by p. Then given any functor $T\colon \Sigma \to \mathsf{Ab}$ there is a composite functor $T \circ p\colon \mathcal{C} \wr \mathcal{B} \to \mathsf{Ab}$. The degree and height of T are the same as the degree and height of $T \circ p$,[4] so the preceding two examples also give twisted coefficient systems, of the stated degree, for any $\mathcal{C}$ and $\pi\colon \mathcal{B} \to \Sigma$.

An aside on Burau representations. So far we have just constructed examples of twisted coefficient systems which factor through the canonical projection to Σ. It would be interesting to have some examples of twisted coefficient systems $\mathcal{B}(\mathbb{R}^2) \to \mathsf{Ab}$, for example, which are not pulled back from a twisted coefficient system $\Sigma \to \mathsf{Ab}$ in this way.

Any collection of representations of the braid groups $\beta_n = \pi_1 C_n(\mathbb{R}^2)$, one for each n, gives a functor $\mathcal{B}(\mathbb{R}^2)_{\mathrm{aut}} \to \mathsf{Ab}$, where $\mathcal{B}(\mathbb{R}^2)_{\mathrm{aut}}$ is the subcategory of all automorphisms of $\mathcal{B}(\mathbb{R}^2)$. The question is then whether the representations extend to a functor on all of $\mathcal{B}(\mathbb{R}^2)$, and if so whether it has finite degree.

[4] One can check that $(\Delta T) \circ p = \Delta(T \circ p)$, using the fact that we assumed that $(-[1]) \circ \pi = \pi \circ (-[1])$ in Definition 4.2.8, and hence by induction $\deg(T \circ p) = \deg(T)$. Also, from the definitions, $(T \circ p)^{(k)}_n = T^{(k)}_n$, so $\mathrm{height}(T \circ p) = \mathrm{height}(T)$.

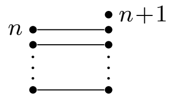
One candidate for this question is the Burau representations. These can be most quickly defined using the presentation

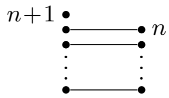
$$\beta_n \;=\; \big\langle \sigma_1, \ldots, \sigma_{n-1} \;\big|\; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2 \big\rangle.$$

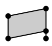
The Burau representation $\beta_n \to \mathrm{Aut}(\mathbb{Z}[t^{\pm 1}]^n) = GL_n(\mathbb{Z}[t^{\pm 1}])$ is defined on generators by

$$\sigma_i \mapsto I_{i-1} \oplus \left(\begin{smallmatrix} 1-t & t \\ 1 & 0 \end{smallmatrix}\right) \oplus I_{n-i-1},$$

where I_k is a $k \times k$ identity matrix. The category $\mathcal{B}(\mathbb{R}^2)_{\mathrm{aut}}$ is generated by the $\sigma_i \in \beta_n = \mathrm{Aut}_{\mathcal{B}(\mathbb{R}^2)}(\underline{n})$ for varying i and n. The full category $\mathcal{B}(\mathbb{R}^2)$ has two extra types of generators, which we can take to be "inclusion braids" $\iota_n \colon \underline{n} \to \underline{n{+}1}$ and "forgetful braids" $\pi_{n+1} \colon \underline{n{+}1} \to \underline{n}$ as follows:

$\iota_n\colon$ 

$\pi_{n+1}\colon$ 

(We are temporarily breaking with our convention of  being the canonical map from $\underline{n}$ to $\underline{n{+}1}$.) To extend the Burau representations to all of $\mathcal{B}(\mathbb{R}^2)$, we would need to define it on these generators, and check the new relations which arise:

$$\left\{ \begin{array}{c} \sigma_i^{\pm 1} \circ \iota_n = \iota_n \circ \sigma_i^{\pm 1} \\ \sigma_i^{\pm 1} \circ \pi_{n+1} = \pi_{n+1} \circ \sigma_i^{\pm 1} \end{array} \right\} \text{(for } i \leq n-1\text{);} \qquad \pi_{n+1} \circ \sigma_n^k \circ \iota_n = \left\{ \begin{array}{ll} \mathrm{id}_{\underline{n}} & k \text{ even} \\ \iota_{n-1} \circ \pi_n & k \text{ odd} \end{array} \right\}$$

However, it is unclear from this combinatorial description whether or not this is possible.[5] As mentioned in §1.2 of Chapter 1, some further work one could do is to investigate this more geometrically. Any representation of the infinite braid group β_∞ is equivalent to an abelian-group-valued functor defined on the subcategory of $\mathcal{B}(\mathbb{R}^2)$ generated by the σ_i and the ι_n. It may be possible to find a geometric condition on such a representation such that the corresponding functor can be extended to the "forgetful braids" π_{n+1} too, i.e. to the whole category $\mathcal{B}(\mathbb{R}^2)$.

4.6 A "twisted stability from untwisted stability" principle

The notation continues as in the previous sections. Suppose we have a sequence of spaces $\{\cdots \to Y_n \xrightarrow{s_n} Y_{n+1} \to \cdots\}$ and a twisted coefficient system $T\colon \mathcal{C} \wr \mathcal{B} \to \mathsf{Ab}$ for this sequence.

Note that we automatically have $\binom{n}{k}$-sheeted covering spaces $\widetilde{Y}_n^{(k)} \to Y_n$ corresponding to the subgroups $G_{k,n-k} \leq G_n = \pi_1 Y_n$. Since $\gamma_n(G_{k,n-k}) \leq G_{k,n+1-k}$ the map s_n lifts to a

[5] For example, one could attempt to extend the Burau representations to $\mathcal{B}(\mathbb{R}^2)$ by sending π_{n+1} to $I_{n-1} \oplus \left(c\; d\right)$ and sending ι_n to $I_{n-1} \oplus \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$, for some Laurent polynomials $a, b, c, d \in \mathbb{Z}[t^{\pm 1}]$. But then applying the right-hand relation above with $k = 1$ yields the contradiction $0 = 1 - t \in \mathbb{Z}[t^{\pm 1}]$.

map

$$\widetilde{s}_n^{(k)} \colon \widetilde{Y}_n^{(k)} \to \widetilde{Y}_{n+1}^{(k)}$$

of covering spaces, which on π_1 is just the restriction of γ_n to $G_{k,n-k} \to G_{k,n+1-k}$. Now, the map $\iota_n \colon T_n^{(k)} \to T_{n+1}^{(k)}$ is equivariant with respect to this, and is also a bijection (Lemma 4.3.7), so the local coefficient system $T_n^{(k)}$ on $\widetilde{Y}_n^{(k)}$ is the pullback along the map $\widetilde{s}_n^{(k)}$ of the local coefficient system $T_{n+1}^{(k)}$ on $\widetilde{Y}_{n+1}^{(k)}$.

The following "twisted stability from untwisted stability principle" is the main technical result of this chapter, and is proved in §4.8.

Theorem 4.6.1 (Twisted stability from untwisted stability) *Let T have (finite) degree d. Suppose that for all $n \geq 0$ there is a map of fibrations*

$$\begin{tikzcd} F_n^k \arrow[r, "t_n^k"] \arrow[d, hook] & F_{n+1}^k \arrow[d, hook] \\ \widetilde{Y}_n^{(k)} \arrow[r, "\widetilde{s}_n^{(k)}"] \arrow[dr] & \widetilde{Y}_{n+1}^{(k)} \arrow[d] \\ & B_k \end{tikzcd} \tag{4.6.1}$$

together with a local coefficient system on the base B_k from which both $T_n^{(k)}$ and $T_{n+1}^{(k)}$ are pulled back. Then if integral (untwisted) homological stability holds for $\{F_n^k\}$ w.r.t. n for all k, then T-twisted homological stability holds for $\{Y_n\}$ w.r.t. n. Quantitatively: if

$$(t_n^k)_* \colon H_*(F_n^k; \mathbb{Z}) \to H_*(F_{n+1}^k; \mathbb{Z})$$

is an isomorphism in the range $* \leq \phi(n,k)$ for some function ϕ, then

$$(s_n; \iota_n)_* \colon H_*(Y_n; T_n) \to H_*(Y_{n+1}; T_{n+1})$$

is an isomorphism in the range $* \leq \min\{\phi(n,k) \mid 0 \leq k \leq d\}$.

4.6.1 Deduction of twisted homological stability for configuration spaces

We now use this to prove the Main Theorem (except the split-injectivity statement, which is proved separately in §4.9).

Proof of the Main Theorem, Part I. Let $\{Y_n\}$ be the sequence of labelled configuration spaces $C_n(M,X)$, with $Y_n \xrightarrow{s_n} Y_{n+1}$ the stabilisation map defined in §4.1. Let x_0 be a basepoint for the label-space X, and let $T \colon \mathcal{B}(M,X) \to \mathsf{Ab}$ be a twisted coefficient system of degree d.

First we will describe the covering spaces $\widetilde{Y}_n^{(k)}$ in this setup. The group G_n is the group of braids in $M \times [0,1]$, with strands labelled by $\pi_1 X$, going from the configuration

$\{(q_1,0),\dots,(q_n,0)\}$ in $M\times\{0\}$ to the configuration $\{(q_1,1),\dots,(q_n,1)\}$ in $M\times\{1\}$. Each such braid induces a permutation of the set $\{q_1,\dots,q_n\}$, and the subgroup $G_{k,n-k}$ consists of those braids whose induced permutation preserves the partition $\{q_1,\dots,q_{n-k}\}\sqcup\{q_{n-k+1},\dots,q_n\}$. Hence the covering space $\widetilde{Y}_n^{(k)}$ corresponding to this subgroup can be thought of as the space $C_{(k,n-k)}(M,X)$ of configurations of n unordered, distinct points in M, with labels in X, and with k of the points coloured red and $n-k$ of them coloured green. Take its basepoint to be the labelled configuration $\{(q_1,x_0),\dots,(q_n,x_0)\}$ with $q_1,\dots,q_{n-k}$ coloured green and $q_{n-k+1},\dots,q_n$ coloured red. Analogously to the stabilisation map $s_n\colon C_n(M,X)\to C_{n+1}(M,X)$, the map $\widetilde{s}_n^{(k)}\colon C_{(k,n-k)}(M,X)\to C_{(k,n+1-k)}(M,X)$ adds a new green point labelled by x_0 near the boundary-component B.

Now we build a map of fibrations as in (4.6.1) together with appropriate coefficients on the base space. Take $B_k=Y_k=C_k(M,X)$, and define $\widetilde{Y}_n^{(k)}=C_{(k,n-k)}(M,X)\to C_k(M,X)=B_k$ to be the map that forgets the green points. This is a fibration (in fact a fibre bundle) with fibre $F_n^k=C_{n-k}(M\smallsetminus\{k\text{ points}\},X)$, and the map $t_n^k\colon F_n^k\to F_{n+1}^k$ is exactly the stabilisation map s for the punctured manifold $M\smallsetminus\{k\text{ points}\}$.

Give $B_k=Y_k$ the local coefficient system $T_k^{(k)}$. Now, the map : $T_n\to T_k$ sends $T_n^{(k)}$ into $T_k^{(k)}$ and moreover restricts to a bijection $T_n^{(k)}\to T_k^{(k)}$. (See Notation 4.3.5 for an explanation of this notation; the claims follow from the duals of Remark 4.3.6 and Lemma 4.3.7.) It is also equivariant w.r.t. π_1 of the map $C_{(k,n-k)}(M,X)\to C_k(M,X)$ which forgets the green points. Hence the coefficients $T_n^{(k)}$ on $C_{(k,n-k)}(M,X)$ are pulled back along this map from the coefficients $T_k^{(k)}$ on $B_k=C_k(M,X)$.

Now we can apply Theorem 4.6.1, using homological stability for configuration spaces with *untwisted* coefficients as input. The map

$$(t_n^k)_*\colon H_*(C_{n-k}(M\smallsetminus\{k\text{ points}\},X);\mathbb{Z})\to H_*(C_{n+1-k}(M\smallsetminus\{k\text{ points}\},X);\mathbb{Z})$$

is an isomorphism in the range $*\leq\frac{n-k}{2}$ by [RW11] (see also [Seg73, McD75, Seg79]). Hence Theorem 4.6.1 implies that

$$(s_n;\iota_n)_*\colon H_*(C_n(M,X);T_n)\to H_*(C_{n+1}(M,X);T_{n+1})$$

is an isomorphism for $*\leq\frac{n-d}{2}$. □

Remark 4.6.2 One can easily see from the proof of Theorem 4.6.1 in §4.8 below that if the twisted coefficient system $T\colon\mathcal{C}\wr\mathcal{B}\to\mathsf{Ab}$ takes values in the subcategory $\mathsf{Vect}_{\mathbb{Q}}$ of Ab, then the hypothesis of Theorem 4.6.1 may be weakened to *rational* (untwisted) homological stability for $\{F_n^k\}$ for each k. Now, the homological stability slope for $C_n(M,X)$ is 1 (rather than just $\frac{1}{2}$) when taking rational coefficients (as long as M is either at least 3-dimensional [RW11, Theorem B] or orientable [Chu12, Corollary 3]). Hence, modifying the last step of the above proof, we see that for *rational* twisted coefficient systems $T\colon\mathcal{B}(M,X)\to\mathsf{Vect}_{\mathbb{Q}}$ we have twisted homological stability for $C_n(M,X)$ in the range $*\leq n-d$.

4.7 A twisted Serre spectral sequence

To prove the "twisted stability from untwisted stability" principle we will need a generalisation of the usual Serre spectral sequence, allowing the base space to be equipped with a local coefficient system. It is a special case of (the homology version of) an *equivariant* generalisation of the Serre spectral sequence constructed by Moerdijk and Svensson [MS93].[6] We will start by describing an alternative basepoint-independent viewpoint on (co)homology with local coefficients (in the non-equivariant setting).

Definition 4.7.1 For a space Y let $\Delta(Y)$ be the category whose objects are all singular simplices in Y, and whose morphisms are simplicial operations (generated by face and degeneracy maps). Denote the fundamental groupoid of Y by $\pi(Y)$, and the standard n-simplex by Δ^n. There is a canonical functor $\upsilon_Y \colon \Delta(Y) \to \pi(Y)$ which takes a singular simplex $\Delta^n \to Y$ to the image of its barycentre b_n. A morphism $\Delta^k \xrightarrow{\alpha} \Delta^n \to Y$ is taken to the image of the straight-line path in Δ^n from $\alpha(b_k)$ to b_n.

A covariant (resp. contravariant) functor $\Delta(Y) \to \mathsf{Ab}$ is a *coefficient system* for homology (resp. cohomology); it is a *local coefficient system* if it factors up to natural isomorphism through υ_Y.

The functor $\upsilon_Y \colon \Delta(Y) \to \pi(Y)$ encapsulates most of the combinatorics needed to define (co)homology with local coefficients. The definition makes sense for any (not necessarily local) coefficient system, but it is only homotopy-invariant for local coefficient systems.

Definition 4.7.2 (Homology) Given a space Y and coefficient system $M \colon \Delta(Y) \to \mathsf{Ab}$, the homology $H_*(Y;M)$ is the homology of the chain complex $C_*(\Delta(Y);M)$:

$$\xrightarrow{\partial_{n+1}} \bigoplus_{\sigma \in N_n\Delta(Y)} M(\sigma_0) \xrightarrow{\partial_n} \bigoplus_{\tau \in N_{n-1}\Delta(Y)} M(\tau_0) \xrightarrow{\partial_{n-1}}$$

where $N_\bullet\Delta(Y)$ denotes the nerve of the category $\Delta(Y)$, and for a chain of singular simplices $\sigma = (\Delta^{k_0} \to \Delta^{k_1} \to \cdots \to \Delta^{k_n} \to Y)$ of $N_n\Delta(Y)$, the 0th one $\Delta^{k_0} \to Y$ is denoted by σ_0. The map ∂_n is the alternating sum of maps ∂_n^i which are defined using the ith face map of $N_\bullet\Delta(Y)$.[7]

Definition 4.7.3 (Cohomology) Given a space Y and coefficient system $M \colon \Delta(Y)^{\mathrm{op}} \to \mathsf{Ab}$, the cohomology $H^*(Y;M)$ is the homology of the cochain complex $C^*(\Delta(Y);M)$:

$$\xrightarrow{\delta_{n-1}} \prod_{\sigma \in N_n\Delta(Y)} M(\sigma_0) \xrightarrow{\delta_n} \prod_{\tau \in N_{n+1}\Delta(Y)} M(\tau_0) \xrightarrow{\delta_{n+1}}$$

[6] See [Kro10] for an extension of this, and [Hon98] for a more geometric construction under some general topological conditions on the spaces involved.

[7] For $\sigma \in N_n\Delta(Y)$, let τ be its ith face. There is a canonical map $\sigma_0 \to \tau_0$ (which is the identity except when $i = 0$) inducing a map $M(\sigma_0) \to M(\tau_0)$. The direct sum of these maps is ∂_n^i.

where the map δ_n is the alternating sum of maps δ_n^i which are defined using the ith face map of $N_\bullet\Delta(Y)$.[8]

This reduces to ordinary (untwisted) homology and cohomology when M is constant. (Although it does not reduce to the usual singular (co)chain complex, one can show that it does compute the same homology as it; cf. [MS93, Theorem 2.2].)

In [MS93] the above is generalised to the equivariant setting: they define $v_Y\colon \Delta_G(Y) \to \pi_G(Y)$ for a G-space Y, and equivariant twisted cohomology $H^*_G(Y;M)$ for any coefficient system $\Delta_G(Y)^{\mathrm{op}} \to \mathsf{Ab}$. Again a coefficient system is *local* if it factors up to natural isomorphism through v_Y. Cohomology with respect to local coefficient systems is G-homotopy invariant [MS93, Theorem 2.3]. Their main theorem is the existence of a twisted equivariant Serre spectral sequence:

Theorem 4.7.4 ([MS93, Theorem 3.2]) *For any G-fibration $f\colon Y \to X$ (i.e. $Y^H \to X^H$ is a fibration for all $H \leq G$) and any local coefficient system M on Y, there is a local coefficient system $H^q_G(f;M)$ on X for each $q \geq 0$ and a spectral sequence*

$$E_2^{p,q} = H^p_G\big(X; H^q_G(f;M)\big) \;\Rightarrow\; H^*_G(Y;M) \tag{4.7.1}$$

with the usual cohomological grading.

Remark 4.7.5 We will describe the local coefficient system $H^q(f;M)$ in the non-equivariant case. As a functor $\Delta(X)^{\mathrm{op}} \to \mathsf{Ab}$ it does the following. A singular simplex $\Delta^k \xrightarrow{\sigma} X$ is taken to the cohomology $H^q(\sigma^*(Y);M)$, where $\sigma^*(Y)$ is the pullback of σ and f, and we denote any pullback of the coefficients M also by M. A morphism $\Delta^l \xrightarrow{\alpha} \Delta^k \xrightarrow{\sigma} X$ induces a map of pullbacks $(\sigma\circ\alpha)^*(Y) \to \sigma^*(Y)$ and hence a map on cohomology.

It is a *local* coefficient system since it factors up to natural isomorphism through v_X by the following functor $\pi(X)^{\mathrm{op}} \to \mathsf{Ab}$. A point $x \in X$ is taken to $H^q(f^{-1}(x);M)$. Given a homotopy class $[I \xrightarrow{p} X]$ of paths from x to y, there are induced maps of pullbacks $f^{-1}(x) \hookrightarrow p^*(Y) \hookleftarrow f^{-1}(y)$. These induce maps on cohomology, and since they are *isomorphisms*[9] the first one can be inverted to get a composite map $H^q(f^{-1}(x);M) \to H^q(f^{-1}(y);M)$. One can check that this map is independent of the choice of representing path p.

In [MS93] the authors point out that there is an analogous version of the spectral sequence (4.7.1) for homology. We will only need the non-equivariant (but twisted) version, which is:[10]

[8] Given an element $\{g_\sigma \in M(\sigma_0) \mid \sigma \in N_n\Delta(Y)\}$, we need to choose an element of $M(\tau_0)$ for each $\tau \in N_{n+1}\Delta(Y)$. Let σ be the ith face of τ, which has a canonical map $\tau_0 \to \sigma_0$ (which is the identity except when $i=0$). Apply M to get a map $M(\sigma_0) \to M(\tau_0)$ and take the image of g_σ under this map.

[9] The inclusion $\{0\} \hookrightarrow [0,1]$ is an acyclic cofibration, so its pullback along the fibration f is again an acyclic cofibration, in particular a weak equivalence.

[10] This was also stated (referencing [MS93]) as Theorem 4.1 of [Han09b].

Theorem 4.7.6 *For any fibration $f\colon Y \to X$ and any local coefficient system M on Y, there is a local coefficient system $H_q(f;M)$ on X for each $q \geq 0$ and a spectral sequence*

$$E^2_{p,q} = H_p\big(X; H_q(f;M)\big) \;\Rightarrow\; H_*(Y;M) \tag{4.7.2}$$

with the usual homological grading.

The description of the local coefficient systems $H_q(f;M)$ is the same as above, replacing cohomology with homology. When the local coefficient system M on Y is pulled back from the base X, they are built out of the *untwisted* homology of each fibre.

We now return to the viewpoint of local coefficient systems as an action of the fundamental group on an abelian group.

Corollary 4.7.7 *For any fibration $f\colon Y \to X$ with fibre F over the basepoint $x_0 \in X$, and any $\pi_1(X)$-module M, there is a spectral sequence*

$$E^2_{p,q} = H_p\big(X; H_q(F;M)\big) \;\Rightarrow\; H_*(Y;M) \tag{4.7.3}$$

with the usual homological grading. Here the action of $\pi_1(Y)$ on M is pulled back from that of $\pi_1(X)$ via f_* and the action of $\pi_1(F)$ on M is trivial.

This is natural for maps of fibrations over a fixed base in the obvious way:

Proposition 4.7.8 *Suppose we have a map of fibrations over a fixed base*

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & Y' \arrow[dl] \\ & X & \end{tikzcd}$$

and a $\pi_1(X)$-module M; denote the fibres over the basepoint $x_0 \in X$ by F and F' respectively. Then there is a map of spectral sequences (4.7.3) where:

- *The map $F \to F'$ induces a map of $\pi_1(X)$-modules $H_q(F;M) \to H_q(F';M)$, which therefore induces a map $H_p(X;H_q(F;M)) \to H_p(X;H_q(F';M))$. This is the map on the E^2 pages.*
- *The action of $\pi_1(Y)$ on M is the pullback of the action of $\pi_1(Y')$ on M, so the map $Y \to Y'$ induces a map $H_*(Y;M) \to H_*(Y';M)$. This is the map in the limit.*

4.8 Proof the principle

We will now prove Theorem 4.6.1, using the twisted Serre spectral sequence of the previous section and the following elementary fact, which is a covering space version of what is usually known as Shapiro's Lemma.

Lemma 4.8.1 (Shapiro for covering spaces) *Suppose we have a based space*[11] X*, a subgroup* H *of* $\pi_1(X)$ *and an* H*-module* A*. Let* $\hat{X}$ *be the (based) covering space corresponding to* H*. Then*

$$H_*(\hat{X}; A) \cong H_*(X; \mathbb{Z}\pi_1(X) \otimes_H A). \tag{4.8.1}$$

Given a map of the above data—namely a (based) map $f\colon X \to X'$ *such that* $f_*(H) \subseteq H'$ *(so that there is a unique based lift* $\hat{f}\colon \hat{X} \to \hat{X}'$*) and a map* $\phi\colon A \to A'$ *which is equivariant w.r.t.* f_**—the identification (4.8.1) is natural in the sense that*

$$\begin{tikzcd} H_*(\hat{X}; A) \arrow[r] \arrow[d, phantom, "\wr\|"] & H_*(\hat{X}'; A') \arrow[d, phantom, "\wr\|"] \\ H_*(X; \mathbb{Z}\pi_1(X) \otimes_H A) \arrow[r] & H_*(X'; \mathbb{Z}\pi_1(X') \otimes_{H'} A') \end{tikzcd} \tag{4.8.2}$$

commutes.

Proof. Denote the singular chain complex functor by $S_*(\ \ )$ and the universal cover of X by $\widetilde{X}$. Then we have an isomorphism of chain complexes

$$S_*(\widetilde{X}) \otimes_H A \longrightarrow S_*(\widetilde{X}) \otimes_{\pi_1(X)} \mathbb{Z}\pi_1(X) \otimes_H A$$

given by $\sigma \otimes a \mapsto \sigma \otimes [c_x] \otimes a$, where c_x is the constant loop at the basepoint x of X. Taking homology gives the identification (4.8.1). Let $\widetilde{f}$ denote the unique (based) lift of f to $\widetilde{X} \to \widetilde{X}'$. The diagram (4.8.2) is induced by

$$\begin{tikzcd} S_*(\widetilde{X}) \otimes_H A \arrow[r] \arrow[d, "\cong"'] & S_*(\widetilde{X}') \otimes_{H'} A' \arrow[d, "\cong"] \\ S_*(\widetilde{X}) \otimes_{\pi_1(X)} \mathbb{Z}\pi_1(X) \otimes_H A \arrow[r] & S_*(\widetilde{X}') \otimes_{\pi_1(X')} \mathbb{Z}\pi_1(X') \otimes_{H'} A' \end{tikzcd}$$

and one can check that both routes around the square send $\sigma \otimes a$ to $\widetilde{f}_\sharp(\sigma) \otimes [c_{x'}] \otimes \phi(a)$. $\square$

Proof of Theorem 4.6.1. We need to show that the map

$$H_*(Y_n; T_n) \longrightarrow H_*(Y_{n+1}; T_{n+1}) \tag{4.8.3}$$

induced by s_n and ι_n is an isomorphism in the stated range. By the decomposition (4.3.1) of Proposition 4.3.1, and the fact that T has degree d, this is the same as the map

$$\bigoplus_{k=0}^{d} H_*(Y_n; \mathbb{Z}G_n \otimes_{G_{k,n-k}} T_n^{(k)}) \longrightarrow \bigoplus_{k=0}^{d} H_*(Y_{n+1}; \mathbb{Z}G_{n+1} \otimes_{G_{k,n+1-k}} T_{n+1}^{(k)}) \tag{4.8.4}$$

induced by s_n, γ_n and ι_n. By Shapiro's Lemma for covering spaces (Lemma 4.8.1) this is

[11] Path-connected, locally path-connected and semilocally simply-connected.

isomorphic to the map

$$\bigoplus_{k=0}^{d} H_*(\widetilde{Y}_n^{(k)}; T_n^{(k)}) \longrightarrow \bigoplus_{k=0}^{d} H_*(\widetilde{Y}_{n+1}^{(k)}; T_{n+1}^{(k)}) \tag{4.8.5}$$

induced by $\widetilde{s}_n^{(k)}$ and ι_n. The map of fibrations (4.6.1) gives a map of twisted Serre spectral sequences (Corollary 4.7.7 and Proposition 4.7.8):

$$\begin{array}{ccc} E^2_{p,q} = H_p(B_k; H_q(F_n^k; A)) & \Rightarrow & H_*(\widetilde{Y}_n^{(k)}; T_n^{(k)}) \\ \downarrow & & \downarrow \\ E^2_{p,q} = H_p(B_k; H_q(F_{n+1}^k; A)) & \Rightarrow & H_*(\widetilde{Y}_{n+1}^{(k)}; T_{n+1}^{(k)}) \end{array} \tag{4.8.6}$$

where A is the local coefficient system on the base B_k which pulls back to $T_n^{(k)}$ and $T_{n+1}^{(k)}$. Note that it is a *constant* coefficient system once it has been pulled back to F_n^k and F_{n+1}^k. The map on E^2 pages is induced by the map t_n^k, and hence is an isomorphism for $q \leq \phi(n,k)$ (and all $p \geq 0$) by the hypothesis of the theorem (and the universal coefficient theorem). Hence by the Zeeman comparison theorem[12] it is an isomorphism in the limit for $* \leq \phi(n,k)$. Hence in the range $* \leq \min\{\phi(n,k) \mid 0 \leq k \leq d\}$ each summand in (4.8.5) is an isomorphism, so (4.8.3) is an isomorphism, as desired. □

4.9 Split-injectivity

To prove the split-injectivity claim of the Main Theorem we will use the following lemma which was used implicitly by Nakaoka in [Nak60] and later written down explicitly by Dold in [Dol62]:

Lemma 4.9.1 ([Dol62, Lemma 2]) *Given a sequence* $0 \to A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} \cdots$ *of abelian groups and homomorphisms, the following is sufficient to imply that each of the maps* ϕ_i *is split-injective: There exist maps* $\tau_{k,n} \colon A_n \to A_k$ *for* $1 \leq k \leq n$ *with* $\tau_{n,n} = \mathrm{id}$ *such that*

$$\mathrm{im}(\tau_{k,n} - \tau_{k,n+1} \circ \phi_n) \leq \mathrm{im}(\phi_{k-1}). \tag{4.9.1}$$

Let $\overline{C}_n(M,X)$ be the universal cover of the configuration space $C_n(M,X)$. There is a natural lift of the stabilisation map to a map $\overline{s}_n \colon \overline{C}_n(M,X) \to \overline{C}_{n+1}(M,X)$, which can be described as follows. The elements of $\overline{C}_n(M,X)$ can be thought of as n-strand 'braids' in $M \times [0,1]$, with strands labelled by the path-space PX, starting at the configuration $\{(q_1,0),\ldots,(q_n,0)\}$ in $M \times \{0\}$ labelled by $\{x_0,\ldots,x_0\}$, and ending at any labelled configuration in $M \times \{1\}$. The map $\overline{s}_n$ pushes the braid in $M \times [0,1]$ inwards, by the self-

[12] The required implication is contained in the proof of Theorem 1 of [Zee57], although stronger hypotheses are stated there. An explicit statement of the comparison theorem which applies to our case is Theorem 1.2 of [Iva93]. It is also written in Remarque 1.8 of [CDG11].

embedding $e \times \mathrm{id}_{[0,1]}$, and adds a trivial strand labelled by the constant path c_{x_0} near the boundary-component B of $\overline{M}$.

Denote $\pi_1 C_n(M,X)$ by G_n, and denote the singular chain complex of a space by $S_*(-)$. Then the map

$$(s_n;\iota_n)_*\colon H_*(C_n(M,X);T_n) \longrightarrow H_*(C_{n+1}(M,X);T_{n+1}). \tag{4.9.2}$$

is induced by the map of chain complexes

$$(\overline{s}_n)_\sharp \otimes \iota_n\colon S_*(\overline{C}_n(M,X)) \otimes_{G_n} T_n \longrightarrow S_*(\overline{C}_{n+1}(M,X)) \otimes_{G_{n+1}} T_{n+1}.$$

Proof of the Main Theorem, Part II. We want to prove that (4.9.2) is split-injective for all $*$ and n. By Dold's Lemma 4.9.1, it is sufficient to construct chain maps

$$t_{k,n}\colon S_*(\overline{C}_n(M,X)) \otimes_{G_n} T_n \longrightarrow S_*(\overline{C}_k(M,X)) \otimes_{G_k} T_k$$

for $1 \leq k \leq n$ such that $t_{n,n} = \mathrm{id}$ and

$$t_{k,n} \simeq t_{k,n+1} \circ ((\overline{s}_n)_\sharp \otimes \iota_n) - ((\overline{s}_{k-1})_\sharp \otimes \iota_{k-1}) \circ t_{k-1,n}. \tag{4.9.3}$$

Let $S \subseteq \underline{n}$. There is a unique partially-defined injection $\{1,\dots,n\} \dashrightarrow \{1,\dots,|S|\}$ which is order-preserving and is defined precisely on S. The chosen isotopy $\mathrm{id} \simeq e\colon M \hookrightarrow M$ gives canonical paths between q_j and q_{j+1} for each j; using these one can lift this partial injection to a partial braid from $\{(q_1,x_0),\dots,(q_n,x_0)\}$ to $\{(q_1,x_0),\dots,(q_{|S|},x_0)\}$. This is a morphism $b_{S,n}$ in $\mathcal{B}(M,X)$ from $\overline{x}_0^n$ to $\overline{x}_0^{|S|}$, so applying T gives a map $\pi_{S,n}\colon T_n \to T_{|S|}$.

We can also define a map $p_{S,n}\colon \overline{C}_n(M,X) \to \overline{C}_{|S|}(M,X)$ as follows. Given an n-strand braid in $\overline{C}_n(M,X)$, forget the strands which start at $(q_i,0)$ for $i \in \underline{n} \smallsetminus S$, and concatenate the resulting partial braid with the reverse of $b_{S,n}$ to get an $|S|$-strand braid in $\overline{C}_{|S|}(M,X)$.

Directly from these definitions one can check that:

(a) If $1 \notin S$ then $\pi_{S,n+1} \circ \iota_n = \pi_{(S-1),n}$ and $p_{S,n+1} \circ \overline{s}_n \simeq p_{(S-1),n}$.
(b) If $1 \in S$ then $\pi_{S,n+1} \circ \iota_n = \iota_{|S|-1} \circ \pi_{(S\smallsetminus\{1\}-1),n}$ and $p_{S,n+1} \circ \overline{s}_n = \overline{s}_{|S|-1} \circ p_{(S\smallsetminus\{1\}-1),n}$.

(The notation $(S-1)$ means $\{s-1 \mid s \in S\}$.) We now define $t_{k,n}$ to be the following chain map:

$$\sigma \otimes x \;\mapsto \sum_{S \subseteq \underline{n},\, |S|=k} (p_{S,n})_\sharp(\sigma) \otimes \pi_{S,n}(x).$$

Clearly $t_{n,n} = \mathrm{id}$, so we just need to check the identity (4.9.3). The right-hand side of this is:

$$\begin{aligned}\sigma \otimes x \quad \mapsto \quad & \sum_{S \subseteq \underline{n+1},\, |S|=k} \big((p_{S,n+1})_\sharp \circ (\overline{s}_n)_\sharp(\sigma)\big) \otimes \big(\pi_{S,n+1} \circ \iota_n(x)\big) \\ & - \sum_{R \subseteq \underline{n},\, |R|=k-1} \big((\overline{s}_{k-1})_\sharp \circ (p_{R,n})_\sharp(\sigma)\big) \otimes \big(\iota_{k-1} \circ \pi_{R,n}(x)\big).\end{aligned} \tag{4.9.4}$$

Using (a) and (b) above, we see that the top line of this decomposition is chain-homotopic to the following:

$$\begin{aligned}\sigma \otimes x \quad \mapsto \quad & \sum_{S \subseteq \underline{n+1},\, |S|=k,\, 1 \in S} \big((\overline{s}_{k-1})_\sharp \circ (p_{(S \smallsetminus \{1\} - 1),n})_\sharp(\sigma)\big) \otimes \big(\iota_{k-1} \circ \pi_{(S \smallsetminus \{1\}-1),n}(x)\big) \\ & + \sum_{S \subseteq \underline{n+1},\, |S|=k,\, 1 \notin S} (p_{(S-1),n})_\sharp(\sigma) \otimes \pi_{(S-1),n}(x).\end{aligned} \tag{4.9.5}$$

The first line of (4.9.5) cancels with the second line of (4.9.4), leaving just the second line of (4.9.5), which is precisely $t_{k,n}$, as required. □

Remark 4.9.2 We did not at any point use the fact that our twisted coefficient system T is finite-degree, so the split-injectivity claim in the Main Theorem also holds for infinite-degree twisted coefficient systems.

4.10 A connection with representation stability[13]

In this last section we briefly describe a link between twisted homological stability for unordered configuration spaces (for a particular twisted coefficient system) and *representation stability* for the cohomology of ordered configuration spaces. For simplicity we will take $X = pt$ in this section.

We have stability for the sequence $H_*\big(C_n(M); \mathbb{Z}[\Sigma_n/\Sigma_{n-k}]\big)$ in the range $n \geq 2* + k$ by the Main Theorem and Example 4.5.4. We can equally well replace $\mathbb{Z}$ by $\mathbb{Q}$ in this example, so we also have stability for the sequence

$$H_*\big(C_n(M); \mathbb{Q}[\Sigma_n/\Sigma_{n-k}]\big). \tag{4.10.1}$$

Aside This special case of twisted homological stability can in fact be deduced from untwisted homological stability fairly quickly, as follows. Let $C_n^{(k)}(M)$ be the space of n disjoint points in M, equipped with an ordering of k of them. Then there is a covering space map $C_n^{(k)}(M) \to C_n(M)$ with fibre Σ_n/Σ_{n-k}, so $H_*\big(C_n(M); \mathbb{Q}[\Sigma_n/\Sigma_{n-k}]\big) \cong H_*\big(C_n^{(k)}(M); \mathbb{Q}\big)$. There is also a fibration sequence $C_{n-k}(M \smallsetminus \{k \text{ points}\}) \to C_n^{(k)}(M) \to \widetilde{C}_k(M)$, and a map of such fibration sequences over $\widetilde{C}_k(M)$ given by stabilisation maps. Homological stability holds for the map of fibres, so, applying the Zeeman Comparison Theorem to the

[13]The observations in this section grew out of several conversations with Oscar Randal-Williams.

corresponding map of Serre spectral sequences, it also holds for the map of total spaces.

The Künneth spectral sequence (for chain complexes of $\mathbb{Q}\Sigma_n$-modules) for the rational singular chain complex of $\widetilde{C}_n(M)$ and the module $\mathbb{Q}[\Sigma_n/\Sigma_{n-k}]$ concentrated in degree 0 is

$$E^2_{p,q} = \mathrm{Tor}^q_{\mathbb{Q}\Sigma_n}\big(H_p(\widetilde{C}_n(M);\mathbb{Q}), \mathbb{Q}[\Sigma_n/\Sigma_{n-k}]\big) \;\Rightarrow\; H_*\big(C_n(M);\mathbb{Q}[\Sigma_n/\Sigma_{n-k}]\big).$$

Moreover $\mathrm{Tor}^q_{\mathbb{Q}\pi} = 0$ for $q \geq 1$ for *finite* groups π, so this spectral sequence degenerates to an isomorphism $H_*\big(\widetilde{C}_n(M);\mathbb{Q}\big) \otimes_{\Sigma_n} \mathbb{Q}[\Sigma_n/\Sigma_{n-k}] \cong H_*\big(C_n(M);\mathbb{Q}[\Sigma_n/\Sigma_{n-k}]\big)$, so we have stability for the sequence

$$H_*\big(\widetilde{C}_n(M);\mathbb{Q}\big) \otimes_{\Sigma_n} \mathbb{Q}[\Sigma_n/\Sigma_{n-k}]. \tag{4.10.2}$$

By Schur's Lemma this has dimension

$$\sum_\lambda a_{n,k}(\lambda).b_n(\lambda)$$

where λ runs over all Young diagrams with n boxes, $a_{n,k}(\lambda)$ is the number of copies of the corresponding irreducible Σ_n-representation $V(\lambda)$ in $\mathbb{Q}[\Sigma_n/\Sigma_{n-k}]$ and $b_n(\lambda)$ is the number of copies of the dual irreducible Σ_n-representation $V(\lambda)^*$ in $H_*\big(\widetilde{C}_n(M);\mathbb{Q}\big)$. Note that the latter is the same as the number of copies of $V(\lambda)$ in the cohomology $H^*\big(\widetilde{C}_n(M);\mathbb{Q}\big)$.

Definition 4.10.1 For a Young diagram λ, denote by λ^+ the Young diagram with one extra box added to the top row. A *stable Young diagram* is a Young diagram with any number of boxes, up to the equivalence relation generated by identifying λ with λ^+.

From now on λ (and μ) will always denote a *stable* Young diagram, and it will be clear from the context which representative (i.e. number of boxes) is meant.

Remark 4.10.2 Note that $\mathbb{Q}[\Sigma_n/\Sigma_{n-k}] = \mathrm{Ind}^{\Sigma_n}_{\Sigma_{n-k}}(\mathbb{Q})$, so by the branching rule for induced modules we have that $a_{n,k}(\lambda)$ is the number of ordered ways of adding k boxes to the Young diagram $\square\square\square\square\square\square\square$ with $n-k$ boxes to obtain λ. From this description we can see that the sequence $a_{n,k}(\lambda)$ is monotone non-decreasing (in n) and is constant once $n \geq 2k$.

Definition 4.10.3 *Multiplicity stability* for $H^*\big(\widetilde{C}_n(M);\mathbb{Q}\big)$, in the language of representation stability (see [CF10]), is the property that each sequence $b_n(\lambda)$ is eventually constant as $n \to \infty$.

Now assume that each sequence $b_n(\lambda)$ is monotone non-decreasing. Using stability for (4.10.2), and this assumption, we can show that multiplicity stability holds for $H^*\big(\widetilde{C}_n(M);\mathbb{Q}\big)$, as follows. For any stable Young diagram μ, the irreducible $V(\mu)$ appears in $\mathbb{Q}[\Sigma_n/\Sigma_{n-k}]$ for sufficiently large k, by the branching rule mentioned above. Fix such a k and let $n \geq k$.

By stability of the sequence (4.10.2) we have that for some N,

$$\sum_{\lambda} a_{n,k}(\lambda).b_n(\lambda) \quad \text{is constant for } n \geq N.$$

Since each summand $a_{n,k}(\lambda).b_n(\lambda)$ is monotone non-decreasing and $a_{n,k}(\mu)$ is positive, the sequence $b_n(\mu)$ must be constant for $n \geq N$.

Remark 4.10.4 The same argument as for split-injectivity in the last section shows that there is a split-injection

$$H_*\big(\widetilde{C}_n(M);\mathbb{Q}\big) \oplus_{\Sigma_n} V(\lambda) \hookrightarrow H_*\big(\widetilde{C}_{n+1}(M);\mathbb{Q}\big) \oplus_{\Sigma_n} V(\lambda). \tag{4.10.3}$$

The dimension of the left-hand side is $b_n(\lambda)$; call the dimension of the right-hand side $c_{n+1}(\lambda)$. So $b_n(\lambda)$ is the number of copies of $V(\lambda)^*$ in $H_*\big(\widetilde{C}_n(M);\mathbb{Q}\big)$, whereas $c_n(\lambda)$ is the number of copies of $V(\lambda)^*$ in the restricted module $\mathrm{Res}^{\Sigma_n}_{\Sigma_{n-1}} H_*\big(\widetilde{C}_n(M);\mathbb{Q}\big)$.

However, this does not help in showing that $b_n(\lambda)$ is non-decreasing, since in general the inequality $c_{n+1}(\lambda) \leq b_{n+1}(\lambda)$ does not hold. For example, taking $M = \mathbb{R}^2$, $n = 3$ and $* = 2$ we have

$$H_2\big(\widetilde{C}_4(\mathbb{R}^2);\mathbb{Q}\big) = V(\,\ydiagram{2,1,1}\,)^2 \,\oplus\, V(\,\ydiagram{3,1}\,) \,\oplus\, V(\,\ydiagram{2,2}\,)$$

as a $\mathbb{Q}\Sigma_4$-module (this is taken from the example on p.5 of [CF10]). Hence by the branching rule for restricted modules,

$$H_2\big(\widetilde{C}_4(\mathbb{R}^2);\mathbb{Q}\big) = V(\,\ydiagram{2,1}\,)^4 \,\oplus\, V(\,\ydiagram{1,1,1}\,)^2 \,\oplus\, V(\,\ydiagram{3}\,)$$

as a $\mathbb{Q}\Sigma_3$-module. So for $\lambda = \ydiagram{2,1}$ we have a counterexample $c_4(\lambda) = 4 \not\leq 2 = b_4(\lambda)$.

CHAPTER 5

Homological stability for configuration spaces of submanifolds

5.1 Introduction

5.1.1 Recollections

Fix a smooth[1] manifold M which is the interior of a connected manifold with non-empty boundary $\bar{M}$ of dimension at least 2. Fix a path-connected space X. Then the n-point unordered configuration space on M with labels in X was defined in §1.1 to be

$$C_n(M,X) \coloneqq \left\{(p_1,\ldots,p_n)\in M^n \;\middle|\; p_i\neq p_j \text{ for } i\neq j\right\} \times_{\Sigma_n} X^n.$$

This can be thought of as the space of n particles floating in M, with internal parameters taking values in X, topologised so that they cannot collide. The configuration spaces $\{C_n(M,X)\}_{n=1}^{\infty}$ enjoy the property of *homological stability*:

Theorem 5.1.1 ([RW11], see also [Seg73, McD75, Seg79]) *In the stable range $n \geq 2*$,*

$$H_*(C_n(M,X);\mathbb{Z}) \;\cong\; H_*(C_{n+1}(M,X);\mathbb{Z}). \tag{5.1.1}$$

There is an explicit map $C_n(M,X)\to C_{n+1}(M,X)$ given by pushing a new particle into M from a chosen boundary-component of $\bar{M}$, and the isomorphism (5.1.1) is induced by this map. Hence in the stable range the homology of the configuration space $C_n(M,X)$ is the same as the homology of the limiting space $C_\infty(M,X) = \mathrm{colim}_{n\to\infty}(C_n(M,X))$. Moreover the limiting space can be identified, up to homology, with a certain section space (see [Seg73, McD75]).

Change of notation. As mentioned in the Introduction, for the remainder of this chapter there is a change of notation: we will denote unordered configuration spaces by Σ (instead of C), ordered configuration spaces by F (instead of $\widetilde{C}$) and oriented configuration spaces by A (instead of C^+).

5.1.2 Results

The aim of this chapter is to generalise Theorem 5.1.1 to "spaces of configurations of submanifolds" under suitable conditions. First we need to define precisely what we mean by a space of configurations of submanifolds.

Definition 5.1.2 Let M be the interior of a connected manifold with non-empty boundary $\bar{M}$ of dimension $d \geq 2$, and let P be a closed, connected submanifold of the boundary $\partial\bar{M}$, contained in some coordinate neighbourhood $U \cong \mathbb{R}^{d-1}$ of $\partial\bar{M}$. Denote the inclusion by $\iota\colon P \hookrightarrow \partial\bar{M}$.

We say that two embeddings $P \rightrightarrows M$ are *unlinked* if there exist two disjoint coordinate neighbourhoods of M, one containing the image of each embedding. So in particular both

[1] Manifolds will always be assumed to be smooth, i.e. C^∞, without further comment.

embeddings must have image contained in some coordinate neighbourhood of M. Let

$$E_n^P(M) \coloneqq \left\{ (\psi_1, \ldots, \psi_n) \in \mathrm{Emb}(\textstyle\coprod_n P, M) \;\middle|\; \begin{array}{c} \text{each } \psi_i \text{ is isotopic to } \iota\colon P \hookrightarrow \bar{M} \\ \text{the embeddings } \psi_i \text{ are pairwise unlinked} \end{array} \right\}.$$

Definition 5.1.3 Fixing M and P as above, we say that an element $g \in \mathrm{Diff}(P)$ is *realisable by isotopy* if the embedding $\iota \circ g$ is isotopic to ι through embeddings $P \hookrightarrow \bar{M}$. A subgroup $G \leq \mathrm{Diff}(P)$ is *realisable by isotopies* if every $g \in G$ is realisable by isotopy.

Under certain codimension conditions this condition on G is automatic:

Proposition 5.1.4 (see [Sko08, Theorem 2.8]) *Let* $\dim(M) = d$ *and* $\dim(P) = k$. *Each of the following conditions implies that any two embeddings* $P \hookrightarrow \mathbb{R}^d$ *are isotopic, so the whole diffeomorphism group* $\mathrm{Diff}(P)$ *is realisable by isotopies for any embedding* $\iota\colon P \hookrightarrow \partial\bar{M}$.

- $k \geq 2$ *and* $d \geq 2k + 1$,
- P *is a homology sphere and* $d \geq \frac{3}{2}k + 2$,
- $\mathrm{hconn}(P) \geq c$ *for some* $c \leq \frac{1}{2}k - 1$ *and* $d \geq 2k - c + 1$,

where "$\mathrm{hconn}(P) \geq c$*" means that* $\widetilde{H}_i(P;\mathbb{Z}) = 0$ *for all* $i \leq c$. *The last condition subsumes the first two.*

Definition 5.1.5 Let $G \leq \mathrm{Diff}(P)$ be a subgroup which is realisable by isotopies, and let X be a path-connected space. Then the subgroup $G \wr \Sigma_n$ of $\mathrm{Diff}(P) \wr \Sigma_n = \mathrm{Diff}(\coprod_n P)$ acts on $E_n^P(M)$ by precomposition, and on X^n by projecting $G \wr \Sigma_n \twoheadrightarrow \Sigma_n$ and permuting the copies of X, so we can define

$$\Sigma_n^P(M, X|G) \;\coloneqq\; E_n^P(M) \times_{G\wr\Sigma_n} X^n.$$

An element of $\Sigma_n^P(M, X|G)$ is therefore a collection of n (pairwise unlinked) submanifolds of M, each homeomorphic to P, labelled by elements of X. Each submanifold is equipped with a parametrisation up to the action of G, and is required to be isotopic to $\iota\colon P \hookrightarrow \bar{M}$. (Note that this last condition is well-defined: the submanifold is equipped with a G-orbit of parametrisations, and since G is realisable by isotopies, either all or none of the parametrisations in the orbit is isotopic to ι.) A typical element will be denoted by

$$\{[\psi_1], \ldots, [\psi_n]; x_1, \ldots, x_n\}, \tag{5.1.2}$$

where $[\psi_j]$ is the orbit of $\psi_j \in E_n^P(M)$ under the action of G and $x_j \in X$. More precisely an element ought to be written $\{([\psi_1], x_1), \ldots, ([\psi_j], x_j)\}$, but (5.1.2) is slightly more readable.

We can of course define configuration spaces in which, as well as being parametrised up to G, the n copies of P are also ordered up to the action of some fixed subgroup $\Gamma_n \leq \Sigma_n$. In particular for the trivial subgroup, we define

$$F_n^P(M, X|G) \;\coloneqq\; E_n^P(M) \times_{G^n} X^n.$$

Notation 5.1.6 We will omit the X if it is just a point, and the G if it is the full group $\mathrm{Diff}(P)$. So the spaces $\Sigma_n^P(M|G)$ consist of *unlabelled* configurations and the spaces $\Sigma_n^P(M,X)$ consist of configurations of *unparametrised* submanifolds. We denote the configuration spaces $\Sigma_n^P(M,X|*)$ of *parametrised* submanifolds by $\widehat{\Sigma}_n^P(M,X)$, and when P is orientable we denote the configuration spaces of *oriented* submanifolds $\Sigma_n^P(M,X|\mathrm{Diff}^+(P))$ by $\mathring{\Sigma}_n^P(M,X)$.

In particular, $F_n^{pt}(M)$ is the classical ordered configuration space of points in M and $\Sigma_n^{pt}(M)$ is the corresponding unordered configuration space.

Remark 5.1.7 Note that $E_n^P(M)$ is clearly path-connected, and hence so is $\Sigma_n^P(M,X|G)$. We would not have this if we had required the weaker condition "each ψ_i is isotopic to $\iota \circ g \colon P \hookrightarrow \bar{M}$ for some $g \in \mathrm{Diff}(P)$" in the definition of $E_n^P(M)$ in Definition 5.1.2. This would have avoided having to worry about realisability by isotopies for $G \leq \mathrm{Diff}(P)$, but path-connectivity is crucial to get the inductive proof of homological stability going, so this really is a necessary consideration.

There is a 'stabilisation' map $\Sigma_n^P(M,X|G) \to \Sigma_{n+1}^P(M,X|G)$, which we will define precisely in §5.2. Intuitively one adds a new copy of P to the configuration by pushing the existing configuration away from the boundary of $\bar{M}$ to vacate some space for the new copy of P.

Main Theorem *Let M and $P \subseteq \partial\bar{M}$ be as in Definition 5.1.2, let X be a path-connected space and $G \leq \mathrm{Diff}(P)$ be a finite or open subgroup. If $\dim(M) \geq 2\dim(P)+3$, then the stabilisation map*

$$\Sigma_n^P(M,X|G) \longrightarrow \Sigma_{n+1}^P(M,X|G) \tag{5.1.3}$$

is an isomorphism on homology up to degree $\frac{n-2}{2}$ and a surjection up to degree $\frac{n}{2}$.

The large codimension condition is used exactly once in the proof, and for particular manifolds P can be avoided, for example for points and spheres:

Extension 5.1.8 *Let M be as above and suppose that P is either a point or a 'standardly' embedded sphere ($\iota \colon P = S^k \hookrightarrow \mathbb{R}^{k+1} \subseteq \mathbb{R}^{d-1} \subseteq \partial\bar{M}$). Again let X be a path-connected space and $G \leq \mathrm{Diff}(P)$ be a finite or open subgroup. If $\dim(M) - \dim(P) \geq 3$ and G is realisable by isotopies,[2] then the stabilisation map (5.1.3) is an isomorphism on homology up to degree $\frac{n-2}{2}$ and a surjection up to degree $\frac{n}{2}$.*

Applying the results of the previous chapter we obtain a twisted version of homological stability:

[2]This was automatic in the Main Theorem by the dimension assumption and Proposition 5.1.4, but not under the weaker dimension assumption in the Extension.

Corollary 5.1.9 *Under the same conditions as the Main Theorem or Extension 5.1.8, if T is a coefficient system for $\{\Sigma_n^P(M,X|G)\}$ of degree d, the stabilisation map (5.1.3) induces isomorphisms*

$$H_*(\Sigma_n^P(M,X|G);T_n) \longrightarrow H_*(\Sigma_{n+1}^P(M,X|G);T_{n+1}) \tag{5.1.4}$$

in the range $* \leq \frac{n-d-2}{2}$.

See §5.8 for precisely what such coefficient systems are, and a deduction of this corollary from the Main Theorem of this chapter and Theorem 4.6.1 of Chapter 4.

Remark 5.1.10 If M and $P \subseteq \partial\bar{M}$ are as in Definition 5.1.2, X is a path-connected space, and $G \leq \mathrm{Diff}(P)$ is realisable by isotopies (so that the definition of $\Sigma_n^P(M,X|G)$ makes sense), then the stabilisation map (5.1.3) always induces a *split-injection* on homology in every degree. This can be proved as for configurations of points (see the proof of Theorem 4.5 in [McD75]) by considering the maps $\Sigma_n^P(M,X|G) \dashrightarrow \Sigma_k^P(M,X|G)$, for $1 \leq k \leq n$, defined only after taking infinite symmetric products, which forget some of the copies of P in the configuration. In §4.9 of Chapter 4 there is an extension of this argument to twisted coefficient systems; this can equally well be extended to configuration spaces of submanifolds, so the map (5.1.4) is also always split-injective.

Remark 5.1.11 Although we have not worked this out in detail, it seems very likely that the methods of this chapter, modified by the ideas of Chapter 2, would prove homological stability for 'oriented' (or, perhaps better terminology in this context, 'alternating') configuration spaces of submanifolds. More precisely there is a stabilisation map

$$A_n^P(M,X|G) \longrightarrow A_{n+1}^P(M,X|G)$$

which induces isomorphisms on homology up to approximately[3] degree $\frac{n}{3}$. Here $A_n^P(M,X|G)$ means $E_n^P(M) \times_{G \wr A_n} X^n$, analogously to Definition 5.1.5.

Remark 5.1.12 A natural next step is to look for a scanning result in this setting, trying to identify the limiting space, as the number of submanifolds goes to infinity, with a more accessible space, and from this explicitly compute the homology in the stable range. This is something the author intends to investigate in the near future.

5.1.3 Discussion of the hypotheses of the Main Theorem and Extension

5.1.3.1 Open subgroups of Diff(P)

An obvious family of open subgroups of $\mathrm{Diff}(P)$ for the Main Theorem is the following. First note that $\mathrm{Diff}(P)$ is locally path-connected (in fact it is locally contractible as it is a Fréchet manifold), so its path-components are all clopen, and so the quotient space

[3] Give or take a small additive constant.

$\pi_0\mathrm{Diff}(P)$ is discrete.[4] Hence any subgroup H of the mapping class group $\pi_0\mathrm{Diff}(P)$ pulls back along the quotient map q to an open subgroup $G = q^{-1}(H) \leq \mathrm{Diff}(P)$. In particular $\mathrm{Diff}_0(P) = q^{-1}(\{e\})$ is open, and when P is orientable $\mathrm{Diff}^+(P)$ is open.

5.1.3.2 Realisability by isotopies

For Extension 5.1.8 we are interested in which subgroups of $\mathrm{Diff}(S^k)$ are realisable by isotopies w.r.t. the embedding $\iota\colon S^k \hookrightarrow \mathbb{R}^{k+1} \subseteq \mathbb{R}^{d-1} \subseteq \partial\bar{M}$. By Proposition 5.1.4 this holds for the whole diffeomorphism group $\mathrm{Diff}(S^k)$ when $\dim(M) \geq \frac{3}{2}k + 2$. Outside this range checking realisability by isotopies is less easy.

Remark 5.1.13 Note that being realisable by isotopy is a locally constant property on $\mathrm{Diff}(P)$. In other words, if $g, g' \in \mathrm{Diff}(P)$ are in the same path-component, then g is realisable by isotopy if and only if g' is. Hence it is enough to check realisability by isotopy for one representative of each element of the mapping class group $\pi_0\mathrm{Diff}(P)$.

Aside on mapping class groups of spheres. There is a decomposition of the diffeomorphism group $\mathrm{Diff}(S^k) \simeq O(k+1) \times \mathrm{Diff}(D^k; \partial D^k)$, so on π_0 we have

$$\pi_0\mathrm{Diff}(S^k) \cong \mathbb{Z}/2 \times \pi_0\mathrm{Diff}(D^k; \partial D^k), \tag{5.1.5}$$

where the generator of the $\mathbb{Z}/2$ summand corresponds to a reflection. In dimensions $k = 1, 2, 3$, the group $\pi_0\mathrm{Diff}(D^k; \partial D^k)$ is trivial (for $k = 2$ see [Sma58] and [EE67]; for $k = 3$ see [Cer68] and [Hat83]). In dimension $k = 4$ nothing is known. In dimensions $k \geq 5$ there is a homomorphism to the group of exotic $(k+1)$-spheres

$$\pi_0\mathrm{Diff}(D^k; \partial D^k) \longrightarrow \Theta_{k+1}$$

given by extending a diffeomorphism $D^k \to D^k$ fixing ∂D^k to a diffeomorphism $S^k \to S^k$ and using it to glue together the boundaries of two copies of D^{k+1} to obtain a (possibly exotic) $(k+1)$-sphere. This map is surjective by the h-cobordism theorem [Sma61] and injective by the pseudoisotopy theorem [Cer70]. The groups Θ_{k+1} are finite abelian by [KM63], and are known for small k (see [Lev85]). For $k = 5, 11$ and 60, the group Θ_{k+1} is trivial, but it is not known to be trivial for any other $k \geq 5$; for example $\Theta_{15} = \mathbb{Z}/2 \oplus \mathbb{Z}/8128$.

It is therefore not in general easy to check realisability by isotopy for an element of each mapping class when Proposition 5.1.4 does not hold. One simple case is the following:

Remark 5.1.14 When $P = S^k$ embedded in $\mathbb{R}^{k+1} \subseteq \mathbb{R}^{d-1} \subseteq \partial\bar{M}$, it is easy to see that a reflection of S^k can be realised by an isotopy. Hence the preimage under $\mathrm{Diff}(S^k) \to$

[4] This is not a vacuous statement as we are thinking of $\pi_0\mathrm{Diff}(P)$ as the quotient *space* by the equivalence relation of being in the same path-component, and as such it may a priori have a non-discrete topology. For example $\pi_0\mathbb{Q} = \mathbb{Q}$ is not discrete.

$\pi_0\mathrm{Diff}(S^k)$ of the $\mathbb{Z}/2$ summand in (5.1.5) is realisable by isotopies, and moreover if $k = 1, 2, 3, 5, 11$ or 60 then this is in fact the full diffeomorphism group $\mathrm{Diff}(S^k)$.

5.1.4 A conjecture

Conjecture 5.1.15 *The codimension assumption* $\dim(M) - \dim(P) \geq 3$ *of Extension 5.1.8 can be reduced to* $\dim(M) - \dim(P) \geq 2$*. In particular we conjecture that homological stability (with stable range* $* \leq \frac{n}{2}$*) holds for the sequences of spaces* $\{\Sigma_n^{S^1}(\mathbb{R}^3)\}_n$ *and* $\{\mathring{\Sigma}_n^{S^1}(\mathbb{R}^3)\}_n$ *consisting of* n *unlinked, unknotted circles in* $\mathbb{R}^3$ *which are unoriented and oriented respectively.*

Some "supporting evidence" for this is as follows. Firstly, when P is a point, homological stability *does* hold in codimension 2, in other words for configurations of points on a surface.[5] Secondly, there are stability results for the sequences of *fundamental groups* $\{\pi_1\Sigma_n^{S^1}(\mathbb{R}^3)\}$ and $\{\pi_1\mathring{\Sigma}_n^{S^1}(\mathbb{R}^3)\}$, as follows.

The group $\pi_1\Sigma_n^{S^1}(\mathbb{R}^3)$ is sometimes called the *circle braid group*, or the *string motion group*, and is isomorphic to the group $\Sigma\mathrm{Aut}(F_n)$ of symmetric automorphisms of the free group F_n on n letters. Homological stability for this sequence of groups was proved by Hatcher-Wahl [HW10, Corollary 1.2] in the range $* \leq \frac{n-2}{2}$.[6] We note that this is not a special case of the conjecture, since $\Sigma_n^{S^1}(\mathbb{R}^3)$ is *not* aspherical, which can be seen as follows. In [BH10] it is proved that the inclusion of the subspace $\mathcal{R}_n$ of *Euclidean* circles[7] into $\Sigma_n^{S^1}(\mathbb{R}^3)$ is a homotopy equivalence. Now $\mathcal{R}_n$ is a $6n$-dimensional manifold but its fundamental group contains torsion (for example it contains a copy of the symmetric group Σ_n), so it cannot be aspherical.

The group $\pi_1\mathring{F}_n^{S^1}(\mathbb{R}^3)$ is the *pure string motion group*, consisting of 'motions' of n disjoint, unlinked, unknotted circles which return each circle to its original position and orientation. It is isomorphic to the group $P\Sigma\mathrm{Aut}(F_n)$ of *pure* symmetric automorphisms of F_n. It was recently proved by Wilson [Wil11] that for fixed $*$ the sequence of $\mathbb{Q}\Sigma_n$-representations $\{H^*(P\Sigma\mathrm{Aut}(F_n);\mathbb{Q})\}$ is *uniformly representation stable* in the range $* \leq \frac{n}{4}$. The notion of uniform representation stability was introduced in [CF10] and in particular implies that the sequence of invariant subgroups $\{H^*(P\Sigma\mathrm{Aut}(F_n);\mathbb{Q})^{\Sigma_n}\}$ is stable (independent of n) in this range. So by a transfer argument we have stability for

$$H^*(\pi_1\mathring{F}_n^{S^1}(\mathbb{R}^3);\mathbb{Q})^{\Sigma_n} \quad \cong \quad H^*(\pi_1\mathring{\Sigma}_n^{S^1}(\mathbb{R}^3);\mathbb{Q})$$

[5] The methods of [RW11], while similar in many repects to the methods of this chapter, involve a more ad hoc argument (special to the case of points) instead of the 'second resolution' of §5.6, which is exactly where the codimension-3 assumption is needed.

[6] Rational homological stability in the larger range $* \leq \frac{2n}{3}$ was proved independently by Zaremsky [Zar12], and in fact something much stronger is true rationally: the reduced homology $\widetilde{H}_*(\Sigma\mathrm{Aut}(F_n);\mathbb{Q})$ is *trivial* for all $*$ and n by [Wil11, Theorem 7.1]. (Zaremsky states that this latter fact also follows from [Gri11].)

[7] Those which are of the form $\{(x, y) \mid (x - x_0)^2 + (y - y_0)^2 = r^2\} \subseteq \mathbb{R}^2 \hookrightarrow \mathbb{R}^3$ for some embedding of $\mathbb{R}^2$ as an affine subspace of $\mathbb{R}^3$.

(see for example [Bro82, Proposition III.10.4]), so rational homological stability holds for the sequence of fundamental groups $\{\pi_1 \mathring{\Sigma}_n^{S^1}(\mathbb{R}^3)\}$ in the range $* \leq \frac{n}{4}$. (This was pointed out as Corollary 8.1 of [Wil11].) Again, this is not a special case of the conjecture since $\mathring{\Sigma}_n^{S^1}(\mathbb{R}^3)$ is *not* aspherical (being a 2^n-sheeted covering space of $\Sigma_n^{S^1}(\mathbb{R}^3)$, which is not aspherical).

5.1.5 A related result

There is a recent result of Federico Cantero and Oscar Randal-Williams [CRW] which is related to the result of this chapter. For a manifold N, with a choice δ of a collection of b circles in its boundary (if it has one), they consider the space $\mathcal{E}_{g,b}(N,\delta)$ of oriented submanifolds Q of N homeomorphic to the compact, connected, orientable surface $\Sigma_{g,b}$ of genus g and with b boundary-components, and with $\partial Q = \delta$. When $b \geq 1$ stabilisation maps can be defined by gluing a collar neighbourhood, containing an embedded surface, onto ∂N. They prove that if N is simply-connected and at least 6-dimensional these spaces satisfy homological stability: the homology $H_*(\mathcal{E}_{g,b}(N,\delta))$ is independent of g and b, once g is sufficiently large (approximately $g \geq \frac{3}{2}*$). Moreover they identify the homology of the limiting space with that of a certain section space over N.

So in a nutshell (and condensing): by [CRW] we have homological stability for spaces of submanifolds w.r.t. the number of *handles*, and by the results of this chapter we have homological stability for spaces of submanifolds w.r.t. the number of *components*.

Conventions

For clarity, we will always use the following notation for manifolds, depending on whether they are assumed to be compact and/or have boundary: M denotes manifolds without boundary, N denotes manifolds possibly with boundary, P denotes compact manifolds without boundary (closed manifolds) and Q denotes compact manifolds possibly with boundary.

We will always take mapping spaces $C^\infty(M_1, M_2)$, $\mathrm{Emb}(P, M)$ etc. to be equipped with the *strong* (Whitney) topology unless otherwise indicated.

Organisation of the chapter

We begin in §5.2 by collecting together some constructions, and giving a precise definition of the stabilisation map. In §5.3 we recall and prove some auxiliary facts about semi-simplicial spaces, transversality and fibre bundles: in particular we prove that certain maps *are* fibre bundles. Since the spectral sequence arguments and the geometric arguments are both somewhat intricate, we separate them out by proving in §5.4 a general, axiomatic homological stability criterion, whose hypotheses we spend the next three sections checking for our specific case of configuration spaces of submanifolds. Checking the hypotheses involves constructing two successive "resolutions" of the stabilisation map, and then constructing factorisations up to homotopy of certain diagrams. This is done in §§5.5, 5.6 and 5.7 respectively. In the final section §5.8 we deduce *twisted* homological stability for configuration spaces of submanifolds from untwisted homological stability and the "twisted stability from untwisted stability" principle §4.6 of Chapter 4.

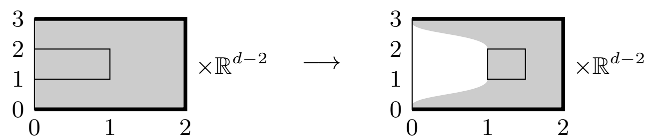
5.2 Constructions

Recall that we have a connected manifold M which is the interior of a manifold $\bar{M}$ with non-empty boundary, and we have a closed, connected submanifold $P \subseteq \partial\bar{M}$. Denote the inclusion $P \hookrightarrow \partial\bar{M}$ by ι, and choose a coordinate neighbourhood $B \cong \mathbb{R}^d_+ = \{(x_1,\ldots,x_d) \mid x_1 \geq 0\}$ containing $\iota(P)$.

We will need to make several constructions within B during the proof, and to precisely define the stabilisation map; we collect them all here for convenience.

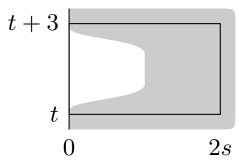
First, choose a smooth self-embedding f of $[0,2]\times[0,3]\times\mathbb{R}^{d-2}$ such that

- $\mathrm{im}(f)$ is disjoint from $[0,1)\times[1,2]\times\mathbb{R}^{d-2}$,
- f is the identity on a neighbourhood of $(([0,2]\times\{0,3\})\cup(\{2\}\times[0,3]))\times\mathbb{R}^{d-2}$,
- f restricts to $(x_1,x_2\ldots)\mapsto(\frac{1}{2}x_1+1,x_2\ldots)$ on $[0,1]\times[1,2]\times\mathbb{R}^{d-2}$.



(5.2.1)

For $s\in[0,1]$ and $t\in\mathbb{R}$ define a smooth map $f_{s,t}\colon\mathbb{R}^d_+\to\mathbb{R}^d_+$ by identifying $[0,2s]\times[t,t+3]\times\mathbb{R}^{d-2}$ with $[0,2]\times[0,3]\times\mathbb{R}^{d-2}$ in the obvious linear way, and then applying f on this subset and the identity everywhere else. This gives a smooth map $f_{s,t}\colon\bar{M}\to\bar{M}$ by identifying $B\cong\mathbb{R}^d_+$ and extending by the identity again. By an abuse of notation we let $f=f_{1,0}$.



(5.2.2)

See Figure 5.2.1 for a picture of the following constructions. Let V and W be the subneighbourhoods $[0,2)\times(0,3)\times\mathbb{R}^{d-2}$ and $[0,2)\times(-3,0)\times\mathbb{R}^{d-2}$ of B respectively, and assume that P is contained in $\{0\}\times(1,2)\times\mathbb{R}^{d-2}\subseteq\partial B\subseteq\partial\bar{M}$. We fix notation for certain embeddings of P and $P\times[0,1]$ by

$$\begin{array}{rcclcrcll}
\iota_s\colon & p & \mapsto & p+(0,s,0,\ldots) & : & P & \hookrightarrow & B & \quad(\text{for } s\in\mathbb{R})\\
\phi_0\colon & p & \mapsto & p+(0,-3,0,\ldots) & : & P & \hookrightarrow & B & \\
\psi_0\colon & p & \mapsto & p+(1,-3,0,\ldots) & : & P & \hookrightarrow & B & \\
e_0\colon & (p,t) & \mapsto & p+(t,-3,0,\ldots) & : & P\times[0,1] & \hookrightarrow & B &
\end{array}$$

Note that $\phi_0=\iota_{-3}$ and $\psi_0=f_{1,-3}\circ\phi_0$.

For a subset $S\subseteq[0,2)$ let W_S be the preimage of the projection onto the first coordinate, so for example W_0 is ∂W. Finally, choose a relatively compact[8] tubular neighbourhood T of $\psi_0(P)$ as a submanifold of W_1.

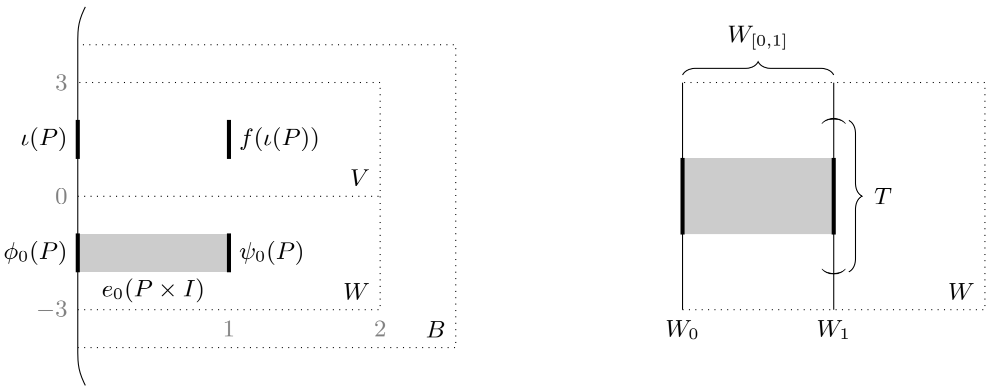


Figure 5.2.1: Left: the coordinate neighbourhood B. Right: the subneighbourhood W in close-up.

Definition 5.2.1 Given a path-connected space X with chosen basepoint x_0, and a subgroup $G \leq \mathrm{Diff}(P)$ which is realisable by isotopies (Definition 5.1.3), the *stabilisation map* $s\colon \Sigma_n^P(M, X|G) \longrightarrow \Sigma_{n+1}^P(M, X|G)$ is defined to be

$$\{[\psi_1], \ldots, [\psi_n]; x_1, \ldots, x_n\} \;\mapsto\; \{[f \circ \psi_1], \ldots, [f \circ \psi_n], [f \circ \iota]; x_1, \ldots, x_n, x_0\},$$

where $[\psi_i]$ denotes the orbit of the embedding ψ_i under the action of G.

5.3 Preliminaries

5.3.1 Semi-simplicial spaces

Definition 5.3.1 A *semi-simplicial space* $X_\bullet$ is a diagram of the form

$$\cdots \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; Y_1 \;\rightrightarrows\; Y_0$$

where the 'face maps' $d_i\colon Y_k \to Y_{k-1}$ $(1 \leq i \leq k+1)$ satisfy the simplicial identities $d_i d_j = d_{j-1} d_i$ whenever $i < j$. An *augmented* semi-simplicial space is a diagram of the form

$$\cdots \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; Y_1 \;\rightrightarrows\; Y_0 \longrightarrow Y_{-1}$$

where again the face maps satisfy the simplicial identities. This is a semi-simplicial space together with an 'augmentation map' $Y_0 \to Y_{-1}$ which equalises the two face maps $d_1, d_2\colon Y_1 \rightrightarrows Y_0$. A map of (augmented) semi-simplicial spaces is a collection of maps, one for each level k, which commutes with d_i for each i.

The (thick) geometric realisation of a Δ-space $Y_\bullet$ is $\|Y_\bullet\| = \big(\coprod_{k\geq 0} Y_k \times \Delta^k\big)/\!\sim$, where $\sim$ is the equivalence relation generated by the face relations $(d_i(y), z) \sim (y, \delta_i(z))$, where

[8]The closure $\overline{T}$ of T in W_1 must be compact.

δ_i is the inclusion of the ith face of Δ^{k+1}. If $Y_\bullet$ is an augmented Δ-space, there is a unique composition of face maps $Y_k \to Y_{-1}$ for each k. These induce a well-defined map $\|Y_\bullet\| \to Y_{-1}$, where $\|Y_\bullet\|$ is the geometric realisation of $Y_\bullet$ as a non-augmented semi-simplicial space (i.e. forgetting Y_{-1}).

Definition 5.3.2 For any space X we can define two semi-simplicial spaces $\mathrm{Map}([\bullet+1], X)$ and $\mathrm{Inj}([\bullet+1], X)$. The space of i-simplices is all maps from $[i+1] \coloneqq \{1, \ldots, i+1\}$ to X in the first case, and is all *injective* maps in the second case. The face maps in both cases are induced by the injective order-preserving maps $[i] \to [i+1]$.

We will make use of the fact that these semi-simplicial spaces are highly-connected:

Lemma 5.3.3 *The geometric realisation* $\|\mathrm{Map}([\bullet+1], X)\|$ *is contractible, and the geometric realisation* $\|\mathrm{Inj}([\bullet+1], X)\|$ *is* $(|X|-2)$*-connected.*

Proof. This is well-known, and can be proved using standard techniques. First, one can construct a semi-simplicial nullhomotopy of the identity $\mathrm{Map}([\bullet+1], X) \to \mathrm{Map}([\bullet+1], X)$. Then one can show that the inclusion $\|\mathrm{Inj}([\bullet+1], X)\| \hookrightarrow \|\mathrm{Map}([\bullet+1], X)\|$ is highly-connected by mapping spheres into the smaller space, extending the map to a disc mapping into the larger space (since it's contractible) and then inductively deforming this extension to land in the smaller space. See [RW11, Proposition 3.2] for an alternative method of proof. □

We also recall here the relative Hurewicz Theorem, which we will use to deduce that maps induce isomorphisms on homology in a range when they are highly-connected.

Fact 5.3.4 (Relative Hurewicz Theorem) *If the homotopy fibre of a map* $f \colon X \to Y$ *is* k*-connected, i.e.* $\pi_i(\mathrm{hofib}(f)) = 0$ *for all* $i \leq k$*, then its homotopy cofibre is* $(k+1)$*-homology-connected, i.e.* $\widetilde{H}_i(\mathrm{hocofib}(f)) = 0$ *for all* $i \leq k+1$*.*

5.3.2 A fibre bundle criterion

There is a useful elementary criterion for checking that a map is a fibre bundle which we will use several times. To give an example of its utility, it was used in [Pal60] to prove that the restriction maps $\mathrm{Emb}(M', M) \to \mathrm{Emb}(P, M)$ and $\mathrm{Diff}(M) \to \mathrm{Emb}(P, M)$ are fibre bundles, for manifolds $P \subseteq M' \subseteq M$ with P compact. Since fibre bundles are Serre fibrations, this implies the (parametrised) Isotopy Extension Theorem. It was also used by [Lim64] to give a shorter proof of these facts, and by [Cer61] to generalise them to manifolds with boundary (and even with corners of arbitrary codimension).

Definition 5.3.5 If G is a topological group and Y is a G-space, then we say that (the action of G on) Y *admits local sections* if, for each $y \in Y$, there is a continuous map

$\gamma_y \colon U_y \to G$ from an open neighbourhood U_y of y such that $\gamma_y(u) \cdot y = u$ holds for each point $u \in U_y$. In other words, the map

$$G \xrightarrow{-\cdot y} Y$$

has a section on some neighbourhood of y. This may be described as saying that the action is "continuously locally transitive".

Lemma 5.3.6 (Fibre bundle criterion; Theorem A of [Pal60]) *Suppose that Y is a G-space which admits local sections, and X is any G-space. Then any G-equivariant map $f \colon X \to Y$ is a fibre bundle.*

Proof. For any $y \in Y$ take $\gamma_y \colon U_y \to G$ as above. Then the map

$$(x, u) \mapsto \gamma_y(u) \cdot x \colon \; f^{-1}(y) \times U_y \longrightarrow f^{-1}(U_y)$$

is a local trivialisation of f over U_y, with inverse given by

$$x \mapsto \big(\gamma_y(f(x))^{-1} \cdot x, f(x)\big) \colon \; f^{-1}(U_y) \longrightarrow f^{-1}(y) \times U_y. \qquad \square$$

We will actually need a slight extension of this.

Lemma 5.3.7 (Second fibre bundle criterion) *As before, let $f \colon X \to Y$ be any G-equivariant map of G-spaces, and assume that the action of G on Y admits local sections. Suppose that X also has an action of another group H, which commutes with the action of G and preserves the fibres of f. Then for any H-space Z, there is a well-defined map*

$$\bar{f} \colon [x, z] \mapsto f(x) \; : \; X \times_H Z \to Y.$$

This map is also a fibre bundle.

Proof. For any $y \in Y$ we have $\gamma_y \colon U_y \to G$ as in the definition of local sections. Since the action of H preserves the fibres of f, we have $\bar{f}^{-1}(S) = f^{-1}(S) \times_H Z$ for any subset $S \subseteq Y$. So to construct a local trivialisation we need to define a homeomorphism

$$\big(f^{-1}(y) \times_H Z\big) \times U_y \;\longrightarrow\; f^{-1}(U_y) \times_H Z$$

over U_y. We can define this to be

$$([x, z], u) \;\mapsto\; [\gamma_y(u) \cdot x, z],$$

which is well-defined since the actions of G and H on X commute, and which has inverse given by

$$([\gamma_y(f(x))^{-1} \cdot x, z], f(x)) \;\leftarrow\!\shortmid\; [x, z]. \qquad \square$$

The main result of [Pal60] (and of [Lim64]), which implies the facts mentioned above, is concerned with the action of $\mathrm{Diff}_{c,0}(M)$ on embedding spaces. Here the 'c' means that the diffeomorphisms are compactly-supported, and the '0' means that they are homotopic to the identity through compactly-supported diffeomorphisms, i.e. in the same path-component of $\mathrm{Diff}_c(M)$ as the identity.

Proposition 5.3.8 ([Pal60, Theorem B] and [Lim64]) *If $P \subseteq M$ are smooth manifolds without boundary, with P compact, then the action of $\mathrm{Diff}_{c,0}(M)$ on $\mathrm{Emb}(P, M)$ admits local sections.*

Remark 5.3.9 In [Cer61, §2.2.1], a similar result is proved for the action of a slightly different group. One chooses an open neighbourhood U of P in M, and instead of the action of $\mathrm{Diff}_{c,0}(M)$ one considers the action of $\mathrm{PDiff}(M; M \setminus U)$ of diffeomorphisms of M supported in U equipped with a *chosen* path (through diffeomorphisms supported in U) to the identity. The result of [Cer61, §2.2.1] is also more general in that M and P may have corners of arbitrary codimension.

We will want a result similar to 5.3.8 for manifolds with boundary. We could use the result of Cerf mentioned above, but it is simpler to just note that the proof of [Lim64] goes through in the specific case that we need.

Definition 5.3.10 Recall that a *neat submanifold* of a manifold with boundary N is a submanifold $N' \subseteq N$ such that $\partial N' = N' \cap \partial N$ and N' is covered by coordinate charts U of N of the form

$$(U, U \cap N') \xrightarrow{\cong} (\mathbb{R}^n, \mathbb{R}^d), \quad (U, U \cap N') \xrightarrow{\cong} (\mathbb{R}^n_+, \mathbb{R}^d_+)$$

in the interior and boundary respectively (where $\dim(N) = n$, $\dim(N') = d$ and $\mathbb{R}^n_+$ is the half-space $\{x_1 \geq 0\} \subseteq \mathbb{R}^n$). A *neat embedding* $N' \hookrightarrow N$ is one whose image is a neat submanifold.

Fact 5.3.11 *The set of neat embeddings $\mathrm{NEmb}(N', N)$ is open in the space (with the strong topology) of smooth maps $C^\infty_\partial(N', N)$ which take $\partial N'$ into ∂N.*

Proposition 5.3.12 *Let N be an open subset of $[0, 1] \times \mathbb{R}^{n-1}$, and let Q be a compact manifold with boundary with $\dim(Q) = q < n$. Then the action of $\mathrm{Diff}_{c,0}(N)$ on $\mathrm{NEmb}(Q, N)$ admits local sections.*

Proof, following [Lim64]. Given $e \in \mathrm{NEmb}(Q, N)$, choose a tubular neighbourhood T of $e(Q)$ in N, with projection $p\colon T \to e(Q)$ onto the zero-section, of finite radius $r > 0$ (i.e. each fibre $p^{-1}(y)$ is isometric to an open $(n - q)$-ball of radius r). Such a tubular neighbourhood exists since $e(Q) \subseteq N$ is neat, by [Hir76, Theorem 6.3] for example. By compactness of Q we may choose r sufficiently small that $T \subseteq N$. For $\varepsilon \in (0, 1]$ we denote by $\varepsilon\overline{T}$ the closed tubular subneighbourhood of T of radius εr, in other words $\varepsilon\overline{T} = \{y \in T \mid |y - p(y)| \leq \varepsilon r\}$. Define an open neighbourhood of e in $\mathrm{NEmb}(Q, N)$ by

$$U'_e \coloneqq \{f \in \mathrm{NEmb}(Q, N) \mid |f(x) - e(x)| < \tfrac{r}{2}\},$$

and choose a smooth function $\lambda\colon \mathbb{R} \to [0, 1]$ which is 1 on $[0, \frac{r}{4}]$ and 0 on $[\frac{r}{2}, \infty)$. We can then define $\gamma_e\colon U'_e \to C^\infty_c(N, N)$ by

$$\begin{aligned} \gamma_e(f)\colon N &\to N \\ y &\mapsto \begin{cases} y + \lambda(|y - p(y)|).\big(fe^{-1}p(y) - p(y)\big) & \text{for } y \in \frac{3}{4}\overline{T}, \\ y & \text{for } y \in N \smallsetminus \frac{3}{4}\overline{T}. \end{cases} \end{aligned}$$

Note that $\gamma_e(f)$ does map y into N, since outside of $\frac{1}{2}\overline{T}$ it is the identity and for $y \in \frac{1}{2}\overline{T}$,

$$\begin{aligned} |\gamma_e(f)(y) - p(y)| &\leq |\gamma_e(f)(y) - y| + |y - p(y)| \\ &\leq |fe^{-1}p(y) - p(y)| + \tfrac{r}{2} \\ &< \tfrac{r}{2} + \tfrac{r}{2} \end{aligned}$$

by definition of U'_e, so $\gamma_e(f)(y) \in T \subseteq N$. Hence $\gamma_e(f)$ is a compactly-supported (supported in $\frac{1}{2}\overline{T}$) smooth map $M \to M$. We also want γ_e to be continuous for the strong topology on $C^\infty_c(N, N)$. Its definition involves (i) precomposition by a fixed map, (ii) pointwise addition and multiplication, and (iii) extending a map from a compact codimension-zero submanifold by the identity. These are all (strongly) continuous operations between mapping spaces—see [GG73, 3.6,3.8,3.9] for example. A slight subtlety is that precomposition (unlike postcomposition) by a fixed map g is only continuous in the strong topology if g is proper. But in our case we are precomposing with $e^{-1} \circ p\colon \frac{3}{4}\overline{T} \to Q$ which is a proper map.

The subset $\mathrm{Diff}_{c,0}(N) \subseteq C^\infty_c(N, N)$ is open and contains $\gamma_e(e) = \mathrm{id}$, so we can define $U_e \coloneqq U'_e \cap \gamma_e^{-1}(\mathrm{Diff}_{c,0}(N))$ to obtain a smaller open neighbourhood of e in $\mathrm{NEmb}(Q, N)$. Restricted to U_e, the map γ_e is the required local section since for $f \in U_e$,

$$\gamma_e(f) \circ e(x) = e(x) + \lambda(0).(f(x) - e(x)) = f(x). \qquad \square$$

5.3.3 Parametrising submanifolds

Given an unparametrised submanifold P of a manifold M, we will need to be able to continuously extend a choice of parametrisation of P to a choice of parametrisation for all submanifolds of type P in a neighbourhood of P. In other words we need the quotient map $\mathrm{Emb}(P, M) \to \mathrm{Emb}(P, M)/\mathrm{Diff}(P)$ which forgets the parametrisation of an embedding to admit local sections. In fact it does more than this: it is a principal $\mathrm{Diff}(P)$-bundle. This was proved for compact P by [BF81], and the compactness assumption was removed by [Mic80a] (see also [Mic80b, §13] and [KM97, §44] for presentations of this result).

We will also need to find local sections in the following more general situation. We have a mod-G-parametrised submanifold P of M (in other words it is given a parametrisation up to the action of a specified subgroup $G \leq \mathrm{Diff}(P)$), and we need to continuously extend a choice of a compatible *full* parametrisation of P (an element of the orbit of its mod-G-parametrisation) to a choice of a compatible full parametrisation for all mod-G-parametrised submanifolds of type P in a neighbourhood of P. More concisely: we need the quotient map $\mathrm{Emb}(P, M) \to \mathrm{Emb}(P, M)/G$ to admit local sections. We prove this directly for closed P, following the method of [BF81], assuming that G is either an open or finite subgroup of $\mathrm{Diff}(P)$.

Proposition 5.3.13 *Suppose P and M are smooth manifolds without boundary, with P closed and $\dim(P) < \dim(M)$. Then for any open subgroup $G \leq \mathrm{Diff}(P)$ the quotient map*

$$\mathrm{Emb}(P, M) \xrightarrow{\pi} \mathrm{Emb}(P, M)/G$$

has local sections (in the sense that for every point e in the domain there is a section s_e of π defined on an open neighbourhood of $\pi(e)$ which sends $\pi(e)$ to e).

Proof, following [BF81]. Fix an element $[e] \in \mathrm{Emb}(P, M)/G$ with chosen representative $e \colon P \hookrightarrow M$. Let T be a tubular neighbourhood of $e(P) \subseteq M$, with projection $p \colon T \to e(P)$ onto the zero-section. Note that $\mathrm{Emb}(P, T)$ is open in $\mathrm{Emb}(P, M)$. There is a continuous map

$$e^{-1} \circ p \circ - \colon \mathrm{Emb}(P, T) \longrightarrow C^\infty(P, P)$$

(since post-composition by a fixed map is continuous by [GG73, Proposition 3.9]); define $\mathrm{Emb}^G(P, T)$ to be the preimage of $G \subseteq \mathrm{Diff}(P) \subseteq C^\infty(P, P)$. Since $\mathrm{Diff}(P)$ is open in $C^\infty(P, P)$ and G was assumed to be open in $\mathrm{Diff}(P)$, $\mathrm{Emb}^G(P, T)$ is open in $\mathrm{Emb}(P, M)$. Another description of it is

$$\mathrm{Emb}^G(P, T) = \{ s \circ e \circ \phi \mid \phi \in G, s \text{ section of } T \}. \tag{5.3.1}$$

Define $V_e := \{ s \circ e \mid s \text{ section of } T \} \subseteq \mathrm{Emb}^G(P, T)$ and note that

(i) $\pi|_{V_e}$ is injective, and
(ii) $\pi^{-1}(\pi(V_e)) = \mathrm{Emb}^G(P, T)$, by the description (5.3.1).

So $\pi(V_e)$ is an open neighbourhood of $[e]$ in the quotient topology on $\mathrm{Emb}(P, M)/G$ and there is a function

$$s_e := (\pi|_{V_e})^{-1} \colon \pi(V_e) \longrightarrow \mathrm{Emb}(P, M)$$

such that $\pi \circ s_e = \mathrm{id}_{\pi(V_e)}$. Hence it is sufficient to check that s_e is continuous; equivalently that the restriction $\pi|_{V_e} \colon V_e \to \pi(V_e)$ is an open map. Let $U \subseteq \mathrm{Emb}(P, M)$ be open. We need to show that $\pi(U \cap V_e)$ is open in $\mathrm{Emb}(P, M)/G$, i.e. that $\pi^{-1}(\pi(U \cap V_e))$ is open in

$\mathrm{Emb}(P, M)$. To see this consider the continuous maps

$$(\mathrm{id}, e^{-1} \circ p \circ -)\colon \mathrm{Emb}^G(P, T) \longrightarrow \mathrm{Emb}^G(P, T) \times G,$$
$$(f, g) \mapsto f \circ g^{-1}\colon \mathrm{Emb}^G(P, T) \times G \longrightarrow \mathrm{Emb}^G(P, T).$$

(The latter is continuous by [GG73, Proposition 3.9] since P is compact.) The image of the composition lies in $V_e \subseteq \mathrm{Emb}^G(P, T)$, and the preimage of $U \cap V_e$ is precisely $\pi^{-1}(\pi(U \cap V_e))$ which is therefore open in $\mathrm{Emb}^G(P, T)$. But this is open in $\mathrm{Emb}(P, M)$, therefore so is $\pi^{-1}(\pi(U \cap V_e))$. □

Remark 5.3.14 In the above proof, for $\pi(V_e)$ to be an open neighbourhood of $[e]$, it is sufficient for G to contain a neighbourhood of the identity in $\mathrm{Diff}(P)$, but to check openness of $\pi|_{V_e}$ we actually need G to be open in $\mathrm{Diff}(P)$.

We also note that the above proposition is also true if G is a *finite* subgroup of $\mathrm{Diff}(P)$:

Proposition 5.3.15 *In the situation of Proposition 5.3.13 above, if $G \leq \mathrm{Diff}(P)$ is a finite, rather than open, subgroup, then again the quotient map*

$$\mathrm{Emb}(P, M) \xrightarrow{\pi} \mathrm{Emb}(P, M)/G$$

has local sections.

Proof. Note that $\mathrm{Emb}(P, M)$ is Hausdorff and the action of $\mathrm{Diff}(P)$ is free. For any free action of a finite group on a Hausdorff space, the associated quotient map is a covering map, and hence has local sections. □

5.3.4 Transversality

Another input we will need is a certain transversality result. First we need to recall some definitions.

Definition 5.3.16 Two smooth maps $f_1\colon N_1 \to N$ and $f_2\colon N_2 \to N$ are said to be *transverse* if for all $y \in f_1(N_1) \cap f_2(N_2)$ and for all preimages $x_1 \in (f_1)^{-1}(y)$ and $x_2 \in (f_2)^{-1}(y)$, the images of the tangent spaces $df_1(T_{x_1}N_1)$ and $df_2(T_{x_2}N_2)$ span the tangent space T_yN. In particular we have a notion of transversality for submanifolds by considering their inclusion maps.

A *residual* subset of a space X is one which can be written as a countable intersection of dense open subsets of X. A space X is a *Baire space* if every residual subset is dense.

For example $C^\infty(M_1, M_2)$ is a Baire space for any smooth manifolds without boundary M_1 and M_2. Thom's Transversality Theorem [Tho54] says that for any fixed submanifold $M_3 \subseteq M_2$, the subset

$$\{f \mid f \text{ is transverse to the inclusion } M_3 \hookrightarrow M_2\} \subseteq C^\infty(M_1, M_2)$$

is residual, and therefore dense. We will need a similar statement, but allowing the manifolds to have boundary. For manifolds N_1 and N_2, possibly with boundary, let $C^\infty_\partial(N_1, N_2)$ be the subspace of $C^\infty(N_1, N_2)$ (with the strong topology) consisting of maps which take ∂N_1 to ∂N_2, and let $C^\infty_{\partial,\mathrm{pr}}(N_1, N_2)$ be the subspace of such maps which are also proper.[9] A transversality theorem for manifolds with boundary was proved in [Ish98] in quite a general setting, using the language of jet bundles; the following is the special case which we will use (mentioning only 0-jets, which are just graphs of maps):

Proposition 5.3.17 (Theorem 1.4 of [Ish98] with $s = 1, r = 0$) *Suppose N_1 and N_2 are smooth manifolds with boundary. Choose a countable set A_{int} of submanifolds of $N_1 \times N_2$ and a countable set A_{bdy} of submanifolds of $(N_1 \times N_2) \smallsetminus (\partial N_1 \times \mathring{N}_2)$. Then there is a residual subset $R \subseteq C^\infty_{\partial,\mathrm{pr}}(N_1, N_2)$ such that for all maps $f \in R$,*

$$\textit{graph of } f|_{\mathring{N}_1} \colon \mathring{N}_1 \longrightarrow N_1 \times N_2$$

is transverse to every manifold in A_{int}, and

$$\textit{graph of } f|_{\partial N_1} \colon \partial N_1 \longrightarrow (N_1 \times N_2) \smallsetminus (\partial N_1 \times \mathring{N}_2)$$

is transverse to every manifold in A_{bdy}.

Moreover, just as in the case without boundary:

Proposition 5.3.18 (Lemma 2.1 of [Ish98]) *For any smooth manifolds with boundary N_1 and N_2, $C^\infty_{\partial,\mathrm{pr}}(N_1, N_2)$ is a Baire space.*

The particular lemma that we will need can be quickly deduced from the above.

Lemma 5.3.19 *Suppose N_1 and N_2 are smooth manifolds with boundary and $W_1, \ldots, W_j$ are submanifolds of N_2 with $\partial W_i \subseteq \partial N_2$, and $\dim(N_2) \geq 1 + \dim(N_1) + \max_i \dim(W_i)$. Then*

$$\{f \mid f(N_1) \textit{ is disjoint from each } W_i\} \subseteq C^\infty_{\partial,\mathrm{pr}}(N_1, N_2)$$

is a dense subset.

Proof. Take $A_{\mathrm{int}} = \{p^{-1}(W_1), \ldots, p^{-1}(W_j)\}$ and $A_{\mathrm{bdy}} = \{q^{-1}(W_1), \ldots, q^{-1}(W_j)\}$, where p, q are the projections $N_1 \times N_2 \twoheadrightarrow N_2$ and $(N_1 \times N_2) \smallsetminus (\partial N_1 \times \mathring{N}_2) \hookrightarrow N_1 \times N_2 \twoheadrightarrow N_2$ respectively. Let $R \subseteq C^\infty_{\partial,\mathrm{pr}}(N_1, N_2)$ be the residual subset from Proposition 5.3.17. By the dimension assumption, "is transverse to" is equivalent to "has disjoint image from" in the conclusion of Proposition 5.3.17, which therefore says precisely that each $f \in R$ has image disjoint from $\bigcup_{i=1}^j W_i$. So R is a dense (by Proposition 5.3.18) subset of $C^\infty_{\partial,\mathrm{pr}}(N_1, N_2)$, contained in $\{f \mid f(N_1) \text{ is disjoint from each } W_i\}$. $\square$

[9] Note that $C^\infty_{\partial,\mathrm{pr}}(N_1, N_2)$ is open in $C^\infty_\partial(N_1, N_2)$, but $C^\infty_\partial(N_1, N_2)$ is not (in general) open in $C^\infty(N_1, N_2)$.

5.4 A criterion for homological stability

Convention All spaces mentioned in this section will be assumed to be path-connected.

In order to separate the geometric part of the proof from the somewhat technical manipulation of spectral sequences, we will axiomatise the latter in this section. The idea for this method of proving homological stability, namely of finding 'resolutions' of the maps one wishes to prove stability for, is from [RW10].

First we need to fix some terminology.

Definition 5.4.1 For a map $f\colon X \to Y$, the number $h\mathrm{conn}(f)$ is the largest integer n such that $f_*\colon H_*(X) \to H_*(Y)$ is an isomorphism for $* \leq n-1$ and surjective for $* = n$. Equivalently it is the largest integer n such that the reduced homology of the mapping cone of f is trivial up to degree n.

Definition 5.4.2 Any augmented semi-simplicial space $X_\bullet = (\cdots X_1 \rightrightarrows X_0 \to X)$ has an associated map $\|X_\bullet\| \to X$, where $\|X_\bullet\|$ is the thick geometric realisation of the unaugmented part of $X_\bullet$ (see Definition 5.3.1). The semi-simplicial space $X_\bullet$ is called a *c-resolution* (of X) if

$$h\mathrm{conn}(\|X_\bullet\| \to X) \geq \lfloor c \rfloor.$$

Definition 5.4.3 If we have a map of augmented semi-simplicial spaces $g_\bullet\colon X_\bullet \to Y_\bullet$,

$$\begin{tikzcd}
\vdots \arrow[d] \arrow[d, shift left] \arrow[d, shift right] & \vdots \arrow[d] \arrow[d, shift left] \arrow[d, shift right] \\
X_1 \arrow[r, "g_1"] \arrow[d, shift left] \arrow[d, shift right] & Y_1 \arrow[d, shift left] \arrow[d, shift right] \\
X_0 \arrow[r, "g_0"] \arrow[d] & Y_0 \arrow[d] \\
X \arrow[r, "g"] & Y
\end{tikzcd} \tag{5.4.1}$$

with $X_\bullet$ a $(c-1)$-resolution and $Y_\bullet$ a c-resolution, then we say that the map $g\colon X \to Y$ *has c-resolution* $g_\bullet\colon X_\bullet \to Y_\bullet$. If additionally there exists for each $i \geq 0$ a map of fibration sequences of the form

$$\begin{tikzcd}
X_i' \arrow[r, "g_i'"] \arrow[d] & Y_i' \arrow[d] \\
X_i \arrow[r, "g_i"] \arrow[dr] & Y_i \arrow[d] \\
 & B_i
\end{tikzcd} \tag{5.4.2}$$

then we say that $g\colon X \to Y$ has c-resolution $g_\bullet\colon X_\bullet \to Y_\bullet$ *approximated by* $\{g_i'\colon X_i' \to Y_i'\}$. Note that we do not require that either $\{X_i'\}$ or $\{Y_i'\}$ admits the structure of a semi-simplicial space.

The strategy for proving that a map $g\colon X \to Y$ is an isomorphism on homology in a range will be to construct a map whose target is the mapping cone Cg, which is both surjective and the zero-map on homology in the required range. The first few lemmas deal with constructing a surjective map of mapping cones.

Lemma 5.4.4 *If we have a map of fibration sequences over a common base space of the form (5.4.2), then* $\mathrm{hconn}(g_i) \geq \mathrm{hconn}(g_i')$ *and the induced map* $Cg_i' \to Cg_i$ *of mapping cones is surjective on homology up to degree* $\mathrm{hconn}(g_i') + 1$.

Proof. There is a first quadrant spectral sequence with second page $E^2_{s,t} \cong H_s(B_i; \widetilde{H}_t(Cg_i'))$, rth differential of bidegree $(-r, r-1)$, and converging to $\widetilde{H}_*(Cg_i)$. The map on homology induced by the map of mapping cones $Cg_i' \to Cg_i$ can be identified with the edge homomorphism on the vertical axis, $\widetilde{H}_t(Cg_i') \cong E^2_{0,t} \twoheadrightarrow E^\infty_{0,t} \hookrightarrow \widetilde{H}_t(Cg_i)$. The existence of this spectral sequence is mentioned in Remark 2 on page 351 of [Swi75] and also as Exercise 5.6 of [McC01]; a construction of it, starting from the usual Serre spectral sequence, is given in Proposition 2.4.1 of Chapter 2.

For $t \leq \mathrm{hconn}(g_i')$ the entries $E^2_{s,t}$ on the second page are trivial, and so the spectral sequence converges to zero in total degree $* \leq \mathrm{hconn}(g_i')$, proving the first claim. For $t \leq \mathrm{hconn}(g_i') + 1$ there are no extension problems in total degree t, so $E^\infty_{0,t} \hookrightarrow \widetilde{H}_t(Cg_i)$ is an isomorphism. Hence the edge homomorphism is surjective in this range. □

Lemma 5.4.5 *If we have a map* $g\colon X \to Y$ *with c-resolution* $g_\bullet\colon X_\bullet \to Y_\bullet$*, then the induced map* $Cg_0 \to Cg$ *of mapping cones is surjective on homology up to degree*

$$\min\bigl(\{c\} \cup \{\mathrm{hconn}(g_s) + s \mid s \geq 1\}\bigr).$$

Proof. There is a spectral sequence in the range $\{s \geq -1, t \geq 0\}$ with first page $E^1_{s,t} \cong \widetilde{H}_t(Cg_s)$, rth differential of bidegree $(-r, r-1)$, and converging to $\widetilde{H}_{*+1}$ of the iterated mapping cone (total homotopy cofibre) of the square

$$\begin{tikzcd} \|X_\bullet\| \arrow[r, "\|g_\bullet\|"] \arrow[d] & \|Y_\bullet\| \arrow[d] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

Since $g_\bullet$ is a c-resolution this is zero for $* + 1 \leq c$. The map on homology induced by the map of mapping cones $Cg_0 \to Cg$ can be identified with the first differential in the leftmost column, $\widetilde{H}_t(Cg) \cong E^1_{-1,t} \leftarrow E^1_{0,t} \cong \widetilde{H}_t(Cg_0)$. See Proposition 2.4.3 of Chapter 2. The construction is fairly standard, and is given in detail in Appendix 2.B of Chapter 2 for example.

For t in the claimed range $E^\infty_{-1,t} = 0$ since $t \leq c$. Also, for $r \geq 2$, $E^1_{r-1,t+1-r} = \widetilde{H}_{t+1-r}(Cg_{r-1}) = 0$ since $t + 1 - r \leq \mathrm{hconn}(g_{r-1})$. So the term $E^1_{-1,t}$ must be killed, but cannot be killed by the differentials on page two or later, since they have domain $E^r_{r-1,t+1-r} = 0$. Hence the first differential $E^1_{-1,t} \leftarrow E^1_{0,t}$ must be surjective. □

Putting together Lemmas 5.4.4 and 5.4.5 we immediately obtain:

Corollary 5.4.6 *If $g\colon X \to Y$ has a c-resolution $g_\bullet\colon X_\bullet \to Y_\bullet$ approximated by $\{g'_i\colon X'_i \to Y'_i\}$, then the induced map $Cg'_0 \to Cg_0 \to Cg$ of mapping cones is surjective on homology up to degree*

$$\min\bigl(\{c, h\mathrm{conn}(g'_0) + 1\} \cup \{h\mathrm{conn}(g'_s) + s \mid s \geq 1\}\bigr).$$

One can iterate this by finding a further resolution and approximation of g'_0:

Corollary 5.4.7 *Suppose $g\colon X \to Y$ has c-resolution $g_\bullet\colon X_\bullet \to Y_\bullet$ approximated by $\{g'_i\colon X'_i \to Y'_i\}$. Let $h\colon Z \to W$ be the map $g'_0\colon X'_0 \to Y'_0$, and suppose it in turn has b-resolution $h\colon Z_\bullet \to W_\bullet$ approximated by $\{h'_i\colon Z'_i \to W'_i\}$. Then the induced map $Ch'_0 \to Ch_0 \to Ch = Cg'_0 \to Cg_0 \to Cg$ of mapping cones is surjective on homology up to degree*

$$\min\bigl(\{c, b, h\mathrm{conn}(g'_0) + 1, h\mathrm{conn}(h'_0) + 1\} \cup \{h\mathrm{conn}(g'_s) + s, h\mathrm{conn}(h'_s) + s \mid s \geq 1\}\bigr).$$

Now that we have a method for constructing maps $? \to Cg$ which are surjective on homology, in a range that we can determine, we need a criterion for such a map to be the *zero-map* on homology in a range.

Suppose we have a square of maps, commuting up to a chosen homotopy:

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "H"] & B \arrow[r] \arrow[d] & Ck \arrow[d] \\ X \arrow[r, "g"'] & Y \arrow[r] & Cg \end{tikzcd} \tag{5.4.3}$$

Note that the map $Ck \to Cg$ depends on the choice of homotopy H; we will call it CH to reflect this. If there is a diagonal map $B \to X$ and homotopies filling both triangles which compose to give H then this determines a nullhomotopy of CH, which is therefore the zero-map on (reduced) homology in all degrees. However, it is often the case that the above is true, except that the homotopies filling the two triangles do *not* compose to give the desired homotopy H. It is therefore useful to have a criterion which in this situation ensures that CH is at least the zero-map on homology in a range of degrees. First we show that there is a natural decomposition of this map on homology.

Lemma 5.4.8 *Suppose the diagram (5.4.3) admits a diagonal map $B \to X$ and homotopies filling the two triangles which compose to give another homotopy H' filling the same square as H. Then the map $\widetilde{H}_*(Ck) \to \widetilde{H}_*(Cg)$ induced by H factorises as*

$$\widetilde{H}_*(Ck) \to \widetilde{H}_{*-1}(A) \to \widetilde{H}_*(S^1 \times A) \to \widetilde{H}_*(Y) \to \widetilde{H}_*(Cg).$$

The first and last maps are from the long exact sequences for k and g respectively, the second map is from the Künneth decomposition for $\widetilde{H}_*(S^1 \times A)$ and the third map is induced by the map $H \cup H' \colon S^1 \times A \to Y$ given by gluing together the two homotopies $H, H' \colon [0,1] \times A \to Y$.

Proof. See Lemma 2.6.2 (the "factorisation lemma") of Chapter 2. □

This decomposition can be used to prove the following sufficient condition for CH to be zero on homology.

Lemma 5.4.9 *Suppose additionally that the space A admits a map $\ell \colon Z \to A$ such that the diagram*

$$\begin{tikzcd}
S^1 \times Z \arrow[r, "\mathrm{id} \times \ell"] \arrow[d, dashed] & S^1 \times A \arrow[d, "H \cup H'"] \\
X \arrow[r, "g"'] & Y
\end{tikzcd} \tag{5.4.4}$$

can be completed by some map $\dashrightarrow$ to a homotopy-commutative square. Then CH is the zero-map on reduced homology up to degree $h\mathrm{conn}(\ell) + 1$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\widetilde{H}_{*-1}(Z) \arrow[r, "\ell_*"] \arrow[d] & \widetilde{H}_{*-1}(A) \arrow[d] & \widetilde{H}_*(Ck) \arrow[l] \arrow[dd] \\
\widetilde{H}_*(S^1 \times Z) \arrow[r] \arrow[d] & \widetilde{H}_*(S^1 \times A) \arrow[d] & \\
\widetilde{H}_*(X) \arrow[r, "g_*"'] & \widetilde{H}_*(Y) \arrow[r] & \widetilde{H}_*(Cg)
\end{tikzcd}$$

The right-hand side is the decomposition of Lemma 5.4.8, the bottom left square is induced by (5.4.4), and the top-left square commutes by the naturality of the Künneth decomposition. The composition along the bottom row is the zero map, since the two maps are consecutive maps in the long exact sequence for g. For $* \leq h\mathrm{conn}(\ell) + 1$, the map ℓ_* in the diagram is surjective. Hence by a diagram chase the right-hand vertical map, which is CH_*, is zero. □

Putting together Corollary 5.4.7 with Lemmas 5.4.8 and 5.4.9, one can prove the following general criterion for homological stability:

Proposition 5.4.10 (Homological stability criterion)

Suppose we are given a set $\mathfrak{X}$ of maps between path-connected topological spaces, graded by a 'weight' function $w \colon \mathfrak{X} \to \mathbb{N}$. Denote by $\mathfrak{X}(n)$ the subset $w^{-1}(\{n\})$ of maps of weight n, and by $\mathfrak{X}_k^n$ the subset $w^{-1}(\{k, \ldots, n\})$ of maps with weight between k and n.

Assume that for $n \geq 2$ each element $g\colon X \to Y$ of $\mathfrak{X}(n)$ can be given the structure assumed in Corollary 5.4.7 above (a twice iterated resolution and approximation), so that the resulting square

$$\begin{tikzcd} Z_0' \arrow[r, "h_0'"] \arrow[d] \arrow[dr, phantom, "I"] & W_0' \arrow[d] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

(I denotes the identity homotopy) admits the structure assumed in Lemmas 5.4.8 and *5.4.9 above (a diagonal factorisation and a map ℓ). If this structure can be chosen so that*

- $g_0', h_0', \ell \in \mathfrak{X}_{n-2}^{n-1}$,
- $g_s', h_s' \in \mathfrak{X}_{n-2s}^{n-1}$ *for all $s \geq 1$, and*
- $b, c \geq \frac{n}{2}$,

then $h\mathrm{conn}(g) \geq \lfloor \frac{n}{2} \rfloor$ for all $g \in \mathfrak{X}(n)$.

Proof. By induction on n. For $n = 0, 1$ the claim is just that g induces a surjection on H_0 for all $g \in \mathfrak{X}(n)$. This follows from path-connectivity of the spaces under discussion, so the base case is done. For $n \geq 2$ we have the structure assumed above. Apply Corollary 5.4.7 and the inductive hypothesis to see that the map $Ch_0' \to Cg$ is surjective on homology up to degree $\frac{n}{2}$, and apply Lemmas 5.4.8 and 5.4.9 and the inductive hypothesis to see that it is the zero-map on reduced homology up to degree $\frac{n}{2}$. □

Remark 5.4.11 There are obviously exactly analogous criteria for homological stability which involve taking resolutions more (and less) than twice, but this is the version of the criterion most convenient for our application to configuration spaces of submanifolds. It is closely related to the notion of '2-triviality' in [RW10].

Remark 5.4.12 The middle $\diamond$ criterion above needs some explanation: it requires that g_s' and h_s' have weight at least $n-2s$ (and at most $n-1$), but we have not yet said what maps of weight n are for negative n. In the proof, the only way in which the property "$f \in \mathfrak{X}(k)$" is ever used is to deduce, when the inductive hypothesis allows it, that $h\mathrm{conn}(f) \geq \lfloor \frac{k}{2} \rfloor$. So we may *define* $\mathfrak{X}(n)$, for negative n, to be the class of all maps f between path-connected spaces such that $h\mathrm{conn}(f) \geq \lfloor \frac{n}{2} \rfloor$, in other words $\widetilde{H}_*(Cf) = 0$ for $* \leq \frac{n}{2}$. So $\mathfrak{X}(-1) = \mathfrak{X}(-2)$ is the class of all maps f between path-connected spaces with $\widetilde{H}_{-1}(Cf) = 0$, which is equivalent to domain$(f) \neq \varnothing$. The condition becomes vacuous for smaller n, so for $n \leq -3$, $\mathfrak{X}(n)$ is just the class of all maps between path-connected spaces.

In our application of this criterion it will turn out that all h_s' are in $\mathfrak{X}(n-1)$, and g_s' is in $\mathfrak{X}(n-s-1)$ for $0 \leq s \leq n-1$. For the remaining g_s' the above convention comes into effect: g_n' is the empty map $\varnothing \to pt$ (which is in $\mathfrak{X}(-2) \subseteq \mathfrak{X}_{-n}^{n-1}$ since $n \geq 2$) and for $s \geq n+1$, g_s' is the empty map $\varnothing \to \varnothing$ (which is in $\mathfrak{X}(-4) \subseteq \mathfrak{X}_{n-2s}^{n-1}$ since $n - 2s \leq -n - 2 \leq -4$).

Strategy The general strategy for proving homological stability by this technique for some particular $\mathbb{N}$-graded set of maps $\mathfrak{X}$ is as follows. For each map $g\colon X \to Y$ in $\mathfrak{X}$ try to construct a highly-connected resolution and approximation (in the sense of Definition 5.4.3), such that the approximating maps belong to $\mathfrak{X}$ and have smaller weight. This gives us a square of maps

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

Then repeat this step for g', and iterate until the resulting square of maps admits a factorisation (up to homotopy) into triangles.

$$\begin{tikzcd} X'^{\cdots\prime} \arrow[r] \arrow[d] & Y'^{\cdots\prime} \arrow[d] \arrow[ddddl] \\ \vdots \arrow[d] & \vdots \arrow[d] \\ X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

Then one also has to check the coherence condition assumed in Lemma 5.4.9 for this factorisation into triangles, which is a certain compatibility requirement between the homotopies filling the above triangles and the set of maps $\mathfrak{X}$.

Outline of the proof of the Main Theorem. The precise setup for the proof of the Main Theorem is as follows. Fix a closed, connected manifold P, an open or finite subgroup $G \leq \mathrm{Diff}(P)$ and a path-connected space X. Also fix an integer $d \geq 2\dim(P) + 3$. Then define

$$\mathfrak{X}(n) \coloneqq \left\{ \Sigma_n^P(M, X|G) \xrightarrow{s} \Sigma_{n+1}^P(M, X|G) \right\}_M \tag{5.4.5}$$

where M runs over all d-dimensional connected manifolds which are the interior of a manifold-with-boundary $\bar{M}$, equipped with an embedding ι of P into a coordinate neighbourhood of the boundary $\partial\bar{M}$.

For Extension 5.1.8 of the Main Theorem, the precise setup is slightly different. Let P be a point or S^k for $k \geq 1$. Fix an open or finite subgroup $G \leq \mathrm{Diff}(P)$, a path-connected space X and an integer $d \geq \dim(P) + 3$. Then $\mathfrak{X}(n)$ is defined to be (5.4.5), where M runs over all d-dimensional connected manifolds which are the interior of a manifold-with-boundary $\bar{M}$, equipped with an embedding ι of P into a coordinate neighbourhood of the boundary $\partial\bar{M}$. The embedding ι is required to be 'standard'[10] when $P = S^k$, and G must be realisable by isotopies with respect to it.

[10] Of the form $S^k \hookrightarrow \mathbb{R}^{k+1} \subseteq \mathbb{R}^{d-1} \subseteq \partial\bar{M}$.

It is immediate from Definitions 5.1.2 and 5.1.5 that the spaces $\Sigma_n^P(M,X|G)$ are always path-connected. We wish to apply the 'homological stability criterion' (Proposition 5.4.10) to this $\mathfrak{X}$. To do the inductive step, we will need to construct, for each such map s, two resolutions and approximations, and then find a factorisation-up-to-homotopy of the resulting square of maps. The first resolution and approximation is constructed in §5.5 (where we also explain why the resulting square does *not* yet factorise into triangles) and the second resolution and approximation is constructed in §5.6. Then in §5.7 we show that the square obtained after the second resolution factorises (coherently) into triangles to complete the proof.

5.5 First resolution

The aim of this section is to prove that for $\mathfrak{X} =$ (5.4.5) the first part of the hypothesis of Proposition 5.4.10 holds:

Proposition 5.5.1 (Step I of the proof of the Main Theorem) *For $n \geq 2$ each $g = s\colon \Sigma_n^P(M,X|G) \to \Sigma_{n+1}^P(M,X|G)$ in $\mathfrak{X}(n)$ admits an n-resolution $g_\bullet$ and approximation $\{g_i'\}$, in the sense of Definition 5.4.3, with $g_i' \in \mathfrak{X}(n-i-1)$ for $i \leq n-1$, $g_n' \in \mathfrak{X}(-2)$ and $g_i' \in \mathfrak{X}(-4)$ for $i \geq n+1$.*

Remark 5.5.2 More concretely, we will have shown (assuming the Main Theorem for smaller values of n by induction) that the square

$$\begin{tikzcd}
\Sigma_{n-1}^P(M',X|G) \arrow[r, "s"] \arrow[d] & \Sigma_n^P(M',X|G) \arrow[r] \arrow[d] & \textit{mapping cone} \arrow[d] \\
\Sigma_n^P(M,X|G) \arrow[r, "s"'] & \Sigma_{n+1}^P(M,X|G) \arrow[r] & \textit{mapping cone}
\end{tikzcd} \tag{5.5.1}$$

induces a homology-surjection up to degree $\frac{n}{2}$ on mapping cones. Here $M' = M \smallsetminus \psi_0(P)$ and the vertical maps add the copy $\psi_0(P)$ of P (labelled by the basepoint $x_0 \in X$) to the configuration; precisely:

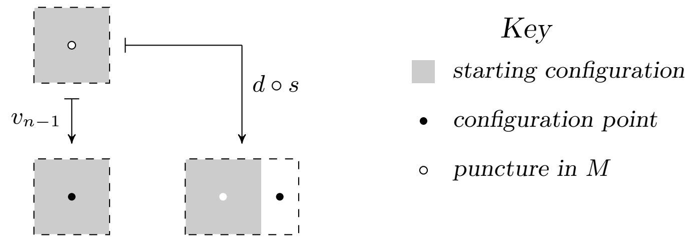
$$\{[\psi_1],\ldots,[\psi_{n-1}];x_1,\ldots,x_{n-1}\} \;\mapsto\; \{[\psi_1],\ldots,[\psi_{n-1}],[\psi_0];x_1,\ldots,x_{n-1},x_0\}.$$

5.5.1 Why this is not yet enough

Before proving 5.5.1 we briefly explain why the square (5.5.1) does *not* factorise into triangles, and so a further resolution and approximation (§5.6) is required in order to find such a factorisation (§5.7).

There is of course an obvious diagonal map $\Sigma_n^P(M',X|G) \to \Sigma_n^P(M,X|G)$ in (5.5.1), given by the inclusion $M' \hookrightarrow M$. Call this map d and the vertical maps v_{n-1} and v_n. Then the problem is that $d \circ s \not\simeq v_{n-1}$ (and $s \circ d \not\simeq v_n$). Schematically (imagine the case $M = \mathbb{R}^2$,

$P = pt$ for concreteness) the first two maps are:



(5.5.2)

A homotopy $d \circ s \rightsquigarrow v_{n-1}$ would have to move the new configuration point $\bullet$ *through* the existing configuration (grey) to where the puncture used to be—but there is no continuous choice for how to do this.

The idea of the second resolution and approximation (§5.6) is to end up with a square as in (5.5.1), but with $M'' = M \smallsetminus e_0(P \times I)$ instead of $M' = M \smallsetminus \psi_0(P)$. Now one *can* find a homotopy $d \circ s \rightsquigarrow v_{n-1}$, essentially by sliding the new copy of P along $e_0(P \times I)$ to reach $\psi_0(P)$. See §5.7 for a precise description of this homotopy.

5.5.2 Construction of the first resolution and approximation

For the following definition it is clearer to use the notation $\{([\psi_1], x_1), \ldots, ([\psi_n], x_n)\}$ (rather than $\{[\psi_1], \ldots, [\psi_n]; x_1, \ldots, x_n\}$) for an element of $\Sigma_n^P(M, X|G)$.

Definition 5.5.3 Let $\Sigma_n^P(M, X|G)^j$ be the space of configurations $\{([\psi_1], x_1), \ldots, ([\psi_n], x_n)\}$ in $\Sigma_n^P(M, X|G)$ each equipped with an injection $\mu\colon \{1, \ldots, j\} \hookrightarrow \{([\psi_1], x_1), \ldots, ([\psi_n], x_n)\}$. In other words there is a given (ordered) choice of j of the copies of P in c. Note that this is the empty space if $j > n$.

Varying j these form an augmented semi-simplicial space $\Sigma_n^P(M, X|G)^{\bullet+1}$ with face maps, induced by the injective order-preserving maps $\{1, \ldots, j\} \to \{1, \ldots, j+1\}$, which correspond to forgetting one of the marked copies of P. The stabilisation map clearly extends to a map of augmented semi-simplicial spaces:

$$\begin{tikzcd}
\Sigma_n^P(M, X|G)^{\bullet+1} \arrow[r, "s^{\bullet+1}"] \arrow[d] & \Sigma_{n+1}^P(M, X|G)^{\bullet+1} \arrow[d] \\
\Sigma_n^P(M, X|G) \arrow[r, "s"'] & \Sigma_{n+1}^P(M, X|G)
\end{tikzcd} \tag{5.5.3}$$

Lemma 5.5.4 *The map of augmented semi-simplicial spaces* (5.5.3) *is an n-resolution of s in the sense of Definition 5.4.3.*

Proof. We claim that the homotopy fibre of the map $\varepsilon\colon \|\Sigma_n^P(M, X|G)^{\bullet+1}\| \to \Sigma_n^P(M, X|G)$ is $\|\mathrm{Inj}([\bullet + 1], [n])\|$ (see Definition 5.3.2). By Lemma 5.3.3 this is $(n-2)$-connected, and so by the relative Hurewicz Theorem (Fact 5.3.4) we deduce that $h\mathrm{conn}(\varepsilon) \geq n - 1$.

So it is enough to prove that ε has the homotopy fibre claimed. Now, each of the (unique) compositions of face maps $\Sigma_n^P(M,X|G)^{i+1} \to \Sigma_n^P(M,X|G)$ is a covering space, so its homotopy fibre is its point-set fibre, which is $\mathrm{Inj}([i+1],[n])$. However, we cannot deduce directly from this that the homotopy fibre of ε is $\|\mathrm{Inj}([\bullet+1],[n])\|$, since the operations $\mathrm{hofib}(-)$ and $\|-\|$ are a homotopy limit and homotopy colimit respectively, which certainly do not commute in general. What we can say is that the *point-set* fibre of ε is $\|\mathrm{Inj}([\bullet + 1],[n])\|$, since taking point-set fibres does commute with $\|-\|$. Hence by Lemma 5.5.5 below we are done. $\square$

Lemma 5.5.5 *The map* $\varepsilon\colon \|\Sigma_n^P(M,X|G)^{\bullet+1}\| \to \Sigma_n^P(M,X|G)$ *is a fibre bundle.*

Proof. Given a point $c = \{([\psi_1],x_1),\ldots,([\psi_n],x_n)\}$ in the base $\Sigma_n^P(M,X|G)$, choose pairwise disjoint open sets V_j containing $\psi_j(P)$. Then

$$U_c \coloneqq \bigl\{\{([\psi_1'],x_1'),\ldots,([\psi_n'],x_n')\} \bigm| \psi_j'(P) \subseteq V_j\bigr\}$$

is an open neighbourhood of c. We will construct a local trivialisation for ε over U_c. Fix a bijection $\{V_1,\ldots,V_n\} \to [n]$, and denote by f_i the unique composition of face maps

$$\Sigma_n^P(M,X|G)^{i+1} \to \Sigma_n^P(M,X|G).$$

Note that any element of $f_i^{-1}(U_c)$ determines an injection $[i+1] \hookrightarrow [n]$ in a canonical way, using the chosen bijection $\{V_1,\ldots,V_n\} \to [n]$ and the fact that each $\psi_j'(P)$ is contained in a unique V_j. This defines a map

$$\mathrm{inj}\colon f_i^{-1}(U_c) \to \mathrm{Inj}([i+1],[n]).$$

We can then easily define a local trivialisation for f_i by

$$\begin{aligned} t_i\colon f_i^{-1}(U_c) &\longrightarrow U_c \times \mathrm{Inj}([i+1],[n]) \\ a &\mapsto (f_i(a),\mathrm{inj}(a)). \end{aligned}$$

This is clearly a homeomorphism and commutes with the projections (i.e. $\mathrm{pr}_1 \circ t_i = f_i$). Of course this is not yet particularly interesting as the f_i are covering space maps and therefore obviously locally trivial. We have a commutative triangle

$$\begin{tikzcd} \coprod_i(\Delta^i \times \Sigma_n^P(M,X|G)^{i+1}) \arrow[r, "f = \coprod_i(f_i \circ \mathrm{pr}_2)"] \arrow[d, "q"'] & \Sigma_n^P(M,X|G) \\ \|\Sigma_n^P(M,X|G)^{\bullet+1}\| \arrow[ur, "\|f_\bullet\| = \varepsilon"'] & \end{tikzcd} \tag{5.5.4}$$

where q is the map which quotients out by the face relations. The t_i fit together to give a local trivialisation for f:

$$\begin{tikzcd} f^{-1}(U_c) \arrow[r, "t"] \arrow[d, equal] & U_c \times \coprod_i \bigl(\Delta^i \times \mathrm{Inj}([i+1],[n])\bigr) \arrow[d, equal] \\ \coprod_i \bigl(\Delta^i \times f_i^{-1}(U_c)\bigr) \arrow[r, "\coprod_i (\mathrm{id}_{\Delta^i} \times t_i)"] & \coprod_i \bigl(\Delta^i \times U_c \times \mathrm{Inj}([i+1],[n])\bigr) \end{tikzcd} \tag{5.5.5}$$

The equivalence relation on the left of (5.5.5) given by the face relations for $\Sigma_n^P(M, X|G)^{\bullet+1}$ is taken by t to precisely the equivalence relation on the right of (5.5.5) given by the face relations for $\mathrm{Inj}([\bullet+1],[n])$. Hence t descends to a local trivialisation

$$\varepsilon^{-1}(U_c) \longrightarrow U_c \times \|\mathrm{Inj}([\bullet+1],[n])\|$$

for ε over U_c. □

Next we need to construct a map of fibrations for each level of the map of augmented semi-simplicial spaces (5.5.3).

Definition 5.5.6 For $0 \le i \le n-1$, let $\pi\colon \Sigma_n^P(M, X|G)^{i+1} \to F_{i+1}^P(M, X|G)$ be the map

$$\bigl(\{([\psi_1], x_1), \ldots, ([\psi_n], x_n)\}, \mu\bigr) \quad\mapsto\quad (\mu(1), \ldots, \mu(i+1))$$

which forgets the unmarked copies of P in a configuration.

Lemma 5.5.7 *The map $\pi\colon \Sigma_n^P(M, X|G)^{i+1} \to F_{i+1}^P(M, X|G)$ is a fibre bundle.*

Proof. This is the same as the map

$$\bar{f} \times \mathrm{id}\colon \bigl(F_n^P(M|G) \times_{\Sigma_{n-i-1}} X^{n-i-1}\bigr) \times X^{i+1} \longrightarrow F_{i+1}^P(M|G) \times X^{i+1}$$

where $\bar{f}$ is induced by the forgetful map

$$f\colon F_n^P(M|G) \longrightarrow F_{i+1}^P(M|G).$$

We will apply the second fibre bundle criterion (Lemma 5.3.7) to show that $\bar{f}$ is a fibre bundle. We take the group called G in Lemma 5.3.7 to be the group $\mathrm{Diff}_{c,0}(M)$.[11] Note that this has a well-defined action on $F_n^P(M|G)$ and $F_{i+1}^P(M|G)$ since $G \le \mathrm{Diff}(P)$ acts on embeddings of P into M by *pre*-composition and $\mathrm{Diff}_{c,0}(M)$ acts by *post*-composition, so the actions commute. The forgetful map f is clearly $\mathrm{Diff}_{c,0}(M)$-equivariant. We take the group H of Lemma 5.3.7 to be Σ_{n-i-1} (whose action on $F_n^P(M|G)$ commutes with that of $\mathrm{Diff}_{c,0}(M)$) and the Σ_{n-i-1}-space Z to be X^{n-i-1}. The action of Σ_{n-i-1} preserves the fibres of the forgetful map $f\colon F_n^P(M|G) \to F_{i+1}^P(M|G)$ since it permutes the copies of P

[11] Henceforth we only ever call it $\mathrm{Diff}_{c,0}(M)$, to avoid confusion.

which are forgotten by f. So by Lemma 5.3.7 it suffices to show that the action

$$\mathrm{Diff}_{c,0}(M) \curvearrowright F^P_{i+1}(M|G)$$

admits local sections. Let $c = ([\psi_1], \ldots, [\psi_{i+1}]) \in F^P_{i+1}(M|G)$ and choose representatives ψ_j of $[\psi_j]$. Choose pairwise disjoint open neighbourhoods V_j of $\psi_j(P)$ in M. By Proposition 5.3.8 ([Pal60, Theorem B]), the action

$$\mathrm{Diff}_{c,0}(V_j) \curvearrowright \mathrm{Emb}(\psi_j(P), V_j)$$

admits local sections, so there is an open subset $U_j \subseteq \mathrm{Emb}(P, V_j)$ containing ψ_j and a map $\gamma_j \colon U_j \to \mathrm{Diff}_{c,0}(V_j)$ satisfying

$$\gamma_j(\psi') \circ \psi_j = \psi' \quad \text{for all } \psi' \in U_j. \tag{5.5.6}$$

Note that $\mathrm{Emb}(P, V_j)$ is an open subset of $\mathrm{Emb}(P, M)$, so we can regard the U_j as open subsets of $\mathrm{Emb}(P, M)$. Recall that by definition we have

$$\mathrm{Emb}(P, M)^{i+1} \supseteq E^P_{i+1}(M) \xrightarrow{q} F^P_{i+1}(M|G),$$

where $E^P_{i+1}(M)$ is a certain G^{i+1}-invariant subspace ($G \leq \mathrm{Diff}(P)$) of $\mathrm{Emb}(P, M)^{i+1}$, and q is the quotient map onto the orbit space of the action of G^{i+1} on $E^P_{i+1}(M)$. On the left we have the open subset $U_1 \times \cdots \times U_{i+1}$, so $U_c \coloneqq q((U_1 \times \cdots \times U_{i+1}) \cap E^P_{i+1}(M))$ is an open subset of $F^P_{i+1}(M|G)$ since the quotient map for a continuous group action is always open. Another description of U_c is

$$U_c = \big\{([\psi'_1], \ldots, [\psi'_{i+1}]) \in F^P_{i+1}(M|G) \;\big|\; \psi'_j \in U_j \text{ for \textit{some} representative } \psi'_j \text{ of } [\psi'_j]\big\}.$$

So we have an open neighbourhood U_c of c and must now construct a section of $\mathrm{Diff}_{c,0}(M) \xrightarrow{-\cdot c} F^P_{i+1}(M|G)$ over U_c or a smaller neighbourhood. By Proposition 5.3.13 or 5.3.15 (since G is either open in $\mathrm{Diff}(P)$ or finite), the quotient map $p\colon \mathrm{Emb}(P, M) \to \mathrm{Emb}(P, M)/G$ has local sections, so for some open neighbourhood W_j of $[\psi_j]$ in $\mathrm{Emb}(P, M)/G$ there is a section

$$s_j \colon W_j \to \mathrm{Emb}(P, M)$$

taking $[\psi_j]$ to ψ_j. There is an inclusion

$$F^P_{i+1}(M|G) \hookrightarrow \big(\mathrm{Emb}(P, M)/G\big)^{i+1}$$

and $\prod_j s_j^{-1}(U_j)$ is an open subset of the right-hand side, so we may define U'_c to be $U_c \cap \prod_j s_j^{-1}(U_j)$ to obtain a smaller open neighbourhood of c in $F^P_{i+1}(M|G)$. We can now define a section s_c of $\mathrm{Diff}_{c,0}(M) \xrightarrow{-\cdot c} F^P_{i+1}(M|G)$ over U'_c by

$$U'_c \xrightarrow{\prod s_j} \prod_j U_j \xrightarrow{\prod \gamma_j} \prod_j \mathrm{Diff}_{c,0}(V_j) = \mathrm{Diff}_{c,0}(\coprod_j V_j) \longrightarrow \mathrm{Diff}_{c,0}(M),$$

where the last map is extension over $M \setminus \coprod_j V_j$ by the identity. This description makes it clear that the map is continuous; more concretely it can be written as

$$s_c([\psi'_1], \ldots, [\psi'_{i+1}]) := \begin{cases} \gamma_j(s_j([\psi'_j])) & \text{on } V_j, \\ \mathrm{id} & \text{on } M \setminus \coprod_j V_j. \end{cases}$$

Finally, one can easily see that this is indeed a section of $\mathrm{Diff}_{c,0}(M) \xrightarrow{-\cdot c} F^P_{i+1}(M|G)$ by noting that

$$[\gamma_j(s_j([\psi'_j])) \circ \psi_j] = [s_j([\psi'_j])] = [\psi'_j],$$

using (5.5.6) and the fact that s_j is a section of p. $\square$

Let M_{i+1} denote M with $i+1$ (unlinked, isotopic to $\iota(P)$) copies of P removed. Then the fibre over any point in the base $F^P_{i+1}(M, X|G)$ is $\Sigma^P_{n-i-1}(M_{i+1}, X|G)$. Hence for any $0 \le i \le n-1$ we have a map of fibrations over a fixed base space:

$$\begin{tikzcd} \Sigma^P_{n-i-1}(M_{i+1}, X|G) \arrow[r, "s_{i+1}"] \arrow[d, hook] & \Sigma^P_{n-i}(M_{i+1}, X|G) \arrow[d, hook] \\ \Sigma^P_n(M, X|G)^{i+1} \arrow[r, "s^{i+1}"] \arrow[dr, "\pi"'] & \Sigma^P_{n+1}(M, X|G)^{i+1} \arrow[d, "\pi"] \\ & F^P_{i+1}(M, X|G) \end{tikzcd} \tag{5.5.7}$$

5.5.3 Proof of Step I

We now have all the ingredients to complete Step I of the proof of the Main Theorem.

Proof of Proposition 5.5.1. For each $g = s\colon \Sigma^P_n(M, X|G \to \Sigma^P_{n+1}(M, X|G))$ in $\mathfrak{X}(n)$ we have constructed an n-resolution $g_\bullet = s^{\bullet+1}$. For $0 \le i \le n-1$ we have constructed an approximation $g'_i = s_{i+1}$ of the ith level s^{i+1} of the resolution. Note that M_{i+1} is a connected manifold since $\dim(M) - \dim(P) \ge 2$, so $g'_i = s_{i+1} \in \mathfrak{X}(n-i-1)$. For $i = n$, the map $g_n = s^{n+1}$ is the empty map $\varnothing \to \Sigma^P_{n+1}(M, X|G)^{n+1}$. The fibre bundle π from $\Sigma^P_{n+1}(M, X|G)^{n+1}$ to $F^P_{n+1}(M, X|G)$ is a homeomorphism, with fibre a point, and the empty map $\varnothing \to F^P_{n+1}(M, X|G)$ is a fibration (in common with all empty maps) with empty fibre. So the analogue of (5.5.7) in this case has $\varnothing \to pt$ as its map of fibres $s_{n+1} = g'_n$. This has nonempty codomain, so $g'_n \in \mathfrak{X}(-2)$ (see Remark 5.4.12). Finally, for $i \ge n+1$, the map $g_i = s^{i+1}$ is the empty map $\varnothing \to \varnothing$, so we may take g'_i to be $\varnothing \to \varnothing$ also, which is vacuously in $\mathfrak{X}(-4)$. □

Verification of Remark 5.5.2. The square of maps resulting from our resolution and approximation has vertical maps equal to the composition of the inclusion of the fibre in (5.5.7) when $i = 0$ followed by the augmentation map of (5.5.3). If we choose the basepoint of $F_1^P(M, X|G)$ to be $([\psi_0]; x_0)$ then this is exactly the description of the vertical maps in the square (5.5.1) of Remark 5.5.2. This square therefore induces a homology-surjection up to degree $\frac{n}{2}$ on mapping cones by applying Corollary 5.4.6 and assuming the Main Theorem for smaller values of n by induction. □

5.6 Second resolution

In this section the aim is to complete Step II of the proof of the Main Theorem, that for $\mathfrak{X} =$ (5.4.5) the second resolution and approximation required by Proposition 5.4.10 can be constructed. Recall that M' denotes $M \smallsetminus \psi_0(P)$.

Proposition 5.6.1 (Step II of the proof of the Main Theorem) *For $n \geq 2$, and any M, P, X, G as in (5.4.5), the map*

$$\Sigma_{n-1}^P(M', X|G) \xrightarrow{s} \Sigma_n^P(M', X|G) \tag{5.6.1}$$

admits an ∞-resolution $h_\bullet$ and approximation $\{h_i'\}$, in the sense of Definition 5.4.3, with $h_i' \in \mathfrak{X}(n-1)$ for all $i \geq 0$.

Remark 5.6.2 More concretely, we will have shown (assuming the Main Theorem for smaller values of n by induction) that the square

$$\begin{tikzcd}
\Sigma_{n-1}^P(M'', X|G) \arrow[r, "s"] \arrow[d] & \Sigma_n^P(M'', X|G) \arrow[r] \arrow[d] & \textit{mapping cone} \arrow[d] \\
\Sigma_{n-1}^P(M', X|G) \arrow[r, "s"'] & \Sigma_n^P(M', X|G) \arrow[r] & \textit{mapping cone}
\end{tikzcd} \tag{5.6.2}$$

induces a homology-surjection up to degree $\frac{n}{2}$ on mapping cones, where $M'' = M \smallsetminus e_0(P \times I)$ and the vertical maps are induced by the inclusion $M'' \hookrightarrow M'$.

5.6.1 Construction of the second resolution and approximation

See §5.2 for the notation used in the following construction (and Figure 5.2.1 for a picture).

Construction 5.6.3 Let $\mathrm{Emb}_\partial(P \times I, W_{[0,1]})$ denote the space of embeddings which take boundary to boundary, and let $\mathrm{NEmb}_{W_0,T}(P \times I, W_{[0,1]})$ be the open subset of *neat* embeddings (see Definition 5.3.10 and Fact 5.3.11) which take $P \times \{0\}$ to W_0 and $P \times \{1\}$ to

T. See Figure 5.6.1 for a schematic picture. Let $E^{P\times I}(M)$ denote the path-component of $\mathrm{NEmb}_{W_0,T}(P\times I, W_{[0,1]})$ containing e_0, which is again an open subset,[12] and let $E^{P\times I}_j(M)$ be the open subset of $(E^{P\times I}(M))^j$ of tuples of embeddings whose images are pairwise disjoint. Finally, we define

$$\Sigma_n^P(M,X|G)^{(j)} \;\subseteq\; \Sigma_n^P(M\smallsetminus\bar{T},X|G)\times E^{P\times I}_j(M)$$

to be the subspace of elements

$$(([\psi_1],\dots,[\psi_n];x_1,\dots,x_n),(e_1,\dots,e_j))$$

such that every $e_k(P\times I)$ is disjoint from every $\psi_i(P)$ (here $\bar{T}$ denotes the closure of the tubular neighbourhood T in M). The collection $\{\Sigma_n^P(M,X|G)^{(i+1)}\}_{i\geq -1}$ forms an augmented semi-simplicial space with face maps given by forgetting one of the embeddings e_k. The (-1)st space is $\Sigma_n^P(M,X|G)^{(0)} = \Sigma_n^P(M\smallsetminus\bar{T},X|G)\cong\Sigma_n^P(M',X|G)$. Clearly the stabilisation map extends to a map of augmented semi-simplicial spaces

$$\begin{tikzcd}
\Sigma_{n-1}^P(M,X|G)^{(\bullet+1)} \arrow[r, "s^{(\bullet+1)}"] \arrow[d] & \Sigma_n^P(M,X|G)^{(\bullet+1)} \arrow[d] \\
\Sigma_{n-1}^P(M\smallsetminus\bar{T},X|G) \arrow[r, "s"'] & \Sigma_n^P(M\smallsetminus\bar{T},X|G)
\end{tikzcd} \tag{5.6.3}$$

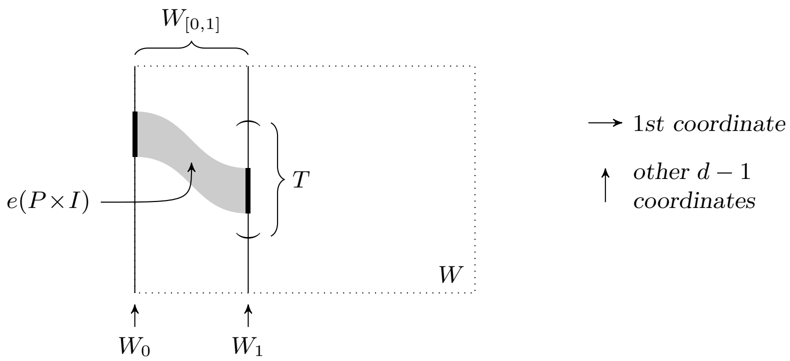


Figure 5.6.1: An element e of $E^{P\times I}(M)$.

We will use the following criteria for an augmented semi-simplicial space to be an ∞-resolution.

Proposition 5.6.4 (Theorem 6.2 of [GRW12], rewritten slightly) *For any augmented semi-simplicial space $Z_\bullet$, the following conditions imply that it is an ∞-resolution, i.e. that the map $\|Z_\bullet\|\to Z_{-1}$ is a weak equivalence:*

[12] Since $\mathrm{NEmb}_{W_0,T}(P\times I, W_{[0,1]})$ is locally path-connected. This is because it is an open subset of $C^\infty_\partial(P\times I, W_{[0,1]})$, which is in fact locally contractible as it can be given the structure of an infinite-dimensional manifold in a suitable sense (q.v. [KM97]).

(a) *The canonical map* $Z_n \to Z_0 \times_{Z_{-1}} \cdots \times_{Z_{-1}} Z_0$ *taking a simplex to its vertices is a homeomorphism onto an open subspace.*

(b) *Under this identification, a set of vertices* $(v_0, \ldots, v_n) \in Z_0 \times_{Z_{-1}} \cdots \times_{Z_{-1}} Z_0$ *is in* Z_n *whenever each pair* $(v_i, v_j) \in Z_0 \times_{Z_{-1}} X_0$ *is in* Z_1.

(c) *The map* $\varepsilon \colon Z_0 \to Z_{-1}$ *is surjective, and for every* $v \in Z_0$ *there is a section* $U \to Z_0$ *on a neighbourhood* U *of* $\varepsilon(v)$ *taking* $\varepsilon(v)$ *to* v.

(d) *For any non-empty finite set* $\{v_1, \ldots, v_j\}$ *in a fibre of* ε *there is another* v *in the same fibre such that* $(v, v_i) \in Z_1$ *for all* i.

Lemma 5.6.5 *The augmented semi-simplicial space* $\Sigma_n^P(M, X|G)^{(\bullet+1)}$ *is an* ∞*-resolution.*

We first prove this in the setup of the Main Theorem, where we assume that $\dim(M) \geq 2\dim(P) + 3$. In §5.6.3 we explain how to modify the proof for Extension 5.1.8 of the Main Theorem, where P is a point or a sphere and we only assume that $\dim(M) \geq \dim(P) + 3$.

Proof of Lemma 5.6.5 in the setup of the Main Theorem. We prove this using the above criteria, with $Z_\bullet = \Sigma_n^P(M, X|G)^{(\bullet+1)}$. In this case, by definition, Z_n is the subset of $Z_0 \times_{Z_{-1}} \cdots \times_{Z_{-1}} Z_0$ consisting of elements which satisfy the condition that the images of the embeddings e_k are pairwise disjoint. This condition is *open* and *pairwise*, so (a) and (b) are satisfied. To see that ε is surjective, note that since the $\psi_i(P)$ are embedded in such a way that they can be enclosed in a coordinate neighbourhood in M, they cannot link with T and obstruct the existence of an embedding of $P \times I$ between T and W_0 which is disjoint from them. For the second half of (c), suppose we have an element $v = (\{[\psi_1], \ldots, [\psi_n]; x_1, \ldots, x_n\}, e_1) \in Z_0$. Choose pairwise disjoint open balls B_i around $\psi_i(P)$ which are all disjoint from $e_1(P \times I)$. Then we can take the open neighbourhood U of $\varepsilon(v)$ to be all $\{[\psi'_1], \ldots, [\psi'_n]; x'_1, \ldots, x'_n\}$ such that $\psi'_i(P) \subseteq B_i$, over which ε has an obvious section:

$$\{[\psi'_1], \ldots, [\psi'_n]; x'_1, \ldots, x'_n\} \;\mapsto\; (\{[\psi'_1], \ldots, [\psi'_n]; x'_1, \ldots, x'_n\}, e_1).$$

For criterion (d) we have a configuration $\{[\psi_1], \ldots, [\psi_n]; x_1, \ldots, x_n\}$ in $\Sigma_n^P(M \smallsetminus \overline{T}, X|G)$ together with embeddings $e_1, \ldots, e_j$ in $E^{P\times I}(M)$ which are disjoint from each $\psi_i(P)$ but not necessarily from each other, and we need to find a new embedding $e \in E^{P\times I}(M)$ which is disjoint from each $\psi_i(P)$ *and* from each $e_k(P \times I)$. This requires slightly more work:

Recall that $E^{P\times I}(M)$ is an open subset of $\mathrm{Emb}_\partial(P \times I, W_{[0,1]})$, which is itself open in $C^\infty_\partial(P \times I, W_{[0,1]})$ since being an embedding is an open property. Hence we have an open subset

$$E^{P\times I}(M) \cap C^\infty_\partial\Big(P \times I, W_{[0,1]} \smallsetminus \textstyle\bigcup_{i=1}^n \psi_i(P)\Big) \quad \subseteq \quad C^\infty_\partial\Big(P \times I, W_{[0,1]} \smallsetminus \textstyle\bigcup_{i=1}^n \psi_i(P)\Big). \tag{5.6.4}$$

The subset in (5.6.4) is also non-empty, because the 'standard' embedding e_0 of $P \times I$ is in $E^{P\times I}(M)$ and, since the $\psi_i(P)$ are contained in coordinate neighbourhoods of M, this embedding can easily be modified to avoid them.

We now apply Lemma 5.3.19 with $N_1 = P \times I$, $N_2 = W_{[0,1]} \smallsetminus \bigcup_{i=1}^n \psi_i(P)$ and $W_k = e_k(P \times I)$ for $1 \leq k \leq j$. The dimension assumption of the lemma is satisfied since we are assuming that $\dim(M) \geq 2\dim(P) + 3$. It tells us that we have a dense subset

$$\{f \mid f(P \times I) \text{ is disjoint from each } e_k(P \times I)\} \quad \subseteq \quad C^\infty_\partial\Big(P \times I, W_{[0,1]} \smallsetminus \textstyle\bigcup_{i=1}^n \psi_i(P)\Big). \tag{5.6.5}$$

The intersection of the subsets in (5.6.4) and (5.6.5) is non-empty, and we may take any element of the intersection to be the required $e \in E^{P\times I}(M)$. $\square$

Now we will construct a map of fibrations for each level of the map of augmented semi-simplicial spaces (5.6.3).

Definition 5.6.6 Let $\pi \colon \Sigma_n^P(M, X|G)^{(i+1)} \to E_{i+1}^{P\times I}(M)$ be the map which forgets the configuration of Ps and just remembers the embedded $P \times I$s. Formally, it is

$$(([\psi_1], \dots, [\psi_n]; x_1, \dots, x_n), (e_1, \dots, e_{i+1})) \;\mapsto\; (e_1, \dots, e_{i+1}).$$

Remark 5.6.7 The point-set fibre of this map is $\Sigma_n^P(M_{(i+1)}, X|G)$, where $M_{(i+1)}$ denotes M with $\bar{T}$ and $i+1$ (pairwise disjoint, embedded by an element of $E^{P\times I}(M)$) copies of $P \times I$ removed.

Lemma 5.6.8 *The map $\pi \colon \Sigma_n^P(M, X|G)^{(i+1)} \to E_{i+1}^{P\times I}(M)$ is a fibre bundle.*

Proof. First we define a continuous map $\Upsilon \colon \mathrm{Diff}_{c,0}(W_{[0,1]}) \to \mathrm{Diff}_{c,0}(\bar{M})$ which extends a diffeomorphism on $W_{[0,1]}$ to all of $\bar{M}$. Denote the projection $W_{[1,2)} \to W_1$ which sets the first coordinate to 1 by p and the projection $W_{[1,2)} \to [1,2)$ onto the first coordinate by q. Also choose a smooth function $\lambda \colon [1,2) \to [0,1]$ which is 1 on $[1, \frac{5}{4}]$ and 0 on $[\frac{7}{4}, 2)$. We can then define, for $\phi \in \mathrm{Diff}_{c,0}(W_{[0,1]})$,

$$\begin{aligned} \Upsilon(\phi) \colon \bar{M} &\to \bar{M} \\ y &\mapsto \begin{cases} \phi(y) & \text{if } y \in W_{[0,1]} \\ y + \lambda(q(y)).\big(\phi(p(y)) - p(y)\big) & \text{if } y \in W_{[1,2)} \\ y & \text{if } y \in \bar{M} \smallsetminus W. \end{cases} \end{aligned}$$

Let $\mathrm{Diff}_{c,0}(W_{[0,1]}; W_0, T)$ be the open (in the strong topology) subgroup of $\mathrm{Diff}_{c,0}(W_{[0,1]})$ of diffeomorphisms which preserve each of W_0, $W_1 \smallsetminus \bar{T}$ and T setwise.[13] There is an action

[13] The condition of sending W_0 to itself is open in the weak topology, since it is equivalent to requiring that a particular point of W_0 is sent into W_0. The condition of sending T to itself is not open in the weak topology, but it *is* open in the strong topology. In the weak topology, the condition $K \rightsquigarrow U$ is an open condition whenever U is open and K is compact, whereas in the strong topology for this to be an open condition it is sufficient that K admits a locally finite open covering $\mathcal{V}$ together with a compact subset $K_V \subseteq V$ for each $V \in \mathcal{V}$ such that $\{K_V\}_{V\in\mathcal{V}}$ also covers K. This condition on K clearly holds for any tubular neighbourhood of a compact submanifold. Similarly, it is not hard to construct a locally finite covering of $W_1 \smallsetminus \bar{T}$ by compact sets, so the condition of sending this to itself is also open in the strong topology.

of $\mathrm{Diff}_{c,0}(W_{[0,1]}; W_0, T)$ on $\Sigma_n^P(M, X|G)^{(i+1)}$ (via Υ) and on $E_{i+1}^{P\times I}(M)$, and the map π is equivariant with respect to these actions. Hence by Lemma 5.3.6 we just need to show that the action

$$\mathrm{Diff}_{c,0}(W_{[0,1]}; W_0, T) \curvearrowright E_{i+1}^{P\times I}(M)$$

admits local sections. Let $c = (e_1, \dots, e_{i+1}) \in E_{i+1}^{P\times I}(M)$ and choose pairwise disjoint open $V_j \subseteq W_{[0,1]}$ containing $e_j(P \times I)$. By Proposition 5.3.12 the action

$$\mathrm{Diff}_{c,0}(V_j) \curvearrowright \mathrm{NEmb}(P \times I, V_j)$$

admits local sections, so there is an open neighbourhood U_j of $e_j \in \mathrm{NEmb}(P \times I, V_j)$ and continuous map $\gamma_j \colon U_j \to \mathrm{Diff}_{c,0}(V_j)$ such that

$$\gamma_j(e') \circ e_j = e' \quad \text{for all } e' \in U_j. \tag{5.6.6}$$

Moreover from the construction of the local section γ_j in the proof of Proposition 5.3.12 we see that $\gamma_j(e_j) = \mathrm{id}$. Now $\mathrm{NEmb}(P\times I, V_j)$ is open in $\mathrm{NEmb}(P\times I, W_{[0,1]})$, so we can consider U_j to be an open subset of $\mathrm{NEmb}(P \times I, W_{[0,1]})$. There is a continuous map $\mathrm{Diff}_{c,0}(V_j) \to \mathrm{Diff}_{c,0}(W_{[0,1]})$ extending a diffeomorphism by the identity; let $\mathrm{Diff}_{c,0}(V_j; W_0, T)$ be the preimage of $\mathrm{Diff}_{c,0}(W_{[0,1]}; W_0, T)$. There is an inclusion $E_{i+1}^{P\times I}(M) \hookrightarrow \mathrm{NEmb}(P\times I, W_{[0,1]})^{i+1}$, and we define

$$U_c \coloneqq E_{i+1}^{P\times I}(M) \cap \prod_j \gamma_j^{-1}\big(\mathrm{Diff}_{c,0}(V_j; W_0, T)\big),$$

which is an open neighbourhood of c in $E_{i+1}^{P\times I}(M)$. We define a local section over U_c by

$$U_c \xrightarrow{\prod \gamma_j} \prod_j \mathrm{Diff}_{c,0}(V_j; W_0, T) \longrightarrow \mathrm{Diff}_{c,0}(W_{[0,1]}; W_0, T),$$

where the second map extends by the identity. Note that by (5.6.6) this is indeed a local section of the map $\mathrm{Diff}_{c,0}(W_{[0,1]}; W_0, T) \xrightarrow{-\cdot c} E_{i+1}^{P\times I}(M)$, as required. □

5.6.2 Proof of Step II

We can now put this all together to complete Step II of the proof of the Main Theorem.

Proof of Proposition 5.6.1. First note that the map (5.6.1) can be identified with the map

$$\Sigma_{n-1}^P(M \smallsetminus \bar{T}, X|G) \longrightarrow \Sigma_n^P(M \smallsetminus \bar{T}, X|G)$$

since $M \smallsetminus \bar{T} \cong M' = M \smallsetminus \psi_0(P)$. By Lemma 5.6.5 we have an ∞-resolution $h_\bullet = s^{(\bullet+1)}$ of this map. By Remark 5.6.7 and Lemma 5.6.8 we have a fibration sequence

$$\Sigma_n^P(M_{(i+1)}, X|G) \hookrightarrow \Sigma_n^P(M, X|G)^{(i+1)} \xrightarrow{\pi} E_{i+1}^{P\times I}(M)$$

for all $i \geq 0$. Note that the map $s^{(i+1)}$ satisfies $\pi \circ s^{(i+1)} = \pi$, since adding a copy of P and then forgetting them all is the same as just forgetting them, so we have a map of fibration sequences

$$\begin{tikzcd}
\Sigma_{n-1}^P(M_{(i+1)}, X|G) \arrow[r, "s_{(i+1)}"] \arrow[d, hook] & \Sigma_n^P(M_{(i+1)}, X|G) \arrow[d, hook] \\
\Sigma_{n-1}^P(M, X|G)^{(i+1)} \arrow[r, "s^{(i+1)}"] \arrow[dr, "\pi"'] & \Sigma_n^P(M, X|G)^{(i+1)} \arrow[d, "\pi"] \\
& E_{i+1}^{P\times I}(M)
\end{tikzcd} \tag{5.6.7}$$

This gives an approximation $\{h_i'\} = \{s_{(i+1)}\}$ of the ∞-resolution $h_\bullet = s^{(\bullet+1)}$ in the sense of Definition 5.4.3. Finally, note that $M_{(i+1)}$ is still a *path-connected* manifold, since we have cut out submanifolds of codimension $\dim(M) - \dim(P) - 1 \geq 2$. Hence $h_i' = s_{(i+1)} \in \mathfrak{X}(n-1)$ for all $i \geq 0$, as required. □

Verification of Remark 5.6.2. The square of maps resulting from the second resolution and approximation has vertical maps equal to the composition of the inclusion of the fibre in (5.6.7) when $i = 0$ followed by the augmentation map of (5.6.3). If we choose the basepoint of $E_1^{P\times I}(M)$ to be (e_0) and take a suitable identification $M \smallsetminus \overline{T} \cong M' = M \smallsetminus \psi_0(P)$, then this is precisely the square of maps (5.6.2). Therefore (5.6.2) induces a homology-surjection up to degree $\frac{n}{2}$ on mapping cones by applying Corollary 5.4.6 and assuming the Main Theorem for $n-1$ by induction. □

5.6.3 Modification for points and spheres

When we are in the setup of Extension 5.1.8 of the Main Theorem, we only assume that $\dim(M) \geq \dim(P) + 3$, so we cannot use a transversality argument (via Lemma 5.3.19) to verify criterion (d) of Proposition 5.6.4 to show that $\Sigma_n^P(M, X|G)^{(\bullet+1)}$ is an ∞-resolution. Instead we can check criterion (d) concretely.

Proof of Lemma 5.6.5 in the setup of Extension 5.1.8 of the Main Theorem. The first three criteria can be checked as before, so we need only consider criterion (d). For this we have a configuration $\{[\psi_1], \ldots, [\psi_n]; x_1, \ldots, x_n\}$ in $\Sigma_n^P(M \smallsetminus \overline{T}, X|G)$ together with embeddings $e_1, \ldots, e_j$ in $E^{P\times I}(M)$ which are disjoint from each $\psi_i(P)$ but not necessarily from each other, and we need to find a new embedding $e \in E^{P\times I}(M)$ which is disjoint from each $\psi_i(P)$ and from each $e_l(P \times I)$.

We will assume that $P = S^k$ with $k \geq 1$ since the case when P is a point is the same idea but easier. We are working entirely in $W_{[0,1]} \cong \mathbb{R}^{d-1} \times [0, 1]$, and since ι was assumed to be a 'standard' embedding of S^k, we may assume that $\psi_0\colon S^k \hookrightarrow W_1$ is simply

$$S^k \hookrightarrow \mathbb{R}^{k+1} \subseteq \mathbb{R}^{d-1} \times \{1\},$$

where the first map is the inclusion of the standard unit sphere in $\mathbb{R}^{k+1}$. Hence we can take the tubular neighbourhood T of $\psi_0(S^k)$ in W_1 to be a standard 'cylindrical' tubular neighbourhood:

$$T \coloneqq \big\{(x_1,\ldots,x_{d-1},1) \;\big|\; (x_1^2+\cdots+x_{k+1}^2)^{\frac{1}{2}} \in (\tfrac{1}{2},2) \text{ and } x_{k+2},\ldots,x_{d-1} \in (-1,1)\big\} \subseteq W_1.$$

See Figure 5.6.2 for a picture of T when $P = S^1$ and $d = \dim(M) = 4$. By compactness of $S^k \times [0,1]$, there exists a function $r\colon [0,1] \to [1,\infty)$, which we may take to be smooth, such that $r(1) < 2$ and for all $t \in [0,1]$ and each $1 \leq l \leq j$,

$$e_l(S^k \times [0,1]) \cap W_t \;\subseteq\; B_{r(t)}(t),$$

where for $r > 1$ and $t \in [0,1]$,

$$B_r(t) \coloneqq \big\{(x_1,\ldots x_{d-1},t) \;\big|\; (x_1^2+\cdots+x_{k+1}^2)^{\frac{1}{2}} < r\big\} \subseteq W_t.$$

Hence we may define $e\colon S^k \times [0,1] \longrightarrow W_{[0,1]} = \mathbb{R}^{d-1} \times [0,1]$ by:

$$(x_1,\ldots,x_{k+1},t) \mapsto (r(t)x_1,\ldots,r(t)x_{k+1},0,\ldots,0,t).$$

Note that $e \in E^{P\times I}(M)$ and its image is disjoint from each $e_l(S^k \times I)$ by construction. It can now easily be modified by an isotopy to also be disjoint from each $\psi_i(S^k)$, since these are pairwise unlinked (and so in particular each contained in a coordinate neighbourhood of M). $\square$

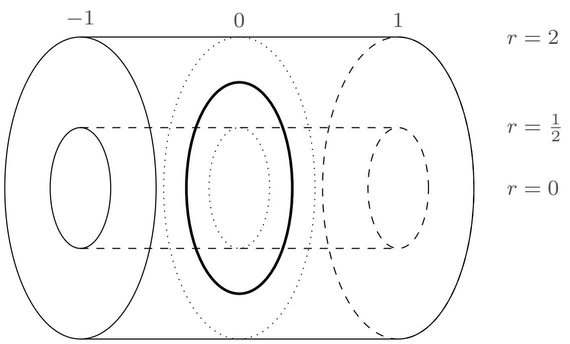


Figure 5.6.2: A cylindrical tubular neighbourhood T for $\psi_0(S^1)$ when $\dim(M) = 4$. The thick circle is $\psi_0(S^1)$.

5.6.4 Why a more naive resolution doesn't work

A simpler and more naive resolution of $\Sigma_n^P(M \smallsetminus \overline{T}, X|G)$ could be obtained by defining it as in Construction 5.6.3, but removing either of the following two conditions:

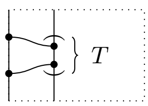
(i) The embeddings of $P \times I$ must have pairwise disjoint images;

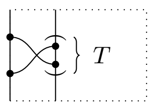
(ii) The embeddings of $P \times I$ must take $P \times \{1\}$ into the tubular neighbourhood T.

Removing either of these conditions still gives a perfectly good ∞-resolution of $\Sigma_n^P(M \smallsetminus \overline{T}, X|G)$; one can verify the criteria (a)–(d) of 5.6.4 in much the same way, and in fact criterion (d) becomes much easier if condition (i) is removed, since there is no need for a transversality argument and therefore no need for any dimension assumption on M and P. The same is also true for removing condition (ii): in this case finding an embedding $e \in E^{P\times I}(M)$ which is disjoint from a given collection $e_1, \ldots, e_j \in E^{P\times I}(M)$ is easily achieved by embedding it "far away" from them. So again there would be no need for any dimension assumption on M and P if condition (ii) were removed.

The problem is that the map $\pi\colon \Sigma_n^P(M, X|G)^{(i+1)} \to E_{i+1}^{P\times I}(M)$ is no longer a fibre bundle (or a fibration or even a homology-fibration) if either of these conditions is removed.

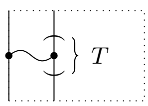
Counterexamples. This can be easily seen in the case $M = \mathbb{R}^2, P = pt$. Suppose we remove condition (i). Then the following are two points of the base space, which is now just $(E^{P\times I}(M))^j$, which are in the same path-component but nevertheless have completely different fibres:

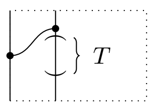




In each case the fibre is the space of unordered configurations of n points in M with $\overline{T}$ and the two indicated arcs removed. But on the left this is the disjoint union of 2 copies of $\mathbb{R}^2$ and on the right it is the disjoint union of 3 copies of $\mathbb{R}^2$.

If instead condition (ii) is removed we can construct a similar simple counterexample:





This time the fibre on the left is $\Sigma_n^{pt}(\mathbb{R}^2)$ and the fibre on the right is $\Sigma_n^{pt}(S^1 \times \mathbb{R}^2)$.

Where conditions (i) and (ii) are used in showing that π is a fibre bundle. It may be illuminating to point out exactly what goes wrong in the proof of Lemma 5.6.8 when either (i) or (ii) is removed. The method of proof involves showing that the action of a certain group of diffeomorphisms $\mathrm{Diff}_{c,0}(W_{[0,1]}; W_0, T)$ on the base space $E_{i+1}^{P\times I}(M)$ has local sections, or is 'continuously locally transitive'. In particular it must be locally transitive, meaning that each point has a neighbourhood which is contained in a single orbit of the action.

If condition (i) is removed, consider a configuration $(e_1, \ldots, e_{i+1})$ in $E^{P\times I}_{i+1}(M)$ in which two of the $e_k(P \times I)$ intersect, but 'only just', so that any open neighbourhood of it contains a configuration $(e'_1, \ldots, e'_{i+1})$ in which the $e_k(P \times I)$ are pairwise disjoint. No diffeomorphism of $W_{[0,1]}$ can change whether two of the copies of $P \times I$ intersect or not, so the action fails to be locally transitive at the point $(e_1, \ldots, e_{i+1})$.

If condition (ii) is removed, consider a configuration $(e_1, \ldots, e_{i+1})$ such that every $e_k(P \times \{1\})$ is contained in $\bar{T}$, and at least one intersects the boundary $\partial\bar{T}$. Then any neighbourhood of it contains a configuration $(e'_1, \ldots, e'_{i+1})$ in which every $e_k(P \times \{1\})$ is contained in T. But the diffeomorphisms in $\mathrm{Diff}_{c,0}(W_{[0,1]}; W_0, T)$ are required to send $W_1 \smallsetminus \bar{T}$ to itself, and so by continuity cannot take any point of $\partial\bar{T}$ into T. Hence the action fails to be locally transitive at the point $(e_1, \ldots, e_{i+1})$.

One could try to instead consider the action of the larger group $\mathrm{Diff}_{c,0}(W_{[0,1]})$, whose diffeomorphisms are not required to take $W_1 \smallsetminus \bar{T}$ to itself. But this group does not act on the domain $\Sigma^P_n(M, X|G)^{(i+1)}$ of π, which it would have to for the method of proof to work (using Lemma 5.3.6). This is because an element of $\Sigma^P_n(M, X|G)^{(i+1)}$ in particular consists of a configuration of copies of P contained in $M \smallsetminus \bar{T}$, and a general diffeomorphism in $\mathrm{Diff}_{c,0}(W_{[0,1]})$ (extended to $\bar{M}$ by Υ) might not send $M \smallsetminus \bar{T}$ to itself.

5.6.5 A red herring

This section is not relevant to the proof of the Main Theorem; it is just a brief aside on a tempting weakening of the hypotheses of the Main Theorem which unfortunately does not work. Rather than assume that $\dim(M) \geq 2\dim(P) + 3$ in order to be able to use a transversality argument to ensure that certain disjoint embeddings can be found, one could instead try assuming that P admits a non-vanishing normal vector field (i.e. the normal bundle $\nu(\iota) \to P$ of the embedding $\iota \colon P \hookrightarrow \mathbb{R}^{d-1} \subseteq \partial\bar{M}$ admits a non-vanishing section).[14]

5.6.5.1 Conditions ensuring existence of non-vanishing normal vector fields

This would somewhat improve the Main Theorem (and Extension 5.1.8), since there are several general conditions which imply the existence of a non-vanishing normal vector field. One obvious one is if ι (is isotopic to an embedding which) factors through the inclusion $\mathbb{R}^{d-2} \hookrightarrow \mathbb{R}^{d-1}$. Some more subtle ones are as follows.

Proposition 5.6.9 *Let P be closed, connected and k-dimensional. Then each of the following conditions is sufficient for any embedding $P \hookrightarrow \mathbb{R}^{d-1}$ to have a non-vanishing section of its normal bundle*:

[14] One now also has to assume that the subgroup $G \leq \mathrm{Diff}(P)$ is realisable by isotopies, since this is no longer guaranteed by a dimension assumption. The weaker dimension assumption $\dim(M) \geq \dim(P) + 3$ is still necessary to ensure that $M_{(i+1)}$ (see Remark 5.6.7) is connected.

- *P is a homology sphere,*
- *$d-1 \geq 2k+1$,*
- *$d-1 = 2k$ and P is orientable,*
- *$d-1 = 2k-1$, P is orientable, $k \geq 5$ and $\bar{w}_2(P) \cup \bar{w}_{k-2}(P) = 0$,*

where $\bar{w}_i \in H^i(P;\mathbb{Z}/2)$ is the ith dual Stiefel-Whitney class of P (which is the same as the ith Stiefel-Whitney class of $\nu(\iota)$ by the Whitney Duality Theorem). In particular this last condition is satisfied when

- *k is* not *of the form $2^s(2^t+1)$ for $s,t \geq 0$, or*
- *k* is *of the form $2(2^t+1)$ for $t \geq 1$ or $2^s.3$ for $s \geq 0$.*

Proof. The first condition is Theorem IV of [Mas61] (see also [Mas59] and [Ker59]). For the second condition we may assume that $k \geq 2$ since the case $P = S^1$ has been taken care of by the first. There exists an embedding $\iota_0 \colon P \hookrightarrow \mathbb{R}^{d-2} \subseteq \mathbb{R}^{d-1}$ by the Whitney Embedding Theorem [Whi36], and any two embeddings $P \hookrightarrow \mathbb{R}^{d-1}$ are isotopic (see for example [Sko08, Theorem 2.5]), so ι is isotopic to ι_0. There is an obvious non-vanishing section of the normal bundle of ι_0. But ι and ι_0 are isotopic (in particular regularly homotopic) so their normal bundles are the same. The third condition is due to [Whi41] (see also [Hir76, Theorem 5.2.11]), and the fourth is Theorem II of [Mas61]. The last fact is observed by Massey in the same paper, and follows from Corollary 2 of [Mas60]. □

Remark 5.6.10 In fact by Theorem II of [Mas61], when $d = 2k$, P is orientable and $k \geq 5$, the characteristic class condition in Proposition 5.6.9 is also *necessary* for the embedding to admit a non-vanishing normal vector field. This is part of a more general necessary condition for the existence of non-vanishing normal vector fields, which is Theorem I of [Mas61].

Example 5.6.11 As an illustration that orientability of P is necessary for the third condition of Proposition 5.6.9, the embedding $\mathbb{RP}^2 \hookrightarrow \mathbb{R}^4$ does not admit a non-vanishing normal vector field. Some further explicit examples and non-examples of embeddings which admit non-vanishing normal vector fields are as follows (all taken from [Mas61]). The existence of these embeddings is due to [Jam59].

With non-vanishing normal vector field:	Without non-vanishing normal vector field:
$\mathbb{RP}^n \hookrightarrow \mathbb{R}^{2n-1}$ ($n \geq 3$ odd)	$\mathbb{CP}^n \hookrightarrow \mathbb{R}^{4n-1}$ ($n = 2^s, n \geq 4$)
$\mathbb{CP}^n \hookrightarrow \mathbb{R}^{4n-1}$ ($n \neq 2^s$)	$\mathbb{HP}^2 \hookrightarrow \mathbb{R}^{13}$, $\mathbb{HP}^4 \hookrightarrow \mathbb{R}^{29}$, $\mathbb{OP}^2 \hookrightarrow \mathbb{R}^{25}$

5.6.5.2 Why this isn't enough

Assuming the non-vanishing normal vector field hypothesis, one could try to argue as follows to prove that criterion (d) of Proposition 5.6.4 holds for $\Sigma_n^P(M,X|G)^{(\bullet+1)}$.

Recall from the proof of Lemma 5.6.5 that essentially what is needed to check this criterion is: given a collection of embeddings $e_1,\ldots,e_j$ in $E^{P\times I}(M)$, we need to find another

$e \in E^{P\times I}(M)$ which is disjoint from all of them. Since e_1 is assumed isotopic to e, and e was constructed in a very simple way from ι, we know that e_1 admits a non-vanishing normal vector field. We can use this to 'push it off itself' (choosing a tubular neighbourhood T_1 of e_1) to obtain a new embedding e which is therefore disjoint from e_1. The problem with this approach is very simple: the original collection of embeddings $e_1, \ldots, e_j$ was *not* assumed to already be pairwise disjoint, so we *cannot* assume that T_1, and therefore e, is disjoint from any of the $e_2, \ldots, e_j$.

5.6.6 Difficulties with codimension 2

This is another 'red herring' section, pointing out why a naive 'fix' of the proof—to extend the result for points and spheres to the codimension-2 case—doesn't work.

The problem with codimension 2 is that the manifold $M_{(i+1)}$ is in general disconnected, since it is obtained from M by cutting out some codimension-1 submanifolds, so the map $s_{(i+1)}$ of §5.6.7 is not necessarily in $\mathfrak{X}(n-1)$. An idea to possibly solve this problem is to pass to a sub-semi-simplicial space of $\Sigma_n^P(M, X|G)^{(\bullet+1)}$ in order to end up with a connected manifold.

Definition 5.6.12 Given $(([\psi_1], \ldots, [\psi_n]; x_1, \ldots, x_n), (e_1, \ldots, e_j)) \in \Sigma_n^P(M, X|G)^{(j)}$, let M_{large} be the ("large") path-component of $M_{(j)} = M \smallsetminus \big(\overline{T} \cup \bigcup_{i=1}^{j} e_i(P \times I)\big)$ which contains $M \smallsetminus W$. Define $\Sigma_n^P(M, X|G)^{[j]}$ to be the subspace of configurations satisfying the additional condition that $\psi_i(P) \subseteq M_{large}$ for all i. (Note that this condition is vacuous if $\dim(M) - \dim(P) \geq 3$, since then $M_{(j)}$ is connected.) The face maps of $\Sigma_n^P(M, X|G)^{(\bullet+1)}$ restrict to the subspaces $\Sigma_n^P(M, X|G)^{[i+1]}$, so they form a augmented semi-simplicial space $\Sigma_n^P(M, X|G)^{[\bullet+1]}$. As before, the stabilisation map extends to a map of augmented semi-simplicial spaces as in (5.6.3).

The advantage of this semi-simplicial space is that the top horizontal map of (5.6.7) becomes the stabilisation map $\Sigma_{n-1}^P(M_{large}, X|G) \to \Sigma_n^P(M_{large}, X|G)$, and M_{large} is of course connected by definition.

Most of the arguments of §§5.6.1 and 5.6.2 go through as before. For example the action of $\mathrm{Diff}_{c,0}(W_{[0,1]}; W_0, T)$ on $\Sigma_n^P(M, X|G)^{(i+1)}$ preserves the subspace $\Sigma_n^P(M, X|G)^{[i+1]}$ setwise, since the diffeomorphisms are all diffeotopic to the identity, so the proof that π is a fibre bundle works as before.

The place where the proof breaks down is again in checking criterion (d) of Proposition 5.6.4 for $\Sigma_n^P(M, X|G)^{[\bullet+1]}$. For example take P to be a point and $\dim(M) = 2$. Then given any collection of arcs and points as in Figure 5.6.3 (arcs disjoint from the points but not necessarily from each other), we need to find a new arc which is disjoint from all of them, *and* such that no point is "cut off" by the new arc together with any one of the old arcs. In other words, if we cut out the new arc, any old arc and $\overline{T}$, then every point should be in the "large" component of M. This final condition is clearly not possible, however: If the collection of arcs is e_1, e_2 in Figure 5.6.3 and ψ_1 is one of the points, then wherever the new arc (dashed) is put, together with either e_1 or e_2 it will cut off ψ_1.

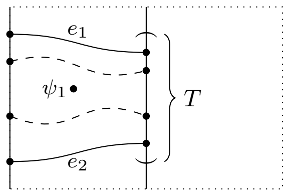


Figure 5.6.3: The point ψ_1 is doomed to be cut off by the new (dashed) arc and one of the old (solid) arcs.

5.7 Final step of the proof

In this section we complete Step III of the proof of the Main Theorem by completing the verification of the hypotheses of the homological stability criterion (Proposition 5.4.10) for $\mathfrak{X} = $ (5.4.5). In §§5.5,5.6 above we constructed a twice-iterated resolution and approximation for each map $\Sigma_n^P(M, X|G) \to \Sigma_{n+1}^P(M, X|G)$ in $\mathfrak{X}(n)$. The resulting square of maps is the vertical composition of the squares (5.5.1) and (5.6.2), which is

$$\begin{tikzcd}
\Sigma_{n-1}^P(M'', X|G) \arrow[r, "s"] \arrow[d, "v"'] & \Sigma_n^P(M'', X|G) \arrow[r] \arrow[d, "v"] & \textit{mapping cone} \arrow[d] \\
\Sigma_n^P(M, X|G) \arrow[r, "s"'] & \Sigma_{n+1}^P(M, X|G) \arrow[r] & \textit{mapping cone}
\end{tikzcd} \tag{5.7.1}$$

where the vertical maps v are given by adding $[\psi_0]$ (labelled by x_0) to the configuration. To finish verifying the hypotheses of Proposition 5.4.10 for $\mathfrak{X} = $ (5.4.5), and therefore the proof of the Main Theorem, we will prove that:

Proposition 5.7.1 (Step III of the proof of the Main Theorem) *The square* (5.7.1) *satisfies the assumptions of Lemmas 5.4.8 and 5.4.9, with the map ℓ in $\mathfrak{X}(n-2)$.*

Discussion 5.7.2 More concretely, what we will prove is the following. Firstly, there is a factorisation of the square (5.7.1) into two triangles

$$\begin{tikzcd}
\Sigma_{n-1}^P(M'', X|G) \arrow[r, "s"] \arrow[d, "v"'] \arrow[dr, phantom, "H_n"] & \Sigma_n^P(M'', X|G) \arrow[d, "v"] \arrow[dl, "J_n"] \\
\Sigma_n^P(M, X|G) \arrow[r, "s"'] & \Sigma_{n+1}^P(M, X|G)
\end{tikzcd} \tag{5.7.2}$$

which commute up to certain homotopies H_n and J_n (the diagonal map is induced by the inclusion $M'' \hookrightarrow M$). This verifies the assumptions of Lemma 5.4.8. The outer square actually commutes on the nose, but the composition of H_n and J_n will *not* be (homotopic to) the identity homotopy. It is a self-homotopy of the map $s \circ v = v \circ s$, so it is a map

$$K_n \colon S^1 \times \Sigma^P_{n-1}(M'', X|G) \longrightarrow \Sigma^P_{n+1}(M, X|G).$$

The second thing that we prove is that the following square commutes up to homotopy:

$$\begin{tikzcd} S^1 \times \Sigma^P_{n-2}(M'', X|G) \arrow[r, "\mathrm{id} \times s"] \arrow[d, "K_{n-1}"'] & S^1 \times \Sigma^P_{n-1}(M'', X|G) \arrow[d, "K_n"] \\ \Sigma^P_n(M, X|G) \arrow[r, "s"'] & \Sigma^P_{n+1}(M, X|G) \end{tikzcd} \tag{5.7.3}$$

This verifies the assumptions of Lemma 5.4.9, with the map ℓ equal to the stabilisation map $\Sigma^P_{n-2}(M'', X|G) \to \Sigma^P_{n-1}(M'', X|G)$, which is in $\mathfrak{X}(n-2)$.

Hence by Lemmas 5.4.8 and 5.4.9, and assuming the Main Theorem for $n-2$ by induction, this shows that the map on mapping cones induced by the square (5.7.1) is trivial on homology up to degree $\frac{n}{2}$.

Proof of Proposition 5.7.1, first half. We prove the first assertion of Discussion 5.7.2, that there is a factorisation up to homotopy of (5.7.2) into triangles. Denote the diagonal map (which is induced by the inclusion $M'' \hookrightarrow M$) by d. We need to construct a homotopy H_n between the two maps

$$\begin{aligned} \Sigma^P_{n-1}(M'', X|G) &\longrightarrow \Sigma^P_n(M, X|G) \\ d \circ s \colon \{\ldots [\psi_j] \ldots\} &\mapsto \{\ldots [f_{1,0} \circ \psi_j] \ldots, [f_{1,0} \circ \iota]\} \\ v \colon \{\ldots [\psi_j] \ldots\} &\mapsto \{\ldots [\psi_j] \ldots, [\psi_0]\} \end{aligned}$$

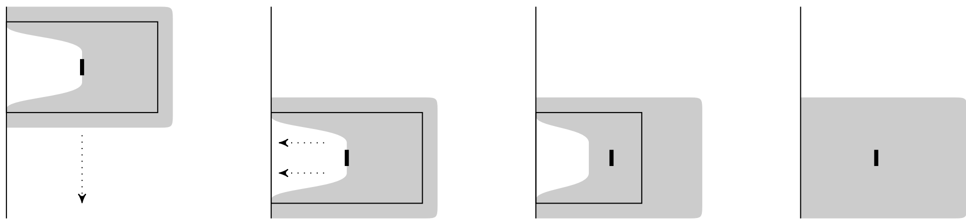
See §5.2 and Figure 5.2.1 for notation and a schematic picture. We omit the labels in X from the notation as they play no role. We construct the homotopy $H_n \colon d \circ s \rightsquigarrow v$ in two steps:

$$\begin{aligned} &\text{step 1} \colon \{\ldots [\psi_j] \ldots\} \mapsto \{\ldots [f_{1,-6t} \circ \psi_j] \ldots, [f_{1,-6t} \circ \iota_{-6t}]\} && t \in [0, \tfrac{1}{2}] \\ &\text{step 2} \colon \{\ldots [\psi_j] \ldots\} \mapsto \{\ldots [f_{2-2t,-3} \circ \psi_j] \ldots, [f_{1,-3} \circ \iota_{-3}]\} && t \in [\tfrac{1}{2}, 1] \end{aligned}$$

(recall that $\psi_0 = f_{1,-3} \circ \iota_{-3}$ and $\iota = \iota_0$).

In words: Step 1 moves the region which the new copy of P is pushed into downwards from V to W. Step 2 keeps the new copy of P fixed while pulling the original configuration of Ps back to where they were before being pushed inwards. This can be done without any of them hitting the *new* copy of P since the original configuration was contained in $M'' = M \smallsetminus e_0(P \times I)$.

In pictures:

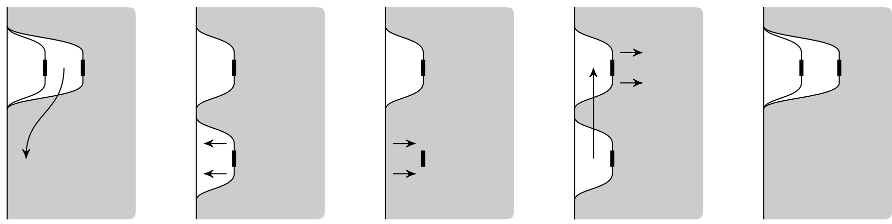


step 1   *step* 2   *part-way through step* 2   *end of step* 2

(5.7.4)

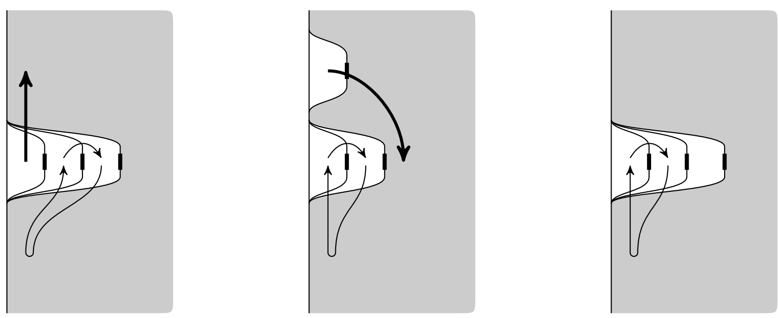
The construction of the other homotopy J_n is exactly the same. $\square$

Composing the homotopies H_n and J_n constructed as in (5.7.4) above we get the following self-homotopy of $s \circ v = v \circ s$ (called K_n in Discussion 5.7.2):



$$s \circ v \xrightarrow[s \cdot H_n]{} s \circ d \circ s \xrightarrow[J_n \cdot s]{} v \circ s \tag{5.7.5}$$

Proof of Proposition 5.7.1, second half. We now prove that the square (5.7.3) commutes up to homotopy. First note that (5.7.5), as a map $S^1 \times \Sigma^P_{n-1}(M'', X|G) \longrightarrow \Sigma^P_n(M, X|G)$, is homotopic to the map depicted in Figure 5.7.1(a). Hence the two ways around the square in (5.7.5) are as depicted in Figure 5.7.1(b) and (c). We just need to construct a homotopy between these two maps; this is given by the following picture:



$$c \circ K_{n-1} \longrightarrow K_n \circ (\mathrm{id} \times s) \tag{5.7.6}$$

What this means: the desired homotopy is a map

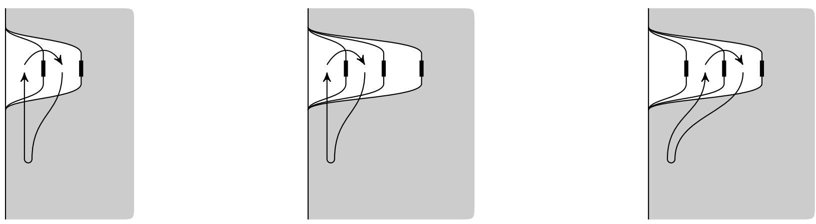
$$[0,1] \times S^1 \times \Sigma^P_{n-2}(M'', X|G) \longrightarrow \Sigma^P_{n+1}(M, X|G),$$

and the diagram above depicts the image of the element (s, t, c) under this map. The original configuration c lives in the shaded region, which is deformed as indicated, and three new copies of P are added where the rectangular blobs in the diagram are. The parameter $t \in S^1$ determines how far along the lighter arrows the unshaded regions have moved, and the parameter $s \in [0,1]$ determines how far along the thick arrows they have moved.

For completeness, here is the formal definition of the homotopy pictured in (5.7.6):

$$
\begin{aligned}
&[0,1] \times S^1 \times \Sigma^P_{n-2}(M'', X|G) \longrightarrow \Sigma^P_{n+1}(M, X|G) \\
&(s, t, \{\ldots [\psi_j] \ldots\}) \mapsto \\
&\quad \{\ldots [f_{6s} \circ f_0 \circ f_{-6t} \circ \psi_j] \ldots, [f_{6s} \circ f_0 \circ \iota_{-6t}], [f_{6s} \circ \iota_0], [\iota_{6s}]\} && s \in [0, \tfrac{1}{2}], t \in [0, \tfrac{1}{2}] \\
&\quad \{\ldots [f_{6s} \circ f_{6t-6} \circ f_0 \circ \psi_j] \ldots, [f_{6s} \circ f_{6t-6} \circ \iota_0], [f_{6s} \circ \iota_{6t-6}], [\iota_{6s}]\} && s \in [0, \tfrac{1}{2}], t \in [\tfrac{1}{2}, 1] \\
&\quad \{\ldots [f_0 \circ f_{-6t} \circ f_{6-6s} \circ \psi_j] \ldots, [f_0 \circ f_{-6t} \circ \iota_{6-6s}], [f_0 \circ \iota_{-6t}], [\iota_0]\} && s \in [\tfrac{1}{2}, 1], t \in [0, \tfrac{1}{2}] \\
&\quad \{\ldots [f_{6t-6} \circ f_0 \circ f_{6-6s} \circ \psi_j] \ldots, [f_{1,6t-6} \circ f_0 \circ \iota_{6-6s}], [f_{6t-6} \circ \iota_0], [\iota_{6t-6}]\} && s \in [\tfrac{1}{2}, 1], t \in [\tfrac{1}{2}, 1]
\end{aligned}
$$

We again omit the labels in X from the notation, use the shorthand $f_u = f_{1,u}$, and identify $S^1 = [0,1]/(0 \sim 1)$. $\square$



(a) (b) (c)

Figure 5.7.1: (a) The map K_n. (b) The composite $\urcorner$ in (5.7.3). (c) The composite $\llcorner$ in (5.7.3).

5.8 Twisted homological stability

In this final section we briefly explain how to deduce *twisted* homological stability for configuration spaces of submanifolds, in other words homological stability w.r.t. a sequence T_n of $\pi_1 \Sigma^P_n(M, X|G)$-modules which forms a "finite-degree twisted coefficient system". We begin by describing exactly what a "twisted coefficient system" for configuration spaces of submanifolds is.

Definition 5.8.1 Let M and $P \subseteq \partial \bar{M}$ be as in Definition 5.1.2, let X be path-connected and let $G \leq \mathrm{Diff}(P)$ be a group of diffeomorphisms of P which is realisable by isotopies (see Definition 5.1.3). There is a "standard" sequence of embeddings of P in M given by $f^n \circ \iota$ (see §5.2), which we denote by q_n.

A *twisted coefficient system* for $\{\Sigma^P_n(M, X|G)\}$ is a functor $\mathcal{B}^P(M, X|G) \to \mathsf{Ab}$, where $\mathcal{B}^P(M, X|G)$ is the following category. It has objects $\coprod_{n \geq 0} X^n$ and a morphism from

$(x_1, \ldots, x_m)$ to $(y_1, \ldots, y_n)$ is a choice of $k \leq \min\{m, n\}$ and a path in $\Sigma_k^P(M, X|G)$ from a k-element subconfiguration of $\{[q_1], \ldots, [q_m]; x_1, \ldots, x_m\}$ to a k-element subconfiguration of $\{[q_1], \ldots, [q_n]; y_1, \ldots, y_n\}$, up to endpoint-preserving homotopy. This may be called the *category of partial braids on M with cross-section P and labels in X.*

This is a special case of the general definition of a twisted coefficient system in Chapter 4 (see Definition 4.2.12), so we get a notion of the *degree* §4.4 of a twisted coefficient system for configuration spaces of submanifolds which is a natural generalisation of the notion for configuration spaces of points §4.2.1.

Now let M, $P \subseteq \partial\bar{M}$, X and $G \leq \mathrm{Diff}(P)$ be as in the Main Theorem or Extension 5.1.8, and let $T \colon \mathcal{B}^P(M, X|G) \to \mathsf{Ab}$ be a twisted coefficient system of degree d. Then we stated in Corollary 5.1.9 that $H_*(\Sigma_n^P(M, X|G); T_n)$ is independent of n in the range $* \leq \frac{n-d-2}{2}$, with isomorphisms given by the stabilisation maps. This can be proved, using the "twisted stability from untwisted stability principle" (Theorem 4.6.1), exactly as for configuration spaces of *points* in §4.6.1. The inputs needed to apply this principle, and prove twisted homological stability for configuration spaces of submanifolds, are that the map

$$\Sigma_{(k,n-k)}^P(M, X|G) \longrightarrow \Sigma_k^P(M, X|G) \tag{5.8.1}$$

is a fibre bundle and that the homology of its fibre, $H_*(\Sigma_{n-k}^P(M \smallsetminus \coprod_{i=1}^k q_i(P), X|G))$, is independent of n in the range $* \leq \frac{n-k-2}{2}$. Here $\Sigma_{(k,n-k)}^P(M, X|G)$ means a configuration space of k red and $n-k$ green copies of P, and the map (5.8.1) forgets the green ones. The first fact can be proved in the same way as Lemma 5.5.7, and the second fact is true by the Main Theorem or Extension 5.1.8, since $M \smallsetminus \coprod_{i=1}^k q_i(P)$ is still connected.

We note that, by Remark 5.1.10, the stabilisation maps $\Sigma_n^P(M, X|G) \to \Sigma_{n+1}^P(M, X|G)$ are always split-injective on homology, so the range in which $H_*(\Sigma_n^P(M, X|G))$ is independent of n is in fact $* \leq \frac{n}{2}$. This improvement goes through the deduction of twisted homological stability, so the range in which $H_*(\Sigma_n^P(M, X|G); T_n)$ is independent of n is in fact $* \leq \frac{n-d}{2}$.

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