Calabi's classification of harmonic maps from S^2 to S^n

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Abstract

In the 1960s E. Calabi classified all minimal isometric immersions, from the 2-sphere into higher dimensional spheres S^n, in terms of holomorphic maps. He showed that such a map can be lifted to a horizontal holomorphic map from $S^2 \cong CP^1$ into a twistor space over S^n. The construction also applies to harmonic maps from S^2 to S^n. The main purpose of this work is to give a detailed presentation of Calabi's constructions and classification result for harmonic maps.

Throughout this work it has been my firm intention to give reference to the stated results and credit to the work of others. All theorems, propositions, lemmas and examples left unmarked are assumed to be too well known for a reference to be given. A bracket [X] *at the beginning of a proof means that the idea was obtained from the reference. A proof left unmarked is written by me, although I make no claim of originality.*

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Chapter 1

Harmonic maps

1.1 Basic properties

In this section we present what is needed to define a harmonic map between Riemannian manifolds. The main source for this chapter is [26].

Let (M, g) and (N, h) be smooth complete connected and orientable Riemannian manifolds of dimension m and n, respectively. We denote the Levi-Civita connections on M and N by ∇^M and ∇^N.

Definition 1.1. Let $\phi : (M, g) \to (N, h)$ be a C^∞-map. We define the **energy density function** $e(\phi) : M \to \mathbb{R}^+$ of ϕ by

$$e(\phi)(x) = \frac{1}{2} \sum_{i=1}^{m} h_{\phi(x)}(d\phi_x(X_i), d\phi_x(X_i)),$$

where $\{X_i\}_{i=1}^m$ is any orthonormal basis of the tangent space T_xM.

That $e(\phi)$ is independent of the choice of the local orthonormal frame is a consequence of it being the trace of the tensor field

$$T(X, Y) = \frac{1}{2} h(d\phi(X), d\phi(Y)).$$

The **trace** at a point $x \in M$ is defined as

$$\operatorname{trace}(T) = \sum_{i=1}^{m} T(X_i, X_i),$$

where $\{X_i\}_{i=1}^m$ is any orthonormal basis for T_xM. That this is independent of the orthonormal basis can be seen by letting $\{Y_j\}_{j=1}^m$ be another orthonormal basis for T_xM. Then there exist an orthogonal matrix $\{a_{ij}\}_{i,j=1}^m$ such that

$$Y_j = \sum_{i=1}^{m} a_{ij} X_i.$$

Since T is bilinear we have

$$\sum_{j=1}^{m} T(Y_j, Y_j) = \sum_{j=1}^{m} T(\sum_{i=1}^{m} a_{ij} X_i, \sum_{k=1}^{m} a_{kj} X_k)$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{m} \sum_{k=1}^{m} a_{ij} a_{kj} T(X_i, X_k)$$
$$= \sum_{i=1}^{m} \sum_{k=1}^{m} \delta_{ik} T(X_i, X_k)$$
$$= \sum_{i=1}^{m} T(X_i, X_i).$$

Hence the trace does not depend on the choice of orthonormal basis.

Definition 1.2. Let $\phi : M \to N$ be a C^∞-map. Then a C^∞-map

$$\Phi : (-\epsilon, \epsilon) \times M \to N$$

is said to be a **smooth variation** of ϕ if $\Phi(0, x) = \phi(x)$. We write $\phi_t(x)$ for $\Phi(t, x)$.

We are now ready to give the definition of a harmonic map.

Definition 1.3. Let M be compact. Given a C^∞-map $\phi : (M, g) \to (N, h)$, the integral

$$E(\phi) = \int_M e(\phi) v_g$$

is called the **energy functional** of ϕ. The map ϕ is said to be a **harmonic map** if it is a **critical point** of E, i.e.

$$\frac{d}{dt}(E(\phi_t))|_{t=0} = 0$$

for any smooth variation $\phi_t : (-\epsilon, \epsilon) \times M \to N$ of ϕ.

1.2 The pull-back bundle

A C^∞-map $\phi : M \to N$ induces a vector bundle $\phi^{-1}TN$ over M. In this section we present the construction of this and show that there is a natural connection ∇^ϕ on $\phi^{-1}TN$ which is compatible with a natural metric h^ϕ on $\phi^{-1}TN$.

Given a smooth variation ϕ_t of $\phi : M \to N$ we define the corresponding **variational vector field** V by

$$V_x = \frac{d}{dt}(\phi_t(x))|_{t=0}$$

for all $x \in M$. Since for each $x \in M$, $\gamma : t \mapsto \phi_t(x)$ is a curve in N with $\gamma(0) = \phi(x)$ we have

$$V_x \in T_{\phi(x)}N$$

for all $x \in M$. For a function $f \in C^\infty(N)$ we have

$$V_x(f) = \frac{d}{dt}(f \circ \phi_t(x))|_{t=0}.$$

This is a smooth function in $x \in M$ so $V \in C^\infty(M, TN)$. Conversely, if $V \in C^\infty(M, TN)$ satisfies

$$V : M \ni x \mapsto V_x \in T_{\phi(x)}N$$

we define

$$\phi_t(x) = \exp_{\phi(x)}(tV_x).$$

The definition makes sense since N is complete and $\phi_t(x)$ satisfies $V_x = \frac{d}{dt}(\phi_t(x))|_{t=0}$.

Proposition 1.4. *Let $\phi : M \to N$ be a C^∞-map. Define the set $\phi^{-1}TN$ by*

$$\phi^{-1}TN = \{(x, u) \,|\, x \in M,\, u \in T_{\phi(x)}N\}.$$

Then $(\phi^{-1}TN, M, \eta)$, where $\eta(x, u) = x$, is a C^∞-vector bundle. We call it the ***pull-back bundle*** *of ϕ over M.*

Proof. We begin by showing that $\phi^{-1}TN$ is a C^∞-manifold. Let (TN, N, π) be the tangent bundle for N. For $(x, u) \in \phi^{-1}TN$ let (U, Ξ) be a chart around $(\phi(x), u)$ with

$$\Xi(y, v) = (\xi(y), \zeta(v)).$$

Let (U_α, x_α) be a chart around $x \in M$ and set $V = \phi^{-1}(\pi(U)) \cap U_\alpha$. If we let $\eta^{-1}(V)$ define a basis for the topology on $\phi^{-1}TN$ then

$$\Pi : \eta^{-1}(V) \ni (x, u) \to (x_\alpha, \zeta(u)) \in \mathbb{R}^{m+n}$$

is a homeomorphism. The transition maps are obviously diffeomorphism so $\phi^{-1}TN$ is a C^∞-manifold.

The map η is obviously a projection. We show that for each $x \in M$ the fiber is a vector space and there exist a local bundle chart.

(i) The fiber

$$\eta^{-1}(x) = \{x\} \times T_{\phi(x)}N$$

is a vector space.

(ii) Let $(\pi^{-1}(V), \Psi)$ where $\phi(x) \in V$ be a bundle chart for TN with

$$\Psi : \pi^{-1}(V) \ni (y, v) \mapsto (y, \psi(v)) \in V \times \mathbb{R}^n$$

then $U = \phi^{-1}(V)$ is a neighborhood of x and $\Phi : \eta^{-1}(U) \to U \times \mathbb{R}^n$ defined by

$$\Phi : (x, v) \mapsto (x, \psi(v))$$

is bundle chart for $\phi^{-1}TN$ since it is obviously a homeomorphism and its restriction to a fiber obviously a vector space isomorphism. That the transition maps are diffeomorphisms is also clear. $\square$

We see that the sections of $\phi^{-1}TN$ are the maps of $C^\infty(M, TN)$ that satisfy $V_x \in T_{\phi(x)}N$ so they are exactly the variational vector fields. Examples of sections of $C^\infty(\phi^{-1}TN)$ are $Z \circ \phi$ defined by

$$x \mapsto Z_{\phi(x)}$$

for $Z \in C^\infty(TN)$ and $d\phi(X)$ defined by

$$x \mapsto d\phi_x(X_x)$$

for $X \in C^\infty(TM)$.

For a curve $\sigma : (-\epsilon, \epsilon) \to M$ on a Riemannian manifold (M, g) we denote by $P^M_{\sigma(t)} : T_xM \to T_{\sigma(t)}M$ the parallel transport along σ from $x = \sigma(0)$ to $\sigma(t)$.

Lemma 1.5. *Let (M, g) be a Riemannian manifold and $\sigma : (-\epsilon, \epsilon) \to M$ be a curve with $\sigma(0) = x \in M$. Then $P^M_{\sigma(t)}$ is a linear isometric isomorphism of $(T_x M, g_x)$ to $(T_{\sigma(t)}M, g_{\sigma(t)})$.*

Proof. It follows from the definition of parallel transport that it is a linear isomorphism. If $X_x, Y_x \in T_x M$ then

$$\frac{d}{dt} g_{\sigma(t)}(P^M_{\sigma(t)}(X_x), P^M_{\sigma(t)}(Y_x)) = 0.$$

Thus

$$g_x(X_x, Y_x) = g_{\sigma(t)}(P^M_{\sigma(t)}(X_x), P^M_{\sigma(t)}(Y_x)).$$

So $P^M_{\sigma(t)}$ isometric. □

Lemma 1.6. *Let (M, g) be a Riemannian manifold and $X, Y \in C^\infty(TM)$. If $\sigma : (-\epsilon, \epsilon) \to M$ is a curve with $\sigma(0) = x \in M$ and $\dot\sigma(0) = X_x$ then*

$$\nabla^M_X Y(x) = \frac{d}{dt}(P^M_{\sigma(t)})^{-1}(Y_{\sigma(t)})|_{t=0}.$$

Proof. [25] Let $Z_x \in T_x M$. Then

$$\begin{aligned}
g_x\Big(\frac{d}{dt}(P^M_{\sigma(t)})^{-1}(Y_{\sigma(t)})|_{t=0}, Z_x\Big) =& \frac{d}{dt} g_x\big((P^M_{\sigma(t)})^{-1}(Y_{\sigma(t)}), Z_x\big)|_{t=0}\\
=& \frac{d}{dt} g_{\sigma(t)}\big(Y_{\sigma(t)}, P^M_{\sigma(t)}(Z_x)\big)|_{t=0}\\
=& g_{\sigma(t)}\Big(\frac{D}{dt}(Y_{\sigma(t)}), P^M_{\sigma(t)}(Z_x)\Big)|_{t=0}\\
&+ g_{\sigma(t)}\Big(Y_{\sigma(t)}, \frac{D}{dt}\big(P^M_{\sigma(t)}(Z_x)\big)\Big)|_{t=0}\\
=& g_x(\nabla^M_X Y(x), Z_x).
\end{aligned}$$

This proves the statement since Z_x is arbitrary. □

Definition 1.7. Let $\phi : (M, g) \to (N, h)$ be a C^∞-map, $X \in C^\infty(TM)$ and $V \in C^\infty(\phi^{-1}TN)$. Then we define the **pull-back connection** by

$$(\nabla^\phi_X V)(x) = \frac{d}{dt}\left((P^N_{\phi\circ\sigma(t)})^{-1}(V_{\sigma(t)})\right)|_{t=0},\ x \in M$$

where $t \mapsto \sigma(t)$ is a C^1-curve in M satisfying $\sigma(0) = x$ and $\dot\sigma(0) = X_x$.

Let $X \in C^\infty(TM)$, $x \in M$ and $\phi : (M, g) \to (N, h)$ be a C^∞-map. If σ is a curve in M with $\sigma(0) = x$ and $\dot\sigma(0) = X_x$ then $\gamma = \phi \circ \sigma$ is a curve with $\gamma(0) = \phi(x)$ and $\dot\gamma(0) = d\phi_x(X_x)$. Thus given a section V of the pull-back bundle we have that

$$\nabla^\phi_X V(x) = 0 \text{ if } d\phi_x(X_x) = 0.$$

Otherwise there exist a open interval I of 0 such that $\gamma : I \to N$ is injective. Thus there exist a vector field $Z \in C^\infty(TN)$ such that $V_{\sigma(t)} = Z_{\phi\circ\sigma(t)}$ for all $t \in I$. This implies that

$$\nabla^\phi_X V(x) = \frac{d}{dt}\left((P^N_{(\phi\circ\sigma)(t)})^{-1}(V_{\sigma(t)})\right)|_{t=0}$$

$$
\begin{aligned}
&= \frac{d}{dt}\left((P^N_{(\phi\circ\sigma)(t)})^{-1}(Z_{\phi\circ\sigma(t)})\right)|_{t=0} \\
&= (\nabla^N_{d\phi(X)} Z)(\phi(x))
\end{aligned}
$$

as vectors in $T_{\phi(x)}N$. By abuse of notation we will write

$$(\nabla^\phi_X V)(x) = (\nabla^N_{d\phi(X)} V)(\phi(x))$$

at a point $x \in M$.

Proposition 1.8. *Let $\phi : (M, g) \to (N, h)$ be a C^∞-map. Then ∇^ϕ the pull-back connection is a connection on the pull-back bundle $\phi^{-1}TN$.*

Proof. [26] Let $X \in C^\infty(TM)$ and $V \in C^\infty(\phi^{-1}TN)$. We show that $\nabla^\phi_X V$ is tensorial in the first argument and is additive and satisfies the product rule in the second.

(i) It is obvious that the pull-back connection only depends on the value X_x of X so it is tensorial in the first argument.

(ii) Now we show that the second argument is additive. Let σ be a curve with $\sigma(0) = x$ and $\dot\sigma(0) = X_x$ and $W \in C^\infty(\phi^{-1}TN)$ then

$$
\begin{aligned}
\nabla^\phi_X (V + W)(x) &= \frac{d}{dt}\left((P^N_{\phi\circ\sigma(t)})^{-1}(V_{\sigma(t)} + W_{\sigma(t)})\right)|_{t=0} \\
&= \frac{d}{dt}\left((P^N_{\phi\circ\sigma(t)})^{-1} V_\sigma(t) + (P^N_{\phi\circ\sigma(t)})^{-1} W_{\sigma(t)}\right)|_{t=0} \\
&= \frac{d}{dt}\left((P^N_{\phi\circ\sigma(t)})^{-1} V_{\sigma(t)}\right)|_{t=0} + \frac{d}{dt}\left((P^N_{\phi\circ\sigma(t)})^{-1} W_{\sigma(t)}\right)|_{t=0} \\
&= \nabla^\phi_X V(x) + \nabla^\phi_X W(x).
\end{aligned}
$$

(iii) Finally we show that the second argument satisfies the product rule. Let $f \in C^\infty(M)$ then

$$
\begin{aligned}
\nabla^\phi_X (fV)(x) &= \frac{d}{dt}\left((P^N_{\phi\circ\sigma(t)})^{-1}(f(\sigma(t))V_{\sigma(t)})\right)|_{t=0} \\
&= \frac{d}{dt}\left(f(\sigma(t))(P^N_{\phi\circ\sigma(t)})^{-1}(V_{\sigma(t)})\right)|_{t=0} \\
&= \Big(\frac{d}{dt}\left(f(\sigma(t))\right)(P^N_{\sigma(t)})^{-1}(V_{\sigma(t)}) \\
&\quad + f(\sigma(t))\frac{d}{dt}\left((P_{\sigma(t)^N})^{-1}(V_{\sigma(t)})\right)\Big)|_{t=0} \\
&= X_x(f)V_x + f(x)\nabla^\phi_X V(x).
\end{aligned}
$$

□

The pull-back bundle $\phi^{-1}TN$ has a natural metric h^ϕ given by

$$h^\phi_x(V_x, W_x) = h_{\phi(x)}(V_x, W_x)$$

where $V, W \in C^\infty(\phi^{-1}TN)$. The metric is well defined since $V_x, W_x \in T_{\phi(x)}N$.

Proposition 1.9. *Let $\phi : (M, g) \to (N, h)$ be a C^∞-map. Then the pull-back connection ∇^ϕ is compatible with the metric h^ϕ on $\phi^{-1}TN$.*

Proof. [26] Let $X \in C^\infty(TM)$ and $V, W \in C^\infty(\phi^{-1}TN)$. At $x \in M$ let σ be a curve such that $\sigma(0) = x$ and $\dot\sigma(0) = X_x$. Then

$$\begin{aligned}
X_x(h^\phi(V,W)) =& \frac{d}{dt}\left(h^\phi_{\sigma(t)}(V_{\sigma(t)}, W_{\sigma(t)})\right)|_{t=0} \\
=& \frac{d}{dt}\left(h_{\phi\circ\sigma(t)}(V_{\sigma(t)}, W_{\sigma(t)})\right)|_{t=0} \\
=& \frac{d}{dt}\left(h_{\phi(x)}((P^N_{\phi\circ\sigma(t)})^{-1}(V_{\sigma(t)}), (P^N_{\phi\circ\sigma(t)})^{-1}(W_{\sigma(t)}))\right)|_{t=0} \\
=& h^\phi_x\left(\frac{d}{dt}\left((P^N_{\phi\circ\sigma(t)})^{-1}(V_{\sigma(t)})\right)|_{t=0}, W_x\right) \\
&+ h^\phi_x\left(V_x, \frac{d}{dt}\left((P^N_{\phi\circ\sigma(t)})^{-1}(W_{\sigma(t)})\right)|_{t=0}\right) \\
=& h^\phi_x(\nabla^\phi_X V(x), W_x) + h^\phi_x(V_x, \nabla^\phi_X W(x)).
\end{aligned}$$

□

1.3 The first variational formula

The variational formulation of a harmonic map is simple to understand but it is not useful for calculations. When it comes to calculations we use the first variational formula instead.

Definition 1.10. Given vector fields $\frac{d}{dt}$ and X on $(-\epsilon, \epsilon)$ and M, respectively, we define their **canonical extensions**, $\frac{\partial}{\partial t}$ and $(0, X)$, to the product $(\epsilon, \epsilon) \times M$ as follows:

Let $f \in C^\infty((-\epsilon, \epsilon) \times M)$ then

$$\frac{\partial}{\partial t}_{(t,x)}(f) = \frac{d}{ds}(f \circ \gamma)|_{s=0}$$

where $\gamma : s \mapsto (\sigma(s), x) \in (-\epsilon, \epsilon) \times M$, $\sigma(0) = t$, $\dot\sigma(0) = (\frac{d}{dt})_t$ and

$$(0, X)_{(t,x)}(f) = \frac{d}{ds}(f \circ \beta)|_{s=0}$$

where $\beta : s \mapsto (t, \tau(s)) \in (-\epsilon, \epsilon) \times M$, $\tau(0) = x$, $\dot\tau(0) = X_x$.

Lemma 1.11. *Let $\phi : M \to N$ be a C^∞-map. If $\Phi : (-\epsilon, \epsilon) \times M \to N$ is a smooth variation for ϕ then at $x \in M$*

$$d\Phi_{(0,x)}\left(\frac{\partial}{\partial t}_{(0,x)}\right) = V_x,$$

where V is the variational vector field for ϕ_t.

Proof. Let $f : N \to \mathbb{R}$ be a smooth function, then

$$\begin{aligned}
V_x(f) &= \frac{d}{dt}(\phi_t(x))|_{t=0}(f) \\
&= \frac{d}{dt}(f \circ \Phi(t, x))|_{t=0} \\
&= (\frac{\partial}{\partial t})_{(0,x)}(f \circ \Phi)
\end{aligned}$$

$$= d\Phi_{(0,x)}\left(\frac{\partial}{\partial t}_{(0,x)}\right)(f). \qquad \square$$

Lemma 1.12. *For any C^∞-map $\phi : (M,g) \to (N,h)$ and $X, Y \in C^\infty(TM)$ the pull-back connection ∇^ϕ on the pull-back bundle $\phi^{-1}TN$ satisfies*

$$\nabla^\phi_X(d\phi(Y)) - \nabla^\phi_Y(d\phi(X)) - d\phi([X,Y]) = 0.$$

Proof. [26] Define $T : C^\infty_2(TM) \to C^\infty(\phi^{-1}TN)$ by

$$T(X,Y) = \nabla^\phi_X(d\phi(Y)) - \nabla^\phi_Y(d\phi(X)) - d\phi([X,Y]).$$

Then for $f, g \in C^\infty(M)$

$$\begin{aligned}
T(fX,gY) =& \nabla^\phi_{fX}(d\phi(gY)) - \nabla^\phi_{gY}(d\phi(fX)) - d\phi([fX,gY]) \\
=& f\nabla^\phi_X(gd\phi(Y)) - g\nabla^\phi_Y(fd\phi(X)) \\
&- d\phi(fg[X,Y] + fX(g)Y - gY(f)X) \\
=& fg\nabla^\phi_X(d\phi(Y)) + fX(g)d\phi(Y) - fg\nabla^\phi_Y(d\phi(X)) - gY(f)d\phi(X) \\
&- fgd\phi([X,Y]) - fX(g)d\phi(Y) + gY(f)d\phi(X) \\
=& fgT(X,Y).
\end{aligned}$$

This means that T is tensorial so it is enough to show that

$$T\left(\frac{\partial}{\partial x_i^\alpha}, \frac{\partial}{\partial x_j^\alpha}\right) = 0$$

for charts (U, x^α) and (V, y^β) around $x \in M$ and $\phi(x) \in N$, respectively. With the notation $\phi^k = y_k^\beta \circ \phi$ and $\phi_i^k = \frac{\partial \phi^k}{\partial x_i^\alpha}$ we obtain

$$\begin{aligned}
T(\frac{\partial}{\partial x_i^\alpha}, \frac{\partial}{\partial x_j^\alpha}) =& \nabla^\phi_{\frac{\partial}{\partial x_i^\alpha}}(d\phi(\tfrac{\partial}{\partial x_j^\alpha})) - \nabla^\phi_{\frac{\partial}{\partial x_j^\alpha}}(d\phi(\tfrac{\partial}{\partial x_i^\alpha})) \\
=& \sum_{k=1}^m \nabla^\phi_{\frac{\partial}{\partial x_i^\alpha}} \phi_j^k \frac{\partial}{\partial y_k^\beta} - \sum_{k=1}^n \nabla^\phi_{\frac{\partial}{\partial x_j^\alpha}} \phi_i^k \frac{\partial}{\partial y_k^\beta} \\
=& \sum_{k=1}^n \left(\phi_{ji}^k \frac{\partial}{\partial y_k^\beta} + \phi_j^k \nabla^\phi_{\frac{\partial}{\partial x_i^\alpha}} \frac{\partial}{\partial y_k^\beta}\right) \\
&- \sum_{k=1}^n \left(\phi_{ij}^k \frac{\partial}{\partial y_k^\beta} + \phi_i^k \nabla^\phi_{\frac{\partial}{\partial x_j^\alpha}} \frac{\partial}{\partial y_k^\beta}\right) \\
=& \sum_{k=1}^n \left(\phi_{ji}^k - \phi_{ij}^k\right) \frac{\partial}{\partial y_k^\beta} \\
&+ \sum_{k=1}^n \left(\phi_j^k \nabla^\phi_{\frac{\partial}{\partial x_i^\alpha}} \frac{\partial}{\partial y_k^\beta} - \phi_i^k \nabla^\phi_{\frac{\partial}{\partial x_j^\alpha}} \frac{\partial}{\partial y_k^\beta}\right) \\
=& \sum_{k=1}^n \left(\phi_j^k \nabla^\phi_{\frac{\partial}{\partial x_i^\alpha}} \frac{\partial}{\partial y_k^\beta} - \phi_i^k \nabla^\phi_{\frac{\partial}{\partial x_j^\alpha}} \frac{\partial}{\partial y_k^\beta}\right)
\end{aligned}$$

since $\phi_{ij}^k = \phi_{ji}^k$. Finally since $\nabla^N_{\frac{\partial}{\partial y_k^\beta}} \frac{\partial}{\partial y_l^\beta} = \nabla^N_{\frac{\partial}{\partial y_l^\beta}} \frac{\partial}{\partial y_k^\beta}$ we have

$$\begin{aligned}\sum_{k=1}^n \left(\phi_j^k \nabla^\phi_{\frac{\partial}{\partial x_i^\alpha}} \frac{\partial}{\partial y_k^\beta} - \phi_i^k \nabla^\phi_{\frac{\partial}{\partial x_j^\alpha}} \frac{\partial}{\partial y_k^\beta} \right) &= \sum_{k=1}^n \sum_{l=1}^n \left(\phi_j^k \phi_i^l \nabla^N_{\frac{\partial}{\partial y_l^\beta}} \frac{\partial}{\partial y_k^\beta} - \phi_i^k \phi_j^l \nabla^N_{\frac{\partial}{\partial y_l^\beta}} \frac{\partial}{\partial y_k^\beta} \right) \\ &= \sum_{k=1}^n \sum_{l=1}^n \left(\phi_j^k \phi_i^l \nabla^N_{\frac{\partial}{\partial y_l^\beta}} \frac{\partial}{\partial y_k^\beta} - \phi_i^l \phi_j^k \nabla^N_{\frac{\partial}{\partial y_l^\beta}} \frac{\partial}{\partial y_k^\beta} \right) \\ &= 0.\end{aligned}$$

□

Given a C^∞-map $\phi : (M, g) \to (N, h)$ we define a tensor field $\hat{\nabla} d\phi$ on $C^\infty(TM)$ by

$$\hat{\nabla} d\phi(X, Y) = \nabla^\phi_X d\phi(Y) - d\phi(\nabla^M_X Y),$$

where $X, Y \in C^\infty(TM)$. Using Lemma 1.12 we see that the tensor field is symmetric.

Definition 1.13. Let $\phi : (M, g) \to (N, h)$ be a C^∞-map. Then we define the **tension field** $\tau(\phi)$ of ϕ by

$$\tau(\phi)(x) = \operatorname{trace}(\hat{\nabla} d\phi) = \sum_{i=1}^m \hat{\nabla} d\phi(X_i, X_i)$$

where $\{X_i\}_{i=1}^m$ is an orthonormal basis for $T_x M$.

Theorem 1.14 (The first variational formula [14, 11])**.** *Let $\phi : (M, g) \to (N, h)$ be a C^∞-map and suppose that M is compact. Then for any smooth variation ϕ_t of ϕ, we have*

$$\frac{d}{dt}(E(\phi_t))|_{t=0} = - \int_M h^\phi(V, \tau(\phi)) v_g,$$

where V is the variational vector field of ϕ_t. Thus ϕ is a harmonic map if and only if $\tau(\phi) \equiv 0$ on M.

Proof. [26] Let $x \in M$ and $\{X_i\}$ be a local orthonormal frame around x then

$$\frac{d}{dt} \left(h^\phi((d\phi_t)_x(X_i)_x, (d\phi_t)_x(X_i)_x) \right) = \left(\frac{\partial}{\partial t} \right)_{(t,x)} \left(h^\Phi(d\Phi(0, X_i), d\Phi(0, X_i)) \right).$$

Now since $[\frac{\partial}{\partial t}, (0, X_i)] = 0$ Lemma 1.12 tells us that

$$\nabla^\Phi_{\frac{\partial}{\partial t}} d\Phi(0, X_i) = \nabla^\Phi_{(0,X_i)} d\Phi \left(\tfrac{\partial}{\partial t} \right).$$

This implies that

$$\begin{aligned}\frac{\partial}{\partial t} \left(h^\Phi \left(d\Phi(0, X_i), d\Phi(0, X_i) \right) \right) =& 2h^\Phi(\nabla^\Phi_{\frac{\partial}{\partial t}} d\Phi(0, X_i), d\Phi(0, X_i)) \\ =& 2h^\Phi(\nabla^\Phi_{(0,X_i)} d\Phi \left(\tfrac{\partial}{\partial t} \right), d\Phi(0, X_i)) \\ =& 2(0, X_i) h^\Phi \left(d\Phi \left(\frac{\partial}{\partial t} \right), d\Phi(0, X_i) \right) \\ & - 2h^\Phi \left(d\Phi \left(\frac{\partial}{\partial t} \right), \nabla^\Phi_{(0,X_i)} d\Phi(0, X_i) \right).\end{aligned}$$

Let $X_t \in C^\infty(TM)$ be given by

$$g(X_t, Y) = h^\Phi(d\Phi\left(\frac{\partial}{\partial t}\right), d\Phi(0, Y))$$

for all $Y \in C^\infty(TM)$. Then

$$\begin{aligned}
\sum_{i=1}^m (0, X_i)\left(h^\Phi\left(d\Phi\left(\frac{\partial}{\partial t}\right), d\Phi(0, X_i)\right)\right) =& \sum_{i=1}^m X_i g(X_t, X_i) \\
=& \sum_{i=1}^m \left(g(\nabla^M_{X_i} X_t, X_i) + g(X_t, \nabla^M_{X_i} X_i)\right) \\
=& \mathrm{div}(X_t) \\
& + \sum_{i=1}^m h^\Phi\left(d\Phi\left(\frac{\partial}{\partial t}\right), d\Phi(0, \nabla^M_{X_i} X_i)\right).
\end{aligned}$$

Now since

$$\frac{d}{dt}(E(\phi_t)) = \frac{1}{2}\int_M \sum_{i=1}^m \frac{d}{dt} h^\phi(d\phi_t(X_i), d\phi_t(X_i)) v_g$$

we get

$$\begin{aligned}
&\frac{d}{dt}(E(\phi_t)) \\
&= \int_M \mathrm{div}(X_t) v_g - \int_M h^\Phi\left(d\Phi\left(\frac{\partial}{\partial t}\right), \sum_{i=1}^m \left(\nabla^\Phi_{(0,X_i)} d\Phi(0, X_i) - d\Phi(0, \nabla^M_{X_i} X_i)\right)\right) v_g.
\end{aligned}$$

The first term is zero by Stokes' theorem, letting $t = 0$ we get

$$\begin{aligned}
&\frac{d}{dt}(E(\phi_t))|_{t=0} \\
&= -\int_M h^{\Phi|_{t=0}}\left(d\Phi\left(\frac{\partial}{\partial t}\right)|_{t=0}, \sum_{i=1}^m \left(\nabla^\Phi_{(0,X_i)} d\Phi(0, X_i)|_{t=0} - d\Phi(0, \nabla^M_{X_i} X_i)|_{t=0}\right)\right) v_g.
\end{aligned}$$

From

$$\begin{aligned}
\Phi|_{t=0} &= \phi, \\
d\Phi\left(\frac{\partial}{\partial t}\right)|_{t=0} &= V, \\
\nabla^\Phi_{(0,X_i)} d\Phi(0, X_i)|_{t=0} &= \nabla^\phi_{X_i} d\phi(X_i) \text{ and} \\
d\Phi(0, \nabla^M_{X_i} X_i)|_{t=0} &= d\phi(\nabla^M_{X_i} X_i)
\end{aligned}$$

we have

$$\frac{d}{dt}(E(\phi_t))|_{t=0} = -\int_M h^\phi(V, \tau(\phi)) v_g.$$

The last part of the statement follows from the fact that V_x can be chosen arbitrary using ϕ_t. $\square$

The condition $\tau(\phi) \equiv 0$ does not depend on M being compact. Thus can $\tau(\phi) \equiv 0$ be used to define harmonic maps for non-compact Riemannian manifolds.

Some further properties of harmonic maps, whose proof are based on the theory for partial differential equations are.

Theorem 1.15 ([11] p. 117)**.** *Let $\phi : (M, g) \to (N, h)$ be a C^2-map such that $\tau(\phi) \equiv 0$. If M and N are real analytic manifolds then ϕ is real analytic.*

Theorem 1.16 ([23] p. 216)**.** *Let $\phi : (M, g) \to (N, h)$ be a harmonic map. If there exist an open subset U of M such that the image $\phi(U)$ is contained in a complete totally geodesic submanifold V of N then $\phi(M) \subset V$.*

1.4 Isometric immersions and Riemannian submersions

In this section we define isometric immersions and Riemannian submersions. We show that a Riemannian submersion has some nice properties that will be important in Chapter 3.

Definition 1.17. A C^∞-map $\phi : (M, g) \to (N, h)$ is said to be an **isometric immersion** if for all $x \in M$ and all $X_x, Y_x \in T_xM$

(i) the tangent map $d\phi_x : T_xM \to T_{\phi(x)}N$ is injective and

(ii) $g_x(X_x, Y_x) = h_{\phi(x)}(d\phi_x(X_x), d\phi_x(Y_x))$.

For each $x \in M$ we have an orthogonal decomposition of the tangent space

$$T_{\phi(x)}N = d\phi(T_xM) \oplus (d\phi(T_xM))^\perp$$

with respect to the metric $h_{\phi(x)}$. Let U be a neighbourhood of x such that $\phi : U \to N$ is an embedding. Then for $X \in C^\infty(TU)$

$$d\phi(X) : \phi(U) \ni \phi(x) \mapsto d\phi_x(X_x) \in T_{\phi(x)}N$$

will be a well-defined vector field in $C^\infty(T\phi(U))$ which we extend to $C^\infty(TN)$. We decompose

$$\nabla^N_{d\phi(X)} d\phi(Y)(\phi(x)) = \left(\nabla^N_{d\phi(X)} d\phi(Y)(\phi(x))\right)^T + B(X, Y)(x)$$

according to the orthogonal decomposition of $T_{\phi(X)}N$ defined above. It is easy to show that B is a symmetric tensor.

Definition 1.18. Let $\phi : (M, g) \to (N, h)$ be an isometric immersion. Then we call the map

$$B : T_xM \times T_xM \to (d\phi(T_xM))^\perp$$

the **second fundamental form** of ϕ.

The fact that B is a tensor insures that the following definition makes sense.

Definition 1.19. An isometric immersion ϕ is said to be **minimal** if for all $x \in M$

$$\operatorname{trace}(B)(x) = \sum_{i=1}^{m} B(X_i, X_i) = 0$$

for any orthonormal basis $\{X_i\}_{i=1}^m$ of T_xM.

Proposition 1.20 ([11])**.** *An isometric immersion $\phi : (M, g) \to (N, h)$ is minimal if and only if it is harmonic.*

Proof. [2] Let $x \in M$ and let U be a neighbourhood such that $\phi : U \to N$ is injective. Also let $X, Y \in C^\infty(TU)$. Then we have

$$\nabla^\phi_X d\phi(Y)(x) = \nabla^N_{d\phi(X)} d\phi(Y)(\phi(x))$$

and since ϕ is an isometric immersion we have

$$d\phi_x(\nabla^M_X Y(x)) = \left(\nabla^N_{d\phi(X)} d\phi(Y)(\phi(x))\right)^T.$$

This implies that

$$\begin{aligned} B(X,Y)(x) =& \nabla^N_{d\phi(X)} d\phi(Y)(\phi(x)) - \left(\nabla^N_{d\phi(X)} d\phi(Y)(\phi(x))\right)^T \\ =& \nabla^\phi_X d\phi(Y)(x) - d\phi_x(\nabla^M_X Y(x)) = \hat{\nabla} d\phi(X,Y)(x). \end{aligned}$$

The statement follows from the fact that

$$\tau(\phi) = \text{trace}(\hat{\nabla}\phi) = \text{trace}(B).$$

□

Given a C^∞-map $\phi : (M, g) \to N$ we define, at a point $x \in M$, the **vertical** space

$$\mathcal{V}_x = \text{Ker}(d\phi_x) = \{X_x \in T_xM \mid d\phi_x(X_x) = 0\}$$

and the **horizontal** space

$$\mathcal{H}_x = \{X_x \in T_xM \mid g(X_x, Y_x) = 0 \text{ for all } Y_x \in \mathcal{V}_x\}$$

as the orthogonal complement $\mathcal{V}_x^\perp$ of $\mathcal{V}_x$. This gives us an orthogonal decomposition

$$T_xM = \mathcal{V}_x \oplus \mathcal{H}_x$$

of the tangent space T_xM.

Definition 1.21. A surjective map $\phi : (M, g) \to (N, h)$ is said to be a **Riemannian submersion** if for all $x \in M$

(i) $d\phi_x : T_xM \to T_{\phi(x)}N$ is surjective and

(ii) $d\phi_x|_{\mathcal{H}_x}$ is an isometric isomorphism of $(\mathcal{H}_x, g_x)$ to $(T_{\phi(x)}N, h_{\phi(x)})$.

Definition 1.22. Let $\phi : (M, g) \to (N, h)$ be a Riemannian submersion. A vector field $\hat{X} \in C^\infty(TM)$ is said to be a **horizontal lift** of a vector field $X \in C^\infty(TN)$ if

$$\hat{X}_x \in \mathcal{H}_x$$

and $\hat{X}$ and X are ϕ-related i.e.

$$d\phi_x(\hat{X}_x) = X_{\phi(x)}.$$

It follows from the definition of a Riemannian submersion that

$$d\phi_x|_{\mathcal{H}_x} : \mathcal{H}_x \to T_{\phi(x)}N$$

is an isomorphism so there exist a unique horizontal lift for each $X \in C^\infty(TN)$.

Lemma 1.23 ([21]). *Let $\phi : (M, g) \to (N, h)$ be a Riemannian submersion and $\hat{X}, \hat{Y} \in C^\infty(TM)$ be horizontal lifts of $X, Y \in C^\infty(TN)$. Then we have*

(i) $g_x(\hat{X}_x, \hat{Y}_x) = h_{\phi(x)}(X_{\phi(x)}, Y_{\phi(x)})$,

(ii) $d\phi([\hat{X}, \hat{Y}]) = [X, Y]$,

(iii) $d\phi(\nabla^M_{\hat{X}}\hat{Y}) = \nabla^N_X Y$.

Proof. [21] (i) The statement follows from the fact that $d\phi_x|_{\mathcal{H}_x}$ is an isometry, i.e.

$$g_x(\hat{X}_x, \hat{Y}_x) = h_{\phi(x)}(d\phi_x(\hat{X}_x), d\phi_x(\hat{X}_x)) = h_{\phi(x)}(X_{\phi(x)}, Y_{\phi(x)}).$$

(ii) This statement follows from the fact that $\hat{X}$ and X are ϕ-related,

$$d\phi([\hat{X}, \hat{Y}]) = [d\phi(\hat{X}), d\phi(\hat{Y})] = [X, Y].$$

(iii) We show that for any vector field $Z \in C^\infty(TN)$ we have

$$h(d\phi(\nabla^M_{\hat{X}}\hat{Y}), Z) \circ \phi = h(\nabla^N_X Y, Z) \circ \phi.$$

For all $Z_1, Z_2, Z_3 \in C^\infty(TN)$ we have

$$\begin{aligned}
(\hat{Z}_1)_x(g(\hat{Z}_2, \hat{Z}_3)) &= (\hat{Z}_1)_x(h(Z_2, Z_3) \circ \phi) = (Z_1)_{\phi(x)}(h(Z_2, Z_3)) \\
g(\hat{Z}_1, [\hat{Z}_2, \hat{Z}_3]) &= g(\hat{Z}_1, \widehat{[Z_2, Z_3]}) = h(Z_1, [Z_2, Z_3]) \circ \phi.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
h(d\phi(\nabla^M_{\hat{X}}\hat{Y}), Z) \circ \phi =& h(d\phi(\nabla^M_{\hat{X}}\hat{Y}), d\phi(\hat{Z})) \circ \phi \\
=& g(\nabla^M_{\hat{X}}\hat{Y}, \hat{Z}) \\
=& \frac{1}{2}\Big(\hat{X}\big(g(\hat{Y}, \hat{Z})\big) + \hat{Y}\big(g(\hat{Z}, \hat{X})\big) - \hat{Z}g\big((\hat{X}, \hat{Y})\big) \\
& + g(\hat{Z}, [\hat{X}, \hat{Y}]) + g(\hat{Y}, [\hat{Z}, \hat{X}]) - g(\hat{X}, [\hat{Y}, \hat{Z}])\Big) \\
=& \frac{1}{2}\Big(X\big(h(Y, Z)\big) + Y\big(h(Z, X)\big) - Z\big(h(X, Y)\big) \\
& + h(Z, [X, Y]) + h(Y, [Z, X]) - h(X, [Y, Z])\Big) \circ \phi \\
=& h(\nabla^N_X Y, Z) \circ \phi.
\end{aligned}$$

□

Definition 1.24. Let $\pi : (Y, g') \to (N, h)$ be a Riemannian submersion. A C^∞-map $\psi : (M, g) \to (Y, g')$ is said to be **horizontal** with respect to π if

$$d\psi_x(T_xM) \subset \mathcal{H}_{\psi(x)}$$

for all $x \in M$ where $T_yY = \mathcal{H}_y \oplus V_y$ for $y \in Y$.

Usually the composition of a harmonic map with another map does not give an harmonic map, even in the case where both maps are harmonic. The next theorem will give us a condition for a Riemannian submersion for preserve the harmonicity under compositions.

Theorem 1.25 ([12])**.** *Let $\pi : (Y, g') \to (N, h)$ be a Riemannian submersion. If $\psi : (M, g) \to (Y, g')$ is a harmonic map which is horizontal with respect to π, then the composition $\phi = \pi \circ \psi : (M, g) \to (N, h)$ is a harmonic map.*

Proof. [26] Let $\{X_i\}_{i=1}^m$ be an orthonormal frame on (M, g). We have

$$\begin{aligned}\tau(\phi)(x) =& \sum_{i=1}^{m} \left(\nabla^{\phi}_{X_i} d\phi(X_i)(x) - d\phi_x \left(\nabla^M_{X_i} X_i(x) \right) \right) \\ =& \sum_{i=1}^{m} \left(\nabla^N_{d\pi(d\psi(X_i))} d\pi(d\psi(X_i))(\phi(x)) - d\pi_{\psi(x)} \left(d\psi_x \left(\nabla^M_{X_i} X_i(x) \right) \right) \right).\end{aligned}$$

Since $d\psi_x((X_i)_x) \in \mathcal{H}_{\psi(x)}$ for all $x \in M$, $d\psi(X_i)$ is horizontal lift of $d\pi(d\psi(X_i))$. By Lemma 1.23 we have

$$\nabla^N_{d\pi(d\psi(X_i))} d\pi(d\psi(X_i)) = d\pi(\nabla^Y_{d\psi(X_i)} d\psi(X_i)) = d\pi(\nabla^{\psi}_{X_i} d\psi(X_i)).$$

Using this for $\tau(\phi)$ we get

$$\begin{aligned}\tau(\phi) &= d\pi \left(\sum_{i=1}^{m} \left(\nabla^{\psi}_{X_i} d\psi(X_i) - d\psi(\nabla^M_{X_i} X_i) \right) \right) \\ &= d\pi(\tau(\psi)) = 0\end{aligned}$$

the last equality follows from the fact that ψ is harmonic. $\square$

Chapter 2

Kähler manifolds

2.1 Basic properties

The concept of a Kähler manifold is an essential part of Calabi's classification. To define Kähler manifolds we need the notion of a complex manifold. We will show some important formulas for Kähler manifolds and give a few examples. The presentation of the theory in this section builds on [3], [18] and [26].

We start with some motivations. Let V be a complex vector space then we define a map $J : V \to V$ by

$$J(v) = iv$$

for $v \in V$. This satisfies $J^2 = -I$. Now instead let V be a real vector space then a map $J : V \to V$ is said to be a **complex structure** if

$$J^2 = -I.$$

J turns V into a complex vector space by defining

$$(a + ib)v = av + bJv.$$

This can only be done if the dimension of V is even.

Definition 2.1. Let M be a $2m$-dimensional manifold. Then M is said to be a **complex manifold** if there exist an atlas of **complex charts**

$$z^\alpha : U_\alpha \to \mathbb{C}^m$$

such that the transition maps

$$z^\alpha \circ (z^\beta)^{-1} : z^\beta(U_\alpha \cap U_\beta) \to \mathbb{C}^m$$

are holomorphic.

Given a complex manifold the complex charts can be decomposed

$$z_k^\alpha = x_k^\alpha + iy_k^\alpha$$

for $k = 1, \ldots, m$. We define real charts by

$$(x_1^\alpha, \ldots, x_m^\alpha, y_1^\alpha, \ldots, y_m^\alpha).$$

Then at a point $p \in M$ the set $\{X_i = \frac{\partial}{\partial x_i^\alpha}, Y_i = \frac{\partial}{\partial y_i^\alpha}, \mid i = 1, \dots, m\}$ is a basis for T_pM as a real manifold. We define $J_p : T_pM \to T_pM$ by

$$J_p(X_i) = Y_i \text{ and } J_p(Y_i) = -X_i.$$

Then this satisfies $J_p^2 = -I_p$. This tensor field is independent of the choice of charts and is called the **complex structure** of M.

Definition 2.2. Let M be a $2m$-dimensional manifold. Then a tensor field $J : C^\infty(TM) \to C^\infty(TM)$ is said to be an **almost complex structure** on M if

$$J^2 = -I,$$

where I is the identity map on $C^\infty(TM)$.

Given an almost complex structure J on a manifold M the **Nijenhuis tensor** N of J is defined by

$$N(X,Y) = 2([JX, JY] - [X,Y] - J[X, JY] - J[JX, Y]).$$

This tensor will let us determine when an almost complex manifold is a complex manifold.

Theorem 2.3 ([20])**.** *Let M be a $2m$-dimensional manifold with almost complex structure J. Then M is a complex manifold with complex structure J if and only if the Nijenhuis tensor of J is zero.*

The proof is complicated and is omitted here. See [18] for a proof in the real analytic case.

We note that the covariant derivate $\nabla^M J : C_2^\infty(TM) \to C^\infty(TM)$ of J is defined by

$$\nabla^M J(X,Y) = \nabla_X^M (JY) - J(\nabla_X^M Y).$$

Definition 2.4. Let M be a complex manifold with complex structure J. Then a Riemannian metric g is said to be **compatible** with J if

$$g(JX, JY) = g(X,Y)$$

for all $X, Y \in C^\infty(TM)$. A complex manifold with a compatible metric is said to be a **Hermitian manifold**.

Example 2.5. On $\mathbb{C}^m$ we have the Hermitian metric $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ defined by

$$\langle u, v \rangle_{\mathbb{C}} = \sum_{i=1}^m u_i \bar{v}_i.$$

Definition 2.6. Let (M, J, g) be a Hermitian manifold. The alternating 2-form

$$\omega(X,Y) = g(JX, Y)$$

is called the **Kähler form** of g. The metric g is said to be a **Kähler metric** if ω is closed i.e. $d\omega = 0$.

Lemma 2.7. *Let (M, J, g) be a Hermitian manifold, with Levi-Civita connection ∇^M. Then*

$$d\omega(X,Y,Z) = g(\nabla^M J(X,Y),Z) + g(\nabla^M J(Y,Z),X) + g(\nabla^M J(Z,X),Y)$$

$$2g(\nabla^M J(X,Y),Z) = d\omega(X,Y,Z) - d\omega(X,JY,JZ)$$

for all $X, Y, Z \in C^\infty(TM)$.

Proof. [3, 18] By the invariant formula for the exterior derivative of differential forms

$$\begin{aligned}
d\omega(X,Y,Z) =& X(\omega(Y,Z)) + Y(\omega(Z,X)) + Z(\omega(X,Y)) \\
&+ \omega(Z,[X,Y]) + \omega(Y,[Z,X]) + \omega(X,[Y,Z]) \\
=& X(g(JY,Z)) + Y(g(JZ,X)) + Z(g(JX,Y)) \\
&+ g(JZ,[X,Y]) + g(JY,[Z,X]) + g(JX,[Y,Z]) \\
=& \big(g(\nabla^M_X JY,Z) + g(JY,\nabla^M_X Z)\big) \\
&+ \big(g(\nabla^M_Y JZ,X) + g(JZ,\nabla^M_Y X)\big) \\
&+ \big(g(\nabla^M_Z JX,Y) + g(JX,\nabla^M_Z Y)\big) \\
&+ g(JZ,[X,Y]) + g(JY,[Z,X]) + g(JX,[Y,Z]) \\
=& \big(g(\nabla^M_X JY,Z) + g(JZ,\nabla^M_Y X) + g(JZ,[X,Y])\big) \\
&+ \big(g(\nabla^M_Y JZ,X) + g(JX,\nabla^M_Z Y) + g(JX,[Y,Z])\big) \\
&+ \big(g(\nabla^M_Z JX,Y) + g(JY,\nabla^M_X Z) + g(JY,[Z,X])\big) \\
=& g(\nabla^M J(X,Y),Z) + g(\nabla^M J(Y,Z),X) + g(\nabla^M J(Z,X),Y).
\end{aligned}$$

Proving the first part. Further

$$\begin{aligned}
d\omega(X,JY,JZ) =& X(\omega(JY,JZ)) + JY(\omega(JZ,X)) + JZ(\omega(X,JY)) \\
&+ \omega(JZ,[X,JY]) + \omega(JY,[JZ,X]) + \omega(X,[JY,JZ]) \\
=& X(g(-Y,JZ)) + JY(g(-Z,X)) + JZ(g(JX,JY)) \\
&+ g(-Z,[X,JY]) + g(-Y,[JZ,X]) + g(JX,[JY,JZ]).
\end{aligned}$$

Which implies that

$$\begin{aligned}
&d\omega(X,Y,Z) - d\omega(X,JY,JZ) = \\
&\quad= \big(X(g(JY,Z)) + Y(g(JZ,X)) + Z(g(JX,Y)) \\
&\qquad + g(JZ,[X,Y]) + g(JY,[Z,X]) + g(JX,[Y,Z])\big) \\
&\qquad - \big(X(g(-Y,JZ)) + JY(g(-Z,X)) + JZ(g(JX,JY)) \\
&\qquad + g(-Z,[X,JY]) + g(-Y,[JZ,X]) + g(JX,[JY,JZ])\big) \\
&\qquad + \big(g(X,[JY,Z]) - g(X,[JY,Z])\big) \\
&\qquad + \big(g(X,[Y,JZ]) - g(X,[Y,JZ])\big) \\
&\quad= \big(X(g(JY,Z)) + JYg(Z,X) - Zg(X,JY) \\
&\qquad - g(X,[JY,Z]) + g(JY,[Z,X]) + g(Z,[X,JY])\big) \\
&\qquad + \big(X(g(Y,JZ)) + Y(g(JZ,X)) - JZ(g(X,Y)) \\
&\qquad - g(X,[Y,JZ]) + g(Y,[JZ,X]) + g(JZ,[X,Y])\big)
\end{aligned}$$

$$
\begin{aligned}
&- \big(g(JX,[JY,JZ]) - g(JX,[Y,Z]) \\
&- g(JX,J[JY,Z]) - g(JX,J[Y,JZ])\big) \\
=&2g(\nabla^M_X JY, Z) + 2g(\nabla^M_X Y, JZ) - \frac{1}{2} g(JX, N(Y,Z)),
\end{aligned}
$$

where $N(Y,Z)$ is the Nijenhuis tensor. The Nijenhuis tensor is zero since J is a complex structure so the theorem follows from the fact that

$$
g(\nabla^M_X Y, JZ) = -g(J\nabla^M_X Y, Z). \qquad \square
$$

Theorem 2.8. *Let (M,J,g) be a Hermitian manifold with Levi-Civita connection ∇^M. Then the metric g is Kähler if and only if the complex structure is parallel i.e. $\nabla^M J = 0$.*

Proof. [3] Assume that g is Kähler i.e. the Kähler form ω satisfies $d\omega = 0$. Then by Lemma 2.7

$$
g(\nabla^M J(X,Y), Z) = 0
$$

for all $X, Y, Z \in C^\infty(TM)$ so $\nabla^M J = 0$.

Now we assume instead that $\nabla^M J = 0$ which by Lemma 2.7 implies that

$$
d\omega(X,Y,Z) = 0
$$

for all $X, Y, Z \in C^\infty(TM)$. Hence the Kähler form is closed. $\square$

Corollary 2.9. *Let (M,J,g) be a Kähler manifold, then*

$$
\nabla^M_X (JY) = J(\nabla^M_X Y)
$$

for all $X, Y \in C^\infty(TM)$.

Proof. The statement follows from the definition of $\nabla^M J$ and Theorem 2.8. $\square$

Definition 2.10. A map $\phi : M \to N$ between complex manifolds is said to be holomorphic if $J^N \circ d\phi = d\phi \circ J^M$.

Proposition 2.11 ([11])**.** *Any holomorphic map $\phi : M \to N$ between two Kähler manifolds is harmonic.*

Proof. [26] Let J^M and J^N be the complex structures on M and N, respectively, and $\{X_i, Y_i\}_{i=1}^m$ be a local orthonormal frame for $C^\infty(TM)$ with $J^M X_i = Y_i$ and $J^M Y_i = -X_i$. Then

$$
\begin{aligned}
\nabla^\phi_{Y_i} d\phi(Y_i) - d\phi(\nabla^M_{Y_i} Y_i) =&\nabla^\phi_{J^M X_i} d\phi(J^M X_i) - d\phi(\nabla^M_{J^M X_i} J^M X_i) \\
=&\nabla^\phi_{J^M X_i} J^N d\phi(X_i) - d\phi(J^M \nabla^M_{J^M X_i} X_i) \\
=&J^N \nabla^\phi_{J^M X_i} d\phi(X_i) - J^N d\phi(\nabla^M_{J^M X_i} X_i) \\
=&J^N (\nabla^\phi_{J^M X_i} d\phi(X_i) - d\phi(\nabla^M_{J^M X_i} X_i)) \\
=&J^N (\nabla^\phi_{X_i} d\phi(J^M X_i) + d\phi([J^M X_i, X_i]) \\
&- d\phi(\nabla^M_{X_i} J^M X_i + [J^M X_i, X_i])) \\
=&J^N (J^N \nabla^\phi_{X_i} d\phi(X_i) - J^N d\phi(\nabla^M_{X_i} X_i)) \\
=&- \nabla^\phi_{X_i} d\phi(X_i) + d\phi(\nabla^M_{X_i} X_i).
\end{aligned}
$$

From this we see that

$$\tau(\phi) = \sum_{i=1}^{m} (\nabla^{\phi}_{Y_i} d\phi(Y_i) - d\phi(\nabla^{M}_{Y_i} Y_i) + \nabla^{\phi}_{X_i} d\phi(X_i) - d\phi(\nabla^{M}_{X_i} X_i)) = 0. \qquad \square$$

Example 2.12. Let V be a vector space. We define an equivalence relation R on $V \backslash \{0\}$ by

$R(u, v)$ if and only if there exists a non-zero scalar λ such that $u = \lambda v$.

The quotient space V/R is called the **projective space** of V and is denoted by $\mathbb{P}(V)$. We define the projection

$$\pi_{\mathbb{P}} : V \backslash \{0\} \ni u \mapsto R(u) \in \mathbb{P}(V).$$

The n-dimensional **complex projective space** $\mathbb{CP}^n$ is defined by $\mathbb{CP}^n = \mathbb{P}(\mathbb{C}^{n+1})$. $\mathbb{CP}^n$ can be shown to be a compact complex manifold. On $\mathbb{CP}^n$ there is a standard Kähler metric called the **Fubini-Study** metric defined to be the unique metric such that the projection $\pi_{\mathbb{P}} : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{CP}^n$ restricted to $S^{2n+1} \subset \mathbb{C}^{n+1}$ is a Riemannian submersion.

Definition 2.13. For $\mathbb{C}^n$ we define the k-**wedge product** $\bigwedge^k \mathbb{C}^n$ by

$$\bigwedge^k \mathbb{C}^n = \{ \sum_{j=1}^{l<\infty} a_1^j \wedge \cdots \wedge a_k^j \,|\, a_i^j \in \mathbb{C}^n \; i = 1, \ldots, k \},$$

where

(i) $(\cdot \wedge \cdots \wedge \cdot)$ is multi-linear,

(ii) $a_1 \wedge \cdots \wedge a_k \neq 0$ if $a_1, \ldots, a_k$ are linearly independent, and

(iii) $a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(k)} = \mathrm{sgn}(\sigma)(a_1 \wedge \cdots \wedge a_k)$ for any permutation σ of $\{1, \ldots, k\}$.

Proposition 2.14. *The wedge product $\bigwedge^k \mathbb{C}^n$ has the following properties.*

(i) The vectors $a_1, \ldots, a_k \in \mathbb{C}^n$ are linearly dependent if and only if $a_1 \wedge \cdots \wedge a_k = 0$.

(ii) $\bigwedge^k \mathbb{C}^n$ is an $\binom{n}{k}$-dimensional complex vector space.

Proof. (i) Suppose that the vectors are linearly dependent. Then for some $j \in \{1, \ldots, k\}$ we have

$$a_j = \sum_{i=1, i \neq j}^{k} \lambda_i a_i.$$

So it is enough to show that $a_1 \wedge \cdots \wedge a_k = 0$ if $a_i = a_j$ for some $i \neq j$. Let σ be the permutation such that $\sigma(i) = j$, $\sigma(j) = i$ and all other integers are left fixed. Then $\mathrm{sgn}(\sigma) = -1$ and

$$\begin{aligned} A =& a_1 \wedge \cdots \wedge a_i \wedge \cdots \wedge a_j \wedge \cdots \wedge a_k \\ =& - a_1 \wedge \cdots \wedge a_{\sigma(i)} \wedge \cdots \wedge a_{\sigma(j)} \wedge \cdots \wedge a_k \\ =& - a_1 \wedge \cdots \wedge a_j \wedge \cdots \wedge a_i \wedge \cdots \wedge a_k \\ =& - a_1 \wedge \cdots \wedge a_i \wedge \cdots \wedge a_j \wedge \cdots \wedge a_k \end{aligned}$$

$$= - A.$$

This implies that $2A = 0$.

Suppose instead that $a_1 \wedge \cdots \wedge a_k = 0$ then by definition $a_1, \ldots, a_k$ are linearly dependent.

(ii) Let $\{e_i\}_{i=1}^n$ be a basis for $\mathbb{C}^n$, if the same basis vector appears more than once in a wedge product the product is zero, and any permutation of a wedge product be linearly dependent with the original so the basis vectors for $\bigwedge^k \mathbb{C}^n$ can be chosen in $\binom{n}{k}$ ways. □

An element of $A \in \bigwedge^k \mathbb{C}^n$ is said to be **decomposable** if $A = a_1 \wedge \cdots \wedge a_n$ for vectors $a_1, \ldots, a_n \in \mathbb{C}^n$. On the decomposable elements of $\bigwedge^k(\mathbb{C}^n)$ we define a scalar product by

$$\langle A, B\rangle = \det\left([\langle a_i, b_j\rangle_{\mathbb{C}}]_{i,j=1}^k\right)$$

where $A = a_1 \wedge \cdots \wedge a_k$ and $B = b_1 \wedge \cdots \wedge b_k$. Since $\bigwedge^k \mathbb{C}^n$ is an $\binom{n}{k}$-dimensional complex vector space we can identify

$$\mathbb{P}(\bigwedge^k \mathbb{C}^n) \cong \mathbb{CP}^{N-1},$$

where $N = \binom{n}{k}$.

Example 2.15. The set

$$G_k(\mathbb{C}^n) = \{V \subset \mathbb{C}^n | \, V \text{ is a } k\text{-dimensional subspace}\}$$

is called the **Grassmannian** of k-planes in $\mathbb{C}^n$. It is known the Grassmannian can be equipped with a compact complex manifold structure. It is also known that the tangent space of $G_k(\mathbb{C}^n)$ at a point V is equal to the space of complex linear maps $\mathrm{Hom}_{\mathbb{C}}(V, V^{\perp})$.

Let $\{a_i\}_{i=1}^k$ and $\{b_j\}_{j=1}^k$ be two different bases for $V \in G_k(\mathbb{C}^n)$ then

$$a_1 \wedge \cdots \wedge a_k = \lambda b_1 \wedge \cdots \wedge b_k$$

for some $\lambda \in \mathbb{C}\backslash 0$ so they generate the same line in $\bigwedge^k \mathbb{C}^n$, thus the map $\mathrm{pl} : G_k(\mathbb{C}^n) \to \mathbb{CP}^{N-1}$, where $N = \binom{n}{k}$, defined by

$$\mathrm{pl}(V) = \pi_{\mathbb{P}}(a_1 \wedge \cdots \wedge a_k)$$

is a well defined injective map.

The map pl can be shown to be a embedding see [13] and is called the **Plücker** embedding.

2.2 Homogeneous spaces

A particularly easy type of manifolds are the homogeneous spaces. On a homogeneous space we define G-invariant structures they are determined by their behavior at a single point of the manifold. We present methods to construct G-invariant metrics and almost complex structures, and to determine when an almost complex structure has a vanishing Nijenhuis tensor. The main sources are [1], [18] and [28].

Throughout this section we identify the Lie algebra $\mathfrak{g}$ of a Lie group G and its tangent space T_eG at the identity element e without reference.

Let G be a group and K be a subgroup. Define the **quotient space** G/K by

$$G/K = \{\, gK \,|\, g \in G \,\}.$$

The map defined by

$$\pi : G \ni g \mapsto gK \in G/K$$

is called the **natural projection**.

Theorem 2.16. *Let G be a Lie group and K a closed Lie subgroup, then the quotient G/K has a unique manifold structure such that*

(i) the natural projection π is a C^∞-map,

(ii) for all $gK \in G/K$ there is a neighbourhood W of gK and a C^∞-map $\sigma : W \to G$ such that $\pi \circ \sigma = \mathrm{id}_{G/K}$.

See [27] for a proof. On each of the neighbourhoods W we have $\pi \circ \sigma = \mathrm{id}_{G/K}$ which implies that

$$d\pi_g \circ d\sigma_{gK} = \mathrm{id}_{T_{gK}G/K}$$

this shows that $d\pi_g : T_gG \to T_{gK}G/K$ is surjective so π is a submersion. The manifold G/K is said to be a **homogeneous space**.

Definition 2.17. Let G be a group, M be a set. Let o_M be an element of M. Then an action $G \times M \to M$ is said to be **transitive** if $M = \{g \cdot o_M | g \in G\}$. The set

$$K = \{\, g \in G \,|\, g \cdot o_M = o_M \,\}$$

is called the **isotropy** subgroup of G at the **origin** o_M.

Theorem 2.18. *Let G be a Lie group and $G \times M \to M$ be a transitive C^∞-action on a manifold M. If K is the isotropy subgroup at o_M then the map $\beta : G/K \to M$ defined by*

$$\beta(gK) = g \cdot o_M$$

is a diffeomorphism.

Again see [27] for a proof.

Definition 2.19. Let G be a Lie group and K be a subgroup of G. Then for $g \in G$ we define the **translation** $\tau_g : G/K \to G/K$ by

$$\tau_g(hK) = (gh)K$$

for $hK \in G/K$.

The map τ_g is a diffeomorphism and if we denote left translation in G by L_g then we have the following commutative diagrams

$$\begin{tikzcd} G \arrow[r, "L_g"] \arrow[d, "\pi"'] & G \arrow[d, "\pi"] \\ G/K \arrow[r, "\tau_g"'] & G/K \end{tikzcd} \qquad \begin{tikzcd} T_hG \arrow[r, "(dL_g)_h"] \arrow[d, "(d\pi)_h"'] & T_{gh}G \arrow[d, "(d\pi)_{gh}"] \\ T_{hK}(G/K) \arrow[r, "(d\tau_g)_{hK}"'] & T_{ghK}(G/K) \end{tikzcd}$$

for all $g, h \in G$.

Definition 2.20. Let $M = G/K$ be a homogeneous space and $x = hK \in G/K$. Then

(i) a metric g on M is said to be G-**invariant** if

$$g_{\tau_g(x)}((d\tau_g)_x X_x, (d\tau_g)_x Y_x) = g_x(X_x, Y_x)$$

and

(ii) an almost complex structure J on M is said to be G-**invariant** if

$$(J_{\tau_g(x)} \circ (d\tau_g)_x)(X_x) = ((d\tau_g)_x \circ J_x)(X_x)$$

for all $g \in G$ and all $X_x, Y_x \in T_x M$.

It should be noted that this implies that τ_g is an isometry for all $g \in G$. Further if G/K is equipped with an G-invariant almost complex structure then the translations τ_g are holomorphic for all $g \in G$.

For the homogeneous space $M = G/K$ we have the natural projection $\pi : G \to G/K$ with $\pi(e) = K \cong o_M$. Let $\exp(tX)$ be a one-parameter subgroup of G where $X \in \mathfrak{g}$. Then

$$\frac{d}{dt}(\exp(tX))|_{t=0} = X$$

and $d\pi_e : \mathfrak{g} \to T_o M$ is given by

$$d\pi_e(X) = \frac{d}{dt}(\pi(\exp(tX)))|_{t=0} = \frac{d}{dt}(\exp(tX)K)|_{t=0}.$$

If $X \in \mathfrak{k}$ then $\exp(tX) \in K$ so $d\pi_e(\mathfrak{k}) = 0$. The map π is a submersion so the quotient space $\mathfrak{g}/\mathfrak{k}$ is isomorphic to the tangent space $T_o M$ at o.

Definition 2.21. Let G be a Lie group and $I_g : G \to G$ be defined by

$$I_g(h) = ghg^{-1}.$$

Then the map defined by

$$\mathrm{Ad}(g) = (dI_g)_e : \mathfrak{g} \to \mathfrak{g}$$

is called the **adjoint** representation of G.

There is another operator $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ defined by

$$\mathrm{ad}(X) = (d\mathrm{Ad})_e(X).$$

It can be shown that $\mathrm{ad}(X)Y = [X, Y]$.

Definition 2.22. Let G be a Lie group and K a closed subgroup. We say that the homogeneous space G/K is **reductive** if there exist a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and $\mathrm{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$. With respect to this decomposition we write $X \in \mathfrak{g}$ as

$$X = X_{\mathfrak{k}} + X_{\mathfrak{m}},$$

where $X_{\mathfrak{k}} \in \mathfrak{k}$ and $X_{\mathfrak{m}} \in \mathfrak{m}$.

For a reductive homogeneous space we have that $d\pi_e|_{\mathfrak{m}}$ is an isomorphism from $\mathfrak{m}$ to T_oM. The condition $\mathrm{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$ implies $\mathrm{ad}(\mathfrak{k})\mathfrak{m} \subset \mathfrak{m}$ and if K is connected they are equivalent.

It is easy to show, see [27], that if $\phi : G \to K$ is a Lie group homomorphism then $d\phi_e$ is a Lie algebra homomorphism and

$$\phi(\exp(X)) = \exp(d\phi_e(X))$$

for $X \in \mathfrak{g}$.

Lemma 2.23. *Let $M = G/K$ be a reductive homogeneous space with $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then*

$$d\pi_e(\mathrm{Ad}(k)X) = (d\tau_k)_o(d\pi_e(X))$$

for all $k \in K$ and all $X \in \mathfrak{m}$.

Proof. [1] The curve $\exp(t\mathrm{Ad}(k)X)$ is a one parameter subgroup of G with

$$\mathrm{Ad}(k)X = \frac{d}{dt}(\exp(t\mathrm{Ad}(k)X))|_{t=0}$$

so

$$\begin{aligned} d\pi_e(\mathrm{Ad}(k)X) =& \frac{d}{dt}\left(\pi(\exp(t\mathrm{Ad}(k)X))\right)|_{t=0} \\ =& \frac{d}{dt}\left(\pi(\exp((dI_k)_e tX))\right)|_{t=0} \\ =& \frac{d}{dt}\left(\pi(I_k(\exp(tX)))\right)|_{t=0} \\ =& \frac{d}{dt}\left(k\exp(tX)k^{-1}K\right)|_{t=0} \\ =& \frac{d}{dt}\left(\tau_k(\exp(tX)K)\right)|_{t=0} \\ =& (d\tau_k)_o(d\pi_e(X)). \end{aligned}$$

□

Theorem 2.24. *Let $M = G/K$ be a reductive homogeneous space. Then there is a natural bijection between the G-invariant metrics on G/K and the $\mathrm{Ad}(K)$-invariant scalar products on $\mathfrak{m}$ i.e. those satisfying*

$$\langle \mathrm{Ad}(k)X, \mathrm{Ad}(k)Y\rangle = \langle X, Y\rangle$$

for all $k \in K$ and all $X, Y \in \mathfrak{m}$.

Proof. [28] Let $T = d\pi_e|_{\mathfrak{m}} : \mathfrak{m} \to T_oM$ and $\langle\cdot,\cdot\rangle$ be $\mathrm{Ad}(K)$-invariant. If $X_{gK}, Y_{gK} \in T_{gK}M$ then since τ_g is a diffeomorphism

$$X_{gK} = (d\tau_g)_o(X_o) \text{ and } Y_{gK} = (d\tau_g)_o(Y_o)$$

for unique vectors $X_o, Y_o \in T_oM$. Define the metric g on M by

$$g_{gK}((d\tau_g)_o(X_o), (d\tau_g)_o(Y_o)) = \left\langle T^{-1}(X_o), T^{-1}(Y_o)\right\rangle.$$

We show that this is well defined. Let $h \in G$ be such that $gK = hK$ then there exist $k \in K$ such that $h = gk$. So $\tau_h = \tau_g \circ \tau_k$ and therefore

$$g_{hK}((d\tau_h)_o(X_o), (d\tau_h)_o(Y_o)) = g_{hK}(((d\tau_g)_o \circ (d\tau_k)_o)(X_o), ((d\tau_g)_o \circ (d\tau_k)_o)(X_o))$$

$$
\begin{aligned}
&= \left\langle T^{-1}((d\tau_k)_o(X_o)), T^{-1}((d\tau_k)_o(Y_o))\right\rangle \\
&= \left\langle \mathrm{Ad}(k)T^{-1}(X_o), \mathrm{Ad}(k)T^{-1}(Y_o)\right\rangle \\
&= \left\langle T^{-1}(X_o), T^{-1}(Y_o)\right\rangle \\
&= g_{gK}((d\tau_g)_o(X_o), (d\tau_g)_o(Y_o)).
\end{aligned}
$$

This shows that g is well defined and it follows by definition that g is G-invariant.

Suppose instead that we have a G-invariant metric g and put

$$\langle X, Y\rangle = g_o(T(X), T(Y)).$$

Then

$$
\begin{aligned}
\langle \mathrm{Ad}(k)X, \mathrm{Ad}(k)Y\rangle &= g_o(T(\mathrm{Ad}(k)X), T(\mathrm{Ad}(k)Y)) \\
&= g_o((d\tau_k)_o(T(X)), (d\tau_k)_o(T(Y))) \\
&= g_o(T(X), T(Y)) \\
&= \langle X, Y\rangle .
\end{aligned}
$$

□

If the subgroup K of G is connected then the $\mathrm{Ad}(K)$-invariance is equivalent to

$$\langle \mathrm{ad}(X)Y, Z\rangle + \langle Y, \mathrm{ad}(X)Z\rangle = 0$$

for all $X \in \mathfrak{k}$ and all $Y, Z \in \mathfrak{m}$.

Proposition 2.25. *Let G/K be a homogeneous space where $\mathfrak{g}$ admits an $\mathrm{Ad}(G)$-invariant scalar product $\langle \cdot, \cdot\rangle$ i.e.*

$$\langle \mathrm{Ad}(g)X, \mathrm{Ad}(g)Y\rangle = \langle X, Y\rangle$$

for all $g \in G$. Then G/K is reductive with respect to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where

$$\mathfrak{m} = \{X \in \mathfrak{g} | \, \langle X, Y\rangle = 0 \text{ for all } Y \in \mathfrak{k}\}$$

and the restriction $\langle \cdot, \cdot\rangle_o = \langle \cdot, \cdot\rangle |_{\mathfrak{m}}$ defines an $\mathrm{Ad}(K)$-invariant scalar product on $\mathfrak{m}$.

Proof. [28] Let $k \in K$ and $X \in \mathfrak{m}$. Then since $\mathrm{Ad}(k)$ is an automorphism of $\mathfrak{k}$ we have that for all $Y \in \mathfrak{k}$ there exist a $Z \in \mathfrak{k}$ such that $\mathrm{Ad}(k)Z = Y$ and

$$\langle \mathrm{Ad}(k)X, Y\rangle = \langle X, Z\rangle = 0$$

so $\mathrm{Ad}(k)X \in \mathfrak{m}$. The rest of the statement is obvious. □

If we have a linear endomorphism L on $\mathfrak{m}$ then

$$\mathrm{Ad}(k) \circ L = L \circ \mathrm{Ad}(k)$$

for all $k \in K$ implies

$$\mathrm{ad}(X) \circ L = L \circ \mathrm{ad}(X)$$

for all $X \in \mathfrak{k}$. And if K is connected they are equivalent. We now state an important result from [18] (p. 219).

Theorem 2.26. *Let G be a Lie group, K be a closed Lie subgroup and $M = G/K$ be a reductive homogeneous space with decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then* *(i) there is a natural bijection between the set of G-invariant almost complex structure on G/K and the set of linear endomorphisms $\tilde{J}$ on $\mathfrak{m}$ satisfying*

$$\tilde{J}^2 = -I \text{ and } \tilde{J} \circ \mathrm{Ad}(k) = \mathrm{Ad}(k) \circ \tilde{J}$$

for every $k \in K$.

(ii) A G-invariant almost complex structure J is a complex structure if and only if the corresponding linear endomorphism $\tilde{J}$ satisfies

$$[\tilde{J}X, \tilde{J}Y]_{\mathfrak{m}} - [X, Y]_{\mathfrak{m}} - \tilde{J}[X, \tilde{J}Y]_{\mathfrak{m}} - \tilde{J}[\tilde{J}X, Y]_{\mathfrak{m}} = 0$$

for all $X, Y \in \mathfrak{m}$.

Proof. We denote by T the isomorphism $T = d\pi_e|_{\mathfrak{m}} : \mathfrak{m} \to T_oM$.

(i) [28] Given an almost complex structure J on M we define $\tilde{J}$ on $\mathfrak{m}$ by

$$\tilde{J} = T^{-1} \circ J_o \circ T.$$

Then obviously $\tilde{J}^2 = -I$ and

$$\begin{aligned}
\mathrm{Ad}(k)\tilde{J}(X) =& \mathrm{Ad}(k)(T^{-1} \circ J_o \circ T)(X) \\
=& (T^{-1} \circ (d\tau_k)_o \circ J_o \circ T)(X) \\
=& (T^{-1} \circ J_o \circ (d\tau_k)_o \circ T)(X) \\
=& (T^{-1} \circ J_o \circ T)(\mathrm{Ad}(k)X) \\
=& \tilde{J}(\mathrm{Ad}(k)X).
\end{aligned}$$

So $\tilde{J} \circ \mathrm{Ad}(k) = \mathrm{Ad}(k) \circ \tilde{J}$ for all $k \in K$.

On the other hand assume that $\tilde{J}$ on $\mathfrak{m}$ satisfies $\mathrm{Ad}(k) \circ \tilde{J} = \tilde{J} \circ \mathrm{Ad}(k)$ for all $k \in K$. Let $x = gK$ and define J_x on T_xM by

$$J_x(X_x) = \left((d\tau_g)_o \circ T \circ \tilde{J} \circ T^{-1} \right) (X_o)$$

where $X_x = (d\tau_g)_o(X_o)$. We show that this is well defined. Let $h \in G$ be such that $gK = hK$ then $h = gk$ for some $k \in K$. If $Y_o \in T_oM$ is the vector such that $X_x = (d\tau_h)_o(Y_o)$ then $X_o = (d\tau_k)_o(Y_o)$ so

$$\begin{aligned}
\left((d\tau_h)_o \circ T \circ \tilde{J} \circ T^{-1} \right) (Y_o) &= \left((d\tau_g)_o \circ (d\tau_k)_o \circ T \circ \tilde{J} \circ T^{-1} \right) (Y_o) \\
&= \left((d\tau_g)_o \circ T \circ \tilde{J} \circ T^{-1} \right) ((d\tau_k)_o(Y_o)) \\
&= \left((d\tau_g)_o \circ T \circ \tilde{J} \circ T^{-1} \right) (X_o).
\end{aligned}$$

So J_x it is well defined and it is obvious that $J_x^2 = -I$ and that J is G-invariant.

(ii) [18] Extend $\tilde{J}$ on $\mathfrak{m}$ to $\mathfrak{g}$ by

$$\tilde{J}(X) = \tilde{J}(X_{\mathfrak{m}}),$$

where $X = X_{\mathfrak{k}} + X_{\mathfrak{m}}$. Define the set $\mathfrak{B}(G)$ by

$$\mathfrak{B}(G) = \{\hat{X} \in C^\infty(TG) \,|\, d\pi_x(\hat{X}_x) = X_{\pi(x)} \text{ for some } X \in C^\infty(TM)\}.$$

Then $\mathfrak{B}(G)$ is a Lie subalgebra of $C^\infty(TG)$ and $d\pi : \mathfrak{B}(G) \to C^\infty(TM)$ is a surjective Lie algebra homomorphism. Let $\hat{U} \in C^\infty(TG)$ and define the tensor field $\hat{J}$ on TG by

$$\hat{J}_g(\hat{U}_g) = ((dL_g)_e \circ \tilde{J} \circ (dL_g)_e^{-1})(\hat{U}_g).$$

There is no ambiguity in the definition and it is obvious that $\hat{J}$ is left-invariant i.e.

$$dL_g \circ \hat{J} = \hat{J} \circ dL_g$$

for all $g \in G$.

We show that if $\hat{X} \in \mathfrak{B}(G)$ then $\hat{J}(\hat{X}) \in \mathfrak{B}(G)$. Since $\hat{J}$ is a tensor field it is enough to show that $J(d\pi_g(\hat{X}_g)) = d\pi_g(\hat{J}_g(\hat{X}_g))$ for $\hat{X}_g \in T_gG$. So let $\hat{Y}_e = (dL_g)_e^{-1}(\hat{X}_g)$ then

$$\begin{aligned} d\pi_g(\hat{J}_g(\hat{X}_g)) =&(d\pi_g \circ (dL_g)_e)(\tilde{J}(\hat{Y}_e)) \\ =&((d\tau_g)_o \circ d\pi_e)(\tilde{J}(\hat{Y}_e)) \\ =&((d\tau_g)_o \circ J)(d\pi_e(\hat{Y}_e)) \\ =&(J \circ (d\tau_g)_e)(d\pi_e(\hat{Y}_e)) \\ =&J((d\pi_g \circ (dL_g)_e)(\hat{Y}_e)) \\ =&J(d\pi_g(\hat{X}_g)). \end{aligned}$$

Define a tensor field S on $C^\infty(TG)$ by

$$S(\hat{U},\hat{V}) = [\hat{J}(\hat{U}),\hat{J}(\hat{V})] + \hat{J}^2([\hat{U},\hat{V}]) - \hat{J}([\hat{U},\hat{J}(\hat{V})]) - \hat{J}([\hat{J}(\hat{U}),\hat{V}]).$$

If $\hat{X},\hat{Y} \in \mathfrak{B}(G)$ then $S(\hat{X},\hat{Y}) \in \mathfrak{B}(G)$ and

$$d\pi(S(\hat{X},\hat{Y})) = \frac{1}{2}N(d\pi(\hat{X}), d\pi(\hat{Y})),$$

where N is the Nijenhuis tensor on G/K. Because $d\pi : \mathfrak{B}(G) \to C^\infty(TM)$ is surjective we have that $N = 0$ if and only if

$$S(\hat{X},\hat{Y})_g \in \ker(d\pi_g)$$

for all $g \in G$. But S is left-invariant so this is equivalent to

$$S(X,Y) \in \mathfrak{k}$$

for all $X, Y \in \mathfrak{g}$. By the $\mathrm{Ad}(K)$ invariance of $\tilde{J}$ and the remarks prior to the theorem we have $\mathrm{ad}(Y) \circ \tilde{J} = \tilde{J} \circ \mathrm{ad}(Y)$ for $Y \in \mathfrak{k}$ i.e.

$$[\tilde{J}X, Y] = \tilde{J}[X,Y]$$

where $X \in \mathfrak{m}$ and $Y \in \mathfrak{k}$. If we decompose $X \in \mathfrak{g}$ as $X_\mathfrak{k} + X_\mathfrak{m}$ and use the relations

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k} \text{ and } \tilde{J}[X_\mathfrak{m}, Y_\mathfrak{k}]_\mathfrak{m} = [\tilde{J}X_\mathfrak{m}, Y_\mathfrak{k}]_\mathfrak{m}$$

we get

$$\begin{aligned} S(X,Y) =&[\tilde{J}(X_\mathfrak{m}), \tilde{J}(Y_\mathfrak{m})]_\mathfrak{k} + [\tilde{J}(X_\mathfrak{m}), \tilde{J}(Y_\mathfrak{m})]_\mathfrak{m} \\ &+ \tilde{J}^2([X_\mathfrak{k} + X_\mathfrak{m}, Y_\mathfrak{k} + Y_\mathfrak{m}]_\mathfrak{m}) - \tilde{J}([X_\mathfrak{k} + X_\mathfrak{m}, \tilde{J}(Y_\mathfrak{m})]_\mathfrak{m}) \end{aligned}$$

$$
\begin{aligned}
&- \tilde{J}([\tilde{J}(X_{\mathfrak{m}}), Y_{\mathfrak{k}} + Y_{\mathfrak{m}}]_{\mathfrak{m}}) \\
=&[\tilde{J}(X_{\mathfrak{m}}), \tilde{J}(Y_{\mathfrak{m}})]_{\mathfrak{k}} + [\tilde{J}(X_{\mathfrak{m}}), \tilde{J}(Y_{\mathfrak{m}})]_{\mathfrak{m}} \\
&- [X_{\mathfrak{k}} + X_{\mathfrak{m}}, Y_{\mathfrak{k}} + Y_{\mathfrak{m}}]_{\mathfrak{m}} - \tilde{J}([X_{\mathfrak{k}} + X_{\mathfrak{m}}, \tilde{J}(Y_{\mathfrak{m}})]_{\mathfrak{m}}) \\
&- \tilde{J}([\tilde{J}(X_{\mathfrak{m}}), Y_{\mathfrak{k}} + Y_{\mathfrak{m}}]_{\mathfrak{m}}) \\
=&[\tilde{J}X_{\mathfrak{m}}, \tilde{J}Y_{\mathfrak{m}}]_{\mathfrak{k}} + [\tilde{J}X_{\mathfrak{m}}, \tilde{J}Y_{\mathfrak{m}}]_{\mathfrak{m}} \\
&- [X_{\mathfrak{k}}, Y_{\mathfrak{m}}]_{\mathfrak{m}} - [X_{\mathfrak{m}}, Y_{\mathfrak{k}}]_{\mathfrak{m}} - [X_{\mathfrak{m}}, Y_{\mathfrak{m}}]_{\mathfrak{m}} \\
&- \tilde{J}[X_{\mathfrak{k}}, \tilde{J}Y_{\mathfrak{m}}]_{\mathfrak{m}} - \tilde{J}[X_{\mathfrak{m}}, \tilde{J}Y_{\mathfrak{m}}]_{\mathfrak{m}} \\
&- \tilde{J}[\tilde{J}X_{\mathfrak{m}}, Y_{\mathfrak{k}}]_{\mathfrak{m}} - \tilde{J}[\tilde{J}X_{\mathfrak{m}}, Y_{\mathfrak{m}}]_{\mathfrak{m}} \\
=&[\tilde{J}X_{\mathfrak{m}}, \tilde{J}Y_{\mathfrak{m}}]_{\mathfrak{k}} + [\tilde{J}X_{\mathfrak{m}}, \tilde{J}Y_{\mathfrak{m}}]_{\mathfrak{m}} \\
&- [X_{\mathfrak{k}}, Y_{\mathfrak{m}}]_{\mathfrak{m}} - [X_{\mathfrak{m}}, Y_{\mathfrak{k}}]_{\mathfrak{m}} - [X_{\mathfrak{m}}, Y_{\mathfrak{m}}]_{\mathfrak{m}} \\
&- [X_{\mathfrak{k}}, \tilde{J}^2 Y_{\mathfrak{m}}]_{\mathfrak{m}} - \tilde{J}[X_{\mathfrak{m}}, \tilde{J}Y_{\mathfrak{m}}]_{\mathfrak{m}} \\
&- [\tilde{J}^2 X_{\mathfrak{m}}, Y_{\mathfrak{k}}]_{\mathfrak{m}} - \tilde{J}[\tilde{J}X_{\mathfrak{m}}, Y_{\mathfrak{m}}]_{\mathfrak{m}} \\
=&[\tilde{J}X_{\mathfrak{m}}, \tilde{J}Y_{\mathfrak{m}}]_{\mathfrak{k}} + [\tilde{J}X_{\mathfrak{m}}, \tilde{J}Y_{\mathfrak{m}}]_{\mathfrak{m}} \\
&- [X_{\mathfrak{k}}, Y_{\mathfrak{m}}]_{\mathfrak{m}} - [X_{\mathfrak{m}}, Y_{\mathfrak{k}}]_{\mathfrak{m}} - [X_{\mathfrak{m}}, Y_{\mathfrak{m}}]_{\mathfrak{m}} \\
&+ [X_{\mathfrak{k}}, Y_{\mathfrak{m}}]_{\mathfrak{m}} - \tilde{J}[X_{\mathfrak{m}}, \tilde{J}Y_{\mathfrak{m}}]_{\mathfrak{m}} \\
&+ [X_{\mathfrak{m}}, Y_{\mathfrak{k}}]_{\mathfrak{m}} - \tilde{J}[\tilde{J}X_{\mathfrak{m}}, Y_{\mathfrak{m}}]_{\mathfrak{m}} \\
=&[\tilde{J}X_{\mathfrak{m}}, \tilde{J}Y_{\mathfrak{m}}]_{\mathfrak{k}} + [\tilde{J}X_{\mathfrak{m}}, \tilde{J}Y_{\mathfrak{m}}]_{\mathfrak{m}} - [X_{\mathfrak{m}}, Y_{\mathfrak{m}}]_{\mathfrak{m}} \\
&- \tilde{J}[X_{\mathfrak{m}}, \tilde{J}Y_{\mathfrak{m}}]_{\mathfrak{m}} - \tilde{J}[\tilde{J}X_{\mathfrak{m}}, Y_{\mathfrak{m}}]_{\mathfrak{m}}.
\end{aligned}
$$

So we see that $S(X, Y) \in \mathfrak{k}$ if and only if

$$[\tilde{J}X_{\mathfrak{m}}, \tilde{J}Y_{\mathfrak{m}}]_{\mathfrak{m}} - [X_{\mathfrak{m}}, Y_{\mathfrak{m}}]_{\mathfrak{m}} - \tilde{J}[X_{\mathfrak{m}}, \tilde{J}Y_{\mathfrak{m}}]_{\mathfrak{m}} - \tilde{J}[\tilde{J}X_{\mathfrak{m}}, Y_{\mathfrak{m}}]_{\mathfrak{m}} = 0$$

for all $X_{\mathfrak{m}}, Y_{\mathfrak{m}} \in \mathfrak{m}$. □

2.3 The isotropic m-space $\mathcal{I}_m$

This section is devoted to the construction of the Kähler manifold $\mathcal{I}_m$. $\mathcal{I}_m$ together with a Riemannian submersion will act as a twistor space over the $2m$-dimensional sphere S^{2m} in Chapter 3. A **twistor space** for a Riemannian manifold (N, h) is an almost complex manifold (Z, J, g) together with submersion $\pi : Z \to N$ such that if M is a cosympletic (see [2]) manifold and $\psi : M \to Z$ is holomorphic then $\phi = \pi \circ \psi$ is harmonic. The sources for this section are [19] and [26].

We extend the standard Euclidean scalar product $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ on $\mathbb{R}^{2m+1}$ to a complex bilinear one $(\cdot, \cdot)$ on $\mathbb{C}^{2m+1}$.

Definition 2.27. A subspace V of $\mathbb{C}^{2m+1}$ is said to be **isotropic** if $(v, w) = 0$ for all $v, w \in V$. The subset $\mathcal{I}_m$ of the complex Grassmannian $G_m(\mathbb{C}^{2m+1})$ consisting of all m-dimensional isotropic subspaces of $\mathbb{C}^{2m+1}$ is called the **isotropic m-space**. We define the manifold structure on $\mathcal{I}_m$ to be the one of a submanifold of $G_m(\mathbb{C}^{2m+1})$.

Note that the standard Hermitian scalar product $\langle u, v \rangle_{\mathbb{C}}$ on $\mathbb{C}^n$ satisfies

$$\langle u, v \rangle_{\mathbb{C}} = (u, \bar{v}).$$

Lemma 2.28. *If $V \in \mathcal{I}_m$ then there exists a unit vector $Z \in \mathbb{R}^{2m+1}$ such that $\mathbb{C}^{2m+1} = V \oplus \bar{V} \oplus \operatorname{span}_{\mathbb{C}}\{Z\}$ is an orthogonal decomposition of $\mathbb{C}^{2m+1}$ with respect to $\langle \cdot, \cdot \rangle_{\mathbb{C}}$.*

Proof. [26] For an element $V \in \mathcal{I}_m$ we can choose an orthonormal basis $\{Z_k\}_{k=1}^m$ for V with respect to $\langle \cdot, \cdot \rangle_{\mathbb{C}}$. If $\overline{W} \in \bar{V}$ and $\overline{W} = \sum_{j=1}^m a_j \overline{Z_j}$, then

$$\left\langle Z_k, \overline{W} \right\rangle_{\mathbb{C}} = \sum_{j=1}^m \overline{a_j}(Z_k, Z_j) = 0$$

for all $k = 1, \ldots, m$ so V and $\bar{V}$ are orthogonal. Now decompose the basis as $Z_k = X_k + iY_k$ with $X_k, Y_k \in \mathbb{R}^{2m+1}$, then

$$\begin{aligned}
0 =& (Z_k, Z_j) \\
=& (\langle X_k, X_j \rangle_{\mathbb{R}} - \langle Y_k, Y_j \rangle_{\mathbb{R}}) \\
& + i(\langle X_k, Y_j \rangle_{\mathbb{R}} + \langle Y_k, X_j \rangle_{\mathbb{R}}) \\
\delta_{kj} =& \langle Z_k, Z_j \rangle_{\mathbb{C}} \\
=& (Z_k, \overline{Z_j}) \\
=& (\langle X_k, X_j \rangle_{\mathbb{R}} + \langle Y_k, Y_j \rangle_{\mathbb{R}}) \\
& + i(- \langle X_k, Y_j \rangle_{\mathbb{R}} + \langle Y_k, X_j \rangle_{\mathbb{R}})
\end{aligned}$$

adding and subtracting the two equalities we obtain

$$\delta_{kj} = \frac{1}{2} \langle X_k, X_j \rangle_{\mathbb{R}} = \frac{1}{2} \langle Y_k, Y_j \rangle_{\mathbb{R}}$$
$$0 = \langle X_k, Y_j \rangle_{\mathbb{R}}.$$

Thus $X_1, \ldots, X_m, Y_1, \ldots, Y_m$ are $2m$ orthogonal vectors in $\mathbb{R}^{2m+1}$ so there exists a $Z \in \mathbb{R}^{2m+1}$ such that Z is orthogonal to all X_k and all Y_k. Thus

$$0 = \langle Z_k, Z \rangle_{\mathbb{C}} = \left\langle \overline{Z_k}, Z \right\rangle_{\mathbb{C}}$$

and $V \oplus \bar{V} \oplus \mathbb{C}Z$ is a orthogonal decomposition of $\mathbb{C}^{2m+1}$. □

We define the origin in $\mathcal{I}_m$ to be the element o given by

$$o = \operatorname{span}_{\mathbb{C}}\{e_j + ie_{j+m}\}_{j=1}^m,$$

where $\{e_j\}_{i=1}^{2m+1}$ is the standard basis for $\mathbb{R}^{2m+1}$.

Lemma 2.29. *There exists a natural transitive C^∞-action of $SO(2m+1)$ on $\mathcal{I}_m$ with isotropy subgroup $U(m)$. So $\mathcal{I}_m$ is diffeomorphic to the homogeneous space $SO(2m+1)/U(m)$.*

Proof. [26] For an element $V \in \mathcal{I}_m$ let $\{X_1, \ldots, X_m, Y_1, \ldots, Y_m, Z\}$ be a positively oriented basis for $\mathbb{R}^{2m+1}$ such that

$$V = \operatorname{span}_{\mathbb{C}}\{X_i + iY_i\}_{j=1}^m.$$

Then

$$h = [X_1, \ldots, X_m, Y_1, \ldots, Y_m, Z] \in SO(2m+1)$$

and

$$gh = [gX_1, \ldots, gX_m, gY_1, \ldots, gY_m, gZ] \in SO(2m+1)$$

for $g \in SO(2m+1)$. We define an action of $SO(2m+1)$ on $\mathcal{I}_m$ by

$$g \cdot V = \operatorname{span}_{\mathbb{C}}\{gX_j + igY_j\}_{j=1}^m.$$

Now we show that the action is transitive. Let $V \in \mathcal{I}_m$ and $g \in SO(2m+1)$ be such that

$$[X_1, \ldots, X_m, Y_1, \ldots, Y_m, Z] = g[e_1, \ldots, e_{2m+1}]$$

then

$$V = \operatorname{span}_{\mathbb{C}}\{X_j + iY_j\}_{j=1}^m = \operatorname{span}_{\mathbb{C}}\{ge_j + ige_{m+j}\}_{j=1}^m = g \cdot o$$

so the action is transitive.

Finally we determine the isotropy subgroup of the action. Let $g \in SO(2m+1)$ be such that $g \cdot o = o$, then

$$\operatorname{span}_{\mathbb{C}}\{ge_j + ige_{m+j}\}_{j=1}^m = \operatorname{span}_{\mathbb{C}}\{e_j + ie_{m+j}\}_{j=1}^m$$

and $ge_{2m+1} = e_{2m+1}$. Form this we get that

$$\begin{aligned} ge_j + ige_{j+m} =& \sum_{k=1}^m ((a_{kj} + ib_{kj})(e_k + ie_{k+m})) \\ ge_{2m+1} =& e_{2m+1} \end{aligned}$$

with $a_{kj}, b_{kj} \in \mathbb{R}$. Hence

$$\begin{aligned} 2 = |ge_j + ige_{j+m}|^2 =& 2(\sum_{k=1}^m (a_{kj}^2 + b_{kj}^2)) \\ ge_j =& \sum_{k=1}^m (a_{kj}e_k - b_{kj}e_{k+m}) \\ ge_{j+m} =& \sum_{k=1}^m (b_{kj}e_k + a_{kj}e_{k+m}) \\ ge_{2m+1} =& e_{2m+1}. \end{aligned}$$

This shows that g must be of the form

$$g = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0^t & 0^t & 1 \end{pmatrix}.$$

By the inclusion $U(m) \subset SO(2m+1)$ given by

$$a + ib \mapsto \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0^t & 0^t & 1 \end{pmatrix}$$

we have $g \in U(m)$. □

Lemma 2.30. *The homogeneous space $\mathcal{I}_m \cong SO(2m+1)/U(m)$ has an $SO(2m+1)$-invariant metric g'.*

Proof. We define a scalar product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{so}(2m+1)$ by

$$\langle X, Y \rangle = \operatorname{trace}(X^t Y).$$

This scalar product is $\mathrm{Ad}(SO(2m+1))$-invariant. So following Proposition 2.25 it induces an $SO(2m+1)$-invariant metric on $\mathcal{I}_m$. $\square$

The Lie algebra $\mathfrak{so}(2m+1)$ has a reductive decomposition $\mathfrak{so}(2m+1) = \mathfrak{u}(m) \oplus \mathfrak{m}$. Since $\mathfrak{m}$ is isomorphic to $\mathfrak{so}(2m+1)/\mathfrak{u}(m)$ they have the same dimension so we need to find a subspace of $\mathfrak{so}(2m+1)$ of dimension $\frac{(2m+1)2m}{2} - m^2 = m(m+1)$ such that

$$\langle X, Y \rangle = 0$$

for all $X \in \mathfrak{m}$ and $Y \in \mathfrak{u}(m)$. We can embed the Lie algebra $\mathfrak{u}(m)$ for $U(m)$ into $\mathfrak{so}(2m+1)$ by

$$\mathfrak{u}(m) = \left\{ \begin{pmatrix} A & B & 0 \\ -B & A & 0 \\ 0^t & 0^t & 0 \end{pmatrix} \mid A, B \in \mathbb{R}^{m \times m} \text{ and } A^t = -A, B^t = B \right\}.$$

We see that $\mathfrak{m}$ is given by

$$\mathfrak{m} = \left\{ \begin{pmatrix} X & Y & U \\ Y & -X & V \\ -U^t & -V^t & 0 \end{pmatrix} \mid X, Y \in \mathbb{R}^{m \times m}, U, V \in \mathbb{R}^m \text{ and } X^t = -X, Y^t = -Y \right\}.$$

The decomposition $\mathfrak{so}(2m+1) = \mathfrak{u}(m) \oplus \mathfrak{m}$ can be made explicitly by

$$\begin{aligned} \begin{pmatrix} X_1 & Y_1 & U \\ -Y_1^t & X_2 & V \\ -U^t & -V^t & 0 \end{pmatrix} =& \frac{1}{2} \begin{pmatrix} X_1 + X_2 & Y_1^t + Y_1 & 0 \\ -Y_1^t - Y_1 & X_1 + X_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &+ \frac{1}{2} \begin{pmatrix} X_1 - X_2 & Y_1 - Y_1^t & 2U \\ Y_1 - Y_1^t & X_2 - X_1 & 2V \\ -2U^t & -2V^t & 0 \end{pmatrix}, \end{aligned}$$

where $X_1, X_2, Y_1 \in \mathbb{R}^{m \times m}$ with $X_1^t = -X_1, X_2^t = -X_2$ and $U, V \in \mathbb{R}^m$.

The sphere S^{2m} in $\mathbb{R}^{2m+1}$ is diffeomorphic to $SO(2m+1)/SO(2m)$. If the origin in S^{2m} is given by e_{2m+1} then $\mathfrak{so}(2m+1) = \mathfrak{so}(2m) \oplus \mathfrak{p}$ where

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & 0 & U \\ 0 & 0 & V \\ -U^t & -V^t & 0 \end{pmatrix} \mid U, V \in \mathbb{R}^m \right\}.$$

The scalar product on $\mathfrak{p}$ is

$$\langle X, Y \rangle = \operatorname{trace}(X^t Y)$$

which gives a $SO(2m+1)$-invariant metric on S^{2m}. This metric coincides with the ordinary metric on S^{2m}.

Lemma 2.31. *The natural projection*

$$\pi : (\mathcal{I}_m, g') \to (S^{2m}, g_{S^{2m}})$$

given by

$$\mathcal{I}_m \ni g \cdot o \cong gU(m) \mapsto gSO(2m) \cong g \cdot e_{2m+1} \in S^{2m},$$

is a Riemannian submersion.

Proof. First of all we notice that $\pi(o) = e_{2m+1}$ and $d\pi_o$ is given by

$$d\pi_o \begin{pmatrix} X & Y & U \\ Y & -X & V \\ -U^t & -V^t & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & U \\ 0 & 0 & V \\ -U^t & -V^t & 0 \end{pmatrix}$$

so we see that π is surjective at o. If we denote the translation by $g \in SO(2m+1)$ in S^{2m} by θ_g we have

$$\pi(\tau_g(o)) = \theta_g(\pi(o))$$

from which we obtain at $x = gU(m)$

$$(d\pi)_x = (d\theta_g)_{\pi(o)} \circ d\pi_o \circ ((d\tau_g)_o)^{-1}$$

so since both τ_g and θ_g are diffeomorphisms and $(d\pi)_o$ is onto $d\pi$ is surjective everywhere and π is a submersion.

It is obvious that $d\pi_o$ is an isometric isomorphism from the horizontal space to $T_{\pi(o)}S^{2m}$. Since the metrics on $\mathcal{I}_m$ and S^{2m} are $SO(2m+1)$-invariant it follows that π is a Riemannian submersion. $\square$

Since $\mathcal{I}_m \subset G_k(\mathbb{C}^{2m+1})$ and the tangent space at $V \in G_k(\mathbb{C}^{2m+1})$ is given by

$$\mathrm{Hom}_{\mathbb{C}}(V, V^{\perp})$$

the horizontal space at $V \in \mathcal{I}_m$ must be a subset of this. The elements of $o = \mathrm{span}_{\mathbb{C}}\{e_j + ie_{j+m}\}_{j=1}^m$ are all of the form

$$v = \begin{pmatrix} A + iB \\ -B + iA \\ 0 \end{pmatrix} \in \mathbb{C}^{2m+1},$$

where $A, B \in \mathbb{R}^m$.

The linear map L defined by

$$L : V \ni v \mapsto \begin{pmatrix} 0 & 0 & U \\ 0 & 0 & V \\ -U^t & -V^t & 0 \end{pmatrix} v \in \mathbb{C}^{2m+1}$$

satisfies

$$\begin{aligned} &\begin{pmatrix} 0 & 0 & U \\ 0 & 0 & V \\ -U^t & -V^t & 0 \end{pmatrix} \begin{pmatrix} A + iB \\ -B + iA \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ (-U^tA + V^tB) + i(-U^tB - V^tA) \end{pmatrix} \in \mathrm{span}_{\mathbb{C}}\{e_{2m+1}\} \end{aligned}$$

So $L \in \mathrm{Hom}_{\mathbb{C}}(o, \mathrm{span}_{\mathbb{C}}\{e_{2m+1}\})$. Since the real dimension of $\mathrm{Hom}_{\mathbb{C}}(o, \mathrm{span}_{\mathbb{C}}\{e_{2m+1}\})$ is $2m$ we have

$$\mathcal{H}_o\mathcal{I}_m = \mathrm{Hom}_{\mathbb{C}}(o, \mathrm{span}_{\mathbb{C}}\{e_{2m+1}\}).$$

As $\mathcal{I}_m$ is a homogeneous space and g' is G-invariant $\mathcal{H}_{g\cdot o}\mathcal{I}_m = \mathrm{Hom}_{\mathbb{C}}(g \cdot o, \mathrm{span}_{\mathbb{C}}\{g \cdot e_{2m+1}\}$.

Lemma 2.32. *There exists a complex structure on $\mathcal{I}_m$ which turns it into a Hermitian manifold with metric g'.*

Proof. Define an endomorphism $\tilde{J} : \mathfrak{m} \to \mathfrak{m}$ by

$$\tilde{J}\begin{pmatrix} X & Y & U \\ Y & -X & V \\ -U^t & -V^t & 0 \end{pmatrix} = \begin{pmatrix} -Y & X & -V \\ X & Y & U \\ V^t & -U^t & 0 \end{pmatrix}$$

We shall show that this linear endomorphism satisfies the necessary requirements of Theorem 2.26.

(i) The first condition translates into

$$\tilde{J}([W,Z]) - [W, \tilde{J}(Z)] = 0, \text{ for all } W \in \mathfrak{u}(m) \text{ and } Z \in \mathfrak{m}.$$

For $W \in \mathfrak{u}(m)$ and $Z \in \mathfrak{m}$ we have

$$W = \begin{pmatrix} A & B & 0 \\ -B & A & 0 \\ 0^t & 0^t & 0 \end{pmatrix} \text{ and } Z = \begin{pmatrix} X & Y & U \\ Y & -X & V \\ -U^t & -V^t & 0 \end{pmatrix}$$

with $A^t = -A$, $B^t = B$ and $X^t = -X$, $Y^t = -Y$ so

$$[W,Z] = \begin{pmatrix} AX - XA + BY + YB & AY - YA - BX - XB & AU + BV \\ AY - YA - BX - XB & XA - AX - BY - YB & -BU + AV \\ U^tA - V^tB & U^tB + V^tA & 0 \end{pmatrix}.$$

Applying $\tilde{J}$ to this we get

$$\begin{pmatrix} YA - AY + BX + XB & AX - XA + BY + YB & BU - AV \\ AX - XA + BY + YB & AY - YA - BX - XB & AU + BV \\ -U^tB - V^tA & U^tA - V^tB & 0 \end{pmatrix}.$$

A similar calculation of $[W, \tilde{J}(Z)]$ shows that this is equal to $\tilde{J}([W,Z])$. Hence there is an almost complex structure on G/K corresponding to $\tilde{J}$.

(ii) Let $Z_1, Z_2 \in \mathfrak{m}$ with

$$Z_1 = \begin{pmatrix} X_1 & Y_1 & U_1 \\ Y_1 & -X_1 & V_1 \\ -U_1^t & -V_1^t & 0 \end{pmatrix} \text{ and } Z_2 = \begin{pmatrix} X_2 & Y_2 & U_2 \\ Y_2 & -X_2 & V_2 \\ -U_2^t & -V_2^t & 0 \end{pmatrix}.$$

Then $[\tilde{J}Z_1, \tilde{J}Z_2]_{\mathfrak{m}}$ is given by

$$\begin{aligned} &\frac{1}{2}\begin{pmatrix} U_1U_2^t - U_2U_1^t & U_1V_2^t - V_2U_1^t & 2(X_1U_2 - X_2U_1) \\ U_1V_2^t - V_2U_1^t & U_2U_1^t - U_1U_2^t & 2(Y_1U_2 - Y_2U_1) \\ 2(-U_1^tX_2 + U_2^tX_1) & 2(-U_1^tY_2 + U_2^tY_1) & 0 \end{pmatrix} \\ &+ \frac{1}{2}\begin{pmatrix} -V_1V_2^t + V_2V_1^t & V_1U_2^t - U_2V_1^t & 2(Y_1V_2 - Y_2V_1) \\ V_1U_2^t - U_2V_1^t & V_1V_2^t - V_2V_1^t & 2(X_2V_1 - X_1V_2) \\ 2(-V_1^tY_2 + V_2^tY_1) & 2(V_1^tX_2 - V_2^tX_1) & 0 \end{pmatrix}, \end{aligned}$$

$[Z_1, Z_2]_{\mathfrak{m}}$ is given by

$$\frac{1}{2}\begin{pmatrix} U_2U_1^t - U_1U_2^t & U_2V_1^t - V_1U_2^t & 2(X_1U_2 - X_2U_1) \\ U_2V_1^t - V_1U_2^t & U_1U_2^t - U_2U_1^t & 2(Y_1U_2 - Y_2U_1) \\ 2(-U_1^tX_2 + U_2^tX_1) & 2(-U_1^tY_2 + U_2^tY_1) & 0 \end{pmatrix}$$

$$+\frac{1}{2}\begin{pmatrix} V_1V_2^t - V_2V_1^t & V_2U_1^t - U_1V_2^t & 2(Y_1V_2 - Y_2V_1) \\ V_2U_1^t - U_1V_2^t & V_2V_1^t - V_1V_2^t & 2(X_2V_1 - X_1V_2) \\ 2(-V_1^tY_2 + V_2^tY_1) & 2(V_1^tX_2 - V_2^tX_1) & 0 \end{pmatrix},$$

$\tilde{J}[Z_1, \tilde{J}Z_2]_{\mathfrak{m}}$ is given by

$$\frac{1}{2}\begin{pmatrix} U_1U_2^t - U_2U_1^t & U_1V_2^t - V_2U_1^t & 2(Y_1V_2 + Y_2V_1) \\ U_1V_2^t - V_2U_1^t & U_2U_1^t - U_1U_2^t & 2(Y_1U_2 + Y_2U_1) \\ 2(U_1^tX_2 + U_2^tX_1) & 2(U_1^tY_2 + U_2^tY_1) & 0 \end{pmatrix}$$
$$+\frac{1}{2}\begin{pmatrix} V_2V_1^t - V_1V_2^t & V_1U_2^t - U_2V_1^t & 2(X_1U_2 + X_2U_1) \\ V_1U_2^t - U_2V_1^t & V_1V_2^t - V_2V_1^t & 2(-X_1V_2 - X_2V_1) \\ 2(V_1^tY_2 + V_2^tY_1) & 2(-V_1^tX_2 - V_2^tX_1) & 0 \end{pmatrix},$$

and finally $\tilde{J}[\tilde{J}Z_1, Z_2]_{\mathfrak{m}}$ is given by

$$\frac{1}{2}\begin{pmatrix} U_1U_2^t - U_2U_1^t & U_1V_2^t - V_2U_1^t & 2(-Y_1V_2 - Y_2V_1) \\ U_1V_2^t - V_2U_1^t & U_2U_1^t - U_1U_2^t & 2(X_1V_1 + X_2V_1) \\ 2(-U_1^tX_2 - U_2^tX_1) & 2(V_1^tX_2 + V_2^tX_1) & 0 \end{pmatrix}$$
$$+\frac{1}{2}\begin{pmatrix} V_2V_1^t - V_1V_2^t & V_1U_2^t - U_2V_1^t & 2(-X_1U_2 - X_2U_1) \\ V_1U_2^t - U_2V_1^t & V_1V_2^t - V_2V_1^t & 2(-Y_1U_2 - Y_2U_1) \\ 2(-V_1^tY_2 - V_2^tY_1) & 2(-U_1^tY_2 - U_2^tY_1) & 0 \end{pmatrix}.$$

We see that $[\tilde{J}Z_1, \tilde{J}Z_2]_{\mathfrak{m}} - [Z_1, Z_2]_{\mathfrak{m}} - \tilde{J}[Z_1, \tilde{J}Z_2]_{\mathfrak{m}} - \tilde{J}[\tilde{J}Z_1, Z_2]_{\mathfrak{m}} = 0$ for all $Z_1, Z_2 \in \mathfrak{m}$ so the almost complex structure J corresponding to $\tilde{J}$ is a complex structure on $\mathcal{I}_m$.

Finally we show that g' is compatible with J. Since g' is $SO(2m+1)$-invariant it is enough to show that it is compatible at $o \in M$. Let $Z_1, Z_2 \in \mathfrak{m}$ be as in (ii). Then using $Z_1^t = -Z_1$ we have

$$\begin{aligned}\langle Z_1, Z_2\rangle =& -\operatorname{trace}(Z_1Z_2) \\ =& \operatorname{trace}(X_1X_2 + Y_1Y_2 - U_1U_2^t) \\ &+ \operatorname{trace}(Y_1Y_2 + X_1X_2 - V_1V_2^t) \\ &+ \operatorname{trace}(-U_1^tU_2 - V_1^tV_2)\end{aligned}$$

and

$$\begin{aligned}\left\langle \tilde{J}Z_1, \tilde{J}Z_2\right\rangle =& -\operatorname{trace}(\tilde{J}Z_1\tilde{J}Z_2) \\ =& \operatorname{trace}(Y_1Y_2 + X_1X_2 - V_1V_2^t) \\ &+ \operatorname{trace}(X_1X_2 + Y_1Y_2 - U_1U_2^t) \\ &+ \operatorname{trace}(-V_1^tV_2 - U_1^tU_2)\end{aligned}$$

so $\langle Z_1, Z_2\rangle = \left\langle \tilde{J}Z_1, \tilde{J}Z_2\right\rangle$ and g' is compatible. □

We also have the following proposition ([26] p. 205).

Proposition 2.33. *$(\mathcal{I}_m, g')$ is a Kähler manifold.*

And by restricting the Plücker embedding $\mathrm{pl} : G_m(\mathbb{C}^{2m+1}) \to \mathbb{CP}^{N-1}$ where $N = \binom{2m+1}{m}$ to $\mathcal{I}_m$ we get an holomorphic embedding of $\mathcal{I}_m$ in $\mathbb{CP}^{N-1}$.

2.4 Riemann surfaces

In Chapter 3 the sphere S^2 will be important. S^2 can aside form being a Riemannian manifold also be seen as a Riemann surface. A Riemann surface is a complex manifold and it will therefore be possible to define holomorphic maps on them. The references for this section are [2], [16], [17] and [28].

Definition 2.34. Let (M, g) be a 2-dimensional Riemannian manifold. Then a chart $(x, y) : U \to R^2$ is said to be **isothermal** if

$$g(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = 0 \text{ and } g(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = g(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}).$$

Theorem 2.35. *Any 2-dimensional Riemannian manifold has an atlas of isothermal charts.*

For a proof of this theorem see [24].

Proposition 2.36. *Let (M, g) be a 2-dimensional Riemannian manifold. If (x, y) is a isothermal chart then*

$$\nabla^M_X X + \nabla^M_Y Y = 0,$$

where $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$.

Proof. From the definition of the Christoffel symbol, we have

$$\begin{aligned}\nabla^M_X X + \nabla^M_Y Y =&(\Gamma^1_{11}X + \Gamma^2_{11}Y) + (\Gamma^1_{22}X + \Gamma^2_{22}Y)\\ =&(\Gamma^1_{11} + \Gamma^1_{22})X + (\Gamma^2_{11} + \Gamma^2_{22})Y.\end{aligned}$$

We calculate the Christoffel symbols using $g_{11} = g_{22}$ and $g_{12} = g_{21} = 0$ and get

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\left(\frac{\partial g_{11}}{\partial x} + \frac{\partial g_{11}}{\partial x} - \frac{\partial g_{11}}{\partial x}\right) = \frac{g^{11}}{2}\frac{\partial g_{11}}{\partial x}$$

and

$$\Gamma^1_{22} = \frac{1}{2}g^{11}\left(\frac{\partial g_{12}}{\partial y} + \frac{\partial g_{12}}{\partial y} - \frac{\partial g_{22}}{\partial x}\right) = -\frac{g^{11}}{2}\frac{\partial g_{11}}{\partial x} = -\Gamma^1_{11}.$$

In the same way $\Gamma^2_{11} = -\Gamma^2_{22}$. □

A 1-dimensional complex manifold is said to be a **Riemann surface**. Let $(U, z = x+iy)$ be a chart on a Riemann surface Σ with compatible metric g. The for $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$ we have $JX = Y$ and $JY = -X$. Thus we obtain

$$g(X, X) = g(JX, JX) = g(Y, Y)$$

and

$$g(X, Y) = g(X, JX) = g(JX, -X) = -g(Y, X)$$

so the chart is isothermal. Since the real dimension of a Riemann surface is 2 any alternating 2-form must be closed. Therefore any compatible metric on a Riemann surface is a Kähler metric.

Definition 2.37. Let Σ be a Riemann surface and $z : U \to \mathbb{C}$ be a local chart. Then for a holomorphic function $f : U \to \mathbb{C}$

$$f dz^k$$

is said to be a **holomorphic k-differential**. The dz^k means that if w is another chart then

$$(\frac{dz}{dw})^k f(z(w))dw^k = f(z)dz^k.$$

We denote the set of all holomorphic k-differentials on Σ by $\mathcal{H}(\Sigma, \Omega^k)$.

The next lemma shows that the topology of a Riemann surface of genus zero imposes severe restrictions on the holomorphic k-differentials. It is known that there exist only Riemann surface of genus zero up to a conformal diffeomorphism. That is the complex projective line $\mathbb{CP}^1$ which is equivalent to the sphere S^2 and to the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Proposition 2.38. *Let $\mathbb{CP}^1$ be a Riemann surface of genus zero. Then all holomorphic k-differentials, $k \geq 1$, are zero.*

Proof. [16] Let $f dz^k \in \mathcal{H}(\mathbb{CP}^1, \Omega^k)$ and $w : \hat{\mathbb{C}} \backslash \{0\} \to \mathbb{C}$ be the chart $z \mapsto w = \frac{1}{z}$ on the Riemann sphere. Then

$$f(z)dz^k = f(z(w)) \left(\frac{dz}{dw}\right)^k dw^k = f(z(w))\frac{1}{w^{2k}} dw^k$$

now Liouville's theorem tells us that $f dz^k = 0$. □

Proposition 2.39. *Let Σ be a compact Riemann surface and $F : \Sigma \to \mathbb{C}^{n+1} \backslash \{0\}$ be a meromorphic map with poles at $\mathcal{R}$. Then $f = \pi_{\mathbb{P}} \circ F : \Sigma \backslash \mathcal{R} \to \mathbb{CP}^n$ is holomorphic and there exist a unique holomorphic map $\hat{f} : \Sigma \to \mathbb{CP}^n$ such that $\hat{f}|_{\Sigma \backslash \mathcal{R}} = f$.*

Proof. [28] Let $p \in \mathcal{R}$ and (U, z) be chart around p with $z(p) = 0$. Since F is meromorphic there exist a $k \in \mathbb{N}$ such that $z(x)^k F(x)$ is nonzero and holomorphic in U. Further

$$\pi_{\mathbb{P}}(F(x)) = \pi_{\mathbb{P}}(z(x)^k F(x))$$

for all $x \in U \backslash p$. Applying the same process to all poles we get the extension to all of Σ. □

Chapter 3

Calabi's classification

This chapter is devoted to the proof of Calabi's classification theorem. Calabi's theorem is a partial converse to Theorem 3.1. Calabi's theorem will be proved in a series of lemmas. But prior to the proof we will need the notion of a map being full. The presentation of the proof will follow [19] closely.

Theorem 3.1. *Let Σ be a compact Riemann surface. Then for any holomorphic map $\psi : \Sigma \to \mathcal{I}_m$ which is horizontal with respect to the submersion $\pi : \mathcal{I}_m \to S^{2m}$, the map $\phi = \pi \circ \psi$ is harmonic.*

Proof. [19] By Theorem 2.11 the holomorphic map ψ is harmonic. Now since ψ is horizontal with respect to π Theorem 1.25 implies that ϕ is harmonic. $\square$

Definition 3.2. A C^∞-map $f : M \to (N, h)$ is said to be **locally full** if there exists no non-empty open subset U of M such that the image $f(U)$ is contained in a complete totally geodesic submanifold of N of lower dimension.

Definition 3.3. A C^∞-map $f : M \to (N, h)$ is said to be **full** if the image $\phi(M)$ is not contained in a complete totally geodesic submanifold of N of lower dimension.

It is obvious that locally full is a stronger condition than full. In [15] full maps are instead said to be **nondegenerate**. To give the reader an understanding of the notion we state the following proposition. It uses the fact that the complete totally geodesic submanifolds of $\mathbb{R}^n$ are the affine subspaces.

Proposition 3.4. *Let $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$ be a locally full C^∞-curve then the vectors*

$$\frac{d\gamma}{dt}(t), \dots, \frac{d^n\gamma}{dt^n}(t)$$

are linearly dependent except for isolated points of $(-\epsilon, \epsilon)$.

Proof. Suppose to the contrary that there exists a set S with a limit point where the vectors are linearly dependent. Since the derivatives vary continuously and the domain is in $\mathbb{R}$ the set S must contain an open set U. Let $U \subset (-\epsilon, \epsilon)$ be such an open set. Let k be the largest integer in $\{1, \dots, n\}$ such that $\frac{d^k\gamma}{dt^k}$ is a linear combination of the other derivatives in U. Then

$$\frac{d^k\gamma}{dt^n}(t) = \sum_{i=1}^{k-1} a(t) \frac{d^i\gamma}{dt^i}(t)$$

for all $t \in U$. This is a linear ODE and it has a unique solution given initial conditions

$$\gamma(t_0) \text{ and } \frac{d^i\gamma}{dt^i}(t_0) = b_i$$

where $b_i \in \mathbb{R}^n$ for $i = 1, \ldots, k-1$. The vectors b_i for $i = 1, \ldots, k-1$ span an affine subspace V of $\mathbb{R}^n$ of at most dimension $k-1$ containing $\gamma(t_0)$. Hence $\gamma(U)$ is contained in V so γ i not locally full. □

Proposition 3.5. *Let $\phi : (M, g) \to (N, h)$ be a harmonic map. Then ϕ is full if and only if it is locally full.*

Proof. If ϕ is locally full then obviously it is also full.

To prove the converse suppose that ϕ is full but not locally full. Since ϕ is not locally full there exist an open subset U of M such that $\phi(U)$ is contained in a complete totally geodesic submanifold V. By Theorem 1.16 we have $\phi(M) \subset V$ which is a contradiction since ϕ is full. □

In a local chart $(U, z = x + iy)$ of $\mathbb{CP}^1$ we define the operators

$$D = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \bar{D} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

which satisfy $D\bar{D} = \bar{D}D$ and $\overline{D\phi} = \bar{D}\phi$ if ϕ maps to $\mathbb{R}^{n+1}$. We will consider $\phi : \mathbb{CP}^1 \to S^n \subset \mathbb{R}^{n+2}$ as a map to both S^n and $\mathbb{R}^{n+1}$ and will use the same symbol in either case.

It is well known that the complete totally geodesic submanifolds of S^n are the intersections of S^n with a linear subspace of $\mathbb{R}^{n+1}$ i.e. the spheres of lower dimension.

Lemma 3.6. *Let $\phi : \mathbb{CP}^1 \to S^n \subset \mathbb{R}^{n+1}$ be a full harmonic map. Then*

$$\mathrm{span}_{\mathbb{C}}\{\phi, \bar{D}^k D^l \phi\}_{1 \le k+l,\, 0 \le k,l} = \mathbb{C}^{n+1}$$

in at least one point.

Proof. [12] Suppose to the contrary that there exist no such point. Then for all $z \in \mathbb{CP}^1$

$$\mathrm{span}_{\mathbb{C}}\{\phi, \bar{D}^k D^l \phi(z)\}_{1 \le k+l,\, 0 \le k,l} = V_z$$

which is a proper subspace of $\mathbb{C}^{n+1}$. Since both $\mathbb{CP}^1$ and S^n are real analytic ϕ is real analytic by Theorem 1.15. So for all points $p \in \mathbb{CP}^1$ there exist a neighbourhood U of p such that for all $z \in U$ we have a convergent series

$$\phi(z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} A_{i,j} (z - p)^i (\bar{z} - \bar{p})^j,$$

where $A_{i,j}$ are constants. Taking derivatives we see that

$$A_{i,j} = \frac{D^i \bar{D}^j \phi(p)}{i! j!}$$

thus $A_{i,j}$ and $D^i \bar{D}^j \phi(p)$ for $0 \le i, j$ have the same span V_p. But since $A_{i,j}$ are constants we have $\phi(U) \subset V$. So ϕ can not be full since it is not locally full. □

It is well known that the dimension of

$$\operatorname{span}_{\mathbb{C}}\{\phi, \bar{D}^k D^l \phi\}_{1\le k+l,\, 0\le k,l}$$

can be less that $n+1$ only at isolated points.

The next theorem is the main theorem of this thesis. It classifies all harmonic maps from S^2 to S^n. In order for us to be able to study one dimension at the time we will assume that ϕ is full. The fact that ϕ is full is not very restrictive, since the complete totally geodesic submanifolds of S^n are the spheres of lower dimension. If we are given a harmonic map $\phi : S^2 \to S^n$ then there exist an $m \le n$ such that $\phi : S^2 \to S^m$ is full.

Theorem 3.7 (Calabi's Theorem [6, 7])**.** *Let $\phi : \mathbb{CP}^1 \to (S^n, g_{S^n})$ be a full harmonic map. Then*

(i) $n = 2m$ for some $m \in \mathbb{Z}^+$, and

(ii) there exists a holomorphic map $\psi : \mathbb{CP}^1 \to (\mathcal{I}_m, g')$ which is horizontal with respect to the natural projection $\pi : (\mathcal{I}_m, g') \to (S^{2m}, g_{S^{2m}})$ and the diagram

$$\begin{tikzcd} & \mathcal{I}_m \arrow[d, "\pi"] \\ \mathbb{CP}^1 \arrow[ur, "\psi"] \arrow[r, "\phi"'] & S^{2m} \end{tikzcd}$$

commutes.

In local coordinates the condition $\tau(\phi) = 0$ is a semi-linear (second order) elliptic partial differential equation. Calabi's theorem tells us that for harmonic maps $\phi : \mathbb{CP}^1 \to S^{2m}$ the differential equation for ϕ can be transformed into a first order complex equation for $\psi : \mathbb{CP}^1 \to \mathcal{I}_m$.

Of course the theorem would be pointless if there did not exist any such harmonic maps. The following theorem together with the fact that S^n is its own universal covering space if $n > 1$ proves that they exist.

Theorem 3.8 ([22] p. 1034)**.** *Let (N, h) be a Riemannian manifold and suppose that the universal covering space of N is not contractible. Then there exists a non-trivial harmonic map $\phi : S^2 \to (N, h)$.*

The next theorem is the first in a series of lemmas that will be used to prove Calabi's Theorem.

Lemma 3.9. *Let $\phi : \mathbb{CP}^1 \to S^n \subset \mathbb{R}^{n+1}$ be a full harmonic map. Then the following conditions are satisfied*

(i) $(\phi, \phi) = \langle \phi, \phi \rangle_{\mathbb{R}} = 1$,

(ii) $D\bar{D}\phi = -(D\phi, \bar{D}\phi)\phi$

(iii) $(D\phi, D\phi) = 0$ and $(D\phi, \bar{D}\phi) \ge 0$ where the zeros are isolated.

Proof. (i) The statement is obvious.

(ii) For all complex charts (U,z) we write $z=x+iy$. Then (x,y) is a isothermal chart for $\mathbb{CP}^1$. So $\{X=\frac{\partial}{\partial x}, Y=\frac{\partial}{\partial y}\}$ is orthogonal basis for $T_{(x,y)}\mathbb{CP}^1$. The vectors are mapped to

$$d\phi_{(x,y)}(X)=\frac{\partial\phi}{\partial x}\in\mathbb{R}^{n+1} \text{ and } d\phi_{(x,y)}(Y)=\frac{\partial\phi}{\partial y}\in\mathbb{R}^{n+1}.$$

Since $\phi(\mathbb{CP}^1)$ is contained in S^n they are orthogonal to $\phi(x,y)$. We have

$$\begin{aligned}
\nabla^{\phi}_X d\phi(X) =&\nabla^{S^n}_{d\phi(X)}d\phi(X)\\
=&\nabla^{\mathbb{R}^{n+1}}_{d\phi(X)}d\phi(X)-\left\langle\nabla^{\mathbb{R}^{n+1}}_{d\phi(X)}d\phi(X),\phi\right\rangle_{\mathbb{R}}\phi\\
=&\frac{\partial^2\phi}{\partial x^2}+\lambda_x\phi,\\
\nabla^{\phi}_Y d\phi(Y) =&\nabla^{S^n}_{d\phi(Y)}d\phi(Y)\\
=&\nabla^{\mathbb{R}^{n+1}}_{d\phi(Y)}d\phi(Y)-\left\langle\nabla^{\mathbb{R}^{n+1}}_{d\phi(Y)}d\phi(Y),\phi\right\rangle_{\mathbb{R}}\phi\\
=&\frac{\partial^2\phi}{\partial y^2}+\lambda_y\phi.
\end{aligned}$$

This implies that

$$\begin{aligned}
0 =&\tau(\phi)\\
=&\left(\frac{\partial^2\phi}{\partial x^2}+\lambda_x\phi-d\phi(\nabla^{\mathbb{CP}^1}_X X)\right)+\left(\frac{\partial^2\phi}{\partial y^2}+\lambda_y\phi-d\phi(\nabla^{\mathbb{CP}^1}_Y Y)\right)\\
=&\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+(\lambda_x+\lambda_y)\phi\right)-d\phi(\nabla^{\mathbb{CP}^1}_X X+\nabla^{\mathbb{CP}^1}_Y Y)\\
=&4D\bar{D}\phi+4\lambda\phi
\end{aligned}$$

where $\lambda=\frac{1}{4}(\lambda_x+\lambda_y)$ is a real valued function. This must satisfy $\lambda=(D\phi,\bar{D}\phi)$, since

$$\begin{aligned}
0 &= D\bar{D}(1)\\
&= D\bar{D}(\phi,\phi)\\
&= 2(D\bar{D}\phi,\phi)+2(D\phi,\bar{D}\phi)\\
&= -2\lambda(\phi,\phi)+2(D\phi,\bar{D}\phi).
\end{aligned}$$

(iii)[19] We see from

$$\begin{aligned}
\bar{D}(D\phi,D\phi) =&2(\bar{D}D\phi,D\phi)\\
=&2(-\lambda\phi,D\phi)\\
=&0
\end{aligned}$$

that $(D\phi,D\phi)$ is holomorphic. Further if $w=w(z)$ is another chart then $(D_w\phi,D_w\phi)=\left(\frac{dz}{dw}\right)^2(D_z\phi,D_z\phi)$. Thus $(D\phi,D\phi)dz^2\in\mathcal{H}(\mathbb{CP}^1,\Omega^2)$ so it must vanish.

Since

$$(D\phi,\bar{D}\phi)=\langle D\phi,D\phi\rangle_{\mathbb{C}}$$

it is obviously non-negative. The zeros are isolated since it is real analytic and must be nonzero at some point since ϕ is full. □

Lemma 3.10 ([7])**.** *Let $\phi : \mathbb{CP}^1 \to S^n$ be a full harmonic map then*

$$(D^j\phi, D^k\phi) = 0$$

for all $j + k \geq 1$ where $j, k \geq 0$.

Proof. [19] Differentiating $(\phi, \phi) = 1$ yields $(\phi, D\phi) = 0$ which corresponds to $j+k = 1$. In the same way from (ii) we have $(D\phi, D\phi) = 0$ now using

$$0 = D(\phi, D\phi) = (D\phi, D\phi) + (\phi, D^2\phi)$$

we get $(\phi, D^2\phi) = 0$ proving the lemma for $j + k = 2$. But $(D\phi, D\phi) = 0$ implies $(D\phi, D^2\phi) = 0$ so

$$0 = D(\phi, D^2\phi) = (D\phi, D^2\phi) + (\phi, D^3\phi)$$

gives us $(\phi, D^3\phi) = 0$ and we have proved the lemma for $j + k = 3$.

Suppose now that $(D^j\phi, D^k\phi) = 0$ for $1 \leq j + k \leq 2p - 1$ we show that it must hold for $1 \leq j + k \leq 2p + 1$ and the lemma follows by induction. By changing local coordinates z to $w(z)$ we have

$$(D^p_w\phi, D^p_w\phi) = \left(\frac{dz}{dw}\right)^{2p} (D^p_z\phi, D^p_z\phi).$$

We also have

$$\begin{aligned}
\bar{D}(D^p\phi, D^p\phi) =& 2(D^p\phi, \bar{D}D^p\phi) \\
=& 2(D^p\phi, D^{p-1}(D\bar{D}\phi)) \\
=& 2(D^p\phi, D^{p-1}(-\lambda\phi)) \\
=& -2(D^p\phi, \sum_{l=0}^{p-1} \left(\binom{p-1}{l}(D^l\lambda)(D^{p-1-l}\phi)\right)) \\
=& -2\sum_{l=0}^{p-1} \left(\binom{p-1}{l}(D^l\lambda)(D^p\phi, D^{p-1-l})\right) \\
=& 0.
\end{aligned}$$

This implies that $\omega_{2p} = (D^p\phi, D^p\phi)dz^{2p} \in \mathcal{H}(\mathbb{CP}^1, \Omega^{2p})$. So according to Proposition 2.38 we have $\omega_{2p} = 0$. All in all we get

$$(D^p\phi, D^p\phi) = 0 \text{ and } (D^{p+1}\phi, D^p\phi) = 0$$

which we use with

$$0 = D(D^j\phi, D^k\phi) = (D^{j+1}\phi, D^k\phi) + (D^j\phi, D^{k+1}\phi)$$

to obtain $(D^j\phi, D^k\phi) = 0$ for $1 \leq j + k \leq 2p + 1$. □

Lemma 3.11 ([6])**.** *Let $\phi : \mathbb{CP}^1 \to S^n$ be a full harmonic map and (U, z) be any local chart. Define the maps*

$$\Phi_p = \bar{D}\phi \wedge \bar{D}^2\phi \wedge \cdots \wedge \bar{D}^p\phi : U \to \bigwedge^p \mathbb{C}^{n+1}.$$

Then the uniquely determined $m \in \mathbb{N}$ such that $\Phi_m \neq 0$ and $\Phi_{m+1} = 0$ must satisfy $n = 2m$.

Proof. [6, 19] We define the map $\mathcal{F} = \phi \wedge \Phi_m \wedge \overline{\Phi_m}$. It satisfies

$$
\begin{aligned}
|\mathcal{F}| =& \det([\langle \mathcal{F}_i, \mathcal{F}_j \rangle_{\mathbb{C}}]_{i,j=1}^{2m+1}) \\
=& \det \begin{pmatrix} \langle \phi, \phi \rangle_{\mathbb{C}} & 0 & 0 \\ 0 & [\langle \bar{D}^i \phi, \bar{D}^j \phi \rangle_{\mathbb{C}}]_{i,j=1}^m & 0 \\ 0 & 0 & [\langle D^i \phi, D^j \phi \rangle_{\mathbb{C}}]_{i,j=1}^m \end{pmatrix} \\
=& \det \begin{pmatrix} [\langle \bar{D}^i \phi, \bar{D}^j \phi \rangle_{\mathbb{C}}]_{i,j=1}^m & 0 \\ 0 & [\langle D^i \phi, D^j \phi \rangle_{\mathbb{C}}]_{i,j=1}^m \end{pmatrix} \\
=& |\Phi_m|^2.
\end{aligned}
$$

In $\mathbb{C}^{n+1}$ the wedge product of $n+2$ vectors must be zero. And since there are points such that $|\Phi_m| \neq 0$, $\mathcal{F}$ which is the wedge product of $2m+1$ vector is non-zero, we must have $2m+1 < n+2$, i.e $2m \leq n$.

Since $\Phi_{m+1} = 0$, $D^{m+1}\phi$ must be a linear combination of the lower derivatives, i.e

$$D^{m+1}\phi = b_1 D\phi + \cdots + b_m D^m \phi$$

where b_i are smooth for $i = 1, \ldots, m$. By applying D we see that $D^n\phi$ is a linear combination of $D\phi, \ldots, D^m\phi$ for all n. By conjugating we see that $\bar{D}^n\phi$ is a linear combination of $\bar{D}\phi, \ldots, \bar{D}^m\phi$ for all n. Now since $D\bar{D}\phi = -\lambda\phi$ we have if, $k \leq l$ that

$$
\begin{aligned}
\bar{D}^k D^l \phi =& D^{l-k}(D\bar{D})^k \phi \\
=& D^{l-k}((-\lambda)^k \phi) \\
=& a_0\phi + a_1 D\phi + \cdots + a_{l-k} D^{l-k}\phi,
\end{aligned}
$$

where a_j are smooth for $j = 1, \ldots, (l-k)$. Conjugating we get a similar relation for $k \geq l$. Thus we obtain

$$\operatorname{span}_{\mathbb{C}}\{\phi, D\phi, \ldots, D^m\phi, \bar{D}\phi, \ldots, \bar{D}^m\phi\} = \operatorname{span}_{\mathbb{C}}\{\phi, \bar{D}^k D^l \phi\}_{1 \leq k+l}.$$

Since ϕ is full

$$\operatorname{span}_{\mathbb{C}}\{\phi, \bar{D}^k D^l \phi\}_{1 \leq k+l} = \mathbb{C}^{n+1}$$

in at least some point. So we must have $2m+1 \geq n+1$. The two inequalities imply $n = 2m$. □

Lemma 3.12 ([7])**.** *Let $\phi : \mathbb{CP}^1 \to S^{2m}$ be a full harmonic map and $\pi : \mathcal{I}_m \to S^{2m}$ be the natural Riemannian submersion. Then there exist a holomorphic map*

$$\psi : \mathbb{CP}^1 \to \mathbb{P}(\bigwedge^m \mathbb{C}^{2m+1})$$

such that $\psi(\mathbb{CP}^1) \subset \mathcal{I}_m$, ψ is horizontal with respect to π and $\phi = \pi \circ \psi : \mathbb{CP}^1 \to S^{2m}$.

Proof. [19] Let Φ_m and $\mathcal{F}$ be as in Lemma 3.11. Then

$$
\begin{aligned}
\bar{D}\Phi_m =& \bar{D}(\bar{D}\phi \wedge \cdots \wedge \bar{D}^m\phi) \\
=& \big(\bar{D}^{1+1}\phi \wedge \bar{D}^2\phi \wedge \bar{D}^3\phi \wedge \cdots \wedge \bar{D}^m\phi\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=2}^{m-1}\left(\bar{D}\phi\wedge\cdots\wedge\bar{D}^{j+1}\phi\wedge\cdots\wedge\bar{D}^{m}\phi\right)\\
&+\left(\bar{D}\phi\wedge\cdots\wedge\bar{D}^{m-1}\phi\wedge\bar{D}^{m+1}\phi\right)\\
=&\bar{D}\phi\wedge\cdots\wedge\bar{D}^{m-1}\phi\wedge(\overline{b_1}\bar{D}\phi+\cdots+\overline{b_m}\bar{D}^{m}\phi)=\overline{b_m}\Phi_m
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{D}\mathcal{F}=&\bar{D}(\phi\wedge\bar{D}\phi\wedge\cdots\wedge\bar{D}^{m}\phi\wedge D\phi\wedge\cdots\wedge D^{m}\phi)\\
=&\left(\bar{D}\phi\wedge\bar{D}\phi\wedge\bar{D}^{2}\phi\wedge\cdots\wedge\bar{D}^{m}\phi\wedge D\phi\wedge\cdots\wedge D^{m}\phi\right)\\
&+\sum_{j=1}^{m-1}\left(\phi\wedge\bar{D}\phi\wedge\cdots\wedge\bar{D}^{j+1}\phi\wedge\cdots\wedge\bar{D}^{m}\phi\wedge D\phi\wedge\cdots\wedge D^{m}\phi\right)\\
&+\left(\phi\wedge\bar{D}\phi\wedge\cdots\wedge\bar{D}^{m-1}\phi\wedge\bar{D}^{m+1}\phi\wedge D\phi\wedge\cdots\wedge D^{m}\phi\right)\\
&+\left(\phi\wedge\bar{D}\phi\wedge\cdots\wedge\bar{D}^{m-1}\phi\wedge\bar{D}^{m}\phi\wedge\bar{D}D\phi\wedge\cdots\wedge D^{m}\phi\right)\\
&+\sum_{j=2}^{m-1}(\phi\wedge\bar{D}\phi\wedge\cdots\wedge\bar{D}^{m}\phi\wedge D\phi\wedge\cdots\wedge\bar{D}D^{j}\phi\wedge\cdots\wedge D^{m}\phi)\\
&+\left(\phi\wedge\bar{D}\phi\wedge\cdots\wedge\bar{D}^{m-1}\phi\wedge\bar{D}^{m}\phi\wedge D\phi\wedge\cdots\wedge\bar{D}D^{m}\phi\right)\\
=&\left(\phi\wedge\bar{D}\phi\wedge\cdots\wedge\bar{D}^{m-1}\phi\wedge\bar{D}^{m+1}\phi\wedge D\phi\wedge\cdots\wedge D^{m}\phi\right)\\
&+\left(\phi\wedge\bar{D}\phi\wedge\cdots\wedge\bar{D}^{m-1}\phi\wedge\bar{D}^{m}\phi\wedge(-\lambda\phi)\wedge\cdots\wedge D^{m}\phi\right)\\
&+\sum_{j=2}^{m-1}(\phi\wedge\bar{D}\phi\wedge\cdots\wedge\bar{D}^{m}\phi\wedge D\phi\wedge\cdots\wedge D^{j-1}(-\lambda\phi)\wedge\cdots\wedge D^{m}\phi)\\
&+\left(\phi\wedge\bar{D}\phi\wedge\cdots\wedge\bar{D}^{m-1}\phi\wedge\bar{D}^{m}\phi\wedge D\phi\wedge\cdots\wedge D^{m-1}(-\lambda\phi)\right)\\
=&\overline{b_m}\mathcal{F}
\end{aligned}
$$

where the b_i are as in Lemma 3.11.

Denote by $\mathcal{R}$ the set of zeros for Φ_m

$$
\mathcal{R}=\{x\in\mathbb{CP}^1|\Phi_m(x)=0\}.
$$

Since $\bigwedge^{2m+1}\mathbb{C}^{2m+1}$ is one dimensional $\mathcal{F}$ is in fact a function from $\mathbb{CP}^1$ to $\mathbb{C}$. Thus in $\mathbb{CP}^1\backslash\mathcal{R}$ we can calculate

$$
\begin{aligned}
\bar{D}(\Phi_m\mathcal{F}^{-1})&=\bar{D}(\Phi_m)\mathcal{F}^{-1}-\Phi_m\mathcal{F}^{-2}\bar{D}(\mathcal{F})\\
&=\overline{b_m}\Phi_m\mathcal{F}^{-1}-\Phi_m\overline{b_m}\mathcal{F}\mathcal{F}^{-2}=0.
\end{aligned}
$$

So $\Psi=\Phi_m\mathcal{F}^{-1}$ is holomorphic in $\mathbb{CP}^1\backslash\mathcal{R}$. The points in $\mathcal{R}$ are isolated so the singularities of Ψ are poles so it is meromorphic. We define the map

$$
\psi=\pi_{\mathbb{P}}\circ\Psi:\mathbb{CP}^1\backslash\mathcal{R}\to\mathbb{P}(\bigwedge^{m}\mathbb{C}^{2m+1}).
$$

The map is well defined since if (U,z) and (V,w) are charts then

$$
\bar{D}_w\phi\wedge\cdots\wedge\bar{D}_w^{m}\phi=\overline{\left(\frac{dz}{dw}\right)}^{m(m+1)/2}\bar{D}_z\phi\wedge\cdots\wedge\bar{D}_z^{m}\phi.
$$

The vectors $\bar{D}\phi, \ldots, \bar{D}^m\phi$ are orthogonal to $\phi, D\phi, \ldots, D^m\phi$ since

$$\left\langle D^j\phi, \bar{D}^k\phi\right\rangle = (D^j\phi, D^k\phi).$$

Thus $\Phi_m = \bar{D}\phi \wedge \cdots \wedge \bar{D}^m\phi$ is the Plücker coordinate of an isotropic m-plane in $\mathbb{C}^{2m+1}$, so

$$\psi(\mathbb{CP}^1 \backslash \mathcal{R}) \subset \mathcal{I}_m.$$

By Proposition 2.39 there is a unique holomorphic extension of ψ to all of $\mathbb{CP}^1$, we denote this extension by ψ also. Since $\mathcal{I}_m$ is compact and hence closed, we have $\psi(\mathbb{CP}^1) \subset \mathcal{I}_m$. Also since ϕ is orthogonal to all $D^j\phi, \bar{D}^j\phi$ for $j = 1, \ldots, m$ we have

$$\pi \circ \psi = \phi.$$

Finally we show that ψ is horizontal with respect to π. Let $p \in \mathbb{CP}^1$. The map ψ could also be given as

$$\psi(p) = \operatorname{span}_{\mathbb{C}}\{\bar{D}\phi(p), \ldots, \bar{D}^m\phi(p)\}.$$

Now the differential at p is a map

$$d\psi_p : T_p\mathbb{CP}^1 \to T_{\psi(p)}\mathcal{I}_m \subset \operatorname{Hom}_{\mathbb{C}}(\psi(p), \psi(p)^\perp).$$

We calculate $d\psi_p$ as

$$\begin{aligned}\psi(p) \ni (c_1\bar{D}\phi(p) + \cdots + c_m\bar{D}^m\phi(p)) &\mapsto D(c_1\bar{D}\phi + \cdots + c_m\bar{D}^m\phi)(p) \\ &\equiv -\big(c_1\lambda + \cdots + c_m(\bar{D}^{m-1}\lambda)\big)\phi \; \big(\operatorname{mod} \psi(p)\big).\end{aligned}$$

Thus $d\psi_p(D)$ is the linear homomorphism that maps the basis element $\bar{D}^k\phi(p)$ for $\psi(p)$ according to

$$d\psi_p(D) : \psi(p) \ni \bar{D}^k\phi \mapsto (\bar{D}^{k-1}\lambda)\phi \in \operatorname{span}_{\mathbb{C}}\{\phi(p)\}.$$

That is

$$d\psi_p : T_p\mathbb{CP}^1 \to \operatorname{Hom}_{\mathbb{C}}(\psi(p), \operatorname{span}_{\mathbb{C}}\{\phi(p)\}) = \mathcal{H}_{\psi(p)}\mathcal{I}_m$$

so ψ is horizontal. □

Proof of Calabi's theorem. (i) This is a direct consequence of Lemma 3.11.

(ii) The statement follows from Lemma 3.12. □

There exist a generalization of Calabi's theorem that use the twistor construction to deals with all harmonic maps $\phi : \mathbb{CP}^1 \to \mathbb{CP}^n$. Both S^n and $\mathbb{CP}^n$ are compact symmetric spaces so it is natural to ask if it is possible to use the twistor construction to classify all harmonic maps $\mathbb{CP}^1 \to G/K$ where G/K is some compact symmetric space. The answer is negative (see [5]), the twistor construction classifies all harmonic maps only in the cases S^n and $\mathbb{CP}^n$.

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