De Rham cohomology twisted by flat complex line bundles. We discuss how complex de Rham cohomology twisted by flat complex line bundles of a smooth manifold X is identified with the untwisted but suitably $\pi_1(\mathrm{X})$-equivariant de Rham cohomology on the universal cover $\widehat{\mathrm{X}}$. This is elementary and ought to be classical, but requires a little care.

We state the following Theorem 0.1 and its proof right away, then further below we recall the basic definitions and facts that are being used and explain our notation. Throughout we assume – for ease of notation and without restriction of generality – that the base manifold X is connected.

Theorem 0.1 (1-Twisted de Rham cohomology as $\pi_1(\mathrm{X})$-invariant de Rham cohomology on universal cover). *Given a flat complex line bundle $(\mathscr{L},\nabla)$ over a smooth manifold* X *(as in Rem. 0.3) the ∇-twisted de Rham cohomology* (16) *of differential forms with coefficients in $\mathscr{L}$* (14) *is isomorphic to the untwisted complex de Rham cohomology of the universal cover $\widehat{X}$ (Ntn. 0.4)*

$$\underbrace{H^\bullet\Big(\Omega^\bullet_{\mathrm{dR}}\big(\mathrm{X};\mathscr{L}\big),\nabla\Big)}_{\text{de Rham cohomology twisted by a flat line bundle}} \;\simeq\; \underbrace{H^\bullet\bigg(\underbrace{\Big(\Omega^\bullet_{\mathrm{dR}}\big(\widehat{\mathrm{X}};\mathbb{C}\big)\Big)^{\pi_1(\mathrm{X})}}_{\textit{forms invariant under } (2)},\mathrm{d}\bigg)}_{\text{un-twisted but } \pi_1(\mathrm{X})\text{-invariant de Rham cohomology on universal cover}} \tag{1}$$

on those differential forms which are invariant under the joint action of the fundamental group $\pi_1(\mathrm{X})$ by Deck transformations (20) *and by multiplication with the the holonomy assignment* (8)*:*

$$\begin{array}{ccc}\pi_1(\mathrm{X})\times\Omega^\bullet_{\mathrm{dR}}\big(\widehat{\mathrm{X}};\mathbb{C}\big) & \longrightarrow & \Omega^\bullet_{\mathrm{dR}}\big(\widehat{\mathrm{X}};\mathbb{C}\big)\\ \big([\lambda],A\big) & \longmapsto & \mathrm{hol}_\nabla(\lambda)\cdot[\lambda]^*(A)\,,\end{array} \tag{2}$$

Here

$$[\lambda]^*(A) := \Big(\mathrm{shr}^*\big(\mathrm{pr}_2^*(A)\big)\Big)(-,[\lambda]) \tag{3}$$

denotes pullback of differential forms under the shear map (21) of the universal cover regarded as a $\pi_1(\mathrm{X})$-principal bundle.

Proof. We may choose Čech cocycle data $(g,\widehat{\omega})$ (11) for the pullback of $(\mathscr{L},\nabla)$ to the universal cover $p_{\widehat{\mathrm{X}}}:\widehat{\mathrm{X}}\to\mathrm{X}$, by Lemma 0.5. Here, due to the assumption that ∇ is flat, the local connection form $\widehat{\omega}\in\Omega^1_{\mathrm{dR}}\big(\widehat{\mathrm{X}};\mathrm{i}\mathbb{R}\big)$ is closed (12). Moreover, since the universal cover $\widehat{\mathrm{X}}$ is simply connected, it follows that $\widehat{\omega}$ is exact (e.g. [dCa94, §2, Prop. 3]). Concretely, a potential is given by the integrals of $\widehat{\omega}$ along the lifted path $\widehat{\gamma}$ (18) from a given basepoint, which will be useful to express in exponential form as follows:

$$\begin{array}{ccccc}\ell & : & \widehat{\mathrm{X}} & \longrightarrow & \mathrm{U}_1 \hookrightarrow \mathbb{C}\,,\\ & & [\gamma] & \longmapsto & \exp\big(-\int_{\widehat{\gamma}}\widehat{\omega}\big)\end{array} \qquad \widehat{\omega} \;=\; -\mathrm{d}\log\ell \;=\; -\frac{1}{\ell}\mathrm{d}\ell\,. \tag{4}$$

The key is now to observe that this function ℓ (4) transforms as follows under Deck transformations:

$$\begin{aligned}[\lambda]^*\ell([\gamma]) &\underset{(3)}{=} \ell\big([\lambda\cdot\gamma]\big) \underset{(4)}{=} \exp\Big(-\textstyle\int_{\widehat{\lambda\cdot\gamma}}\widehat{\omega}\Big) = \exp\Big(-\textstyle\int_{\gamma\cdot\widehat{(\overline{\gamma}\cdot\lambda\cdot\gamma)}}\widehat{\omega}\Big) = \exp\Big(-\textstyle\int_{\widehat{\gamma}}\widehat{\omega}\Big)\cdot\frac{1}{\exp\big(\int_{\widehat{\overline{\gamma}\cdot\lambda\cdot\gamma}}\widehat{\omega}\big)}\\ &\underset{(13)}{=} \ell([\gamma])\cdot\frac{1}{\mathrm{hol}_\nabla([\lambda])\cdot g\big([\gamma],[\lambda\cdot\gamma]\big)}\\ &\underset{(21)}{=} \ell([\gamma])\cdot\frac{1}{\mathrm{hol}_\nabla([\lambda])\cdot g\big(\mathrm{shr}(-,[\lambda])\big)}\,,\end{aligned}$$

whereas the local representative $\widehat{\alpha}$ (14) of any $\mathscr{L}$-valued form α transforms like this:

$$\begin{aligned}[\lambda]^*\widehat{\alpha} &\underset{(3)}{=} \Big(\mathrm{shr}^*\big(\mathrm{pr}_2^*(\widehat{\alpha})\big)\Big)(-,[\lambda]) \underset{(14)}{=} \Big(\mathrm{shr}^*\big(g\cdot\mathrm{pr}_1^*(\widehat{\alpha})\big)\Big)(-,[\lambda])\\ &\underset{(21)}{=} g\Big(\mathrm{shr}\big(-,[\lambda]\big)\Big)\cdot\widehat{\alpha}\,.\end{aligned}$$

This implies that in products of these two expressions the contribution of the transition functions g drops out and the result is fixed by the above joint action (2):

$$\widehat{\alpha}\in\Big(\Omega^\bullet_{\mathrm{dR}}\big(\widehat{\mathrm{X}};\mathbb{C}\big)\Big)^g \quad\Leftrightarrow\quad \underset{[\lambda]\in\pi_1(\mathrm{X})}{\forall}\;[\lambda]^*\big(\ell\cdot\widehat{\alpha}\big) = \frac{1}{\mathrm{hol}_\nabla([\lambda])}\cdot\widehat{\alpha} \quad\Leftrightarrow\quad \ell\cdot\widehat{\alpha}\in\Big(\Omega^\bullet_{\mathrm{dR}}\big(\widehat{\mathrm{X}};\mathbb{C}\big)\Big)^{\pi_1(\mathrm{X})}. \tag{5}$$

Moreover, taking the product with ℓ evidently trivializes the twist in the local expression (15) for the covariant derivative:

$$
\begin{aligned}
\widehat{\alpha} \;&=\; \big(\mathrm{d} - \widehat{\omega} \wedge\big)\widehat{\beta}
\\
\underset{(4)}{\Leftrightarrow}\;\; \ell \cdot \widehat{\alpha} \;&=\; \mathrm{d}\big(\ell \cdot \widehat{\beta}\big)\,.
\end{aligned}
\tag{6}
$$

In summary this means that we have an isomorphic cochain maps of this form:

$$
\begin{array}{ccccc}
\Big(\Omega^\bullet_{\mathrm{dR}}\big(\mathrm{X}; \mathscr{L}\big), \nabla\Big) & \underset{(16)}{\xrightarrow{\sim}} & \Big(\Big(\Omega^\bullet_{\mathrm{dR}}\big(\widehat{\mathrm{X}}; \mathbb{C}\big)\Big)^g, \mathrm{d} - \widehat{\omega}\Big) & \underset{(5)\,(6)}{\xrightarrow{\sim}} & \Big(\Big(\Omega^\bullet_{\mathrm{dR}}\big(\widehat{\mathrm{X}}; \mathbb{C}\big)\Big)^{\pi_1(\mathrm{X})}, \mathrm{d}\Big)
\\
\alpha & \longmapsto & \widehat{\alpha} & \longmapsto & \ell \cdot \widehat{\alpha}
\end{array}
\tag{7}
$$

Under passage to cochain cohomology, this yields the claim (1). ☐

We now recall the basic facts that enter the above proof.

Proposition 0.2 (Flat bundles equivalent to their holonmy representation of the fundamental group). *Let* X *be a smooth manifold and* $(\mathscr{L}, \nabla)$ *a smooth complex line bundle with flat connection* ∇ *over a connected smooth manifold* X *with any base point* $x_0 \in \mathrm{X}$*. Then the operation which sends smooth loops in* X *to their holonomy under* ∇ *constitutes a bijection between isomorphism classes of flat complex line bundles and group homomorphisms from the fundamental group of* X*:*

$$
\begin{array}{ccc}
\big(\mathrm{FltCplxLineBund}_{\mathrm{X}}\big)_{/\sim} & \xrightarrow{\quad\sim\quad} & \mathrm{Hom}\big(\pi_1(\mathrm{X}), \mathrm{U}_1\big)
\\
(\mathscr{L}, \nabla) & \longmapsto & \big([\lambda] \mapsto \mathrm{hol}_\nabla(\lambda)\big)
\end{array}
\tag{8}
$$

Remark 0.3 (Flat connections and their twist of de Rham cohomology in terms of Čech cocycles). Given a surjective submersion $\widehat{\mathrm{X}} \xrightarrow{p_{\widehat{\mathrm{X}}}} \mathrm{X}$, the smooth complex line bundles $\mathscr{L}$ over X such that their pullback $p^*_{\widehat{\mathrm{X}}}\big(\mathscr{L}\big)$ admits a trivialization are equivalent to transition functions on the fiber product space

$$
g \in C^\infty\Big(\widehat{\mathrm{X}} \underset{\mathrm{X}}{\times} \widehat{\mathrm{X}}, \mathrm{U}_1\Big) \hookrightarrow C^\infty\Big(\widehat{\mathrm{X}} \underset{\mathrm{X}}{\times} \widehat{\mathrm{X}}, \mathbb{C}\Big) \qquad \text{such that on } \widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \text{ we have:} \qquad \big(\mathrm{pr}^*_{1,2}(g)\big) \cdot \big(\mathrm{pr}^*_{1,2}(g)\big) \;=\; \mathrm{pr}^*_{1,3}(g)\,. \tag{9}
$$

An isomorphism $g \xrightarrow[\sim]{h} g'$ of such cocycle data is:

$$
h \in C^\infty\big(\widehat{\mathrm{X}}, \mathrm{U}_1\big) \qquad \text{such that on } \widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \text{ we have} \qquad g \cdot \mathrm{pr}^*_1(h) \;=\; \mathrm{pr}^*_2(h) \cdot g'\,. \tag{10}
$$

Given cocycle data (9) for a complex line bundle $\mathscr{L}$, there are the following cocycle incarnations of various related notions:

- A *connection* ∇ on $\mathscr{L}$ is equivalently:

$$
\widehat{\omega} \;\in\; \Omega^1_{\mathrm{dR}}(\widehat{\mathrm{X}}; \mathrm{i}\mathbb{R}) \qquad \text{such that on } \widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \text{ we have:} \qquad \mathrm{pr}^*_2(\widehat{\omega}) - \mathrm{pr}^*_1(\widehat{\omega}) \;=\; \mathrm{d}\log g \;:=\; \frac{1}{g} \cdot \mathrm{d}g\,, \tag{11}
$$

  and flatness means:

$$
\nabla \circ \nabla = 0 \qquad \Leftrightarrow \qquad \mathrm{d}\widehat{\omega} = 0\,. \tag{12}
$$

- The *holonomy* of ∇ along a smooth curve $\lambda : [0,1] \to \mathrm{X}$ (i.e.: $\lambda(0) = \lambda(1)$) is[1]

$$
\mathrm{hol}_\nabla(\lambda) \;=\; \exp\Big(\int_{\widehat{\lambda}} \omega_1\Big) \cdot g\Big(\widehat{\lambda}(1), \widehat{\lambda}(0)\Big) \qquad \text{for any lift}^2\ \widehat{\lambda} : [0,1] \to \widehat{\mathrm{X}}, \text{ i.e. such that } p_{\widehat{\mathrm{X}}} \circ \lambda \;=\; \lambda\,. \tag{13}
$$

- A differential form on X *with coefficients* in $\mathscr{L}$, denoted $\alpha \in \Omega^\bullet_{\mathrm{dR}}\big(\mathrm{X}; \mathscr{L}\big)$, is equivalently:

$$
\widehat{\alpha} \in \Omega^\bullet_{\mathrm{dR}}\big(\widehat{\mathrm{X}}; \mathbb{C}\big) \qquad \text{such that on } \widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \text{ we have} \qquad \mathrm{pr}^*_2(\alpha) \;=\; g \cdot \mathrm{pr}^*_1(\alpha)\,. \tag{14}
$$

[1] The positive sign in the exponent of (13) is a convention. If we adopted a minus sign, as in $\exp\big(-\int_{\widehat{\gamma}\widehat{\omega}}\big)$, then also the relative sign in (15) and that in the exponent of spring changes.

- The *covariant derivative* ∇ is, under this identification (14), given by

$$\nabla\alpha \in \Omega^\bullet_{\mathrm{dR}}(\mathrm{X};\mathscr{L}) \qquad \leftrightarrow \qquad (\mathrm{d} - \widehat{\omega}\wedge)\widehat{\alpha} \in \Omega^\bullet_{\mathrm{dR}}\big(\widehat{\mathrm{X}};\mathbb{C}\big). \tag{15}$$

- The ∇-*twisted de Rham cohomology* of $\mathscr{L}$-valued forms on X, for flat ∇, is equivalently the cohomology of $\mathrm{d} - \widehat{\omega}\wedge$ (15) on those $\mathbb{C}$-valued differential forms over $\widehat{\mathrm{X}}$ which satisfy their cocycle condition (14), in fact the respective cochain complexes are isomorphic:

$$\big(\Omega^\bullet_{\mathrm{dR}}(\mathrm{X};\mathscr{L}),\nabla\big) \simeq \Big(\underbrace{\big(\Omega^\bullet_{\mathrm{dR}}(\widehat{\mathrm{X}};\mathbb{C})\big)^g}_{\text{forms satisfying (14)}}, \mathrm{d} - \widehat{\omega}\wedge\Big) \tag{16}$$

Notice that this is consistent, in that $\mathrm{d} - \widehat{\omega}$ respects the condition (14), because:

$$\begin{aligned}
\mathrm{pr}_2^*\Big(\big(\mathrm{d} - \widehat{\omega}\wedge\big)\widehat{\alpha}\Big) &\simeq \mathrm{d}\Big(\mathrm{pr}_2^*(\widehat{\alpha})\Big) - \mathrm{pr}_2^*(\widehat{\omega})\wedge\mathrm{pr}_2^*(\widehat{\alpha}) \\
&= \underbrace{\mathrm{d}\big(g\cdot\mathrm{pr}_1^*(\widehat{\alpha})\big)}_{g\left(\mathrm{d}\left(\mathrm{pr}_1^*(\widehat{\alpha})\right)+\frac{1}{g}\cdot\mathrm{d}g\wedge\left(\mathrm{pr}_1^*(\widehat{\alpha})\right)\right)} - \big(\mathrm{pr}_1^*(\widehat{\omega}) + \tfrac{1}{g}\cdot\mathrm{d}g\big)\wedge\big(g\cdot\mathrm{pr}_1^*(\widehat{\alpha})\big) \qquad \text{by (11) \& (14)} \\
&= g\cdot\mathrm{d}\Big(\mathrm{pr}_1^*(\widehat{\alpha})\Big) - g\cdot\mathrm{pr}_1^*(\widehat{\omega})\wedge\mathrm{pr}_1^*(\widehat{\alpha}) \\
&= g\cdot\mathrm{pr}_1^*\Big(\big(\mathrm{d} - \widehat{\omega}\wedge\big)\widehat{\alpha}\Big).
\end{aligned}$$

We now consider the general situation of Rem. 0.3 for the case that the surjective submersion is the universal cover (Ntn. 0.4) which is sufficient for the local trivialization of flat bundles (by Lem. 0.5 below).

Notation 0.4 (Universal cover of smooth manifold). Let $\mathrm{X} \in \mathrm{SmthMfd}$ be connected smooth manifold, we denote its universal cover by

$$\begin{array}{ccccc}
[\gamma] & \in & \widehat{X} & := & \Big\{[\gamma] \,\Big|\, \gamma : [0,1] \xrightarrow{\text{smooth}} \mathrm{X},\ \gamma(0) = x_0\Big\} \\
\downarrow & & \downarrow & & \\
\gamma(1) & \in & \mathrm{X}. & &
\end{array} \tag{17}$$

Here $x_0 \in \mathrm{X}$ is a fixed base point, and square brackets $[-]$ denote homotopy classes of paths fixing their endpoints (equivalently: smooth homotopy classes, e.g. by [Lee12, Prop. 6.29]). Given such a path γ in X, we have a canonical lift to a path in $\widehat{\mathrm{X}}$ by the following formula:

$$\begin{array}{rcl}
\widehat{\gamma} : [0,1] & \longrightarrow & \widehat{\mathrm{X}} \\
t & \mapsto & \big[\gamma(t\cdot(-))\big]
\end{array} \tag{18}$$

Accordingly, with the the fundamental group of X denoted by

$$\pi_1(\mathrm{X}) = \Big\{[\lambda] \,\Big|\, \lambda : [0,1] \to \mathrm{X}, \lambda(0) = \lambda(1) = x_0\Big\}, \tag{19}$$

the canonical left action of $\pi_1(\mathrm{X})$ on $\widehat{\mathrm{X}}$ (by Deck transformations) is given by concatenation of paths

$$\begin{array}{rcl}
\pi_1(\mathrm{X}) \times \widehat{\mathrm{X}} & \longrightarrow & \widehat{\mathrm{X}} \\
([\lambda],[\gamma]) & \mapsto & [\lambda\cdot\gamma],
\end{array} \tag{20}$$

Since $\widehat{\mathrm{X}} \to \mathrm{X}$ is a $\pi_1(\mathrm{X})$-principal bundle, this action induces a *shear map* isomorphism:

$$\begin{array}{rcl}
\widehat{\mathrm{X}} \times p_1(\mathrm{X}) & \xrightarrow[\sim]{\text{shr}} & \widehat{\mathrm{X}} \times_{\mathrm{X}} \widehat{\mathrm{X}} \\
([\lambda],[\gamma]) & \longmapsto & ([\gamma],[\lambda\cdot\gamma])
\end{array} \tag{21}$$

Using this notation, one readily checks that the universal cover is given by the following homotopy pullback in $\mathrm{SmthGrpd}_\infty$:

$$\begin{array}{ccccc} \widehat{\mathrm{X}} & \longrightarrow & \int\widehat{\mathrm{X}} & \longrightarrow & * \\ {\scriptstyle p_{\widehat{\mathrm{X}}}}\downarrow & {\scriptstyle (\mathrm{pb})} & \downarrow & {\scriptstyle (\mathrm{pb})} & \downarrow \\ \mathrm{X} & \underset{\eta^{\int}_{\mathrm{X}}}{\longrightarrow} & \int\mathrm{X} & \underset{\eta^{[-]_1}_{\int\mathrm{X}}}{\longrightarrow} & \mathbf{B}\pi_1(\mathrm{X}) \end{array} \tag{22}$$

Lemma 0.5 (Flat bundle trivializes over universal cover of the base space)**.** *A complex line bundle with* $\mathscr{L}$ *which carries a flat connection* ∇ *admits a trivialization after pullback to the universal cover* $\widehat{\mathrm{X}}$ *(Ntn. 0.4) of the base manifold* X.

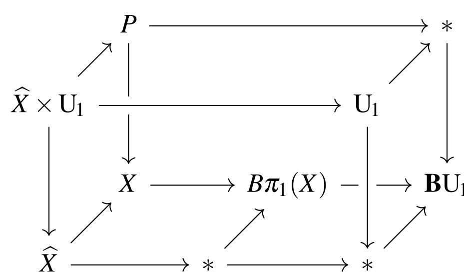
Proof. Here is a quick way to see this, using the higher topos theory of $\mathrm{SmthGrpd}_\infty$:

The flat complex line bundle $\mathscr{L}$ and its underlying U_1-principal bundle P are given by the following homotopy pullbacks, respectively:

$$\begin{array}{ccccccc} \mathscr{L} & \longrightarrow & \mathbb{C}/\!\!/\flat\mathrm{U}_1 & \longrightarrow & \mathbb{C}/\!\!/\mathrm{U}_1 \\ \downarrow & {\scriptstyle (\mathrm{pb})} & \downarrow & {\scriptstyle (\mathrm{pb})} & \downarrow \\ \mathrm{X} & \underset{\eta^{[-]_1}_{\int\mathrm{X}}\circ\eta^{\int}_{\mathrm{X}}}{\longrightarrow} \mathbf{B}\pi_1(\mathrm{X}) \underset{\mathbf{B}\mathrm{hol}_\nabla}{\longrightarrow} & \mathbf{B}\flat\mathrm{U}_1 & \underset{\varepsilon^\flat_{\mathbf{B}\mathrm{U}_1}}{\longrightarrow} & \mathbf{B}\mathrm{U}_1 \end{array} \qquad \begin{array}{ccccc} P & \longrightarrow & \mathrm{U}_1/\!\!/\flat\mathrm{U}_1 & \longrightarrow & * \\ \downarrow & {\scriptstyle (\mathrm{pb})} & \downarrow & {\scriptstyle (\mathrm{pb})} & \downarrow \\ \mathrm{X} & \underset{\eta^{[-]_1}_{\int\mathrm{X}}\circ\eta^{\int}_{\mathrm{X}}}{\longrightarrow} \mathbf{B}\pi_1(\mathrm{X}) \underset{\mathbf{B}\mathrm{hol}_\nabla}{\longrightarrow} & \mathbf{B}\flat\mathrm{U}_1 & \underset{\varepsilon^\flat_{\mathbf{B}\mathrm{U}_1}}{\longrightarrow} & \mathbf{B}\mathrm{U}_1\,. \end{array} \tag{23}$$

In the bottom row of these diagrams we are using (1.) that the moduli stack for these bundles is $\mathbf{B}\mathrm{U}_1$, (2.) that a flat connection is equivalently a lift of the modulating map to locally constant coefficients and (3.) that the their moduli stack $\mathbf{B}\flat\mathrm{U}_1$ is (i) geometrically discrete and (ii) 1-truncated, which implies that any map to it factors (i) through the shape unit and (2) through the truncation unit.

Notice then the following homotopy-commuting diagram:



Here the bottom square commutes by (22), the rear square is the above homotopy pullback (23), the right square is the homotopy pullback exhibiting the delooping property of the moduli stack, and the from square is a pullback by definition of Cartesian products. Therefore, the pasting law implies that also the left square is a pullback, which exhibits the local trivialization to be shown. □

References

[dCa94] M. P. do Carmo, *Differential Forms and Applications*, Springer (1994) [doi:10.1007/978-3-642-57951-6]

[Lee12] J. Lee, *Introduction to Smooth Manifolds*, Springer (2012) [doi:10.1007/978-1-4419-9982-5]