Higher Categorical Structures in Geometry

General Theory and Applications to Quantum Field Theory

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Thomas Nikolaus

aus Esslingen am Neckar

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und Prof. Dr. Chenchang Zhu

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Prof. Dr. Ingenuin Gasser
Leiter des Fachbereichs Mathematik

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Introduction

Higher categorical structures in geometry

The following situation arises frequently in mathematics and mathematical physics: for a given smooth, finite dimensional manifold M we want to consider certain classes of geometric objects on M. The reader should keep in mind structures like metrics or symplectic forms or, more important for this thesis, objects like bundles. There are many reasons that one is interested in such objects, let us list two here:

- One wants to gather information about the structure of M as a manifold. For example one can use a metric to compute holonomy groups and thereby better understand the global and local behavior of M. Another typical situation is to compute the set of isomorphism classes of G-bundles over M for a fixed Lie group G. This turns out to be an invariant of the homotopy type of M, hence can be used to distinguish manifolds that are not homotopy equivalent.

- One is interested in the objects over M itself. This situation especially occurs in mathematical physics. For example in general relativity the object of interest is not the mere spacetime manifold M but a Lorentzian metric on M. Another class of examples is given by gauge theories, such as Yang-Mills-theory. The fields are given by connections on (non-abelian) bundles over M. Such fields can also play the role of background fields. For example the electromagnetic field in classical electromagnetism is given by a U(1)-bundle with connection over M that determines the equations of motion for charged particles moving through M.

For bundles it is very important not only to consider the geometric objects over M, but also to take the morphisms into account, i.e. the gauge transformations. This shows that we really associate *categories* of objects to M.

Now we do not want to restrict ourselves to one fixed manifold M, but allow different manifolds. Therefore we have to take the transformation-behavior of the geometric objects into account. More precisely we want to specialize to geometric objects that behave like bundles in so far as they can be pulled back along smooth maps $f : N \longrightarrow M$. The mathematical structure that formalizes this behavior is called a *stack*, see [Met03, Hei05] for a definition in the differentiable setting. Apart from associating categories to smooth manifolds and pullback functors to smooth maps, a stack has another important defining property that turns out to be crucial for geometry and central for this thesis. Namely it has to satisfy a 'locality condition' called the *descent property*. Roughly speaking this property ensures that the geometric objects can be glued together from locally defined objects. If we think of bundles again this property is clearly satisfied and can be seen as a guiding principle since the local behavior of bundles is prescribed by definition, i.e. locally they look like a product of M with a vector space, manifold, torsor etc. For a more precise discussion in the case of U(1)-bundles see section 1.2.1.

In the past years it has turned out that there are certain geometric objects over M for which we do not only have to take morphisms into account, but also 2-morphisms, i.e. gauge transformations between gauge transformations. Let us give two guiding examples here:

- An important class of such objects is given by bundle gerbes and bundle gerbes with connection [Bry93, Mur96, Ste00, Wal07]. See also section 1.2.2 and 2.4.1 of this thesis. In particular bundle gerbes and related objects are needed in two-dimensional non-linear sigma models with Wess-Zumino term. The role they play is analogous to the role of U(1)-bundles with connection in electromagnetism. From the mathematical side, the feature of bundle gerbes (resp. Jandl gerbes) entering here is that they allow to define surface holonomy (resp. unoriented surface holonomy). We will explain that in more detail in the next part of this introduction and in chapter 1.

- Another class of examples is given by 2-principal bundles for 2-groups [Bar04, Woc08]. See also section 3.6 for a slightly different approach. These 2-bundles are classified by non-abelian cohomology as considered in [Gir71, Bre94], see also section 3.3. One of the most important 2-groups is the string 2-group, see [BCSS07] and section 4.5 for another model. Geometric string structures are needed in supersymmetric sigma models to cancel certain anomalies in the fermionic functional integral, see [Wal09, Bun09] and also later in this introduction.

To treat such 2-categorical examples we cannot use ordinary stacks but have to consider 2-stacks. A 2-stack assigns 2-categories (or more generally bicategories) to each smooth manifold M and pullback 2-functors to smooth maps $f : N \longrightarrow M$ (section 2.2.2). Still a 2-categorical analogue of the *descent condition* has to be imposed in order to make the objects behave geometrically (definition 2.2.12). It turns out that again, as for 1-stacks, it suffices to control the local behavior of the objects in order to produce the 2-categories of global objects by means of a 2-stackification procedure, see section 2.3.

For the examples given above (bundle gerbes, 2-bundles...) we make the definition and the structure explicit in terms of 2-stacks. This allows us to give a systematic treatment of surface holonomy and unoriented surface holonomy from first principles (section 1.2.3 and 2.4.3). Moreover it allows to compare several approaches to 2-bundles and non-abelian gerbes that have appeared in the literature, see chapter 3. Finally it allows to take symmetries into account properly. More precisely it allows to give a consistent definition of equivariant objects from the mere description as a 2-stack (section 2.2.2). This definition is given very generally in terms of Lie groupoids but agrees with previously introduced concepts in special cases. Finally it can be shown to be well-behaved with respect to Morita equivalence of groupoids (Theorem 2.2.16). This for example allows to simplify bundle gerbes which are equivariant under the action of a Lie group G on a manifold M in terms of central extensions of stabilizers and gluing isomorphisms [Mei03, Nik09].

So far we have emphasized the importance of 2-stacks in geometry and will explain their role in quantum field theory later. But let us first come to another related occurrence of categorical structures in low-dimensional geometry. It goes by the name of three-dimensional topological field theory. Topological field theory is a mathematical structure that has been inspired by physical theories [Wit89]. A three-dimensional topological field theory, more specifically, assigns complex invariants to 3-manifolds. It contains more structure that allows to compute the 3-manifold invariants by cutting the 3-manifold along 2-dimensional submanifolds, see [Ati88]. This additional structure can again be seen as a 'locality condition' like the descent property of stacks. It is now a natural idea to cut these 2-manifolds along 1-dimensional submanifolds to further simplify the computation. The structure needed to make this additional step well-behaved is a so-called *extended* three-dimensional field theory [Law93, Lur09b]. An extended topological field theory is defined as a 2-functor between a geometric 2-category and an algebraic 2-category, see definition 5.2.8. In particular, it assigns $\mathbb{C}$-linear categories to 1-manifolds.

Let us note here that three-dimensional extended topological field theories are related to the higher categorical geometric structures such as bundle gerbes described above. We will explain this relationship in more detail below. For the purpose of this introduction, we just mention that there is a notion of equivariant topological field theory ([Kir04, Tur10] and section 5.3.3) which is closely related to our concept of equivariance for 2-stacks. We demonstrate this relation in section 5.3 where we use the geometric and physical intuition from the rest of the thesis to construct and explicitly describe equivariant extensions of a particularly nice class of topological field theories called Dijkgraaf-Witten theories [DW90].

Finally from extended three-dimensional field theories one can extract interesting algebraic data, called modular tensor categories, see [BK01] and section 5.2.5. Conversely one can construct a three-dimensional topological field theory from a modular tensor category. Therefore the study of three-dimensional topological field theories can be understood as the study of modular tensor categories. Analogously there is a concept of equivariant modular tensor category [Kir04, Tur10], and the study of equivariant three-dimensional field theories can be seen as the study of equivariant modular tensor categories. This allows to reinterpret the equivariant Dijkgraaf-Witten theory constructed in section 5.3 in purely algebraic terms. We find an equivariant Hopf-algebra which, as a byproduct, solves a purely algebraic problem which arose independently [Ban05, MS10]. See also section 5.1.1 for a motivation from this point of view.

Surface Holonomy and the Wess-Zumino Term

Two-dimensional conformal field theories (CFTs) have been a source for several interesting developments and for deep relations between mathematics and physics.

We concentrate here on conformal field theories (or, more generally, on two-dimensional quantum field theories) that admit a classical description by a sigma model, at least heuristically. Such a (non-linear) sigma model assigns to any smooth map $\phi : \Sigma \longrightarrow M$ between a surface Σ, called the world-sheet, and a manifold M, called the target space, a Feynman amplitude: that is a complex number $\mathcal{A}(\phi)$. This complex number serves heuristically as the integrand in the functional integral of the quantum theory. Such sigma models in particular play a role in string theory, where the map ϕ describes the string moving through M, i.e. Σ parametrizes the surface swept out by the moving string.

Now connections on gerbes over M contribute a factor to the definition of the Amplitude $\mathcal{A}(\phi)$. More precisely they provide a topological term in the action, called the Wess-Zumino term, by virtue of the surface holonomy around Σ.

Let us explain this in more detail here. Usually the amplitude consists of a so-called *kinetic amplitude* $\mathcal{A}^{\text{kin}}(\phi)$, which can be defined using a metric g on M as follows:

$$\mathcal{A}^{\text{kin}}(\phi) := \exp\big(2\pi \mathrm{i} S^{\text{kin}}(\phi)\big)$$

where the *kinetic action term* $S^{\text{kin}}(\phi)$ is defined by

$$S^{\text{kin}}(\phi) := \frac{1}{2}\int_\Sigma g\big(d\phi \wedge \star d\phi\big).$$

Now it turns out that one has to add another term $\mathcal{A}^{\text{WZ}}(\phi)$ to the amplitude in order to obtain conformal invariance of the quantum theory. This additional term has first been introduced in the case that the target space is given by a compact, simple, simply connected Lie group G [Wit84].

Let us review Witten's definition of the Wess-Zumino term. We shall thus explain how to obtain a complex number $\mathcal{A}^{\text{WZ}}(\phi)$ for a smooth map $\phi : \Sigma \longrightarrow G$. The definition relies on topological properties of the Lie group G. As a first step choose an oriented three dimensional manifold $\widetilde{\Sigma}$ whose boundary is Σ. Such a manifold exists but is not unique. Now we use the fact that $\pi_2(G) = 0$ for G, which is true for all finite dimensional Lie groups, to extend the map $\phi : \Sigma \longrightarrow G$ to a map $\widetilde{\phi} : \widetilde{\Sigma} \longrightarrow G$. For a compact, simple, simply connnected Lie group G we have $\mathrm{H}^3(G, \mathbb{Z}) = \mathbb{Z}$ and there is a canonical bi-invariant 3-form H over G given by

$$H = \frac{1}{6}\langle \theta, [\theta, \theta] \rangle \tag{1}$$

where $\langle \quad , \quad \rangle$ is an invariant metric on G and θ is the left invariant Maurer-Cartan form on G. The 3-form H has integral periods and coincides with the image of the generator $1 \in \mathrm{H}^3(G, \mathbb{Z})$ in $\mathrm{H}^3_{\mathrm{dR}}(G) \cong \mathrm{H}^3(G, \mathbb{Z}) \otimes \mathbb{R}$. With this form and the extension $\widetilde{\phi} : \widetilde{\Sigma} \longrightarrow G$ Witten defined

$$S^{\mathrm{WZ}}\left(\widetilde{\Sigma}, \widetilde{\phi}\right) := \int_{\widetilde{\Sigma}} \widetilde{\phi}^* H$$

and showed that the amplitude

$$\mathcal{A}^{\mathrm{WZ}}(\phi) := \exp\left(2\pi \mathrm{i} S^{\mathrm{WZ}}\big(\widetilde{\Sigma}, \widetilde{\phi}\big)\right)$$

is well-defined, i.e. independent of the choice of $\widetilde{\Sigma}$ and $\widetilde{\phi}$. Moreover he indicated that the full Feynman amplitude $\mathcal{A}(\phi) := \mathcal{A}^{\mathrm{kin}}(\phi) \cdot \mathcal{A}^{\mathrm{WZ}}(\phi)$ leads to a conformally invariant two-dimensional quantum field theory (i.e. a CFT) which is called the Wess-Zumino-Witten model.

At this point one can try to generalize Witten's description of the Wess-Zumino term for an arbitrary target space M equipped with a metric g and a 3-form H. But if M is not 2-connected, there are in general obstructions against the extension of a smooth map $\phi : \Sigma \longrightarrow M$ to a smooth map $\widetilde{\phi} : \widetilde{\Sigma} \longrightarrow M$. It is then a better strategy to find local 2-forms B_i on open subsets U_i of M such that $dB_i = H$. Locally the integral of B_i over Σ can serve as a substitute for the integral of H over $\widetilde{\Sigma}$ by means of Stokes' theorem. Hence the choice of locally defined 2-forms B_i over M allows to define a local contribution to the amplitude. However in order to turn this into a globally well-defined amplitude we have to take local gauge transformations into account which are here 1-forms A_{ij} defined on double overlaps $U_i \cap U_j$. Since a 1-form can itself be a derivative there are even gauge transformations between these gauge transformations, i.e. U(1)-valued functions g_{ijk} defined on triple overlaps. This can then be combined into a well-defined expression for the amplitude $\mathcal{A}^{\mathrm{WZ}}(\phi)$ which has first been discovered in terms of Deligne-cohomology [Gaw88].

The local description given above in terms of 2-forms B_i, 1-forms A_{ij} and U(1)-valued functions g_{ijk} suggests that again a 2-categorical structure is present. Indeed, one can define bicategories associated to each smooth manifold M and then apply the general stackification construction given in section 2.3. In this way we obtain global objects which allow for a surface holonomy, see section 1.2. These objects have been introduced before under the name *bundle gerbes with connection* [Mur96, MS00]. Isomorphism classes of bundle gerbes $\mathcal{G}$ over a manifold M are classified by a characteristic class $DD(\mathcal{G}) \in \mathrm{H}^3(M, \mathbb{Z})$, called the Dixmier-Douady class. Moreover a connection on a bundle gerbe provides a curvature three form with integral periods, which agrees with the image of the Dixmier-Douady class in $H^3(M, \mathbb{R})$.

Now we revisit the case of a compact, simple, simply connected Lie group G. There is a canonical gerbe, which realizes the generator $1 \in H^3(G, \mathbb{Z}) = \mathbb{Z}$ [GR02, Mei03]. This gerbe moreover admits a unique connection with curvature given by the bi-invariant three form $H \in \Omega^3(G)$, which was given in equation (1). Finally it is basically an application of Stokes' theorem to show that the holonomy of the gerbe around a smooth map $\phi : \Sigma \longrightarrow G$ agrees with the Wess-Zumino term $\mathcal{A}^{\mathrm{WZ}}(\phi)$ defined by Witten. Therefore we see that bundle gerbes with connection provide a global framework for the definition of the Wess-Zumino term which is not bound to compact, simply connected Lie groups.

Our systematic introduction of bundle gerbes, building only on the knowledge of the local description needed for a consistent definition of surface holonomy, allows us to easily generalize resp. adapt to different cases. For example we give a definition of a Jandl gerbe (section 1.4 and section 2.4.2) generalizing and clarifying earlier work [SSW07]. Jandl gerbes allow for a definition of surface holonomy around unoriented, possibly not even orientable, surfaces. Thereby, they provide the Wess-Zumino term in unoriented WZW models, see section 2.4.3. These unoriented world sheets arise e.g. in type I string theories.

String structures and supersymmetric sigma models

So far we have described field theories where the 'fields' are given by smooth maps $\phi : \Sigma \longrightarrow M$. From the perspective of string theory ϕ describes the worldsheet of a string moving through the target space M. But it only describes the bosonic string. Hence these theories are called bosonic sigma models. A general superstring theory should clearly also incorporate worldsheet fermions. This can be done using supersymmetric sigma models. In such a supersymmetric sigma model we need in addition a spin-structure on the world sheet Σ. Such a spin structure can equivalently be considered as an $N = 1$ superconformal structure on Σ [MM91].

Remember that $\mathrm{Spin}(n)$ is a compact, connected Lie group which is a $\mathbb{Z}/2$-covering of $\mathrm{SO}(n)$. A spin structure is then by definition a lift of the frame bundle $P_{\mathrm{SO}(n)}$ of an oriented Riemannian manifold X to a $\mathrm{Spin}(n)$-bundle $P_{\mathrm{Spin}(n)}$. In general such a lift does not need to exist, and if it exists, it is only unique up to an element in $H^1(X, \mathbb{Z}/2)$.

For a given spin structure on Σ we construct the *spinor bundle* $S\Sigma$ and moreover for each map $\phi : \Sigma \longrightarrow M$ we obtain a *twisted Dirac operator*

$$D_\phi : \quad \Gamma(S\Sigma \otimes \phi^* TM) \longrightarrow \Gamma(S\Sigma \otimes \phi^* TM).$$

Furthermore the family D_ϕ has a determinant line bundle $\mathrm{Det}(D)$, which is a line bundle over the space $C^\infty(\Sigma, M)$. This line bundle admits a canonical square root $\mathrm{Pfaff}(D)$, the *Pfaffian*. For these facts see [Fre87, Bun09].

Now given a spin-structure on Σ we not just take into account a bosonic field, which is a map $\phi : \Sigma \longrightarrow M$, but additionally a fermionic worldsheet field, which is a section

$$\psi \in \Gamma(S\Sigma \otimes \phi^* TM).$$

Again, as before, we want to define a Feynman amplitude $\mathcal{A}(\phi, \psi) \in \mathbb{C}$ for each pair (ϕ, ψ). It consists of the bosonic kinetic term $\mathcal{A}^{\mathrm{kin}}(\phi)$ which only depends on ϕ and a *fermionic amplitude* $\mathcal{A}^{\mathrm{fer}}(\phi, \psi) := \exp(2\pi \mathrm{i} S^{\mathrm{fer}}(\phi, \psi))$, with the fermionic action term

$$S^{\mathrm{fer}}(\phi, \psi) := \int_\Sigma \langle \psi, D_\phi \psi \rangle \ \mathrm{dvol}_\Sigma.$$

The idea is now to perform the *fermionic path integral*, i.e. integrate over the space of all fermions for a given map $\phi : \Sigma \longrightarrow M$. In [FM06] it is explained why this heuristic integral should not yield a complex number but an element in the Pfaffian line bundle:

$$\hat{\mathcal{A}}^{\mathrm{fer}}(\phi) = \text{“} \int \mathrm{d}\psi \ \mathcal{A}^{\mathrm{fer}}(\phi, \psi) \text{ ”} \quad \in \mathrm{Pfaff}(D).$$

This element is then rigourosly defined using spectral theory of Dirac operators. Moreover the assignment $\hat{\mathcal{A}}^{\mathrm{fer}}$ turns out to be a section of the Pfaffian line bundle $\mathrm{Pfaff}(D) \longrightarrow C^\infty(\Sigma, M)$.

Now the next step is motivated by the idea that the *effective amplitude*

$$\hat{\mathcal{A}}(\phi) := \mathcal{A}^{\mathrm{kin}}(\phi) \cdot \hat{\mathcal{A}}^{\mathrm{fer}}(\phi) \quad \in \Gamma(\mathrm{Pfaff(D)})$$

should be subject to another functional integral, this time over the bosonic degrees of freedom. Therefore we need a trivialization of the Pfaffian line bundle over $C^\infty(\Sigma, M)$. By work of Bunke [Bun09] such a trivialization for all choices of Σ is provided by a geometric string structure on the target space M.

Let us explain also from the mathematical side what string structures are. First of all, the topological group $\mathrm{String}(n)$ is required to be an object in the *Whitehead Tower* of the Lie group $\mathrm{O}(n)$:

$$\cdots \longrightarrow \mathrm{String}(n) \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \longrightarrow \mathrm{O}(n).$$

More precisely $\mathrm{String}(n)$ is a 3-connected cover of $\mathrm{Spin}(n)$, which fixes $\mathrm{String}(n)$ up to homotopy. For concrete constructions see [Sto96, ST04]. It is a natural question whether String(n) can also be realized as a (necessarily infinite dimensional) Lie group. We give an affirmative answer in chapter 4. Then a *string structure* on an oriented Riemannian manifold M is a lift of the frame bundle $P_{\mathrm{SO}(n)}$ to a String(n)-bundle $P_{\mathrm{String}(n)}$. This is the initial point for string geometry on M, which is closely related to spin geometry on the free loop space LM [Wit88, Sto96].

There have been other approaches to the string group using 2-group models [BCSS07, SP10]. They have a number of advantages, in particular imposing tighter constraints on the models. We define and explain what this means in section 4.4. However, if one replaces groups by 2-groups one also has to replace bundles by 2-bundles. There have been different approaches and definitions of 2-bundles [Jur05, Bar04, Woc08]. In this thesis, we repeat and improve these definitions from our general higher categorical perspective on geometry and provide direct comparisons between them in chapter 3. Moreover we give a new 2-group model for the string group which allows to compare 2-bundle definitions of string structures to ordinary string structures in section 4.5.

This comparison of ordinary string structures and higher-categorical string structures, presented in section 4.6 allows to make contact to other work: geometric string structures have been defined and studied in [Wal09]. Based on these results, Bunke [Bun09] produced the trivialization of the Pfaffian line bundle whose importance has been explained above.

Chiral CFT and Dijkgraaf-Witten theory

We now take a different approach to conformal field theories. Remember that sigma models, as described above, are a source of examples for quantum field theories, at least on a heuristic level. Or to put it another way, one can see a sigma model as a classical limit of a quantum field theory.

We are in this thesis more specifically interested in two dimensional conformal field theories. Among these, a particularly tractable subclass is given by rational conformal field theories (RCFTs) for which a rigorous approach via representation theory exists. In this case one obtains a rational conformal vertex algebra $\mathcal{V}$, which conversely encodes the chiral part of the RCFT (see [FBZ04] and section 1.3.2 for more details). The representation category of $\mathcal{V}$ is a modular tensor category, see definition 5.2.20. In this situation we can use the tools of three-dimensional topological quantum field theory (TFT) to obtain information about the full CFT, in particular to compute the correlations functions, see [FRS02, FRS04, FRS05, FFRS06] and section 1.3.3 for a short review. The TFT that is important in this situation can be built out of the representation category of $\mathcal{V}$ by a construction of Resehtikin and Turaev [RT91]. As mentioned earlier, modular tensor categories are even in 1-1 correspondence with extended three dimensional TFTs (up to some hard technicalities). See also section 5.2.4 for a description how to obtain a modular category from a TFT.

Now let us come to the chiral RCFT given by the Wess-Zumino-Witten model. In this specific situation, the relevant TFT is called Chern-Simons theory and has been introduced in [Wit89], see also [Fre95]. Chern-Simons theory admits a classical description as a 3-dimensional sigma model. Therefore let M be a closed manifold of dimension 3 and G be a compact, simple Lie group. As the 'space' of field configurations, we choose principal G-bundles with connection,

$$\mathcal{A}_G(M) := \mathcal{B}un_G^{\nabla}(M).$$

Now assume G is simply connected. In this situation, each G-bundle P over M is globally of the form $P \cong G \times M$, which follows by $\pi_0(G) = \pi_1(G) = \pi_2(G) = 0$ and standard obstruction theory. Hence a field configuration is given by a connection on the trivial bundle which is a 1-form $A \in \Omega^1(M, \mathfrak{g})$ with values in the Lie algebra of G. The Chern-Simons action can then be defined by

$$S[A] := \int_M \langle A \wedge dA \rangle - \frac{1}{6} \langle A \wedge A \wedge A \rangle$$

where $\langle \cdot, \cdot \rangle$ is the basic invariant inner product on the Lie algebra $\mathfrak{g}$.

Now, we want to drop the condition that the group G is simply connected. In this case the situation changes crucially, since we may have topologically nontrivial G-bundles over M. In order to apply the results from above we consider the simply connected cover $\tilde{G}$ of G which turns out to be an extension by a discrete group (the fundamental group of G). Hence we first try to understand the theory for a discrete group G and the general case is a combination of the discrete case and the simply connected case. For the case that G is even finite the theory has been defined and investigated in [DW90, FQ93] and is called Dijkgraaf-Witten theory. The advantage of Dijkgraaf-Witten theory is that one can rigorously obtain the quantum theory from the classical description due to the finiteness of G. We review this process in section 5.2. Moreover one can even explicitly determine the modular category and describe it algebraically via a Hopf algebra $\mathcal{D}(G)$, the Drinfel'd double of G [BK01].

Inspired by our discussion of equivariance in sigma models (chapter 2) we investigate the corresponding notion for Dijkgraaf-Witten models in section 5.3. We give a construction of equivariant Dijkgraaf-Witten theory based on an action of another finite group J which acts on G. As in the non-equivariant case we obtain an extended topological field theory which is equivariant under to action. This leads us to the equivariant Drinfel'd double $\mathcal{D}^J(G)$ (section 5.4.3) whose representation category is equivariant modular.

Summary of results

Now we give a short description of what we consider to be the main results of this thesis.

The main novelty in the first chapter is the descent perspective on the definition of bundle gerbes and Jandl structures. This is the basis for the theory of 2-stacks we develop in chapter 2. In particular we extend 2-stacks on manifolds to 2-stacks on Lie groupoids. The central technical result is that stacks are invariant under Morita equivalences of Lie groupoids. This result allows us to give a general stackification procedure and to recognize bundle gerbes and Jandl gerbes as special instances of this general construction.

In chapter 3, we set up a precise framework for four versions of non-abelian gerbes: Čech cocycles, classifying maps, bundle gerbes, and principal 2-bundles. We present structural results and results relating these four frameworks in a very precise sense. The proofs rely on the results on 2-stacks presented in chapter 2.

In the chapter 4 we present a concrete construction of the string 2-group. More precisely we present an (infinite dimensional) smooth model of string group as a 1-group and enlarge this to a model as a 2-group. This 2-group can serve as structure group for the general 2-bundle theory developed in chapter 3.

In the last chapter we present an equivariant generalization of extended Dijkgraaf-Witten theory based on a weak action of a finite group J on another finite group G. From this geometric construction of the TFT 2-functor we extract the algebraic data of an equivariant modular category.

Outline of the thesis

We now want to give a more detailed description of how this thesis is organized and briefly list the main results of the chapters.

Chapter 1: In section 1.1 we shortly review hermitian line bundles and their holonomy with special emphasis on the local descriptions. We show that line bundles can be glued together from the local data and make explicit the structure of a stack in section 1.2.1. We then give a similar definition of bundle gerbes as descent objects in section 1.2.2 and how this leads to a consistent notion of surface holonomy 1.2.3. This surface holonomy enters as the Wess-Zumino term in non-linear sigma models. The description using gerbes allows to classify Wess-Zumino-Witten models and explain some facts such as discrete torsion, see section 1.2.4.

Section 1.3 is devoted to the representation theoretic description of conformal field theories. We explain in more detail the relation between sigma models and CFTs (section 1.3.1), the relation of RCFTs and TFTs (section 1.3.2) and finally the TFT construction for a full RCFT (section 1.3.3). In particular, the algebraic results serve as a guide for geometric structures and constructions in sigma models.

In section 1.4 we review the definition of Jandl structures on gerbes from a local perspective. Then we show that they allow to define surface holonomy around unoriented surfaces and give a local formula. In the rest of the chapter the notions of D-Branes and Bibranes are reviewed and it is demonstrated how they lead to Wess-Zumino terms for boundary conditions and defects.

Chapter 2: In this chapter we develop the theory of stacks and equivariance which is behind the descent considerations for gerbes and Jandl structures.

In section 2.2 we first define Lie groupoids and Presheaves in bicategories on Lie groupoids. Then we give the definition of equivariant objects (definition 2.2.5) and use this to define the 2-stack property (definition 2.2.12). In particular we obtain for each 2-stack $\mathfrak{X}$ and each Lie groupoid Γ a bicategory $\mathfrak{X}(\Gamma)$ (proposition 2.2.8). We introduce the notion of weak equivalence between Lie groupoids and state our first main theorem.

Theorem (Theorem 2.2.16)**.** Suppose that Γ and Λ are Lie groupoids and $\Gamma \longrightarrow \Lambda$ is a weak equivalence of Lie groupoids. For a 2-stack $\mathfrak{X}$ the induced functor

$$\mathfrak{X}(\Lambda) \longrightarrow \mathfrak{X}(\Gamma)$$

given by pullback is an equivalence of bicategories.

The proof of the theorem is given in sections 2.5 - 2.8. In section 2.2.3 we use the theorem to demonstrate that the stack conditions for open coverings and surjective submersions are equivalent.

In section 2.3 we define the plus construction $\mathfrak{X}^+$ for a pre-2-stack $\mathfrak{X}$ (definition 2.3.1) and state the next theorem:

Theorem (Theorem 2.3.3)**.** If $\mathfrak{X}$ is a pre-2-stack, then $\mathfrak{X}^+$ is a 2-stack. Furthermore the canonical embedding $\mathfrak{X}(M) \longrightarrow \mathfrak{X}^+(M)$ is fully faithful for each M.

The proof is given in section 2.9 and uses theorem 2.2.16 again.

The fact that the plus construction essentially consists of descent objects allows to exhibit bundle gerbes as special instances of this general construction (section 2.4.1). In particular this shows that bundle gerbes form a 2-stack. We can use the plus construction to define *Jandl gerbes* in section 2.4.2. Moreover we define the orientation bundle of a Jandl gerbe (definition 2.4.8) and demonstrate how this is related to reductions of a Jandl gerbes to a bundle gerbe (proposition 2.4.9). As a next step proposition 2.4.12 precisely states in which way Jandl gerbes generalize Jandl structures (as reviewed in section 1.4). This can be used to define unoriented surface holonomy for Jandl gerbes in a very general setting as done in section 2.4.3. Finally we sketch another application of the plus construction to 2-vector bundles 2.4.4.

For a more detailed overview of the chapter see section 2.

Chapter 3: The aim of this chapter is to define and compare four versions of non-abelian gerbes for a Lie-2-group Γ, namely: Čech cocycles, classifying maps, bundle gerbes, and principal 2-bundles, see also section 3.1 for an outline.

We start in section 3.2 by reviewing some preliminaries about Lie groupoids (section 3.2.1), principle groupoid bundles (section 3.2.2), anafunctors (section 3.2.3) and Lie 2-groups (section 3.2.4).

In section 3.3 we review the definition of non-abelian Čech cohomology $\check{\mathrm{H}}^1(M,\Gamma)$ for a Lie 2-group Γ (as given in [Gir71] and [Bre90]). In the next section 3.4 we proceed with classifying maps. That are maps into the classifying space $\mathfrak{B}|\Gamma|$ of the 2-group Γ. We introduce the notion of smoothly separable 2-group and show:

Theorem (Theorem 3.4.6)**.** For M a smooth manifold and Γ a smoothly separable Lie 2-group, there is a bijection

$$\check{\mathrm{H}}^1(M,\Gamma) \cong \big[M, \mathfrak{B}|\Gamma|\big].$$

The proof is based on results of Baez and Stevenson [BS09] and a comparison result between smooth and continuous non-abelian Čech cohomology (Proposition 3.4.1).

In section 3.5 we define the third version: Γ-bundle gerbes. The definition is based on Γ-bundles, and similar to bundle gerbes it uses the plus construction (definition 3.5.1). We explicitly unwind the definition in this specific case and compare it to abelian gerbes and other definitions of non-abelian gerbes in the literature, see section 3.5.1.

In section 3.5.2 we provide some properties of Γ-bundle gerbes. In particular for a homomorphism $\Gamma \longrightarrow \Omega$ of 2-groups we obtain an induced 2-functor $\mathcal{G}rb_\Gamma \longrightarrow \mathcal{G}rb_\Omega$, see proposition 3.5.11. The systematic definition of Γ-bundle gerbes and our general theory from chapter 2 then allows us to show:

Theorem (Theorem 3.5.5 and Theorem 3.5.12)**.** The pre-2-stack $\mathcal{G}rb_\Gamma$ of Γ-bundle gerbes is a 2-stack. For a weak equivalence $\Gamma \longrightarrow \Omega$ between Lie 2-groups the induced morphism $\mathcal{G}rb_\Gamma \longrightarrow \mathcal{G}rb_\Omega$ is an equivalence of 2-stacks.

Finally the local nature of Γ-bundle gerbes and some of the established properties are then used to make contact to non-abelian cohomology:

Theorem (Theorem 3.5.20)**.** Let M be a smooth manifold and let Γ be a Lie 2-group. There is a canonical bijection

$$\left\{ \begin{array}{c} \text{Isomorphism classes of } \Gamma\text{-bundle} \\ \text{gerbes over } M \end{array} \right\} \cong \check{\mathrm{H}}^1(M,\Gamma).$$

In the following section 3.6 we come to the definition of principal 2-bundles (definition 3.6.5) based on earlier work of Bartels [Bar04] and Wockel [Woc08]. These 2-bundles also form a pre-2-stack denoted 2-$\mathcal{B}un_\Gamma$ for a 2-group Γ. From section 3.7 on the rest of the chapter is devoted to prove the following comparison statement:

Theorem (Theorem 3.7.1)**.** There is an equivalence of pre-2-stacks

$$\mathcal{G}rb_\Gamma \cong 2\text{-}\mathcal{B}un_\Gamma.$$

We use this theorem to extend all the statements above to 2-bundles: they form a 2-stack (Theorem 3.6.9), for smoothly weak equivalent 2-groups these 2-stacks are equivalent (Theorem 3.6.11) and they are classified by non-abelian Čech cohomology or classifying maps, respectively.

Chapter 4: In this chapter we construct a model for the string group as an infinite-dimensional Lie group. In fact we present a construction not only for $\mathrm{Spin}(n)$ but for any compact, simple, simply connected Lie group G. In a second step we extend this model by a contractible Lie group to a Lie 2-group model.

In section 4.2 we review the fact [Woc08] that the gauge group of a principal bundle is an infinite dimensional Lie group. Now let $P \longrightarrow G$ be a basic smooth principal $PU(\mathcal{H})$-bundle. Basic means that $[P] \in [G, BPU(\mathcal{H})] \cong H^3(G,\mathbb{Z}) = \mathbb{Z}$ is a generator. The main result of Section 4.3 is then

Theorem (Theorem 4.3.6)**.** Let G be a simple, simply connected and compact Lie group, then there exists a smooth string group model $\mathcal{S}tring_G$. It is constructed as an infinite dimensional extension of G by the gauge group of P.

We also show that $\mathcal{S}tring_G$ is metrizable and Fréchet.

In Section 4.4 we introduce the concept of infinite dimensional Lie 2-group models (Definition 4.4.10). An important construction in this context is the geometric realization that produces topological groups from Lie 2-groups (Definition 4.4.2). We show that geometric realization is well-behaved under mild technical conditions, such as metrizability (Lemma 4.4.4, Proposition 4.4.5 and Proposition 4.4.7).

In Section 4.5 we construct a $U(1)$-central extension $\widehat{\mathcal{G}au}(P)$ of the gauge group of P. We show that $\widehat{\mathcal{G}au}(P)$ is contractible and promote the pair $(\widehat{\mathcal{G}au}(P), \mathcal{S}tring_G)$ to a smooth crossed module. Crossed modules are a source for Lie 2-groups (Example 4.4.3). In that way we obtain a Lie 2-group $\mathtt{STRING}_G$.

Theorem (Theorem 4.5.6)**.** $\mathtt{STRING}_G$ is a String-2-group model in the sense of Definition 4.4.10.

The proof of this theorem relies on a comparison of the model $\mathcal{S}tring_G$ with the geometric realization of $\mathtt{STRING}_G$. This direct comparison allows to show that the corresponding bundle theories and string structures are equivalent, see Section 4.6. This explicit comparison is a distinctive feature of our 2-group model that is not available for the other 2-group models.

Chapter 5: In this last chapter we give an equivariant version of Dijkgraaf-Witten theory. For a motivation from two different angles see section 5.1.

We begin in section 5.2 by reviewing ordinary (i.e. non-equivariant) Dijkgraaf-Witten theory: in section 5.2.1, 5.2.2 and 5.2.3 we define Dijkgraaf-Witten theory as an *extended* TFT from first principles based on a construction of Morton [Mor10] (which is inspired by [FQ93]); in section 5.2.4 we explain how to extract a braided monoidal category out of extended TFTs and compute it explicitly; in section 5.2.5 we exhibit this category as the representation category of a Hopf algebra $\mathcal{D}(G)$ (the Drinfel'd double of G) and thereby see that it is a modular tensor category.

In section 5.3 we turn to new results about the equivariant case. There we first define the notion of weak action of a group J on a group G (Definition 5.3.1) and use it to define *twisted bundles* in section 5.3.2. We show how to classify and describe these twisted bundles using the fundamental group (Proposition 5.3.8) and Čech cohomology (in section 5.6.1).

In the following section 5.3.3 we introduce the concept of equivariant TFT and then state:

Theorem (Theorem 5.3.16)**.** For a finite group G and a weak J-action on G, there is an extended 3d J-TFT Z_G^J which is an equivariant extension of Dijkgraaf-Witten theory.

The theorem is proved by explicitly constructing Z_G^J in section 5.3.4 and relies on the notion of twisted bundles. Due to this explicit nature we can compute the category $\mathcal{C}^J(G)$ assigned to the circle together with fusion product and braiding in section 5.3.5.

The next section 5.4 is devoted to the algebraic study of the category $\mathcal{C}^J(G)$. In subsection 5.4.1 we review the concept of equivariant fusion category. In the next subsection we introduce the concept of (weakly) equivariant ribbon algebra. We closely follow [Tur10] except for the fact that we have to consider weak actions as well in order to accommodate our examples. We then show that the representation category of an (weakly) equivariant ribbon algebra is an equivariant fusion category (Proposition 5.4.19). In the next subsection we introduce a J-equivariant ribbon algebra $\mathcal{D}^J(G)$ given a weak J action on G. We show that the representation category of $\mathcal{D}^J(G)$ is equivalent to our geometrically obtained category $\mathcal{C}^J(G)$ (Proposition 5.4.25). In particular this shows that $\mathcal{C}^J(G)$ is an equivariant *fusion* category. The main result about this category is :

Theorem (Theorem 5.4.35)**.** The category $\mathcal{C}^J(G)$ is a J-modular tensor category.

The proof relies on a result of Kirillov [Kir04] which allows to check modularity on the level of orbifold categories. Therefore we carry out the orbifold construction on the level of ribbon algebras in section 5.4.4. Finally the proof reduces to a direct algebraic comparison of two ribbon algebras (Proposition 5.4.34).

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Chapter 2: T. Nikolaus and C. Schweigert. Equivariance in higher geometry. *Adv. Math.* , 226(4):3367–3408, 2011

Chapter 3: T. Nikolaus and K. Waldorf. Four Equivalent Versions of Non-Abelian Gerbes. Preprint arxiv: 1103.4815, 2011

Chapter 4: T. Nikolaus, C. Sachse, and C. Wockel. A smooth model for the string group. Preprint arxiv: 1104.4288, 2011

Chapter 5: J. Maier, T. Nikolaus, and C. Schweigert. Equivariant modular categories via Dijkgraaf-Witten theory. Preprint arxiv: 1103.2963, 2011

Another publication that is independent of this thesis is:

T. Nikolaus. Algebraic models for higher categories. to appear in Indag. Math., arxiv: 1003.1342, 2010

Chapter 1

Bundle Gerbes and Surface Holonomy

Two-dimensional quantum field theories have been a rich source of relations between different mathematical disciplines. A prominent class of examples of such theories are the two-dimensional rational conformal field theories, which admit a mathematically precise description (see [SFR06] for a summary of progress in the last decade). A large subclass of these also have a classical description in terms of an action, in which a term given by a surface holonomy enters.

The appropriate geometric object for the definition of surface holonomies for oriented surfaces with empty boundary are hermitian bundle gerbes. In this chapter we systematically introduce bundle gerbes by first defining a pre-stack of trivial bundle gerbes, in such a way that surface holonomy can be defined, and then closing this pre-stack under descent. This construction constitutes in fact a generalization of the geometry of line bundles, their holonomy and their applications to classical particle mechanics.

Inspired by results in a representation theoretic approach to rational conformal field theories, we then introduce geometric structure that allows to define surface holonomy in more general situations: Jandl gerbes for unoriented surfaces, D-branes for surfaces with boundaries, and bi-branes for surfaces with defect lines.

This chapter has introductory character. Important objects of study are introduced. Later, in chapter 2, we clarify the mathematical structure behind these objects.

1.1 Hermitian line bundles and holonomy

Before discussing bundle gerbes, it is appropriate to summarize some pertinent aspects of line bundles.

One of the basic features of a (complex) line bundle L over a smooth manifold M is that it is locally trivializable. This means that M can be covered by open sets U_α

such that there exist isomorphisms $\phi_\alpha\colon L|_{U_\alpha} \longrightarrow \mathbf{1}_{U_\alpha}$, where $\mathbf{1}_{U_\alpha}$ denotes the trivial line bundle $\mathbb{C}\times U_\alpha$. A choice of such maps ϕ_α defines gluing isomorphisms

$$g_{\alpha\beta}: \quad \mathbf{1}_{U_\alpha}\big|_{U_\alpha\cap U_\beta} \longrightarrow \mathbf{1}_{U_\beta}\big|_{U_\alpha\cap U_\beta} \quad \text{with} \quad g_{\beta\gamma}\circ g_{\alpha\beta} = g_{\alpha\gamma} \quad \text{on} \quad U_\alpha\cap U_\beta\cap U_\gamma\,.$$

Isomorphisms between trivial line bundles are just smooth functions. Given a set of gluing isomorphisms one can obtain as additional structure the total space as the manifold

$$L := \bigsqcup_\alpha \mathbf{1}_{U_\alpha} /\!\sim\,, \tag{1.1}$$

with the relation $\sim$ identifying an element ℓ of $\mathbf{1}_{U_\alpha}$ with $g_{\alpha\beta}(\ell)$ of $\mathbf{1}_{U_\beta}$. In short, every bundle is glued together from trivial bundles.

In the following all line bundles will be equipped with a hermitian metric, and all isomorphisms are supposed to be isometries. Such line bundles form categories, denoted $\mathcal{B}un(M)$. The trivial bundle $\mathbf{1}_M$ defines a full, one-object subcategory $\mathcal{B}untriv(M)$ whose endomorphism set is the monoid of U(1)-valued functions on M. Denoting by $\pi_0(\mathcal{C})$ the set of isomorphism classes of a category $\mathcal{C}$ and by $H^\bullet(M,\underline{\mathrm{U}(1)})$ the sheaf cohomology of M with coefficients in the sheaf of U(1)-valued functions, we have the bijection

$$\pi_0(\mathcal{B}un(M)) \;\cong\; H^1(M,\underline{\mathrm{U}(1)}) \;\cong\; H^2(M,\mathbb{Z})\,, \tag{1.2}$$

under which the isomorphism class of the trivial bundle is mapped to zero.

Another basic feature of line bundles is that they pull back along smooth maps: for L a line bundle over M and $f\colon M' \longrightarrow M$ a smooth map, the pullback f^*L is a line bundle over M', and this pullback f^* extends to a functor

$$f^*: \quad \mathcal{B}un(M) \longrightarrow \mathcal{B}un(M')\,.$$

Furthermore, there is a unique isomorphism $g^*(f^*L) \longrightarrow (f\circ g)^*L$ for composable maps f and g.

As our aim is to discuss holonomies, we should in fact consider a different category, namely line bundles equipped with (metric) connections. These form again a category, denoted by $\mathcal{B}un^\nabla(M)$, and there is again a full subcategory $\mathcal{B}untriv^\nabla(M)$ of trivial line bundles with connection. But now this subcategory has more than one object: every 1-form $\omega\in\Omega^1(M)$ can serve as a connection on a trivial line bundle $\mathbf{1}$ over M; the so obtained objects are denoted by $\mathbf{1}_\omega$. The set $\mathrm{Hom}(\mathbf{1}_\omega,\mathbf{1}_{\omega'})$ of connection-preserving isomorphisms $\eta\colon \mathbf{1}_\omega \longrightarrow \mathbf{1}_{\omega'}$ is the set of smooth functions $g\colon M \longrightarrow \mathrm{U}(1)$ satisfying

$$\omega' - \omega = -\,\mathrm{i}\,\mathrm{dlog}\, g\,. \tag{1.3}$$

Just like in (1.1), every line bundle L with connection can be glued together from line bundles $\mathbf{1}_{\omega_\alpha}$ along connection-preserving gluing isomorphisms $\eta_{\alpha\beta}$.

The curvature of a trivial line bundle $\mathbf{1}_\omega$ is $\mathrm{curv}(\mathbf{1}_\omega) := \mathrm{d}\omega \in \Omega^2(M)$, and is thus invariant under connection-preserving isomorphisms. It follows that the curvature of any line bundle with connection is a globally well-defined, closed 2-form. We recall that the cohomology class of this 2-form in real cohomology coincides with the characteristic class in (1.2).

In order to introduce the holonomy of line bundles with connection, we say that the holonomy of a trivial line bundle $\mathbf{1}_\omega$ over S^1 is

$$\mathrm{Hol}_{\mathbf{1}_\omega} := \exp\Big(2\pi\mathrm{i}\int_{S^1}\omega\Big) \in \mathrm{U}(1)\,.$$

If $\mathbf{1}_\omega$ and $\mathbf{1}_{\omega'}$ are trivial line bundles over S^1, and if there exists a morphism η in $\mathrm{Hom}(\mathbf{1}_\omega, \mathbf{1}_{\omega'})$, we have $\mathrm{Hol}_{\mathbf{1}_\omega} = \mathrm{Hol}_{\mathbf{1}_{\omega'}}$ because

$$\int_{S^1}\omega' - \int_{S^1}\omega = \int_{S^1} -\,\mathrm{i}\,\mathrm{dlog}\,\eta \;\in \mathbb{Z}\,.$$

More generally, if L is any line bundle with connection over M, and $\Phi\colon S^1 \longrightarrow M$ is a smooth map, then the pullback bundle Φ^*L is trivial since $H^2(S^1,\mathbb{Z}) = 0$, and hence one can choose an isomorphism $\mathcal{T}\colon \Phi^*L \xrightarrow{\sim} \mathbf{1}_\omega$ for some $\omega \in \Omega^1(S^1)$. We then set

$$\mathrm{Hol}_L(\Phi) := \mathrm{Hol}_{\mathbf{1}_\omega}\,.$$

This is well-defined because any other trivialization $\mathcal{T}'\colon \Phi^*L \longrightarrow 1_{\omega'}$ provides a transition isomorphism $\eta := \mathcal{T}' \circ \mathcal{T}^{-1}$ in $\mathrm{Hom}(\mathbf{1}_\omega, \mathbf{1}_{\omega'})$. But as we have seen above, the holonomies of isomorphic trivial line bundles coincide.

Let us also mention an elementary example of a physical application of line bundles and their holonomies: the action functional S for a charged point particle. For (M,g) a (pseudo-)Riemannian manifold and $\Phi\colon \mathbb{R} \supset [t_1,t_2] \longrightarrow (M,g)$ the trajectory of a point particle of mass m and electric charge e, one commonly writes the action $S[\Phi]$ as the sum of the kinetic term

$$S_{\mathrm{kin}}[\Phi] = \frac{m}{2}\int_{t_1}^{t_2} g\left(\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)$$

and a term

$$-e\int_{t_1}^{t_2}\Phi^*A\,,$$

with A the electromagnetic gauge potential. However, this formulation is inappropriate when the electromagnetic field strength F is not exact, so that a gauge potential A with $\mathrm{d}A = F$ exists only locally. As explained above, keeping track of such local 1-forms A_α and local 'gauge transformations', i.e. connection-preserving isomorphisms between those, leads to the notion of a line bundle L with connection. For a closed trajectory, i.e. $\Phi(t_1) = \Phi(t_2)$, the action should be defined as

$$\mathrm{e}^{\mathrm{i}S[\Phi]} = \mathrm{e}^{\mathrm{i}S_{\mathrm{kin}}[\Phi]}\,\mathrm{Hol}_L(\Phi)\,. \tag{1.4}$$

An important feature of bundles in physical applications is the 'Dirac quantization' condition on the field strength F: the integral of F over any closed surface Σ in M gives an integer. This follows from the coincidence of the cohomology class of F with the characteristic class in (1.2). Another aspect is a neat explanation of the Aharonov-Bohm effect. A line bundle over a non-simply connected manifold can have vanishing curvature and yet non-trivial holonomies. In the quantum theory holonomies are observable, and thus the gauge potential A contains physically relevant information even if its field strength is zero. Both aspects, the quantization condition and the Aharonov-Bohm effect, persist in the generalization of line bundles to bundle gerbes, which we discuss next.

1.2 Gerbes and surface holonomy

In this section we formalize the procedure of Section 1.1 that has lead us from local 1-form gauge potentials to line bundles with connection: we will explain that it is the closure of the category of trivial bundles with connection under descent. We then apply the same principle to locally defined 2-forms, whereby we arrive straightforwardly at the notion of bundle gerbes with connection. We describe the notion of surface holonomy of such gerbes and their applications to physics analogously to Section 1.1.

1.2.1 Descent of bundles

As a framework for structures with a category assigned to every manifold and consistent pullback functors we consider presheaves of categories. Let $\mathcal{M}an$ be the category of smooth manifolds and smooth maps, and let $\mathcal{C}at$ be the 2-category of categories, with functors between categories as 1-morphisms and natural transformations between functors as 2-morphisms. Then a presheaf of categories is a lax functor

$$\mathcal{F}: \quad \mathcal{M}an^{\mathrm{opp}} \longrightarrow \mathcal{C}at$$

It assigns to every manifold M a category $\mathcal{F}(M)$, and to every smooth map f from M' to M a functor $\mathcal{F}(f)\colon \mathcal{F}(M) \longrightarrow \mathcal{F}(M')$. By the qualification 'lax' we mean that the composition of maps must only be preserved up to coherent isomorphisms.

In Section 1.1 we have already encountered four examples of presheaves: the presheaf $\mathcal{B}un$ of line bundles, the presheaf $\mathcal{B}un^{\nabla}$ of line bundles with connection, and their sub-presheaves of trivial bundles.

To formulate a gluing condition for presheaves of categories we need to specify coverings. Here we choose *surjective submersions* $\pi\colon Y \longrightarrow M$. We remark that every cover of M by open sets U_α provides a surjective submersion with Y the disjoint union of the U_α; thus surjective submersions generalize open coverings. This generalization proves to be important for many examples of bundle gerbes, such as the lifting of bundle gerbes and the canonical bundle gerbes of compact simple Lie groups.

With hindsight, a choice of coverings endows the category $\mathcal{M}an$ of smooth manifolds with a Grothendieck topology. Both surjective submersions and open covers define a Grothendieck topology, and since every surjective submersion allows for local sections, the resulting two Grothendieck topologies are equivalent. And in fact the submersion topology is the maximal one equivalent to open coverings.

Along with a covering $\pi\colon Y \longrightarrow M$ there comes a simplicial manifold

$$\cdots \underset{\partial_3}{\overset{\partial_0}{\rightrightarrows\!\!\rightrightarrows}} Y^{[3]} \underset{\partial_2}{\overset{\partial_0}{\rightrightarrows\!\!\rightarrow}} Y^{[2]} \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} Y \xrightarrow{\ \pi\ } M\,.$$

Here $Y^{[n]}$ denotes the n-fold fibre product of Y over M,

$$Y^{[n]} := \{(y_0, \ldots, y_{n-1}) \in Y^n \mid \pi(y_0) = \ldots = \pi(y_{n-1})\}\,,$$

and the map $\partial_i\colon Y^{[n]} \longrightarrow Y^{[n-1]}$ omits the ith entry. In particular $\partial_0\colon Y^{[2]} \longrightarrow Y$ is the projection to the second factor and $\partial_1\colon Y^{[2]} \longrightarrow Y$ the one to the first. All fibre products $Y^{[k]}$ are smooth manifolds, and all maps ∂_i are smooth. Now let L be a line bundle over M. By pullback along π we obtain:

(BO1) An object $\tilde{L} := \pi^*L$ in $\mathcal{B}un(Y)$.

(BO2) A morphism

$$\phi: \quad \partial_0^*\tilde{L} \cong \partial_0^*\pi^*L \xrightarrow{\ \sim\ } \partial_1^*\pi^*L \cong \partial_1^*\tilde{L}$$

in $\mathcal{B}un(Y^{[2]})$ induced from the identity $\pi \circ \partial_0 = \pi \circ \partial_1$. in $\mathcal{B}un(Y^{[2]})$ induced from the identity $\pi \circ \partial_0 = \pi \circ \partial_1$.

(BO3) A commutative diagram

$$\partial_1^*\partial_0^*\tilde{L} = \partial_0^*\partial_0^*\tilde{L} \xrightarrow{\partial_0^*\phi} \partial_0^*\partial_1^*\tilde{L} = \partial_2^*\partial_0^*\tilde{L} \xrightarrow{\partial_2^*\phi} \partial_2^*\partial_1^*\tilde{L} = \partial_1^*\partial_1^*\tilde{L}$$

$$\partial_1^*\partial_0^*\tilde{L} \xrightarrow{\partial_1^*\phi} \partial_1^*\partial_1^*\tilde{L}$$

of morphisms in $\mathcal{B}un(Y^{[3]})$; or in short, an equality $\partial_2^*\phi \circ \partial_0^*\phi = \partial_1^*\phi$.

We call a pair $(\tilde{L}, \phi)$ as in (BO1) and (BO2) which satisfies (BO3) a *descent object* in the presheaf $\mathcal{B}un$. Analogously we obtain for a morphism $f\colon L \longrightarrow L'$ of line bundles over M

(BM1) A morphism $\tilde{f} := \pi^* f \colon \tilde{L} \longrightarrow \tilde{L}'$ in $\mathcal{B}un(Y)$.

(BM2) A commutative diagram

$$\phi' \circ \partial_0^* \tilde{f} = \partial_1^* \tilde{f} \circ \phi$$

of morphisms in $\mathcal{B}un(Y^{[2]})$.

Such a morphism $\tilde{f}$ as in (BM1) obeying (BM2) is called a *descent morphism* in the presheaf $\mathcal{B}un$.

Descent objects and descent morphisms for a given covering π form a category $\mathcal{D}esc(\pi\colon Y \longrightarrow M)$ of descent data. What we described above is a functor

$$\iota_\pi : \ \mathcal{B}un(M) \longrightarrow \mathcal{D}esc(\pi\colon Y \longrightarrow M)\,.$$

The question arises whether every 'local' descent object corresponds to a 'global' object on M, i.e. whether the functor ι_π is an equivalence of categories.

The construction generalizes straightforwardly to any presheaf of categories $\mathcal{F}$, and if the functor ι_π is an equivalence for all coverings $\pi\colon Y \longrightarrow M$, the presheaf $\mathcal{F}$ is called a *sheaf of categories* (or stack). Extending the gluing process from (1.1) to non-trivial bundles shows that the presheaves $\mathcal{B}un$ and $\mathcal{B}un^\nabla$ are sheaves. In contrast, the presheaves $\mathcal{B}untriv$ and $\mathcal{B}untriv^\nabla$ of trivial bundles are not sheaves, since gluing of trivial bundles does in general not result in a trivial bundle. In fact the gluing process (1.1) shows that every bundle can be obtained by gluing trivial ones. In short, the sheaf $\mathcal{B}un^\nabla$ of line bundles with connection is obtained by closing the presheaf $\mathcal{B}untriv^\nabla$ under descent.

1.2.2 Bundle gerbes

Our construction of line bundles started from trivial line bundles with connection which are just 1-forms on M, and the fact that 1-forms can be integrated along curves has lead us to the notion of holonomy. To arrive at a notion of *surface* holonomy, we now consider a category of 2-forms, or rather a 2-category:

- An object is a 2-form $\omega \in \Omega^2(M)$, called a *trivial bundle gerbe with connection* and denoted by $\mathcal{I}_\omega$.
- A 1-morphism $\eta\colon \omega \longrightarrow \omega'$ is a 1-form $\eta \in \Omega^1(M)$ such that $\mathrm{d}\eta = \omega' - \omega$.
- A 2-morphism $\phi\colon \eta \Longrightarrow \eta'$ is a smooth function $\phi : M \longrightarrow \mathrm{U}(1)$ such that $-\mathrm{i}\,\mathrm{dlog}(\phi) = \eta' - \eta$.

There is also a natural pullback operation along maps, induced by pullback on differential forms. The given data can be rewritten as a presheaf of 2-categories, as there is a 2-category attached to each manifold. This presheaf should now be closed under descent to obtain a sheaf of 2-categories. As a first step we complete the morphism categories under descent. Since these are categories of trivial line bundles with connections, we set

$$\mathrm{Hom}(\mathcal{I}_\omega, \mathcal{I}_{\omega'}) := \mathcal{B}un^{\nabla}_{\omega'-\omega}(M)\,,$$

the category of hermitian line bundles with connection of fixed curvature $\omega' - \omega$. The horizontal composition is given by the tensor product in the category of bundles. Finally, completing the 2-category under descent, we get the definition of a bundle gerbe:

Definition 1.2.1. A bundle gerbe $\mathcal{G}$ (with connection) over M consists of the following data: a covering $\pi\colon Y \longrightarrow M$, and for the associated simplicial manifold

$$\ldots\, Y^{[4]} \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; Y^{[3]} \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; Y^{[2]} \;\underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}}\; Y \xrightarrow{\;\pi\;} M$$

(GO1) an object $\mathcal{I}_\omega$ of $\mathcal{G}rbtriv^{\nabla}(Y)$: a 2-form $\omega \in \Omega^2(Y)$;

(GO2) a 1-morphism

$$L\colon\; \partial_0^*\mathcal{I}_\omega \longrightarrow \partial_1^*\mathcal{I}_\omega$$

in $\mathcal{G}rbtriv^{\nabla}(Y^{[2]})$: a line bundle L with connection over $Y^{[2]}$;

(GO3) a 2-isomorphism

$$\mu\colon\; \partial_2^*L \otimes \partial_0^*L \;\Longrightarrow\; \partial_1^*L$$

in $\mathcal{G}rbtriv^{\nabla}(Y^{[3]})$: a connection-preserving morphism of line bundles over $Y^{[3]}$;

(GO4) an equality

$$\partial_2^*\mu \circ (\mathrm{id} \otimes \partial_0^*\mu) = \partial_1^*\mu \circ (\partial_3^*\mu \otimes \mathrm{id})$$

of 2-morphisms in $\mathcal{G}rbtriv^{\nabla}(Y^{[4]})$.

For later applications it will be necessary to close the morphism categories under a second operation, namely direct sums. Closing the category of line bundles with connection under direct sums leads to the category of complex vector bundles with connection, i.e. we set

$$\mathrm{Hom}(\mathcal{I}_\omega, \mathcal{I}_{\omega'}) := \mathcal{V}ect\mathcal{B}un^{\nabla}_{\omega'-\omega}(M)\,, \tag{1.5}$$

where the curvature of these vector bundles is constrained to satisfy

$$\frac{1}{n}\,\mathrm{Tr}(\mathrm{curv}(L)) = \omega' - \omega\,,$$

with n the rank of the vector bundle. Notice that this does not affect the definition of a bundle gerbe, since the existence of the 2-isomorphism μ restricts the rank of L to be one.

As a next step, we need to introduce 1-morphisms and 2-morphisms between bundle gerbes. 1-morphisms have to compare two bundle gerbes $\mathcal{G}$ and $\mathcal{G}'$. We assume first that both bundle gerbes have the same covering $Y \longrightarrow M$.

Definition 1.2.2.
i) A 1-morphism between bundle gerbes $\mathcal{G} = (Y, \omega, L, \mu)$ and $\mathcal{G}' = (Y, \omega', L', \mu')$ over M with the same surjective submersion $Y \longrightarrow M$ consists of the following data on the associated simplicial manifold

$$\dots Y^{[4]} \rightrightarrows\!\!\rightrightarrows Y^{[3]} \rightrightarrows\!\!\rightarrow Y^{[2]} \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} Y \xrightarrow{\ \pi\ } M\,.$$

(G1M1) a 1-morphism $A \colon \mathcal{I}_\omega \longrightarrow \mathcal{I}_{\omega'}$ in $\mathcal{G}rbtriv^\nabla(Y)$: a rank-$n$ hermitian vector bundle A with connection of curvature $\frac{1}{n}\,\mathrm{Tr}(\mathrm{curv}(L)) = \omega' - \omega$;

(G1M2) a 2-isomorphism $\alpha \colon L' \otimes \partial_0^* A \Longrightarrow \partial_1^* A \otimes L$ in $\mathcal{G}rbtriv^\nabla(Y^{[2]})$: a connection-preserving morphism of hermitian vector bundles;

(G1M3) a commutative diagram

$$(\mathrm{id} \otimes \mu') \circ (\partial_2^* \alpha \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \partial_0^* \alpha) = \partial_1^* \alpha \circ (\mu \otimes id)$$

of 2-morphisms in $\mathcal{G}rbtriv^\nabla(Y^{[3]})$.

ii) A 2-morphism between two such 1-morphisms (A, α) and (A', α') consists of

(G2M1) a 2-morphism $\beta \colon A \Longrightarrow A'$ in $\mathcal{G}rbtriv^\nabla(Y)$: a connection-preserving morphism of vector bundles;

(G2M2) a commutative diagram

$$\alpha' \circ (\mathrm{id} \otimes \partial_0^* \beta) = (\partial_1^* \beta \otimes \mathrm{id}) \circ \alpha$$

of 2-morphisms in $\mathcal{G}rbtriv^\nabla(Y^{[2]})$.

Since 1-morphisms are composed by taking tensor products of vector bundles, a 1-morphism is invertible if and only if its vector bundle is of rank one.

In order to define 1-morphisms and 2-morphisms between bundle gerbes with possibly different coverings $\pi \colon Y \longrightarrow M$ and $\pi' \colon Y' \longrightarrow M$, we pull all the data back to a common refinement of these coverings and compare them there. We call a covering $\zeta \colon Z \longrightarrow M$ a common refinement of π and π' iff there exist maps $s \colon Z \longrightarrow Y$ and $s' \colon Z \longrightarrow Y'$ such that

$$\begin{array}{ccccc} Y & \xleftarrow{\ s\ } & Z & \xrightarrow{\ s'\ } & Y' \\ & {\scriptstyle \pi}\searrow & \downarrow{\scriptstyle \zeta} & \swarrow{\scriptstyle \pi'} & \\ & & M & & \end{array}$$

commutes. An important example of such a common refinement is the fibre product $Z := Y \times_M Y' \longrightarrow M$, with the maps $Z \longrightarrow Y$ and $Z \longrightarrow Y'$ given by the projections.

The important point about a common refinement $Z \twoheadrightarrow M$ is that the maps s and s' induce simplicial maps

$$Y^\bullet \longleftarrow Z^\bullet \longrightarrow Y'^\bullet .$$

For bundle gerbes $\mathcal{G} = (Y, \omega, L, \mu)$ and $\mathcal{G}' = (Y', \omega', L', \mu')$ we obtain new bundle gerbes with surjective submersion Z by pulling back all the data along the simplicial maps s and s'. Explicitly, $\mathcal{G}_Z := (Z, s_0^*\omega, s_1^*L, s_2^*\mu)$ and $\mathcal{G}'_Z = (Z, s_0'^*\omega', s_1'^*L', s_2'^*\mu')$. Also morphisms can be refined by pulling them back.

Definition 1.2.3. i) A *1-morphism* between bundle gerbes $\mathcal{G} = (Y, \omega, L, \mu)$ and $\mathcal{G}' = (Y', \omega', L', \mu')$ consists of a common refinement $Z \twoheadrightarrow M$ of the coverings $Y \twoheadrightarrow M$ and $Y' \twoheadrightarrow M$ and a morphism (A, α) of the two refined gerbes $\mathcal{G}_Z$ and $\mathcal{G}'_Z$.
ii) A 2-morphism between 1-morphisms $\mathfrak{m} = (Z, A, \alpha)$ and $\mathfrak{m}' = (Z', A', \alpha')$ consists of a common refinement $W \twoheadrightarrow M$ of the coverings $Z \twoheadrightarrow M$ and $Z' \twoheadrightarrow M$ (respecting the projections to Y and Y', respectively) and a 2-morphism β of the refined morphisms $\mathfrak{m}_W$ and $\mathfrak{m}'_W$. In addition two such 2-morphisms (W, β) and (W', β') must be identified iff there exists a further common refinement $V \twoheadrightarrow M$ of $W \twoheadrightarrow M$ and $W' \twoheadrightarrow M$, compatible with the other projections, such that the refined 2-morphisms agree on V.

Remark 1.2.4. *The fact, that this really is the right thing to do, i.e. that the so obtained categories are really closed under descent will be shown in chapter 2. More precisely we will first formalize descent and stack conditions and then show that this naturally leads to the category of gerbes given here in section 2.4.1.*

For a gerbe $\mathcal{G} = (Y, \omega, L, \mu)$ and a refinement $Z \twoheadrightarrow M$ of Y the refined gerbe $\mathcal{G}_Z$ is isomorphic to $\mathcal{G}$. This implies that every gerbe is isomorphic to a gerbe defined over an open covering $Z := \bigsqcup_{i \in I} U_i$. Furthermore we can choose the covering in such a way that the line bundle over double intersections is trivial as well. When doing so we obtain the familiar description of gerbes in terms of local data, reproducing formulas by [Alv85, Gaw88]. Extending this description to morphisms it is straightforward to show that gerbes are classified by the so-called Deligne cohomology $H^k(M, \mathcal{D}(2))$ in degree two:

$$\pi_0(\mathcal{G}rb^\nabla(M)) \;\cong\; H^2(M, \mathcal{D}(2)) .$$

Analogously we get the classification of gerbes without connection as

$$\pi_0(\mathcal{G}rb(M)) \;\cong\; H^2(M, \underline{\mathrm{U}(1)}) \;\cong\; H^3(M, \mathbb{Z}) .$$

1.2.3 Surface holonomy

The holonomy of a trivial bundle gerbe $\mathcal{I}_\omega$ over a closed oriented surface Σ is by definition

$$\mathrm{Hol}_{\mathcal{I}_\omega} := \exp\Big(2\pi\mathrm{i} \int_\Sigma \omega\Big) \;\in \mathrm{U}(1) .$$

If $\mathcal{I}_\omega$ and $\mathcal{I}_{\omega'}$ are two trivial bundle gerbes over Σ such that there exists a 1-isomorphism $\mathcal{I}_\omega \longrightarrow I_{\omega'}$, i.e. a vector bundle L of rank one, we have an equality $\mathrm{Hol}_{\mathcal{I}_\omega} = \mathrm{Hol}_{\mathcal{I}_{\omega'}}$ because

$$\int_\Sigma \omega' - \int_\Sigma \omega = \int_\Sigma \mathrm{curv}(L) \in \mathbb{Z}\,.$$

More generally, consider a bundle gerbe $\mathcal{G}$ with connection over a smooth manifold M, and a smooth map

$$\Phi: \quad \Sigma \longrightarrow M$$

defined on a closed oriented surface Σ. Since $H^3(\Sigma, \mathbb{Z}) = 0$, the pullback $\Phi^*\mathcal{G}$ is isomorphic to a trivial bundle gerbe. Hence one can choose a trivialization, i.e. a 1-isomorphism

$$\mathcal{T}: \quad \Phi^*\mathcal{G} \xrightarrow{\sim} \mathcal{I}_\omega$$

and define the holonomy of $\mathcal{G}$ around Φ by

$$\mathrm{Hol}_{\mathcal{G}}(\Phi) := \mathrm{Hol}_{\mathcal{I}_\omega}\,.$$

In the same way as for the holonomy of a line bundle with connection, this definition is independent of the choice of the 1-isomorphism $\mathcal{T}$. Namely, if $\mathcal{T}'\colon \Phi^*\mathcal{G} \xrightarrow{\sim} \mathcal{I}_{\omega'}$ is another trivialization, we have a transition isomorphism

$$L := \mathcal{T}' \circ \mathcal{T}^{-1}: \quad \mathcal{I}_\omega \xrightarrow{\sim} \mathcal{I}_{\omega'}\,, \tag{1.6}$$

which shows the independence.

1.2.4 Wess-Zumino terms

As we have seen in Section 1.1, the holonomy of a line bundle with connection supplies a term in the action functional of a classical charged particle, describing the coupling to a gauge field whose field strength is the curvature of the line bundle. Analogously, the surface holonomy of a bundle gerbe with connection defines a term in the action of a classical charged string. Such a string is described in terms of a smooth map $\Phi\colon \Sigma \longrightarrow M$. The exponentiated action functional of the string is (compare (1.4))

$$\mathrm{e}^{\mathrm{i}S[\Phi]} = \mathrm{e}^{\mathrm{i}S_{\mathrm{kin}}[\Phi]}\,\mathrm{Hol}_{\mathcal{G}}(\Phi)\,,$$

where $S_{\mathrm{kin}}[\Phi]$ is a kinetic term which involves a conformal structure on Σ. Physical models whose fields are maps defined on surfaces are called (non-linear) *sigma models*, and the holonomy term is called a *Wess-Zumino term*. Such terms are needed in certain models in order to obtain quantum field theories that are conformally invariant.

A particular class of sigma models with Wess-Zumino term is given by WZW (Wess-Zumino-Witten) models. For these the target space M is a connected compact simple Lie group G, and the curvature of the bundle gerbe $\mathcal{G}$ is an integral multiple of the canonical 3-form

$$H = \langle \theta \wedge [\theta \wedge \theta] \rangle \in \Omega^3(G)$$

(θ is the left-invariant Maurer-Cartan form on G, and $\langle \cdot , \cdot \rangle$ the Killing form of the Lie algebra $\mathfrak{g}$ of G). WZW models have been a distinguished arena for the interplay between Lie theory and the theory of bundle gerbes [Gaw88, GR02]. This has lead to new insights both in the physical applications and in the underlying mathematical structures. Some of these will be discussed in the following sections.

Defining Wess-Zumino terms as the holonomy of a bundle gerbe with connection allows one in particular to explain the following two facts.

- *The Aharonov-Bohm effect*: This occurs when the bundle gerbe has a *flat* connection, i.e. its curvature $H \in \Omega^3(M)$ vanishes. This does not mean, though, that the bundle gerbe is trivial, since its class in $H^3(M, \mathbb{Z})$ may be pure torsion. In particular, it can still have non-constant holonomy, and thus a non-trivial Wess-Zumino term.

  An example for the Aharonov-Bohm effect is the sigma model on the 2-torus $T = S^1 \times S^1$. By dimensional reasons, the 3-form H vanishes. Nonetheless, since $H^2(T, \mathrm{U}(1)) = \mathrm{U}(1)$, there exists a whole family of Wess-Zumino terms parameterized by an angle, of which only the one with angle zero is trivial.

- *Discrete torsion*: The set of isomorphism classes of bundle gerbes with connection that have the same curvature H is parameterized by $H^2(M, \mathrm{U}(1))$ via the map

  $$H^2(M, \mathrm{U}(1)) \longrightarrow \mathrm{Tors}(H^3(M, \mathbb{Z}))\,.$$

  If this group is non-trivial, there exist *different* Wess-Zumino terms for one and the same field strength H; their difference is called 'discrete torsion'.

  An example for discrete torsion is the level-k WZW model on the Lie group $\mathrm{PSO}(4n)$. Since $H^2(\mathrm{PSO}(4n), \mathrm{U}(1)) = \mathbb{Z}_2$, there exist two non-isomorphic bundle gerbes with connection having equal curvature.

1.3 The representation theoretic formulation of RCFT

1.3.1 Sigma models

Closely related to surface holonomies are novel geometric structures that have been introduced for unoriented surfaces, for surfaces with boundary, and for surfaces with defect lines. These structures constitute the second theme of this contribution, extending the construction of gerbes and surface holonomy via descent; they will be discussed in Sections 1.4, 1.5 and 1.6.

These geometric developments were in fact strongly inspired by algebraic and representation theoretic results in two-dimensional quantum field theories. To appreciate this connection we briefly review in this section the relation between spaces of maps $\Phi\colon \Sigma \longrightarrow M$, as they appear in the treatment of holonomies, and quantum field theories.

As already indicated in Section 1.2.4, a classical field theory, the (non-linear) *sigma model*, on a two-dimensional surface Σ, called the *world sheet*, can be associated to the space of smooth maps Φ from Σ to some smooth manifold M, called the *target space*. Appropriate structure on the target space determines a Lagrangian for the field theory on Σ. Geometric structure on M, e.g. a (pseudo-) Riemannian metric G, becomes, from this point of view, for any given map Φ a background function $G(\Phi(x))$ for the field theory on Σ.

Three main issues will then lead us to a richer structure related to surface holonomies:

- In string theory (where the world sheet Σ arises as the surface swept out by a string moving in M) and in other applications as well, one also encounters sigma models on world sheets Σ that have *non-empty boundary*. We will explain how the geometric data relevant for encoding boundary conditions – so called D-branes – can be derived from geometric principles.

- String theories of type I, which form an integral part of string dualities, involve *unoriented* world sheets. In string theory it is therefore a fundamental problem to exhibit geometric structure on the target space that provides a notion of holonomy for unoriented surfaces.

- An equally natural structure present in quantum field theory are *topological defect lines*, along which correlation functions of bulk fields can have a branch-cut. In specific models these can be understood, just like boundary conditions, as continuum versions of corresponding structures in lattice models of statistical mechanics. (For instance, in the lattice version of the Ising model a topological defect is produced by changing the coupling along all bonds that cross a specified line from ferromagnetic to antiferromagnetic.)

Sigma models have indeed been a significant source of examples for quantum field theories, at least on a heuristic level. Conversely, having a sigma model interpretation for a given quantum field theory allows for a geometric interpretation of quantum field theoretic quantities.

A distinguished subclass of theories in which this relationship between quantum field theory and geometry can be studied are two-dimensional conformal field theories, or CFTs, for short, and among these in particular the rational conformal field theories for which there exists a rigorous representation theoretic approach. The structures appearing in that approach in the three situations mentioned above suggest new geometric notions for conformal sigma models. Below we will investigate these notions with the help of standard geometric principles. Before doing so we formulate, in representation theoretic terms, the relevant aspects of the quantum field theories in question.

1.3.2 Rational conformal field theory

The conformal symmetry, together with further, so-called chiral, symmetries of a CFT can be encoded in the structure of a conformal vertex algebra $\mathcal{V}$. For any conformal vertex algebra one can construct (see e.g. [FBZ04]) a chiral CFT; in mathematical terms, a chiral CFT is a system of *conformal blocks*, i.e. sheaves over the moduli spaces of curves with marked points. These sheaves of conformal blocks are endowed with a projectively flat connection, the Knizhnik-Zamolodchikov connection, which in turn furnishes representations of the fundamental groups of the moduli spaces, i.e. of the mapping class groups.

Despite the physical origin of its name, a chiral conformal field theory is mathematically rigorous. On the other hand, from the two-dimensional point of view it is, despite its name, not a conventional quantum field theory, as one deals with (sections of) bundles instead of local correlation functions. In particular, it must not be confused with a full local conformal field theory, which is the relevant structure to enter our discussion of holonomies.

Chiral conformal field theories are particularly tractable when the vertex algebra $\mathcal{V}$ is rational in the sense of [Hua05, thm 2.1]. Then the representation category $\mathcal{C}$ of $\mathcal{V}$ is a modular tensor category, and the associated chiral CFT is a *rational chiral CFT*, or chiral RCFT. In this situation, we can use the tools of three-dimensional topological quantum field theory (TFT). A TFT is, in short, a monoidal functor $\mathtt{tft}_\mathcal{C}$ [Tur94, chap. IV.7] that associates a finite-dimensional vector space $\mathtt{tft}_\mathcal{C}(\mathrm{E})$ to any (extended) surface E, and a linear map from $\mathtt{tft}_\mathcal{C}(\mathrm{E})$ to $\mathtt{tft}_\mathcal{C}(\mathrm{E}')$ to any (extended) cobordism $\mathcal{M}\colon \mathrm{E} \rightarrow \mathrm{E}'$.

More precisely, a three-dimensional TFT is a projective monoidal functor from a category $\mathcal{C}ob_\mathcal{C}$ of decorated cobordisms to the category of finite-dimensional complex vector spaces. The modular tensor category $\mathcal{C}$provides the decoration data for $\mathcal{C}ob_\mathcal{C}$. Specifically, the objects E of $\mathcal{C}ob_\mathcal{C}$ are extended surfaces, i.e. [1] compact closed oriented two-manifolds with a finite set of embedded arcs, and each of these arcs is marked by an object of $\mathcal{C}$. A morphism $\mathrm{E} \rightarrow \mathrm{E}'$ is an *extended cobordism*, i.e. a compact oriented three-manifold $\mathcal{M}$ with $\partial\mathcal{M} = (-\mathrm{E}) \sqcup \mathrm{E}'$, together with an oriented ribbon graph $\Gamma_\mathcal{M}$ in $\mathcal{M}$ such that at each marked arc of $(-\mathrm{E}) \sqcup \mathrm{E}'$ a ribbon of $\Gamma_\mathcal{M}$ is ending. Each ribbon of $\Gamma_\mathcal{M}$ is labeled by an object of $\mathcal{C}$, while each coupon of $\Gamma_\mathcal{M}$ is labeled by an element of the morphism space of $\mathcal{C}$that corresponds to the objects of the ribbons which enter and leave the coupon. Composition in $\mathcal{C}ob_\mathcal{C}$ is defined by gluing, the

[1] Here various details are suppressed. Detailed information, e.g. the precise definition of a ribbon graph or the reason why $\mathtt{tft}_\mathcal{C}$ is only projective, can be found in many places, such as [Tur94, BK01, KRT97] or [FFFS02, sect. 2.5-2.7].

identity morphism id_{E} is the cylinder over E, and the tensor product is given by disjoint union of objects and cobordisms.

A topological field theory furnishes, for any extended surface, a representation of the mapping class group. Our approach relies on the fundamental conjecture (which is largely established for a broad class of models) that, for $\mathcal{C}$the representation category of a rational vertex algebra $\mathcal{V}$, the mapping class group representation given by $\mathtt{tft}_{\mathcal{C}}$ is equivalent to the one provided by the Knizhnik-Zamolodchikov connection on the conformal blocks for the vertex algebra $\mathcal{V}$.

1.3.3 The TFT construction of full RCFT

Let us now turn to the discussion of full local conformal field theories, which are the structures to be compared to holonomies. A *full* CFT is, by definition, a consistent system of *local correlation functions* that satisfy all sewing constraints (see e.g. [FFRS08, def. 3.14]). According to the principle of holomorphic factorization, every full RCFT can be understood with the help of a corresponding chiral CFT. The relevant chiral CFT is, however, not defined on world sheets Σ (which may be unoriented or have a non-empty boundary), but rather on their *complex doubles* $\widehat{\Sigma}$, which can be given the structure of extended surfaces; this affords a geometric separation of left- and right-movers. The double $\widehat{\Sigma}$ of Σ is, by definition, the orientation bundle over Σ modulo identification of the two points in the fibre over each boundary point of Σ. The world sheet Σ can be obtained from $\widehat{\Sigma}$ as the quotient by an orientation-reversing involution τ. To give some examples, when Σ is closed and orientable, then $\widehat{\Sigma}$ is just the disconnected sum $\widehat{\Sigma} = \Sigma \sqcup -\Sigma$ of two copies of Σ with opposite orientation, and the involution τ just exchanges these two copies; the double of both the disk and the real projective plane is the two-sphere (with τ being given, in standard complex coordinates, by $z \longmapsto \overline{z}^{-1}$ and by $z \longmapsto -\overline{z}^{-1}$, respectively); and the double of both the annulus and the Möbius strip is a two-torus. Further, when Σ comes with field insertions, that is, embedded arcs labeled by objects of either $\mathcal{C}$(for arcs on $\partial\Sigma$) or pairs of objects of $\mathcal{C}$(for arcs in the interior of Σ), then corresponding arcs labeled by objects of $\mathcal{C}$are present on $\widehat{\Sigma}$.

Given this connection between the surfaces relevant to chiral and full CFT, the relationship between the chiral and the full CFT can be stated as follows: A correlation function $C(\Sigma)$ of the full CFT on Σ is a specific element in the appropriate space of conformal blocks of the chiral CFT on the double $\widehat{\Sigma}$. A construction of such elements has been accomplished in [FRS02, FRS04, FRS05, FFRS06]. The first observation is that they can be computed with the help of the corresponding TFT, namely as

$$C(\Sigma) = \mathtt{tft}_{\mathcal{C}}(\mathcal{M}_\Sigma)\,1 \;\in \mathtt{tft}_{\mathcal{C}}(\widehat{\Sigma})\,. \tag{1.7}$$

Here $\mathcal{M}_\Sigma \equiv \emptyset \xrightarrow{\mathcal{M}_\Sigma} \widehat{\Sigma}$, the *connecting manifold* for the world sheet Σ, is an extended cobordism that is constructed from the data of Σ. Besides the category $\mathcal{C}$, the specification of the vector $C(\Sigma)$ needs a second ingredient: a (Morita class of a) symmetric special Frobenius algebra A in $\mathcal{C}$.

Let us give some details [2] of the construction of $C(\Sigma)$.

- As a three-manifold, $\mathcal{M}_\Sigma$ is the interval bundle over Σ modulo a $\mathbb{Z}_2$-identification of the intervals over $\partial\Sigma$. Explicitly,

$$\mathcal{M}_\Sigma = \big(\widehat{\Sigma}\times[-1,1]\big)/\sim \qquad \text{with} \qquad ([x,\mathrm{or}_2],t) \sim ([x,-\mathrm{or}_2],-t)\,. \tag{1.8}$$

  It follows in particular that $\partial\mathcal{M}_\Sigma = \widehat{\Sigma}$ and that Σ is naturally embedded in $\mathcal{M}_\Sigma$ as $\imath\colon \Sigma \xrightarrow{\cong} \Sigma\times\{t{=}0\} \hookrightarrow \mathcal{M}_\Sigma$. Indeed, $\imath(\Sigma)$ is a deformation retract of $\mathcal{M}_\Sigma$, so that the topology of $\mathcal{M}_\Sigma$ is completely determined by the one of Σ.

- A crucial ingredient of the construction of the ribbon graph $\Gamma_{\mathcal{M}_\Sigma}$ in $\mathcal{M}_\Sigma$ is a (dual) oriented triangulation Γ of the submanifold $\imath(\Sigma)$ of $\mathcal{M}_\Sigma$. This triangulation is labeled by objects and morphisms of $\mathcal{C}$. It is here that the Frobenius algebra A enters: Each edge of $\Gamma\setminus\imath(\partial\Sigma)$ is covered with a ribbon labeled by the object A of $\mathcal{C}$, while each (three-valent) vertex is covered with a coupon labeled by the multiplication morphism $m\in\mathrm{Hom}_{\mathcal{C}}(A\otimes A,A)$. In addition, whenever these assignments in themselves would be in conflict with the orientations of the edges, a coupon with morphism in either $\mathrm{Hom}_{\mathcal{C}}(A\otimes A,\mathbf{1})$ or $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},A\otimes A)$ is inserted. Such morphisms are part of the data for a Frobenius structure on A. Assuming, for now, that the world sheet Σ is oriented, independence of $C(\Sigma)$ from the choice of triangulation Γ amounts precisely to the statement that the object A carries the structure of a symmetric special Frobenius algebra.

- If Σ has *non-empty boundary*, the prescription for Γ is amended as follows. Each edge e of $\Gamma\cap\imath(\partial\Sigma)$ is covered with a ribbon labeled by a (left, say) A-module $N=N(e)$, while each vertex lying on $\imath(\partial\Sigma)$ is covered with a coupon that has incoming N- and A-ribbons as well as an outgoing N-ribbon and that is labeled by the representation morphism $\rho_N\in\mathrm{Hom}_{\mathcal{C}}(A\otimes N,N)$. The physical interpretation of the A-module N is as the *boundary condition* that is associated to a component of $\partial\Sigma$. That the object N of $\mathcal{C}$labeling a boundary condition carries the structure of an A-module and that the morphism ρ_N is the corresponding representation morphism is precisely what is required (in addition to A being a symmetric special Frobenius algebra) in order to get independence of $C(\Sigma)$ from the choice of triangulation Γ.

- If Σ is *unoriented*, then as an additional feature one must ensure independence of $C(\Sigma)$ from the choice of local orientations of Σ. As shown in [FRS04], this

[2] For another brief summary, with different emphasis, see Section 7 of [FRS07]. An in-depth exposition, including for instance the relevance of various orientations, can e.g. be found in Appendix B of [FFRS06].

requires an additional structure on the algebra A, namely the existence of a morphism $\sigma \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ that is an algebra isomorphism from the opposite algebra A^{opp} to A and squares to the twist of A, i.e. satisfies

$$\sigma \circ \eta = \eta \,, \qquad \sigma \circ m = m \circ c_{A,A} \circ (\sigma \otimes \sigma) \,, \qquad \sigma \circ \sigma = \theta_A \,, \tag{1.9}$$

where $\eta \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A)$, $\theta_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ and $c_{A,A} \in \mathrm{Hom}_{\mathcal{C}}(A \otimes A, A \otimes A)$ denote the unit morphism, the twist, and the self-braiding of A, respectively. This way A becomes a braided version of an algebra with involution. A symmetric special Frobenius algebra endowed with a morphism σ satisfying (1.9) is called a *Jandl algebra*.

- In the presence of *topological defect lines* on Σ a further amendment of the prescription is in order. The defect lines partition Σ into disjoint regions, and to the regions to the left and to the right of a defect line one may associate different (symmetric special Frobenius) algebras A_l and A_r, such that the part of the triangulation Γ in one region is labeled by the algebra A_l, while the part of Γ in the other region is labeled by A_r. The defect lines are to be regarded as forming a subset Γ^D of Γ themselves; each edge of Γ^D is covered with a ribbon labeled by some object B of $\mathcal{C}$, while each vertex of Γ lying on Γ^D is covered with a coupon labeled by a morphism $\rho \in \mathrm{Hom}_{\mathcal{C}}(A_l \otimes B, B)$, respectively $\varrho \in \mathrm{Hom}_{\mathcal{C}}(B \otimes A_r, B)$. Consistency requires that these morphisms endow the object B of $\mathcal{C}$ that labels a defect line with the structure of an A_l-A_r-bimodule. (Below we will concentrate on the case $A_l = A_r =: A$, so that we deal with A-bimodules.)
- There are also rules for the morphisms of $\mathcal{C}$ that label bulk, boundary and defect fields, respectively.

The prescription summarized above allows one to construct the correlator (1.7) for any arbitrary world sheet Σ. The so obtained correlators can be proven [FFRS06] to satisfy all consistency conditions that the correlators of a CFT must obey. Thus, specifying the algebra A is sufficient to obtain a consistent system of correlators. The assignment of a (suitably normalized) correlator $C(\Sigma)$ to Σ actually depends only on the Morita class of the symmetric special Frobenius algebra A. Conversely, any consistent set of correlators can be obtained this way [FFRS08].

Topological defects admit a number of interesting operations. In particular, they can be fused – on the algebraic side this corresponds to the tensor product $B \otimes_A B'$ of bimodules. The bimodule morphisms $\mathrm{Hom}_{A|A}(B \otimes_A B', B'')$ appear as labels of vertices of defect lines. Defect lines can also be fused to boundaries; depending on the relative situation of the defect line and the boundary, this is given on the algebraic side by the tensor product $B \otimes_A N$ of a bimodule with a left module, or by the tensor product $N \otimes_A B$ with a right module, respectively.

In the following table we collect some pertinent aspects of the construction and exhibit the geometric structures on the sigma model target space M that correspond to them.

geometric situation	algebraic structure in $\mathcal{C}$	geom. structure on M
Σ closed oriented	symm. special Frob. algebra A	bundle gerbe $\mathcal{G}$
Σ unoriented	Jandl structure $\sigma\colon A^{\mathrm{opp}} \longrightarrow A$	Jandl gerbe
boundary condition	A-module	$\mathcal{G}$-D-brane
topological defect line	A-bimodule	$\mathcal{G}$-bi-brane

Jandl gerbes, D-branes and bi-branes will be presented in Sections 1.4, 1.5 and 1.6, respectively.

1.4 Jandl gerbes: Holonomy for unoriented surfaces

We have defined trivial bundle gerbes with connection as 2-forms because 2-forms can be integrated over oriented surfaces. Closing the 2-category of trivial bundle gerbes under descent has lead us to bundle gerbes. *Jandl gerbes* are bundle gerbes with additional structure, whose holonomy is defined for closed surfaces without orientation, even for unorientable surfaces [SSW07]. In particular, Jandl gerbes provide Wess-Zumino terms for unoriented surfaces. Comparing the geometric data with the representation theoretic ones from Section 1.3, bundle gerbes with connection correspond to Frobenius algebras, while Jandl gerbes correspond to Jandl algebras.

The appropriate quantity that has to replace 2-forms in order to make integrals over an unoriented surface well-defined is a *2-density*. Every surface Σ has an oriented double covering $\mathrm{pr}\colon \hat{\Sigma} \longrightarrow \Sigma$ that comes with an orientation-reversing involution $\sigma\colon \hat{\Sigma} \longrightarrow \hat{\Sigma}$ which exchanges the two sheets and preserves the fibres. A 2-density on Σ is a 2-form $\omega \in \Omega^2(\hat{\Sigma})$ such that

$$\sigma^*\omega = -\omega\,. \tag{1.10}$$

A 2-density on Σ can indeed be integrated without requiring Σ to be oriented. One chooses a dual triangulation Γ of Σ and, for each face f of Γ, one of its two preimages under $\mathrm{pr}\colon \hat{\Sigma} \longrightarrow \Sigma$, denoted f_{or}. Then one sets

$$\int_\Sigma \omega := \sum_f \int_{f_{\mathrm{or}}} \omega\,. \tag{1.11}$$

Owing to the equality (1.10) the so defined integral does not depend on the choice of the preimages f_{or} nor on the choice of triangulation Γ. If Σ can be endowed with an orientation, the preimages $f_{\rm or}$ can be chosen in such a way that $\mathrm{pr}|_{f_{\rm or}}: f_{\rm or} \longrightarrow f$ is orientation-preserving. Then the integral of a 2-density ω_ρ coincides with the ordinary integral of the 2-form ρ.

Next we want to set up a 2-category whose objects are related to 2-densities. To this end we use the 2-category of trivial bundle gerbes introduced in Section 1.2.2. Thus, one datum specifying an object is a 2-form $\omega \in \Omega^2(\hat\Sigma)$. In the context of 2-categories, demanding strict equality as in (1.10) is unnatural. Instead, we replace equality by a 1-morphism

$$\eta: \quad \sigma^*\omega \longrightarrow -\omega\,, \tag{1.12}$$

i.e. a 1-form $\eta \in \Omega^1(\hat\Sigma)$ such that $\sigma^*\omega = -\omega + \mathrm{d}\eta$. As we shall see in a moment, we must impose equivariance of the 1-morphism up to some 2-morphism, i.e. we need in addition a 2-isomorphism

$$\phi: \quad \sigma^*\eta \Longrightarrow \eta\,, \tag{1.13}$$

in other words a smooth function $\phi: M \longrightarrow \mathrm{U}(1)$ such that $\eta = \sigma^*\eta - \mathrm{i}\,\mathrm{dlog}\,\phi$. This 2-isomorphism, in turn, must satisfy the equivariance relation

$$\sigma^*\phi = \phi^{-1}. \tag{1.14}$$

Thus the objects of the 2-category are triples (ω, η, ϕ). Let us verify that they still lead to a well-defined notion of holonomy. We choose again a dual triangulation Γ of Σ as well as a preimage $f_{\rm or}$ for each of its faces. The expression (1.11) is now no longer independent of these choices, because every change creates a boundary term in the integrals of the 1-form η. To resolve this problem, we involve *orientation-reserving edges*: these are edges in Γ whose adjacent faces have been lifted to opposite sheets. Since Γ is a dual triangulation, its orientation-reversing edges form a disjoint union of piecewise smooth circles $c \subset \Sigma$. For each of these circles, we choose again a preimage $c_{\rm or}$. It may not be possible to choose $c_{\rm or}$ to be closed, in which case there exists a point $p^c \in \Sigma$ which has two preimages in $c_{\rm or}$. We choose again one of these preimages, denoted $p^c_{\rm or}$. Then

$$\mathrm{Hol}_{\omega,\eta,\phi} := \exp\left(2\pi\mathrm{i}\Big(\sum_f \int_{f_{\rm or}} \omega + \sum_c \int_{c_{\rm or}} \eta\Big)\right) \prod_c \phi(p^c_{\rm or}) \tag{1.15}$$

is independent of the choice of the lifts $f_{\rm or}$, $c_{\rm or}$ and $p_{\rm or}$, and is independent of the choice of the triangulation.

More generally, let $\mathcal{M}an_+$ be the category of smooth manifolds with involution, whose morphisms are equivariant smooth maps. (The involution is not required to act freely.) In a first step, we want to define a presheaf

$$\mathcal{J}antriv^\nabla: \quad \mathcal{M}an_+^{\rm opp} \longrightarrow \mathcal{C}at$$

of trivial Jandl gerbes. For (M,k) a smooth manifold with involution $k: M \longrightarrow M$, a trivial Jandl gerbe involves as a first datum a trivial bundle gerbe $\mathcal{I}_\omega$, but as explained in Section 1.2.1 we replace the 1-morphism η from (1.12) by a line bundle L over M with connection of curvature

$$\mathrm{curv}(L) = -\,\omega - k^*\omega\,, \tag{1.16}$$

and we replace the 2-isomorphism ϕ from (1.13) by an isomorphism $\phi\colon k^*L \longrightarrow L$ of line bundles with connection, still subject to the condition (1.14). Notice that the pair (L,ϕ) is nothing but a k-equivariant line bundle with connection over M. After this step, we still have the holonomy (1.15), which now looks like

$$\mathrm{Hol}_{\mathcal{I}_\omega,L,\phi} = \exp\Big(2\pi\mathrm{i}\sum_f \int_{f_{\mathrm{or}}}\omega\Big)\prod_c \mathrm{Hol}_{\bar{L}}(c)\,,$$

where we have used the fact that, since the action of $\langle k\rangle$ on c_{or} is free, the k-equivariant line bundle (L,ϕ) descends to a line bundle $\bar{L}$ with connection over the quotient $c = c_{\mathrm{or}}/\langle k\rangle$. This formula is now manifestly independent of the choices of c_{or} and p^c_{or}. Its independence under different choices of faces f_{or} is due to (1.16).

Now we close the presheaf $\mathcal{J}antriv^\nabla(M)$ under descent to allow for non-trivial bundle gerbes. To do so, we need to introduce duals of bundle gerbes, 1-morphisms and 2-isomorphisms see [Wal07]; for the sake of brevity we omit these definitions here.

Definition 1.4.1. Let M be a smooth manifold with involution $k\colon M \longrightarrow M$. A *Jandl gerbe* is a bundle gerbe $\mathcal{G}$ over M together with a 1-isomorphism $\mathcal{A}\colon k^*\mathcal{G} \longrightarrow \mathcal{G}^*$ to the dual gerbe and a 2-isomorphism $\varphi\colon k^*\mathcal{A} \Longrightarrow \mathcal{A}^*$ that satisfies $k^*\varphi = \varphi^{*-1}$.

Jandl gerbes form a sheaf

$$\mathcal{J}an^\nabla:\quad \mathcal{M}an_+^{\mathrm{opp}} \longrightarrow \mathcal{C}at\,.$$

The gluing axiom for this sheaf has been proved in [GSW08a]. We remark that the 1-isomorphism $\mathcal{A}$ may be regarded as the counterpart of a Jandl structure σ on the Frobenius algebra A that corresponds to the bundle gerbe $\mathcal{G}$, if one accepts that the dual gerbe plays the role of the opposed algebra.

Suppose we are given a Jandl gerbe $\mathcal{J}$ over a smooth manifold M with involution k. If Σ is a closed surface, possibly unoriented and possibly unorientable, and

$$\Phi:\quad (\hat{\Sigma},\sigma) \longrightarrow (M,k)$$

is a morphism in $\mathcal{M}an_+$, we can pull back the Jandl gerbe $\mathcal{J}$ from M to $\hat{\Sigma}$. As in the case of ordinary surface holonomy, it then becomes trivial as a gerbe for dimensional reasons, and we can choose an isomorphism

$$\mathcal{T}:\quad \Phi^*\mathcal{J} \xrightarrow{\sim} (\mathcal{I}_\omega, L, \phi)\,.$$

Then we define

$$\mathrm{Hol}_{\mathcal{J}}(\Phi) := \mathrm{Hol}_{\mathcal{I}_\omega, L, \phi} \,.$$

This is independent of the choice of $\mathcal{T}$, because any other choice $\mathcal{T}'$ gives rise to an isomorphism $\mathcal{T}' \circ \mathcal{T}^{-1}$ in $\mathcal{J}antriv^{\nabla}(\hat{\Sigma}, \sigma)$ under which the holonomy stays unchanged.

We have now seen that every Jandl gerbe $\mathcal{J}$ over a smooth manifold M with involution k has holonomies for unoriented closed surfaces and equivariant smooth maps $\Phi\colon \hat{\Sigma} \longrightarrow M$. We thus infer that sigma models on M whose fields are such maps, are defined by Jandl gerbes $\mathcal{J}$ over M rather than by ordinary bundle gerbes $\mathcal{G}$. This makes it an interesting problem to classify Jandl gerbes.

Concerning the existence of a Jandl gerbe $\mathcal{J}$ with underlying bundle gerbe $\mathcal{G}$, the 1-isomorphism $\mathcal{A}\colon k^*\mathcal{G} \longrightarrow \mathcal{G}^*$ requires the curvature H of $\mathcal{G}$ to satisfy

$$k^*H = -H \,. \tag{1.17}$$

Apart from this necessary condition, there is a sequence of obstructions against the existence of Jandl structures [GSW08a]. Reduced to the case that M is 2-connected, there is one obstruction class $o(\mathcal{G})$ in $H^3(\mathbb{Z}_2, \mathrm{U}(1))$, the group cohomology of $\mathbb{Z}_2$ with coefficients in $\mathrm{U}(1)$, on which $\mathbb{Z}_2$ acts by inversion. If $o(\mathcal{G})$ vanishes, then inequivalent Jandl gerbes with the same underlying bundle gerbe $\mathcal{G}$ are parameterized by $H^2(\mathbb{Z}_2, \mathrm{U}(1))$.

These results can be made very explicit in the case of WZW models, for which the object in $\mathcal{M}an_+$ is a connected compact simple Lie group G equipped with an involution $k\colon G \longrightarrow G$ acting as

$$k\colon \quad g \longmapsto (zg)^{-1}$$

for a fixed 'twist element' $z \in Z(G)$. It is easy to see that the 3-form $H_k \in \Omega^3(G)$, which is the curvature of the level-k bundle gerbes $\mathcal{G}$ over G, satisfies the necessary condition (1.17). All obstruction classes $o(\mathcal{G})$ and all parameterizing groups have been computed in dependence of the twist element z and the level k [GSW08b]. The numbers of inequivalent Jandl gerbes range from two (for simply connected G, per level and involution) to sixteen (for $\mathrm{PSO}(4n)$, for every even level).

Most prominently, there are two involutions on $\mathrm{SU}(2)$, namely $g \longmapsto g^{-1}$ and $g \longmapsto -g^{-1}$, and for each of them two inequivalent Jandl gerbes per level. On $\mathrm{SO}(3)$ there is only a single involution, but the results of [SSW07, GSW08b] exhibit four inequivalent Jandl gerbes per even level. This explains very nicely why $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ have the same number of orientifolds, despite a different number of involutions. These results reproduce those of the algebraic approach (see e.g. [FRS04]); for the precise comparison, Jandl structures related by the action of the trivial line bundle with either of its two equivariant structures have to be identified.

1.5 D-branes: Holonomy for surfaces with boundary

We now introduce the geometric structure needed to define surface holonomies and Wess-Zumino terms for surfaces with boundary. When one wants to define holonomy along a curve that is not closed, one way to make the parallel transport group-valued is to choose trivializations at the end points. To incorporate these trivializations into the background, one can choose a submanifold $\dot{\mathcal{D}} \subset M$ together with a trivialization $E|_{\dot{\mathcal{D}}} \longrightarrow \mathbf{1}_A$. Admissible paths $\gamma\colon [0,1] \longrightarrow M$ are then required to start and end on this submanifold, $\gamma(0), \gamma(1) \in \dot{\mathcal{D}}$. The same strategy has proven to be successful for *surfaces* with boundary.

Definition 1.5.1. Let $\mathcal{G}$ be a bundle gerbe with connection over M. A *$\mathcal{G}$-D-brane* is a submanifold $\dot{\mathcal{D}} \subset M$ together with a 1-morphism

$$\mathcal{D}\colon \quad \mathcal{G}|_{\dot{\mathcal{D}}} \longrightarrow \mathcal{I}_\omega \tag{1.18}$$

to a trivial bundle gerbe $\mathcal{I}_\omega$ given by a two-form ω on $\dot{\mathcal{D}}$.

The morphism $\mathcal{D}$ is called a $\mathcal{G}$-module, or twisted vector bundle. Notice that if H is the curvature of $\mathcal{G}$, the 1-morphism $\mathcal{D}$ enforces the identity

$$H|_{\dot{\mathcal{D}}} = \mathrm{d}\omega\,.$$

This equality restricts the possible choices of the world volume $\dot{\mathcal{D}}$ of the $\mathcal{G}$-D-brane.

Suppose that Σ is an oriented surface, possibly with boundary, and $\Phi\colon \Sigma \longrightarrow M$ is a smooth map. We require that $\Phi(\partial\Sigma) \subset \dot{\mathcal{D}}$. As described in Section 1.2.3, we choose a trivialization $\mathcal{T}\colon \Phi^*\mathcal{G} \longrightarrow \mathcal{I}_\rho$. Its restriction to $\partial\Sigma$ and the $\mathcal{G}$-module $\mathcal{D}$ define a 1-morphism

$$\mathcal{I}_\rho\big|_{\partial\Sigma} \xrightarrow{\ \mathcal{T}^{-1}|_{\partial\Sigma}\ } \Phi^*\mathcal{G}\big|_{\partial\Sigma} = \Phi^*(\mathcal{G}\big|_{\dot{\mathcal{D}}}) \xrightarrow{\ \Phi^*(\mathcal{D})\ } \Phi^*(\mathcal{I}_\omega)\,.$$

According to the definition (1.5), this 1-morphism is nothing but a hermitian vector bundle E with connection over $\partial\Sigma$ and its curvature is $\mathrm{curv}(E) = \omega - \rho$. Then we consider

$$\mathrm{Hol}_{\mathcal{G},\mathcal{D}}(\Phi) := \exp\Big(2\pi\mathrm{i}\int_\Sigma \rho\Big)\,\mathrm{Tr}(\mathrm{Hol}_E(\partial\Sigma))\,,$$

where the trace makes the holonomy of E independent of the choice of a parameterization of $\partial\Sigma$. This expression is independent of the choice of the trivialization $\mathcal{T}$: if $\mathcal{T}'\colon \mathcal{G} \longrightarrow \mathcal{I}_{\rho'}$ is another one and E' is the corresponding vector bundle, we have the transition isomorphism L from (1.6) with curvature $\rho' - \rho$, and an isomorphism $E' \otimes L \cong E$. It follows that

$$\exp\Big(2\pi\mathrm{i}\int_\Sigma \rho\Big)\,\mathrm{Tr}(\mathrm{Hol}_E(\partial\Sigma)) = \exp\Big(2\pi\mathrm{i}\Big(\int_\Sigma \rho' - \mathrm{curv}(L)\Big)\Big)\;\mathrm{Tr}(\mathrm{Hol}_{E'\otimes L}(\partial\Sigma))\,,$$

and on the right hand side the unprimed quantities cancel by Stokes' theorem.

Important results on D-branes concern in particular two large classes of models, namely free field theories and again WZW theories. The simplest example of a free field theory is the one of a compactified free boson, in which M is a circle $S^1_R \cong \mathbb{R} \bmod 2\pi R\,\mathbb{Z}$ of radius R. As is well known, there are then in particular two distinct types of D-branes: D0-branes $\mathcal{D}^{(0)}_x$, whose support is localized at a position $x \in S^1_R$, and D1-branes $\mathcal{D}^{(1)}_\alpha$, whose world volume is all of S^1_R and which are characterized by a Wilson line $\alpha \in \mathbb{R} \bmod \frac{1}{2\pi R}\,\mathbb{Z}$, corresponding to a flat connection on S^1_R.

For WZW theories, which are governed by a bundle gerbe $\mathcal{G}$ over a connected compact simple Lie group G, preserving the non-abelian current symmetries puts additional constraints on the admissible D-branes: their support $\dot{\mathcal{D}}$ must be a conjugacy class $\mathcal{C}_h$ of a group element $h \in G$. This can e.g. be seen by studying the scattering of bulk fields in the presence of the D-brane. On such conjugacy classes one finds a canonical 2-form $\omega_h \in \Omega^2(\mathcal{C}_h)$. Additionally, the 1-morphism $\mathcal{D}\colon \mathcal{G}|_{\mathcal{C}_h} \longrightarrow \mathcal{I}_{\omega_h}$ of a symmetric D-brane must satisfy a certain equivariance condition [Gaw05]. Interestingly, only on those conjugacy classes $\mathcal{C}_h$ for which

$$h = \exp\left(2\pi\mathrm{i}\,\tfrac{\alpha+\rho}{k+g^\vee}\right) ,$$

with α an *integrable* highest weight, admit such 1-morphisms. Here ρ denotes the Weyl vector and $g^\vee$ the dual Coxeter number of the Lie algebra $\mathfrak{g}$ of G. Thus in particular the possible world volumes of symmetric D-branes form only a discrete subset of conjugacy classes.

We finally remark that the concepts of D-branes and Jandl gerbes can be merged [GSW08a]. The resulting structures provide holonomies for unoriented surfaces with boundary, and can be used to define D-branes in WZW orientifold theories.

1.6 Bi-branes: Holonomy for surfaces with defect lines

1.6.1 Gerbe bimodules and bi-branes

In the representation theoretic approach to rational conformal field theory, boundary conditions and defect lines are described as modules and bimodules, respectively. The fact that the appropriate target space structure for describing boundary conditions, D-branes, is related to gerbe modules, raises the question of what the appropriate target space structure for defect lines should be. The following definition turns out to be appropriate.

Definition 1.6.1. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be bundle gerbes with connection over M_1 and M_2, respectively. A $\mathcal{G}_1$-$\mathcal{G}_2$-*bi-brane* is a submanifold $\dot{\mathcal{B}} \subset M_1 \times M_2$ together with a $(p_1^*\mathcal{G}_1)|_{\dot{\mathcal{B}}}$-$(p_2^*\mathcal{G}_2)|_{\dot{\mathcal{B}}}$-bimodule, i.e. with a 1-morphism

$$\mathcal{B}: \quad (p_1^*\mathcal{G}_1)|_{\dot{\mathcal{B}}} \longrightarrow (p_2^*\mathcal{G}_2)|_{\dot{\mathcal{B}}} \otimes \mathcal{I}_\varpi \tag{1.19}$$

with $\mathcal{I}_\varpi$ a trivial bundle gerbe given by a two-form ϖ on $\dot{\mathcal{B}}$.

Similarly as in (1.5) it follows that the two-form ϖ on $\dot{\mathcal{B}}$ obeys

$$p_1^*H|_{\dot{\mathcal{B}}} = p_2^*H|_{\dot{\mathcal{B}}} + \mathrm{d}\varpi\,. \tag{1.20}$$

We call $\dot{\mathcal{B}}$ the world volume and ϖ the curvature of the bimodule. With the appropriate notion of duality for bundle gerbes (see Section 1.4 of [Wal07]), a $\mathcal{G}_1$-$\mathcal{G}_2$-bimodule is the same as a $(\mathcal{G}_1 \otimes \mathcal{G}_2^*)$-module. For a formulation in terms of local data, see (B.8) of [FSW08].

As an illustration, consider again the free boson and WZW theories, restricting attention to the case $M_1 = M_2$. For the free boson compactified on a circle S^1_R of radius R, one finds that the world volume of a bi-brane is a submanifold $\dot{\mathcal{B}}_x \subset S^1_R \times S^1_R$ of the form

$$\dot{\mathcal{B}}_{x,\alpha} := \{(y, y-x) \,|\, y \in S^1_R\} \tag{1.21}$$

with $x \in S^1_R$. The submanifold $\dot{\mathcal{B}}_{x,\alpha}$ has the topology of a circle and comes with a flat connection, i.e. with a Wilson line α. Thus the bi-branes of a compactified free boson are naturally parameterized by a pair (x, α) taking values in two dual circles that describe a point on S^1_R and a Wilson line.

In the WZW case, for which the target space is a compact connected simple Lie group G, a scattering calculation [FSW08] similar to the one performed for D-branes indicates that the world volume of a (maximally symmetric) bi-brane is a *biconjugacy class*

$$\dot{\mathcal{B}}_{h,h'} := \big\{ (g, g') \in G \times G \,|\, \exists\, x_1, x_2 \in G\colon\; g = x_1 h x_2^{-1},\, g' = x_1 h' x_2^{-1} \big\} \;\subset G \times G$$

of a pair (h, h') of group elements satisfying $h\,(h')^{-1} \in \mathcal{C}_{h_\alpha}$ with h_α as given in (1.5). The biconjugacy classes carry two commuting G-actions, corresponding to the presence of two independent conserved currents in the field theory. Further, a biconjugacy class can be described as the preimage

$$\dot{\mathcal{B}}_{h,h'} = \tilde{\mu}^{-1}(\mathcal{C}_{hh'^{-1}}) = \big\{ (g, g') \in G \times G \,|\, g g'^{-1} \in \mathcal{C}_{hh'^{-1}} \big\}$$

of the conjugacy class $\mathcal{C}_{hh'^{-1}}$ under the map

$$\tilde{\mu}: \quad G \times G \ni (g_1, g_2) \longmapsto g_1 g_2^{-1} \in G\,.$$

Finally, the relevant two-form on $\dot{\mathcal{B}}_{h,h'}$ is

$$\varpi_{h,h'} := \tilde{\mu}^* \omega_{hh'^{-1}} - \frac{k}{2} \langle p_1^* \theta \wedge p_2^* \theta \rangle \, . \tag{1.22}$$

Here k is the level, θ is the left-invariant Maurer-Cartan form, p_i are the projections to the factors of $G \times G$, and ω_h is the canonical 2-form (see Section 1.5) on the conjugacy class $\mathcal{C}_h$. One checks that $\varpi_{h,h'}$ is bi-invariant and satisfies (1.20).

Examples of symmetric bi-branes can be constructed from symmetric D-branes using a multiplicative structure on the bundle gerbe $\mathcal{G}$ [Wal10]. Another important class of examples are Poincaré line bundles. These describe T-dualities; an elementary relation between T-duality and Poincaré line bundles is provided [SS09] by the equation of motion [RS09] in the presence of defects.

1.6.2 Holonomy and Wess-Zumino term for defects

The notion of bi-brane allows one in particular to define holonomy also for surfaces with defect lines.

The simplest world sheet geometry involving a defect line consists of a closed oriented world sheet Σ together with an embedded oriented circle $S \subset \Sigma$ that separates the world sheet into two components, $\Sigma = \Sigma_1 \cup_S \Sigma_2$. Assume that the defect S separates regions that support conformally invariant sigma models with target spaces M_1 and M_2, respectively, and consider maps ϕ_i: $\Sigma_i \longrightarrow M_i$ for $i \in \{1, 2\}$ such that the image of

$$\begin{array}{rccl} \phi_S : & S & \longrightarrow & M_1 \times M_2 \\ & s & \longmapsto & (\phi_1(s), \phi_2(s)) \end{array}$$

is contained in the submanifold $\dot{\mathcal{B}}$ of $M_1 \times M_2$. The orientation of Σ_i is the one inherited from the orientation of Σ, and without loss of generality we take $\partial\Sigma_1 = S$ and $\partial\Sigma_2 = -S$.

We wish to find the Wess-Zumino part of the sigma model action, or rather the corresponding holonomy $\mathrm{Hol}_{\mathcal{G}_1,\mathcal{G}_2,\mathcal{B}}$, that corresponds to having bundle gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$ over M_1 and M_2 and a $\mathcal{G}_1$-$\mathcal{G}_2$-bi-brane $\mathcal{B}$. The pullback of the bimodule (1.19) along the map ϕ_S: $S \longrightarrow \dot{\mathcal{B}}$ gives a $(\phi_1^*\mathcal{G}_1)|_S$-$(\phi_2^*\mathcal{G}_2)|_S$-bimodule

$$\phi_S^*\mathcal{B} : \quad (\phi_1^*\mathcal{G}_1)|_S \longrightarrow (\phi_2^*\mathcal{G}_2)|_S \otimes \mathcal{I}_{\phi_S^*\varpi} \, .$$

The pullback bundle gerbes $\phi_i^*\mathcal{G}_i$ over Σ_i are trivializable for dimensional reasons, and a choice $\mathcal{T}_i$: $\phi_i^*\mathcal{G}_i \longrightarrow \mathcal{I}_\rho$ of trivializations for two-forms ρ_i on Σ_i produces a vector bundle E over S. We then define

$$\mathrm{Hol}_{\mathcal{G}_1,\mathcal{G}_2,\mathcal{B}}(\Sigma, S) := \exp\Big(2\pi\mathrm{i} \int_{\Sigma_1} \rho_1\Big) \, \exp\Big(2\pi\mathrm{i} \int_{\Sigma_2} \rho_2\Big) \, \mathrm{Tr}(\mathrm{Hol}_E(S)) \, \in \mathbb{C}$$

to be the holonomy in the presence of the bi-brane $\mathcal{B}$. As shown in Appendix B.3 of [FSW08], for similar reasons as in the case of D-branes the number $\mathrm{Hol}_{\mathcal{G}_1,\mathcal{G}_2,\mathcal{B}}(\Sigma, S)$ is independent of the choice of the trivializations $\mathcal{T}_1$ and $\mathcal{T}_2$.

1.6.3 Fusion of defects

In the field theory context of section 1.3 there are natural notions of the fusion of a defect (an A-bimodule) with a boundary condition (a left A-module), yielding another boundary condition, and of the fusion of two defects, yielding another defect. Both of these are provided by the tensor product over the relevant Frobenius algebra A. These representation theoretic notions of fusion have a counterpart on the geometric side as well.

Consider first the fusion of a defect with a boundary condition. We allow for the general situation of a defect described by an M_1-M_2-bi-brane with different target spaces M_1 and M_2. Thus take an M_1-M_2-bi-brane with world volume $\dot{\mathcal{B}} \subseteq M_1 \times M_2$ and an M_2-D-brane with world volume $\dot{\mathcal{D}} \subseteq M_2$. The action of correspondences on sheaves suggests the following ansatz for the world volume of the fusion product:

$$(\mathcal{B} \star \mathcal{D})^{\cdot} := p_1\Big(\dot{\mathcal{B}} \cap p_2^{-1}(\dot{\mathcal{D}})\Big) \tag{1.23}$$

with p_i the projection $M_1 \times M_2 \longrightarrow M_i$. The corresponding ansatz for the fusion of an M_1-M_2-bi-brane $\mathcal{B}$ of world volume $\dot{\mathcal{B}}$ with an M_2-M_3-bi-brane $\mathcal{B}'$ of world volume $\dot{\mathcal{B}}'$ uses projections p_{ij} from $M_1 \times M_2 \times M_3$ to $M_i \times M_j$:

$$(\mathcal{B} \star \mathcal{B}')^{\cdot} := p_{13}\big(p_{12}^{-1}(\dot{\mathcal{B}}) \cap p_{23}^{-1}(\dot{\mathcal{B}}')\big)\,. \tag{1.24}$$

In general one obtains this way only subsets, rather than submanifolds, of M_1 and $M_1 \times M_3$, respectively. On a heuristic level one would, however, expect that owing to quantization of the branes a finite superposition of branes is selected, which should then reproduce the results obtained in the field theory setting.

We illustrate this again with the two classes of models already considered, i.e. free bosons and WZW theories, again restricting attention to the case $M_1 = M_2$. First, for the theory of a compactified free boson, the D-brane is of one of the types $\mathcal{D}_x^{(0)}$ or $\mathcal{D}_\alpha^{(1)}$ (see Section 1.5) and the bi-brane world volume is of the form $\dot{\mathcal{B}}_{x,\alpha}$ given in (1.21). For D-branes of type $\mathcal{D}_x^{(0)}$ the prescription (1.23) thus yields

$$\mathcal{B}_{(x,\alpha)} \star \mathcal{D}_y^{(0)} = \mathcal{D}_{x+y}^{(0)}\,.$$

For the fusion of a bi-brane $\mathcal{B}_{(x,\alpha)}$ and a D1-brane $\mathcal{D}_\beta^{(1)}$, one must take the flat line bundle on the bi-brane into account. We first pull back the line bundle on $\dot{\mathcal{D}}_\beta^{(1)}$ along p_2 to a line bundle on $S_R^1 \times S_R^1$, then restrict it to $\dot{\mathcal{B}}_{(x,\alpha)}$, and finally tensor this restriction with the line bundle on $\dot{\mathcal{B}}_{(x,\alpha)}$ described by the Wilson line α. This results in a line bundle with Wilson line $\alpha+\beta$ on the bi-brane world volume, which in turn can be pushed down along p_1 to a line bundle on S_R^1, so that

$$\mathcal{B}_{(x,\alpha)} \star \mathcal{D}_\beta^{(1)} = \mathcal{D}_{\alpha+\beta}^{(1)}\,.$$

In short, the fusion with a defect $\mathcal{B}_{(x,\alpha)}$ acts on D0-branes as a translation by x in position space, and on D1-branes as a translation by α in the space of Wilson lines. Similarly, the prescription (1.24) leads to

$$\mathcal{B}_{(x,\alpha)} \star \mathcal{B}_{(x',\alpha')} = \mathcal{B}_{(x+x',\alpha+\alpha')}$$

for the fusion of two bi-branes $\mathcal{B}_{(x,\alpha)}$ and $\mathcal{B}_{(x',\alpha')}$, i.e. both the position and the Wilson line variable of the bi-branes add up.

For WZW theories, besides the quantization of the positions of the branes another new phenomenon is that multiplicities other than zero or one appear in the field theory approach. In that context they arise from the decomposition of a tensor product $B_\alpha \otimes_A B_\beta$ of simple A-bimodules into a finite direct sum $\bigoplus_\gamma \mathcal{N}_{\alpha\beta}{}^\gamma B_\gamma$ of simple A-bimodules, and analogously for the case of mixed fusion (in rational CFT, both the category of A-modules and the category of A-bimodules are semisimple). Moreover, for simply connected groups, the multiplicities appearing in both types of fusion are in fact the same as the chiral fusion multiplicities which are given by the Verlinde formula.

By analogy with the field theory situation we expect fusion rules

$$\mathcal{B}_\alpha \star \mathcal{B}_\beta = \sum_\gamma \mathcal{N}_{\alpha\beta}{}^\gamma \mathcal{B}_\gamma \tag{1.25}$$

of bi-branes, and analogously for mixed fusion of bi-branes and D-branes. In the particular case of WZW theories on simply connected Lie groups one can in addition invoke the duality $\alpha \longmapsto \alpha^\vee$ which in that case exists on the sets of branes as well as defects that preserve all current symmetries, so as to work instead with fusion coefficients of type $\mathcal{N}_{\alpha\beta\gamma} = \mathcal{N}_{\alpha\beta}{}^{\gamma^\vee}$. Then for the case of two D-branes $\mathcal{D}_\alpha$ and $\mathcal{D}_\gamma$ with world volumes given by conjugacy classes $\mathcal{C}_{h_\alpha}$ and $\mathcal{C}_{h_\gamma}$ of G, as well as a bi-brane $\mathcal{B}_\beta$ whose world volume is the biconjugacy class $\tilde{\mu}^{-1}(\mathcal{C}_{h_\beta})$, one is lead to consider the subset

$$\Pi_{\alpha\beta\gamma} := p_1^{-1}(\mathcal{C}_\alpha) \cap \tilde{\mu}^{-1}(\mathcal{C}_\beta) \cap p_2^{-1}(\mathcal{C}_\gamma) = \{(g,g') \in G \times G \mid g \in \mathcal{C}_\alpha,\ g' \in \mathcal{C}_\gamma,\ gg'^{-1} \in \mathcal{C}_\beta\}$$

of $G \times G$. Combining the adjoint action on g and on g' gives a natural G-action on $\Pi_{\alpha\beta\gamma}$. And since both D-branes and the bi-brane are equipped with two-forms ω_α, ω_γ and ϖ_β, $\Pi_{\alpha\beta\gamma}$ comes with a natural two-form as well, namely with

$$\omega_{\alpha\beta\gamma} := p_1^*\omega_\alpha|_{\Pi_{\alpha\beta\gamma}} + p_2^*\omega_\gamma|_{\Pi_{\alpha\beta\gamma}} + \varpi_\beta|_{\Pi_{\alpha\beta\gamma}}\,. \tag{1.26}$$

By comparison with the field theory approach, this result should be linked to the fusion rules of the chiral WZW theory and thereby provide a physically motivated realization of the Verlinde algebra. To see how such a relation can exist, notice that fusion rules are dimensions of spaces of conformal blocks and as such can be obtained by geometric quantization from suitable moduli spaces of flat connections which arise in the quantization of Chern-Simons theories (see e.g. [SSE91]). The moduli space relevant to us is the one for the three-punctured sphere $S^2_{(3)}$, for which the monodromy of the flat connection around the punctures takes values in conjugacy classes $\mathcal{C}_\alpha$, $\mathcal{C}_\beta$ and $\mathcal{C}_\gamma$, respectively. The relations in the fundamental group of $S^2_{(3)}$ imply the condition $g_\alpha g_\beta g_\gamma = 1$ on the monodromies $g_\alpha \in \mathcal{C}_\alpha$, $g_\beta \in \mathcal{C}_\beta$ and $g_\gamma \in \mathcal{C}_\gamma$. Since monodromies are defined only up to simultaneous conjugation, the moduli space that matters in classical Chern-Simons theory is isomorphic to the quotient $\Pi_{\alpha\beta\gamma}/G$.

It turns out that the range of bi-branes appearing in the fusion product is correctly bounded already before geometric quantization. Indeed, the relevant product of conjugacy classes is

$$\mathcal{C}_h * \mathcal{C}_{h'} := \{ gg' \mid g \in \mathcal{C}_h,\ g' \in \mathcal{C}_{h'} \} ,$$

and for the case of $G = \mathrm{SU}(2)$ it is easy to see that this yields the correct upper and lower bounds for the SU(2) fusion rules [JW92, FSW08]. A full understanding of fusion can, however, only be expected after applying geometric quantization to the so obtained moduli space: this space must be endowed with a two-form, which is interpreted as the curvature of a line bundle, and the holomorphic sections of this bundle are what results from geometric quantization. In view of this need for quantization it is a highly non-trivial observation that the two-form (1.26) furnished by the two branes and the bi-brane is exactly the same as the one that arises from classical Chern-Simons theory.

In terms of defect lines, the decomposition (1.25) of the fusion product of bi-branes corresponds to the presence of a *defect junction*, which constitutes a particular type of *defect field*. A sigma model description for world sheets with such embedded defect junctions has been proposed in [RS09].

We have demonstrated how structural analogies between the geometry of bundle gerbes and the representation theoretic approach to rational conformal field theory lead to interesting geometric structure, including a physically motivated realization of the Verlinde algebra. The precise form of the latter of supersymmetric conformal field theory [FHT07] remain to be understood. and its relation with the realization of the Verlinde algebra in the context But in any case the parallelism between classical actions and full quantum theory exhibited above remains intriguing and raises the hope that some of the structural aspects discussed in this contribution are generic features of quantum field theories.

Chapter 2

Equivariance in Higher Geometry

The main goal of this chapter is to provide the correct mathematical framework to treat bundle gerbes and Jandl structures, as introduced in chapter 1 and non-abelian bundle gerbes as to be investigate in chapter 3. The results and techniques developed here are of independent interest.

In this chapter more precisely, we study (pre-)sheaves in bicategories on geometric categories: smooth manifolds, manifolds with a Lie group action and Lie groupoids. We present three main results: we describe equivariant descent, we generalize the plus construction to our setting and show that the plus construction yields a 2-stackification for 2-prestacks. Finally we show that, for a 2-stack, the pullback functor along a Morita-equivalence of Lie groupoids is an equivalence of bicategories.

We then discuss direct applications of our results to gerbes and 2-vector bundles. For instance, they allow to construct equivariant gerbes from local data and can be used to simplify the description of the local data. We illustrate the usefulness of our results in a systematic discussion of Jandl gerbes, which we have already encountered in section 1.4.

2.1 Overview

In a typical geometric situation, one selects a category of geometric spaces, e.g. smooth manifolds, and then considers for every geometric space M a category $\mathfrak{X}(M)$ of geometric objects on M, e.g. complex line bundles or principal G-bundles, with G a Lie group. The categories for different geometric spaces are related by pullback functors: they form a presheaf in categories.

In this chapter, the category of geometric spaces we consider is the category LieGrpd of Lie groupoids. This category has crucial advantages: it contains Čech groupoids and thus provides a convenient setting to discuss local data. Moreover, it contains action groupoids and thus allows us to deal with equivariant geometric objects as well.

We show that any presheaf $\mathfrak{X}$ on manifolds can be naturally extended to a presheaf on Lie groupoids. We also generalize the structure we associate to a geometric space M by considering a bicategory $\mathfrak{X}(M)$. This choice is motivated by the fact that bundle gerbes and bundle gerbes with connection on a given manifold have the structure of a bicategory [Ste00, Wal07]. Hence we will work with a presheaf in bicategories on the geometric category LieGrpd of Lie groupoids. Our theory extends the theory for (pre-)sheaves in categories on smooth manifolds presented in [Met03, Hei05].

A hallmark of any geometric theory is a procedure to obtain global objects from locally defined objects by a gluing procedure. To this end, one considers open covers which are, in the category of smooth manifolds, just a special class τ_{open} of morphisms. More generally, we endow the category of manifolds with a Grothendieck topology, although we will not directly use this language to keep this article at a more elementary level. The two prime examples for choices of τ for the category of smooth manifolds are τ_{open}, i.e. open covers, and τ_{sub}, i.e. surjective submersions.

Having fixed a choice for τ, we get a notion of τ-essential surjectivity of Lie functors and of τ-weak equivalence of Lie groupoids Γ and Λ. (τ_{sub}-weak equivalent Lie groupoids are also called *Morita equivalent*; some authors also call a τ_{sub}-weak equivalence a Morita equivalence.) Imposing different gluing conditions on the presheaf $\mathfrak{X}$ on LieGrpd for morphisms in τ, we get the notion of a τ-2-prestack on LieGrpd and of a τ-2-stack on LieGrpd, respectively. To simplify the notation, we refer to a 2-prestack as a prestack and to a 2-stack as a stack.

These basic definitions are the subject of section 2.2. At the end of this section, we can state our first main theorem 2.2.16:

Theorem. *Suppose, Γ and Λ are Lie groupoids and $\Gamma \to \Lambda$ is a τ-weak equivalence of Lie groupoids.*

1. *Let $\mathfrak{X}$ be a τ-prestack on* LieGrpd*. Then the functor*

$$\mathfrak{X}(\Lambda) \to \mathfrak{X}(\Gamma)$$

given by pullback is fully faithful, i.e. an equivalence on the Hom categories.

2. *Let $\mathfrak{X}$ be a τ-stack on* LieGrpd*. Then the functor*

$$\mathfrak{X}(\Lambda) \to \mathfrak{X}(\Gamma)$$

given by pullback is an equivalence of bicategories.

This theorem, or more precisely its first assertion, is a central ingredient for our second main result which we explain in section 2.3. In analogy to the sheafification of a presheaf, we associate to any prestack $\mathfrak{X}$ a presheaf in bicategories $\mathfrak{X}^+$ where the objects of the bicategory $\mathfrak{X}^+(M)$ consist of a cover $Y \to M$ and an object in the descent bicategory $\mathcal{D}\mathrm{esc}_{\mathfrak{X}}(Y \to M)$. We call this construction the plus construction. We then state theorem 2.3.3:

Theorem. *Let $\mathfrak{X}$ be a prestack on $\mathcal{M}an$. Then the presheaf in bicategories $\mathfrak{X}^+$ on $\mathcal{M}an$ obtained by the plus construction is a stack. Furthermore the canonical embedding $\mathfrak{X}(M) \longrightarrow \mathfrak{X}^+(M)$ is fully faithful for each manifold M.*

The plus construction is a powerful tool to construct geometric objects. In section 2.4, we show this in the example of bundle gerbes with connection: we introduce a bicategory $\mathcal{G}rbtriv^\nabla$ of trivial bundle gerbes with connection whose objects are given by 2-forms. A brief check reveals that the plus construction yields bundle gerbes,

$$\mathcal{G}rb^\nabla = \left(\mathcal{G}rbtriv^\nabla\right)^+ \ .$$

Theorem 2.3.3 then immediately implies that bundle gerbes form a stack.

Bundle gerbes give rise to a notion of surface holonomy. We then apply the reasoning leading to the definition of bundle gerbes to the definition of surface holonomy for unoriented surfaces and find the notion of a Jandl gerbe. In section 2.4.3, we also compare this notion to the notion of a Jandl structure on a gerbe that has been introduced earlier [SSW07] (see also section 1.4 of this thesis for a review). Based on the notion of Jandl gerbe, we introduce in section 2.4.3 the notion of an orientifold background on a Lie groupoid Λ. Theorem 2.2.16 allows us to define a surface holonomy for any Hilsum-Skandalis morphism [Met03, definition 62] from the unoriented worldsheet Σ to Λ.

It should be stressed that our results apply to general higher geometric objects, in particular to non-abelian gerbes and 2-vector bundles. To illustrate this point, subsection 2.4.4 contains a short discussion of 2-vector bundles. In all cases, theorem 2.1 immediately ensures that these higher geometric objects form a stack over the category of manifolds (and even of Lie groupoids).

Together, these results provide us with tools to construct concrete geometric objects: theorem 2.1 allows us to glue together geometric objects like e.g. gerbes from locally defined geometric object. Applications frequently require not only gerbes, but equivariant gerbes. Here, it pays off that our approach is set off for Lie groupoids rather than for manifolds only, since the latter combine equivariance and local data on the same footing. In particular, we are able to formulate in this framework theorem 2.7.5 on *equivariant* descent. One application of this theorem is to obtain equivariant gerbes from locally defined equivariant gerbes.

Theorem 2.1 and theorem 2.7.5 can then be combined with standard results on the action of Lie groups or Lie groupoids [DK00, Wei02] to obtain a simplified description of the local situation in terms of stabilizer groups. This strategy provides, in particular, an elegant understanding of equivariant higher categorical geometric objects, see e.g. [Nik09] for the construction of gerbes on compact Lie groups [Mei03, GR04] that are equivariant under the adjoint action.

We have collected the proofs of the theorems in the second part of this chapter in sections 2.5 – 2.9.

2.2 Sheaves on Lie groupoids

2.2.1 Lie groupoids

We start our discussion with an introduction to Lie groupoids. Groupoids are categories in which all morphisms are isomorphisms. A small groupoid, more specifically, consists of a set Γ_0 of objects and a set Γ_1 of morphisms, together with maps $s, t : \Gamma_1 \longrightarrow \Gamma_0, \iota : \Gamma_0 \longrightarrow \Gamma_1$ that associate to a morphism $f \in \Gamma_1$ its source $s(f) \in \Gamma_0$ and its target $t(f) \in \Gamma_0$ and to an object $m \in \Gamma_0$ the identity $\mathrm{id}_m \in \Gamma_1$. Finally, there is an involution $in : \Gamma_1 \longrightarrow \Gamma_1$ that obeys the axioms of an inverse. Concatenation is a map $\circ : \Gamma_1 \times_{\Gamma_0} \Gamma_1 \longrightarrow \Gamma_1$ where it should be appreciated that in the category of sets the pullback $\Gamma_1 \times_{\Gamma_0} \Gamma_1 = \{(f_1, f_2) \in \Gamma_1 \times \Gamma_1 | t(f_1) = s(f_2)\}$ exists. It is straightforward to translate the usual axioms of a category into commuting diagrams.

A Lie groupoid is groupoid object in the category of smooth manifolds:

Definition 2.2.1.
A groupoid in the category $\mathcal{M}an$ or a *Lie-groupoid* consists of two smooth manifolds Γ_0 and Γ_1 together with the following collection of smooth maps:

- Source and target maps $s, t : \Gamma_1 \longrightarrow \Gamma_0$.

To be able to define compositions, we need the existence of the pullback $\Gamma_1 \times_{\Gamma_0} \Gamma_1$. To ensure its existence, we require s and t to be surjective submersions.

The other structural maps are:

- A composition map $\circ : \Gamma_1 \times_{\Gamma_0} \Gamma_1 \longrightarrow \Gamma_1$
- A neutral map $\iota : \Gamma_0 \longrightarrow \Gamma_1$ providing identities
- A map $in : \Gamma_1 \longrightarrow \Gamma_1$ giving inverses

such that the usual diagrams commute.

Examples 2.2.2.

1. *For any manifold, we have the trivial Lie groupoid $M \rightrightarrows M$ in which all structure maps are identities. We use this to embed $\mathcal{M}an$ into* LieGrpd.

2. *Given any Lie group G, we consider the Lie groupoid BG with structure maps $G \rightrightarrows pt$ with pt the smooth zero-dimensional manifold consisting of a single point. The neutral map $pt \longrightarrow G$ is given by the map to the neutral element and composition $G \times G \longrightarrow G$ is group multiplication. Hence Lie groupoids are also a generalization of Lie groups.*

3. *More generally, if a Lie group G is acting smoothly on a smooth manifold M, the* action groupoid *$M//G$ has $\Gamma_0 := M$ as objects and the manifold $\Gamma_1 := G \times M$ as morphisms. The source map s is projection to M, the target map*

t is given by the action $t(g,m) := g \cdot m$. The neutral map is the injection $m \longmapsto (1,m)$ and composition is given by the group product, $(g,m) \circ (h,n) := (gh,n)$. Action Lie groupoids frequently are the appropriate generalizations of quotient spaces.

4. *For any covering $(U_i)_{i \in I}$ of a manifold M by open sets $U_i \subset M$, we consider the disjoint union $Y := \sqcup_{i \in I} U_i$ with the natural local homeomorphism $\pi : Y \twoheadrightarrow M$. Consider the two natural projections $Y \times_M Y \rightrightarrows Y$ with the composition map $(Y \times_M Y) \times_Y (Y \times_M Y) \cong Y^{[3]} \longrightarrow Y^{[2]}$ given by omission of the second element. The neutral map is the diagonal map $Y \longrightarrow Y \times_M Y$. This defines a groupoid $\check{C}(Y)$, the Čech-groupoid.*

The last two examples show that Lie groupoids provide a convenient framework to unify "local data" and equivariant objects.

We next need to introduce morphisms of Lie groupoids.

Definition 2.2.3.
A morphism of Lie groupoids or *Lie functor* $F : (\Gamma_1 \rightrightarrows \Gamma_0) \longrightarrow (\Omega_1 \rightrightarrows \Omega_0)$ consists of smooth maps $F_0 : \Gamma_0 \longrightarrow \Omega_0$ and $F_1 : \Gamma_1 \longrightarrow \Omega_1$ that are required to commute with the structure maps. For example, for the source map s, we have the commuting diagram

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{F_1} & \Omega_1 \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\ \Gamma_0 & \xrightarrow{F_0} & \Omega_0 \end{array}$$

Examples 2.2.4.

1. *Given two smooth manifolds M, N, every Lie functor $F : (M \rightrightarrows M) \longrightarrow (N \rightrightarrows N)$ is given by a smooth map $f : M \longrightarrow N$ with $F_0 = F_1 = f$. Hence $M \longmapsto (M \rightrightarrows M)$ is a fully faithful embedding and we identify the manifold M with the Lie groupoid $M \rightrightarrows M$.*

2. *Given two Lie groups G and H, the Lie functors $F : BG \longrightarrow BH$ between the corresponding Lie groupoids are given by smooth group homomorphisms $f : G \longrightarrow H$. Thus the functor $G \longmapsto BG$ is a fully faithful embedding of Lie groups into Lie groupoids.*

3. *For any two action groupoids $M//G$ and $N//G$, a G-equivariant map $f : M \longrightarrow N$ provides a Lie functor via $F_0 := f$ and $F_1 := f \times id : M \times G \longrightarrow N \times G$. The previous example with $M = N = pt$ shows that not all Lie functors between action groupoids are of this form.*

4. *Consider a refinement $Z \twoheadrightarrow M$ of a covering $Y \twoheadrightarrow M$ together with the refinement map $s : Z \longrightarrow Y$. This provides a Lie functor $S : \check{C}(Z) \longrightarrow \check{C}(Y)$ of Čech groupoids which acts on objects by $S_0 := s : Z \longrightarrow Y$ and on morphisms $S_1 : Z \times_M Z \longrightarrow Y \times_M Y$ by $S_1(z_1, z_2) := (s(z_1), s(z_2)) \in Y \times_M Y$.*

5. *As a special case, any covering* $Y \twoheadrightarrow M$ *is a refinement of the trivial covering* $id : M \twoheadrightarrow M$ *and we obtain a Lie functor* $\Pi^Y : \check{C}(Y) \longrightarrow M$.

2.2.2 Presheaves in bicategories on Lie groupoids

A presheaf in bicategories $\mathfrak{X}$ on the category $\mathcal{M}an$ of manifolds consist of a bicategory [Bén67] $\mathfrak{X}(M)$ for each manifold M, a pullback functor $f^* : \mathfrak{X}(N) \longrightarrow \mathfrak{X}(M)$ for each smooth map $f : M \longrightarrow N$ and natural isomorphisms $f^* \circ g^* \cong (g \circ f)^*$ for composable smooth maps f and g. Moreover, we need higher coherence isomorphisms satisfying the obvious, but lengthy conditions. More precisely, $\mathfrak{X}$ is a weak functor

$$\mathfrak{X} : \mathcal{M}an^{op} \longrightarrow \mathcal{B}i\mathcal{C}at.$$

Furthermore we impose the technical condition that $\mathfrak{X}$ preserves products, i.e. for a disjoint union $M = \coprod_{i \in I} M_i$ of manifolds indexed by a set I the following equivalence holds:

$$\mathfrak{X}(M) \cong \prod_{i \in I} \mathfrak{X}(M_i) \ . \tag{2.1}$$

Our next step is to extend such a presheaf in bicategories on $\mathcal{M}an$ to a presheaf in bicategories on Lie groupoids. For a Lie groupoid Γ finite fiber products $\Gamma_1 \times_{\Gamma_0} \cdots \times_{\Gamma_0} \Gamma_1$ exist in $\mathcal{M}an$ and we introduce the notation $\Gamma_2 = \Gamma_1 \times_{\Gamma_0} \Gamma_1$ and Γ_n analogously.

We can then use the nerve construction to associate to a Lie groupoid a simplicial manifold

$$\left(\cdots \overset{\partial_0}{\underset{\partial_3}{\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}}} \Gamma_2 \overset{\partial_0}{\underset{\partial_2}{\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}}} \Gamma_1 \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} \Gamma_0 \right) =: \Gamma_\bullet \ .$$

We can think of Γ_n as n-tuples of morphisms in Γ_1 that can be concatenated. The map $\partial_i : \Gamma_n \longrightarrow \Gamma_{n-1}$ is given by composition of the i-th and $i+1$-th morphism. Thus

$$\begin{aligned} \partial_i(f_1, \ldots, f_n) &:= (f_1, \ldots, f_i \circ f_{i+1}, \ldots, f_n) \\ \partial_0(f_1, \ldots, f_n) &:= (f_2, \ldots, f_n) \\ \partial_n(f_1, \ldots, f_n) &:= (f_1, \ldots, f_{n-1}) \ . \end{aligned}$$

In particular, $\partial_1, \partial_0 : \Gamma_1 \longrightarrow \Gamma_0$ are the source and target map of the groupoid. One easily verifies the simplicial identities $\partial_i \partial_{j+1} = \partial_j \partial_i$ for $i \leq j$. (We suppress the discussion of the degeneracy maps $\sigma_i : \Gamma_n \longrightarrow \Gamma_{n+1}$ which are given by insertion of an identity morphism at the i-the position.)

The nerve construction can also be applied to Lie functors and provides an embedding of Lie groupoids into simplicial manifolds. Suppose we are given a Lie functor

$F : (\Gamma_1 \rightrightarrows \Gamma_0) \longrightarrow (\Omega_1 \rightrightarrows \Omega_0)$. Consider the nerves $\Gamma_\bullet$ and $\Omega_\bullet$ and define a family $F_\bullet = (F_i)$ of maps, a *simplicial map*

$$F_i : \Gamma_i \longrightarrow \Omega_i$$

for all $i = 0, 1, 2, \dots$ with F_0, F_1 given by the Lie functor and maps given for $i > 1$ by

$$\begin{array}{rrcl} F_i : & \Gamma_1 \times_{\Gamma_0} \dots \times_{\Gamma_0} \Gamma_1 & \longrightarrow & \Omega_1 \times_{\Omega_0} \dots \times_{\Omega_0} \Omega_1 \\ & (f_1, \dots, f_n) & \longmapsto & \big(F_1(f_1), \dots F_1(f_n)\big) \ . \end{array}$$

By definition, the maps F_i commute with the maps ∂_j and σ_k that are part of the simplicial object. We summarize this in the following diagram:

$$\begin{array}{ccccccc} \cdots & \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} & \Gamma_2 & \substack{\rightarrow\\\rightarrow\\\rightarrow} & \Gamma_1 & \rightrightarrows & \Gamma_0 \\ & & \downarrow{\scriptstyle F_2} & & \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_0} \\ \cdots & \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} & \Omega_2 & \substack{\rightarrow\\\rightarrow\\\rightarrow} & \Omega_1 & \rightrightarrows & \Omega_0 \end{array}$$

Definition 2.2.5.
Let $\mathfrak{X}$ be a presheaf in bicategories on $\mathcal{M}an$ and Γ a Lie groupoid or, more generally, a simplicial manifold. A Γ-*equivariant object* of $\mathfrak{X}$ consists of

(O1) an object $\mathcal{G}$ of $\mathfrak{X}(\Gamma_0)$;

(O2) a 1-isomorphism

$$P\colon\ \partial_0^*\mathcal{G} \longrightarrow \partial_1^*\mathcal{G}$$

in $\mathfrak{X}(\Gamma_1)$;

(O3) a 2-isomorphism

$$\mu\colon\ \partial_2^*P \otimes \partial_0^*P \Longrightarrow \partial_1^*P$$

in $\mathfrak{X}(\Gamma_2)$, where we denote the horizontal product by $\otimes$;

(O4) a coherence condition

$$\partial_2^*\mu \circ (\mathrm{id} \otimes \partial_0^*\mu) = \partial_1^*\mu \circ (\partial_3^*\mu \otimes \mathrm{id})$$

on 2-morphisms in $\mathfrak{X}(\Gamma_3)$.

We next introduce 1-morphisms and 2-morphisms of Γ-equivariant objects:

Definition 2.2.6.

1. A 1-morphism between Γ-equivariant objects $(\mathcal{G}, P, \mu)$ and $(\mathcal{G}', P', \mu')$ in $\mathfrak{X}$ consists of the following data on the simplicial manifold

$$\left(\dots\ \Gamma_3 \ \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}\ \Gamma_2 \ \substack{\rightarrow\\\rightarrow\\\rightarrow}\ \Gamma_1 \ \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}}\ \Gamma_0 \right) = \Gamma_\bullet$$

(1M1) A 1-morphism $A\colon \mathcal{G} \longrightarrow \mathcal{G}'$ in $\mathfrak{X}(\Gamma_0)$;

(1M2) A 2-isomorphism $\alpha\colon P' \otimes \partial_0^* A \Longrightarrow \partial_1^* A \otimes P$ in $\mathfrak{X}(\Gamma_1)$;

(1M3) A commutative diagram

$$(\mathrm{id} \otimes \mu') \circ (\partial_2^* \alpha \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \partial_0^* \alpha) = \partial_1^* \alpha \circ (\mu \otimes id)$$

of 2-morphisms in $\mathfrak{X}(\Gamma_2)$.

2. A 2-morphism between two such 1-morphisms (A, α) and (A', α') consists of

(2M1) A 2-morphism $\beta\colon A \Longrightarrow A'$ in $\mathfrak{X}(\Gamma_0)$;

(2M2) a commutative diagram

$$\alpha' \circ (\mathrm{id} \otimes \partial_0^* \beta) = (\partial_1^* \beta \otimes \mathrm{id}) \circ \alpha$$

of 2-morphisms in $\mathfrak{X}(\Gamma_1)$.

We define the composition of morphisms using simplicial identities and composition in the bicategories $\mathfrak{X}(\Gamma_i)$, see e.g. [Wal07]. The relevant definitions are lengthy but straightforward, and we refrain from giving details.

One can check that in this way, one obtains the structure of a bicategory.

Remarks 2.2.7.

1. *Similar descent bicategories have been introduced in [Bre94] and [Dus89]. For a related discussion of equivariance in presheaves in bicategories, see also [Sko09].*

2. *We have defined Γ-equivariant objects for a presheaf $\mathfrak{X}$ in bicategories. Any presheaf $\mathfrak{X}$ in categories can be considered as a presheaf in bicategories with trivial 2-morphisms. We thus obtain a definition for $\mathfrak{X}(\Gamma)$ for presheaves in categories as well, where the 2-morphisms in (O3) on Γ_3 become identities and the condition (O4) is trivially fulfilled. Similar remarks apply to morphisms. All 2-morphisms are identities, hence $\mathfrak{X}(\Gamma_\bullet)$ can be identified with a category. This allows us to deal with presheaves in categories as special cases of our more general results on presheaves in bicategories and to recover part of the results of [Met03, Hei05].*

One can check that the following proposition holds:

Proposition 2.2.8.
Our construction provides for any Lie groupoid Γ a bicategory $\mathfrak{X}(\Gamma)$. The bicategories form a presheaf in bicategories on the category LieGrpd *of Lie groupoids.*

To make contact with existing literature, we introduce for the special case of an action groupoid $N//G$ as in example 2.2.2.3 the alternative notation

$$\mathfrak{X}_G(N) := \mathfrak{X}(N//G).$$

Remarks 2.2.9.

1. *For the convenience of the reader, we spell out the definition of a G-equivariant object of a presheaf in bicategories $\mathfrak{X}$ for the special case of a discrete group G. A G-equivariant object on a G-manifold N consists of*

   - *An object $\mathcal{G} \in \mathfrak{X}(N)$.*
   - *For every group element $g \in G$ a morphism $g^*\mathcal{G} \xrightarrow{\varphi_g} \mathcal{G}$.*
   - *A coherence 2-isomorphism for every pair of group elements $g, h \in G$,*

$$\begin{array}{ccc} g^*h^*\mathcal{G} & \xrightarrow{g^*\varphi_h} & g^*\mathcal{G} \\ & \searrow^{\varphi_{hg}} & \downarrow{\scriptstyle \varphi_g} \\ & & \mathcal{G} \end{array}$$

   - *A coherence condition.*

2. *We also show how to obtain the usual definition of equivariant bundles on a G-manifold N, where G is a Lie group. We denote the action by $w : N \times G \longrightarrow N$. An equivariant bundle on N consists of the following data: a bundle $\pi : P \longrightarrow N$ on N. The simplicial map $\partial_0 : N \times G \longrightarrow N$ is projection, $\partial_1 = w$ is the action. Hence $\partial_0^* P = P \times G$ and $\partial_1^* P = w^* P$. The second data is a morphism $P \times G \longrightarrow w^* P = (N \times G) \times_N P$. A morphism to a fibre product is a commuting diagram*

$$\begin{array}{ccc} P \times G & \longrightarrow & P \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ N \times G & \xrightarrow{w} & N \end{array}$$

   The left vertical map is bound to be $\pi \times \mathrm{id}_G$. The coherence condition of the equivariant object tells us that $\tilde{w} : P \times G \longrightarrow P$ is in fact a G-action that covers the G-action on N.

Corollary 2.2.10.
Let G be a Lie group. Then the functor $\mathfrak{X}_G$ forms a presheaf in bicategories on the category $\mathcal{M}\mathrm{an}_G$ of smooth manifolds with G action.

By abuse of notation, we denote the presheaf in bicategories on LieGrpd introduced in proposition 2.2.8 by $\mathfrak{X}$. This is justified by the fact that for a constant Lie groupoid $M \rightrightarrows M$ one has the equivalence $\mathfrak{X}(M \rightrightarrows M) \cong \mathfrak{X}(M)$.

We next wish to impose generalizations of the sheaf conditions on a presheaf. To this end, we have to single out a collection τ of morphisms in $\mathcal{M}an$. Technically, such a collection should form a Grothendieck (pre-)topology. This means essentially that the collection τ of morphisms is closed under compositions, pullbacks and contains all identities. See [Met03] for a detailed introduction. For our purposes, two families are important:

- The family τ_{sub} of surjective submersions.
- The family τ_{open} that consists of morphisms obtained from an open covering $(U_i)_{i\in I}$ of a manifold M by taking the local homeomorphism $\pi:\ Y \twoheadrightarrow M$ with $Y := \sqcup_{i\in I} U_i$.

From now on, two-headed arrows will be reserved for morphisms in the relevant topology ρ. Whenever, in the sequel, no explicit topology is mentioned, we refer to τ_{sub} as our standard (pre-)topology.

Remarks 2.2.11.
Let ρ be a topology on $\mathcal{M}an$.

- *For any morphism $\pi : Y \twoheadrightarrow M$ in ρ, we can form a Čech groupoid $\check{C}(Y)$ as in example 2.2.2.4 which we again call the Čech groupoid.*
- *Given a morphism $\pi : Y \twoheadrightarrow M$ of ρ, we define the descent bicategory by*

  $$\mathcal{D}esc_{\mathfrak{X}}(Y \twoheadrightarrow M) := \mathfrak{X}\big(\check{C}(Y)\big)\ .$$

  Recall the Lie functor $\Pi^Y :\ \check{C}(Y) \longrightarrow M$ for the Čech cover $Y \twoheadrightarrow M$ introduced in example 2.2.4.5. Applying the presheaf functor $\mathfrak{X}$ to this Lie functor, gives the functor of bicategories

  $$\tau_Y :\ \mathfrak{X}(M) \longrightarrow \mathfrak{X}(\check{C}(Y)) = \mathcal{D}esc_{\mathfrak{X}}(Y \twoheadrightarrow M) \qquad (2.2)$$

We are now ready for two definitions:

Definition 2.2.12.
Let $\mathfrak{X}$ be a presheaf in bicategories on $\mathcal{M}an$ and τ a topology on $\mathcal{M}an$.

1. A presheaf $\mathfrak{X}$ is called a *τ-prestack, if for every covering $Y \twoheadrightarrow M$ in τ the functor τ_Y of bicategories in (2.2) is fully faithful. (A functor of bicategories is called fully faithful, if all functors on Hom categories are equivalences of categories.)*
2. *A presheaf $\mathfrak{X}$ is called a τ*-stack, if for every covering $Y \twoheadrightarrow M$ in τ the functor τ_Y of bicategories is an equivalence of bicategories.

Generalizing the discussion of [MM03, Section 5.4] for submersions, we use the topology τ to single out certain morphisms of Lie groupoids that we call τ-weak equivalences of Lie groupoids. To motivate our definition, we discuss equivalences of small categories $\mathcal{C}, \mathcal{D}$. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is an equivalence, if it is fully faithful and essentially surjective. The latter condition means that for any object $d \in \mathcal{D}$, there exists an object $c \in \mathcal{C}$ and a isomorphism $F(c) \xrightarrow{f} d$ in $\mathcal{D}$. If the category $\mathcal{D}$ is a groupoid, this amounts to the requirement that the map from

$$\mathcal{C}_0 \times_{\mathcal{D}_0} \mathcal{D}_1 = \{(c, f) | c \in \mathcal{C}_0 = Ob(\mathcal{C}), f \in \mathcal{D}_1 = Mor(\mathcal{D}) \text{ with } F(c) = s(f) \ \}$$

to $\mathcal{D}_0$ induced by the target map is surjective. In the context of Lie groupoids, we will require this map to be in τ.

Definition 2.2.13.

1. A morphism of Lie groupoids $\Gamma \longrightarrow \Lambda$ is called *fully faithful, if the diagram*

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{F_1} & \Lambda_1 \\ {\scriptstyle s\times t}\downarrow & & {\scriptstyle s\times t}\downarrow \\ \Gamma_0 \times \Gamma_0 & \xrightarrow[F_0 \times F_0]{} & \Lambda_0 \times \Lambda_0 \end{array}$$

   is a pull back diagram.

2. *A morphism of Lie groupoids $\Gamma \longrightarrow \Lambda$ is called* τ-essentially surjective*, if the smooth map*

$$\Gamma_0 \times_{\Lambda_0} \Lambda_1 \longrightarrow \Lambda_0$$

   induced by the target map in Λ is in τ.

3. *A Lie functor is called a τ-weak equivalence of Lie groupoids, if it is fully faithful and τ-essentially surjective. If we omit the prefix τ, we always refer to τ_{sub}-weak equivalences.*

Remark 2.2.14.
Despite its name, τ-weak equivalence is not an equivalence relation. The equivalence relation generated by τ_{sub}-weak equivalences is called Morita equivalence or, for general τ-weak equivalences τ-Morita equivalence. Explicitly, two Lie groupoids Γ and Λ are Morita equivalent, if there exists a third Lie groupoid Ω and τ-weak equivalences $\Gamma \longrightarrow \Omega$ and $\Lambda \longrightarrow \Omega$.

Example 2.2.15.
The Lie functor $\Pi^Y : \check{C}(Y) \longrightarrow M$ is a τ-weak equivalence for all τ-covers.

The stack axiom just asserts that for all τ-weak equivalences of this type, the induced functor on bicategories $\tau_Y : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(\check{C}(Y))$ is an equivalence of bicategories. The first theorem of this chapter generalizes this statement to all τ-weak equivalences of Lie groupoids:

Theorem 2.2.16. *Suppose, Γ and Λ are Lie groupoids and $\Gamma \longrightarrow \Lambda$ is a τ-weak equivalence of Lie groupoids.*

1. *Let* $\mathfrak{X}$ *be a* τ*-prestack on* LieGrpd*. Then the functor*

$$\mathfrak{X}(\Lambda) \twoheadrightarrow \mathfrak{X}(\Gamma)$$

given by pullback is fully faithful.

2. *Let* $\mathfrak{X}$ *be a* τ*-stack on* LieGrpd*. Then the functor*

$$\mathfrak{X}(\Lambda) \twoheadrightarrow \mathfrak{X}(\Gamma)$$

given by pullback is an equivalence of bicategories.

Roughly speaking, τ-covers of manifolds can be thought of as being dense enough in τ-weak equivalences of Lie groupoids to allow an extension of the (pre-)stack condition.

We defer the proof of the theorem to section 2.5 - 2.8 and first present some applications.

2.2.3 Open coverings versus surjective submersions

We have already introduced two Grothendieck (pre-)topologies τ_{open} and τ_{sub} on the category of smooth manifolds. Since open covers are special examples of surjective submersions, any τ_{sub}-(pre)stack is obviously a τ_{open}-(pre)stack. From theorem 2.2.16, we deduce the converse:

Proposition 2.2.17.
A presheaf in bicategories on LieGrpd *is a* τ_{open}*-(pre)stack if and only if it is a* τ_{sub}*-(pre)stack.*

The proposition implies in particular that it is enough to check the stack condition on open covers.

Proof. It remains to be shown that any τ_{open}-stack $\mathfrak{X}$ is also a τ_{sub}-stack. We fix a surjective submersion $\pi : Y \twoheadrightarrow M$ and obtain a functor

$$\tau_Y : \ \mathfrak{X}(M) \ \longrightarrow \mathcal{D}esc_{\mathfrak{X}}(Y \twoheadrightarrow M) = \mathfrak{X}(\check{\mathrm{C}}\,(Y)) \ .$$

For the surjective submersion π, we can find local sections

$$s_i : \ U_i \longrightarrow Y$$

for an open cover $(U_i)_{i \in I}$ of M. We glue together these sections to a map s on the disjoint union of the open subsets. Then the diagram

$$\begin{array}{ccc} \sqcup_{i\in I} U_i & \xrightarrow{s} & Y \\ & \searrow & \downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

commutes. Here the unlabeled arrow is the inclusion of open subsets. This diagram induces a commuting diagram of Lie groupoids

$$\begin{array}{ccc} \check{\mathrm{C}}\,(\sqcup_{i\in I}U_i) & \xrightarrow{s} & \check{\mathrm{C}}\,(Y) \\ & \searrow & \downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

in which s is an τ_{open}-weak equivalence of Lie groupoids. Since $\mathfrak{X}$ is a τ_{open}-stack, the application of $\mathfrak{X}$ yields a diagram that commutes up to a 2-cell,

$$\begin{array}{ccc} \mathcal{D}esc_{\mathfrak{X}}(\sqcup_{i\in I}U_i) & \xleftarrow{s^*} & \mathcal{D}esc_{\mathfrak{X}}(Y) \\ & \nwarrow & \uparrow{\scriptstyle \pi^*} \\ & & \mathfrak{X}(M) \end{array}$$

We wish to show that the vertical arrow is an equivalence of bicategories. The lower left arrow is an equivalence of bicategories, since $\mathfrak{X}$ is assumed to be a τ_{open}-stack. Since s is a τ_{open}-weak equivalence of Lie groupoids, theorem 2.2.16 implies that s^* is an equivalence of bicategories and the assertion follows. □

Since presheaves in categories are particular examples, an immediate corollary is:

Corollary 2.2.18.
A presheaf in categories on LieGrpd *is a τ_{open}-(pre)stack if and only if it is a τ_{sub}-(pre)stack.*

After one further decategorification, we also obtain

Corollary 2.2.19.
A presheaf on LieGrpd *is a τ_{open}-separated presheaf if and only if it is a τ_{sub}-separated presheaf.*
A presheaf on LieGrpd *is a τ_{open}-sheaf if and only if it is a τ_{sub}-sheaf.*

Let us discuss an application of this result: $U(1)$ principal bundles form a stack on $\mathcal{M}an$ with respect to the open topology τ_{open}, see section 1.2.1. As a consequence of corollary 2.2.18, $U(1)$ bundles also form a stack with respect to surjective submersions. Hence we can glue bundles also with respect of surjective submersions. In this way, we recover the following well-known

Proposition 2.2.20.
Consider a free action groupoid $M//G$ so that the quotient space M/G has a natural structure of a smooth manifold and the canonical projection is a submersion. (This is, e.g., the case if the action of G on M is proper and discontinuous.) Then the category of smooth $U(1)$-bundles on M/G is equivalent to the category of G-equivariant $U(1)$-bundles on M.

Proof. Since the action is free, the canonical projection $\pi : \ M \longrightarrow M/G$ is a submersion that induces a τ_{sub}-weak equivalence of Lie groupoids. We have seen that $U(1)$-bundles form a τ_{sub}-stack, and hence by theorem 2.2.16 the canonical projection π induces an equivalence of categories. □

We have formulated this result for the special case of $U(1)$ bundles. Obviously, the same argument applies to any stack on $\mathcal{M}an$, and we obtain similar equivalences of categories for G-equivariant principal bundles, and associated bundles for any structure group.

2.3 The plus construction

In this section we describe a general procedure for 2-stackification. More precisely, we show how to obtain a 2-stack $\mathfrak{X}^+$ on $\mathcal{M}an$ starting from 2-prestack $\mathfrak{X}$ on $\mathcal{M}an$. In analogy to the case of sheaves, we call this construction the plus construction. The idea is to complement the bicategories $\mathfrak{X}(M)$ by adding objects in descent bicategories. The main result is then that the 2-presheaf in bicategories obtained in this way is closed under descent.

We first describe the bicategory $\mathfrak{X}^+(M)$ for a manifold M.

Definition 2.3.1. An object of $\mathfrak{X}^+(M)$ consists of a covering $Y \twoheadrightarrow M$ and an object G in the descent bicategory $\mathcal{D}esc_{\mathfrak{X}}(Y)$.

In order to define 1-morphisms and 2-morphisms between objects with possibly different coverings $\pi : Y \twoheadrightarrow M$ and $\pi' : Y' \twoheadrightarrow M$, we pull all the data back to a common refinement of these coverings and compare them there. We call a covering $\zeta : Z \twoheadrightarrow M$ a *common refinement* of π and π' iff there exist coverings $s\colon Z \twoheadrightarrow Y$ and $s'\colon Z \twoheadrightarrow Y'$ such that the diagram

$$\begin{array}{ccccc} Y & \overset{s}{\twoheadleftarrow} & Z & \overset{s'}{\twoheadrightarrow} & Y' \\ & {\scriptstyle\pi}\searrow & \downarrow{\scriptstyle\zeta} & \swarrow{\scriptstyle\pi'} & \\ & & M & & \end{array} \tag{2.3}$$

commutes. An important example of such a common refinement is the fibre product $Z := Y \times_M Y' \twoheadrightarrow M$, with the maps $Z \twoheadrightarrow Y$ and $Z \twoheadrightarrow Y'$ given by the projections. We call this the *canonical common refinement*. The maps s and s' of a common refinement $Z \twoheadrightarrow M$ induce Lie functors on the Lie groupoids

$$\check{\mathrm{C}}\,(Y) \longleftarrow \check{\mathrm{C}}\,(Z) \longrightarrow \check{\mathrm{C}}\,(Y').$$

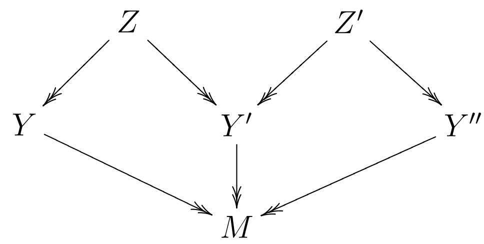
Hence we have *refinement functors* s^* and s'^*:

$$\mathcal{D}esc_{\mathfrak{X}}(Y) \xrightarrow{s^*} \mathcal{D}esc_{\mathfrak{X}}(Z) \xleftarrow{(s')^*} \mathcal{D}esc_{\mathfrak{X}}(Y').$$

For an object G in $\mathcal{D}esc_{\mathfrak{X}}(Y)$ we denote the refinement $s^*(G)$ by G_Z.

Definition 2.3.2.
- A *1-morphism* between objects $\mathcal{G} = (Y, G)$ and $\mathcal{G}' = (Y', G')$ of $\mathfrak{X}^+(M)$ consists of a common refinement $Z \twoheadrightarrow M$ of the coverings $Y \twoheadrightarrow M$ and $Y' \twoheadrightarrow M$ and a 1-morphism $A : G_Z \longrightarrow G'_Z$ of the two refinements in $\mathcal{D}esc_{\mathfrak{X}}(Z)$.
- A 2-morphism between 1-morphisms $\mathfrak{m} = (Z, A)$ and $\mathfrak{m}' = (Z', A')$ consists of a common refinement $W \twoheadrightarrow M$ of the coverings $Z \twoheadrightarrow M$ and $Z' \twoheadrightarrow M$ (respecting the projections to Y and Y', respectively) and a 2-morphism $\beta : \mathfrak{m}_W \Longrightarrow \mathfrak{m}'_W$ of the refined morphisms in $\mathcal{D}esc_{\mathfrak{X}}(W)$. In addition two such 2-morphisms (W, β) and (W', β') must be identified iff there exists a further common refinement $V \twoheadrightarrow M$ of $W \twoheadrightarrow M$ and $W' \twoheadrightarrow M$ such that the refined 2-morphisms agree on V.

Now that we have defined objects, morphisms and 2-morphisms in $\mathfrak{X}^+(M)$ it remains to define compositions and identities. We will just indicate how this is done. For example let $\mathcal{G} = (Y, G)$, $\mathcal{G}' = (Y', G')$ and $\mathcal{G}'' = (Y'', G'')$ be objects and $\mathfrak{m} = (Z, A) : \mathcal{G} \longrightarrow \mathcal{G}'$ and $\mathfrak{m}' = (Z', A') : \mathcal{G}' \longrightarrow \mathcal{G}''$ be morphisms. The covers can then be arranged to the diagram



Now let $Z'' := Z \times_{Y'} Z'$ be the pullback of the upper diagram. This exists in $\mathcal{M}an$ and is evidently a common refinement of Y and Y''. The composition $\mathfrak{m}' \circ \mathfrak{m}$ is then defined to be the tuple $(Z'', A'_{Z''} \circ A_{Z''})$ where $A'_{Z''} \circ A_{Z''}$ denotes the composition of the refined morphisms in $\mathcal{D}esc_{\mathfrak{X}}(Z'')$.

Finally one can check that this defines the structure of a bicategory $\mathfrak{X}^+(M)$. See [Wal07] for a very detailed treatment of a related bicategory. In order to turn the bicategories $\mathfrak{X}^+(M)$ into a stack we have to define the pullback functors

$$f^* : \mathfrak{X}^+(N) \longrightarrow \mathfrak{X}^+(M)$$

for all smooth maps $f : M \longrightarrow N$. This is done in the obvious way using the pullback of covers and the pullback functors of the prestack $\mathfrak{X}$.

Theorem 2.3.3. *If $\mathfrak{X}$ is a prestack, then $\mathfrak{X}^+$ is a stack. Furthermore the canonical embedding $\mathfrak{X}(M) \longrightarrow \mathfrak{X}^+(M)$ is fully faithful for each M.*

We relegate the proof of this theorem to section 2.9.

Remark 2.3.4. *1. If we choose the covers in definition 2.3.1 and 2.3.2 to be in the topology τ_{open} we obtain a slightly different stack $\mathfrak{X}^+_{open}$. Argument similar to the ones used in section 2.2.3 show that $\mathfrak{X}^+(M) \cong \mathfrak{X}^+_{open}(M)$ for each smooth manifold M.*

2. As in remark 2.2.7.2, one can specialize to presheaves in categories and obtains the stackification process for 1-prestacks.

2.4 Applications of the plus construction

We next present several applications of the plus construction.

2.4.1 Bundle gerbes

In this section we make contact to the earlier ad-hoc definition of bundle gerbes given in section 1.2.2. We more ore less repeat what we have done there in the light of the abstract machinery.

The input for the plus construction is a presheaf in bicategories on $\mathcal{M}an$. In the same way a monoid is the simplest example of a category (with one object), any monoidal category gives a bicategory with a single object. An example for a bicategory can thus be obtained from the monoidal category of principal A-bundles, where A is any abelian Lie group. This way, we get a presheaf $\mathcal{G}rbtriv_A$ of of trivial A-gerbes. Since bundles can be glued together, the homomorphism categories are closed under descent. The presheaf $\mathcal{G}rbtriv_A$ is thus a prestack. The plus construction yields the stack

$$\mathcal{G}rb_A := (\mathcal{G}rbtriv_A)^+$$

of gerbes (without connection). Our general result implies that gerbes form a sheaf on $\mathcal{M}an$. Together with theorem 2.2.16 and theorem 2.7.5 of this chapter, this provides a local construction of gerbes and the definition of equivariant gerbes.

Let us next construct gerbes with connection; for simplicity, we restrict to the abelian group $A = U(1)$ and suppress the index A. The guiding principle for our construction is the requirement that gerbes should lead to a notion of surface holonomy (see section 1.2.3). Remember from 1.2.2 the bicategory whose objects are two-forms. It is convenient to close them first under descent. This way, we obtain the prestack $\mathcal{G}rbtriv^\nabla$ of trivial bundle gerbes with connection where the bicategories $\mathcal{G}rbtriv^\nabla(M)$ are defined by:

- An object is a 2-form $\omega \in \Omega^2(M)$, called a *trivial bundle gerbe with connection* and denoted by $\mathcal{I}_\omega$.
- A 1-morphism $\mathcal{I}_\omega \longrightarrow \mathcal{I}_{\omega'}$ is a U(1) bundle L with connection of curvature $\omega' - \omega$.

- A 2-morphism $\phi : L \longrightarrow L'$ is a morphism of bundles with connection.

Together with the natural operation of pullback of bundles and forms one can easily checks that $\mathcal{G}rbtriv^\nabla$ is a prestack. By theorem 2.3.3, the plus construction yields a stack

$$\mathcal{G}rb^\nabla := \left(\mathcal{G}rbtriv^\nabla\right)^+$$

on $\mathcal{M}an$ and even a stack on the category of Lie groupoids. This agrees with the one given in section 1.2.1 except for the fact that we have additionally allowed higher rank bundles there, i.e. non-invertivle morphisms. In particular now, definition 2.2.5 provides a natural notion of an equivariant gerbe. Theorem 2.2.16 then implies:

Corollary 2.4.1.
For an equivalence $F : \Gamma \longrightarrow \Lambda$ of Lie groupoids, the pullback functor

$$F^* : \mathcal{G}rb(\Lambda) \longrightarrow \mathcal{G}rb(\Gamma) \qquad \mathcal{G}rb^\nabla(\Lambda) \longrightarrow \mathcal{G}rb^\nabla(\Gamma)$$

is an equivalence of bicategories. In particular, for a free, proper and discontinuous action of a Lie group G on a smooth manifold M we have the following equivalences of bicategories

$$\mathcal{G}rb_G(M) \cong \mathcal{G}rb(M/G) \qquad \textit{respectively} \qquad {\mathcal{G}rb^\nabla}_G(M) \cong \mathcal{G}rb^\nabla(M/G)\ .$$

We compare this new stack $\mathcal{G}rb^\nabla$ with objects introduced in the literature. An object in $\mathcal{G}rb^\nabla(M)$ consists by definition of a covering $Y \longrightarrow M$ and an object G in $\mathcal{D}esc_{\mathcal{G}rbtriv^\nabla}(Y)$. Spelling out the data explicitly, one verifies that objects are just bundle gerbes in the sense of [Mur96] and [Ste00]. For the special case of an open cover $Y := \bigsqcup U_i$, an object in $\mathcal{D}esc_{\mathcal{G}rbtriv^\nabla}(Y)$ is an Chaterjee-Hitchin gerbe, see [Cha98].

To compare different morphisms introduced in the literature, we first need a definition:

Definition 2.4.2.
i) A morphism $\mathcal{A} : (Y, G) \longrightarrow (Y', G')$ in $\mathfrak{X}^+(M)$ is called a *stable isomorphism*, if it is defined on the canonical common refinement

$$Z := Y \times_M Y'\ .$$

ii) A *stable 2-isomorphism* in $\mathfrak{X}(M)$ between stable isomorphisms (Z, A) and (Z, A') is a morphism in $\mathcal{D}esc_{\mathfrak{X}}(Z \twoheadrightarrow M)$, i.e. a morphism on the canonical common refinement $Z = Y \times_M Y'$.
iii) Two objects $(Y, \mathcal{G})$ and $(Y', \mathcal{G}')$ are called *stably isomorphic* if there is a stable isomorphism $(Y, \mathcal{G}) \longrightarrow (Y', \mathcal{G}')$.

For bundle gerbes (Y, G) and (Y', G'), stable morphisms are a subcategory,

$$\mathrm{Hom}_{\mathrm{Stab}}\Big((Y,G),(Y',G')\Big) \subset \mathrm{Hom}_{\mathcal{G}rb^{\nabla}}\Big((Y,G),(Y',G')\Big) .$$

We next show that the these categories are in fact equivalent. We start with the following observation:

Lemma 2.4.3.
Let $Z \twoheadrightarrow M$ be a common refinement of $Y \twoheadrightarrow M$ and $Y' \twoheadrightarrow M$ with morphisms $s : Z \twoheadrightarrow Y$ and $s' : Z \twoheadrightarrow Y'$ as in (2.3). Then the morphism $s \times_M s' : Z \longrightarrow Y \times_M Y'$ induces a τ-weak equivalence of Čech groupoids

$$\check{C}(Z) \xrightarrow{\sim} \check{C}(Y \times_M Y').$$

Proof. Spelling out the definition of τ-essential surjectivity for the relevant Lie functor $\check{\mathrm{C}}\,(Z) \longrightarrow \check{\mathrm{C}}\,(Y \times_M Y')$, we see that we have to show that the smooth map

$$Z \times_{Y \times_M Y'} (Y \times_M Y' \times_M Y \times_M Y') \equiv Z \times_M Y \times_M Y' \longrightarrow Y \times_M Y'$$

is in τ. This follows at once from the pullback diagram:

$$\begin{array}{ccc} Z \times_M Y \times_M Y' & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y \times_M Y' & \longrightarrow & M \end{array}$$

It remains to show that the Lie functor is fully faithful. From example 2.2.15 we know that the vertical morphisms in the diagram

$$\begin{array}{ccc} \check{\mathrm{C}}\,(Z) & \longrightarrow & \check{\mathrm{C}}\,(Y \times_M Y') \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

are τ-weak equivalences, and thus in particular fully faithful. Elementary properties of pullback diagrams then imply that the horizontal morphism is fully faithful as well. □

Hence the induced morphism of Lie groupoids

$$\check{\mathrm{C}}\,(Z) \longrightarrow \check{\mathrm{C}}\,(Y \times_M Y')$$

is fully faithful and τ-essentially surjective. Since $\mathfrak{X}$ is a prestack, we deduce from the first assertion of theorem 2.2.16

Proposition 2.4.4.
For any two objects $\mathcal{O} = (Y, \mathcal{G})$ and $\mathcal{O}' = (Y', \mathcal{G}')$ in $\mathfrak{X}^+(M)$, the morphism 1-category $\mathrm{Hom}(\mathcal{O}, \mathcal{O}')$ *is equivalent to the subcategory of stable isomorphisms and stable 2-isomorphisms.*

In particular, two objects are isomorphic in $\mathfrak{X}^+(M)$, if and only if they are stably isomorphic.

Remark 2.4.5.

- *Stable isomorphisms have been introduced in [Ste00, MS00]; proposition 2.4.4 shows that our bicategory is equivalent to the one in that paper. With our definition of morphisms, composition has a much simpler structure.*
- *In [Wal07] a further different choice of common refinement was made. The bicategory introduced in [Wal07] has as morphisms categories that are contained in our morphism categories and contain the morphism categories of [MS00]. Hence all three bicategories are equivalent.*

Finally, recall from section 1.2.3 that for bundle gerbes there is a notion of surface holonomy. More precisely let $\mathcal{G}$ be a gerbe with connection over a smooth oriented manifold M, and

$$\Phi : \quad \Sigma \longrightarrow M$$

be a smooth map defined on a closed oriented surface Σ. Then the holonomy of $\mathcal{G}$ around Φ is an element

$$\mathrm{Hol}_{\mathcal{G}}(\Phi) \in \mathrm{U}(1)$$

which is defined using the local 2-forms and descent properties.

2.4.2 Jandl gerbes

It is instructive to apply the same reasoning to the construction of Jandl gerbes. The slightly less general notion of gerbes with a Jandl structure has been introduced in [SSW07] to obtain a notion of surface holonomy for unoriented surfaces. In this subsection, we introduce the more general notion of a Jandl gerbe. To this end, we follow the general pattern from section 2.4.1 and first define Jandl bundles:

Definition 2.4.6.
A *Jandl bundle* over M is a pair, consisting of a $U(1)$-bundle P with connection over M and a smooth smooth map $\sigma : M \longrightarrow \mathbb{Z}/2 = \{1, -1\}$. Morphisms of Jandl bundles $(P, \sigma) \longrightarrow (Q, \mu)$ only exist if $\sigma = \mu$. In this case they are morphisms $P \longrightarrow Q$ of bundles with connection. We denote the category of Jandl bundles by $\mathcal{JB}un^\nabla(M)$

We need the covariant functor

$$(\quad)^{-1} : \mathcal{B}un^\nabla(M) \longrightarrow \mathcal{B}un^\nabla(M)$$

which sends a bundle P to its dual bundle P^*. A morphism $f : P \longrightarrow Q$ is sent to $\left(f^*\right)^{-1} : P^* \longrightarrow Q^*$. This functor is well defined since all morphisms in $\mathcal{B}un^\nabla(M)$ are isomorphisms. It squares to the identity and thus defines an $\mathbb{Z}/2$ action on the category $\mathcal{B}un^\nabla(M)$.

Smooth maps $\sigma : M \longrightarrow \mathbb{Z}/2$ are constant on connected components of M. For each such map σ, we get a functor by letting $(\quad)^{-1}$ acting on each connected component by the power given by the value of σ on that connected component. For each map σ we thus have a functor

$$(\quad)^\sigma : \mathcal{B}un^\nabla(M) \longrightarrow \mathcal{B}un^\nabla(M).$$

For our construction, we need a monoidal category of morphisms of trivial objects. Hence we endow $\mathcal{J}\mathcal{B}un^\nabla(M)$ with a monoidal structure;

$$(P, \sigma) \otimes (Q, \mu) := (P \otimes Q^\sigma, \sigma\mu) \; .$$

Now we are ready to define the prestack $\mathcal{J}\mathcal{G}rbtriv^\nabla$ of trivial Jandl gerbes. Again the guiding principle is the definition of holonomies, this time for unoriented surfaces (for more details, see section 2.4.3).

- An object is a 2-form $\omega \in \Omega^2(M)$, called a *trivial Jandl gerbe with connection* and denoted by $\mathcal{I}_\omega$.
- A 1-morphism $\mathcal{I}_\omega \longrightarrow \mathcal{I}_{\omega'}$ is a Jandl bundle (P, σ) of curvature $\mathrm{curv}P = \sigma\cdot\omega' - \omega$.
- A 2-morphism $\phi : (P, \sigma) \longrightarrow (Q, \mu)$ is a morphism of Jandl bundles with connection.

Composition of morphisms is defined as the tensor product of Jandl bundles. It is easy to see that $\mathcal{J}\mathcal{G}rbtriv^\nabla$ is a prestack. We define Jandl gerbes by applying the plus construction:

$$\mathcal{J}\mathcal{G}rb^\nabla := \left(\mathcal{J}\mathcal{G}rbtriv^\nabla\right)^+.$$

By theorem 2.3.3, this defines a stack.

Remark 2.4.7.

1. *We relegate the discussion of the relation between Jandl gerbes and gerbes with a Jandl structure introduced in [SSW07] to the next section 2.4.3, see proposition 2.4.12. In the same section, we discuss holonomy for unoriented surfaces.*
2. *In terms of descent data, we can describe a Jandl gerbe on M by a cover $Y \twoheadrightarrow M$, a two-form $\omega \in \Omega^2(Y)$, a Jandl bundle (P, σ) on $Y^{[2]}$ such that $\sigma\partial_1^*\omega - \partial_0^*\omega = \mathrm{curv}(P)$ and a 2-morphism*

$$\mu : \partial_2^*(P, \sigma) \otimes \partial_0^*(P, \sigma) \Longrightarrow \partial_1^*(P, \sigma)$$

of Jandl bundles on $Y^{[3]}$. The definition of morphisms of Jandl bundles implies that such a morphism only exists, if the identity

$$\partial_2^*\sigma \cdot \partial_0^*\sigma = \partial_1^*\sigma \tag{2.4}$$

holds. Under this condition, the data on $Y^{[3]}$ reduce to a morphism of $U(1)$-bundles

$$\mu : \ \partial_2^* P \otimes \partial_0^* P \Longrightarrow \partial_1^* P$$

that obeys the same associativity condition on $Y^{[4]}$ as ordinary gerbes.

3. *Both trivial Jandl gerbes and trivial bundle gerbes are given, as objects, by 2-forms; hence they are locally the same. The crucial difference between Jandl gerbes and bundle gerbes is the fact that there are more 1-morphisms between Jandl gerbes: apart from the morphisms $(P, 1)$, we also have "odd" morphisms $(P, -1)$.*

We have the inclusion $j : \mathcal{B}un(M) \longrightarrow \mathcal{JB}un(M)$ where we identify a bundle $P \in \mathcal{B}un(M)$ with a Jandl bundle $(P, 1) \in \mathcal{JB}un(M)$. Here $1 : M \longrightarrow \mathbb{Z}/2$ is the constant function to the neutral element. The category $\mathcal{B}un(M)$ is thus a full subcategory of $\mathcal{JB}un(M)$. The inclusion functor is clearly monoidal and thus yields an inclusion $\mathcal{G}rbtriv^\nabla(M) \longrightarrow \mathcal{JG}rbtriv^\nabla(M)$ of bicategories. Finally this induces an inclusion functor

$$\mathcal{J} : \mathcal{G}rb^\nabla(M) \longrightarrow \mathcal{JG}rb^\nabla(M) \ .$$

In terms of descent data, the functor $\mathcal{J}$ maps

$$(Y, \omega, P, \mu) \longmapsto (Y, \omega, (P, 1), \mu) \ .$$

The inclusion functor $\mathcal{J}$ is faithful, but neither full nor essentially surjective. Hence we have to understand its essential image.

Given a Jandl bundle (P, σ), we can forget P and just keep the smooth map σ. Since morphisms in $\mathcal{JB}un(M)$ preserve σ by definition, this yields a functor

$$o : \mathcal{JB}un^\nabla(M) \longrightarrow C^\infty(M, \mathbb{Z}/2) \tag{2.5}$$

where the category on the right hand side has $\mathbb{Z}/2$-valued smooth functions as objects and only identities as morphisms.

The functor o is monoidal, i.e. $(P, \sigma) \otimes (Q, \mu) \longmapsto \sigma \cdot \mu$. We denote the category of $\mathbb{Z}/2$ bundles on M by $\mathcal{B}un_{\mathbb{Z}/2}(M)$. It contains the full subcategory $\mathcal{B}untriv_{\mathbb{Z}/2}(M)$ of trivial $\mathbb{Z}/2$-bundles:

- The category $\mathcal{B}untriv_{\mathbb{Z}/2}(M)$ has exactly one object, the trivial $\mathbb{Z}/2$ bundle $M \times \mathbb{Z}/2 \longrightarrow M$.

- The endomorphisms of $M \times \mathbb{Z}/2$ are given by elements in $C^\infty(M, \mathbb{Z}/2)$.
- Composition of endomorphisms is pointwise multiplication of smooth maps $M \longrightarrow \mathbb{Z}/2$

Together with this observation the functor (2.5) yields a functor

$$\mathcal{J}\mathcal{G}\mathrm{rbtriv}^\nabla(M) \longrightarrow \mathcal{B}\mathrm{untriv}_{\mathbb{Z}/2}(M).$$

Applying the plus construction, that functor induces a functor

$$\mathcal{O} : \mathcal{J}\mathcal{G}\mathrm{rb}^\nabla(M) \longrightarrow \mathcal{B}\mathrm{un}_{\mathbb{Z}/2}(M).$$

In terms of descent data, the functor $\mathcal{O}$ maps

$$(Y, \omega, (P, \sigma), \mu) \longmapsto (Y, \sigma) \ .$$

Equation (2.4) implies that the cocycle condition holds on $Y^{[3]}$ so that the pair (Y, σ) indeed describes a $\mathbb{Z}/2$-bundle in terms of local data. For later use, we note that a section of the bundle (Y, σ) is described in local data by a function $s : Y \longrightarrow \mathbb{Z}/2$ such that the identity $\sigma = \partial_0^* s \partial_1^* s$ holds on $Y^{[2]}$.

We are now ready for the next definition:

Definition 2.4.8.

1. We call $\mathcal{O}(\mathcal{G})$ the *orientation bundle* of the Jandl gerbe $\mathcal{G}$.
2. A global section s of the orientation bundle $\mathcal{O}(\mathcal{G})$ is called an *orientation* of the Jandl gerbe $\mathcal{G}$.
3. A morphism $\varphi : \mathcal{G} \longrightarrow \mathcal{G}'$ of oriented Jandl gerbes is called *orientation preserving*, if the morphism $\mathcal{O}(\varphi)$ of $\mathbb{Z}/2$-covers preserves the global sections, $\mathcal{O}(\varphi) \circ s = s'$.
4. Together with all 2-morphisms of Jandl gerbes, we obtain the bicategory of oriented Jandl gerbes $\mathcal{J}\mathcal{G}\mathrm{rb}^\nabla_{\mathrm{or}}(M)$.

Proposition 2.4.9.

1. *For any gerbe $\mathcal{G}$, the induced Jandl gerbe $\mathcal{J}(\mathcal{G})$ is canonically oriented. For any morphism $\varphi : \mathcal{G} \longrightarrow \mathcal{G}'$ of gerbes, the induced morphism $\mathcal{J}(\varphi) : \mathcal{J}(\mathcal{G}) \longrightarrow \mathcal{J}(\mathcal{G}')$ is orientation preserving.*
2. *The functor $\mathcal{J}$ induces an equivalence of bicategories*

$$\mathcal{G}\mathrm{rb}^\nabla(M) \longrightarrow \mathcal{J}\mathcal{G}\mathrm{rb}^\nabla_{or}(M) \ .$$

Hence the choice of an orientation reduces a Jandl gerbe to a gerbe.

Proof. 1. Let $\mathcal{G}$ be an ordinary gerbe with connection in $\mathcal{G}rb^{\nabla}(M)$. By definition of the functors $\mathcal{J}$ and $\mathcal{O}$, the bundle $\mathcal{O}(\mathcal{J}(\mathcal{G}))$ is given by the trivial $\mathbb{Z}/2$ cocycle on the covering of $\mathcal{G}$. Hence it admits a canonical section $s_{\mathcal{G}}$. This section is preserved by $\mathcal{O}(\mathcal{J}(\varphi))$ for any morphism $\varphi : \mathcal{G} \longrightarrow \mathcal{G}'$ of gerbes. This shows part 1 of the claim.

2. By looking at the local data, we find that data and conditions of a Jandl gerbe $(Y, \mathcal{I}_{\omega}, (P, \sigma), \mu)$ with $\sigma : Y^{[2]} \longrightarrow \mathbb{Z}/2$ the constant map to 1 are precisely the local data of a gerbe with connection. Since the orientation bundle $(Y, 1)$ of such a Jandl gerbe is trivial, we choose the trivial section $1 : Y \longrightarrow \mathbb{Z}/2$ as the canonical orientation. Similarly, one sees that morphisms of such Jandl gerbes preserving the canonical orientation are described by exactly the same local data as morphisms of gerbes with connection. The 2-morphisms between two such morphisms are the same anyway. Hence, the functor $\mathcal{J}$ is an isomorphism from the bicategory $\mathcal{G}rb^{\nabla}(M)$ to the full subbicategory of $\mathcal{J}\mathcal{G}rb^{\nabla}_{\mathrm{or}}(M)$ with trivial map σ.

   It remains to show that any oriented Jandl gerbe with connection is isomorphic within $\mathcal{J}\mathcal{G}rb^{\nabla}_{\mathrm{or}}(M)$ to an object in the full subbicategory with trivial map σ. To this end, we apply to a general Jandl gerbe $(Y, \mathcal{I}_{\omega}, (P, \sigma), \mu)$ with orientation $s : Y \longrightarrow \mathbb{Z}/2$ the isomorphism $m = (Y, (\mathrm{triv}, s), \mathrm{id})$. Here triv is the trivial $U(1)$-bundle on Y. The target of this isomorphism is a trivially oriented Jandl gerbe of the form $(Y, \mathcal{I}_{s\omega}, (P^{\partial_0^* s}, 1), \tilde{\mu})$ and thus in the full subbicategory of $\mathcal{J}\mathcal{G}rb^{\nabla}_{\mathrm{or}}(M)$ described in the preceding paragraph.

□

The last assertion crucially enters in the discussion of unoriented surface holonomy which we will give now.

2.4.3 Unoriented surface holonomy

Let M be a smooth manifold and $\mathcal{J}$ a Jandl gerbe on M. In this section, we discuss the definition of a holonomy for $\mathcal{J}$ around an unoriented, possibly even unorientable, closed surface Σ. Such a definition is in particular needed to write down Wess-Zumino terms for two-dimensional field theories on unoriented surfaces which arise, e.g. as worldsheets in type I string theories.

We will define surface holonomy for any pair consisting of a smooth map $\varphi : \Sigma \longrightarrow M$ and an isomorphism of $\mathbb{Z}/2$-bundles

$$\begin{array}{ccc} \mathcal{O}(\varphi^* \mathcal{J}) & \xrightarrow{\ \sim\ } & \hat{\Sigma} \\ & \searrow \quad \swarrow & \\ & \Sigma & \end{array} \tag{2.6}$$

where we denote the orientation bundle of Σ by $\hat{\Sigma}$. This is a canonically oriented two-dimensional manifold [BG88]. In particular, the orientation bundle introduced in definition 2.4.8.1 of the pulled back gerbe $\varphi^*\mathcal{J}$ must be isomorphic to the orientation bundle of the surface.

Let us first check that this setting allows us to recover the notion of holonomy from section 1.2.3 if the surface Σ is oriented. An orientation of Σ is just a global section of the orientation bundle $\hat{\Sigma} \longrightarrow \Sigma$. Due to the isomorphism (2.6), such a global section gives a global section $\Sigma \longrightarrow \mathcal{O}(\varphi^*\mathcal{J})$, i.e. an orientation of the Jandl gerbe $\varphi^*\mathcal{J}$. By proposition 2.4.9.2 an oriented Jandl gerbe amounts to a gerbe on Σ, for which we can define a holonomy as in 1.2.3. We will see that the isomorphism in (2.6) is the correct weakening of the choice of an orientation of a Jandl gerbe to the case of unoriented surfaces.

Our first goal is to relate this discussion to the one in [SSW07]. In that paper, a smooth manifold N together with an involution k was considered. This involution was not required to act freely, hence we describe the situation by looking at the action groupoid $N//(\mathbb{Z}/2)$. Since Jandl gerbes define a stack on $\mathcal{M}an$ and since any stack on $\mathcal{M}an$ can be extended by definition 2.2.5 to a stack on Lie groupoids, the definition of a Jandl gerbe on the Lie groupoid $N//(\mathbb{Z}/2)$ is clear.

We now need a few facts about $\mathbb{Z}/2$-bundles on quotients. For transparency, we formulate them for the action of an arbitrary Lie group G. Consider a free G-action on a smooth manifold N such that N/G is a smooth manifold and such that the canonical projection $N \longrightarrow N/G$ is a surjective submersion. (This is, e.g., the case if the action of G on M is proper and discontinuous.) It is an important fact that then $N \longrightarrow N/G$ is a smooth G-bundle.

If we wish to generalize this situation to the case where the action of G is not free any longer, we have to replace the quotient N/G by the Lie groupoid $N//G$. This Lie groupoid can be considered for a free action action as well, and then the Lie groupoids N/G and $N//G$ are τ-weak equivalent. By theorem 2.2.16, the categories of G-bundles over N/G and $N//G$ are equivalent.

This raises the question whether there is a natural G-bundle on the Lie groupoid $N//G$ generalizing the G-bundle $N \longrightarrow N/G$. In fact, any action Lie groupoid $N//G$ comes with a canonical G-bundle $\mathfrak{Can}_G$ over $N//G$ which we describe as in remark 2.2.9. As a bundle over N, it is the trivial bundle $N \times G$, but it carries a non-trivial G-equivariant structure. Namely $g \in G$ acts on $N \times G$ by diagonal multiplication, i.e.

$$g \cdot (n, h) := (gn, gh) \ .$$

The following lemma shows that the G-bundle $\mathfrak{Can}_G$ has the desired property:

Lemma 2.4.10.
Consider a smooth G-manifold with a free G-action such that N/G is a smooth manifold and such that the canonical projection $N \longrightarrow N/G$ is a surjective submersion.

Then the pullback of the G-bundle $N \longrightarrow N/G$ to the action Lie groupoid $N//G$ is just $\mathfrak{Can}_G$.

Proof. The proof of the lemma consists of a careful unwinding of the definitions. The most subtle aspect concerns the G-bundle over N contained in the pullback: this bundle is $N \times_{N/G} N \longrightarrow N$ which has the diagonal as a canonical section. □

We are now ready to define the target space structure corresponding to (2.6).

Definition 2.4.11.
An *orientifold background* consists of an action groupoid $N//(\mathbb{Z}/2)$, a Jandl gerbe $\mathcal{J}$ on $N//(\mathbb{Z}/2)$ and an isomorphism of equivariant $\mathbb{Z}/2$-bundles

$$\begin{array}{ccc} \mathcal{O}(\mathcal{J}) & \xrightarrow{\quad\sim\quad} & \mathfrak{Can}_{\mathbb{Z}/2} \\ & \searrow \quad \swarrow & \\ & N//(\mathbb{Z}/2) & \end{array} \tag{2.7}$$

Proposition 2.4.12.
An orientifold background is the same as a gerbe with Jandl structure from [SSW07, Definition 5]. More precisely we have an equivalence of bicategories between the bicategory of orientifold backgrounds over the Lie groupoid $N//(\mathbb{Z}/2)$ and the bicategory of gerbes over the manifold N with Jandl structure with involution $k : N \longrightarrow N$ given by the action of $-1 \in \mathbb{Z}/2$.

Proof. We concentrate on how to extract a gerbe with a Jandl structure from the orientifold background. Let us first express from remark 2.2.9 the data of a Jandl gerbe on the Lie groupoid $N//(\mathbb{Z}/2)$ in terms of data on the manifold N. We have just to keep one isomorphism $\varphi = \varphi_k$ and a single coherence 2-isomorphism, for the non-trivial element $-1 \in \mathbb{Z}/2$. We thus get:

- A Jandl gerbe $\mathcal{J}_N$ on N.
- A morphism $\varphi : k^*\mathcal{J}_N \longrightarrow \mathcal{J}_N$ of Jandl gerbes.
- A coherence 2-isomorphism c in the diagram

$$\begin{array}{ccc} \mathcal{J}_N & \xrightarrow{k^*\varphi} & k^*\mathcal{J}_N \\ & \searrow \;\; \overset{c}{\swarrow\!\!\swarrow} & \big\downarrow \varphi \\ & & \mathcal{J}_N \end{array}$$

- A coherence condition on the 2-isomorphism c.

Similarly, we extract the data in the isomorphism

$$\mathcal{O}(\mathcal{J}_N) \longrightarrow \mathfrak{Can}_{\mathbb{Z}/2}$$

of $\mathbb{Z}/2$-bundles over the Lie groupoid $N//(\mathbb{Z}/2)$ that is the second piece of data in an orientifold background. It consists of

(i) An isomorphism

$$\mathcal{O}(\mathcal{J}_N) \xrightarrow{\sim} N \times \mathbb{Z}/2$$

of $\mathbb{Z}/2$-bundles over the smooth manifold N.

(ii) A commuting diagram

$$\begin{array}{ccc} \mathcal{O}(k^*\mathcal{J}_N) & \xrightarrow{\mathcal{O}(\varphi)} & \mathcal{O}(\mathcal{J}_N) \\ {\scriptstyle k^*s}\downarrow & & \downarrow{\scriptstyle s} \\ N \times \mathbb{Z}/2 & \xrightarrow[\mathrm{id}_N \times m_{-1}]{} & N \times \mathbb{Z}/2 \end{array}$$

where m_{-1} is multiplication by $-1 \in \mathbb{Z}/2$.

Now the data in part (i) are equivalent to a section of the orientation bundle $\mathcal{O}(\mathcal{J}_N)$, i.e. an orientation of the Jandl gerbe $\mathcal{J}_N$. By proposition 2.4.9.2, our Jandl gerbe is thus equivalent to an ordinary gerbe $\mathcal{G}$ on N. Part (ii) expresses the condition that φ is an orientation reversing morphism of Jandl gerbes. We summarize the data: we get

- A bundle gerbe $\mathcal{G}$ on N.
- The odd morphism φ gives, in the language of [SSW07], a morphism $A : k^*\mathcal{G} \longrightarrow \mathcal{G}^*$ of bundle gerbes.
- Similarly, the coherence isomorphism

$$c : \varphi \circ k^*\varphi \Longrightarrow \mathrm{id}$$

is in that language a 2-isomorphism

$$A \otimes (k^*A)^* \Longrightarrow \mathrm{id}$$

of gerbes which is expressed in [SSW07] by a $\mathbb{Z}/2$-equivariant structure on A.

- Finally, one gets the coherence conditions of [SSW07].

We have thus recovered all data of [SSW07, definition 5]. □

Corollary 2.4.13.
The bicategory formed by Jandl gerbes $\mathcal{J}$ over Σ together with an isomorphism $f : O(\mathcal{J}) \xrightarrow{\sim} \hat{\Sigma}$ is equivalent to the bicategory of orientifold backgrounds over $\hat{\Sigma}//(\mathbb{Z}/2)$.

Proof. Pull back along the τ-weak equivalence $\hat{\Sigma}//(\mathbb{Z}/2) \longrightarrow \Sigma$ gives by theorem 2.2.16 an equivalence of bicategories

$$\mathcal{J}\mathcal{G}rb^{\nabla}(\Sigma) \xrightarrow{\sim} \mathcal{J}\mathcal{G}rb^{\nabla}(\hat{\Sigma}//(\mathbb{Z}/2)) \ .$$

Concatenating f with the isomorphism $\hat{\Sigma} \longrightarrow \mathfrak{C}\mathrm{an}_{\mathbb{Z}/2}$ from lemma 2.4.10 provides the second data in the definition 2.4.11 of an orientifold background. □

The formula for the holonomy $\mathrm{Hol}_{\mathcal{J}}(f)$ of such an orientifold background over $\hat{\Sigma}//(\mathbb{Z}/2)$ is given in [SSW07] and section 1.4 along the lines of holonomy for ordinary gerbes, see section 1.2.3. We refrain from giving details here. We then define

Definition 2.4.14.
Let M be smooth manifold and $\mathcal{J}$ a Jandl gerbe on M. Let Σ be an unoriented closed surface. Given a smooth map $\varphi : \Sigma \longrightarrow M$ and a morphism $f : \mathcal{O}(\varphi^*\mathcal{J}) \longrightarrow \hat{\Sigma}$ of $\mathbb{Z}/2$-bundles over Σ, we define the surface holonomy to be

$$\mathrm{Hol}_{\mathcal{J}}(\varphi, f) := \mathrm{Hol}_{(\varphi^*\mathcal{J})}(f) \ .$$

Remarks 2.4.15.

1. *This holonomy enters as the exponentiated Wess-Zumino term in a Lagrangian description of two-dimensional sigma models on unoriented surfaces with target space M which are relevant e.g. for type I string theories.*

2. *More generally, one considers target spaces which are Lie groupoids. If the target is a Lie groupoid Γ, the smooth map φ has to be replaced by a Hilsum-Skandalis morphism $\Phi : \Sigma \longrightarrow \Lambda$ which is a special span of Lie groupoids*

$$\begin{array}{ccccc} & & \Lambda & & \\ & \overset{\sim}{\swarrow} & & \searrow & \\ \Sigma & & & & \Gamma \end{array}$$

where $\Lambda \longrightarrow \Sigma$ is a τ-weak equivalence. (For a definition and discussion, see [Met03, definition 62]).

Theorem 2.2.16 ensures that the pullback along $\Lambda \longrightarrow \Gamma$ is an equivalence of bicategories. Using its inverse, we can pull back a Jandl gerbe over Γ along Φ to Σ.

3. *In particular, we get in this situation a notion of holonomy $\mathrm{Hol}_{\mathcal{J}}(\Phi, f)$ for a Hilsum-Skandalis morphism Φ and an isomorphism f of $\mathbb{Z}/2$-bundles over Σ as before.*

4. *Consider an orientifold background,* $\Gamma = N//(\mathbb{Z}/2)$. *Then each* $\mathbb{Z}/2$*-equivariant map* $\tilde{\varphi} : \hat{\Sigma} \longrightarrow N$ *provides a special Hilsum-Skandalis morphism*

$$\begin{array}{ccc} & \hat{\Sigma}//(\mathbb{Z}/2) & \\ \swarrow & & \searrow \\ \Sigma & & N//(\mathbb{Z}/2) \end{array}$$

The pullback of $\mathfrak{Can}_{\mathbb{Z}/2}$ *on* $N//(\mathbb{Z}/2)$ *to* $\hat{\Sigma}//(\mathbb{Z}/2)$ *gives again the canonical bundle which by Lemma 2.4.10 is mapped to the* $\mathbb{Z}/2$*-bundle* $\hat{\Sigma} \longrightarrow \Sigma$. *Thus pulling back the isomorphism of* $\mathbb{Z}/2$*-bundles in the orientifold background to an isomorphism of bundles on* Σ *gives us just the data needed in definition 2.4.14 to define holonomy.*

This way, we obtain holonomies $\mathrm{Hol}_{\mathcal{J}}(\tilde{\varphi}) \in \mathrm{U}(1)$ *which have been introduced in [SSW07] and enter e.g. in orientifolds of the WZW models, see [GSW08a].*

2.4.4 Kapranov-Voevodsky 2-vector bundles

As a further application of the plus construction, we investigate a version of 2-vector bundles, more precisely 2-vector bundles modeled on the notion of Kapranov-Voevodsky 2-vector spaces [KV94]. The bicategory of complex KV 2-vector spaces is (equivalent to) the following bicategory:

- Objects are given by non negative integers $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. This is a shortcut for the category $\mathrm{Vect}_{\mathbb{C}}^n = \mathrm{Vect}_{\mathbb{C}} \times \ldots \mathrm{Vect}_{\mathbb{C}}$, where we have the product of categories.

- 1-morphisms $n \longrightarrow m$ are given by $m \times n$ matrices $\left(V_{ij}\right)_{i,j}$ of complex vector spaces. This encodes an exact functor $\mathrm{Vect}_{\mathbb{C}}^n \longrightarrow \mathrm{Vect}_{\mathbb{C}}^m$.

- 2-morphisms $\left(V_{ij}\right)_{i,j} \Longrightarrow \left(W_{ij}\right)_{i,j}$ are given by families $\left(\varphi_{ij}\right)_{i,j}$ of linear maps. This encodes a natural transformation between functors $\mathrm{Vect}_{\mathbb{C}}^n \longrightarrow \mathrm{Vect}_{\mathbb{C}}^m$.

The 1-*iso*morphisms in this bicategory are exactly those $n \times n$ square matrices (V_{ij}) for which the $n \times n$ matrix with non-negative integral entries $(\dim_{\mathbb{C}} V_{ij})$ is invertible in the ring $M(n \times n, \mathbb{N})$ of matrices with integral entries.

Based on this bicategory we define for a smooth manifold M the bicategories $\mathcal{V}ect_2triv(M)$ of trivial Kapranov-Voevodsky 2-vector bundles:

- Objects are given by non negative integers $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$.

- 1-morphisms $n \longrightarrow m$ are given by $m \times n$ matrices $\left(E_{ij}\right)_{i,j}$ of complex vector bundles over M.

- 2-morphisms $\big(E_{ij}\big)_{i,j} \Longrightarrow \big(F_{ij}\big)_{i,j}$ are given by families $\phi_{ij} : E_{ij} \longrightarrow F_{ij}$ of vector bundle morphisms.

The pullback of vector bundles turns this into a presheaf in bicategories. Since vector bundles can be glued together, the presheaf $\mathcal{V}ect_2triv$ is even a prestack. Hence we can apply the plus construction:

$$\mathcal{V}ect_2 := \Big(\mathcal{V}ect_2triv\Big)^+.$$

By theorem 2.3.3, we obtain a stack of 2-vector bundles. Thus we have properly defined bicategories of $\mathcal{V}ect_2(M)$ of 2-vector bundles over a manifold M and even over Lie groupoids and thus obtained a notion of equivariant 2-vector bundles.

In [BDR04] a notion of 2-vector bundles on the basis of Kapranov-Voevodsky 2-vector spaces has been introduced under the name of charted 2-vector bundles. They are defined on ordered open covers to accomodate more 1-isomorphisms and thus yield a richer setting for 2-vector bundles.

2.5 Proof of theorem 2.2.16, part 1: Factorizing morphisms

Sections 2.5–2.8 are devoted to the proof of theorem 2.2.16. For this proof, we factor any fully faithful and τ-essentially surjective Lie functor $F : \Gamma \longrightarrow \Omega$ into two morphisms of Lie groupoids belonging to special classes of morphisms of Lie groupoids: τ-surjective equivalences and strong equivalences. We first discuss these two classes of morphisms.

2.5.1 Strong equivalences

We start with the definition of strong equivalences [MM03]. To this end, we introduce natural transformations of Lie groupoids: Consider the free groupoid on a single morphism, the interval groupoid:

$$\mathrm{I} := (\mathrm{I}_1 \rightrightarrows \mathrm{I}_0)$$

It has two objects $\mathrm{I}_0 := \{a, b\}$ and the four isomorphisms $\mathrm{I}_1 := \{id_a, id_b, \ell, \ell^{-1}\}$ with $s(\ell) = a, t(\ell) = b$. Consider two functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ for two categories $\mathcal{C}, \mathcal{D}$. For any category Γ, we consider the cylinder category $\Gamma \times \mathrm{I}$ with the canonical inclusion functors $i_0, i_1 : \Gamma \longrightarrow \Gamma \times \mathrm{I}$.

It is an easy observation that natural isomorphisms $\eta : F \Longrightarrow G$ are in bijection to functors $\tilde{\eta} : \mathcal{C} \times \mathrm{I} \longrightarrow \mathcal{D}$ with $\tilde{\eta} \circ i_0 = F$ and $\tilde{\eta} \circ i_1 = G$. (The bijection maps $\eta_c : F(c) \longrightarrow G(c)$ to $\tilde{\eta}(\mathrm{id}_c \times \ell)$.)

This observation allows us to reduce smoothness conditions on natural transformations to smoothness conditions on functors. Hence, we consider the interval groupoid I as a discrete Lie groupoid and obtain for any Lie groupoid Γ the structure of a Lie groupoid on the cylinder groupoid $\Gamma \times \mathrm{I}$.

Definition 2.5.1.

1. A *Lie transformation* η between two Lie functors $F, G : \Gamma \longrightarrow \Omega$ is a Lie functor $\eta : \Gamma \times \mathrm{I} \longrightarrow \Omega$ with $\eta \circ i_0 = F$ and $\eta \circ i_1 = G$.

2. Two Lie functors F and G are called *naturally isomorphic*, $F \simeq G$, if there exists a Lie transformation between F and G.

3. A Lie functor $F : \Gamma \longrightarrow \Omega$ is called an *strong equivalence*, if there exists a Lie functor $G : \Omega \longrightarrow \Gamma$ such that $G \circ F \simeq \mathrm{id}_\Gamma$ and $F \circ G \simeq \mathrm{id}_\Omega$.

We need the following characterization of strong equivalences, which is completely analogous to a well-known statement from category theory:

Proposition 2.5.2.
A Lie functor $F : \Gamma \longrightarrow \Omega$ is an strong equivalence if and only if it is fully faithful and split essential surjective. The latter means that the map in definition 2.2.13.2

$$\Gamma_0 \times_{\Omega_0} \Omega_1 \longrightarrow \Omega_0$$

induced by the target map has a section.

Proof. The proof is roughly the same as in classical category theory c.f. [Kas95] Prop. XI.1.5. We only have to write down everything in diagrams, e.g. the condition fully faithful in terms of pullback diagram as in definition 2.2.13. Note that the proof in [Kas95] needs the axiom of choice; in our context, we need a section of the map $\Gamma_0 \times_{\Omega_0} \Omega_1 \longrightarrow \Omega_0$. □

Lemma 2.5.3.
If a Lie functor $F : \Gamma \longrightarrow \Omega$ admits a fully faithful retract, i.e. a fully faithful left inverse, it is an strong equivalence.

Proof. Let P be the fully faithful left inverse of P, hence

$$P \circ F = \mathrm{id}_\Gamma \quad .$$

It remains to find a Lie transformation

$$\eta : F \circ P \Longrightarrow \mathrm{id}_\Omega \quad .$$

Since the functor P is fully faithful, the diagram

$$\begin{array}{ccc} \Omega_1 & \xrightarrow{P_1} & \Gamma_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ \Omega_0 \times \Omega_0 & \xrightarrow{P_0 \times P_0} & \Gamma_0 \times \Gamma_0 \end{array}$$

is by definition 2.2.13 a pullback diagram. Define $\eta : \Omega_0 \longrightarrow \Omega_1 \cong \Omega_0 \times_{\Gamma_0} \Gamma_1 \times_{\Gamma_0} \Omega_0$ by

$$\eta(\omega) = \big(F_0 P_0(\omega)\,, \mathrm{id}_{P_0(\omega)},\, \omega\big) \quad .$$

The identities $P_0(w) = s(\mathrm{id}_{P_0(\omega)})$ and $t(\mathrm{id}_{P_0(\omega)}) = P_0(\omega) = P_0 F_0 P_0(\omega)$ imply that this is well-defined; one also checks naturality. The two identities

$$s\eta(\omega) = F_0 P_0(\omega) \quad \text{and} \quad t\eta(\omega) = \omega$$

imply that η is indeed a Lie transformation $F \circ P \implies \mathrm{id}_\Omega$. One verifies that it has also the other properties we were looking for. □

2.5.2 τ-surjective equivalences

For any choice of topology τ, we introduce the notion of τ-surjective equivalence. This is called hypercover in [Zhu09]. In contrast to τ-weak equivalences, τ-surjective equivalences are required to be τ-surjective, rather than only τ-essentially surjective, as in definition 2.2.13.

Definition 2.5.4.
A *τ-surjective equivalence* is a fully faithful Lie functor $F : \Lambda \longrightarrow \Gamma$ such that $F_0 : \Lambda_0 \longrightarrow \Gamma_0$ is a morphism in τ.

Proposition 2.5.5.
Let $F : \Lambda \longrightarrow \Gamma$ be a fully faithful Lie functor and $F_\bullet : \Lambda_\bullet \longrightarrow \Gamma_\bullet$ the associated simplicial map. Then F is a τ-surjective equivalence, if and only if all maps $F_i : \Lambda_i \longrightarrow \Gamma_i$ are in τ.

The proof is based on

Lemma 2.5.6.
For any two τ-covers $\pi : Y \twoheadrightarrow M$ and $\pi' : Y' \twoheadrightarrow M'$ in $\mathcal{M}$an, the product $\pi \times \pi' : Y \times Y' \longrightarrow M \times M'$ is in τ as well.

Proof. Writing $\pi \times \pi' = (\pi \times id) \circ (id \times \pi')$ and using the fact that the composition of τ-covers is a τ-cover, we can assume that $\pi' = id : M' \twoheadrightarrow M'$. The assertion then follows from the observation that the diagram

$$\begin{array}{ccc} Y \times M' & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ M \times M' & \longrightarrow & M \end{array}$$

is a pullback diagram and that τ is closed under pullbacks. $\square$

Proof. of proposition 2.5.5. Since F is fully faithful, all diagrams

$$\begin{array}{ccc} \Lambda_n & \xrightarrow{F_n} & \Gamma_n \\ \downarrow & & \downarrow \\ \underbrace{\Lambda_0 \times \cdots \times \Lambda_0}_{n+1} & \xrightarrow{F_0 \times \cdots \times F_0} & \underbrace{\Gamma_0 \times \cdots \times \Gamma_0}_{n+1} \end{array}$$

are pullback diagrams. Then F_n is a τ-cover since $F_0 \times \cdots \times F_0$ is, by lemma 2.5.6 a τ-cover. $\square$

2.5.3 Factorization

Proposition 2.5.7 (Factorization of Lie functors)**.**
Let Γ and Ω be Lie groupoids. Every fully faithful and τ-essentially surjective Lie functor $F : \Gamma \longrightarrow \Omega$ factors as

$$\begin{array}{ccccc} & & \Lambda & & \\ & {}^{G}\nearrow & & \searrow^{H} & \\ \Gamma & & \xrightarrow[F]{} & & \Omega \end{array}$$

where H is a τ-surjective equivalence and G an strong equivalence.

Proof. We ensure the surjectivity of H by defining

$$\Lambda_0 := \Gamma_0 \ {}_{F_0}\!\times_s \Omega_1.$$

Then $H_0 : \Lambda_0 \longrightarrow \Omega_0$ is given on objects by the target map of Ω. This is a τ-covering by the definition of τ-essential surjectivity. On objects, we define $G_0 : \Gamma_0 \longrightarrow \Lambda_0$ by $\gamma \longmapsto \big(\gamma, \mathrm{id}_{F_0(\gamma)}\big)$. This gives the commutative diagram

$$\begin{array}{ccccc} & & \Lambda_0 & & \\ & {}^{G_0}\nearrow & & \searrow^{H_0} & \\ \Gamma_0 & & \xrightarrow[F_0]{} & & \Omega_0 \end{array}$$

on the level of objects. We combine the maps in the diagram

$$\begin{array}{ccccc} \Gamma_1 & & \xrightarrow{F_1} & & \Omega_1 \\ {\scriptstyle (s,t)}\downarrow & & & & \downarrow{\scriptstyle (s,t)} \\ \Gamma_0 \times \Gamma_0 & \xrightarrow{G_0 \times G_0} & \Lambda_0 \times \Lambda_0 & \xrightarrow{H_0 \times H_0} & \Omega_0 \times \Omega_0 \end{array}$$

which is a pull back diagram by definition 2.2.13, since F is fully faithful. To define the Lie functor H such that it is fully faithful, we have to define Λ_1 as the pull back of the right half of the diagram, i.e. $\Lambda_1 := \Lambda_0 \times_{\Omega_0} \Omega_1 \times_{\Omega_0} \Lambda_0$. The universal property of pull backs yields a diagram

$$\begin{array}{ccccc} \Gamma_1 & \xrightarrow{G_1} & \Lambda_1 & \xrightarrow{H_1} & \Omega_1 \\ \downarrow{\scriptstyle (s,t)} & & \downarrow{\scriptstyle (s,t)} & & \downarrow{\scriptstyle (s,t)} \\ \Gamma_0 \times \Gamma_0 & \xrightarrow{G_0 \times G_0} & \Lambda_0 \times \Lambda_0 & \xrightarrow{H_0 \times H_0} & \Omega_0 \times \Omega_0 \end{array} \tag{2.8}$$

in which all squares are pullbacks. The groupoid structure on $\Omega = (\Omega_1 \rightrightarrows \Omega_0)$ induces a groupoid structure on $\Lambda = (\Lambda_1 \rightrightarrows \Lambda_0)$ in such a way that G and H become Lie functors.

By construction of this factorization, H is a τ-surjective equivalence. It remains to be shown that G is an strong equivalence. According to proposition 2.5.2, it suffices to show that G is fully faithful and split essential surjective. The left diagram in (2.8) is a pullback diagram. Hence G is fully faithful. It remains to give a section of the map

$$\Gamma_0 \times_{\Lambda_0} \Lambda_1 \longrightarrow \Lambda_0 \tag{2.9}$$

Since we have defined $\Lambda_1 = \Lambda_0 \times_{\Omega_0} \Omega_1 \times_{\Omega_0} \Lambda_0$, we have

$$\Gamma_0 \times_{\Lambda_0} \Lambda_1 \cong \Gamma_0 \times_{\Omega_0} \Omega_1 \times_{\Omega_0} \Lambda_0.$$

Thus a section of (2.9) is given by three maps

$$\Lambda_0 \longrightarrow \Gamma_0 \qquad \Lambda_0 \longrightarrow \Omega_1 \qquad \Lambda_0 \longrightarrow \Lambda_0$$

that agree on Ω_0, when composed with the source and the target map of Ω_0, respectively. By definition $\Lambda_0 = \Gamma_0 \, {}_{F_0}\!\times_s \Omega_1$, and we can define the three maps by projection to the first factor, projection to the second factor and the identity. □

The factorization lemma allows to isolate the violation of τ-surjectivity in an strong equivalence and to work with τ-surjective equivalences rather than only τ-essentially surjective equivalences. Hence it suffices to prove theorem 2.2.16 for τ-surjective equivalences and for strong equivalences. This will be done in sections 2.6 and 2.8, respectively.

2.6 Proof of theorem 2.2.16, part 2: Sheaves and strong equivalences

Lemma 2.6.1.
Let $\mathfrak{X}$ be a presheaf that preserves products, cf. equation (2.1). *Let Γ be a Lie groupoid*

and D be a discrete Lie groupoids i.e. D_0 and D_1 are discrete manifolds. Then D can also be regarded as a bicategory and we have natural equivalences

$$\mathfrak{X}(\Gamma \times D) \cong \Big[D, \mathfrak{X}(\Gamma)\Big]$$

where $[D, \mathfrak{X}(\Gamma)]$ denotes the bicategory of functors $D \longrightarrow \mathfrak{X}(\Gamma)$.

Proof. The claim is merely a consequence the requirement (2.1) that $\mathfrak{X}$ preserves products: In the case that Γ is a manifold M considered as a Lie groupoid and D a set I considered as a discrete groupoid we have $M \times I = \bigsqcup_{i \in I} M$. Thus the left hand is equal to $\mathfrak{X}(\bigsqcup M)$ and the right hand side to $\mathfrak{X}(M)^I = \prod_{i \in I} \mathfrak{X}(M)$. In this case (2.1) directly implies the equivalence.

In the case of a general Lie groupoid, the product $\Gamma \times D$ decomposes levelwise into a disjoint union. Using this fact and explicitly spelling out $\mathfrak{X}(\Gamma \times D)$ and $[D, \mathfrak{X}(\Gamma)]$ according to definition, 2.2.5 it is straightforward to see that the two bicategories are equivalent. □

Proposition 2.6.2.
Let $\mathfrak{X}$ be a presheaf in bicategories. Any Lie transformation $\eta : F \Longrightarrow G$ of Lie functors $F, G : \Gamma \longrightarrow \Omega$ induces a natural isomorphism of the pullback functors $F^*, G^* : \mathfrak{X}(\Omega) \longrightarrow \mathfrak{X}(\Gamma)$.

Proof. Recall from definition 2.5.1 that the Lie transformation η is by definition a Lie functor

$$\Gamma \times I \longrightarrow \Omega \ ,$$

where I is the interval groupoid. Applying the presheaf $\mathfrak{X}$ to this functor yields a functor

$$\mathfrak{X}(\Omega) \longrightarrow \mathfrak{X}(\Gamma \times I).$$

Since I is discrete the preceding lemma 2.6.1 shows that this is a functor

$$\mathfrak{X}(\Omega) \longrightarrow \Big[I, \mathfrak{X}(\Gamma)\Big].$$

That is the same as a functor

$$\mathfrak{X}(\Omega) \times I \longrightarrow \mathfrak{X}(\Gamma)$$

i.e. a natural isomorphism of bifunctors. □

Corollary 2.6.3.
For any presheaf $\mathfrak{X}$ in bicategories, the pull back along an strong equivalence $\Gamma \longrightarrow \Omega$ induces an equivalence $\mathfrak{X}(\Omega) \longrightarrow \mathfrak{X}(\Gamma)$ of bicategories.

2.7 Proof of theorem 2.2.16, part 3: Equivariant descent

To deal with τ-surjective equivalences, we need to consider simplicial objects in the category of simplicial objects, i.e. bisimplicial objects. In the course of our investigations, we obtain results about bisimplicial objects that are of independent interest, in particular theorem 2.7.5 and corollary 2.7.6 on equivariant descent.

We first generalize the definition of equivariant objects as follows: If we evaluate a presheaf in bicategories $\mathfrak{X}$ on a simplicial object $\Gamma_\bullet$, we obtain the following diagram in $\mathcal{B}i\mathcal{C}at$:

$$\mathfrak{X}(\Gamma_0) \underset{\partial_1^*}{\overset{\partial_0^*}{\rightrightarrows}} \mathfrak{X}(\Gamma_1) \underset{\partial_2^*}{\overset{\partial_0^*}{\rightrightarrows}} \mathfrak{X}(\Gamma_2) \underset{\partial_3^*}{\overset{\partial_0^*}{\rightrightarrows}} \cdots$$

in which the cosimplicial identities are obeyed up to natural isomorphism,

$$\partial_j^* \partial_i^* \quad \cong \quad \partial_i^* \partial_{j-1}^* \qquad \text{for } i < j \ .$$

The coherence cells turn this into a weak functor $\Delta \longrightarrow \mathcal{B}i\mathcal{C}at$ from the simplicial category Δ to $\mathcal{B}i\mathcal{C}at$. Such a functor will be called a (weak) cosimplicial bicategory.

The equivariant objects can be constructed in this framework by selecting objects in $\mathfrak{X}(\Gamma_0)$, 1-morphisms in $\mathfrak{X}(\Gamma_1)$ and so on. This leads us to the following definition:

Definition 2.7.1.
Given a cosimplicial bicategory $C_\bullet$, we introduce the category

$$\mathrm{holim}_{i\in\Delta}\, C_i \equiv \mathrm{holim} \left(C_0 \underset{\partial_1^*}{\overset{\partial_0^*}{\rightrightarrows}} C_1 \underset{\partial_2^*}{\overset{\partial_0^*}{\rightrightarrows}} C_2 \underset{\partial_3^*}{\overset{\partial_0^*}{\rightrightarrows}} \cdots \right)$$

with objects given by the following data:

(O1) An object $\mathcal{G}$ in the bicategory C_0;

(O2) A 1-isomorphism in the bicategory C_1;

$$P\colon\ \partial_0^*\mathcal{G} \longrightarrow \partial_1^*\mathcal{G}$$

(O3) A 2-isomorphism in the bicategory C_2;

$$\mu\colon\ \partial_2^* P \otimes \partial_0^* P \Longrightarrow \partial_1^* P$$

(O4) A coherence condition of 2-morphisms in the bicategory C_3:

$$\partial_2^*\mu \circ (\mathrm{id} \otimes \partial_0^*\mu) = \partial_1^*\mu \circ (\partial_3^*\mu \otimes \mathrm{id})$$

Morphisms and 2-morphisms are defined as in definition 2.2.6.

In this notation, the extension of a prestack $\mathfrak{X}$ to an equivariant object $\Gamma_\bullet$ described in definition 2.2.5 is given by

$$\mathfrak{X}(\Gamma_\bullet) = \mathrm{holim}_{i\in\Delta}\, \mathfrak{X}(\Gamma_i) \ . \tag{2.10}$$

In the special case of a τ-covering $Y \twoheadrightarrow M$, we can write the descent object as

$$\mathcal{D}esc_{\mathfrak{X}}(Y \twoheadrightarrow M) = \mathrm{holim}_{i\in\Delta}\, \mathfrak{X}(Y^{[i+1]}).$$

For the constant simplicial bicategory $C_\bullet$, with $C_i = C$ for all i, one checks that $\mathrm{holim}_{i\in\Delta} C_i = C$.

We next need to extend the notion of a τ-covering to a simplicial object:

Definition 2.7.2.

1. Let $\Lambda_\bullet$ and $\Gamma_\bullet$ be simplicial manifolds and $\Pi_\bullet : \Lambda_\bullet \longrightarrow \Gamma_\bullet$ a simplicial map. Then $\Pi_\bullet$ is called a τ-cover, if all maps $\Pi_i : \Lambda_i \longrightarrow \Gamma_i$ are τ-covers.

2. A Lie functor $\Pi : (\Lambda_1 \rightrightarrows \Lambda_0) \longrightarrow (\Gamma_1 \rightrightarrows \Gamma_0)$ is called a τ-cover, if the associated simplicial map $\Pi_\bullet : \Lambda_\bullet \longrightarrow \Gamma_\bullet$ of the nerves is a τ-cover of simplicial manifolds.

Remark 2.7.3.

1. *Proposition 2.5.5 shows that for a τ-surjective equivalence the associated simplicial map is τ-cover.*

2. *For any τ-covering $\pi : Y \twoheadrightarrow M$, the simplicial map induced by the Lie functor $\check{C}(Y) \longrightarrow M$ is an example of a τ-cover of simplicial manifolds.*

Given a τ-cover $\Pi_\bullet : \Lambda_\bullet \longrightarrow \Gamma_\bullet$ of simplicial manifolds, we can construct the simplicial manifold

$$\Lambda_\bullet^{[2]} := \Lambda_\bullet \times_{\Gamma_\bullet} \Lambda_\bullet := \left(\cdots \overset{\partial_0}{\underset{\partial_3}{\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}}} \Lambda_2 \times_{\Gamma_2} \Lambda_2 \overset{\partial_0}{\underset{\partial_2}{\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}}} \Lambda_1 \times_{\Gamma_1} \Lambda_1 \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} \Lambda_0 \times_{\Gamma_0} \Lambda_0 \right)$$

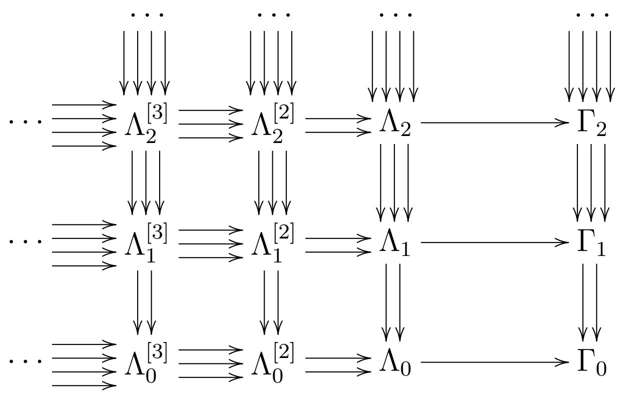
with obvious maps ∂_i. One verifies that the two projections $\delta_0, \delta_1 : \Lambda_\bullet^{[2]} \longrightarrow \Lambda_\bullet$ are simplicial maps. Similarly, we form simplicial manifolds

$$\Lambda_\bullet^{[n]} := \underbrace{\Lambda_\bullet \times_{\Gamma_\bullet} \ldots \times_{\Gamma_\bullet} \Lambda_\bullet}_{n}$$

and simplicial maps $\delta_i : \Lambda_\bullet^{[n]} \longrightarrow \Lambda_\bullet^{[n-1]}$ with $i = 0, \ldots, n-1$. We thus obtain an (augmented) simplicial object

$$(\Lambda_\bullet)^{[\bullet]} := \left(\cdots \overset{\delta_0}{\underset{\delta_3}{\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}}} \Lambda_\bullet^{[3]} \overset{\delta_0}{\underset{\delta_2}{\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}}} \Lambda_\bullet^{[2]} \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} \Lambda_\bullet \right) \longrightarrow \Gamma_\bullet$$

in the category of simplicial manifolds. A simplicial object in the category of simplicial manifolds will also be called a *bisimplicial manifold*. In full detail, a bisimplicial manifold consists of the following data:

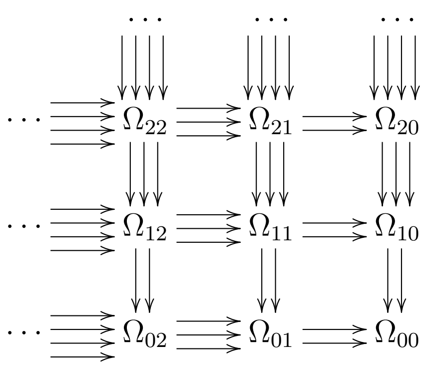


The rows are, by construction, nerves of Čech groupoids. This fact will enter crucially in the proof of our main result on equivariant descent. Before turning to this, we need the following

Proposition 2.7.4.
Let $\mathfrak{X}$ be a presheaf in bicategories and $\Omega_{\bullet\bullet}$ a bisimplicial manifold. Then

$$\mathrm{holim}_{i\in\Delta}\,\mathrm{holim}_{j\in\Delta}\,\mathfrak{X}\,(\Omega_{ij}) = \mathrm{holim}_{j\in\Delta}\,\mathrm{holim}_{i\in\Delta}\,\mathfrak{X}\,(\Omega_{ij})$$

Proof. We first discuss what data of the bisimplicial manifold



enter in an object of $\mathrm{holim}_{i\in\Delta}\,\mathrm{holim}_{j\in\Delta}\,\mathfrak{X}\,(\Omega_{ij})$. To this end, we denote horizontal boundary maps by δ and vertical boundary maps by ∂. Then such an object is given by

- An object in $\mathrm{holim}_{j\in\Delta}\,\mathfrak{X}\,(\Omega_{0j})$ which in turn consists of
  - An object $\mathcal{G}$ on Ω_{00}
  - An isomorphism $A_{01} : \delta_0^*\mathcal{G} \longrightarrow \delta_1^*\mathcal{G}$ on Ω_{01}

- A 2-isomorphism $\mu_{02} : \delta_2^* A_{01} \otimes \delta_0^* A_{01} \Longrightarrow \delta_1^* A_{01}$ on Ω_{02}
- A coherence condition on Ω_{03}

- A morphism $\partial_0^*(\mathcal{G}, A_{0,1}, \mu_{0,2}) \longrightarrow \partial_1^*(\mathcal{G}, A_{0,1}, \mu_{02})$ in $\operatorname{holim}_{j\in\Delta} \mathfrak{X}(\Omega_{1j})$ which in turn consists of
  - An isomorphism $A_{10} : \partial_0^*\mathcal{G} \longrightarrow \partial_1^*\mathcal{G}$ on Ω_{10}
  - A 2-isomorphism $\mu_{11} : \partial_1^* A_{01} \otimes \delta_0^* A_{10} \Longrightarrow \delta_1^* A_{10} \otimes \partial_0^* A_{01}$ on Ω_{11}.
  - A coherence condition on Ω_{12}.
- A 2-isomorphism $\partial_2^*(A_{10}, \mu_{11}) \otimes \partial_0^*(A_{10}, \mu_{11}) \Longrightarrow \partial_1^*(A_{10}, \mu_{11})$ in $\operatorname{holim}_{j\in\Delta} \mathfrak{X}(\Omega_{2j})$:
  - A 2-isomorphism $\mu_{20} : \partial_2^* A_{10} \otimes \partial_0^* A_{10} \Longrightarrow \partial_1^* A_{10}$ on Ω_{20}.
  - A coherence condition on Ω_{21}.
- A condition on the 2-morphisms in $\operatorname{holim}_{j\in\Delta} \mathfrak{X}(\Omega_{3j})$ which is just
  - A coherence condition on Ω_{30}.

To summarize, we get an object $\mathcal{G} \in \mathfrak{X}(\Omega_{00})$ in the lower right corner of the diagram, two isomorphisms $A_{01} \in \mathfrak{X}(\Omega_{01})$, $A_{10} \in \mathfrak{X}(\Omega_{01})$ on the diagonal, three 2-isomorphisms $\mu_{02} \in \mathfrak{X}(\Omega_{02}), \mu_{11} \in \mathfrak{X}(\Omega_{11}), \mu_{20} \in \mathfrak{X}(\Omega_{20})$ on the first translate of the diagonal and four conditions on the second translate of the diagonal.

For an object in $\operatorname{holim}_{j\in\Delta} \operatorname{holim}_{i\in\Delta} \mathfrak{X}(\Omega_{ij})$, we get the same data, as can be seen by exchanging the roles of i and j. Since we interchange the roles of ∂ and δ, we have to replace the 2-isomorphism $\mu_{11} : \partial_1^* A_{01} \otimes \delta_0^* A_{10} \Longrightarrow \delta_1^* A_{10} \otimes \partial_0^* A_{01}$ by its inverse. For all other isomorphisms and conditions, the objects remain unchanged.

By analogous considerations, one also checks that the morphisms and 2-morphisms in both bicategories coincide. □

Theorem 2.7.5 (Equivariant descent).
Let $\Pi : \Lambda_\bullet \longrightarrow \Gamma_\bullet$ be a τ-covering of simplicial manifolds.

1. *Let $\mathfrak{X}$ be a τ-stack on $\mathcal{M}\mathrm{an}$. Then we have the following equivalence of bicategories:*

$$\mathfrak{X}(\Gamma_\bullet) \xrightarrow{\sim} \operatorname{holim}\left(\mathfrak{X}(\Lambda_\bullet) \underset{\delta_1^*}{\overset{\delta_0^*}{\rightrightarrows}} \mathfrak{X}\left(\Lambda_\bullet^{[2]}\right) \underset{\delta_2^*}{\overset{\delta_0^*}{\rightrightarrows}} \mathfrak{X}\left(\Lambda_\bullet^{[3]}\right) \underset{\delta_3^*}{\overset{\delta_0^*}{\rightrightarrows}} \cdots \right)$$

   In other words, we have extended $\mathfrak{X}$ to a τ-stack on the category of simplicial manifolds.

2. *If $\mathfrak{X}$ is a τ-prestack on $\mathcal{M}\mathrm{an}$, this functor is still fully faithful, i.e. an equivalence of the* Hom*-categories.*

Proof. By definition, we have $\mathfrak{X}(\Gamma_\bullet) = \mathrm{holim}_{i\in\Delta}\,\mathfrak{X}(\Gamma_i)$. Since $\mathfrak{X}$ is supposed to be a τ-stack and since all $\Pi_i : \Lambda_i \twoheadrightarrow \Gamma_i$ are τ-covers, we have the following equivalence of bicategories:

$$\mathfrak{X}(\Gamma_i) \xrightarrow{\sim} \mathcal{D}esc_{\mathfrak{X}}(\Lambda_i \twoheadrightarrow \Gamma_i) = \mathrm{holim}_{j\in\Delta}\,\mathfrak{X}(\Lambda_i^{[j]}) \ .$$

Altogether, we have the equivalence of bicategories

$$\mathfrak{X}(\Gamma_\bullet) \xrightarrow{\sim} \mathrm{holim}_{i\in\Delta}\,\mathrm{holim}_{j\in\Delta}\,\mathfrak{X}\left(\Lambda_i^{[j]}\right)$$

By proposition 2.7.4, we can exchange the homotopy limits and get

$$\begin{aligned}\mathrm{holim}_{i\in\Delta}\,\mathrm{holim}_{j\in\Delta}\,\mathfrak{X}\left(\Lambda_i^{[j]}\right) &= \mathrm{holim}_{j\in\Delta}\,\mathrm{holim}_{i\in\Delta}\,\mathfrak{X}\left(\Lambda_i^{[j]}\right)\\ &\overset{(2.10)}{=} \mathrm{holim}_{j\in\Delta}\,\mathfrak{X}\left(\Lambda_\bullet^{[j]}\right)\end{aligned}$$

and thus the assertion for stacks. The assertion in the case when $\mathfrak{X}$ is a prestack follows by an analogous argument.

□

By restriction, we obtain a τ-stack on the full subcategory of Lie groupoids. By a further restriction, we get a τ-stack on the full subcategory of G-manifolds. For convenience, we state our result in the special case of G-manifolds:

Corollary 2.7.6.
Let M be a G-manifold and $\{U_i\}_{i\in I}$ be a G-invariant covering. Denote, as usual $Y := \sqcup_{i\in I} U_i$. Then we have:

$$\mathfrak{X}_G(M) \xrightarrow{\sim} \mathrm{holim}\left(\mathfrak{X}_G(Y) \underset{\delta_1^*}{\overset{\delta_0^*}{\rightrightarrows}} \mathfrak{X}_G\left(Y^{[2]}\right) \underset{\delta_2^*}{\overset{\delta_0^*}{\Rrightarrow}} \mathfrak{X}_G\left(Y^{[3]}\right) \underset{\delta_3^*}{\overset{\delta_0^*}{\rightrightarrows\!\rightrightarrows}} \cdots \right)$$

2.8 Proof of theorem 2.2.16, part 4: Sheaves and τ - surjective equivalences

We are now ready to prove theorem 2.2.16 in the special case of τ-surjective equivalences. This actually finishes the proof of theorem 2.2.16, since by the factorization lemma 2.5.7 we have to consider only the two cases of τ-surjective equivalences and strong equivalences. The latter case has already been settled with corollary 2.6.3. We start with the following

Lemma 2.8.1.
Let $F : \Gamma \longrightarrow \Lambda$ be a τ-surjective equivalence of Lie groupoids. By remark 2.7.3.1, the functor F induces a τ-cover of Lie groupoids.

(i) For any n, we have a canonical functor $M^n : \Gamma^{[n]} \longrightarrow \Lambda$ which is given by arbitrary compositions in the augmented simplicial manifold

$$\cdots \overset{\delta_0}{\underset{\delta_3}{\rightrightarrows\!\rightrightarrows}} \Gamma^{[3]} \overset{\delta_0}{\underset{\delta_2}{\rightrightarrows\!\rightarrow}} \Gamma^{[2]} \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} \Gamma \xrightarrow{F} \Lambda$$

Then the functor M^n is a τ-surjective equivalence.

(ii) The diagonal functors $\Gamma \longrightarrow \Gamma^{[n]}$ are strong equivalences.

Proof. (i) As compositions of τ-coverings, all functors M^n are τ-coverings. The functor $F : \Gamma \longrightarrow \Lambda$ is in particular fully faithful. Hence,

$$\Gamma_1 = \Gamma_0 \times_{\Lambda_0} \Lambda_1 \times_{\Lambda_0} \Gamma_0 \ .$$

We now calculate

$$\begin{aligned} \Gamma_1^{[n]} &= \Gamma_1 \times_{\Lambda_1} \cdots \times_{\Lambda_1} \Gamma_1 \\ &= \Big(\Gamma_0 \times_{\Lambda_0} \Lambda_1 \times_{\Lambda_0} \Gamma_0\Big) \times_{\Lambda_1} \cdots \times_{\Lambda_1} \Big(\Gamma_0 \times_{\Lambda_0} \Lambda_1 \times_{\Lambda_0} \Gamma_0\Big) \end{aligned}$$

and find by reordering that this equals

$$\Big(\Gamma_0 \times_{\Lambda_1} \cdots \times_{\Lambda_1} \Gamma_0\Big) \times_{\Lambda_0} \Lambda_1 \times_{\Lambda_0} \Big(\Gamma_0 \times_{\Lambda_1} \cdots \times_{\Lambda_1} \Gamma_0\Big) \ .$$

Hence the diagram

$$\begin{array}{ccc} \Gamma_1^{[n]} & \xrightarrow{M_1^n} & \Lambda_1 \\ \downarrow & & \downarrow \\ \Gamma_0^{[n]} \times \Gamma_0^{[n]} & \xrightarrow{M_0^n} & \Lambda_0 \times \Lambda_0 \end{array} \tag{2.11}$$

is a pullback diagram and thus the functor M^n is fully faithful.
(ii) Take any of the n possible projection functors $P^n : \Gamma^{[n]} \longrightarrow \Gamma$ and consider the diagram

$$\begin{array}{ccccc} \Gamma_1^{[n]} & \xrightarrow{P_1^n} & \Gamma_1 & \xrightarrow{F_1} & \Lambda_1 \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma_0^{[n]} \times \Gamma_0^{[n]} & \xrightarrow{P_0^n \times P_0^n} & \Gamma_0 \times \Gamma_0 & \xrightarrow{F_0 \times F_0} & \Lambda_0 \times \Lambda_0 \end{array}$$

The right diagram is by our assumptions on F a pullback diagram. The external diagram is just the diagram (2.11) considered in part (i) of the lemma and thus a pullback diagram, as well. Hence also the left part of the diagram is a pullback diagram and thus the functor P^n is fully faithful. The functor P^n is a left inverse of of the diagonal functor $\Lambda \longrightarrow \Lambda^{[n]}$. Lemma 2.5.3 now implies that the diagonal functors are strong equivalences.

□

Proposition 2.8.2.
Let $\mathfrak{X}$ be a presheaf in bicategories and $\Gamma \longrightarrow \Lambda$ be a τ-surjective equivalence. Then we have the following equivalences of bicategories

$$\mathfrak{X}(\Gamma_\bullet) \cong \operatorname{holim}\left(\mathfrak{X}(\Gamma_\bullet) \underset{\delta_1^*}{\overset{\delta_0^*}{\rightrightarrows}} \mathfrak{X}\left(\Gamma_\bullet^{[2]}\right) \underset{\delta_2^*}{\overset{\delta_0^*}{\Rrightarrow}} \mathfrak{X}\left(\Gamma_\bullet^{[3]}\right) \underset{\delta_3^*}{\overset{\delta_0^*}{\Rrightarrow}} \cdots \right)$$

$$\cong \operatorname{holim}\left(\mathcal{D}esc_{\mathfrak{X}}(\Gamma_0 \twoheadrightarrow \Lambda_0) \underset{\partial_1^*}{\overset{\partial_0^*}{\rightrightarrows}} \mathcal{D}esc_{\mathfrak{X}}(\Gamma_1 \twoheadrightarrow \Lambda_1) \underset{\partial_2^*}{\overset{\partial_0^*}{\Rrightarrow}} \mathcal{D}esc_{\mathfrak{X}}(\Gamma_2 \twoheadrightarrow \Lambda_2) \underset{\partial_3^*}{\overset{\partial_0^*}{\Rrightarrow}} \cdots \right)$$

Proof. The diagonal functors $\Gamma \longrightarrow \Gamma^n$ give a morphism of simplicial manifolds

$$\begin{array}{ccccccccc}
\cdots \underset{\delta_3}{\overset{\delta_0}{\Rrightarrow}} & \Gamma^{[2]} & \underset{\delta_2}{\overset{\delta_0}{\Rrightarrow}} & \Gamma^{[1]} & \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} & \Gamma & \xrightarrow{F} & \Lambda \\
& \uparrow & & \uparrow & & \uparrow & & \| \\
\cdots \underset{\delta_3}{\overset{\delta_0}{\Rrightarrow}} & \Gamma & \underset{\delta_2}{\overset{\delta_0}{\Rrightarrow}} & \Gamma & \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} & \Gamma & \xrightarrow{F} & \Lambda
\end{array}$$

which is by lemma 2.8.1(ii) in each level an strong equivalence. Using corollary 2.6.3, we get the following equivalence of bicategories

$$\operatorname{holim}\left(\mathfrak{X}(\Gamma_\bullet) \underset{\delta_1^*}{\overset{\delta_0^*}{\rightrightarrows}} \mathfrak{X}\left(\Gamma_\bullet^{[2]}\right) \underset{\delta_2^*}{\overset{\delta_0^*}{\Rrightarrow}} \mathfrak{X}\left(\Gamma_\bullet^{[3]}\right) \underset{\delta_3^*}{\overset{\delta_0^*}{\Rrightarrow}} \cdots \right)$$

$$\xrightarrow{\sim} \operatorname{holim}\left(\mathfrak{X}(\Gamma_\bullet) \underset{\delta_1^*}{\overset{\delta_0^*}{\rightrightarrows}} \mathfrak{X}(\Gamma_\bullet) \underset{\delta_2^*}{\overset{\delta_0^*}{\Rrightarrow}} \mathfrak{X}(\Gamma_\bullet) \underset{\delta_3^*}{\overset{\delta_0^*}{\Rrightarrow}} \cdots \right) \cong \mathfrak{X}(\Gamma_\bullet)$$

The second equivalence is now a direct consequence of Proposition 2.7.4. □

We are now ready to take the final step and prove theorem 2.2.16 for τ-surjective equivalences:

Proposition 2.8.3.
Let $F : \Gamma \longrightarrow \Lambda$ be a τ-surjective equivalence of Lie groupoids.

1. *If $\mathfrak{X}$ is stack, then the functor $F^* : \mathfrak{X}(\Lambda) \longrightarrow \mathfrak{X}(\Gamma)$ is an equivalence of bicategories.*

2. *If $\mathfrak{X}$ is a prestack, then the functor $F^* : \mathfrak{X}(\Lambda) \longrightarrow \mathfrak{X}(\Gamma)$ is fully faithful.*

Proof. Theorem 2.7.5 about equivariant descent implies

$$\mathfrak{X}(\Lambda_\bullet) \xrightarrow{\sim} \operatorname{holim}\left(\mathfrak{X}(\Gamma_\bullet) \underset{\delta_1^*}{\overset{\delta_0^*}{\rightrightarrows}} \mathfrak{X}\left(\Gamma_\bullet^{[2]}\right) \underset{\delta_2^*}{\overset{\delta_0^*}{\Rrightarrow}} \mathfrak{X}\left(\Gamma_\bullet^{[3]}\right) \underset{\delta_3^*}{\overset{\delta_0^*}{\Rrightarrow}} \cdots \right)$$

The preceding proposition 2.8.2 implies that this bicategory is equivalent to $\mathfrak{X}(\Gamma_\bullet)$, which shows part (i). The second statement is proven by a similar argument, using part (ii) of theorem 2.7.5. □

2.9 Proof of the theorem 2.3.3

The central ingredient in the proof of theorem 2.3.3 is an explicit description of descent objects

$$\mathcal{D}esc_{\mathfrak{X}^+}(Y \twoheadrightarrow M) = \mathfrak{X}^+\left(\check{C}(Y)\right) .$$

Instead of specializing to the Čech groupoid $\check{C}(Y)$, we rather describe $\mathfrak{X}^+(\Gamma)$ for a general groupoid Γ. The plus construction involves the choice of a cover of Γ_0 and a descent object for that cover. For a cover $Y \twoheadrightarrow \Gamma_0$, we consider the *covering groupoid* Γ^Y which is defined by

$$\Gamma_0^Y := Y \qquad \text{and} \qquad \Gamma_1^Y := Y \times_{\Gamma_0} \Gamma_1 \times_{\Gamma_0} Y \ .$$

By definition, the diagram

$$\begin{array}{ccc} \Gamma_1^Y & \longrightarrow & \Gamma_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ \Gamma_0^Y \times \Gamma_0^Y & \xrightarrow{\pi\times\pi} & \Gamma_0 \times \Gamma_0 \end{array}$$

is a pullback diagram; hence the map $\Pi : \Gamma_Y \longrightarrow \Gamma$ is fully faithful and thus a τ-weak equivalence. All other structure on Γ^Y is induced from the groupoid structure on Γ. We thus have:

Proposition 2.9.1.
Let $\mathfrak{X}$ be a prestack and Γ be a groupoid. Then the bicategory $\mathfrak{X}^+(\Gamma)$ is equivalent to the following bicategory:

- *Objects are pairs, consisting of a covering $Y \twoheadrightarrow \Gamma_0$ and an object $\mathcal{G}$ in $\mathfrak{X}\left(\Gamma^Y\right)$.*

- *Morphisms between $(Y, \mathcal{G})$ and $(Y', \mathcal{G}')$ consist of a common refinement $Z \twoheadrightarrow \Gamma_0$ of $Y \twoheadrightarrow \Gamma_0$ and $Y' \twoheadrightarrow \Gamma_0$ and a morphism A between the refined objects $\mathcal{G}_Z$ and $\mathcal{G}'_Z$ in $\mathfrak{X}\left(\Gamma^Z\right)$.*

- *2-Morphisms between one-morphisms (Z, A) and (Z', A') are described by pairs consisting of a common refinement $W \twoheadrightarrow \Gamma_0$ of Z and Z' that is compatible with all projections and a morphism of the refinements A_W and A'_W in $\mathfrak{X}\left(\Gamma^W\right)$.*

- *We identify 2-morphisms (W, g) and (W', g'), if there exists a common refinement $V \twoheadrightarrow \Gamma_0$ such that the refined 2-morphisms g_V and g'_V in $\mathfrak{X}\left(\Gamma^V\right)$ are equal.*

Proof. We describe explicitly an object of the bicategory $\mathfrak{X}^+(\Gamma)$: the first piece of data is an object in $\mathfrak{X}^+(\Gamma_0)$. This is just a covering $Y \twoheadrightarrow M$ and

- an object in the descent bicategory $\mathcal{D}esc_{\mathfrak{X}}(Y \twoheadrightarrow \Gamma_0)$.

The second piece of data is a morphism that relates the two pullbacks to $\mathfrak{X}^+(\Gamma_1)$. Such a morphism contains the coverings $Y \times_{\Gamma_0} \Gamma_1 \twoheadrightarrow \Gamma_1$ and $\Gamma_1 \times_{\Gamma_0} Y \twoheadrightarrow \Gamma_1$ where one pullback is along the source map and one pullback along the target map of Γ.

Proposition 2.4.4 allows us to describe this morphism as a stable morphism on the canonical common refinement

$$Y \times_{\Gamma_0} \Gamma_1 \times_{\Gamma_0} Y \twoheadrightarrow \Gamma_1 \ ,$$

i.e.

- A morphism of pullbacks in $\mathcal{D}esc_{\mathfrak{X}}(\Gamma_1^Y \twoheadrightarrow \Gamma_1)$.

Further data and axioms can be transported to the canonical common refinement:

- A 2-morphism of pullbacks in $\mathcal{D}esc_{\mathfrak{X}}(\Gamma_2^Y \twoheadrightarrow \Gamma_2)$.

- A condition on the pullbacks in $\mathcal{D}esc_{\mathfrak{X}}(\Gamma_3^Y \twoheadrightarrow \Gamma_3)$.

Altogether, we have an object in

$$\operatorname{holim}\left(\mathcal{D}esc_{\mathfrak{X}}(\Gamma_0^Y \twoheadrightarrow \Gamma_0) \underset{\partial_1^*}{\overset{\partial_0^*}{\rightrightarrows}} \mathcal{D}esc_{\mathfrak{X}}(\Gamma_1^Y \twoheadrightarrow \Gamma_1) \underset{\partial_2^*}{\overset{\partial_0^*}{\substack{\rightarrow\\\rightarrow\\\rightarrow}}} \mathcal{D}esc_{\mathfrak{X}}(\Gamma_2^Y \twoheadrightarrow \Gamma_2) \underset{\partial_3^*}{\overset{\partial_0^*}{\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}}} \cdots \right)$$

This bicategory is, according to proposition 2.8.2 equivalent to $\mathfrak{X}(\Gamma^Y)$. This shows our assertion for objects; the argument for morphisms and 2-morphisms closely parallels the argument for objects. □

Remark 2.9.2.
We comment on the relation of the three equivalent descriptions of G-equivariant objects like e.g. bundle gerbes to objects described in the literature:

1. *Definition 2.2.5, which has the advantage of being conceptually simple. This definition is used for action groupoids of finite groups in [GR04].*

2. *The definition as a G-equivariant descent objects, using a G-equivariant open covering, cf. corollary 2.7.6. This definition is used in the construction [Mei03] of gerbes on compact Lie groups.*

3. *The characterization in proposition 2.9.1, which has the advantage that invariance under τ-weak equivalences is almost immediate from the definition. Such a definition is used in [BX06].*

We are now ready for the proof of theorem 2.3.3:

Proof. We have to show that the presheaf $\mathfrak{X}^+$ in bicategories is a stack. We thus consider for any cover $Z \twoheadrightarrow M$ the bicategory

$$\mathcal{D}\mathrm{esc}_{\mathfrak{X}^+}(Z \twoheadrightarrow M) = \mathfrak{X}^+\big(\check{\mathrm{C}}\,(Z)\big) \ .$$

By proposition 2.9.1, this bicategory is given by objects, morphisms and 2-morphisms on covering groupoids $\check{\mathrm{C}}\,(Z)^Y$ for covering $Y \twoheadrightarrow Z$. We write out such a groupoid explicitly:

$$\begin{aligned} \check{\mathrm{C}}\,(Z)^Y_0 &= Y = \check{\mathrm{C}}\,(Y)_0 \\ \check{\mathrm{C}}\,(Z)^Y_1 &= Y \times_Z (Z \times_M Z) \times_Z Y \\ &= Y \times_M Y = \check{\mathrm{C}}\,(Y)_1 \ . \end{aligned}$$

We find $\check{\mathrm{C}}\,(Z)^Y = \check{\mathrm{C}}\,(Y)$. Thus $\mathcal{D}esc_{\mathfrak{X}^+}(Z \twoheadrightarrow M) = \mathfrak{X}^+\big(\check{\mathrm{C}}\,(Z)\big)$ is equivalent to the subbicategory of objects of $\mathfrak{X}^+(M)$ which are defined on coverings $Y \twoheadrightarrow Z \twoheadrightarrow M$. This subbicategory is obviously equivalent to the bicategory $\mathfrak{X}^+(M)$. □

Chapter 3

Four Equivalent Versions of Non-Abelian Gerbes

Building on the plus construction and the descent techniques from the last chapter, we review and partially improve four versions of smooth, non-abelian gerbes: Čech cocycles, classifying maps, bundle gerbes, and principal 2-bundles. We prove that all these four versions are equivalent, and so establish new relations between recent developments. Prominent partial results we prove are a bijection between continuous and smooth non-abelian cohomology, and an explicit equivalence between the 2-stacks of bundle gerbes and of 2-bundles. These non-abelian gerbes generalize the notion of abelian gerbes and Jandl gerbes that have been discussed in Chapter 1 and 2. Another class of 2-bundles with geometric applications is the one of string-2-bundles. We will treat them in chapter 4.

3.1 Outline of the chapter

Let G be a Lie group and M be a smooth manifold. There are (among others) the following four ways to say what a "smooth G-bundle" over M is:

(1) *Čech 1-Cocycles*: an open cover $\{U_i\}$ of M, and for each non-empty intersection $U_i \cap U_j$ a smooth map $g_{ij} : U_i \cap U_j \longrightarrow G$ satisfying the cocycle condition

$$g_{ij} \cdot g_{jk} = g_{ik}.$$

(2) *Classifying maps*: a continuous map

$$f : M \longrightarrow \mathfrak{B}G$$

to the classifying space $\mathfrak{B}G$ of the group G.

(3) *Bundle 0-gerbes*: a surjective submersion $\pi : Y \longrightarrow M$ and a smooth map $g : Y \times_M Y \longrightarrow G$ satisfying

$$\pi_{12}^* g \cdot \pi_{23}^* g = \pi_{13}^* g,$$

where $\pi_{ij} : Y \times_M Y \times_M Y \longrightarrow Y \times_M Y$ denotes the projection to the ith and the jth factor.

(4) *Principal bundles*: a surjective submersion $\pi : P \longrightarrow M$ with a smooth action of G on P that preserves π, such that the map

$$P \times G \longrightarrow P \times_M P : (p, g) \longmapsto (p, p.g)$$

is a diffeomorphism.

It is well-known that these four versions of "smooth G-bundles" are all equivalent. Indeed, (1) forms the smooth G-valued Čech cohomology in degree one, whereas (2) is known to be equivalent to continuous G-valued Čech cohomology, which in turn coincides with the smooth one. Further, (3) and (4) form equivalent categories; and isomorphism classes of the objects (3) are in bijection with equivalence classes of the cocycles (1).

In this article we provide an analogous picture for "smooth Γ-gerbes", where Γ is a strict Lie 2-group. In particular, Γ can be the automorphism 2-group of an ordinary Lie group G, in which case the term "non-abelian G-gerbe" is commonly used. We compare the following four versions:

Version I: *Smooth, non-abelian Čech* Γ*-cocycles* (Definition 3.3.6). These form the classical, smooth groupoid-valued cohomology $\check{\mathrm{H}}^1(M, \Gamma)$ in the sense of Giraud [Gir71] and Breen [Bre94], [Bre90, Ch. 4].

Version II: *Classifying maps* (Definition 3.4.4). These are continuous maps $f : M \longrightarrow \mathfrak{B}|\Gamma|$ to the classifying space of the geometric realization of Γ; such maps have been introduced and studied by Baez and Stevenson [BS09].

Version III: Γ*-bundle gerbes* (Definition 3.5.1). These have been developed by Aschieri, Cantini and Jurco [ACJ05] as a generalization of the abelian bundle gerbes of Murray [Mur96]. Here we present an equivalent but more conceptual definition by applying the plus construction (section 2.3) to the monoidal pre-2-stack of principal Γ-bundles.

Version IV: *Principal* Γ*-2-bundles* (Definition 3.6.5). These have been introduced by Bartels [Bar04]; their total spaces are Lie groupoids on which the Lie 2-group Γ acts in a certain way. Compared to Bartels' definition, ours uses a stricter and easier notion of such an action.

Apart from improving the existent definitions of Versions III and IV, the main contribution of this article is to prove that all four versions listed above are equivalent. We follow the same line of arguments as in the case of G-bundles outlined before:

- Baez and Stevenson have shown that homotopy classes of classifying maps of Version II are in bijection with the continuous groupoid-valued Čech cohomology $\check{\mathrm{H}}^1_c(M,\Gamma)$. We prove (Proposition 3.4.1) that the inclusion of *smooth* into *continuous* Čech Γ-cocycles induces a bijection $\check{\mathrm{H}}^1_c(M,\Gamma) \cong \check{\mathrm{H}}^1(M,\Gamma)$. These two results establish the equivalence between our Versions I and II (Theorem 3.4.6).

- Γ-bundle gerbes and principal Γ-2-bundles over M form bicategories. We prove (Theorem 3.7.1) that these bicategories are equivalent, and so establish the equivalence between Versions III and IV in the strongest possible sense. Our proof provides explicit 2-functors in both directions.

- We prove the equivalence between Versions I and III by showing that non-abelian Γ-bundle gerbes are classified by the non-abelian cohomology $\check{\mathrm{H}}^1(M,\Gamma)$ (Theorem 3.5.20).

The aim of this chapter is to simplify and clarify the notion of a non-abelian gerbe, and to make it possible to compare and transfer available results between the various versions. As an example, we use Theorem 3.7.1 – the equivalence between Γ-bundle gerbes and principal Γ-2-bundles – in order to carry two facts about Γ-bundle gerbes over to principal Γ-2-bundles. We prove:

1. Principal Γ-2-bundles form a 2-stack over smooth manifolds (Theorem 3.6.9). This is a new and evidently important result, since it explains precisely in which way one can *glue* 2-bundles from local patches.

2. If Γ and Ω are weakly equivalent Lie 2-groups, the 2-stacks of principal Γ-2-bundles and principal Ω-2-bundles are equivalent (Theorem 3.6.11). This is another new result that generalizes the well-known fact that principal G-bundles and principal H-bundles form equivalent stacks, whenever G and H are isomorphic Lie groups.

The two facts about Γ-bundle gerbes (Theorems 3.5.5 and 3.5.12) on which these results are based are proved in an outmost abstract way: the first is a mere consequence of the definition of Γ-bundle gerbes that we give, namely via a 2-stackification procedure for principal Γ-bundles. The second follows from the fact that principal Γ-bundles and principal Ω-bundles form equivalent monoidal pre-2-stacks, which is in turn a simple corollary of their description by anafunctors that we frequently use.

The present chapter is part of a larger program. In a forthcoming paper, we address the discussion of non-abelian lifting problems, in particular string structures. In a second forthcoming paper we will present the picture of four equivalent versions in a setting *with connections*, based on the results of the present chapter. Our motivation is to understand the role of 2-bundles with connections in higher gauge theories, where they serve as "B-fields". Here, two (non-abelian) 2-groups are especially important, namely the string group [BCSS07] and the Jandl group (which is secretly behind chapter 2.4.2) . More precisely, string-2-bundles appear in supersymmetric sigma models that describe fermionic string theories [Bun09]; while Jandl-2-bundles appear in unoriented sigma models that describe e.g. bosonic type I string theories [SSW07].

This chapter is organized as follows. In Section 3.2 we recall and summarize the theory of principal groupoid bundles and their description by anafunctors. The rest of the chapter is based on this theory. In Sections 3.3, 3.4, 3.5 and 3.6 we introduce our four versions of smooth Γ-gerbes, and establish all but one equivalence. The remaining equivalence, the one between bundle gerbes and principal 2-bundles, is discussed in Section 3.7.

3.2 Preliminaries

There is no claim of originality in this section. Our sources are Lerman [Ler08], Metzler [Met03], Heinloth [Hei05] and Moerdijk-Mrčun [MM03]. A slightly different but equivalent approach is developed in [MRS11].

3.2.1 Lie Groupoids and Groupoid Actions on Manifolds

Here the definition of Lie groupoids, smooth natural transformation from 2.2.1 is important. Since we make a slight shift of notation, we recall the examples that play a role in this chapter.

Example 3.2.1. *(a) Every smooth manifold M defines a Lie groupoid denoted M_{dis} whose objects and morphisms are M, and all whose structure maps are identities.*

(b) Every Lie group G defines a Lie groupoid denoted $\mathcal{B}G$, with one object, with G as its smooth manifold of morphisms, and with the composition $g_2 \circ g_1 := g_2 g_1$.

(c) Suppose X is a smooth manifold and $\rho : H \times X \longrightarrow X$ is a smooth left action of a Lie group H on X. Then, a Lie groupoid $X /\!/ H$ is defined with objects X and morphisms $H \times X$, and with

$$s(h, x) := x \quad , \quad t(h, x) := \rho(h, x) \quad \text{and} \quad \mathrm{id}_x := (1, x).$$

The composition is

$$(h_2, x_2) \circ (h_1, x_1) := (h_2 h_1, x_1),$$

where $x_2 = \rho(h_1, x_1)$. The Lie groupoid $X /\!/ H$ is called the action groupoid *of the action of H on X.*

(d) Let $t : H \longrightarrow G$ be a homomorphism of Lie groups. Then,

$$\rho : H \times G \longrightarrow G : (h, g) \longmapsto (t(h)g)$$

defines a smooth left action of H on G. Thus, we have a Lie groupoid $G /\!/ H$.

(e) *To every Lie groupoid* Γ *one can associate an* opposite Lie groupoid Γ^{op} *which has the source and the target map exchanged.*

We say that a *right action* of a Lie groupoid Γ on a smooth manifold M is a pair (α, ρ) consisting of smooth maps $\alpha : M \longrightarrow \Gamma_0$ and $\rho : M\,{}_\alpha\!\times_t \Gamma_1 \longrightarrow M$ such that

$$\rho(\rho(x,g),h) = \rho(x, g \circ h) \quad , \quad \rho(x, \mathrm{id}_{\alpha(x)}) = x \quad \text{and} \quad \alpha(\rho(x,g)) = s(g)$$

for all possible $g, h \in \Gamma_1$, $p \in \Gamma_0$ and $x \in M$. The map α is called *anchor*. Later on we will replace the letter ρ for the action by the symbol $\circ$ that denotes the composition of the groupoid. A *left action* of Γ on M is a right action of the opposite Lie groupoid Γ^{op}. A smooth map $f : M \longrightarrow M'$ between Γ-spaces with actions (α, ρ) and (α', ρ') is called Γ-*equivariant*, if

$$\alpha' \circ f = \alpha \quad \text{and} \quad f(\rho(x,g)) = \rho'(f(x), g).$$

Example 3.2.2. *(a) Let* Γ *be a Lie groupoid. Then,* Γ *acts on right on its morphisms* Γ_1 *by* $\alpha := s$ *and* $\rho := \circ$ *. It acts on the left on its morphisms by* $\alpha := t$ *and* $\rho := \circ$ *.*

(b) *Let* G *be a Lie group. Then, a right/left action of the Lie groupoid* $\mathcal{B}G$ *(see Example 3.2.1 (b)) on* M *is the same as an ordinary smooth right/left action of* G *on* M*.*

(c) *Let* X *be a smooth manifold. A right/left action of* X_{dis} *(see Example 3.2.1 (a)) on* M *is the same as a smooth map* $\alpha : M \longrightarrow X$*.*

3.2.2 Principal Groupoid Bundles

We give the definition of a principal bundle in exactly the same way as we are going to define principal *2*-bundles in Section 3.6.

Definition 3.2.3. Let M be a smooth manifold, and let Γ be a Lie groupoid.

1. A *principal* Γ-*bundle over* M is a smooth manifold P with a surjective submersion $\pi : P \longrightarrow M$ and a right Γ-action (α, ρ) that respects the projection π, such that
$$\tau : P\,{}_\alpha\!\times_t \Gamma_1 \longrightarrow P \times_M P : (p, g) \longmapsto (p, \rho(p,g))$$
is a diffeomorphism.

2. Let P_1 and P_2 be principal Γ-bundles over M. A *morphism* $\varphi : P_1 \longrightarrow P_2$ is a Γ-equivariant smooth map that respects the projections to M.

Principal Γ-bundles over M form a category $\mathcal{B}un_\Gamma(M)$. In fact, this category is a groupoid, i.e. all morphisms between principal Γ-bundles are invertible. There is an evident notion of a pullback f^*P of a principal Γ-bundle P over M along a smooth map $f : X \longrightarrow M$, and similarly, morphisms between principal Γ-bundles pull back. These define a functor

$$f^* : \mathcal{B}un_\Gamma(M) \longrightarrow \mathcal{B}un_\Gamma(X).$$

These functors make principal Γ-bundles a prestack over smooth manifolds. One can easily show that this prestack is a stack (for the Grothendieck topology of surjective submersions).

Example 3.2.4 (Ordinary principal bundles)**.** *For G a Lie group, we have an equality of categories*

$$\mathcal{B}un_{\mathcal{B}G}(M) = \mathcal{B}un_G(M),$$

i.e. Definition 3.2.3 reduces consistently to the definition of an ordinary principal G-bundle.

Example 3.2.5 (Trivial principal groupoid bundles)**.** *For M a smooth manifold and $f : M \longrightarrow \Gamma_0$ a smooth map, $P := M\,{}_f\!\times_t \Gamma_1$ and $\pi(m,g) := m$ define a surjective submersion, and $\alpha(m,g) := s(g)$ and $\rho((m,g),h) := (m, g \circ h)$ define a right action of Γ on P that preserves the fibers. The map τ we have to look at has the inverse*

$$\tau^{-1} : P \times_M P \longrightarrow P\,{}_\pi\!\times_t \Gamma_1 : ((m,g_1),(m,g_2)) \longmapsto ((m,g_1), g_1^{-1} \circ g_2),$$

which is smooth. Thus we have defined a principal Γ-bundle, which is denoted $\mathbf{I}_f$ and called the trivial bundle for the map f. *Any bundle that is isomorphic to a trivial bundle is called* trivializable.

Example 3.2.6 (Discrete structure groupoids)**.** *For X a smooth manifold, we have an equivalence of categories*

$$\mathcal{B}un_{X_{dis}}(M) \cong C^\infty(M,X)_{dis}.$$

Indeed, for a given principal X_{dis}-bundle P one observes that the anchor $\alpha : P \longrightarrow X$ descends along the bundle projection to a smooth map $f : M \longrightarrow X$, and that isomorphic bundles determine the same map. Conversely, one associates to a smooth map $f : M \longrightarrow X$ the trivial principal X_{dis}-bundle $\mathbf{I}_f$ over M.

Example 3.2.7 (Exact sequences)**.** *Let*

$$1 \longrightarrow H \xrightarrow{t} G \xrightarrow{p} K \longrightarrow 1 \tag{3.1}$$

be an exact sequence of Lie groups, and let $\Gamma := G /\!\!/ H$ be the action groupoid associated to the Lie group homomorphism $t : H \longrightarrow G$ as explained in Example 3.2.1 (d). In this situation, $p : G \longrightarrow K$ is a surjective submersion, and

$$\alpha : G \longrightarrow \Gamma_0 : g \longmapsto g \quad \text{and} \quad \rho : G\,{}_\alpha\!\times_t \Gamma_1 \longrightarrow G : (g,(h,g')) \longmapsto g'$$

define a smooth right action of Γ *on* G *that preserves* p*. The inverse of the map* τ *is*

$$\tau^{-1} : G \times_K G \longrightarrow G\,{}_{\alpha}\!\times_t \Gamma_1 : (g_1, g_2) \longmapsto (g_1, (t^{-1}(g_1 g_2^{-1}), g_2)),$$

which is smooth because t *is an embedding. Thus,* G *is a principal* Γ*-bundle over* K*.*

Next we provide some elementary statements about trivial principal Γ-bundles.

Lemma 3.2.8. *A principal* Γ*-bundle over* M *is trivializable if and only if it has a smooth section.*

Proof. A trivial bundle $\mathbf{I}_f$ has the section

$$s_f : M \longrightarrow \mathbf{I}_f : x \longmapsto (x, \mathrm{id}_{f(x)});$$

and so any trivializable bundle has a section. Conversely, suppose a principal Γ-bundle P has a smooth section $s : M \longrightarrow P$. Then, with $f := \alpha \circ s$,

$$\varphi : \mathbf{I}_f \longrightarrow P : (m, g) \longmapsto \rho(s(m), g)$$

is an isomorphism. □

The following consequence shows that principal Γ-bundles of Definition 3.2.3 are locally trivializable in the usual sense.

Corollary 3.2.9. *Let* P *be a principal* Γ*-bundle over* M*. Then, every point* $x \in M$ *has an open neighborhood* U *over which* P *has a trivialization: a smooth map* $f : U \longrightarrow \Gamma_0$ *and a morphism* $\varphi : \mathbf{I}_f \longrightarrow P|_U$*.*

Proof. One can choose U such that the surjective submersion $\pi : U \longrightarrow P$ has a smooth section. Then, Lemma 3.2.8 applies to the restriction $P|_U$. □

We determine the Hom-set $\mathcal{H}om(\mathbf{I}_{f_1}, \mathbf{I}_{f_2})$ between trivial principal Γ-bundles defined by smooth maps $f_1, f_2 : M \longrightarrow \Gamma_0$. To a bundle morphism $\varphi : \mathbf{I}_{f_1} \longrightarrow \mathbf{I}_{f_2}$ one associates the smooth function $g : M \longrightarrow \Gamma_1$ which is uniquely defined by the condition

$$(\varphi \circ s_{f_1})(x) = s_{f_2}(x) \circ g(x).$$

for all $x \in M$. It is straightforward to see that

Lemma 3.2.10. *The above construction defines a bijection*

$$\mathcal{H}om(\mathbf{I}_{f_1}, \mathbf{I}_{f_2}) \longrightarrow \{g \in C^{\infty}(M, \Gamma_1) \mid s \circ g = f_1 \text{ and } t \circ g = f_2\},$$

under which identity morphisms correspond to constant maps and the composition of bundle morphisms corresponds to the point-wise composition of functions.

Finally, we consider the case of principal bundles for action groupoids.

Lemma 3.2.11. *Suppose $X/\!\!/H$ is a smooth action groupoid. Then the category $\mathcal{B}un_{X/\!\!/H}(M)$ is equivalent to a category with*

> *Objects: principal H-bundles P_H over M together with a smooth, H-anti-equivariant map $f : P_H \longrightarrow X$, i.e. $f(p \cdot h) = h^{-1} f(p)$.*
>
> *Morphisms: bundle morphisms $\varphi_H : P_H \longrightarrow P'_H$ that respect the maps f and f'.*

Proof. For a principal $X/\!\!/H$-bundle (P, α, ρ) we set $P_H := P$ with the given projection to M. The action of H on P_H is defined by

$$p \star h := \rho(p, (h, h^{-1} \cdot \alpha(p))).$$

This action is smooth, and it follows from the axioms of the principal bundle P that it is principal. The map $f : P_H \longrightarrow X$ is the anchor α. The remaining steps are straightforward and left as an exercise. □

3.2.3 Anafunctors

An anafunctor is a generalization of a smooth functor between Lie groupoids, similar to a Morita equivalence, and also known as a Hilsum-Skandalis morphism. The idea goes back to Benabou [Bén73], also see [Joh77]. The references for the following definitions are [Ler08, Met03].

Definition 3.2.12. Let $\mathcal{X}$ and $\mathcal{Y}$ be Lie groupoids.

1. An *anafunctor* $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is a smooth manifold F, a left action (α_l, ρ_l) of $\mathcal{X}$ on F, and a right action (α_r, ρ_r) of $\mathcal{Y}$ on F such that the actions commute and $\alpha_l : F \longrightarrow \mathcal{X}_0$ is a principal $\mathcal{Y}$-bundle over $\mathcal{X}_0$.
2. A *transformation* between anafunctors $f : F \Longrightarrow F'$ is a smooth map $f : F \longrightarrow F'$ which is $\mathcal{X}$-equivariant, $\mathcal{Y}$-equivariant, and satisfies $\alpha'_l \circ f = \alpha_l$ and $\alpha'_r \circ f = \alpha_r$.

The smooth manifold F of an anafunctor is called its *total space*. Notice that the condition that the two actions on F commute implies that each respects the anchor of the other. For fixed Lie groupoids $\mathcal{X}$ and $\mathcal{Y}$, anafunctors $F : \mathcal{X} \longrightarrow \mathcal{Y}$ and transformations form a category $\mathcal{A}na^\infty(\mathcal{X}, \mathcal{Y})$. Since transformations are in particular morphisms between principal $\mathcal{Y}$-bundles, every transformation is invertible so that $\mathcal{A}na^\infty(\mathcal{X}, \mathcal{Y})$ is in fact a groupoid.

Example 3.2.13 (Anafunctors from ordinary functors)**.** *Given a smooth functor $\phi : \mathcal{X} \longrightarrow \mathcal{Y}$, we obtain an anafunctor in the following way. We set $F := \mathcal{X}_0 \,{}_\phi\!\times_t \mathcal{Y}_1$ with anchors $\alpha_l : F \longrightarrow \mathcal{X}_0$ and $\alpha_r : F \longrightarrow \mathcal{Y}_0$ defined by $\alpha_l(x, g) := x$ and $\alpha_r(x, g) := s(g)$, and actions*

$$\rho_l : \mathcal{X}_1 \,{}_s\!\times_{\alpha_l} F \longrightarrow F \quad \textit{and} \quad \rho_r : F \,{}_{\alpha_r}\!\times_t \mathcal{Y}_1 \longrightarrow F$$

defined by $\rho_l(f,(x,g)) := (t(f), \phi(f) \circ g)$ *and* $\rho_r((x,g),f) := (x, g \circ f)$. *In the same way, a smooth natural transformation* $\eta : \phi \Longrightarrow \phi'$ *defines a transformation* $f_\eta : F \Longrightarrow F'$ *by* $f_\eta(x,g) := (x, \eta(x) \circ g)$. *Conversely, one can show that an anafunctor comes from a smooth functor, if its principal* Γ*-bundle has a smooth section.*

Example 3.2.14 (Anafunctors with discrete source)**.** *For* M *a smooth manifold and* Γ *a Lie groupoid, we have an equality of categories*

$$\mathcal{B}un_\Gamma(M) = \mathcal{A}na^\infty(M_{dis}, \Gamma).$$

Further, trivial *principal* Γ*-bundles correspond to smooth* functors. *In particular, with Example 3.2.4 we have,*

(a) For G *a Lie group and* M *a smooth manifold, an anafunctor* $F : M_{dis} \longrightarrow \mathcal{B}G$ *is the same as an ordinary principal* G*-bundle over* M*.*

(b) For M *and* X *smooth manifolds, an anafunctor* $F : M_{dis} \longrightarrow X_{dis}$ *is the same as a smooth map.*

Example 3.2.15 (Anafunctors with discrete target)**.** *For* Γ *a Lie groupoid and* M *a smooth manifold, we have an equivalence of categories*

$$C^\infty(\Gamma_0, M)^\Gamma_{dis} \cong \mathcal{A}na^\infty(\Gamma, M_{dis})$$

where $C^\infty(\Gamma_0, M)^\Gamma$ *denotes the set of smooth maps* $f : \Gamma_0 \longrightarrow M$ *such that* $f \circ s = f \circ t$ *as maps* $\Gamma_1 \longrightarrow M$. *The equivalence is induced by regarding a map* $f \in C^\infty(\Gamma_0, M)^\Gamma$ *as a smooth functor* $f : \Gamma \longrightarrow M_{dis}$*, which in turn induces an anafunctor. Conversely, an anafunctor* $F : \Gamma \longrightarrow M_{dis}$ *is in particular an* M_{dis}*-bundle over* Γ_0*, which is nothing but a smooth function* $f : \Gamma_0 \longrightarrow M$ *by Example 3.2.6. The additional* Γ*-action assures the* Γ*-invariance of* f*.*

For the following definition, we suppose $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are Lie groupoids, and $F : \mathcal{X} \longrightarrow \mathcal{Y}$ and $G : \mathcal{Y} \longrightarrow \mathcal{Z}$ are anafunctors given by $F = (F, \alpha_l, \rho_l, \alpha_r, \rho_r)$ and $G = (G, \beta_l, \tau_l, \beta_r, \tau_r)$.

Definition 3.2.16. The *composition* $G \circ F : \mathcal{X} \longrightarrow \mathcal{Z}$ is the anafunctor defined in the following way:

1. Its total space is
$$E := (F \,{}_{\alpha_r}\!\times_{\beta_l} G)/\sim$$
where $(f, \tau_l(h,g)) \sim (\rho_r(f,h), g)$ for all $h \in \mathcal{Y}_1$ with $\alpha_r(f) = t(h)$ and $\beta_l(g) = s(h)$.

2. The anchors are $(f,g) \longmapsto \alpha_l(f)$ and $(f,g) \longmapsto \beta_r(g)$.

3. The actions $\mathcal{X}_1 {}_s\!\times_\alpha E \longrightarrow E$ and $E {}_\beta\!\times_t \mathcal{Z}_1 \longrightarrow E$ are given, respectively, by

$$(\gamma, (f, g)) \longmapsto (\rho_l(\gamma, f), g) \quad \text{and} \quad ((f, g), \gamma) \longmapsto (f, \tau_r(g, \gamma)).$$

Remark 3.2.17. *Lie groupoids, anafunctors and transformations form a bicategory. This bicategory is equivalent to the bicategory of differentiable stacks (also known as geometric stacks) [Pro96].*

In this article, anafunctors serve for two purposes. The first is that one can use conveniently the composition of anafunctors to define *extensions* of principal groupoid bundles:

Definition 3.2.18. If $P : M_{dis} \longrightarrow \Gamma$ is a principal Γ-bundle over M, and $\Lambda : \Gamma \longrightarrow \Omega$ is an anafunctor, then the principal Ω-bundle

$$\Lambda P := \Lambda \circ P : M_{dis} \longrightarrow \Omega$$

is called the *extension of P along Λ*.

Unwinding this definition, the principal Ω-bundle ΛP has the total space

$$\Lambda P = (P {}_\alpha\!\times_{\alpha_l} \Lambda) \,/ \sim \tag{3.2}$$

where $(p, \rho_l(\gamma, \lambda)) \sim (\rho(p, \gamma), \lambda)$ for all $p \in P$, $\lambda \in \Lambda$ and $\gamma \in \Gamma_1$ with $\alpha(p) = t(\gamma)$ and $\alpha_l(\lambda) = s(\gamma)$. Here α is the anchor and ρ is the action of P, and $\Lambda = (\Lambda, \alpha_l, \alpha_r, \rho_l, \rho_r)$. The bundle projection is $(p, \lambda) \longmapsto \pi(p)$, where π is the bundle projection of P, the anchor is $(p, \lambda) \longmapsto \alpha_r(\lambda)$, and the action is $(p, \lambda) \circ \omega = (p, \rho_r(\lambda, \omega))$.

Extensions of bundles are accompanied by extensions of bundle morphisms. If $\varphi : P_1 \longrightarrow P_2$ is a morphism between Γ-bundles, a morphism $\Lambda\varphi : \Lambda P_1 \longrightarrow \Lambda P_2$ is defined by $\Lambda\varphi(p_1, \lambda) := (\varphi(p_1), \lambda)$ in terms of (3.2). Summarizing, we have

Lemma 3.2.19. *Let M be a smooth manifold and $\Lambda : \Gamma \longrightarrow \Omega$ be an anafunctor. Then, extension along Λ is a functor*

$$\Lambda : \mathcal{B}un_\Gamma(M) \longrightarrow \mathcal{B}un_\Omega(M).$$

Moreover, it commutes with pullbacks and so extends to a morphism between stacks.

Next we suppose that $t : H \longrightarrow G$ is a Lie group homomorphism, and $G/\!/H$ is the associated action groupoid of Example 3.2.1 (d). We look at the functor $\Theta : G/\!/H \longrightarrow \mathcal{B}H$ which is defined by $\Theta(h, g) := h$. Combining Lemma 3.2.11 with the extension along Θ, we obtain

Lemma 3.2.20. *The category $\mathcal{B}un_{G/\!/H}(M)$ of principal $G/\!/H$-bundles over a smooth manifold M is equivalent to a category with*

Objects: principal H-bundles P_H over M together with a section of ΘP_H.

Morphisms: morphisms φ of H-bundles so that $\Theta\varphi$ preserves the sections.

The second motivation for introducing anafunctors is that they provide the inverses to certain smooth functors which are not necessarily equivalences of categories. We redefine for a second what a weak equivalence between Lie groupoids is, from the perspective of anafunctors, but Theorem 3.2.23 shows that this is equivalent to the definition of τ_{sub}-weak equivalence as defined in Definition 2.2.13.

Definition 3.2.21. A smooth functor or anafunctor $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is called a *weak equivalence*, if there exists an anafunctor $G : \mathcal{Y} \longrightarrow \mathcal{X}$ together with transformations $G \circ F \cong \mathrm{id}_{\mathcal{X}}$ and $F \circ G \cong \mathrm{id}_{\mathcal{Y}}$.

We have the following immediate consequence for the stack morphisms of Lemma 3.2.19.

Corollary 3.2.22. *Let $\Lambda : \Gamma \longrightarrow \Omega$ be a weak equivalence between Lie groupoids. Then, extension of principal bundles along Λ is an equivalence of categories Λ : $\mathcal{B}un_\Gamma(M) \longrightarrow \mathcal{B}un_\Omega(M)$. Moreover, these define an equivalence between the stacks $\mathcal{B}un_\Gamma$ and $\mathcal{B}un_\Omega$.*

Concerning the claimed generalization of invertibility, we have the following well-known theorem, see [Ler08, Lemma 3.34], [Met03, Proposition 60].

Theorem 3.2.23. *A smooth functor $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is a weak equivalence if and only if the following two conditions are satisfied:*

(a) it is smoothly essentially surjective: the map

$$s \circ \mathrm{pr}_2 : \mathcal{X}_0 \, {}_{F_0}\!\times_t \mathcal{Y}_1 \longrightarrow \mathcal{Y}_0$$

is a surjective submersion.

(b) it is smoothly fully faithful: the diagram

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{F} & \mathcal{Y}_1 \\ {\scriptstyle s\times t}\downarrow & & \downarrow{\scriptstyle s\times t} \\ \mathcal{X}_0 \times \mathcal{X}_0 & \xrightarrow[F\times F]{} & \mathcal{Y}_0 \times \mathcal{Y}_0 \end{array}$$

is a pullback diagram.

Remark 3.2.24. *One can show that any smooth functor $F : \mathcal{X} \longrightarrow \mathcal{Y}$ that is a weak equivalence actually has a* canonical *inverse anafunctor.*

3.2.4 Lie 2-Groups and crossed Modules

A (strict) *Lie 2-group* is a Lie groupoid Γ whose objects and morphisms are Lie groups, and all whose structure maps are Lie group homomorphisms. One can conveniently bundle the multiplications and the inversions into smooth functors

$$m : \Gamma \times \Gamma \longrightarrow \Gamma \quad \text{and} \quad i : \Gamma \longrightarrow \Gamma.$$

Example 3.2.25. *For A an abelian Lie group, the Lie groupoid $\mathcal{B}A$ from Example 3.2.1 (b) is a Lie 2-group. The condition that A is abelian is necessary.*

Example 3.2.26. *Let $t : H \longrightarrow G$ be a homomorphism of Lie groups, and let $G/\!/H$ be the corresponding Lie groupoid from Example 3.2.1 (d). This Lie groupoid becomes a Lie 2-group if the following structure is given: a smooth left action of G on H by Lie group homomorphisms, denoted $(g, h) \longmapsto {}^{g}h$, satisfying*

$$t({}^{g}h) = gt(h)g^{-1} \quad \text{and} \quad {}^{t(h)}x = hxh^{-1}$$

for all $g \in G$ and $h, x \in H$. Indeed, the objects G of $G/\!/H$ already form a Lie group, and the multiplication on the morphisms $H \times G$ of $G/\!/H$ is the semi-direct product

$$(h_2, g_2) \cdot (h_1, g_1) = (h_2\, {}^{g_2}h_1, g_2g_1). \tag{3.3}$$

The homomorphism $t : H \longrightarrow G$ together with the action of G on H is called a smooth crossed module. *Summarizing, every smooth crossed module defines a Lie 2-group.*

Remark 3.2.27. *Every Lie 2-group Γ can be obtained from a smooth crossed module. Indeed, one puts $G := \Gamma_0$ and $H := \ker(s)$, equipped with the Lie group structures defined by the multiplication functor m of Γ. The homomorphism $t : H \longrightarrow G$ is the target map $t : \Gamma_1 \longrightarrow \Gamma_0$, and the action of G on H is given by the formula ${}^{g}\gamma := \mathrm{id}_g \cdot \gamma \cdot \mathrm{id}_{g^{-1}}$ for $g \in \Gamma_0$ and $\gamma \in \ker(s)$. These two constructions are inverse to each other (up to canonical Lie group isomorphisms and strict Lie 2-group isomorphisms, respectively).*

Example 3.2.28. *Consider a connected Lie group H, so that its automorphism group $\mathrm{Aut}(H)$ is again a Lie group [OV91]. Then, we have a smooth crossed module $(\mathrm{Aut}(H), H, i, \mathrm{ev})$, where $i : H \longrightarrow \mathrm{Aut}(H)$ is the assignment of inner automorphisms to group elements, and $\mathrm{ev} : \mathrm{Aut}(H) \times H \longrightarrow H$ is the evaluation action. The associated Lie 2-group is denoted $\mathrm{AUT}(H)$ and is called the* automorphism 2-group *of H.*

Example 3.2.29. *Let*

$$1 \longrightarrow H \xrightarrow{t} G \xrightarrow{p} K \longrightarrow 1$$

be an exact sequence of Lie groups, i.e. an exact sequence in which p is a submersion and t is an embedding. The homomorphisms $t : H \longrightarrow G$ and $p : G \longrightarrow K$ define *action groupoids $G/\!/H$ and $K/\!/G$ as explained in Example 3.2.1. The first one is even a Lie 2-group: the action of G on H is defined by ${}^g h := t^{-1}(gt(h)g^{-1})$. This is well-defined: since*

$$p(gt(h)g^{-1}) = p(g)p(t(h))p(g^{-1}) = p(g)p(g)^{-1} = 1,$$

the element $gt(h)g^{-1}$ lies in the image of t, and has a unique preimage. The action is smooth because t is an embedding. The axioms of a crossed module are obviously satisfied.

If a Lie groupoid Γ is a Lie 2-group in virtue of a multiplication functor $m : \Gamma \times \Gamma \longrightarrow \Gamma$, then the category $\mathcal{B}un_\Gamma(M)$ of principal Γ-bundles over a smooth manifold M is monoidal:

Definition 3.2.30. Let $P : M_{dis} \longrightarrow \Gamma$ and $Q : M_{dis} \longrightarrow \Gamma$ be principal Γ-bundles. The *tensor product* $P \otimes Q$ is the anafunctor

$$M_{dis} \xrightarrow{\text{diag}} M_{dis} \times M_{dis} \xrightarrow{P\times Q} \Gamma \times \Gamma \xrightarrow{m} \Gamma.$$

Example 3.2.31. *(a) Since trivial principal Γ-bundles $\mathbf{I}_f$ correspond to* smooth *functors $f : M_{dis} \longrightarrow \Gamma$ (Example 3.2.14), it is clear that $\mathbf{I}_f \otimes \mathbf{I}_g = \mathbf{I}_{fg}$.*

(b) Unwinding Definition 3.2.30 in the general case, the tensor product of two principal Γ-bundles P_1 and P_2 with anchors α_1 and α_2, respectively, and actions ρ_1 and ρ_2, respectively, is given by

$$P_1 \otimes P_2 = ((P_1 \times_M P_2)\, {}_{m\circ(\alpha_1\times\alpha_2)}\!\times_t \Gamma_1) \,/ \sim, \tag{3.4}$$

where

$$(p_1, p_2, m(\gamma_1, \gamma_2) \circ \gamma) \sim (\rho_1(p_1, \gamma_1), \rho_2(p_2, \gamma_2), \gamma) \tag{3.5}$$

for all $p_1 \in P_1$, $p_2 \in P_2$ and morphisms $\gamma, \gamma_1, \gamma_2 \in \Gamma_1$ satisfying $t(\gamma_i) = \alpha_i(p_i)$ for $i = 1, 2$ and $s(\gamma_1)s(\gamma_2) = t(\gamma)$. The bundle projection is $\tilde{\pi}(p_1, p_2, \gamma) := \pi_1(p_1) = \pi_2(p_2)$, the anchor is $\tilde{\alpha}(p_1, p_2, \gamma) := s(\gamma)$, and the Γ-action is given by $\tilde{\rho}((p_1, p_2, \gamma), \gamma') := (p_1, p_2, \gamma \circ \gamma')$.

As a consequence of Lemma 3.2.19 and the fact that the composition of anafunctors is associative up to coherent transformations, we have

Proposition 3.2.32. *For M a smooth manifold and Γ a Lie 2-group, the tensor product*

$$\otimes : \mathcal{B}un_\Gamma(M) \times \mathcal{B}un_\Gamma(M) \longrightarrow \mathcal{B}un_\Gamma(M)$$

equips the groupoid of principal Γ-bundles over M with a monoidal structure. Moreover, it turns the stack $\mathcal{B}un_\Gamma$ into a monoidal stack.

Notice that the tensor unit of the monoidal groupoid $\mathcal{B}un_{\Gamma}(M)$ is the trivial principal Γ-bundle $\mathbf{I}_1$ associated to the constant map $1 : M \longrightarrow \Gamma_0$, or, in terms of anafunctors, the one associated to the constant functor $1 : M \longrightarrow \Gamma$.

A (weak) *Lie 2-group homomorphism* between Lie 2-groups (Γ, m_Γ) and (Ω, m_Ω) is an anafunctor $\Lambda : \Gamma \longrightarrow \Omega$ together with a transformation

$$\begin{array}{ccc} \Gamma \times \Gamma & \xrightarrow{m_\Gamma} & \Gamma \\ {\scriptstyle \Lambda\times\Lambda}\downarrow & \eta & \downarrow{\scriptstyle \Lambda} \\ \Omega \times \Omega & \xrightarrow[m_\Omega]{} & \Omega \end{array} \tag{3.6}$$

satisfying the evident coherence condition. A Lie 2-group homomorphism is called *weak equivalence*, if the anafunctor Λ is a weak equivalence. Since extensions and tensor products are both defined via composition of anafunctors, we immediately obtain

Proposition 3.2.33. *Extension along a Lie 2-group homomorphism* $\Lambda : \Gamma \longrightarrow \Omega$ *between Lie 2-groups is a monoidal functor*

$$\Lambda : \mathcal{B}un_{\Gamma}(M) \longrightarrow \mathcal{B}un_{\Omega}(M)$$

between monoidal categories. Moreover, these form a monoidal morphism between monoidal stacks.

Since a monoidal functor is an equivalence of monoidal categories if it is an equivalence of the underlying categories, Corollary 3.2.22 implies:

Corollary 3.2.34. *For* $\Lambda : \Gamma \longrightarrow \Omega$ *a weak equivalence between Lie 2-groups, the monoidal functor of Proposition 3.2.33 is an equivalence of monoidal categories. Moreover, these form a monoidal equivalence between monoidal stacks.*

If we represent the Lie 2-group Γ by a smooth crossed module $t : H \longrightarrow G$ as described in Example 3.2.26, we want to determine explicitly how the tensor product looks like under the correspondence of $G/\!\!/H$-bundles and principal H-bundles with anti-equivariant maps to G, see Lemma 3.2.11.

Lemma 3.2.35. *Let* $t : H \longrightarrow G$ *be a crossed module and let* P *and* Q *be* $G/\!\!/H$*-bundles over* M. *Let* (P_H, f) *and* (Q_H, g) *be the principal* H*-bundles together with their* H*-anti-equivariant maps that belong to* P *and* Q*, respectively, under the equivalence of Lemma 3.2.11. Then, the principal* H*-bundle that corresponds to the tensor product* $P \otimes Q$ *is given by*

$$(P \otimes Q)_H = \big(P \times_M Q\big)/\sim \qquad \textit{where} \qquad (p \star h, q) \sim (p, q \star ({}^{f(p)^{-1}}h)).$$

The action of H *on* $(P \otimes Q)_H$ *is* $[(p,q)] \star h = [(p \star h, q)]$*, and the* H*-anti-equivariant map of* $(P \otimes Q)_H$ *is* $[(p,q)] \longmapsto f(p) \cdot g(q)$.

Similar to the tensor product of principal Γ-bundles, the dual $P^\vee$ of a principal Γ-bundle P over M is the extension of P along the inversion $i : \Gamma \longrightarrow \Gamma$ of the 2-group, $P^\vee := i(P)$. The equality $m \circ (\mathrm{id}, i) = 1$ of functors $M \longrightarrow \Gamma$ induces a "death map" $d : P \otimes P^\vee \longrightarrow \mathbf{I}_1$. We are going to use this bundle morphism in Section 3.5.2, but omit a further systematical treatment of duals for the sake of brevity.

3.3 Version I: Groupoid-valued Cohomology

We have already mentioned group valued Čech 1-cocycles in the introduction. They consist of an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of M and smooth functions $g_{ij} : U_i \cap U_j \longrightarrow G$ satisfying the cocycle condition $g_{ij} \cdot g_{jk} = g_{ik}$. Segal realized [Seg68] that this is the same as a smooth functor

$$g : \check{\mathcal{C}}(\mathcal{U}) \longrightarrow \mathcal{B}G$$

where $\mathcal{B}G$ denotes the one-object groupoid introduced in Example 3.2.1 (b) and $\check{\mathcal{C}}(\mathcal{U})$ denotes the *Čech groupoid* corresponding to the cover $\mathcal{U}$. It has objects $\bigsqcup_{i\in I} U_i$ and morphisms $\bigsqcup_{i,j\in I} U_i \cap U_j$, and its structure maps are

$$s(x,i,j) = (x,i),\ t(x,i,j) = (x,j),\ \mathrm{id}_{(x,i)} = (x,i,i) \text{ and } (x,j,k) \circ (x,i,j) = (x,i,k).$$

In the same way, a smooth natural transformations between smooth functors $\check{\mathcal{C}}(\mathcal{U}) \longrightarrow \mathcal{B}G$ gives rise to a Čech coboundary. Thus the set $\big[\check{\mathcal{C}}(\mathcal{U}), \mathcal{B}G\big]$ of equivalence classes of smooth functors equals the usual first Čech cohomology with respect to the cover $\mathcal{U}$. The classical first Čech-cohomology $\check{\mathrm{H}}^1(M,G)$ of M is hence given by the colimit over all open covers $\mathcal{U}$ of M

$$\check{\mathrm{H}}^1(M,G) = \varinjlim_{\mathcal{U}} \big[\check{\mathcal{C}}(\mathcal{U}), \mathcal{B}G\big].$$

We use this coincidence in order to define the 0-th Čech cohomology with coefficients in a general Lie groupoid Γ:

Definition 3.3.1. If Γ is a Lie groupoid we set

$$\check{\mathrm{H}}^0(M,\Gamma) := \varinjlim_{\mathcal{U}} \big[\check{\mathcal{C}}(\mathcal{U}), \Gamma\big]$$

where the colimit is taken over all covers $\mathcal{U}$ of M and $\big[\check{\mathcal{C}}(\mathcal{U}), \Gamma\big]$ denotes the set of equivalence classes of smooth functors $\check{\mathcal{C}}(\mathcal{U}) \longrightarrow \Gamma$.

Remark 3.3.2. *The choice of the degree is such that* $\check{\mathrm{H}}^0(M,\Gamma)$ *agrees in the case* $\Gamma = G_{dis}$ *(Example 3.2.1* (a)*) with the classical 0-th Čech-cohomology* $\check{\mathrm{H}}^0(M,G)$ *of* M *with values in* G*.*

The geometrical meaning of the set is given in the following well-known theorem, which can be proved e.g. using Lemma 3.2.10.

Theorem 3.3.3. *There is a bijection*

$$\check{\mathrm{H}}^0(M,\Gamma) \cong \left\{ \begin{array}{c} \textit{Isomorphism classes of} \\ \textit{principal } \Gamma\textit{-bundles over} \\ M \end{array} \right\}.$$

If Γ is not only a Lie 2-groupoid but a Lie 2-group one can also define a *first* cohomology group $\check{\mathrm{H}}^1(M,\Gamma)$. Indeed, in this case one can consider the Lie 2-groupoid $\mathcal{B}\Gamma$ with one object, morphisms Γ_0 and 2-morphisms Γ_1. Multiplication in Γ gives the composition of morphisms in $\mathcal{B}\Gamma$. Let $\left[\check{\mathcal{C}}(\mathcal{U}),\mathcal{B}\Gamma\right]$ denote the set of equivalence classes of smooth, weak 2-functors from the Čech-groupoid $\check{\mathcal{C}}(\mathcal{U})$ to the Lie 2-groupoid $\mathcal{B}\Gamma$. For the definition of weak functors see [Bén67] – below we will determine this set explicitly.

Definition 3.3.4. For a 2-group Γ we set

$$\check{\mathrm{H}}^1(M,\Gamma) := \varinjlim_{\mathcal{U}} \left[\check{\mathcal{C}}(\mathcal{U}),\mathcal{B}\Gamma\right].$$

Remark 3.3.5. *This agrees for* $\Gamma = G_{dis}$ *with the classical* $\check{\mathrm{H}}^1(M,G)$*. Furthermore, for an abelian Lie group* A *the Lie groupoid* $\mathcal{B}A$ *is even a 2-group and* $\check{\mathrm{H}}^1(M,\mathcal{B}A)$ *agrees with the classical Čech-cohomology* $\check{\mathrm{H}}^2(M,A)$*.*

Unwinding the above definition, we get Version I of smooth Γ-gerbes:

Definition 3.3.6. Let Γ be a Lie 2-group, and let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be an open cover of M.

1. A Γ-*1-cocycle* with respect to $\mathcal{U}$ is a pair $(f_{\alpha\beta}, g_{\alpha\beta\gamma})$ of smooth maps

   $$f_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \Gamma_0 \quad \text{and} \quad g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\beta \longrightarrow \Gamma_1$$

   satisfying $s(g_{\alpha\beta\gamma}) = f_{\beta\gamma} \cdot f_{\alpha\beta}$ and $t(g_{\alpha\beta\gamma}) = f_{\alpha\gamma}$, and

   $$g_{\alpha\beta\delta} \circ (g_{\beta\gamma\delta} \cdot \mathrm{id}_{f_{\alpha\beta}}) = g_{\alpha\gamma\delta} \circ (\mathrm{id}_{f_{\gamma\delta}} \cdot g_{\alpha\beta\gamma}). \tag{3.7}$$

   Here, the symbols $\circ$ and $\cdot$ stand for the composition and multiplication of Γ, respectively.

2. Two Γ-1-cocycles $(f_{\alpha\beta}, g_{\alpha\beta\gamma})$ and $(f'_{\alpha\beta}, g'_{\alpha\beta\gamma})$ are *equivalent*, if there exist smooth maps $h_\alpha : U_\alpha \longrightarrow \Gamma_0$ and $s_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \Gamma_1$ with

   $$s(s_{\alpha\beta}) = g'_{\alpha\beta} \cdot h_\alpha \quad , \quad t(s_{\alpha\beta}) = h_\beta \cdot g_{\alpha\beta}$$
   $$\text{and} \quad (\mathrm{id}_{h_\gamma} \cdot g_{\alpha\beta\gamma}) \circ (s_{\beta\gamma} \cdot \mathrm{id}_{f_{\alpha\beta}}) \circ (\mathrm{id}_{f_{\beta\gamma}} \cdot s_{\alpha\beta}) = s_{\alpha\gamma} \circ (g'_{\alpha\beta\gamma} \cdot \mathrm{id}_{h_\alpha}).$$

Remark 3.3.7. *For a crossed module $t : H \longrightarrow G$ and $\Gamma := G/\!/H$ the associated Lie 2-group (Example 3.2.26) one can reduce Γ-1-cocycles to pairs*

$$\tilde{f}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G \quad \text{and} \quad \tilde{g}_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\beta \longrightarrow H$$

which satisfies then a cocycle condition similar to (3.7). Analogously, coboundaries can be reduced to pairs

$$\tilde{h}_\alpha : U_\alpha \longrightarrow G \quad \text{and} \quad \tilde{s}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow H.$$

This yields the common definition of non-abelian cocycles, which can for example be found in [Bre90] or [BS09].

Example 3.3.8. *In case of the crossed module $i : H \longrightarrow \mathrm{Aut}(H)$ with $\Gamma = \mathrm{AUT}(H)$ (see Example 3.2.28) Γ-1-cocycles consist of pairs $\tilde{f}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Aut}(H)$ and $\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \cap U_\gamma \longrightarrow H$. Cocycles of this kind classify so-called "Lie groupoid H-extensions" [LGSX09, Proposition 3.14], which can hence be seen as another equivalent version for $\mathrm{AUT}(H)$-gerbes.*

3.4 Version II: Classifying Maps

It is well known that for a Lie group G the *smooth* Čech-cohomology $\check{\mathrm{H}}^1(M, G)$ and the *continuous* Čech-cohomology $\check{\mathrm{H}}^1_c(M, G)$ agree if M is a smooth manifold (in particular paracompact). This can e.g. be shown by locally approximating continuous cocycles by smooth ones without changing the cohomology class – see [MW09] (even for G infinite-dimensional). Below we generalize this fact to non-abelian cohomology for certain Lie 2-groups Γ. Here the continuous Čech-cohomology $\check{\mathrm{H}}^1_c(M, \Gamma)$ is defined in the same way as the smooth one (Definition 3.3.4) but with all maps continuous instead of smooth. A Lie groupoid Γ is called *smoothly separable*, if the set $\underline{\pi}_0\Gamma$ of isomorphism classes of objects is a smooth manifold for which the projection $\Gamma_0 \longrightarrow \underline{\pi}_0\Gamma$ is a submersion.

Proposition 3.4.1. *Let M be a smooth manifold and let Γ be a smoothly separable Lie 2-group. Then, the inclusion*

$$\check{\mathrm{H}}^1(M, \Gamma) \longrightarrow \check{\mathrm{H}}^1_c(M, \Gamma)$$

of smooth into continuous Čech cohomology is a bijection.

Remark 3.4.2. *It is possible that the assumption of being smoothly separable is not necessary, but a proof not assuming this would certainly be more involved than ours. Anyway,* all *Lie 2-groups we are interested in are smoothly separable.*

Proof of Proposition 3.4.1. We denote by $\underline{\pi}_1\Gamma$ the Lie subgroup of Γ_1 consisting of automorphisms of $1 \in \Gamma_0$. Since it has two commuting group structures – composition and multiplication – it is abelian. The idea of the proof is to reduce the statement via long exact sequences to statements proved in [MW09]. The exact sequence we need can be found in [Bre90]:

$$\check{\mathrm{H}}^0(M, (\underline{\pi}_0\Gamma)_{dis}) \longrightarrow \check{\mathrm{H}}^1(M, \mathcal{B}\underline{\pi}_1\Gamma) \longrightarrow \check{\mathrm{H}}^1(M, \Gamma) \longrightarrow \check{\mathrm{H}}^1(M, (\underline{\pi}_0\Gamma)_{dis}) \longrightarrow \check{\mathrm{H}}^2(M, \mathcal{B}\underline{\pi}_1\Gamma).$$

Note that $\check{\mathrm{H}}^1(M,\Gamma)$ and $\check{\mathrm{H}}^1(M, (\underline{\pi}_0\Gamma)_{dis})$ do not have group structures, hence exactness is only meant as exactness of pointed sets. But we actually have more structure, namely an action of $\check{\mathrm{H}}^1(M, B\underline{\pi}_1\Gamma)$ on $\check{\mathrm{H}}^1(M,\Gamma)$. This action factors to an action of

$$C := \mathrm{coker}\Big(\check{\mathrm{H}}^0(M, (\underline{\pi}_0\Gamma)_{dis}) \longrightarrow \check{\mathrm{H}}^1(M, \mathcal{B}\underline{\pi}_1\Gamma)\Big).$$

In fact on the non-empty fibres of the morphism $\check{\mathrm{H}}^1(M,\Gamma) \longrightarrow \check{\mathrm{H}}^1(M, (\underline{\pi}_0\Gamma)_{dis})$ this action is simply transitive. In other words: $\check{\mathrm{H}}^1(M,\Gamma)$ is a C-Torsor over

$$K := \ker\Big(\check{\mathrm{H}}^1(M, (\underline{\pi}_0\Gamma)_{dis}) \longrightarrow \check{\mathrm{H}}^2(M, \mathcal{B}\underline{\pi}_1\Gamma)\Big).$$

The same type of sequence also exists in continuous cohomology

$$\check{\mathrm{H}}^0_c(M, (\underline{\pi}_0\Gamma)_{dis}) \longrightarrow \check{\mathrm{H}}^1_c(M, \mathcal{B}\underline{\pi}_1\Gamma) \longrightarrow \check{\mathrm{H}}^1_c(M, \Gamma) \longrightarrow \check{\mathrm{H}}^1_c(M, (\underline{\pi}_0\Gamma)_{dis}) \longrightarrow \check{\mathrm{H}}^2_c(M, \mathcal{B}\underline{\pi}_1\Gamma).$$

With

$$\begin{aligned} C' &:= \mathrm{coker}\Big(\check{\mathrm{H}}^0_c(M, (\underline{\pi}_0\Gamma)_{dis}) \longrightarrow \check{\mathrm{H}}^1_c(M, \mathcal{B}\underline{\pi}_1\Gamma)\Big) \\ K' &:= \ker\Big(\check{\mathrm{H}}^1_c(M, (\underline{\pi}_0\Gamma)_{dis}) \longrightarrow \check{\mathrm{H}}^2_c(M, \mathcal{B}\underline{\pi}_1\Gamma)\Big), \end{aligned}$$

we exhibit $\check{\mathrm{H}}^1_c(M,\Gamma)$ as a C'-Torsor over K'.

The natural inclusions of smooth into continuous cohomology form a chain map between the two sequences. From [MW09] we know that they are isomorphisms on the second, fourth and fifth factor. In particular we have an induced isomorphism $K \xrightarrow{\sim} K'$. Lemma 3.4.3 below additionally shows that also the induced morphism $C \longrightarrow C'$ is an isomorphisms. Using these isomorphisms we see that $\check{\mathrm{H}}^1(M,\Gamma)$ and $\check{\mathrm{H}}^1_c(M,\Gamma)$ are both C-torsors over K and that the natural map

$$\check{\mathrm{H}}^1(M,\Gamma) \longrightarrow \check{\mathrm{H}}^1_c(M,\Gamma)$$

is a morphism of torsors. But each morphism of group torsors is bijective, which concludes the proof. □

Lemma 3.4.3. *The images of*

$$f : \check{\mathrm{H}}^0(M, (\underline{\pi}_0\Gamma)_{dis}) \longrightarrow \check{\mathrm{H}}^1(M, \mathcal{B}\underline{\pi}_1\Gamma) \quad \textit{and} \quad f' : \check{\mathrm{H}}^0_c(M, (\underline{\pi}_0\Gamma)_{dis}) \longrightarrow \check{\mathrm{H}}^1_c(M, \mathcal{B}\underline{\pi}_1\Gamma)$$

are isomorphic.

Proof. $\check{\mathrm{H}}^0(M, (\underline{\pi}_0\Gamma)_{dis})$ is the group of smooth maps $s : M \longrightarrow \underline{\pi}_0\Gamma$ and the group $\check{\mathrm{H}}^0_c(M, (\underline{\pi}_0\Gamma)_{dis})$ consists of continuous maps $t : M \longrightarrow \underline{\pi}_0\Gamma$. We already know that the groups $\check{\mathrm{H}}^1(M, \mathcal{B}\underline{\pi}_1\Gamma) = \check{\mathrm{H}}^2(M, \underline{\pi}_1\Gamma)$ and $\check{\mathrm{H}}^1_c(M, \mathcal{B}\underline{\pi}_1\Gamma) = \check{\mathrm{H}}^2_c(M, \underline{\pi}_1\Gamma)$ are isomorphic by the result of [MW09]. Under the connecting homomorphism

$$\check{\mathrm{H}}^0(\underline{\pi}_0\Gamma, (\underline{\pi}_0\Gamma)_{dis}) \longrightarrow \check{\mathrm{H}}^1(\underline{\pi}_0\Gamma, \mathcal{B}\underline{\pi}_1\Gamma)$$

the identity $\mathrm{id}_{\underline{\pi}_0\Gamma}$ is send to a class ξ_Γ with the property that $f(s) = s^*\xi_\Gamma$ and $f'(t) = t^*\xi_\Gamma$. Hence it suffices to show that for each continuous map $t : M \longrightarrow \underline{\pi}_0\Gamma$ there is a smooth map $s : M \longrightarrow \underline{\pi}_0\Gamma$ with $s^*\xi_\Gamma = t^*\xi_\Gamma$. It is well know that for each continuous map t between smooth manifolds a homotopic smooth map s exists. It remains to show that the pullback $\check{\mathrm{H}}^1(\underline{\pi}_0\Gamma, \mathcal{B}\underline{\pi}_1\Gamma) \longrightarrow \check{\mathrm{H}}^1(M, \mathcal{B}\underline{\pi}_1\Gamma)$ along smooth maps is homotopy invariant. This can e.g. be seen by choosing smooth (abelian) $\mathcal{B}\underline{\pi}_1\Gamma$-bundle gerbes as representatives, in which case the homotopy invariance can be deduced from the existence of connections. □

It is a standard result in topology that the continuous G-valued Čech cohomology of paracompact spaces is in bijection with homotopy classes of maps to the classifying space $\mathfrak{B}G$ of the group G. A model for the classifying space $\mathfrak{B}G$ is for example the geometric realization of the nerve of the groupoid $\mathcal{B}G$, or Milnor's join construction [Mil56].

Now let Γ be a Lie 2-group, and let $|\Gamma|$ denote the geometric realization of the nerve of Γ. Since the nerve is a simplicial topological group, $|\Gamma|$ is a topological group. Version II for smooth Γ-gerbes is:

Definition 3.4.4 ([BS09])**.** A classifying map for a smooth Γ-gerbe is a continuous map

$$f : M \longrightarrow \mathfrak{B}|\Gamma|.$$

We denote by $\left[M, \mathfrak{B}|\Gamma|\right]$ the set of homotopy classes of classifying maps.

Proposition 3.4.5 ([BS09, Theorem 1])**.** *Let Γ be a Lie 2-group. Then there is a bijection*

$$\check{\mathrm{H}}^1_c(M, \Gamma) \cong \left[M, \mathfrak{B}|\Gamma|\right]$$

where the topological group $|\Gamma|$ is the geometric realization of the nerve of Γ.

Note that the assumption of [BS09, Theorem 1] that Γ is well-pointed is automatically satisfied because Lie groups are well-pointed. Propositions 3.4.1 and 3.4.5 imply the following equivalence theorem between Version I and Version II.

Theorem 3.4.6. *For M a smooth manifold and Γ a smoothly separable Lie 2-group, there is a bijection*

$$\check{\mathrm{H}}^1(M, \Gamma) \cong \left[M, \mathfrak{B}|\Gamma|\right].$$

Remark 3.4.7. *Baez and Stevenson argue in [BS09, Section 5.2.] that the space* $\mathfrak{B}|\Gamma|$ *is homotopy equivalent to a certain geometric realization of the Lie 2-groupoid* $|\mathcal{B}\Gamma|$ *from Section 3.3. Baas, Böstedt and Kro have shown [BBK06] that* $|\mathcal{B}\Gamma|$ *classifies concordance classes of "charted* Γ*-2-bundles". In particular, charted* Γ*-2-bundles are a further equivalent version of smooth* Γ*-gerbes.*

3.5 Version III: Groupoid Bundle Gerbes

Several definitions of "non-abelian bundle gerbes" have appeared in literature so far [ACJ05, Jur05, MRS11]. The approach we give here not only shows a conceptually clear way to define non-abelian bundle gerbes, but also produces systematically a whole bicategory. Moreover, these bicategories form a 2-stack over smooth manifolds (with the Grothendieck topology of surjective submersions).

3.5.1 Definition via the Plus Construction

Recall that the stack $\mathcal{B}un_\Gamma$ of principal Γ-bundles is monoidal if Γ is a Lie 2-group (Proposition 3.2.32). Associated to the monoidal stack $\mathcal{B}un_\Gamma$ we have a pre-2-stack

$$\mathcal{T}riv\mathcal{G}rb_\Gamma := \mathcal{B}(\mathcal{B}un_\Gamma)$$

of *trivial* Γ*-gerbes*. Explicitly, there is one trivial Γ-gerbe $\mathcal{I}$ over every smooth manifold M. The 1-morphisms from $\mathcal{I}$ to $\mathcal{I}$ are principal Γ-bundles over M, and the 2-morphisms between those are morphisms of principal Γ-bundles. Horizontal composition is given by the tensor product of principal Γ-bundles, and vertical composition is the ordinary composition of Γ-bundle morphisms.

Now we apply the *plus construction* of section 2.3 in order to stackify this pre-2-stack. The resulting 2-stack is by definition the *2-stack of* Γ*-bundle gerbes*, i.e.

$$\mathcal{G}rb_\Gamma := (\mathcal{T}riv\mathcal{G}rb_\Gamma)^+ .$$

Unwinding the details of the plus construction, we obtain the following definitions:

Definition 3.5.1. Let M be a smooth manifold. A Γ*-bundle gerbe over* M is a surjective submersion $\pi : Y \twoheadrightarrow M$, a principal Γ-bundle P over $Y^{[2]}$ and an associative morphism

$$\mu : \pi_{23}^* P \otimes \pi_{12}^* P \longrightarrow \pi_{13}^* P$$

of Γ-bundles over $Y^{[3]}$.

The morphism μ is called the *bundle gerbe product*. Its associativity is the evident condition for bundle morphisms over $Y^{[4]}$.

In order to proceed with the 1-morphisms, we say that a *common refinement* of two surjective submersions $\pi_1 : Y_1 \longrightarrow M$ and $\pi_2 : Y_2 \longrightarrow M$ is a smooth manifold Z together with surjective submersions $Z \longrightarrow Y_1$ and $Z \longrightarrow Y_2$ such that the diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & & \searrow & \\ Y_1 & & & & Y_2 \\ & {\scriptstyle \pi_1}\searrow & & \swarrow{\scriptstyle \pi_2} & \\ & & M & & \end{array}$$

is commutative.

We fix the following convention: suppose P_1 and P_2 are Γ-bundles over surjective submersions U_1 and U_2, respectively, and V is a common refinement of U_1 and U_2. Then, a bundle morphism $\varphi : P_1 \longrightarrow P_2$ is understood to be a bundle morphism between the pullbacks of P_1 and P_2 to the common refinement V. For example, in the following definition this convention applies to $U_1 = Y_1^{[2]}$, $U_2 = Y_2^{[2]}$ and $V = Z^{[2]}$.

Definition 3.5.2. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be Γ-bundle gerbes over M. A *1-morphism* $\mathcal{A} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ is a common refinement Z of the surjective submersions of $\mathcal{G}_1$ and $\mathcal{G}_2$ together with a principal Γ-bundle Q over Z and a morphism

$$\beta : P_2 \otimes \zeta_1^*Q \longrightarrow \zeta_2^*Q \otimes P_1$$

of Γ-bundles over $Z^{[2]}$, where $\zeta_1, \zeta_2 : Z^{[2]} \longrightarrow Z$ are the two projections, such that α is compatible with the bundle gerbe products μ_1 and μ_2.

The compatibility of α with μ_1 and μ_2 means that the diagram

$$\begin{array}{ccc} \pi_{23}^*P_2 \otimes \pi_{12}^*P_2 \otimes \zeta_1^*Q & \xrightarrow{\mu_2 \otimes \mathrm{id}} & \pi_{13}^*P_2 \otimes \zeta_1^*Q \\ {\scriptstyle \mathrm{id} \otimes \zeta_{12}^*\beta}\downarrow & & \\ \pi_{23}^*P_2 \otimes \zeta_2^*Q \otimes \pi_{12}^*P_1 & & \downarrow{\scriptstyle \zeta_{13}^*\beta} \\ {\scriptstyle \zeta_{23}^*\beta \otimes \mathrm{id}}\downarrow & & \\ \zeta_3^*Q \otimes \pi_{23}^*P_1 \otimes \pi_{12}^*P_1 & \xrightarrow[\mathrm{id} \otimes \mu_1]{} & \zeta_3^*Q \otimes \pi_{13}^*P_1 \end{array} \tag{3.8}$$

of morphisms of Γ-bundles over $Z^{[3]}$ is commutative.

If $\mathcal{A}_{12} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ and $\mathcal{A}_{23} : \mathcal{G}_2 \longrightarrow \mathcal{G}_3$ are 1-morphisms between bundle gerbes over M, the composition $\mathcal{A}_{23} \circ \mathcal{A}_{12} : \mathcal{G}_1 \longrightarrow \mathcal{G}_3$ is given by the fibre product $Z := Z_{23} \times_{Y_2} Z_{12}$, the principal Γ-bundle $Q := Q_{23} \otimes Q_{12}$ over Z, and the morphism

$$P_3 \otimes \zeta_1^*Q \xrightarrow{\beta_{23} \otimes \mathrm{id}} \zeta_2^*Q_{23} \otimes P_2 \otimes \zeta_1^*Q_{12} \xrightarrow{\mathrm{id} \otimes \beta_{12}} \zeta_2^*Q \otimes P_1.$$

The identity 1-morphism $\mathrm{id}_{\mathcal{G}}$ associated to a Γ-bundle gerbe $\mathcal{G}$ is given by Y regarded as a common refinement of $\pi : Y \longrightarrow M$ with itself, the trivial Γ-bundle $\mathbf{I}_1$ (the tensor unit of $\mathcal{B}un_\Gamma(Y)$), and the evident morphism $\mathbf{I}_1 \otimes P \longrightarrow P \otimes \mathbf{I}_1$.

In order to define 2-morphisms, suppose that $\pi_1 : Y_1 \longrightarrow M$ and $\pi_2 : Y_2 \longrightarrow M$ are surjective submersions, and that Z and Z' are common refinements of π_1 and π_2. Let W be a common refinement of Z and Z' with surjective submersions $r : W \longrightarrow Z$ and $r' : W \longrightarrow Z'$. We obtain two maps

$$s_1 : W \xrightarrow{r} Z \longrightarrow Y_1 \qquad \text{and} \qquad t_1 : W \xrightarrow{r'} Z' \longrightarrow Y_1,$$

and analogously, two maps $s_2, t_2 : W \longrightarrow Y_2$. These patch together to maps

$$x_W := (s_1, t_1) \; : \; W \longrightarrow Y_1 \times_M Y_1 \qquad \text{and} \qquad y_W := (s_2, t_2) \; : \; W \longrightarrow Y_2 \times_M Y_2.$$

Definition 3.5.3. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be Γ-bundle gerbes over M, and let $\mathcal{A}, \mathcal{A}' : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ be 1-morphisms. A *2-morphism* $\varphi : \mathcal{A} \Longrightarrow \mathcal{A}'$ is a common refinement W of the common refinements Z and Z', together with a morphism

$$\varphi : y_W^* P_2 \otimes r^* Q \longrightarrow r'^* Q' \otimes x_W^* P_1$$

of Γ-bundles over W that is compatible with the morphisms β and β'.

The compatibility means that a certain diagram over $W^{[2]}$ commutes. Fibrewise over a point $(w, w') \in W \times_M W$ this diagram looks as follows:

$$\begin{array}{ccc}
P_2|_{s_2(w'),t_2(w')} \otimes P_2|_{s_2(w),s_2(w')} \otimes Q|_{r(w)} & \xrightarrow{\mathrm{id}\otimes\beta} & P_2|_{s_2(w'),t_2(w')} \otimes Q|_{r(w')} \otimes P_1|_{s_1(w),s_1(w')} \\
\Big\downarrow{\scriptstyle \mu_2\otimes\mathrm{id}} & & \Big\downarrow{\scriptstyle \varphi\otimes\mathrm{id}} \\
P_2|_{s_2(w),t_2(w')} \otimes Q|_{r(w)} & & Q'|_{r'(w')} \otimes P_1|_{s_1(w'),t_1(w')} \otimes P_1|_{s_1(w),s_1(w')} \\
\Big\downarrow{\scriptstyle \mu_2^{-1}\otimes\mathrm{id}} & & \Big\downarrow{\scriptstyle id\otimes\mu_1} \\
P_2|_{t_2(w),t_2(w')} \otimes P_2|_{s_2(w),t_2(w)} \otimes Q|_{r(w)} & & Q'|_{r'(w')} \otimes P_1|_{s_1(w),t_1(w')} \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes\varphi} & & \Big\downarrow{\scriptstyle id\otimes\mu_1^{-1}} \\
P_2|_{t_2(w),t_2(w')} \otimes Q'|_{r'(w)} \otimes P_1|_{s_1(w),t_1(w)} & \xrightarrow[\beta'\otimes\mathrm{id}]{} & Q'|_{r'(w')} \otimes P_1|_{t_1(w),t_1(w')} \otimes P_1|_{s_1(w),t_1(w)}
\end{array} \tag{3.9}$$

Finally we identify two 2-morphisms $(W_1, r_1, r_1', \varphi_1)$ and $(W_2, r_2, r_2', \varphi_2)$ if the pullbacks of φ_1 and φ_2 to $W \times_{Z \times_\times Z'} W'$ agree. Explicitly, this condition means that for all $w_1 \in W_1$ and $w_2 \in W_2$ with $r_1(w_1) = r_2(w_2)$ and $r_1'(w_1) = r_2'(w_2)$, and for all $p_2 \in y_{W_1}^* P_2 = y_{W_2}^* P_2$ and $q \in r_1^* Q = r_2^* Q$ we have $\varphi_1(p_2, q) = \varphi_2(p_2, q)$.

Remark 3.5.4.
- *In the above situation of a common refinement W of two common refinements Z, Z' of surjective submersions Y_1, Y_2, the diagram*

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & \uparrow{\scriptstyle r} & \searrow & \\ Y_1 & & W & & Y_2 \\ & \nwarrow & \downarrow{\scriptstyle r'} & \nearrow & \\ & & Z' & & \end{array} \tag{3.10}$$

is not *necessarily commutative. In fact, diagram (3.10) commutes if and only if the two maps $x_W : W \longrightarrow Y_1 \times_M Y_1$ and $y_W : W \longrightarrow Y_2 \times_M Y_2$ factor through the diagonal maps $Y_1 \longrightarrow Y_1 \times_M Y_1$ and $Y_2 \longrightarrow Y_2 \times_M Y_2$, respectively.*

- *In the case that a 2-morphism φ is defined on a common refinement Z for which diagram (3.10)* does *commute, Definition 3.5.3 can be simplified. As remarked before, the two maps x_W and y_W factor through the diagonals, over which the bundles P_1 and P_2 have canonical trivializations (see Corollary 3.5.16). Under these trivializations, φ can be identified with a bundle morphism*

$$\varphi : Q \longrightarrow Q'.$$

Furthermore, the compatibility diagram (3.9) simplifies to the diagram

$$\begin{array}{ccc} P_2 \otimes \eta_1^* Q & \xrightarrow{\beta} & \eta_2^* Q \otimes P_1 \\ {\scriptstyle \mathrm{id} \otimes \eta_1^* \varphi}\downarrow & & \downarrow{\scriptstyle \eta_2^* \varphi \otimes \mathrm{id}} \\ P_2 \otimes \eta_1^* Q' & \xrightarrow[\beta']{} & \eta_2^* Q' \otimes P_1 . \end{array} \tag{3.11}$$

Next we define the vertical composition $\varphi_{23} \bullet \varphi_{12} : \mathcal{A}_1 \Longrightarrow \mathcal{A}_3$ of 2-morphisms $\varphi_{12} : \mathcal{A}_1 \Longrightarrow \mathcal{A}_2$ and $\varphi_{23} : \mathcal{A}_2 \Longrightarrow \mathcal{A}_3$. The refinement is the fibre product $W := W_{12} \times_{Z_2} W_{23}$ of the covers of φ_{12} and φ_{23}. The bundle gerbe products induce isomorphisms

$$x_W^* P_1 \cong x_{W_{23}}^* P_1 \otimes x_{W_{12}}^* P_1 \qquad \text{and} \qquad y_W^* P_2 \cong y_{W_{23}}^* P_2 \otimes y_{W_{12}}^* P_2$$

over W. Under these identifications, the morphism $y_W^* P_2 \otimes Q_1 \longrightarrow Q_3 \otimes x_W^* P_1$ for the 2-morphism $\varphi_{23} \bullet \varphi_{12}$ is defined as

$$y_{W_{23}}^* P_2 \otimes y_{W_{12}}^* P_2 \otimes Q_1 \xrightarrow{\mathrm{id} \otimes \varphi_{12}} y_{W_{23}}^* P_2 \otimes Q_2 \otimes x_{W_{12}}^* P_1 \xrightarrow{\varphi_{23} \otimes \mathrm{id}} Q_3 \otimes x_{W_{23}}^* P_1 \otimes x_{W_{12}}^* P_1 .$$

The identity for vertical composition is just the identity refinement and the identity morphism. Finally we come to the horizontal composition

$$\varphi_{23} \circ \varphi_{12} : \mathcal{A}_{23} \circ \mathcal{A}_{12} \Longrightarrow \mathcal{A}'_{23} \circ \mathcal{A}'_{12}$$

of 2-morphisms $\varphi_{12} : \mathcal{A}_{12} \Longrightarrow \mathcal{A}'_{12}$ and $\varphi_{23} : \mathcal{A}_{23} \Longrightarrow \mathcal{A}'_{23}$: its refinement W is given by $W_{12} \times_{(Y_2 \times Y_2)} W_{23}$. We look at the three relevant maps $x_W : W \longrightarrow Y_1 \times_M Y_1$, $y_W : W \longrightarrow Y_2 \times_M Y_2$ and $z_W : W \longrightarrow Y_3 \times_M Y_3$. The morphism φ of the 2-morphism $\varphi_{23} \circ \varphi_{12}$ is defined as the composition

$$z_W^* P_3 \otimes Q_{23} \otimes Q_{12} \xrightarrow{\varphi_{23} \otimes \mathrm{id}} Q'_{23} \otimes y_W^* P_2 \otimes Q_{12} \xrightarrow{\mathrm{id} \otimes \varphi_{12}} Q'_{23} \otimes Q'_{12} \otimes x_W^* P_1.$$

It follows from the properties of the plus construction that (a) these definitions fit together into a bicategory $\mathcal{G}rb_\Gamma(M)$, and that (b) these form a pre-2-stack $\mathcal{G}rb_\Gamma$ over smooth manifolds. That means, there are *pullback 2-functors*

$$f^* : \mathcal{G}rb_\Gamma(N) \longrightarrow \mathcal{G}rb_\Gamma(M)$$

associated to smooth maps $f : M \longrightarrow N$, and that these are compatible with the composition of smooth maps. Pullbacks of Γ-bundle gerbes, 1-morphisms, and 2-morphisms are obtained by just taking the pullbacks of all involved data. Finally, Theorem 2.3.3 implies (c):

Theorem 3.5.5. *The pre-2-stack $\mathcal{G}rb_\Gamma$ of Γ-bundle gerbes is a 2-stack.*

Remark 3.5.6. *Every 2-stack over smooth manifolds defines a 2-stack over Lie groupoids by Proposition 2.2.8. This way, our approach produces automatically bicategories $\mathcal{G}rb_\Gamma(\mathcal{X})$ of Γ-bundle gerbes over a Lie groupoid $\mathcal{X}$. In particular, for an action groupoid $\mathcal{X} = M/\!\!/G$ we have a bicategory $\mathcal{G}rb_\Gamma(M/\!\!/G)$ of G-*equivariant Γ-bundle gerbes *over M.*

In the remainder of this section we give some examples and describe relations between the definitions given here and existing ones.

Example 3.5.7. *Let A be an abelian Lie group, for instance* $\mathrm{U}(1)$. *Then, $\mathcal{B}A$-bundle gerbes are the same as the well-known A-bundle gerbes [Mur96]. For more details see Remark 3.5.10 below.*

Example 3.5.8. *Let (G, H, t, α) be a smooth crossed module, and let $G/\!\!/H$ the associated action groupoid. Then, a $(G/\!\!/H)$-bundle gerbe is the same as a* crossed module bundle gerbe *in the sense of Jurco [Jur05]. The equivalence relation "stably isomorphic" of [Jur05] is given by "1-isomorphic" in terms of the bicategory constructed here. These coincidences come from the equivalence between $(G/\!\!/H)$-bundles and so-called G-H-bundles used in [Jur05, ACJ05] expressed by Lemma 3.2.20. In particular, in case of the automorphism 2-group* $\mathrm{AUT}(H)$ *of a connected Lie group H, a* $\mathrm{AUT}(H)$*-bundle gerbe is the same as a H-bibundle gerbe in the sense of Aschieri, Cantini and Jurco [ACJ05].*

Example 3.5.9. *Let G be a Lie group, so that G_{dis} is a Lie 2-group. Then, there is an equivalence of 2-categories*

$$\mathcal{G}rb_{G_{dis}}(M) \cong \mathcal{B}un_{\mathcal{B}G}(M)_{dis}.$$

Indeed, if $\mathcal{G}$ is a G_{dis}-bundle gerbe over M, its principal G_{dis}-bundle over $Y^{[2]}$ is by Example 3.2.6 just a smooth map $\alpha : Y^{[2]} \longrightarrow G$, and its bundle gerbe product degenerates to an equality $\pi_{23}^*\alpha \cdot \pi_{12}^*\alpha = \pi_{13}^*\alpha$ for functions on $Y^{[3]}$. In other words, a G_{dis}-bundle gerbe is the same as a so-called "G-bundle 0-gerbe". These form a category that is equivalent to the one of ordinary principal G-bundles, as pointed out in Section 3.1.

Remark 3.5.10. *There are two differences between the definitions given here (for $\Gamma = \mathcal{B}A$) and the ones of Murray et al. [Mur96, MS00, Ste00]. Firstly, we have a slightly different ordering of tensor products of bundles. These orderings are not essential in the case of abelian groups because the tensor category of ordinary A-bundles is symmetric. In the non-abelian case, a consistent theory requires the conventions we have chosen here. Secondly, the definitions of 1-morphisms and 2-morphisms have been generalized step by step:*

1. *In [Mur96], 1-morphisms did not include a* common *refinement, but rather required that the surjective submersion of one bundle gerbe refines the other. This definition is too restrictive in the sense that e.g.* $\mathrm{U}(1)$*-bundle gerbes are not classified by* $\mathrm{H}^3(M,\mathbb{Z})$*, as intended.*

2. *In [MS00], 1-morphisms were defined on the canonical refinement $Z := Y_1 \times_M Y_2$ of the surjective submersions of the bundle gerbes. This definition solves the previous problems concerning the classification of bundle gerbes, but makes the composition of 1-morphisms quite involved [Ste00].*

3. *In [Wal07], 1-morphisms were defined on refinements $\zeta : Z \longrightarrow Y_1 \times_M Y_2$. This generalization allows the same elegant definition of composition we have given here, and results in the same isomorphism classes of bundle gerbes. Moreover, 2-morphisms are defined with* commutative *diagrams (3.10) – this makes the structure of the bicategory outmost simple (see Remark 3.5.4).*

4. *In the present article we have allowed for a yet more general refinement in the definition of 1-morphisms. Its achievement is that bundle gerbes come out as an example of a more general concept – the plus construction – and we get e.g. Theorem 3.5.5 for free.*

Despite of these different definitions of 1-morphisms and 2-morphisms, the resulting bicategories of $\mathcal{B}A$-bundle gerbes in 2., 3. and 4. are all equivalent (see [Wal07, Theorem 1], Remark 2.4.5 and Lemma 3.5.18 below).

3.5.2 Properties of Groupoid Bundle Gerbes

We recall that a homomorphism $\Lambda : \Gamma \longrightarrow \Omega$ between Lie 2-groups is an anafunctor and a transformation (3.6) describing its compatibility with the multiplications. We recall further from Proposition 3.2.33 that extension along Λ is a 1-morphism

$$\Lambda : \mathcal{B}un_\Gamma \longrightarrow \mathcal{B}un_\Omega$$

between monoidal stacks over smooth manifolds. That is, extension along Λ is compatible with pullbacks, tensor products, and morphisms between principal Γ-bundles. Applying it to the principal Γ-bundle P of a Γ-bundle gerbe $\mathcal{G}$, and also to the bundle gerbe product μ, we obtain immediately an Ω-bundle gerbe $\Lambda\mathcal{G}$. The same is evidently true for morphisms and 2-morphisms. Summarizing, we get:

Proposition 3.5.11. *Extension of bundle gerbes along a homomorphism $\Lambda : \Gamma \longrightarrow \Omega$ between Lie 2-groups defines a 1-morphism*

$$\Lambda : \mathcal{G}rb_\Gamma \longrightarrow \mathcal{G}rb_\Omega$$

of 2-stacks over smooth manifolds.

We recall that a weak equivalence between Lie 2-groups is a homomorphism $\Lambda : \Gamma \longrightarrow \Omega$ that is a weak equivalence (see Definition 3.2.21). We have:

Theorem 3.5.12. *Suppose $\Lambda : \Gamma \longrightarrow \Omega$ is a weak equivalence between Lie 2-groups. Then, the 1-morphism $\Lambda : \mathcal{G}rb_\Gamma \longrightarrow \mathcal{G}rb_\Omega$ of Proposition 3.5.11 is an equivalence of 2-stacks.*

Proof. The monoidal equivalence $\Lambda : \mathcal{B}un_\Gamma \longrightarrow \mathcal{B}un_\Omega$ between the monoidal stacks (Corollary 3.2.34) induces an equivalence $\mathcal{T}riv\mathcal{G}rb_\Gamma(M) \longrightarrow \mathcal{T}riv\mathcal{G}rb_\Lambda(M)$ between pre-2-stacks. Since the plus construction is functorial, this induces in turn the claimed equivalence of 2-stacks. □

Next we generalize a couple of well-known result from abelian to non-abelian bundle gerbes. We say a *refinement* of a surjective submersion $\pi : Y \longrightarrow M$ is another surjective submersion $\omega : W \longrightarrow M$ together with a smooth map $f : W \longrightarrow Y$ such that $\zeta = \pi \circ f$. Notice that such a refinement induces smooth maps $f_k : W^{[k]} \longrightarrow Y^{[k]}$ that commute with the various projections $\omega_{i_1 \dots i_k}$ and $\pi_{i_1 \dots i_k}$.

Lemma 3.5.13. *Suppose $\mathcal{G}_1 = (Y_1, P_1, \mu_1)$ and $\mathcal{G}_2 = (Y_2, P_2, \mu_2)$ are Γ-bundle gerbes over M, $f : Y_1 \longrightarrow Y_2$ is a refinement of surjective submersions, and $\varphi : f_2^* P_2 \longrightarrow P_1$ is an isomorphism of Γ-bundles over $Y_1^{[2]}$ that is compatible with the bundle gerbe products μ_1 and μ_2 in the sense that the diagram*

$$\begin{array}{ccc} \pi_{23}^* f_2^* P_2 \otimes \pi_{12}^* f_2^* P_2 & \xrightarrow{f_3^*\mu} & \pi_{13}^* f_2^* P_2 \\ {\scriptstyle \pi_{23}^*\varphi \otimes \pi_{12}^*\varphi}\downarrow & & \downarrow{\scriptstyle \pi_{13}^*\varphi} \\ \pi_{23}^* P_1 \otimes \pi_{12}^* P_1 & \xrightarrow[\mu]{} & \pi_{13}^* P_1 \end{array}$$

is commutative. Then, $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic.

The proof works just the same way as in the abelian case: one constructs the 1-isomorphism over the common refinement $Z := Y_1 \times_M Y_2$ in a straightforward way. As a consequence of Lemma 3.5.13 we have

Proposition 3.5.14. *Let $\mathcal{G} = (Y, P, \mu)$ be a Γ-bundle gerbe over M, and let $f : W \longrightarrow Y$ be a refinement of its surjective submersion $\pi : Y \longrightarrow M$. Then, the refined gerbe $(W, f_2^*P, f_3^*\mu)$ is a Γ-bundle gerbe over M, and is isomorphic to $\mathcal{G}$.*

Lemma 3.5.15. *Let $\mathcal{G} = (Y, P, \mu)$ be a Γ-bundle gerbe over M. Then, there exist unique smooth maps $i : P \longrightarrow P$ and $t : Y \longrightarrow P$ such that*

(i) the diagrams

$$\begin{array}{ccc} P & \xrightarrow{\;i\;} & P \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi} \\ Y^{[2]} & \xrightarrow[\text{flip}]{} & Y^{[2]} \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} & & P \\ & {\scriptstyle t}\nearrow & \downarrow{\scriptstyle\chi} \\ Y & \xrightarrow[\text{diag}]{} & Y^{[2]} \end{array}$$

are commutative.

(ii) the map t is neutral with respect to the bundle gerbe product μ, i.e.

$$\mu(t(y_2), p) = p = \mu(p, t(y_1)).$$

for all $p \in P$ with $\chi(p) = (y_1, y_2)$.

(iii) the map i provides inverses with respect to the bundle gerbe product μ, i.e.

$$\mu(i(p), p) = t(y_1) \quad \textit{and} \quad \mu(p, i(p)) = t(y_2)$$

for all $p \in P$ with $\chi(p) = (y_1, y_2)$.

Moreover, $\alpha(t(y)) = 1$ and $\alpha(i(p)) = \alpha(p)^{-1}$ for all $p \in P$ and $y \in Y$.

Proof. Concerning uniqueness, suppose (t, i) and (t', i') are pairs of maps satisfying (i), (ii) and (iii). Firstly, we have $t'(y) = \mu(t(y), t'(y)) = t(y)$ and so $t = t'$. Then, $\mu(i(p), p) = t(y_1) = t'(y_1) = \mu(i'(p), p)$ implies $i(p) = i'(p)$, and so $i = i'$. In order to see the existence of t and i, denote by $Q := \mathrm{diag}^*P$ the pullback of P to Y, denote by $Q^\vee$ the dual bundle and by $d : Q \otimes Q^\vee \longrightarrow \mathbf{I}_1$ the death map. Consider the smooth map

$$Y \xrightarrow{s} \mathbf{I}_1 \xrightarrow{d^{-1}} Q \otimes Q^\vee \xrightarrow{\mu^{-1} \otimes \mathrm{id}_{Q^\vee}} Q \otimes Q \otimes Q^\vee \xrightarrow{\mathrm{id} \otimes d} \mathbf{I}_1 \otimes Q \cong Q \xrightarrow{\mathrm{diag}} P$$

where $s : Y \longrightarrow \mathbf{I}_1$ is the canonical section (see the proof of Lemma 3.2.8). It is straightforward to see that this satisfies the properties of the map t. Since all maps in the above sequence are (anchor-preserving) bundle morphisms, it is clear that $t \circ \alpha = 1$. □

Corollary 3.5.16. *Let $\mathcal{G} = (Y, P, \mu)$ be a Γ-bundle gerbe over M, and let t and i be the unique maps of Lemma 3.5.15. Then,*

(i) t is a section of diag^*P, and defines a trivialization $\mathrm{diag}^*P \cong \mathbf{I}_1$.

(ii) i is a bundle isomorphism $i : P^\vee \longrightarrow \mathrm{flip}^*P$.

(iii) $\mathcal{C}_0 := Y$ and $\mathcal{C}_1 := P$ define a Lie groupoid with source and target maps $\pi_1 \circ \chi$ and $\pi_2 \circ \chi$, respectively, composition μ, identities t and inversion i.

The following statement is well-known for abelian gerbes; the general version can be proved by a straightforward generalization of the constructions given in the proof of [Wal07, Proposition 3].

Lemma 3.5.17. *Every 1-morphism $\mathcal{A} : \mathcal{G} \longrightarrow \mathcal{H}$ between Γ-bundle gerbes over M is invertible.*

The last statement of this section shows a way to bring 1-morphisms and 2-morphisms into a simpler form (see Remark 3.5.10). For bundle gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$ with surjective submersions $\pi_1 : Y_1 \longrightarrow M$ and $\pi_2 : Y_2 \longrightarrow M$ we denote by $\mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)$ the Hom-category in the bicategory $\mathcal{G}rb_\Gamma(M)$, and by $\mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)^{FP}$ the category whose objects are those 1-morphisms whose common refinement is $Z := Y_1 \times_M Y_2$, and whose 2-morphisms are those 2-morphisms whose refinement is $W := Y_1 \times_M Y_2$ with the maps $r, r' : W \longrightarrow Z$ the identity maps. In principle the lemma has already be prooven in proposition 2.4.4 using the abstract machinery but we need to give a more explicit construction here for later purposes.

Lemma 3.5.18. *The inclusion $\mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)^{FP} \longrightarrow \mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)$ is an equivalence of categories.*

Proof. First we show that it is essentially surjective. We assume $\mathcal{A} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ is a general 1-morphism with a principal Γ-bundle Q over a common refinement Z of the surjective submersions $\pi_1 : Y_1 \longrightarrow M$ and $\pi_2 : Y_2 \longrightarrow M$ of the two bundle gerbes. We look at the principal Γ-bundle

$$\tilde{Q} := \kappa_2^* P_2 \otimes \mathrm{pr}_2^* Q \otimes \kappa_1^* P_1$$

over $\tilde{Z} := Y_1 \times_M Z \times_M Y_2$, where

$\kappa_1 : \tilde{Z} \longrightarrow Y_1^{[2]} : (y_1, z, y_2) \longmapsto (y_1, y_1(z))$ and $\kappa_2 : \tilde{Z} \longrightarrow Y_2^{[2]} : (y_1, z, y_2) \longmapsto (y_2(z), y_2)$.

The projection $\mathrm{pr}_{13} : \tilde{Z} \longrightarrow Y_1 \times_M Y_2$ is a surjective submersion, and over $\tilde{Z} \times_{Y_1 \times_M Y_2} \tilde{Z}$ we have a bundle morphism $\alpha : \mathrm{pr}_1^* \tilde{Q} \longrightarrow \mathrm{pr}_2^* \tilde{Q}$ defined over a point $(\tilde{z}, \tilde{z}')$ with

$\tilde{z} = (y_1, z, y_2)$ and $\tilde{z}' = (y_1, z', y_2)$ by

$$\begin{array}{c} \tilde{Q}_{\tilde{z}} = P_2|_{y_2(z),y_2} \otimes Q_z \otimes P_1|_{y_1,y_1(z)} \\ \downarrow \mu_2^{-1}\otimes\mathrm{id}\otimes\mathrm{id} \\ P_2|_{y_2(z'),y_2} \otimes P_2|_{y_2(z),y_2(z')} \otimes Q_z \otimes P_1|_{y_1,y_1(z)} \\ \downarrow \mathrm{id}\otimes\beta\otimes\mathrm{id} \\ P_2|_{y_2(z'),y_2} \otimes Q_{z'} \otimes P_1|_{y_1(z),y_1(z')} \otimes P_1|_{y_1,y_1(z)} \\ \downarrow \mathrm{id}\otimes\mathrm{id}\otimes\mu_1 \\ P_2|_{y_2(z'),y_2} \otimes Q_{z'} \otimes P_1|_{y_1,y_1(z')} = \tilde{Q}_{\tilde{z}'}. \end{array}$$

The compatibility condition (3.8) implies a cocycle condition for α over the three-fold fibre product of $\tilde{Z}$ over $Y_1 \times_M Y_2$, and since principal Γ-bundles form a stack, the pair $(\tilde{Q}, \alpha)$ defines a principal Γ-bundle Q^{FP} over $Z^{FP} := Y_1 \times_M Y_2$. It is now straightforward to show that the bundle isomorphism β itself descends to a bundle isomorphism β^{FP} over $Z^{FP} \times_M Z^{FP}$ in such a way that the triple $(Z^{FP}, Q^{FP}, \beta^{FP})$ forms a 1-morphism $\mathcal{A}^{FP} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$.

In order to show that $\mathcal{A}^{FP}$ is an essential preimage of $\mathcal{A}$, it remains to construct a 2-morphism $\varphi_{\mathcal{A}}^{FP} : \mathcal{A} \Longrightarrow \mathcal{A}^{FP}$. In the terminology of Definition 3.5.3, we choose $W = \tilde{Z}$ with $r := \mathrm{pr}_2 : W \longrightarrow Z$ and $r' := \mathrm{pr}_{13} : W \longrightarrow Z^{FP}$. Note that diagram (3.10) does not commute. The maps $x_W : W \longrightarrow Y_1^{[2]}$ and $y_W : W \longrightarrow Y_2^{[2]}$ are given by $x_W = s \circ \kappa_1$ and $y_W = \kappa_2$, where $s : Y_1^{[2]} \longrightarrow Y_1^{[2]}$ switches the factors. Now, the bundle isomorphism of the 2-morphism $\varphi_{\mathcal{A}}^{FP}$ we want to construct is a bundle isomorphism

$$\varphi : y_W^* P_2 \otimes r^* Q \longrightarrow \tilde{Q} \otimes x_W^* P_1$$

over W, and is fibrewise over a point $w = (y_1, z, y_2)$ given by

$$\begin{array}{c} P_2|_{y_2(z),y_2} \otimes Q_z \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes t^{-1}} P_2|_{y_2(z),y_2} \otimes Q_z \otimes P_{y_1(z),y_1(z)} \\ \downarrow \mathrm{id}\otimes\mathrm{id}\otimes\mu_1^{-1} \\ P_2|_{y_2(z),y_2} \otimes Q_z \otimes P_1|_{y_1,y_1(z)} \otimes P_1|_{y_1(z),y_1} = \tilde{Q}_w \otimes P_1|_{s(y_1,y_1(z))}, \end{array}$$

where t is the trivialization of $\mathrm{diag}^* P$ of Corollary 3.5.16. The compatibility condition (3.9) is straightforward to check.

Now we show that the inclusion $\mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)^{FP} \longrightarrow \mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)$ is full and faithful. Since it is clearly faithful, it only remains to show that it is full. Given a morphism $\mathcal{A} \longrightarrow \mathcal{A}'$ in $\mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)$, i.e. a common refinement W of $Y_1 \times_M Y_2$ with itself and a bundle morphism φ, we have to find a morphism in $\mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)^{FP}$ such that the two morphisms are identified under the equivalence relation on bundle gerbe 2-morphisms. We denote the bundles over $Y_1 \times_M Y_2$ corresponding to $\mathcal{A}$ and $\mathcal{A}'$ by Q and Q'. The refinement maps are denoted as before by $r = (s_1, s_2) : W \longrightarrow Y_1 \times_M Y_2$

and $r' = (t_1, t_2) : W \longrightarrow Y_1 \times_M Y_2$. Then we obtain an isomorphism $r^*Q \longrightarrow r^*Q'$ fibrewise over a point $w \in W$ by

$$\begin{array}{ccl} Q|_{s_1(w),s_2(w)} & \xrightarrow{d^{-1}\otimes \mathrm{id}} & P_2^\vee|_{s_2(w),t_2(w)} \otimes P_2|_{s_2(w),t_2(w)} \otimes Q|_{s_1(w),s_2(w)} \\ & & \Big\downarrow \mathrm{id}\otimes\varphi \\ & & P_2^\vee|_{s_2(w),t_2(w)} \otimes Q'|_{t_1(w),t_2(w)} \otimes P_1|_{s_1(w),t_1(w)} \\ & & \Big\downarrow \mathrm{id}\otimes\beta'^{-1} \\ & & P_2^\vee|_{s_2(w),t_2(w)} \otimes P_2|_{s_2(w),t_2(w)} \otimes Q'|_{s_1(w),s_2(w)} \xrightarrow{d\otimes\mathrm{id}} Q'|_{s_1(w),s_2(w)} \end{array} \tag{3.12}$$

where $d : P_2^\vee|_{s_2(w),t_2(w)} \otimes P_2|_{s_2(w),t_2(w)} \longrightarrow \mathbf{I}_1$ is the death map. One can use the compatibility condition for φ to show that this morphism descents to a morphism $\psi : Q \longrightarrow Q'$ which is a morphism in $\mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)^{FP}$. The two morphisms (W, ψ) and $(Y_1 \times_M Y_2, \varphi)$ are identified if their pullbacks to

$$W \times_{(Y_1 \times_M Y_2 \times_M Y_1 \times_M Y_2)} (Y_1 \times_M Y_2) = \{w \in W \mid r(w) = r'(w)\} =: W_0$$

are equal. On the one side, the map $W_0 \longrightarrow W$ is the inclusion and the map $W_0 \longrightarrow Y_1 \times_M Y_2$ is equal to r. The pullback of ψ along r is by construction the map $r^*Q \longrightarrow r^*Q'$ from (3.12). On the other side, bundles $x_W^*P_1$ and $y_W^*P_2$ over W_0 have canonical trivializations (Lemma 3.5.16 (i)) under which φ becomes also equal to the morphism (3.12). □

3.5.3 Classification by Čech Cohomology

In this section we prove that Versions I (Čech Γ-1-cocycles) and III (Γ-bundle gerbes) are equivalent. For this purpose, we extract a Čech cocycle from a Γ-bundle gerbe $\mathcal{G}$ over M, and prove that this procedure defines a bijection on the level of equivalence classes (Theorem 3.5.20 below). First we have to assure the existence of appropriate open covers.

Lemma 3.5.19. *For every Γ-bundle gerbe $\mathcal{G} = (Y, P, \mu)$ over M there exists an open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of M with sections $\sigma_i : U_i \longrightarrow Y$, such that the principal Γ-bundles $(\sigma_i \times \sigma_j)^*P$ over $U_i \cap U_j$ are trivializable.*

Proof. One can choose an open cover such that the 2-fold intersections $U_i \cap U_j$ are contractible. Since every Lie 2-group is a crossed module $G/\!\!/H$ (Remark 3.2.27), and $G/\!\!/H$-bundles are ordinary H-bundles (Lemma 3.2.11), these admit sections over contractible smooth manifolds. But a section is enough to trivialize the original Γ-bundle (Lemma 3.2.8). □

Let $\mathcal{G}$ be a Γ-bundle gerbe over M, and let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open cover with the properties of Lemma 3.5.19. We denote by $M_\mathfrak{U}$ the disjoint union of all the open sets U_i, and by $\sigma : M_\mathfrak{U} \longrightarrow Y$ the union of the sections σ_i. Then, σ is a refinement of

$\pi : Y \longrightarrow M$, and we have a Γ-bundle gerbe $\mathcal{G}_{\mathcal{U},\sigma}$ that is isomorphic to $\mathcal{G}$ (Proposition 3.5.14).

The principal Γ-bundle P_{ij} of $\mathcal{G}_{\mathcal{U},\sigma}$ over the component $U_i \cap U_j$ is by assumption trivializable. Thus there exists a trivialization $t_{ij} : P_{ij} \longrightarrow \mathbf{I}_{f_{ij}}$ for smooth functions $f_{ij} : U_i \cap U_j \longrightarrow \Gamma_0$. We define an isomorphism μ_{ijk} between trivial bundles such that the diagram

$$\begin{array}{ccc} P_{jk} \otimes P_{ij} & \xrightarrow{\mu} & P_{ik} \\ {\scriptstyle t_{jk}\otimes t_{ij}}\downarrow & & \downarrow{\scriptstyle t_{ik}} \\ \mathbf{I}_{f_{jk}} \otimes \mathbf{I}_{f_{ij}} & \xrightarrow[\mu_{ijk}]{} & \mathbf{I}_{f_{ik}} \end{array}$$

is commutative. Now we are in the situation of Lemma 3.5.13, which implies that the Γ-bundle gerbe $\mathcal{G}_{\mathcal{U},\sigma,t} := (M_{\mathcal{U}}, \mathbf{I}_{f_{ij}}, \mu_{ijk})$ is still isomorphic to $\mathcal{G}$.

Combining Lemma 3.2.10 with Example 3.2.31 (a), we see that the isomorphisms μ_{ijk} correspond to smooth maps $g_{ijk} : U_i \cap U_j \cap U_k \longrightarrow \Gamma_1$ such that $s(g_{ijk}) = f_{jk} \cdot f_{ij}$ and $t(g_{ijk}) = f_{ik}$. The associativity condition for μ_{ijk} implies moreover that

$$g_{\alpha\gamma\delta} \circ (g_{\alpha\beta\gamma} \cdot \mathrm{id}_{f_{\gamma\delta}}) = g_{\alpha\beta\delta} \circ (\mathrm{id}_{f_{\alpha\beta}} \cdot g_{\beta\gamma\delta}).$$

Hence, the collection $\{f_{ij}, g_{ijk}\}$ is a Γ-1-cocycle on M with respect to the open cover $\mathcal{U}$.

Theorem 3.5.20. *Let M be a smooth manifold and let Γ be a Lie 2-group. The above construction defines a bijection*

$$\left\{ \begin{array}{c} \textit{Isomorphism classes of } \Gamma\textit{-bundle} \\ \textit{gerbes over } M \end{array} \right\} \cong \check{\mathrm{H}}^1(M, \Gamma).$$

Proof. In order to prove that we have a well-defined map, we make the claim that Γ-bundle gerbes $(M_{\mathcal{U}}, \mathbf{I}_{f_{ij}}, \mu_{ijk})$ and $(M_{\mathcal{V}}, \mathbf{I}_{h_{ij}}, \nu_{ijk})$ are isomorphic if and only if the corresponding Γ-1-cocycles are equivalent. This proves at the same time that the choices of open covers and sections we have made during the construction do not matter, that the resulting map is well-defined on isomorphism classes, and that this map is injective. Surjectivity follows by assigning to a Γ-1-cocycle (f_{ij}, g_{ijk}) with respect to some cover $\mathcal{U}$ the Γ-bundle gerbe $(M_{\mathcal{U}}, \mathbf{I}_{f_{ij}}, \mu_{ijk})$ with μ_{ijk} determined by Lemma 3.2.10.

It remains to prove that claim. We assume $\mathcal{A} = (Z, Q, \alpha)$ is a 1-isomorphism between the Γ-bundle gerbes $(M_{\mathcal{U}}, \mathbf{I}_{f_{ij}}, \mu_{ijk})$ and $(M_{\mathcal{V}}, \mathbf{I}_{h_{ij}}, \nu_{ijk})$. Similarly to Lemma 3.5.19 one can show that there exists a cover $\mathcal{W}$ of M by open sets W_i that refines both $\mathcal{U}$ and $\mathcal{V}$, and that allows smooth sections $\omega_i : W_i \longrightarrow Z$ for which the Γ-bundle $\omega_i^* Q$ is trivializable. In the terminology of the above construction, choosing a trivialization $t : \omega^* Q \longrightarrow \mathbf{I}_{h_i}$ with smooth maps $h_i : W_i \longrightarrow \Gamma_0$ over $M_{\mathcal{W}}$ converts the isomorphism α into smooth functions $s_{ij} : W_i \cap W_j \longrightarrow \Gamma_1$ satisfying $s(s_{ij}) = g'_{ij} \cdot h_i$ and $t(s_{ij}) = h_j \cdot g_{ij}$. The compatibility diagram (3.8) implies the remaining condition that makes (h_i, s_{ij}) an equivalence between the Γ-2-cocycles (f_{ij}, g_{ijk}) and (f'_{ij}, g'_{ijk}). $\square$

3.6 Version IV: Principal 2-Bundles

The basic idea of a smooth 2-bundle is that it gives for every point x in the base manifold M a Lie groupoid $\mathcal{P}_x$ varying smoothly with x. Numerous different versions have appeared so far in the literature, e.g [Bar04, BS07, Woc08, SP10]. The main objective of *our* version of principal 2-bundles is to make the definition of the objects (i.e. the 2-bundles) *as simple as possible*, while keeping their isomorphism classes in bijection with non-abelian cohomology. Thus, our principal 2-bundles will be defined using *strict* actions of Lie 2-groups on Lie groupoids, and *not* using anafunctors. The necessary "weakness" will be pushed into the definition of 1-morphisms.

3.6.1 Definition of Principal 2-Bundles

As an important prerequisite for principal 2-bundles we have to discuss actions of Lie 2-groups on Lie groupoids, and equivariant anafunctors.

Definition 3.6.1. Let $\mathcal{P}$ be a Lie groupoid, and let Γ be a Lie 2-group. A *smooth right action of* Γ *on* $\mathcal{P}$ is a smooth functor $R : \mathcal{P} \times \Gamma \longrightarrow \mathcal{P}$ such that $R(p,1) = p$ and $R(\rho, \mathrm{id}_1) = \rho$ for all $p \in \mathcal{P}_0$ and $\rho \in \mathcal{P}_1$, and the diagram

$$\begin{array}{ccc} \mathcal{P} \times \Gamma \times \Gamma & \xrightarrow{\mathrm{id} \times m} & \mathcal{P} \times \Gamma \\ {\scriptstyle R \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle R} \\ \mathcal{P} \times \Gamma & \xrightarrow[m]{} & \mathcal{P} \end{array}$$

of smooth functors is commutative (strictly, on the nose).

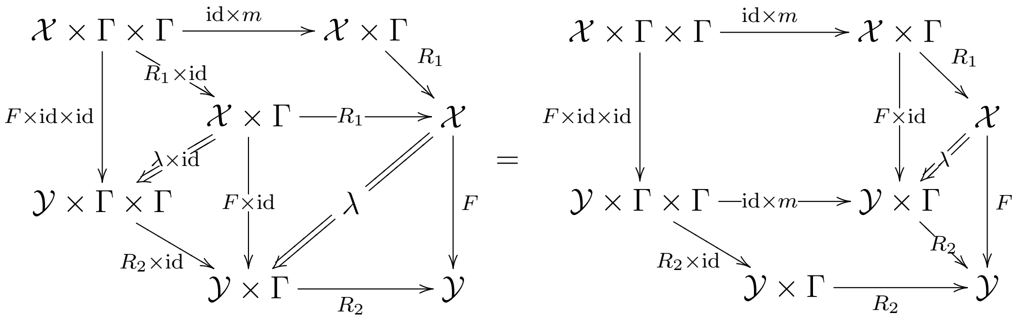
For example, every Lie 2-group acts on itself via multiplication. Note that due to strict commutativity, one has $R(R(p,g),g^{-1}) = p$ and $R(R(\rho,\gamma),i(\gamma)) = \rho$ for all $g \in \Gamma_0$, $p \in \mathcal{P}_0$, $\gamma \in \Gamma_1$ and $\rho \in \mathcal{P}_1$.

Remark 3.6.2. *This definition could be weakened in two steps. First, one could allow a natural transformation in the above diagram instead of commutativity. Secondly, one could allow R to be an anafunctor instead of an ordinary functor. It turns out that for our purposes the above definition is sufficient.*

Definition 3.6.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be Lie groupoids with smooth actions (R_1, ρ_1), (R_2, ρ_2) of a Lie 2-group Γ. An *equivariant structure* on an anafunctor $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is a transformation

$$\begin{array}{ccc} \mathcal{X} \times \Gamma & \xrightarrow{R_1} & \mathcal{X} \\ {\scriptstyle F \times \mathrm{id}} \downarrow & \overset{\lambda}{\swarrow\!\!\swarrow} & \downarrow {\scriptstyle F} \\ \mathcal{Y} \times \Gamma & \xrightarrow[R_2]{} & \mathcal{Y} \end{array}$$

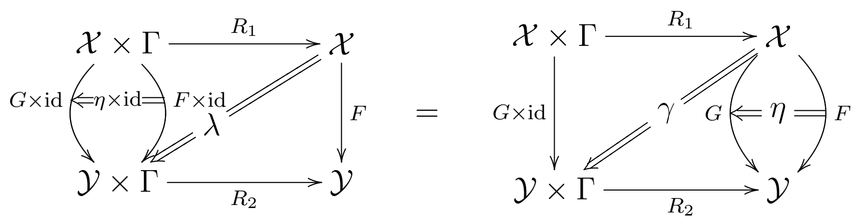
satisfying the following condition:



An anafunctor together with a Γ-equivariant structure is called Γ-*equivariant anafunctor*.

In Appendix 3.8.1 we translate this abstract (but evidently correct) definition of equivariance into more concrete terms involving a Γ_1-action on the total space of the anafunctor.

Definition 3.6.4. If $(F, \lambda) : \mathcal{X} \longrightarrow \mathcal{Y}$ and $(G, \gamma) : \mathcal{X} \longrightarrow \mathcal{Y}$ are Γ-equivariant anafunctors, a transformation $\eta : F \Longrightarrow G$ is called Γ-*equivariant*, if the following equality of transformation holds:



It follows from abstract nonsense in the bicategory of Lie groupoids, anafunctors and transformations that we have another bicategory with

- objects: Lie groupoids with smooth right Γ-actions.
- 1-morphisms: Γ-equivariant anafunctors.
- 2-morphisms: Γ-equivariant transformations.

We need three further notions for the definition of a principal 2-bundle. Let M be a smooth manifold, and let $\mathcal{P}$ be a Lie groupoid. We say that a smooth functor $\pi : \mathcal{P} \longrightarrow M_{dis}$ is a *surjective submersion functor*, if $\pi : \mathcal{P}_0 \longrightarrow M$ is a surjective submersion. Let $\pi : \mathcal{P} \longrightarrow M_{dis}$ be a surjective submersion functor, and let $\mathcal{Q}$ be a Lie groupoid with some smooth functor $\chi : \mathcal{Q} \longrightarrow M_{dis}$. Then, the fibre product $\mathcal{P} \times_M \mathcal{Q}$ is defined to be the full subcategory of $\mathcal{P} \times \mathcal{Q}$ over the submanifold $\mathcal{P}_0 \times_M \mathcal{Q}_0 \subset \mathcal{P}_0 \times \mathcal{Q}_0$.

Definition 3.6.5. Let M be a smooth manifold and let Γ be a Lie 2-group.

(a) A *principal* Γ*-2-bundle over* M is a Lie groupoid $\mathcal{P}$, a surjective submersion functor $\pi : \mathcal{P} \longrightarrow M_{dis}$, and a smooth right action R of Γ on $\mathcal{P}$ that preserves π, such that the smooth functor

$$\tau := (\mathrm{pr}_1, R) : \mathcal{P} \times \Gamma \longrightarrow \mathcal{P} \times_M \mathcal{P}$$

is a weak equivalence.

(b) A *1-morphism* between principal Γ-2-bundles is a Γ-equivariant anafunctor

$$F : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$$

that respects the surjective submersion functors to M.

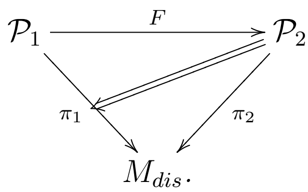
(c) A *2-morphism* between 1-morphisms is a Γ-equivariant transformation between these.

Remark 3.6.6. *(a) The condition in (a) that the action R preserves the surjective submersion functor π means that the diagram of functors*

$$\begin{array}{ccc} \mathcal{P} \times \Gamma & \xrightarrow{R} & \mathcal{P} \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{P} & \xrightarrow[\pi]{} & M_{dis} \end{array}$$

is commutative.

(b) The condition in (b) that the anafunctor F respects the surjective submersion functors means in the first place that there exists a transformation



However, since the target of the anafunctors π_1 and $\pi_2 \circ F$ is the discrete *groupoid M_{dis}, the equivalence of Example 3.2.15 applies, and implies that if such a transformation exists, it is unique. Indeed, it is easy to see that an anafunctor $F : \mathcal{P} \longrightarrow \mathcal{Q}$ with anchors $\alpha_l : F \longrightarrow \mathcal{P}_0$ and $\alpha_r : F \longrightarrow \mathcal{Q}_0$ respects smooth functors $\pi : \mathcal{P} \longrightarrow M_{dis}$ and $\chi : \mathcal{Q} \longrightarrow M_{dis}$ if and only if $\pi \circ \alpha_l = \chi \circ \alpha_r$.*

Example 3.6.7. *The* trivial Γ-2-bundle *over M is defined by*

$$\mathcal{P} := M_{dis} \times \Gamma \quad , \quad \pi := \mathrm{pr}_1 \quad , \quad R := \mathrm{id}_M \times m.$$

Here, the smooth functor τ even has a smooth inverse functor. *In the following we denote the trivial* Γ*-2-bundle by $\mathcal{I}$.*

Remark 3.6.8. *The principal* Γ*-2-bundles of Definition 3.6.5 are very similar to those of Bartels [Bar04] and Wockel [Woc08], in the sense that their fibres are groupoids with a* Γ*-action. They only differ in the strictness assumptions for the action, and in the formulation of principality. Opposed to that, the "principal 2-group bundles" introduced in [GS08] are quite different: their fibres are Lie 2-groupoids equipped with a certain Lie 2-groupoid morphism to $B\Gamma$.*

3.6.2 Properties of Principal 2-Bundles

Principal Γ-2-bundles over M form a bicategory denoted $2\text{-}\mathcal{B}un_\Gamma(M)$. There is an evident pullback 2-functor

$$f^* : 2\text{-}\mathcal{B}un_\Gamma(N) \longrightarrow 2\text{-}\mathcal{B}un_\Gamma(M)$$

associated to smooth maps $f : M \longrightarrow N$, and these make $2\text{-}\mathcal{B}un_\Gamma$ a pre-2-stack over smooth manifolds. We deduce the following important two theorems about this pre-2-stack. The first asserts that it actually is a 2-stack:

Theorem 3.6.9. *Principal* Γ*-2-bundles form a 2-stack $2\text{-}\mathcal{B}un_\Gamma$ over smooth manifolds.*

Proof. This follows from Theorem 3.5.5 (Γ-bundle gerbes form a 2-stack) and Theorem 3.7.1 (the equivalence $\mathcal{G}rb_\Gamma \cong 2\text{-}\mathcal{B}un_\Gamma$) we prove in Section 3.7. □

Remark 3.6.10. *Similar to Remark 3.5.6, we obtain automatically bicategories $2\text{-}\mathcal{B}un_\Gamma(\mathcal{X})$ of principal* Γ*-2-bundles over Lie groupoids $\mathcal{X}$, including bicategories of* equivariant principal Γ-2-bundles*.*

The second concerns a homomorphism $\Lambda : \Gamma \longrightarrow \Omega$ of Lie 2-groups, which induces the extension $\Lambda : \mathcal{G}rb_\Gamma \longrightarrow \mathcal{G}rb_\Omega$ between 2-stacks of bundle gerbes (Proposition 3.5.11). Combined with the equivalence $\mathcal{G}rb_\Gamma \cong 2\text{-}\mathcal{B}un_\Gamma$ of Theorem 3.7.1, it defines a 1-morphism

$$\Lambda : 2\text{-}\mathcal{B}un_\Gamma \longrightarrow 2\text{-}\mathcal{B}un_\Omega$$

between 2-stacks of principal 2-bundles. Now we get as a direct consequence of Theorem 3.5.12:

Theorem 3.6.11. *If $\Lambda : \Gamma \longrightarrow \Omega$ is a weak equivalence between Lie 2-groups, then the 1-morphism $\Lambda : 2\text{-}\mathcal{B}un_\Gamma \longrightarrow 2\text{-}\mathcal{B}un_\Omega$ is an equivalence of 2-stacks.*

A third consequence of the equivalence of Theorem 3.7.1 in combination with Lemma 3.5.17 is

Corollary 3.6.12. *Every 1-morphism* $F : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ *between principal* Γ*-2-bundles over* M *is invertible.*

The following discussion centers around *local trivializability* that is implicitly contained in Definition 3.6.5. A principal Γ-2-bundle that is isomorphic to the trivial Γ-2-bundle $\mathcal{I}$ introduced in Example 3.6.7 is called *trivializable*. A *section* of a principal Γ-2-bundle $\mathcal{P}$ over M is an anafunctor $S : M_{dis} \longrightarrow \mathcal{P}$ such that $\pi \circ S = \mathrm{id}_{M_{dis}}$ (recall that an anafunctor $\pi \circ S : M \longrightarrow M$ is the same as a smooth map). One can show that every point $x \in M$ has an open neighborhood U together with a section $s : U_{dis} \longrightarrow \mathcal{P}|_U$. Such sections can even be chosen to be smooth functors, rather then anafunctors, namely simply as ordinary sections of the surjective submersion $\pi : (\mathcal{P}|_U)_0 \longrightarrow U_{dis}$.

Lemma 3.6.13. *A principal* Γ*-2-bundle over* M *is trivializable if and only if it has a smooth section.*

Proof. The trivial Γ-2-bundle $\mathcal{I}$ has the section $S(m) := (m, 1)$, where 1 denotes the unit of Γ_0. If $\mathcal{P}$ is trivializable, and $F : \mathcal{I} \longrightarrow \mathcal{P}$ is an isomorphism, then, $F \circ S$ is a section of $\mathcal{P}$. Conversely, suppose $\mathcal{P}$ has a section $S : M_{dis} \longrightarrow \mathcal{P}$. Then, we get the anafunctor

$$\mathcal{I} = M_{dis} \times \Gamma \xrightarrow{S \times \mathrm{id}} \mathcal{P} \times \Gamma \xrightarrow{\quad R \quad} \mathcal{P}. \tag{3.13}$$

It has an evident Γ-equivariant structure and respects the projections to M. According to Corollary 3.6.12, this is sufficient to have a 1-isomorphism. □

Corollary 3.6.14. *Every principal* Γ*-2-bundle is locally trivializable, i.e. every point* $x \in M$ *has an open neighborhood* U *and a 1-morphism* $T : \mathcal{I} \longrightarrow \mathcal{P}|_U$.

Remark 3.6.15. *In Wockel's version [Woc08] of principal 2-bundles, local trivializations are required to be smooth functors and to be invertible as smooth functors, rather than allowing anafunctors. This version turns out to be too restrictive in the sense that the resulting bicategory receives no 2-functor from the bicategory* $\mathcal{G}rb_\Gamma(M)$ *of* Γ*-bundle gerbes that establishes an equivalence.*

It is also possible to reformulate our definition of principal 2-bundles in terms of local trivializations. This reformulation gives us criteria which might be easier to check than the actual definition, similar to the case of ordinary principal bundles.

Proposition 3.6.16. *Let* $\mathcal{P}$ *be a Lie groupoid,* $\pi : \mathcal{P} \longrightarrow M_{dis}$ *be a surjective submersion functor, and* R *be a smooth right action of* Γ *on* $\mathcal{P}$ *that preserves* π*. Suppose every point* $x \in M$ *has an open neighborhood* U *together with a* Γ*-equivariant anafunctor* $T : \mathcal{I} \longrightarrow \mathcal{P}|_U$ *that respects the projections. Then,* $\pi : \mathcal{P} \longrightarrow M_{dis}$ *is a principal* Γ*-2-bundle over* M*.*

Proof. We only have to prove that the functor τ is a weak equivalence, and we use Theorem 3.2.23. Since all morphisms of $\mathcal{P}$ have source and target in the same fibre of $\pi : \mathcal{P}_0 \longrightarrow M_{dis}$, we may check the two conditions of Theorem 3.2.23 locally, i.e. for $\mathcal{P}|_{U_i}$ where U_i is an open cover of M. Using local trivializations $\mathcal{T}_i : \mathcal{I} \longrightarrow \mathcal{P}|_{U_i}$, the smooth functor τ translates into the smooth functor $(\mathrm{id}, \mathrm{pr}_1, m) : M_{dis} \times \Gamma \times \Gamma \longrightarrow (M_{dis} \times \Gamma) \times_M (M_{dis} \times \Gamma)$. This functor is an isomorphism of Lie groupoids, and hence essentially surjective and fully faithful. □

3.7 Equivalence between Bundle Gerbes and 2-Bundles

In this section we show that Versions III and IV of smooth Γ-gerbes are equivalent in the strongest possible sense:

Theorem 3.7.1. *For M a smooth manifold and Γ a Lie 2-group, there is an equivalence of bicategories*

$$\mathcal{G}rb_\Gamma(M) \cong 2\text{-}\mathcal{B}un_\Gamma(M).$$

between the bicategories of Γ-bundle gerbes and principal Γ-2-bundles over M. This equivalence is natural in M, i.e. it is an equivalence between pre-2-stacks.

Since the definitions of the bicategories $\mathcal{G}rb_\Gamma(M)$ and $2\text{-}\mathcal{B}un_\Gamma(M)$, *and* the above equivalence are all natural in M, we obtain automatically an induced equivalence for the induced bicategories over Lie groupoids (see Remarks 3.5.6 and 3.6.10).

Corollary 3.7.2. *For $\mathcal{X}$ a Lie groupoid and Γ a Lie 2-group, there is an equivalence*

$$\mathcal{G}rb_\Gamma(\mathcal{X}) \cong 2\text{-}\mathcal{B}un_\Gamma(\mathcal{X}).$$

The following outlines the proof of Theorem 3.7.1. In Section 3.7.1 we construct explicitly a 2-functor

$$\mathcal{E}_M : 2\text{-}\mathcal{B}un_\Gamma(M) \longrightarrow \mathcal{G}rb_\Gamma(M).$$

We then use a general criterion assuring that $\mathcal{E}_M$ is an equivalence of bicategories. This criterion is stated in Lemma 3.8.4: it requires (A) that $\mathcal{E}_M$ is fully faithful on Hom-categories, and (B) to choose certain preimages of objects and 1-morphisms under $\mathcal{E}_M$. Under these circumstances, Lemma 3.8.4 constructs an inverse 2-functor $\mathcal{R}_M$ together with the required pseudonatural transformations assuring that $\mathcal{E}_M$ and $\mathcal{R}_M$ form an equivalence of bicategories. Condition (A) is proved as Lemma 3.7.9 in Section 3.7.1. The choices (B) are constructed in Section 3.7.2.

In order to prove that the 2-functors $\mathcal{E}_M$ extend to the claimed equivalence between pre-2-stacks, we use another criterion stated in Lemma 3.8.6. The only additional assumption of Lemma 3.8.6 is that the given 2-functors $\mathcal{E}_M$ form a 1-morphism of pre-2-stacks; this is proved in Proposition 3.7.10. Then, the inverse 2-functors $\mathcal{R}_M$ obtained before automatically form an inverse 1-morphism between pre-2-stacks.

3.7.1 From Principal 2-Bundles to Bundle Gerbes

In this section we define the 2-functor $\mathcal{E}_M : 2\text{-}\mathcal{B}un_\Gamma(M) \longrightarrow \mathcal{G}rb_\Gamma(M)$.

Definition of $\mathcal{E}_M$ on objects

Let $\mathcal{P}$ be a principal Γ-2-bundle over M, with projection $\pi : \mathcal{P} \longrightarrow M$ and right action R of Γ on $\mathcal{P}$. The first ingredient of the Γ-bundle gerbe $\mathcal{E}_M(\mathcal{P})$ is the surjective submersion $\pi : \mathcal{P}_0 \longrightarrow M$. The second ingredient is a principal Γ-bundle P over $\mathcal{P}_0^{[2]}$. We put

$$P := \mathcal{P}_1 \times \Gamma_0.$$

Bundle projection, anchor and Γ-action are given, respectively, by

$$\chi(\rho, g) := (t(\rho), R(s(\rho), g^{-1})) \quad , \quad \alpha(\rho, g) := g \quad \text{and} \quad (\rho, g) \circ \gamma := (R(\rho, \mathrm{id}_{g^{-1}} \cdot \gamma), s(\gamma)). \tag{3.14}$$

These definitions are motivated by Remark 3.7.4 below.

Lemma 3.7.3. *This defines a principal* Γ*-bundle over* $\mathcal{P}_0^{[2]}$.

Proof. First we check that $\chi : P \longrightarrow \mathcal{P}_0^{[2]}$ is a surjective submersion. Since the functor $\tau = (\mathrm{id}, R)$ is a weak equivalence, we know from Theorem 3.2.23 that

$$f : (\mathcal{P}_0 \times \Gamma_0)\, {}_\tau\!\times_{t\times t} \mathcal{P}_1^{[2]} \longrightarrow \mathcal{P}_0^{[2]} : (p, g, \rho_1, \rho_2) \longmapsto (s(\rho_1), s(\rho_2))$$

is a surjective submersion. Now consider the smooth surjective map

$$g : (\mathcal{P}_0 \times \Gamma_0)\, {}_\tau\!\times_{t\times t} \mathcal{P}_1^{[2]} \longrightarrow \mathcal{P}_1 \times \Gamma_0 : (p, g, \rho_1, \rho_2) \longmapsto (\rho_1^{-1} \circ R(\rho_2, \mathrm{id}_{g^{-1}}), g^{-1}).$$

We have $\chi \circ g = f$; thus, χ is a surjective submersion. Next we check that we have defined an action. Suppose $(\rho, g) \in P$ and $\gamma \in \Gamma_1$ such that $\alpha(\rho, g) = g = t(\gamma)$. Then, $\alpha((\rho, g) \circ \gamma) = s(\gamma)$. Moreover, suppose $\gamma_1, \gamma_2 \in \Gamma_1$ with $t(\gamma_1) = g$ and $t(\gamma_2) = s(\gamma_1)$. Then,

$$\begin{aligned} ((\rho, g) \circ \gamma_1) \circ \gamma_2 &= (R(\rho, \mathrm{id}_{g^{-1}} \cdot \gamma_1), s(\gamma_1)) \circ \gamma_2 \\ &= (R(\rho, \mathrm{id}_{g^{-1}} \cdot \gamma_1 \cdot \mathrm{id}_{s(\gamma_1)^{-1}} \cdot \gamma_2), s(\gamma_2)) = (\rho, g) \circ (\gamma_1 \circ \gamma_2), \end{aligned}$$

where we have used that $\gamma_1 \circ \gamma_2 = \gamma_1 \cdot \mathrm{id}_{s(\gamma_1)^{-1}} \cdot \gamma_2$ in any 2-group. It remains to check that the smooth map

$$\tilde{\tau} : P\, {}_\alpha\!\times_t \Gamma_1 \longrightarrow P\, {}_\chi\!\times_\chi P : ((\rho, g), \gamma) \longmapsto ((\rho, g), (\rho, g) \circ \gamma)$$

is a diffeomorphism. For this purpose, we consider the diagram

$$\begin{array}{ccc} & & \mathcal{P}_1^{[2]} \\ & & \big\downarrow{\scriptstyle s\times t} \\ (\mathcal{P}_0 \times \Gamma_0) \times (\mathcal{P}_0 \times \Gamma_0) & \xrightarrow[\tau\times\tau]{} & \mathcal{P}_0^{[2]} \times \mathcal{P}_0^{[2]} \end{array} \tag{3.15}$$

and claim that (a) $N_1 := P\,{}_\alpha\!\times_t \Gamma_1$ is a pullback of (3.15), (b) $N_2 := P\,{}_\chi\!\times_\chi P$ is a pullback of (3.15), and (c) that the unique map $N_1 \longrightarrow N_2$ is $\tilde{\tau}$. Thus, $\tilde{\tau}$ is a diffeomorphism.

In order to prove claim (a) we use again that the functor $\tau = (\mathrm{id}, R)$ is a weak equivalence, so that by Theorem 3.2.23 the triple $(\mathcal{P}_1 \times \Gamma_1, \tau, s \times t)$ is a pullback of (3.15). We consider the smooth map

$$\xi : N_1 \longrightarrow \mathcal{P}_1 \times \Gamma_1 : ((\rho, g), \gamma) \longmapsto (R(\rho, \mathrm{id}_{g^{-1}}), \gamma)$$

which is a diffeomorphism because $(\rho, \gamma) \longmapsto ((R(\rho, \mathrm{id}_{t(\gamma)}), t(\gamma)), \gamma)$ is a smooth map which is inverse to ξ. Thus, putting $f_1 := \tau \circ \xi$ and $g_1 := (s \times t) \circ \xi$ we see that (N_1, f_1, g_1) is a pullback of (3.15). In order to prove claim (b), we put

$$\begin{aligned} f_2((\rho_1, g_1), (\rho_2, g_2)) &:= (R(\rho_1, \mathrm{id}_{g_1^{-1}}), \rho_2) \\ g_2((\rho_1, g_1), (\rho_2, g_2)) &:= (R(s(\rho), g_1^{-1}), g_2, R(t(\rho_1), g_1^{-1}), g_1), \end{aligned}$$

and it is straightforward to check that the cone (N_2, f_2, g_2) makes (3.15) commutative. The triple (N_2, f_2, g_2) is also universal: in order to see this suppose N' is any smooth manifold with smooth maps $f' : N' \longrightarrow \mathcal{P}_1^{[2]}$ and $g' : N' \longrightarrow (\mathcal{P}_0 \times \Gamma_0) \times (\mathcal{P}_0 \times \Gamma_0)$ so that (3.15) is commutative. For $n \in N'$, we write $f'(n) = (\rho_1, \rho_2)$ and $g'(n) = (p_1, g_1, p_2, g_2)$. Then, $\sigma(n) := ((R(\rho_1, \mathrm{id}_{g_2^{-1}}), g_2), (\rho_2, g_1))$ defines a smooth map $\sigma : N' \longrightarrow P\,{}_\chi\!\times_\chi P$. One checks that $f_2 \circ \sigma = f'$ and $g_2 \circ \sigma = g'$, and that σ is the only smooth map satisfying these equations. This proves that (N_2, f_2, g_2) is a pullback. We are left with claim (c). Here one only has to check that $\tau : N_1 \longrightarrow N_2$ satisfies $f_2 = f_1 \circ \tau$ and $g_2 = g_1 \circ \tau$. □

Remark 3.7.4. *The smooth functor* $\tau = (\mathrm{id}, R) : \mathcal{P} \times \Gamma \longrightarrow \mathcal{P} \times_M \mathcal{P}$ *is a weak equivalence, and so has a canonical inverse anafunctor* τ^{-1} *(Remark 3.2.24). The anafunctor*

$$\mathcal{P}_0^{[2]} \xrightarrow{\iota} \mathcal{P} \times_M \mathcal{P} \xrightarrow{c} \mathcal{P} \times_M \mathcal{P} \xrightarrow{\tau^{-1}} \mathcal{P} \times \Gamma \xrightarrow{\mathrm{pr}_2} \Gamma,$$

where c *is the functor that switches the factors, corresponds to a principal* Γ*-bundle over* $\mathcal{P}_0^{[2]}$ *that is canonically isomorphic to the bundle* P *defined above.*

It remains to provide the bundle gerbe product

$$\mu : \pi_{23}^* P \otimes \pi_{12}^* P \longrightarrow \pi_{13}^* P,$$

which we define by the formula

$$\mu((\rho_{23}, g_{23}), (\rho_{12}, g_{12})) := (\rho_{12} \circ R(\rho_{23}, \mathrm{id}_{g_{12}}), g_{23} g_{12}). \tag{3.16}$$

Lemma 3.7.5. *Formula (3.16) defines an associative isomorphism $\mu : \pi_{23}^* P \otimes \pi_{12}^* P \longrightarrow \pi_{13}^* P$ of principal Γ-bundles over $\mathcal{P}_0^{[3]}$.*

Proof. First of all, we recall from Example 3.2.31 (b) that an element in the tensor product $\pi_{23}^* P \otimes \pi_{12}^* P$ is represented by a triple (p_{23}, p_{12}, γ) where $p_{23}, p_{12} \in P$ with $\pi_1(\chi(p_{23})) = \pi_2(\chi(p_{12}))$, and $\alpha(p_{23}) \cdot \alpha(p_{12}) = t(\gamma)$. In (3.16) we refer to triples where $\gamma = \mathrm{id}_{g_{23} g_{12}}$, and this definition extends to triples with general $\gamma \in \Gamma_1$ by employing the equivalence relation

$$(p_1, p_2, \gamma) \sim (p_1 \circ (\gamma \cdot \mathrm{id}_{\alpha(p_2)^{-1}}), p_2, \mathrm{id}_{s(\gamma)}). \tag{3.17}$$

The complete formula for μ is then

$$\mu((\rho_{23}, g_{23}), (\rho_{12}, g_{12}), \gamma) = (\rho_{12} \circ R(\rho_{23}, \mathrm{id}_{g_{23}^{-1}} \cdot \gamma), s(\gamma)). \tag{3.18}$$

Next we check that (3.18) is well-defined under the equivalence relation (3.17):

$$\begin{aligned}
&\mu(((\rho_{23}, g_{23}), (\rho_{12}, g_{12}), \gamma)) \\
&\qquad = (\rho_{12} \circ R(\rho_{23}, \mathrm{id}_{g_{23}^{-1}} \cdot \gamma), s(\gamma)) \\
&\qquad = (\rho_{12} \circ R(\rho_{23} \circ R(\mathrm{id}_{R(s(\rho_{23}), g_{23}^{-1})}, \gamma \cdot \mathrm{id}_{g_{12}^{-1}}), \mathrm{id}_{g_{12}}), s(\gamma)) \\
&\qquad = \mu((\rho_{23} \circ R(\mathrm{id}_{R(s(\rho_{23}), g_{23}^{-1})}, \gamma \cdot \mathrm{id}_{g_{12}^{-1}}), s(\gamma) g_{12}^{-1}), (\rho_{12}, g_{12}), \mathrm{id}_{s(\gamma)})) \\
&\qquad = \mu(((\rho_{23}, g_{23}) \circ (\gamma \cdot \mathrm{id}_{g_{12}^{-1}}), (\rho_{12}, g_{12}), \mathrm{id}_{s(\gamma)})).
\end{aligned}$$

Now we have shown that μ is a well-defined map from $\pi_{23}^* P \otimes \pi_{12}^* P$ to $\pi_{13}^* P$, and it remains to prove that it is a bundle morphism. Checking that it preserves fibres and anchors is straightforward. It remains to check that (3.18) preserves the Γ-action. We calculate

$$\begin{aligned}
&\mu(((\rho_{23}, g_{23}), (\rho_{12}, g_{12}), \gamma) \circ \tilde{\gamma}) \\
&\qquad = \mu((\rho_{23}, g_{23}), (\rho_{12}, g_{12}), \gamma \circ \tilde{\gamma}) \\
&\qquad = (\rho_{23} \circ R(\rho_{12}, \mathrm{id}_{g_{12}} \cdot i(\gamma \circ \tilde{\gamma})), s(\tilde{\gamma})) \\
&\qquad = (\rho_{23} \circ R(R(\rho_{12}, \mathrm{id}_{g_{12}}), i(\gamma) \circ i(\tilde{\gamma})), s(\tilde{\gamma})) \\
&\qquad = (\rho_{23} \circ R(R(\rho_{12}, \mathrm{id}_{g_{12}}), i(\gamma))) \circ R(\mathrm{id}_{R(s(\rho_{12}), g)}, i(\tilde{\gamma})), s(\tilde{\gamma})) \\
&\qquad = (\rho_{23} \circ R(\rho_{12}, \mathrm{id}_{g_{12}} \cdot i(\gamma)) \circ R(\mathrm{id}_{R(s(\rho_{12}), g)}, i(\tilde{\gamma})), s(\tilde{\gamma})) \\
&\qquad = (\rho_{23} \circ R(\rho_{12}, \mathrm{id}_{g_{12}} \cdot i(\gamma)), s(\gamma)) \circ \tilde{\gamma} \\
&\qquad = \mu((\rho_{23}, g_{23}), (\rho_{12}, g_{12}), \gamma) \circ \tilde{\gamma}.
\end{aligned}$$

Summarizing, μ is a morphism of Γ-bundles over $\mathcal{P}_0^{[3]}$. The associativity of μ follows directly from the definitions.

$\square$

Definition of $\mathscr{E}_M$ on 1-morphisms

We define a 1-morphism $\mathscr{E}_M(F) : \mathscr{E}_M(\mathcal{P}) \longrightarrow \mathscr{E}_M(\mathcal{P}')$ between Γ-bundle gerbes from a 1-morphism $F : \mathcal{P} \longrightarrow \mathcal{P}'$ between principal Γ-2-bundles. The refinement of the surjective submersions $\pi : \mathcal{P} \longrightarrow M$ and $\pi' : \mathcal{P}' \longrightarrow M$ is the fibre product $Z := \mathcal{P}_0 \times_M \mathcal{P}'_0$. Its principal Γ-bundle has the total space

$$Q := F \times \Gamma_0,$$

and its projection, anchor and Γ-action are given, respectively, by

$$\chi(f,g) := (\alpha_l(f), R(\alpha_r(f), g^{-1})), \quad \alpha(f,g) := g \quad \text{and} \quad (f,g) \circ \gamma := (\rho(f, \mathrm{id}_{g^{-1}} \cdot \gamma), s(\gamma)), \tag{3.19}$$

where $\rho : F \times \Gamma_1 \longrightarrow F$ denotes the Γ_1-action on F that comes from the given Γ-equivariant structure on F (see Appendix 3.8.1).

Lemma 3.7.6. *This defines a principal* Γ*-bundle* Q *over* Z*.*

Proof. We show first the the projection $\chi : Q \longrightarrow Z$ is a surjective submersion. Since the functor $\tau' : \mathcal{P}' \times \Gamma \longrightarrow \mathcal{P} \times_M \mathcal{P}$ is a weak equivalence, we have by Theorem 3.2.23 a pullback

$$\begin{array}{ccc} X & \longrightarrow & (\mathcal{P}'_0 \times \Gamma_0)\, {}_R\!\times_t (\mathcal{P}'_1 \times_M \mathcal{P}'_1) \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle s \circ \mathrm{pr}_2} \\ F\, {}_{\pi' \circ \alpha_l(f)}\!\times_{\pi'} \mathcal{P}'_0 & \longrightarrow & \mathcal{P}'_0 \times_M \mathcal{P}'_0 \end{array}$$

along the bottom map $(f,p') \longmapsto (\alpha_r(f), p')$, which is well-defined because the ana-functor F preserves the projections to M (see Remark 3.6.6 (b)). In particular, the map ξ is a surjective submersion. It is easy to see that the smooth map

$$k : X \longrightarrow F \times \Gamma_0 : ((f,p'), (p'_0, g, \rho, \tilde{\rho})) \longmapsto (f \circ \rho^{-1} \circ R(\tilde{\rho}, \mathrm{id}_{g^{-1}}), g^{-1})$$

is surjective. Now we consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\;k\;} & F \times \Gamma_0 \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \chi} \\ F\, {}_{\pi' \circ \alpha_l(f)}\!\times_{\pi'} \mathcal{P}_0 & \xrightarrow[\alpha_l \times \mathrm{id}]{} & \mathcal{P}_0 \times_M \mathcal{P}'_0. \end{array}$$

The surjectivity of k and the fact that ξ and $\alpha_l \times \mathrm{id}$ are surjective submersions shows that χ is one, too.

Next, one checks (similarly as in the proof of Lemma 3.7.3) that the Γ-action on Q defined above is well-defined and preserves the projection. Then it remains to check that the smooth map

$$\xi \;:\; Q\,{}_{\alpha}\!\times_t \Gamma_1 \longrightarrow Q \times_{\mathcal{P}_0 \times_M \mathcal{P}_0'} Q : (f,g,\gamma) \longmapsto (f,g,\rho(f,\mathrm{id}_{g^{-1}} \cdot \gamma), s(\gamma))$$

is a diffeomorphism. An inverse map is given as follows. For a given element (f_1,g_1,f_2,g_2) on the right hand side, we have $\alpha_l(f_1) = \alpha_l(f_2)$, so that there exists a unique element $\rho' \in \mathcal{P}_1'$ such that $f_1 \circ \rho' = f_2$. One calculates that (ρ', g_2) and $(\mathrm{id}_{\alpha_r(f_1)}, g_1)$ are elements of the principal Γ-bundle $\mathcal{P}' \times \Gamma_0$ over $\mathcal{P}_0'^{[2]}$ of Lemma 3.7.3. Thus, there exists a unique element $\gamma \in \Gamma_1$ such that $(\rho', g_2) = (\mathrm{id}_{\alpha_r(f_1)}, g_1) \circ \gamma$. Clearly, $t(\gamma) = g_1$ and $s(\gamma) = g_2$, and we have $\rho' = R(\mathrm{id}_{\alpha_r(f_1)}, \mathrm{id}_{g_1^{-1}} \cdot \gamma)$. We define $\xi^{-1}(f_1,g_1,f_2,g_2) := (f_1,g_1,\gamma)$. The calculation that ξ^{-1} is an inverse for ξ uses property (ii) of Definition 3.8.1 for the action ρ, and is left to the reader. $\square$

The next step in the definition of the 1-morphism $\mathcal{E}(F)$ is to define the bundle morphism

$$\beta : P' \otimes \zeta_1^* Q \longrightarrow \zeta_2^* Q \otimes P$$

over $Z \times_M Z$. We use the notation of Example 3.2.31 (b) for elements of tensor products of principal Γ-bundles; in this notation, the morphism β in the fibre over a point $((p_1,p_1'),(p_2,p_2')) \in Z \times_M Z$ is given by

$$\beta : ((\rho',g'),(f,g),\gamma) \longmapsto ((\tilde{f}, g'gh), (\tilde{\rho}, h^{-1}), \gamma),$$

where $h \in \Gamma_0$ and $\tilde{\rho} \in \mathcal{P}_1'$ are chosen such that $s(\tilde{\rho}) = R(p_2, h^{-1})$ and $t(\tilde{\rho}) = p_1$, and

$$\tilde{f} := \rho(\tilde{\rho}^{-1} \circ f \circ R(\rho', \mathrm{id}_g), \mathrm{id}_h). \tag{3.20}$$

Lemma 3.7.7. *This defines an isomorphism between principal* Γ*-bundles.*

Proof. The existence of choices of $\tilde{\rho}, h$ follows because the functor $\tau' : \mathcal{P}' \times \Gamma \longrightarrow \mathcal{P}' \times_M \mathcal{P}'$ is smoothly essentially surjective (Theorem 3.2.23); in particular, one can choose them locally in a smooth way. We claim that the equivalence relation on $\zeta_2^* Q \otimes P$ identifies different choices; thus, we have a well-defined smooth map. In order to prove this claim, we assume other choices $\tilde{\rho}', h'$. The pairs $(\tilde{\rho}, h^{-1})$ and $(\tilde{\rho}', h'^{-1})$ are elements in the principal Γ-bundle P' over $\mathcal{P}_0' \times_M \mathcal{P}_0'$ and sit over the same fibre; thus, there exists a unique $\tilde{\gamma} \in \Gamma_1$ such that $(\tilde{\rho}, h^{-1}) \circ \tilde{\gamma} = (\tilde{\rho}', h'^{-1})$, in particular, $R(\tilde{\rho}, \mathrm{id}_h \cdot \tilde{\gamma}) = \tilde{\rho}'$. Now we have

$$\begin{aligned} ((\tilde{f}, g'gh), (\tilde{\rho}, h^{-1}), \gamma) &= ((\tilde{f}, g'gh), (\tilde{\rho}, h^{-1}), (\mathrm{id}_{t(\gamma)} \cdot i(\tilde{\gamma}) \cdot \tilde{\gamma}) \circ \gamma) \\ &\sim ((\tilde{f}, g'gh) \circ (\mathrm{id}_{t(\gamma)} \cdot i(\tilde{\gamma})), (\tilde{\rho}, h^{-1}) \circ \tilde{\gamma}, \gamma) \end{aligned}$$

so that it suffices to calculate

$$\begin{aligned}(\tilde{f}, g'gh) \circ (\mathrm{id}_{t(\gamma)} \cdot i(\tilde{\gamma})) &= (\rho(\tilde{f}, \mathrm{id}_{h^{-1}} \cdot i(\tilde{\gamma})), g'gh') \\ &= (\rho(\tilde{\rho}^{-1} \circ f \circ R(\rho', \mathrm{id}_g), i(\tilde{\gamma})), g'gh') \\ &= (\rho(R(\tilde{\rho}^{-1}, i(\tilde{\gamma}) \cdot \mathrm{id}_{h'^{-1}}) \circ f \circ R(\rho', \mathrm{id}_g), \mathrm{id}_{h'}), g'gh'),\end{aligned}$$

where the last step uses the compatibility condition for ρ from Definition 3.8.1 (ii). In any 2-group, we have $i(\tilde{\gamma}) \cdot \mathrm{id}_{s(\tilde{\gamma})} = (\mathrm{id}_{t(\tilde{\gamma})^{-1}} \cdot \tilde{\gamma})^{-1}$, in which case the last line is exactly the formula (3.20) for the pair $(\tilde{\rho}', h')$.

Next we check that β is well-defined under the equivalence relation on the tensor product $P' \otimes \zeta_1^* Q$. We have

$$x := ((\rho', g'), (f, g), (\gamma_1 \cdot \gamma_2) \circ \gamma) \sim ((\rho', g') \circ \gamma_1, (f, g) \circ \gamma_2, \gamma) =: x'$$

for $\gamma_1, \gamma_2 \in \Gamma_1$ such that $t(\gamma_1) = g'$, $t(\gamma_2) = g$ and $s(\gamma_1)s(\gamma_2) = t(\gamma)$. Taking advantage of the fact that we can make the same choice of $(\tilde{\rho}, h)$ for both representatives x and x', it is straightforward to show that $\beta(x) = \beta(x')$. Finally, it is obvious from the definition of β that it is anchor-preserving and Γ-equivariant. □

In order to show that the triple (Z, Q, β) defines a 1-morphism between bundle gerbes, it remains to verify that the bundle isomorphism β is compatible the with the bundle gerbe products μ_1 and μ_2 in the sense of diagram (3.8). This is straightforward to do and left for the reader.

Definition of $\mathcal{E}_M$ on 2-morphisms, compositors and unitors

Let $F_1, F_2 : \mathcal{P} \longrightarrow \mathcal{P}'$ be 1-morphisms between principal Γ-bundles over M, and let $\eta : F \Longrightarrow G$ be a 2-morphism. Between the Γ-bundles Q_1 and Q_2, which live over the same common refinement $Z = \mathcal{P}_0 \times_M \mathcal{P}'_0$, we find immediately the smooth map

$$\eta : Q_1 \longrightarrow Q_2 : (f_1, g) \longmapsto (\eta(f_1), g)$$

which is easily verified to be a bundle morphism. Its compatibility with the bundle morphisms β_1 and β_2 in the sense of the simplified diagram (3.11) is also easy to check. Thus, we have defined a 2-morphism $\mathcal{E}_M(\eta) : \mathcal{E}_M(F_1) \Longrightarrow \mathcal{E}_M(F_2)$.

The compositor for 1-morphisms $F_1 : \mathcal{P} \longrightarrow \mathcal{P}'$ and $F_2 : \mathcal{P}' \longrightarrow \mathcal{P}''$ is a bundle gerbe 2-morphism

$$c_{F_1,F_2} : \mathcal{E}_M(F_2 \circ F_1) \longrightarrow \mathcal{E}_M(F_2) \circ \mathcal{E}_M(F_1).$$

Employing the above constructions, the 1-morphism $\mathcal{E}_M(F_2 \circ F_1)$ is defined on the common refinement $Z_{12} := \mathcal{P}_0 \times_M \mathcal{P}''_0$ and has the Γ-bundle $Q_{12} = (F_1 \times_{\mathcal{P}'_0} F_2)/\mathcal{P}'_1 \times \Gamma_0$, whereas the 1-morphism $\mathcal{E}_M(F_2) \circ \mathcal{E}_M(F_1)$ is defined on the common refinement $Z := \mathcal{P}_0 \times_M \mathcal{P}'_0 \times_M \mathcal{P}''_0$ and has the Γ-bundle $Q_2 \otimes Q_1$ with $Q_k = F_k \times \Gamma_0$. The compositor c_{F_1,F_2} is defined over the refinement Z with the obvious refinement maps $\mathrm{pr}_{13} : Z \longrightarrow Z_{12}$ and $\mathrm{id} : Z \longrightarrow Z$ making diagram (3.10) commutative. It is thus a bundle morphism $c_{F_1,F_2} : \mathrm{pr}_{13}^* Q_{12} \longrightarrow Q_2 \otimes Q_1$. For elements in a tensor product of Γ-bundles we use the notation of Example 3.2.31 (b). Then, we define c_{F_1,F_2} by

$$((p,p',p''),(f_1,f_2,g)) \longmapsto ((\rho_2(\tilde{\rho}^{-1} \circ f_2, \mathrm{id}_h), gh), (f_1 \circ \tilde{\rho}, h^{-1}), \mathrm{id}_g), \tag{3.21}$$

where $h \in \Gamma_0$ and $\tilde{\rho} : R(p', h^{-1}) \longrightarrow \alpha_r(f_1) = \alpha_l(f_2)$ are chosen in the same way as in the proof of Lemma 3.7.7. The assignment (3.21) does not depend on the choices of h and $\tilde{\rho}$, and also not on the choice of the representative (f_1, f_2) in $(F_1 \times_{\mathcal{P}_0'} F_2)/\mathcal{P}_1'$. It is obvious that (3.21) is anchor-preserving, and its Γ-equivariance can be seen by choosing $(\tilde{\rho}, h)$ in order to compute $c_{F_1,F_2}((p,p',p''),(f_1,f_2,g))$ and $(\tilde{\rho}', h)$ with $\tilde{\rho}' := R(\tilde{\rho}, \mathrm{id}_{g^{-1}} \cdot \gamma^{-1})$ in order to compute $c_{F_1,F_2}(((p,p',p''),(f_1,f_2,g)) \circ \gamma)$. In order to complete the construction of the bundle gerbe 2-morphism c_{F_1,F_2} we have to prove that the bundle morphism c_{F_1,F_2} is compatible with the isomorphisms β_{12} of $\mathscr{E}_M(F_2 \circ F_1)$ and $(\mathrm{id} \otimes \beta_1) \circ (\beta_2 \otimes \mathrm{id})$ of $\mathscr{E}_M(F_2) \circ \mathscr{E}_M(F_1)$ in the sense of diagram (3.11). We start with an element $((\rho'', g''), (f_{12}, g)) \in \mathscr{E}_M(\mathcal{P}'') \otimes \zeta_1^* Q_{12}$, where $f_{12} = (f_1, f_2)$. We have

$$\beta_{12}((\rho'', g''), (f_{12}, g)) = (\widetilde{f_{12}}, g''gh, \tilde{\rho}, h^{-1})$$

upon choosing $(\tilde{\rho}, h)$ as required in the definition of $\mathscr{E}_M(F_2 \circ F_1)$. Writing $\widetilde{f_{12}} = (\tilde{f}_1, \tilde{f}_2)$ further we have

$$(\zeta_2^* c_{F_1,F_2} \otimes \mathrm{id})((\widetilde{f_{12}}, g''gh, \tilde{\rho}, h^{-1})) = (\rho_2(\tilde{\rho}_2^{-1} \circ \tilde{f}_2, \mathrm{id}_{h_2}), g''ghh_2, \tilde{f}_1 \circ \tilde{\rho}_2, h_2^{-1}, \tilde{\rho}, h^{-1}) \tag{3.22}$$

upon choosing appropriate $(\tilde{\rho}_2, h_2)$ as required in the definition of c_{F_1,F_2}. This is the result of the clockwise composition of diagram (3.11). Counter-clockwise, we first get

$$(\mathrm{id} \otimes \zeta_1^* c_{F_1,F_2})((\rho'', g''), (f_{12}, g)) = (\rho'', g'', f'', gh_1, f', h_1^{-1})$$

for choices $(\tilde{\rho}_1, h_1)$, where $f'' := \rho_2(\tilde{\rho}_1^{-1} \circ f_2, \mathrm{id}_{h_1})$ and $f' := f_1 \circ \tilde{\rho}_1$. Next we apply the isomorphism β_2 of $\mathscr{E}_M(F_2)$ and get

$$(\beta_2 \otimes \mathrm{id})(\rho'', g'', f'', gh_1, f_1', h_1^{-1}) = (\widetilde{f''}, g''ghh_2, \hat{\rho}, \hat{h}^{-1}, f_1', h_1^{-1})$$

where we have used the choices $(\hat{\rho}, \hat{h})$ defined by $\hat{\rho} := R(\tilde{\rho}_1^{-1}, h_1) \circ R(\tilde{\rho}_2, h^{-1}h_1)$ and $\hat{h} := h_1^{-1} h h_2$. The last step is to apply the isomorphism β_1 of $\mathscr{E}_M(F_2)$ which gives

$$(\mathrm{id} \otimes \beta_1)(\widetilde{f''}, g''ghh_2, \hat{\rho}, \hat{h}^{-1}, f_1', h_1^{-1}) = (\widetilde{f''}, g''ghh_2, \widetilde{f'}, h_2^{-1}, \tilde{\rho}, h^{-1}), \tag{3.23}$$

where we have used the choices $(\tilde{\rho}, h)$ from above. Comparing (3.22) and (3.23), we have obviously coincidence in all but the first and the third component. For these remaining factors, coincidence follows from the definitions of the various variables.

Finally, we have to construct unitors. The unitor for a principal Γ-2-bundle $\mathcal{P}$ over M is a bundle gerbe 2-morphism

$$u_{\mathcal{P}} : \mathcal{E}_M(\mathrm{id}_{\mathcal{P}}) \Longrightarrow \mathrm{id}_{\mathcal{E}_M(\mathcal{P})}.$$

Abstractly, one can associate to $\mathrm{id}_{\mathcal{E}_M(\mathcal{P})}$ the 1-morphism $\mathrm{id}^{FP}_{\mathcal{E}_M(\mathcal{P})}$ constructed in the proof of Lemma 3.5.18, and then notice that $\mathrm{id}^{FP}_{\mathcal{E}_M(\mathcal{P})}$ and $\mathcal{E}_M(\mathrm{id}_{\mathcal{P}})$ are canonically 2-isomorphic. In more concrete terms, the unitor $u_{\mathcal{P}}$ has the refinement $W := \mathcal{P}_0^{[3]}$ with the surjective submersions $r := \mathrm{pr}_{12}$ and $r' := \mathrm{pr}_3$ to the refinements $Z = \mathcal{P}_0^{[2]}$ and $Z' = \mathcal{P}_0$ of the 1-morphisms $\mathcal{E}_M(\mathrm{id}_{\mathcal{P}})$ and $\mathrm{id}_{\mathcal{E}_M(\mathcal{P})}$, respectively. The relevant maps x_W and y_W are pr_{13} and pr_{23}, respectively. The principal Γ-bundle of the 1-morphism $\mathrm{id}_{\mathcal{E}_M(\mathcal{P})}$ is the trivial bundle $Q' = \mathbf{I}_1$. We claim that the principal Γ-bundle Q of $\mathcal{E}_M(\mathrm{id}_{\mathcal{P}})$ is the bundle P of the bundle gerbe $\mathcal{E}_M(\mathcal{P})$. Indeed, the formulae (3.19) reduce for the identity anafunctor $\mathrm{id}_{\mathcal{P}}$ to those of (3.14). Now, the bundle isomorphism of the unitor $u_{\mathcal{P}}$ is

$$y_W^*P \otimes r^*Q = \mathrm{pr}_{23}^*P \otimes \mathrm{pr}_{12}^*P \xrightarrow{\mu} \mathrm{pr}_{13}^*P \cong r'^*Q' \otimes x_W^*P,$$

where μ is the bundle gerbe product of $\mathcal{E}_M(\mathcal{P})$. The commutativity of diagram (3.9) follows from the associativity of μ.

Proposition 3.7.8. *The assignments $\mathcal{E}_M$ for objects, 1-morphisms and 2-morphisms, together with the compositors and unitors defined above, define a 2-functor*

$$\mathcal{E}_M : 2\text{-}\mathcal{B}un_\Gamma(M) \longrightarrow \mathcal{G}rb_\Gamma(M).$$

Proof. A list of axioms for a 2-functor with the same conventions as we use here can be found in [SW08, Appendix A]. The first axiom requires that the 2-functor $\mathcal{E}_M$ respects the vertical composition of 2-morphisms – this follows immediately from the definition.

The second axiom requires that the compositors respect the horizontal composition of 2-morphisms. To see this, let $F_1, F_1' : \mathcal{P} \longrightarrow \mathcal{P}'$ and $F_2, F_2' : \mathcal{P}' \longrightarrow \mathcal{P}''$ be 1-morphisms between principal Γ-2-bundles, and let $\eta_1 : F_1 \Longrightarrow F_1'$ and $\eta_2 : F_2 \Longrightarrow F_2'$ be 2-morphisms. Then, the diagram

$$\begin{array}{ccc} \mathcal{E}_M(F_2 \circ F_1) & \xRightarrow{\mathcal{E}_M(\eta_1 \circ \eta_2)} & \mathcal{E}_M(F_2' \circ F_1') \\ {\scriptstyle c_{F_1,F_2}} \Big\Downarrow & & \Big\Downarrow {\scriptstyle c_{F_1',F_2'}} \\ \mathcal{E}_M(F_2) \circ \mathcal{E}_M(F_1) & \xRightarrow[\mathcal{E}_M(\eta_1) \circ \mathcal{E}_M(\eta_2)]{} & \mathcal{E}_M(F_2') \circ \mathcal{E}_M(F_1') \end{array}$$

has to commute. Indeed, in order to compute c_{F_1,F_2} and $c_{F_1',F_2'}$ one can make the same choice of $(\tilde{\rho}, h)$, because the transformations η and η_2 preserve the anchors. Then, commutativity follows from the fact that η_1 and η_2 commute with the groupoid actions and the Γ_1-action according to Definition 3.8.1.

The third axiom describes the compatibility of the compositors with the composition of 1-morphisms in the sense that the diagram

$$\begin{array}{ccc} \mathcal{E}_M(F_3 \circ F_2 \circ F_1) & \xrightarrow{c_{F_2\circ F_1, F_3}} & \mathcal{E}_M(F_3) \circ \mathcal{E}_M(F_2 \circ F_1) \\ {\scriptstyle c_{F_3\circ F_2, F_1}} \Downarrow & & \Downarrow {\scriptstyle \mathrm{id}\circ c_{F_2,F_1}} \\ \mathcal{E}_M(F_3 \circ F_2) \circ \mathcal{E}_M(F_1) & \xrightarrow[c_{F_3,F_2}\circ \mathrm{id}]{} & \mathcal{E}_M(F_3) \circ \mathcal{E}_M(F_2) \circ \mathcal{E}_M(F_1). \end{array}$$

is commutative. In order to verify this, one starts with an element (f_1, f_2, f_3, g) in $\mathcal{E}_M(F_3 \circ F_2 \circ F_1)$. In order to go clockwise, one chooses pairs $(\tilde{\rho}_{12,3}, h_{12,3})$ and $(\tilde{\rho}_{1,2}, h_{1,2})$ and gets from the definitions

$$\mathrm{CW} = ((\rho_3(\tilde{\rho}_{12,3}^{-1} \circ f_3, \mathrm{id}_{h_{12,3}}), gh_{12,3}), (\rho_2(\tilde{\rho}_{1,2}^{-1} \circ f_2 \circ \tilde{\rho}_{12,3}, \mathrm{id}_{h_{1,2}}), h_{12,3}^{-1} h_{1,2}), (f_1 \circ \tilde{\rho}_{1,2}, h_{1,2}^{-1})).$$

Counter-clockwise, one can choose firstly again the pair $(\tilde{\rho}_{1,2}, h_{1,2})$ and then the pair $(\tilde{\rho}_{2,3}, h_{2,3})$ with $\tilde{\rho}_{2,3} = R(\tilde{\rho}_{12,3}, \mathrm{id}_{h_{1,2}})$ and $h_{2,3} = h_{1,2}^{-1} h_{12,3}$. Then, one gets

$$\begin{aligned} \mathrm{CCW} = ((\rho_3(\tilde{\rho}_{2,3}^{-1} \circ \rho_3(f_3, \mathrm{id}_{h_{1,2}}), \mathrm{id}_{h_{2,3}}), gh_{1,2}h_{2,3}), \\ (\rho_2(\tilde{\rho}_{1,2}^{-1} \circ f_2, \mathrm{id}_{h_{1,2}}) \circ \tilde{\rho}_{2,3}, h_{2,3}^{-1}), (f_1 \circ \tilde{\rho}_{1,2}, h_{1,2}^{-1})), \end{aligned}$$

where one has to use formula (3.33) for the Γ_1-action on the composition of equivariant anafunctors. Using the definitions of $h_{2,3}$ and $\tilde{\rho}_{2,3}$ as well as the axiom of Definition 3.8.1 (ii) one can show that CW = CCW.

The fourth and last axiom requires that compositors and unitors are compatible with each other in the sense that for each 1-morphism $F : \mathcal{P} \longrightarrow \mathcal{P}'$ the 2-morphisms

$$\mathcal{E}_M(F) \cong \mathcal{E}_M(F \circ \mathrm{id}_{\mathcal{P}}) \xRightarrow{c_{\mathrm{id}_{\mathcal{P}}, F}} \mathcal{E}_M(F) \circ \mathcal{E}_M(\mathrm{id}_{\mathcal{P}}) \xRightarrow{\mathrm{id}\circ u_{\mathcal{P}}} \mathcal{E}_M(F) \circ \mathrm{id}_{\mathcal{E}_M(\mathcal{P})} \cong \mathcal{E}_M(F)$$

and

$$\mathcal{E}_M(F) \cong \mathcal{E}_M(\mathrm{id}_{\mathcal{P}'} \circ F) \xRightarrow{c_{F, \mathrm{id}_{\mathcal{P}'}}} \mathcal{E}_M(\mathrm{id}_{\mathcal{P}'}) \circ \mathcal{E}_M(F) \xRightarrow{u_{\mathcal{P}'}\circ \mathrm{id}} \mathrm{id}_{\mathcal{E}_M(\mathcal{P}')} \circ \mathcal{E}_M(F) \cong \mathcal{E}_M(F)$$

are the identity 2-morphisms. We prove this for the first one and leave the second as an exercise. Using the definitions, we see that the 2-morphism has the refinement $W := \mathcal{P}_0 \times_M \mathcal{P}_0 \times_M \mathcal{P}'_0$ with $r = \mathrm{pr}_{13}$ and $r' = \mathrm{pr}_{23}$. The maps $x_W : W \longrightarrow \mathcal{P}_0 \times_M \mathcal{P}_0$ and $y_W : W \longrightarrow \mathcal{P}'_0 \times_M \mathcal{P}'_0$ are pr_{12} and $\Delta \circ \mathrm{pr}_3$, respectively, where Δ is the diagonal map. Its bundle morphism is a morphism

$$\varphi : \mathrm{pr}_{13}^* Q \longrightarrow \mathrm{pr}_{23}^* Q \otimes \mathrm{pr}_{12}^* P,$$

where $Q = F \times \Gamma_0$ is the principal Γ-bundle of $\mathcal{E}_M(F)$, and $P = \mathcal{P}_1 \times \Gamma_0$ is the principal Γ-bundle of $\mathcal{E}_M(\mathcal{P})$. Over a point (p_1, p_2, p') and $(f, g) \in \mathrm{pr}_{13}^* Q$, i.e. $\alpha_l(f) = p_1$ and $R(\alpha_r(f), g^{-1}) = p'$, the bundle morphism φ is given by

$$(f, g) \longmapsto (\rho(\tilde{\rho}^{-1} \circ f, \mathrm{id}_h), gh, \tilde{\rho}, h^{-1}),$$

where $h \in \Gamma_0$, and $\tilde{\rho} \in \mathcal{P}_1$ with $s(\tilde{\rho}) = R(p_2, h^{-1})$ and $t(\tilde{\rho}) = \alpha_l(f)$. We have to compare (W, φ) with the identity 2-morphism of $\mathcal{E}_M(F)$, which has the refinement Z with $r = r' = \mathrm{id}$ and the identity bundle morphism. According to the equivalence relation on bundle gerbe 2-morphisms we have to evaluate φ over a point $w \in W$ with $r(w) = r'(w)$, i.e. w is of the form $w = (p, p, p')$. Here we can choose $h = 1$ and $\tilde{\rho} = \mathrm{id}_p$, in which case we have $\varphi(f, g) = ((f, g), (\mathrm{id}_p, 1))$. This is indeed the identity on Q. □

Properties of the 2-functor $\mathcal{E}_M$

For the proof of Theorem 3.7.1 we provide the following two statements.

Lemma 3.7.9. *The 2-functor $\mathcal{E}_M$ is fully faithful on Hom-categories.*

Proof. Let $\mathcal{P}, \mathcal{P}'$ be principal Γ-2-bundles over M, let $F_1, F_2 : \mathcal{P} \longrightarrow \mathcal{P}'$ be 1-morphisms. By Lemma 3.5.18 every 2-morphism $\eta : \mathcal{E}_M(F_1) \Longrightarrow \mathcal{E}_M(F_2)$ can be represented by one whose refinement is $\mathcal{P}_0 \times_M \mathcal{P}'_0$, so that its bundle isomorphism is $\eta : Q_1 \longrightarrow Q_2$, where $Q_k := F_k \times \Gamma$ for $k = 1, 2$. We can read of a map $\eta : F_1 \longrightarrow F_2$, and it is easy to see that this is a 2-morphism $\eta : F_1 \Longrightarrow F_2$. This procedure is clearly inverse to the 2-functor $\mathcal{E}_M$ on 2-morphisms. □

Proposition 3.7.10. *The 2-functors $\mathcal{E}_M$ form a 1-morphism between pre-2-stacks.*

Proof. For a smooth map $f : M \longrightarrow N$, we have to look at the diagram

$$\begin{array}{ccc} 2\text{-}\mathcal{B}un_\Gamma(N) & \xrightarrow{f^*} & 2\text{-}\mathcal{B}un_\Gamma(M) \\ {\scriptstyle \mathcal{E}_N}\downarrow & & \downarrow{\scriptstyle \mathcal{E}_M} \\ \mathcal{G}rb_\Gamma(N) & \xrightarrow[f^*]{} & \mathcal{G}rb_\Gamma(M) \end{array}$$

of 2-functors. For $\mathcal{P}$ a principal Γ-2-bundle over N, the Γ-bundle gerbe $\mathcal{E}_M(f^*\mathcal{P})$ has the surjective submersion $\mathrm{pr}_1 : Y := M \times_N \mathcal{P}_0 \longrightarrow M$, the principal Γ-bundle $P := M \times_N \mathcal{P}_1 \times \Gamma_0$ over $Y^{[2]}$, and a bundle gerbe product μ defined as in (3.16) that ignores the M-factor. On the other hand, the Γ-bundle gerbe $f^*\mathcal{E}_N(\mathcal{P})$ has the same surjective submersion, and – up to canonical identifications between fibre products – the same Γ-bundle and the same bundle gerbe product. These canonical identifications make up a pseudonatural transformation that renders the above diagram commutative. □

3.7.2 From Bundle Gerbes to Principal 2-Bundles

In this section we provide the data we will feed into Lemma 3.8.4 in order to produce a 2-functor $\mathcal{R}_M : \mathcal{G}rb_\Gamma(M) \longrightarrow 2\text{-}\mathcal{B}un_\Gamma(M)$ that is inverse to the 2-functor $\mathcal{E}_M$ constructed in the previous section. These data are:

1. A principal Γ-2-bundle $\mathcal{R}_\mathcal{G}$ for each Γ-bundle gerbe $\mathcal{G}$ over M.
2. A 1-isomorphism $\mathcal{A}_\mathcal{G} : \mathcal{G} \longrightarrow \mathcal{E}_M(\mathcal{R}_\mathcal{G})$ for each Γ-bundle gerbe $\mathcal{G}$ over M.
3. A 1-isomorphism $\mathcal{R}_\mathcal{A} : \mathcal{P} \longrightarrow \mathcal{P}'$ and a 2-isomorphism $\eta_\mathcal{A} : \mathcal{A} \Longrightarrow \mathcal{E}_M(\mathcal{R}_\mathcal{A})$ for all principal Γ-2-bundles $\mathcal{P}, \mathcal{P}'$ over M and all bundle gerbe 1-morphisms $\mathcal{A} : \mathcal{E}_M(\mathcal{P}) \longrightarrow \mathcal{E}_M(\mathcal{P}')$.

Construction of the principal Γ-2-bundle $\mathcal{R}_\mathcal{G}$

We assume that $\mathcal{G}$ consists of a surjective submersion $\pi : Y \longrightarrow M$, a principal Γ-bundle P over $Y^{[2]}$ and a bundle gerbe product μ. Let $\alpha : P \longrightarrow \Gamma_0$ be the anchor of P, and let $\chi : P \longrightarrow Y^{[2]}$ be the bundle projection.

The Lie groupoid $\mathcal{P}$ of the principal 2-bundle $\mathcal{R}_\mathcal{G}$ is defined by

$$\mathcal{P}_0 := Y \times \Gamma_0 \quad \text{and} \quad \mathcal{P}_1 := P \times \Gamma_0;$$

source map, target maps, and composition are given by, respectively,

$$s(p,g) := (\pi_2(\chi(p)), g) \quad , \quad t(p,g) := (\pi_1(\chi(p)), \alpha(p)^{-1} \cdot g)$$
$$\text{and} \quad (p_2, g_2) \circ (p_1, g_1) := (\mu(p_1, p_2), g_1). \tag{3.24}$$

The identity morphism of an object $(y,g) \in \mathcal{P}_0$ is $(t_y, g) \in \mathcal{P}_1$, where t_y denotes the unit element in P over the point (y,y), see Lemma 3.5.15. The inverse of a morphism $(p,g) \in \mathcal{P}_1$ is $(i(p), \alpha(p)^{-1}g)$, where $i : P \longrightarrow P$ is the map from Lemma 3.5.15. The bundle projection is $\pi(y,g) := \pi(y)$. The action is given on objects and morphisms by

$$R_0((y,g),g') := (y, gg') \quad \text{and} \quad R_1((p,g),\gamma) := \Big(p \circ \big(\mathrm{id}_g \cdot \gamma \cdot \mathrm{id}_{t(\gamma)^{-1}g^{-1}\alpha(p)}\big), g \cdot s(\gamma)\Big). \tag{3.25}$$

Lemma 3.7.11. *This defines a functor $R : \mathcal{P} \times \Gamma \longrightarrow \mathcal{P}$, and R is an action of* Γ *on* $\mathcal{P}$.

Proof. We assume that $t : H \longrightarrow G$ is a smooth crossed module, and that Γ is the Lie 2-group associated to it, see Example 3.2.26 and Remark 3.2.27. Then we use the correspondence between principal Γ-bundles and principal H-bundles with H-anti-equivariant maps to G of Lemma 3.2.11. Writing $\gamma = (h, g')$, we have

$$R_1((p,g),\gamma) = (p \star {}^g h, gg').$$

With this simple formula at hand it is straightforward to show that R respects source and target maps and satisfies the axiom of an action. For the composition, we assume composable $(p_2, g_2), (p_1, g_1) \in \mathcal{P}_1$, i.e. $g_2 = \alpha(p_1)^{-1}g_1$, and composable $(h_2, g_2'), (h_1, g_1') \in \Gamma_1$, i.e. $g_2' = t(h_1)g_1'$. Then we have

$$\begin{aligned} R((p_2, g_2) \circ (p_1, g_1), (h_2, g_2') \circ (h_1, g_1')) &= R((\mu(p_1, p_2), g_1), (h_2h_1, g_1')) \\ &= (\mu(p_1, p_2) \star {}^{g_1}(h_2h_1), g_1g_1') \\ &= (\mu(p_1 \star {}^{g_1}h_2, p_2) \star {}^{g_1}h_1, g_1g_1') \\ &= (\mu(p_1, p_2 \star {}^{g_2}h_2) \star {}^{g_1}h_1, g_1g_1') \\ &= (\mu(p_1 \star {}^{g_1}h_1, p_2 \star {}^{g_2}h_2), g_1g_1') \\ &= (p_2 \star {}^{g_2}h_2, g_2g_2') \circ (p_1 \star {}^{g_1}h_1, g_1g_1') \\ &= R((p_2, g_2), (h_2, g_2')) \circ R((p_1, g_1), (h_1, g_1')), \end{aligned}$$

finishing the proof. □

It is obvious that the action R preserves the projection π. Thus, in order to complete the construction of the principal 2-bundle $\mathcal{R}_{\mathcal{G}}$ it remains to show that the functor $\tau = (\mathrm{pr}_1, R)$ is a weak equivalence. This is the content of the following two lemmata in connection with Theorem 3.2.23.

Lemma 3.7.12. *τ is smoothly essentially surjective.*

Proof. The condition we have to check is whether or not the map

$$(Y \times \Gamma_0 \times \Gamma_0)\,{}_\tau\!\times_t ((P \times \Gamma_0) \times_M (P \times \Gamma_0)) \xrightarrow{(s\times s)\circ \mathrm{pr}_2} (Y \times \Gamma_0) \times_M (Y \times \Gamma_0)$$

is a surjective submersion. The left hand side is diffeomorphic to $(P \times \Gamma_0)\,{}_{\pi_1}\!\times_{\pi_1} (P \times \Gamma_0)$ via pr_2, so that this is equivalent to checking that

$$s \times s : (P \times \Gamma_0)\,{}_{\pi_1\circ\chi}\!\times_{\pi_1\circ\chi} (P \times \Gamma_0) \longrightarrow (Y \times \Gamma_0) \times_M (Y \times \Gamma_0)$$

is a surjective submersion. Since the Γ_0-factors are just spectators, this is in turn equivalent to checking that

$$(\pi_2 \times \pi_2) \circ (\chi \times \chi) : P\,{}_{\pi_1\circ\chi}\!\times_{\pi_1\circ\chi} P \longrightarrow Y^{[2]}$$

is a surjective submersion. It fits into the pullback diagram

$$\begin{array}{ccc} P\,{}_{\pi_1\circ\chi}\!\times_{\pi_1\circ\chi} P & \hookrightarrow & P \times P \\ \downarrow{\scriptstyle \chi\times\chi} & & \downarrow{\scriptstyle \chi\times\chi} \\ Y^{[2]}\,{}_{\pi_1}\!\times_{\pi_1} Y^{[2]} & \hookrightarrow & Y^{[2]} \times Y^{[2]} \\ \downarrow{\scriptstyle \pi_2\times\pi_2} & & \downarrow{\scriptstyle \pi_2\times\pi_2} \\ Y^{[2]} & \hookrightarrow & Y \times Y \end{array}$$

which has a surjective submersion on the right hand side; hence, also the map on the left hand side must be a surjective submersion. ☐

Lemma 3.7.13. *τ is smoothly fully faithful.*

Proof. We assume a smooth manifold N with two smooth maps

$$f : N \longrightarrow (\mathcal{P}_0 \times \Gamma_0) \times (\mathcal{P}_0 \times \Gamma_0) \quad \text{and} \quad g : N \longrightarrow \mathcal{P}_1 \times_M \mathcal{P}_1$$

such that the diagram

$$\begin{array}{ccc} N & \xrightarrow{\quad g \quad} & \mathcal{P}_1 \times_M \mathcal{P}_1 \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle s\times t} \\ (\mathcal{P}_0 \times \Gamma_0) \times (\mathcal{P}_0 \times \Gamma_0) & \xrightarrow[\tau\times\tau]{} & (\mathcal{P}_0 \times_M \mathcal{P}_0) \times (\mathcal{P}_0 \times_M \mathcal{P}_0) \end{array}$$

is commutative. For a fixed point $n \in N$ we put

$$((p_1, g_1), (p_2, g_2)) := g(n) \in (P \times \Gamma_0) \times_M (P \times \Gamma_0)$$

and

$$((y, g, \tilde{g}), (y', g', \tilde{g}')) := f(n) \in (Y \times \Gamma_0 \times \Gamma_0) \times (Y \times \Gamma_0 \times \Gamma_0).$$

The commutativity of the diagram implies $\chi(p_1) = \chi(p_2) = (y', y)$, so that there exists $\gamma' \in \Gamma_1$ with $p_2 = p_1 \circ \gamma'$. We define $\gamma := \mathrm{id}_{g_1^{-1}} \cdot \gamma' \cdot \mathrm{id}_{\alpha(p_2)^{-1}g_2}$, which yields a morphism $\gamma \in \Gamma_1$ satisfying $\tau(p_1, g_1, \gamma) = (p_1, g_1, p_2, g_2) = g(n)$. On the other hand, we check that

$$(s(p_1, g_1, \gamma), t(p_1, g_1, \gamma)) = (\pi_2(p_1), g_1, s(\gamma), \pi_1(p_1), \alpha(p_1)^{-1}g_1, t(\gamma)) = f(n),$$

using that $s(\gamma) = g_1^{-1}g_2$ and $t(\gamma) = g_1^{-1}\alpha(p_1)\alpha(p_2)^{-1}g_2$. Summarizing, we have defined a smooth map

$$\sigma : N \longrightarrow \mathcal{P}_1 \times \Gamma_1 : n \longmapsto (p_1, g_1, \gamma)$$

such that $\tau \circ \sigma = g$ and $(s \times t) \circ \sigma = f$. Now let $\sigma' : N \longrightarrow \mathcal{P}_1 \times \Gamma_1$ be another such map, and let $\sigma'(n) =: (p_1', g_1', \gamma')$. The condition that $\tau(\sigma(n)) = g(n) = \tau(\sigma'(n))$ shows immediately that $p_1 = p_1'$ and $g_1 = g_1'$, and then that $p_1 \circ \gamma = p_1 \circ \gamma'$. But since the Γ-action on P is principal, we have $\gamma = \gamma'$. This shows $\sigma = \sigma'$. Summarizing, $\mathcal{P}_1 \times \Gamma_1$ is a pullback. ☐

Example 3.7.14. *Suppose $\Gamma = \mathcal{B}\mathrm{U}(1)$, see Example 3.2.1 (b), and suppose $\mathcal{G}$ is a Γ-bundle gerbe over M, also known as a $\mathrm{U}(1)$-bundle gerbe, see Example 3.5.7. Then, the associated principal $\mathcal{B}\mathrm{U}(1)$-2-bundle $\mathcal{R}_{\mathcal{G}}$ has the groupoid $\mathcal{P}$ with $\mathcal{P}_0 = Y$ and $\mathcal{P}_1 = P$, source and target maps $s = \pi_2 \circ \chi$ and $t = \pi_1 \circ \chi$, and composition $p_2 \circ p_1 = \mu(p_1, p_2)$. The action of $\mathcal{B}\mathrm{U}(1)$ on $\mathcal{P}$ is trivial on the level of objects and the given $\mathrm{U}(1)$-action on P on the level of morphisms. The same applies for general abelian Lie groups A instead of $\mathrm{U}(1)$.*

Construction of the 1-isomorphism $\mathcal{A}_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathcal{E}_M(\mathcal{R}_{\mathcal{G}})$

The Γ-bundle gerbe $\mathcal{E}_M(\mathcal{R}_{\mathcal{G}})$ has the surjective submersion $\tilde{Y} := Y \times \Gamma_0$ with $\tilde{\pi}(y, g) := \pi(y)$. The total space of its Γ-bundle $\tilde{P}$ is $\tilde{P} := P \times \Gamma_0 \times \Gamma_0$; it has the anchor $\alpha(p, g, h) = h$, the bundle projection

$$\tilde{\chi} : \tilde{P} \longrightarrow \tilde{Y}^{[2]} : (p, g, h) \longmapsto ((\pi_1(\chi(p)), \alpha(p)^{-1}g), (\pi_2(\chi(p)), gh^{-1})),$$

the Γ-action is

$$\begin{aligned} (p, g, h) \circ \gamma &\overset{(3.14)}{=} ((p, g) \circ R((t_{\pi_2(\chi(p))}, gh^{-1}), \gamma), s(\gamma)) \\ &\overset{(3.25)}{=} ((p, g) \circ (t_{\pi_2(\chi(p))} \circ (\mathrm{id}_{gh^{-1}} \cdot \gamma \cdot \mathrm{id}_{g^{-1}}), gh^{-1}s(\gamma)), s(\gamma)) \\ &\overset{(3.24)}{=} (\mu(t_{\pi_2(\chi(p))} \circ \big(\mathrm{id}_{gh^{-1}} \cdot \gamma \cdot \mathrm{id}_{g^{-1}}\big), p), gh^{-1}s(\gamma), s(\gamma)) \\ &\overset{(3.5)}{=} (p \circ \big(\mathrm{id}_{gh^{-1}} \cdot \gamma \cdot \mathrm{id}_{g^{-1}\alpha(p)}\big), gh^{-1}s(\gamma), s(\gamma)), \end{aligned}$$

and its bundle gerbe product $\tilde{\mu}$ is given by

$$\begin{aligned} \tilde{\mu}((p_{23}, g_{23}, h_{23}), (p_{12}, g_{12}, h_{12})) &\overset{(3.16)}{=} ((p_{12}, g_{12}) \circ R((p_{23}, g_{23}), \mathrm{id}_{h_{12}}), h_{23}h_{12}) \\ &\overset{(3.25)}{=} ((p_{12}, g_{12}) \circ (p_{23}, g_{23}h_{12}), h_{23}h_{12}) \\ &\overset{(3.24)}{=} (\mu(p_{23}, p_{12}), g_{23}h_{12}, h_{23}h_{12}). \end{aligned}$$

In order to compare the bundle gerbes $\mathcal{G}$ and $\mathcal{E}_M(\mathcal{R}_{\mathcal{G}})$ we consider the smooth maps $\sigma : Y \longrightarrow Y \times \Gamma_0$ and $\tilde{\sigma} : P \longrightarrow \tilde{P}$ that are defined by $\sigma(y) := (y, 1)$ and $\tilde{\sigma}(p) := (p, \alpha(p), \alpha(p))$.

Lemma 3.7.15. *$\tilde{\sigma}$ defines an isomorphism $\tilde{\sigma} : P \longrightarrow (\sigma \times \sigma)^*\tilde{P}$ of Γ-bundles over $Y^{[2]}$. Moreover, the diagram*

$$\begin{array}{ccc} \pi_{23}^*P \otimes \pi_{12}^*P & \xrightarrow{\tilde{\sigma}\otimes\tilde{\sigma}} & \tilde{\pi}_{23}^*\tilde{P} \otimes \tilde{\pi}_{12}^*\tilde{P} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \tilde{\mu}} \\ \pi_{13}^*P & \xrightarrow[\tilde{\sigma}]{} & \tilde{\pi}_{13}^*\tilde{P} \end{array}$$

is commutative.

Proof. For the first part it suffices to prove that $\tilde{\sigma}$ is Γ-equivariant, preserves the anchors, and that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\tilde{\sigma}} & \tilde{P} \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \tilde{\chi}} \\ Y^{[2]} & \xrightarrow[\sigma\times\sigma]{} & \tilde{Y}^{[2]} \end{array}$$

is commutative. Indeed, the commutativity of the diagram is obvious, and also that the anchors are preserved. For the Γ-equivariance, we have

$$\tilde{\sigma}(p \circ \gamma) = (p \circ \gamma, s(\gamma), s(\gamma)) = (p, \alpha(p), \alpha(p)) \circ \gamma = \tilde{\sigma}(p) \circ \gamma.$$

Finally, we calculate

$$\begin{aligned} \tilde{\mu}((p_{23}, \alpha(p_{23}), \alpha(p_{23})) \quad , \quad & (p_{12}, \alpha(p_{12}), \alpha(p_{12}))) \\ &= (\mu(p_{23}, p_{12}), \alpha(p_{23})\alpha(p_{12}), \alpha(p_{23})\alpha(p_{12})) \\ &= (\mu(p_{23}, p_{12}), \alpha(\mu(p_{23}, p_{12})), \alpha(\mu(p_{23}, p_{12}))) \end{aligned}$$

which shows the commutativity of the diagram. □

Via Lemma 3.5.17 the bundle morphism $\tilde{\sigma}$ defines the required 1-morphism $\mathcal{A}_{\mathcal{G}}$, and Lemma 3.5.13 guarantees that $\mathcal{A}_{\mathcal{G}}$ is a 1-*iso*morphism.

Construction of the 1-morphism $\mathcal{R}_{\mathcal{A}} : \mathcal{P} \longrightarrow \mathcal{P}'$

Let $\mathcal{A} : \mathcal{E}_M(\mathcal{P}) \longrightarrow \mathcal{E}_M(\mathcal{P}')$ be a 1-morphism between Γ-bundle gerbes obtained from principal Γ-2-bundles $\mathcal{P}$ and $\mathcal{P}'$ over M. By Lemma 3.5.18 we can assume that $\mathcal{A}$ consists of a principal Γ-bundle $\chi : Q \longrightarrow Z$ with $Z = \mathcal{P}_0 \times_M \mathcal{P}'_0$, and some isomorphism β over $Z^{[2]}$. For preparation, we consider the fibre products $Z_r := \mathcal{P}_0 \times_M \mathcal{P}'^{[2]}_0$ and $Z_l := \mathcal{P}^{[2]}_0 \times_M \mathcal{P}'_0$ with the obvious embeddings $\iota_l : Z_l \longrightarrow Z$ and $\iota_r : Z_r \longrightarrow Z$ obtained by doubling elements. Together with the trivialization of Corollary 3.5.16, the pullbacks of β along ι_l and ι_r yield bundle morphisms

$$\beta_l := \iota_l^*\beta : \mathrm{pr}_{13}^*Q \longrightarrow \mathrm{pr}_{23}^*Q \otimes \mathrm{pr}_{12}^*P \quad \text{and} \quad \beta_r := \iota_r^*\beta : \mathrm{pr}_{23}^*P' \otimes \mathrm{pr}_{12}^*Q \longrightarrow \mathrm{pr}_{13}^*Q,$$

where $P := \mathcal{P}_1 \times \Gamma_0$ and $P' := \mathcal{P}' \times \Gamma_0$ are the principal Γ-bundles of the Γ-bundle gerbes $\mathcal{E}_M(\mathcal{P})$ and $\mathcal{E}_M(\mathcal{P}')$, respectively.

Lemma 3.7.16. *The bundle morphisms β_l and β_r have the following properties:*

(i) They commute with each other in these sense that the diagram

$$\begin{array}{ccc} P'_{p'_1,p'_2} \otimes Q_{p_1,p'_1} & \xrightarrow{\beta_r} & Q_{p_1,p'_2} \\ {\scriptstyle \mathrm{id}\otimes\beta_l}\downarrow & \searrow{\scriptstyle \beta} & \downarrow{\scriptstyle \beta_l} \\ P'_{p'_1,p'_2} \otimes Q_{p_2,p'_1} \otimes P_{p_1,p_2} & \xrightarrow[\beta_r\otimes\mathrm{id}]{} & Q_{p_2,p'_2} \otimes P_{p_1,p_2} \end{array}$$

is commutative for all $((p_1, p'_1), (p_2, p'_2)) \in Z^{[2]}$.

(ii) β_l *is compatible with the bundle gerbe product* μ *in the sense that*

$$\beta_l|_{p_1,p_3,p'} = (\mathrm{id} \otimes \mu_{p_1,p_2,p_3}) \circ (\beta_l|_{p_2,p_3,p'} \otimes \mathrm{id}) \circ \beta_l|_{p_1,p_2,p'}$$

for all $(p_1, p_2, p_3, p') \in \mathcal{P}_0^{[3]} \times \mathcal{P}'_0$.

(iii) β_r *is compatible with the bundle gerbe product* μ' *in the sense that*

$$\beta_r|_{p,p'_1,p'_3} \circ (\mu'_{p'_1,p'_2,p'_3} \otimes \mathrm{id}) = \beta_r|_{p,p'_2,p'_3} \circ (\mathrm{id} \otimes \beta_r|_{p,p'_1,p'_2})$$

for all $(p, p'_1, p'_2, p'_3) \in \mathcal{P}_0 \times \mathcal{P}'^{[3]}_0$.

Proof. The identities (ii) and (iii) follow by restricting the commutative diagram (3.8) to the submanifolds $\mathcal{P}_0^{[3]} \times \mathcal{P}'_0$ and $\mathcal{P}_0 \times \mathcal{P}'^{[3]}_0$ of $Z^{[3]}$, respectively. Similarly, the commutativity of the two triangular subdiagrams in (i) follows by restricting (3.8) along appropriate embeddings $Z^{[2]} \longrightarrow Z^{[3]}$. □

Now we are in position to define the anafunctor $\mathcal{R}_{\mathcal{A}}$. First, we consider the left action

$$\beta_0 : \Gamma_0 \times Q \longrightarrow Q : (g, q) \longmapsto \beta_r((\mathrm{id}, g), q)$$

that satisfies $\alpha(\beta_0(g, q)) = g\alpha(q)$. The action β_0 is properly discontinuous and free because β_r is a bundle isomorphism. The quotient $F := Q/\Gamma_0$ is the total space of the anafunctor $\mathcal{R}_{\mathcal{A}}$ we want to construct. Left and right anchor of an element $q \in F$ with $\chi(q) = (p, p')$ are given by

$$\alpha_l(q) := p \quad \text{and} \quad \alpha_r(q) := R(p', \alpha(q)).$$

The actions are defined by

$$\rho_l(\rho, q) := \beta_l^{-1}(q, (\rho, 1)) \quad \text{and} \quad \rho_r(q, \rho') := \beta_r((R(\rho', \mathrm{id}_{\alpha(q)^{-1}}), 1), q).$$

The left action is invariant under the action β_0 because of Lemma 3.7.16 (i). For the right action, invariance follows from Lemma 3.7.16 (ii) and the identity

$$\mu'((R(\rho', \mathrm{id}_{\alpha(q)^{-1}g^{-1}}), 1), (\mathrm{id}, g)) \overset{(3.16)}{=} \mu'((\mathrm{id}, g), (R(\rho', \mathrm{id}_{\alpha(q)^{-1}}), 1)).$$

Lemma 3.7.17. *The above formulas define an anafunctor* $F : \mathcal{P} \longrightarrow \mathcal{P}'$.

Proof. The compatibility between anchors and actions is easy to check. The axiom for the actions ρ_l and ρ_r follows from Lemma 3.7.16 (ii) and (iii). Lemma 3.7.16 (i) shows that the actions commute. It remains to prove that $\alpha_l : F \longrightarrow \mathcal{P}_0$ is a principal $\mathcal{P}'$-bundle. Since α_l is a composition of surjective submersions, we only have to show that the map

$$\tau : F \;{}_{\alpha_r}\!\times_t \mathcal{P}' \longrightarrow F \;{}_{\alpha_l}\!\times_{\alpha_l} F : (q, \rho') \longmapsto (q, \rho_r(q, \rho'))$$

is a diffeomorphism. We construct an inverse map τ^{-1} as follows. For (q_1, q_2) with $\chi(q_1) = (p, p')$ and $\chi(q_2) = (p, \tilde{p}')$, choose a representative

$$((\tilde{\rho}', g'), \tilde{q}) := \beta_r|^{-1}_{p,p',\tilde{p}'}(q_2).$$

Such choices can be made locally in a smooth way, and the result will not depend on them. We have $\chi(\tilde{q}) = (p, p')$ that that there exists a unique $\gamma \in \Gamma_1$ such that $q_1 = \tilde{q} \circ \gamma$. Now we put

$$\tau^{-1}(q_1, q_2) := (q_1, R(\tilde{\rho}', \gamma^{-1})).$$

The calculation of $\tau^{-1} \circ \tau$ is straightforward. For the calculation of $(\tau \circ \tau^{-1})(q_1, q_2)$ we have to compute in the second component

$$\begin{aligned} \beta_r((R(\tilde{\rho}', \gamma^{-1} \cdot \mathrm{id}_{\alpha(q_1)^{-1}}), 1), q_1) &= \beta_r((R(\tilde{\rho}', \gamma^{-1} \cdot \mathrm{id}_{\alpha(q_1)^{-1}}), 1) \circ (\gamma \cdot \mathrm{id}_{\alpha(\tilde{q})^{-1}}), \tilde{q}) \\ &= \beta_r((\tilde{\rho}', \alpha(q_1)\alpha(\tilde{q})^{-1}), \tilde{q}) \\ &= \beta_0(\alpha(q_1)\alpha(\tilde{q})^{-1}g'^{-1}, \beta_r((\tilde{\rho}', g'), \tilde{q})) \\ &= \beta_0(\alpha(q_1)\alpha(\tilde{q})^{-1}g'^{-1}, q_2), \end{aligned}$$

and this is equivalent to q_2. □

In order to promote the anafunctor F to a 1-morphism between principal 2-bundles, we have to do two things: we have to check that F commutes with the projections of the bundle $\mathcal{P}_1$ and $\mathcal{P}_2$, and we have to construct a Γ-equivariant structure on F. For the first point we use Remark 3.6.6 (b), whose criterion $\pi \circ \alpha_l = \pi \circ \alpha_r$ is clearly satisfied. For the second point we provide a smooth action $\rho : F \times \Gamma_1 \longrightarrow F$ in the sense of Definition 3.8.1 and use Lemma 3.8.2, which provides a construction of a Γ-equivariant structure. The action is defined by

$$\rho(q, \gamma) := \beta_l^{-1}(q \circ (\mathrm{id}_{\alpha(q)} \cdot \gamma \cdot \mathrm{id}_{t(\gamma)^{-1}}), (\mathrm{id}_{R(\alpha_l(q), t(\gamma))}, t(\gamma))). \tag{3.26}$$

Lemma 3.7.18. *This defines a smooth action of Γ_1 on F in the sense of Definition 3.8.1.*

Proof. Smoothness is clear from the definition. The identity

$$\rho(\rho(q, \gamma_1), \gamma_2) = \beta_l^{-1}(q \circ (\mathrm{id}_{\alpha(q)} \cdot \gamma_1 \cdot \gamma_2 \cdot \mathrm{id}_{t(\gamma_2)^{-1}t(\gamma_1)^{-1}}), (\mathrm{id}, t(\gamma_1 \cdot \gamma_2))) = \rho(q, \gamma_1 \cdot \gamma_2)$$

follows from the definition and the two identities

$$\alpha(\rho(q, \gamma)) = \alpha(q)s(\gamma) \quad \text{and}$$
$$(\gamma_1 \cdot \mathrm{id}_{t(\gamma_1)^{-1}}) \cdot (\mathrm{id}_{s(\gamma_1)} \cdot \gamma_2 \cdot \mathrm{id}_{t(\gamma_2)^{-1}t(\gamma_1)^{-1}}) = \gamma_1 \cdot \gamma_2 \cdot \mathrm{id}_{t(\gamma_2)^{-1}t(\gamma_1)^{-1}}. \tag{3.27}$$

The latter can easily be verified upon substituting a crossed module for Γ. Checking condition (i) of Definition 3.8.1 just uses the definitions. We check condition (ii) in two steps. First we prove the identity

$$\rho(\rho_l(\rho, q), \gamma_l \circ \gamma) = \rho_l(R(\rho, \gamma_l), \rho(q, \gamma)).$$

Main ingredient is the decomposition

$$\mathrm{id}_{\alpha(q)} \cdot (\gamma_l \circ \gamma) \cdot \mathrm{id}_{t(\gamma_l)^{-1}} = (\mathrm{id}_{\alpha(q)} \cdot \gamma \cdot \mathrm{id}_{t(\gamma)^{-1}}) \circ (\mathrm{id}_{\alpha(q)s(\gamma)t(\gamma)^{-1}} \cdot \gamma_l \cdot \mathrm{id}_{t(\gamma_l)^{-1}}) \quad (3.28)$$

that can e.g. be verified in the crossed module language. Now we compute

$$\begin{aligned} \rho(\rho_l(\rho, q), \gamma_l \circ \gamma) &= \beta_l^{-1}(q \circ (\mathrm{id}_{\alpha(q)} \cdot (\gamma_l \circ \gamma) \cdot \mathrm{id}_{t(\gamma_l)^{-1}}), (R(\rho, t(\gamma_l)), t(\gamma_l))) \\ &\overset{(3.28)}{=} \beta_l^{-1}(q \circ (\mathrm{id}_{\alpha(q)} \cdot \gamma \cdot \mathrm{id}_{t(\gamma)^{-1}}), (R(\rho, \gamma_l), t(\gamma_l)) \\ &= \rho_l(R(\rho, \gamma_l), \rho(q, \gamma)). \end{aligned}$$

The second step is to show the identity

$$\rho(\rho_r(q, \rho'), \gamma \circ \gamma_r) = \rho_r(\rho(q, \gamma), R(\rho', \gamma_r)).$$

Here we use the decomposition

$$\mathrm{id}_{\alpha(q)} \cdot (\gamma \circ \gamma_r) \cdot \mathrm{id}_{t(\gamma)^{-1}} = (\mathrm{id}_{\alpha(q)} \cdot \gamma \cdot \mathrm{id}_{t(\gamma)^{-1}}) \circ (\mathrm{id}_{\alpha(q)} \cdot \gamma_r \cdot \mathrm{id}_{t(\gamma)^{-1}}). \quad (3.29)$$

Then we compute

$$\begin{aligned} &\rho(\rho_r(q, \rho'), \gamma \circ \gamma_r) \\ &\quad = \beta_l^{-1}(\beta_r((R(\rho', \mathrm{id}_{\alpha(q)^{-1}}), 1), q \circ (\mathrm{id}_{\alpha(q)} \cdot (\gamma \circ \gamma_r) \cdot \mathrm{id}_{t(\gamma)^{-1}})), (\mathrm{id}, t(\gamma))) \\ &\quad \overset{(3.29)}{=} \beta_l^{-1}(\beta_r((R(\rho', \gamma_r \cdot \mathrm{id}_{s(\gamma)^{-1}\alpha(q)^{-1}}), 1), \\ &\qquad\qquad \beta_0(\alpha(q)s(\gamma_r)s(\gamma)^{-1}\alpha(q)^{-1}, q \circ (\mathrm{id}_{\alpha(q)} \cdot \gamma \cdot \mathrm{id}_{t(\gamma)^{-1}}))), (\mathrm{id}, t(\gamma))) \\ &\quad \overset{(3.27)}{=} \beta_l^{-1}(\beta_r((R(\rho', \gamma_r \cdot \mathrm{id}_{\alpha(\rho(q,\gamma))^{-1}})), q \circ (\mathrm{id}_{\alpha(q)} \cdot \gamma \cdot \mathrm{id}_{t(\gamma)^{-1}})), (\mathrm{id}, t(\gamma))) \\ &\quad = \rho_r(\rho(q, \gamma), R(\rho', \gamma_r)), \end{aligned}$$

where we have employed the equivalence relation on F that was generated by the action of β_0. □

Construction of a 2-isomorphism $\eta_{\mathcal{A}} : \mathcal{A} \Longrightarrow \mathscr{E}_M(\mathscr{R}_{\mathcal{A}})$

We may again assume that the common refinement of $\mathcal{A}$ is the fibre product $\mathcal{P}_0 \times_M \mathcal{P}_0'$; otherwise, the proof of Lemma 3.5.18 provides a 2-isomorphism between $\mathcal{A}$ and one of these. Now, $\mathcal{A}$ and $\mathscr{E}_M(\mathscr{R}_{\mathcal{A}})$ have the same common refinement, and $\eta_{\mathcal{A}}$ is given by the map

$$\eta : Q \longrightarrow F \times \Gamma_0 : q \longmapsto (q, \alpha(q)).$$

This is obviously smooth and respects the projections to the base: if $\chi(q) = (p, p')$, then

$$\chi(q, \alpha(q)) \overset{(3.19)}{=} (\alpha_l(q), R(\alpha_r(q), \alpha(q)^{-1})) = (p, p').$$

Further, it respects the Γ-actions:

$$\eta(q \circ \gamma) = (q \circ \gamma, s(\gamma)) = \beta_l^{-1}(q \circ \gamma, (\mathrm{id}, 1)) \overset{(3.26)}{=} (\rho(q, \mathrm{id}_{\alpha(q)^{-1}} \cdot \gamma), s(\gamma)) \overset{(3.19)}{=} \eta(q) \circ \gamma,$$

so that η is a bundle morphism. It remains to verify the commutativity of the compatibility diagram (3.11). Let $((\rho', g'), q') \in P' \otimes \zeta_1^* Q$, and let $(q, (\rho, g)) \in \zeta_2^* Q \otimes P$ be a representative for $\beta((\rho', g'), q')$. In particular, we have $\alpha(q) g = g' \alpha(q')$, since β_r is anchor-preserving. Then, we get clockwise

$$(\eta \otimes \mathrm{id})(\beta((\rho', g'), q')) = ((q, \alpha(q)), (\rho, g)). \tag{3.30}$$

Counter-clockwise, we have to use the isomorphism of Lemma 3.7.7 that we call $\tilde{\beta}$ here. Then,

$$\tilde{\beta}((\mathrm{id} \otimes \eta)((\rho', g'), q')) = \tilde{\beta}((\rho', g'), (q', \alpha(q'))) = ((\tilde{q}, g' \alpha(q') g^{-1}), (\rho, g)) \tag{3.31}$$

where the choices $(\tilde{\rho}, h)$ we have to make for the definition of $\tilde{\beta}$ are here (ρ, g^{-1}), and $\tilde{q}$ is defined in (3.20), which gives here

$$\tilde{q} = \beta_l^{-1}(\beta_r((\rho', 1), q'), (R(\rho^{-1}, \mathrm{id}_{g^{-1}}), g^{-1})).$$

Comparing (3.30) and (3.31) it remains to prove $q = \tilde{q}$ in F. As F was the quotient of Q by the action β_0, it suffices to have

$$\begin{aligned}
\beta_0(g', \tilde{q}) &\overset{\text{(i)}}{=} \beta_l^{-1}(\beta_r((\mathrm{id}, g'), \beta_r((\rho', 1), q')), (R(\rho^{-1}, \mathrm{id}_{g^{-1}}), g^{-1})) \\
&\overset{\text{(iii)}}{=} \beta_l^{-1}(\beta_r((\rho', g'), q'), (R(\rho^{-1}, \mathrm{id}_{g^{-1}}), g^{-1})) \\
&= \beta_l^{-1}(\beta_l^{-1}(q, (\rho, g)), (R(\rho^{-1}, \mathrm{id}_{g^{-1}}), g^{-1})) \\
&\overset{\text{(ii)}}{=} \beta_l^{-1}(q, (\mathrm{id}, 1)) \\
&= q.
\end{aligned}$$

This finishes the construction of the 2-isomorphism $\eta_{\mathcal{A}}$.

3.8 Appendix

3.8.1 Appendix: Equivariant Anafunctors and Group Actions

In this section we are concerned with a Lie 2-group Γ and Lie groupoids $\mathcal{X}$ and $\mathcal{Y}$ with actions $R_1 : \mathcal{X} \times \Gamma \longrightarrow \mathcal{X}$ and $R_2 : \mathcal{Y} \times \Gamma \longrightarrow \mathcal{Y}$.

Definition 3.8.1. An *action* of the 2-group Γ on an anafunctor $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is an ordinary smooth action $\rho : F \times \Gamma_1 \longrightarrow F$ of the group Γ_1 on the total space F that

(i) preserves the anchors in the sense that the diagrams

$$\begin{array}{ccc} F \times \Gamma_1 & \xrightarrow{\rho} & F \\ {\scriptstyle \alpha_l \times t}\downarrow & & \downarrow{\scriptstyle \alpha_l} \\ \mathcal{X}_0 \times \Gamma_0 & \xrightarrow[R_1]{} & \mathcal{X}_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} F \times \Gamma_1 & \xrightarrow{\rho} & F \\ {\scriptstyle \alpha_r \times s}\downarrow & & \downarrow{\scriptstyle \alpha_r} \\ \mathcal{Y}_0 \times \Gamma_0 & \xrightarrow[R_2]{} & \mathcal{Y}_0 \end{array}$$

are commutative.

(ii) is compatible with the Γ-actions in the sense that the identity

$$\rho(\chi \circ f \circ \eta, \gamma_l \circ \gamma \circ \gamma_r) = R_1(\chi, \gamma_l) \circ \rho(f, \gamma) \circ R_2(\eta, \gamma_r)$$

holds for all appropriately composable $\chi \in \mathcal{X}_1$, $\eta \in \mathcal{Y}_1$, $f \in F$, and $\gamma_l, \gamma, \gamma_r \in \Gamma_1$.

If $F_1, F_2 : \mathcal{X} \longrightarrow \mathcal{Y}$ are anafunctors with Γ-action, a transformation $\eta : F_1 \Longrightarrow F_2$ is called Γ-equivariant if the map $\eta : F_1 \longrightarrow F_2$ between total spaces is Γ_1-equivariant in the ordinary sense.

Anafunctors $\mathcal{X} \longrightarrow \mathcal{Y}$ with Γ-actions together with Γ-equivariant transformations form a groupoid $\mathcal{A}na^{\infty}_{\Gamma}(\mathcal{X}, \mathcal{Y})$. On the other hand, there is another groupoid Γ-$\mathcal{A}na^{\infty}(\mathcal{X}, \mathcal{Y})$ consisting of Γ-equivariant anafunctors (Definition 3.6.3) and Γ-equivariant transformations (Definition 3.6.4).

Lemma 3.8.2. *The categories $\mathcal{A}na^{\infty}_{\Gamma}(\mathcal{X}, \mathcal{Y})$ and Γ-$\mathcal{A}na^{\infty}(\mathcal{X}, \mathcal{Y})$ are canonically isomorphic.*

Proof. We construct a functor

$$\mathcal{E} : \mathcal{A}na^{\infty}_{\Gamma}(\mathcal{X}, \mathcal{Y}) \longrightarrow \Gamma\text{-}\mathcal{A}na^{\infty}(\mathcal{X}, \mathcal{Y}). \tag{3.32}$$

Let $F : \mathcal{X} \longrightarrow \mathcal{Y}$ be an anafunctor with Γ-action ρ. We shall define a transformation

$$\lambda_\rho : F \circ R_1 \Longrightarrow R_2 \circ (F \times \mathrm{id}).$$

First of all, the composite

$$\mathcal{X} \times \Gamma \xrightarrow{R_1} \mathcal{X} \xrightarrow{F} \mathcal{Y}$$

is given by the total space $(\mathcal{X}_0 \times \Gamma_0)\, {}_{R_1}\!\times_{\alpha_l} F$, left and right anchors send an element (x, g, f) to (x, g) and $\alpha_r(f)$, respectively, and the actions are

$$(\chi, \gamma) \circ (x, g, f) = (t(\chi), t(\gamma), R_1(\chi, \gamma) \circ f) \quad \text{and} \quad (x, g, f) \circ \eta = (x, g, f \circ \eta).$$

On the other hand, the composite

$$\mathcal{X} \times \Gamma \xrightarrow{F\times\mathrm{id}} \mathcal{Y} \times \Gamma \xrightarrow{R_2} \mathcal{Y}$$

is given by the total space $((F \times \Gamma_1)\,{}_{R_2\circ(\alpha_r\times s)}\!\times_t \mathcal{Y}_1)/\sim$ with the equivalence relation

$$(f \circ \eta', \gamma \circ \gamma', \eta) \sim (f, \gamma, R_2(\eta', \gamma') \circ \eta).$$

Left and right anchor send an element (f, γ, η) to $(\alpha_l(f), t(\gamma))$ and $s(\eta)$, respectively, and the actions are

$$(\chi, \gamma') \circ (f, \gamma, \eta) = (\chi \circ f, \gamma' \circ \gamma, \eta) \quad \text{and} \quad (f, \gamma, \eta) \circ \eta' = (f, \gamma, \eta \circ \eta').$$

The inverse of the following map will define the transformation λ:

$$(F \times \Gamma_1)\,{}_{R_2\circ(\alpha_r\times s)}\!\times_t \mathcal{Y}_1 \longrightarrow (\mathcal{X}_0 \times \Gamma_0)\,{}_{R_1}\!\times_{\alpha_l} F : (f, \gamma, \eta) \longmapsto (\alpha_l(f), t(\gamma), \rho(f, \gamma) \circ \eta).$$

Condition (i) assures that this map ends in the correct fibre product, and condition (ii) assures that it is well-defined under the equivalence relation $\sim$. The left anchors are automatically respected, and the right anchors require condition (i). Similarly, the left action is respected automatically, and the right actions due to condition (ii). The axiom for a transformation is satisfied because ρ is a group action. This defines the functor $\mathcal{E}$ on objects. On morphisms, it is straightforward to check that the conditions on both hand sides coincide; in particular, $\mathcal{E}$ is full and faithful.

In order to prove that the functor $\mathcal{E}$ is an isomorphism, we start with a given Γ-equivariant structure λ on the anafunctor F. Then, an action $\rho : F \times \Gamma_1 \longrightarrow F$ is defined by

$$(f, \gamma) \longmapsto \mathrm{pr}_3(\lambda^{-1}(f, \gamma, \mathrm{id}_{R_2(\alpha_r(f), s(\gamma))}))$$

with $\mathrm{pr}_3 : (\mathcal{X}_0 \times \Gamma_0)\,{}_{R_1}\!\times_{\alpha_l} F \longrightarrow F$ the projection. The axiom for an action is satisfied due to the identity λ obeys. It is straightforward to verify conditions (i) and (ii) of Definition 3.8.1. To close the proof it suffices to notice that the two procedures we have defined are (strictly) inverse to each other. □

We are also concerned with the composition of anafunctors with Γ-action. Suppose that $\mathcal{Z}$ is a third Lie groupoid with a Γ-action R_3, and $F : \mathcal{X} \longrightarrow \mathcal{Y}$ and $G : \mathcal{Y} \longrightarrow \mathcal{Z}$ are anafunctors with Γ-actions $\rho : F \times \Gamma_1 \longrightarrow F$ and $\tau : G \times \Gamma_1 \longrightarrow G$. Then, the composition $G \circ F$ is equipped with the Γ-action defined by

$$(F \times_{\mathcal{Y}_0} G) \times \Gamma_1 \longrightarrow (F \times_{\mathcal{Y}_0} G) : ((f, g), \gamma) \longmapsto (\rho(f, \gamma), \tau(g, \mathrm{id}_{s(\gamma)})). \tag{3.33}$$

We leave it to the reader to check

Lemma 3.8.3. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be Lie groupoids with Γ-actions R_1, R_2 and R_3.*

(a) *Let $F : \mathcal{X} \longrightarrow \mathcal{Y}$ and $G : \mathcal{Y} \longrightarrow \mathcal{Z}$ be Γ-equivariant anafunctors. If Γ-equivariant structures on F and G correspond to Γ_1-actions under the isomorphism of Lemma 3.8.2, then the Γ-equivariant structure on the composite $F \circ G$ corresponds to the Γ_1-action defined above.*

(b) *The isomorphism of Lemma 3.8.2 identifies the trivial Γ-equivariant structure on the identity anafunctor $\mathrm{id} : \mathcal{X} \longrightarrow \mathcal{X}$ with the Γ_1-action $R_1 : \mathcal{X}_1 \times \Gamma_1 \longrightarrow \mathcal{X}_1$ on its total space $\mathcal{X}$.*

3.8.2 Appendix: Equivalences between 2-Stacks

Let $\mathcal{C}$ be a bicategory (we assume that associators and unifiers are *invertible* 2-morphisms). We fix the following terminology: a 1-*iso*morphism $f : X_1 \longrightarrow X_2$ in $\mathcal{C}$ always includes the data of an inverse 1-morphism $\bar{f} : X_2 \longrightarrow X_1$ and of 2-isomorphisms $i : \bar{f} \circ f \Longrightarrow \mathrm{id}$ and $j : \mathrm{id} \Longrightarrow f \circ \bar{f}$ satisfying the zigzag identities. Let $\mathcal{D}$ be another bicategory. A 2-functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is assumed to have *invertible* compositors and unitors.

The following lemma is certainly "well-known", although we have not been able to find a reference for exactly this statement.

Lemma 3.8.4. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a 2-functor that is fully faithful on Hom-categories. Suppose one has chosen:*

1. *for every object $Y \in \mathcal{D}$ an object $G_Y \in \mathcal{C}$ and a 1-isomorphism $\xi_Y : Y \longrightarrow F(G_Y)$.*
2. *for all objects $X_1, X_2 \in \mathcal{C}$ and all 1-morphisms $g : F(X_1) \longrightarrow F(X_2)$, a 1-morphism $G_g : X_1 \longrightarrow X_2$ in $\mathcal{C}$ together with a 2-isomorphism $\eta_g : g \Longrightarrow F(G_g)$.*[1]

Then, there is a 2-functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ and pseudonatural equivalences

$$a : \mathrm{id}_{\mathcal{D}} \Longrightarrow F \circ G \quad \text{and} \quad b : G \circ F \Longrightarrow \mathrm{id}_{\mathcal{C}}.$$

In particular, F is an equivalence of bicategories.

Proof. We recall our convention concerning 1-isomorphisms: the 1-isomorphisms ξ_Y include choices of inverse 1-morphisms $\bar{\xi}_Y$ together with 2-isomorphisms i_Y : $\bar{\xi}_Y \circ \xi_Y \Longrightarrow \mathrm{id}$ and $j_Y : \mathrm{id} \Longrightarrow \xi_Y \circ \bar{\xi}_Y$ satisfying the zigzag identities.

First we explicitly construct the 2-functor G. On objects, we put $G(Y) := G_Y$. We use the notation $\tilde{g} := (\xi_{Y_2} \circ g) \circ \bar{\xi}_{Y_1}$ for all 1-morphisms $g : Y_1 \longrightarrow Y_2$ in $\mathcal{D}$, and define $G(g) = G_{\tilde{g}}$. If $g, g' : Y_1 \longrightarrow Y_2$ are 1-morphisms, and $\psi : g \Longrightarrow g'$ is a 2-morphism, we consider the 2-morphism $\tilde{\psi}$ defined by

$$F(G_{\tilde{g}}) \xRightarrow{\eta_{\tilde{g}}^{-1}} (\xi_{Y_2} \circ g) \circ \bar{\xi}_{Y_1} \xRightarrow{(\mathrm{id} \circ \psi) \circ \mathrm{id}} (\xi_{Y_2} \circ g') \circ \bar{\xi}_{Y_1} \xRightarrow{\eta_{\tilde{g}'}} F(G_{\tilde{g}'}).$$

[1] More accurately we should write $G_{X_1,X_2,g}$ and $\eta_{X_1,X_2,g}$, but we will suppress X_1 and X_2 in the notation.

Since F is fully faithful on 2-morphisms, we may choose the unique 2-morphism $G(\psi) : G(g) \Longrightarrow G(g')$ such that $F(G(\psi)) = \tilde{\psi}$. In order to define the compositor of G we look at 1-morphisms $g_{12} : Y_1 \longrightarrow Y_2$ and $g_{23} : Y_2 \longrightarrow Y_3$. We consider the 2-morphism

$$\begin{array}{l} F(G(g_{23}) \circ G(g_{12})) \xRightarrow{c^{-1}_{G(g_{12}),G(g_{23})}} F(G_{\tilde{g}_{23}}) \circ F(G_{\tilde{g}_{12}}) \\ \qquad \Big\Downarrow {\scriptstyle \eta^{-1}_{\tilde{g}_{23}} \circ \eta^{-1}_{\tilde{g}_{12}}} \\ ((\xi_{Y_3} \circ g_{23}) \circ \bar{\xi}_{Y_2}) \circ ((\xi_{Y_2} \circ g_{12}) \circ \bar{\xi}_{Y_1}) \\ \qquad \Big\Downarrow {\scriptstyle a, i_{Y_2}} \\ (\xi_{Y_3} \circ (g_{23} \circ g_{12})) \circ \bar{\xi}_{Y_1} \xRightarrow[\eta_{\widetilde{g_{23} \circ g_{12}}}]{} F(G(g_{23} \circ g_{12})); \end{array}$$

its unique preimage under the 2-functor F is the compositor

$$c_{g_{12},g_{23}} : G(g_{23}) \circ G(g_{12}) \Longrightarrow G(g_{23} \circ g_{12}).$$

In order to define the unitor of G we consider an object $Y \in \mathcal{D}$ and look at the 2-morphism

$$F(G(\mathrm{id}_Y)) \xRightarrow{\eta^{-1}_{\widetilde{\mathrm{id}_Y}}} (\xi_Y \circ \mathrm{id}_Y) \circ \bar{\xi}_Y \xRightarrow{l_{\xi_Y}, j^{-1}_Y} \mathrm{id}_{F(G(Y))} \xRightarrow{u^{-1}_{G(Y)}} F(\mathrm{id}_{G(Y)}).$$

Its unique preimage under the 2-functor F is the unitor $u_Y : G(\mathrm{id}_Y) \Longrightarrow \mathrm{id}_{G(Y)}$. The second step is to verify the axioms of a 2-functor. This is simple but extremely tedious and can only be left as an exercise. The third step is to construct the pseudonatural transformation

$$a : \mathrm{id}_{\mathcal{D}} \Longrightarrow F \circ G.$$

Its component at an object Y in $\mathcal{D}$ is the 1-morphism $a(Y) := \xi_Y : Y \longrightarrow F(G(Y))$. Its component at a 1-morphism $g : Y_1 \longrightarrow Y_2$ is the 2-morphism $a(g)$ defined by

$$\begin{array}{l} a(Y_2) \circ g = \xi_{Y_2} \circ g \\ \qquad \Big\Downarrow {\scriptstyle \mathrm{id} \circ l^{-1}_{\xi_{Y_2} \circ g}} \\ (\xi_{Y_2} \circ g) \circ \mathrm{id} \\ \qquad \Big\Downarrow {\scriptstyle a, i^{-1}_{Y_2}} \\ ((\xi_{Y_2} \circ g) \circ \bar{\xi}_{Y_1}) \circ \xi_{Y_1} \\ \qquad \Big\Downarrow {\scriptstyle \eta_{\tilde{g}} \circ \mathrm{id}} \\ F(G_{\tilde{g}}) \circ \xi_{Y_1} = F(G(g)) \circ a(Y_1). \end{array}$$

There are two axioms a pseudonatural transformation has to satisfy, and their proofs are again left as an exercise. It is easy to see that a is a pseudonatural *equivalence*, with an inverse transformation given by $\bar{a}(Y) := \bar{\xi}_Y$. The fourth and last step is to construct the pseudonatural transformation

$$b : G \circ F \Longrightarrow \mathrm{id}_{\mathcal{C}}.$$

Its component at an object X is $b(X) := G_{\bar{\xi}_{F(X)}} : G(F(X)) \longrightarrow X$. Its component at a 1-morphism $f : X_2 \longrightarrow X_2$ is the 2-morphism

$$b(f) : b(X_2) \circ G(F(f)) \Longrightarrow f \circ b(X_1)$$

given as the unique preimage under F of the 2-morphism

$$\begin{array}{l} F(b(X_2) \circ G(F(f))) \xRightarrow{c^{-1}} F(b(X_2)) \circ F(G(F(f))) \\ \qquad \Big\Downarrow \eta^{-1}_{\bar{\xi}_{F(X_2)}} \circ \eta^{-1}_{F(f)} \\ \bar{\xi}_{F(X_2)} \circ ((\xi_{F(X_2)} \circ F(f)) \circ \xi_{F(X_1)}) \\ \qquad \Big\Downarrow a, i_{F(X_2)}, r \\ F(f) \circ \bar{\xi}_{F(X_1)} \\ \qquad \Big\Downarrow \mathrm{id}_{F(f)} \circ \eta_{\bar{\xi}_{F(X_1)}} \\ F(f) \circ F(b(X_1)) \xRightarrow[c]{} F(f \circ b(X_1)). \end{array}$$

The proofs of the axioms are again left for the reader, and again it is easy to see that b is a pseudonatural *equivalence* with an inverse transformation given by $\bar{b}(X) := G_{\xi_{F(X)}}$. □

As a consequence of Lemma 3.8.4 we obtain the certainly well-known

Corollary 3.8.5. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be essentially surjective, and an equivalence on all Hom-categories. Then, F is an equivalence of bicategories.*

Since we work with 2-stacks over manifolds, we need the following "stacky" extension of Lemma 3.8.4. For a pre-2-stack $\mathcal{C}$, we denote by $\mathcal{C}_M$ the 2-category it associates to a smooth manifold M, and by $\psi^* : \mathcal{C}_N \longrightarrow \mathcal{C}_M$ the 2-functor it associates to a smooth map $\psi : M \longrightarrow N$. The pseudonatural equivalences $\psi^* \circ \varphi^* \cong (\varphi \circ \psi)^*$ will be suppressed from the notation in the following. If $\mathcal{C}$ and $\mathcal{D}$ are pre-2-stacks, a *1-morphism* $F : \mathcal{C} \longrightarrow \mathcal{D}$ associates 2-functors $F_M : \mathcal{C}_M \longrightarrow \mathcal{D}_M$ to a smooth manifold M, pseudonatural equivalences

$$F_\psi : \psi^* \circ F_N \longrightarrow F_M \circ \psi^*$$

to smooth maps $\psi : M \longrightarrow N$, and certain modifications $F_{\psi,\varphi}$ that control the relation between F_ψ and F_φ for composable maps ψ and φ.

Lemma 3.8.6. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are pre-2-stacks over smooth manifolds, and $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a 1-morphism. Suppose that for every smooth manifold M*

1. *the assumptions of Lemma 3.8.4 for the 2-functor F_M are satisfied and*
2. *the data (G_Y, ξ_Y) and (G_g, η_g) is chosen for all objects Y and 1-morphisms g in $\mathcal{D}_M$.*

Then, there is a 1-morphism $G : \mathcal{D} \longrightarrow \mathcal{C}$ of pre-2-stacks together with 2-isomorphisms

$$a : F \circ G \Longrightarrow \mathrm{id}_{\mathcal{D}} \quad \text{and} \quad b : G \circ F \Longrightarrow \mathrm{id}_{\mathcal{C}}$$

such that for every smooth manifold M the 2-functor G_M and the pseudonatural transformations a_M and b_M are the ones of Lemma 3.8.4. In particular, F is an equivalence of pre-2-stacks.

For the proof one constructs the required pseudonatural equivalences G_ψ and the modifications $G_{\psi,\varphi}$ from the given ones, F_ψ and $F_{\psi,\varphi}$, respectively, in a similar way as explained in the proof of Lemma 3.8.4. Since these constructions are straightforward to do but would consume many pages, and the statement of the lemma is not too surprising and certainly well-known to many people, we decided to leave these constructions for the interested reader.

Chapter 4

A Smooth Model for the String Group

In this chapter we discuss an example of a Lie 2-group that can be used as structure 2-group for our general theory of 2-bundles (resp. non-abelian gerbes). The string-2-group can be used to define string-structures, like the spin-group is used to define spin-structures. These string-structures are needed to cancel certain anomalies in supersymmetric sigma models as mentioned in the introduction. In fact we not only construct a string-2-group model but also a model as an infinite dimensional Lie group. Therefore we have to adapt our general setting (Lie groups, Lie groupoids, bundles, 2-bundles, ...) to the infinite dimensional setting. This is why we repeat some definitions with special emphasis on the infinite dimensional context.

4.1 Recent and new models

String structures and the string group play an important role in algebraic topology [Hen08b, Lur09a, BN09], string theory [Kil87, FM06] and geometry [Wit88, Sto96]. The group $\mathcal{String}$ is defined to be a 3-connected cover of the spin group or, more generally of any simple simply connected compact Lie group G [ST04]. This definition fixes only its homotopy type and makes abstract homotopy theoretic constructions possible. But for geometric applications these models are not very well suited, one is rather interested in concrete models that carry, for instance, topological or even Lie-group structures.

There is a direct cohomological argument showing that $\mathcal{String}_G$ cannot be a finite CW-complex or a finite-dimensional manifold (see Corollary 4.3.3), so the best thing one can hope for is a topological group or an infinite-dimensional Lie group. There have been various constructions of models of $\mathcal{String}_G$ as A_∞-spaces or topological groups, but the question whether an infinite-dimensional Lie group model is also possible has been open so far. One of the main contributions of the present chapter is to give an affirmative answer to this question and provide an explicit Lie group model, based on a topological construction of Stolz [Sto96].

Something that is not directly apparent from the setting of the problem is that string group models as Lie 2-groups are something more natural to expect when taking the perspective of string theory or higher homotopy theory into account. However, the notion of a Lie 2-group model deserves a thorough clarification itself. We discuss this notion carefully by establishing the relevant homotopy theoretic facts about infinite-dimensional Lie 2-groups and promote our Lie group model $\mathcal{S}tring_G$ to such a Lie 2-group model $\mathtt{STRING}_G$.

Before we outline our construction let us briefly summarize the existing ones. One model for $\mathcal{S}tring_G$ is the pullback of the path fibration $PK(\mathbb{Z},3) \to K(\mathbb{Z},3)$ along a characteristic map $u\colon G \to K(\mathbb{Z},3)$. This is a standard construction of the Whitehead tower and leads to a model of $\mathcal{S}tring_G$ as a space. Since this construction also works for a characteristic map $BG \to K(\mathbb{Z},4)$, each 3-connected cover is homotopy equivalent to a loop space and thus admits an A_∞-structure. Taking a functorial construction of the Whitehead tower one even obtains a model as a topological group. Unfortunately, these models are not very tractable.

There are more geometric constructions of $\mathcal{S}tring_G$, for instance the one by Stolz in [Sto96]. The model given there has as an input the basic principal $PU(\mathcal{H})$-bundle P over G, where $\mathcal{H}$ is a separable Hilbert space. Stolz then defines a model for $\mathcal{S}tring_G$ as a topological group together with a homomorphism $\mathcal{S}tring_G \to G$ whose kernel is the group of continuous gauge transformations of the bundle P. Our constructions will be based on this idea. In [ST04] Stolz and Teichner construct a model for $\mathcal{S}tring_G$ as an extension of G by $PU(\mathcal{H})$. It is a natural idea to equip this model with a smooth structure. But this does not work since this extension is constructed as a pushout along a positive energy representation of the loop group of G which is not smooth.

We now come to Lie 2-group models. One construction has been given by Henriques [Hen08a], based on work of Getzler [Get09]. Its basic idea is to apply a general integration procedure for L_∞-algebras to the string Lie 2-algebra. To make this construction work one has to weaken the naive notion of a Lie 2-group and besides that work in the category of Banach spaces. Similarly, the model of Schommer-Pries [SP10] realizes $\mathcal{S}tring_G$ as a stacky Lie 2-group, but it has the advantage of being finite-dimensional. This model is constructed from a cocycle in Segal's Cohomology for G [Seg70].

A common thing about the above Lie 2-group models is that they are not strict, i.e., not associative on the nose but only up to an additional coherence. This complication is not present in the strict 2-group model of Baez, Crans, Schreiber and Stevenson from [BCSS07]. It is constructed from a crossed module $\widehat{\Omega G} \to P_eG$, built out of the level one Kac-Moody central extension $\widehat{\Omega G}$ of the loop group of G and its path space P_eG. The price to pay is that the model is infinite dimensional, but the strictness makes the corresponding bundle theory more tractable (as explained in chapter 3.

Summarizing, quite some effort has been made in constructing models for $\mathcal{S}tring_G$ that are as close as possible to finite-dimensional Lie groups. However, one of the most natural questions, namely whether there exists an infinite-dimensional *Lie group* model for $\mathcal{S}tring_G$ is still open. We answer this question by the following result.

Let $P \to G$ be a basic smooth principal $PU(\mathcal{H})$-bundle. Basic here means that $[P] \in [G, BPU(\mathcal{H})] \cong H^3(G,\mathbb{Z}) = \mathbb{Z}$ is a generator. In Section 4.2 we review the fact that $\mathcal{G}au(P)$ is a Lie group modeled on the infinite-dimensional space of vertical vector fields on P. The main result of Section 4.3 is then

Theorem (Theorem 4.3.6)**.** Let G be a simple, simply connected and compact Lie group, then there exists a smooth string group model $\mathcal{S}tring_G$ turning

$$\mathcal{G}au(P) \to \mathcal{S}tring_G \to G$$

into an extension of Lie groups. It is uniquely determined up to isomorphism by this property.

From now on $\mathcal{S}tring_G$ will always refer to this particular model. The proof of the theorem is based on [Sto96] and [Woc08]. We also show that $\mathcal{S}tring_G$ is metrizable and Fréchet. This metrizability makes the homotopy theory that we use in the sequel work due to results of Palais [Pal66].

In Section 4.4 we introduce the concept of Lie 2-group models culminating in Definition 4.4.10. An important construction in this context is the geometric realization that produces topological groups from Lie 2-groups. We show that geometric realization is well-behaved under mild technical conditions, such as metrizability.

In Section 4.5 we then construct a central extension $U(1) \to \widehat{\mathcal{G}au}(P) \to \mathcal{G}au(P)$ with contractible $\widehat{\mathcal{G}au}(P)$. We define an action of $\mathcal{S}tring_G$ on $\widehat{\mathcal{G}au}(P)$ such that $\widehat{\mathcal{G}au}(P) \to \mathcal{S}tring_G$ is a smooth crossed module. Crossed modules are a source for Lie 2-groups (Example 4.4.3) and in that way we obtain a Lie 2-group $\texttt{STRING}_G$.

Theorem (Theorem 4.5.6)**.** $\texttt{STRING}_G$ is a Lie 2-group model in the sense of Definition 4.4.10.

The proof of this theorem relies on a comparison of the model $\mathcal{S}tring_G$ with the geometric realization of $\texttt{STRING}_G$. Moreover, this direct comparison allows to derive a comparison between the corresponding bundle theories and string structures, see Section 4.6. This explicit comparison is a distinct feature of our 2-group model that is not available for the other 2-group models.

In an appendix we have collected some elementary facts about infinite dimensional manifolds and Lie groups. A second appendix gives a useful characterization of smooth weak equivalences between Lie 2-groups.

4.2 Preliminaries on gauge groups

Throughout this chapter Lie groups are permitted to be infinite-dimensional. More precisely, a Lie group is a group, together with the structure of a locally convex manifold such that the group operations are smooth, see Appendix 4.7. The term topological group throughout refers to a group in compactly generated spaces.

In this section the setting will be as follows:

- M is a compact manifold.
- K is a metrizable Banach–Lie group (or equivalently a paracompact Banach–Lie group).
- P is a smooth principal K-bundle over M.

Note that if P is only a continuous principal bundle, then we always find a smooth principal bundle which is equivalent to it [MW09].

Definition 4.2.1. The group $\operatorname{Aut}(P)$ denotes the group of K-equivariant diffeomorphisms $f\colon P \longrightarrow P$. Identifying M with P/K we have a natural homomorphism

$$Q\colon \operatorname{Aut}(P) \longrightarrow \operatorname{Diff}(M), \quad Q(f)([p]) = [f(p)]$$

and we define the *gauge group* by $\mathcal{G}au(P) := \ker(Q)$.

It will be convenient to identify $\mathcal{G}au(P)$ with $C^\infty(P,K)^K$, the smooth K-equivariant maps $P \longrightarrow K$, via

$$C^\infty(P,K)^K \ni f \longmapsto (p \longmapsto p \cdot f(p)) \in \mathcal{G}au(P).$$

If P is topologically trivial, then the left hand side $C^\infty(P,K)^K$ is isomorphic to $C^\infty(M,K)$. In [Woc08] it is shown how to exploit the local triviality of P in order to construct a Lie group structure on $C^\infty(P,K)^K$ similar to the one on $C^\infty(M,K)$.

Proposition 4.2.2. *The group $\mathcal{G}au(P) \cong C^\infty(P,K)^K$ admits the structure of a Fréchet Lie-group modeled on the gauge algebra $\mathfrak{gau}(P) := C^\infty(P,\mathfrak{k})^K$ of smooth equivariant maps $P \longrightarrow \mathfrak{k}$. If* $\exp\colon \mathfrak{k} \longrightarrow K$ *is the exponential function of K, then*

$$\exp_*\colon C^\infty(P,\mathfrak{k})^K \longrightarrow C^\infty(P,K)^K, \quad \xi \longmapsto \exp\circ\xi \tag{4.1}$$

is an exponential function and a local diffeomorphism.

Proof. The proof of this proposition can be found in [Woc08, Theorem 1.11 and Lemma 1.14(c)]. We will therefore only sketch the arguments that become important in the sequel.

Let N be a manifold with boundary (the boundary might be empty) modeled on a locally convex space. The space $C^\infty(N,K)$ can be given a topology by pulling back the compact open topology along

$$C^\infty(N,K) \longrightarrow \prod_{i=0}^{\infty} C^0(T^iN, T^iK).$$

We refer to this topology as the C^∞-topology. This also applies to the Lie algebra $\mathfrak{k}$ of K and induces a locally convex vector space topology on $C^\infty(N,\mathfrak{k})$. Moreover, $C^\infty(N,\mathfrak{k})$ is a Fréchet space if N is finite-dimensional [Glö02]. If we now restrict to the case where N is compact and if $\varphi\colon U \subset K \longrightarrow W \subset \mathfrak{k}$ is a chart satisfying $\varphi(e) = 0$, then $C^\infty(N,W)$ is in particular open in $C^\infty(N,\mathfrak{k})$ and thus

$$\varphi_*\colon C^\infty(N,U) \longrightarrow C^\infty(N,W), \quad \gamma \longmapsto \varphi \circ \gamma \tag{4.2}$$

defines a manifold structure on $C^\infty(N,U)$. It can be shown that the (point-wise) group structures are compatible with this smooth structure and that it may be extended to a Lie group structure on $C^\infty(N,K)$. Details of this construction can be found in [Woc06] and [GN11].

The topologies mentioned aboved applied to the case $N = P$ also endow the subspaces $C^\infty(P,K)^K$ and $C^\infty(P,\mathfrak{k})^K$ with the structure of topological groups and $C^\infty(P,\mathfrak{k})^K$ with the structure of a topological Lie algebra, both with respect to point-wise operations. The exponential function $\exp\colon \mathfrak{k} \longrightarrow K$ is K-equivariant and, by the inverse function theorem for Banach spaces, a local diffeomorphism. It thus defines in particular a map

$$\exp_*\colon C^\infty(P,\mathfrak{k})^K \longrightarrow C^\infty(P,K)^K, \quad \xi \longmapsto \exp\circ\xi$$

Like in the case of a compact manifold with boundary N, it can be shown that this map restricts to a bijection on some open subset of $C^\infty(P,\mathfrak{k})^K$, which then gives rise to a manifold structure around the identity in $C^\infty(P,K)^K$ that can be enlarged to a Lie group structure. The details of this are spelled out in [Woc08, Propositions 1.4 and 1.8]. □

Lemma 4.2.3. *The topology underlying $\mathcal{G}au(P)$ is metrizable.*

Proof. We first note that $C^\infty(N,K)$ is metrizable for finite-dimensional N since $C^0(T^iN,T^iK)$ is so [Bou98a, X.3.3] and countable products of metrizable spaces are metrizable. From [Woc08, Proposition 1.8] it follows that $\mathcal{G}au(P)$ is identified with a closed subspace of $C^\infty(\coprod \overline{V_i}, K)$, where $V_i, ..., V_n$ is a cover of M such that $\overline{V_i}$ is a manifold with boundary and $P|_{\overline{V_i}}$ is trivial. Since $C^\infty(\coprod \overline{V_i}, K)$ is metrizable, $\mathcal{G}au(P)$ is so as well. □

Remark 4.2.4. *([Woc08, Remark 1.18]) There also is a continuous version of the gauge group, namely the group of K-equivariant homeomorphisms $P \longrightarrow P$ covering the identity on M. This group will be denoted $\mathcal{G}au^c(P)$. As above, we have that $\mathcal{G}au^c(P) \cong C(P,K)^K$ and since $C(X,K)$ is a Lie group modeled on $C(X,\mathfrak{k})$ for each compact topological space X (with respect to the compact-open topology, cf. [GN11]) the above proof carries over to show that $\mathcal{G}au^c(P)$ is also a metrizable Lie group modeled on $C(P,\mathfrak{k})^K$.*

Now [Woc08, Proposition 1.20] and Theorem 4.7.5 imply

Proposition 4.2.5. *The canonical inclusion*

$$\mathcal{G}au(P) \hookrightarrow \mathcal{G}au^c(P). \tag{4.3}$$

is a homotopy equivalence.

In the sequel we will also need the following slight variation. Consider a central extension

$$Z \longrightarrow \widehat{K} \longrightarrow K$$

of Banach–Lie groups admitting smooth local sections. Similar to the gauge group $C^\infty(P,K)^K$, the groups $C^\infty(G,Z)$ and $C^\infty(P,\widehat{K})^K$ possess Lie group structures, modeled on $C^\infty(G,\mathfrak{z})$ and $C^\infty(P,\widehat{\mathfrak{k}})^K$ [NW09, Appendix A], [Woc08, Theorem 1.11]. As in Proposition 4.2.2, charts can be obtained from the exponential map

$$\exp_*\colon C^\infty(P,\widehat{\mathfrak{k}})^K \longrightarrow C^\infty(P,\widehat{K})^K, \quad \xi \longmapsto \exp\circ\xi.$$

Moreover this is a central extension, as we show in proposition 4.2.7.

Lemma 4.2.6. *([EG54]) If $F \longrightarrow E \longrightarrow B$ is a fiber bundle with F and B metrizable, then E is metrizable.*

Proposition 4.2.7. *Let $Z \longrightarrow \widehat{K} \xrightarrow{q} K$ be a central extension of Banach–Lie groups, admitting a local smooth section. Then the exact sequence of Fréchet–Lie groups*

$$C^\infty(M,Z) \longrightarrow C^\infty(P,\widehat{K})^K \longrightarrow C^\infty(P,K)^K \tag{4.4}$$

admits a smooth local section. Moreover, $C^\infty(M,\widehat{K})^K$ is metrizable if Z and K are so.

Proof. We have to recall some facts on the construction of the Lie group structure from [NW09, Appendix A] and [Woc08, Proposition 1.11]. Let $V_1,...,V_n$ be an open cover of G such that each $\overline{V_i}$ is a manifold (with boundary) and such that there exist smooth sections $\sigma_i\colon \overline{V_i} \longrightarrow P$. These give rise to smooth transition functions $k_{ij}\colon \overline{V_i}\cap\overline{V_j} \longrightarrow K$ and we have that

$$\gamma \longmapsto \Sigma(\gamma) := (\gamma\circ\sigma_i)_{i=1,\dots,n}$$

induces an isomorphism

$$C^{\infty}(P,K)^{K}\cong\{(\gamma_i)_{i=1,\ldots,n}\in\prod_{i=1}^{n}C^{\infty}(\overline{V_i},K)\mid\gamma_i=k_{ij}\cdot\gamma_j\cdot k_{ji}\text{ on }\overline{V_i}\cap\overline{V_j}\}$$

If now $\exp\colon\mathfrak{k}\longrightarrow K$ restricts to a diffeomorphism $\exp\colon W\longrightarrow U$, then we have that

$$\mathfrak{W}:=\{(\gamma_i)_{i=1,\ldots,n}\in\prod_{i=1}^{n}C^{\infty}(\overline{V_i},W)\mid\gamma_i=k_{ij}\cdot\gamma_j\cdot k_{ji}\text{ on }\overline{V_i}\cap\overline{V_j}\}$$

maps under Σ^{-1} to an identity neighborhood $\Sigma^{-1}(\mathfrak{W})$ on which $\exp_*$ restricts to a diffeomorphism (cf. [Woc08, Proposition 1.11]). Note that we may also assume w.l.o.g. that there exists a smooth section $\tau\colon U\longrightarrow\widehat{K}$ of q satisfying $\tau(1_K)=1_{\widehat{K}}$.

Next we choose a smooth partition of unity $\lambda_i\colon V_i\longrightarrow[0,1]$. For $\gamma\in\Sigma^{-1}(\mathfrak{W})$ we then set

$$\Lambda_i(\gamma):=\exp_*\left(\sum_{j\leq i}\lambda_i\cdot\log_*(\gamma)\right)\cdot\exp_*\left(\sum_{j<i}\lambda_i\cdot\log_*(\gamma)\right)^{-1}$$

and note that we have

$$\gamma=\Lambda_n(\gamma)\cdot\Lambda_{n-1}(\gamma)\cdots\Lambda_1(\gamma).$$

Moreover, $\lambda_i(\pi(p))=0$ implies $\Lambda_i(\gamma)(p)=1$ and thus $\operatorname{supp}(\Lambda_i(\gamma))\subset V_i$. Moreover, we have $\Sigma(\Lambda_i(\gamma))_i\in C^{\infty}(\overline{V_i},W)$ by the definition of $\mathfrak{W}$.

We now use all the data that we collected so far to define lifts of each $\Lambda_i(\gamma)$. To this end we first introduce functions $k_i\colon\ P|_{\overline{V_i}}\longrightarrow K$, defined by $p=\sigma_i(\pi(p)).k_i(p)$. Then the assignment

$$P|_{V_i}\ni p\longmapsto k_i(p).\tau\left(\Sigma(\Lambda_i(\gamma))_i(\pi(p))\right)\tag{4.5}$$

is smooth since τ and $\Sigma(\Lambda_i(\gamma))_i$ are so and equivariant since k_i is so. Moreover, (4.5) vanishes on a neighborhood of each point in $\partial\overline{V_i}$ since λ_i and thus $\tau\circ\Sigma(\Lambda_i(\gamma))_i$ do so. Consequently, we may extend (4.5) by $e_{\widehat{K}}$ to all of P, defining a lift $\Theta_i(\gamma)$ of $\Lambda_i(\gamma)$. Indeed, we have for $p\in\pi^{-1}(V_i)$

$$\begin{aligned}q(\Theta_i(\gamma)(p))&=q\left(k_i(p).\tau\left(\Sigma(\Lambda_i(\gamma))_i(\pi(p))\right)\right)=k_i(p).q\left(\tau(\Sigma(\Lambda_i(\gamma))_i(\pi(p)))\right)=\\&k_i(p).\Sigma(\Lambda_i(\gamma))_i(\pi(p))=k_i(p).\Lambda_i(\gamma)(\sigma_i(\pi(p)))=\Lambda_i(\sigma_i(\pi(p)).k_i(p))=\Lambda_i(\gamma)(p)\end{aligned}$$

and for $p\notin\pi^{-1}(V_i)$ we have $q(\Theta_i(\gamma)(p))=q(e_{\widehat{K}})=e_K=\Lambda_i(\gamma)(p)$. Eventually,

$$\Theta(\gamma):=\Theta_n(\gamma)\cdot\Theta_{n-1}(\gamma)\cdots\Theta_1(\gamma)$$

defines a lift of γ, since we have

$$\begin{aligned}q_*(\Theta_n(\gamma)\cdot\Theta_{n-1}(\gamma)\cdots\Theta_1(\gamma))&=q_*(\Theta_n(\gamma))\cdot q_*(\Theta_{n-1}(\gamma))\cdots q_*(\Theta_1(\gamma))=\\&\Lambda_n(\gamma)\cdots\Lambda_{n-1}(\gamma)\cdots\Lambda_1(\gamma)=\gamma.\end{aligned}$$

Since $\Theta_i(\gamma)$ is constructed in terms of push-forwards of smooth maps, it depends smoothly on γ and so does $\Theta(\gamma)$.

The previous argument shows in particular that (4.4) is a fiber bundle (cf. 4.7.1). As in Lemma 4.2.3 one sees that $C^\infty(M, Z)$ is metrizable if Z is so, and thus the last claim follows from Lemma 4.2.6. □

Remark 4.2.8. *Note that all results of this section remain valid in more general situations. For instance, if we replace K by an arbitrary Lie group with exponential function that is a local diffeomorphism, then $\mathcal{G}\mathfrak{au}(P)$ is a Lie group, modeled on $\mathfrak{gau}(P)$. Moreover,* (4.1) *still defines an exponential function which itself is a local diffeomorphism. If, in addition, K is metrizable, then the proof of Lemma 4.2.3 shows that $\mathcal{G}\mathfrak{au}(P)$ is also metrizable.*

Proposition 4.2.7 generalizes to the situation where $Z \longrightarrow \widehat{K} \longrightarrow K$ is a central extension of Lie groups for which $\widehat{K}$ and K have exponential functions that are local diffeomorphisms. Since its proof only uses the fact that $\widehat{K} \longrightarrow K$ has a smooth local section, (4.4) *still admits a smooth local section in this case.*

This shows in particular that the construction applies to the smooth principal bundle $\Omega G \longrightarrow PG \longrightarrow G$, where ΩG denotes the group of smooth loops (as for instance in [BCSS07, Section 3]) and the universal central extension $U(1) \longrightarrow \widehat{\Omega G} \longrightarrow \Omega G$.

4.3 The string group as a smooth extension of G

In this section we want the give a smooth model for the string group. Our construction is mainly based on [Sto96, Section 5]. By smooth model of the string group we mean a smooth 3-connected cover of a compact Lie-group G which is a Lie group itself. We are mainly interested in the case $G = Spin(n)$ but we define more generally:

Definition 4.3.1. Let G be a compact, simple and simply connected Lie group. A *smooth string group model* for G is a Lie group $\widehat{G}$ together with a smooth homomorphism

$$\widehat{G} \xrightarrow{q} G$$

such that q is a Serre fibration, $\pi_k(\widehat{G}) = 0$ for $k \leq 3$ and that $\pi_i(q)$ is an isomorphism for $i > 3$.

Proposition 4.3.2 (Cartan [Car36])**.** *Let G be a compact, simple and simply-connected Lie group. Then*

$$\pi_2(G) = 0 \quad \textit{and} \quad \pi_3(G) \cong H^3(G, \mathbb{Z}) \cong \mathbb{Z}.$$

Corollary 4.3.3. *If $\widehat{G} \xrightarrow{q} G$ is a smooth string group model, then*

1. $\ker(q)$ *is a $K(\mathbb{Z}, 2)$ (i.e., $\pi_k(\ker(q)) \cong \mathbb{Z}$ for $k = 2$ and vanishes for $k \neq 2$);*

2. $\widehat{G}$ cannot be finite-dimensional.

Proof. 1. This follows from the long exact homotopy sequence.

2. If $\widehat{G}$ were finite-dimensional, then it would have $\ker(q)$ as a closed Lie subgroup. But by 1. we have $H^{2n}(\ker(q),\mathbb{Z}) \cong H^{2n}(K(\mathbb{Z},2),\mathbb{Z}) \cong \mathbb{Z}$, a contradiction.

□

Now we come to the construction of our string group model. Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space. Then it is well known that the projective unitary group $PU(\mathcal{H})$, together with the norm topology is a $K(\mathbb{Z},2)$ [Kui65], so that $BPU(\mathcal{H})$ is a $K(\mathbb{Z},3)$. Thus isomorphism classes of $PU(\mathcal{H})$-bundles over a manifold M are in bijection with $H^3(M,\mathbb{Z})$.

Now there is a canonical generator $1 \in H^3(G,\mathbb{Z})$. Let $P \longrightarrow G$ be a principal $PU(\mathcal{H})$-bundle over G that represents this generator. Note that $PU(\mathcal{H})$ is a Banach–Lie group (see [GN03] and references therein) which is paracompact by [Dug66, Theorem VIII.2.4] and [Bre72, Theorem I.3.1]. In particular, it is metrizable. We can choose P to be smooth [MW09] and apply the results from Section 4.2. Recall in particular the map

$$Q : \mathrm{Aut}(P) \longrightarrow \mathrm{Diff}(G)$$

that sends a bundle automorphism to its underlying diffeomorphism of the base.

Definition 4.3.4. Let G be connected, simple and simply connected and $P \longrightarrow G$ represent the generator $1 \in H^3(G,\mathbb{Z})$. Then we set

$$\mathcal{S}tring_G := \{f \in \mathrm{Aut}(P) \mid Q(f) \in G \subset \mathrm{Diff}(G)\}$$

where the inclusion $G \hookrightarrow \mathrm{Diff}(G)$ sends g to left multiplication with g. In other words: $\mathcal{S}tring_G$ is the group consisting of bundle automorphisms that cover left multiplication in G.

Note that there is also a continuous version of $\mathcal{S}tring_G$, given by

$$\mathcal{S}tring^c_G := \{f \in \mathrm{Homeo}(P) \mid f \text{ is } K\text{-equivariant and } Q(f) \in G \subset \mathrm{Diff}(G)\}.$$

The motivation for constructing a smooth model for the String group as in the present chapter now comes from the following fact [Sto96]. For the sake of completeness we include (a part of) the proof here.

Proposition 4.3.5 (Stolz). *The fibration $Q : \mathcal{S}tring^c_G \longrightarrow G$ is a 3-connected cover of G, i.e. $\pi_i(\mathcal{S}tring^c_G) = 0$ for $i \leq 3$ and $\pi_i(Q)$ is an isomorphism for $i > 3$.*

Proof. Pick a point in the fiber $p \in P$ over $1 \in G$. Let ev be the evaluation that sends a bundle automorphisms f to $f(p)$. Then we obtain a diagram

$$\begin{array}{ccccc} \mathcal{G}au^c(P) & \longrightarrow & \mathcal{S}tring_G^c & \xrightarrow{Q} & G \\ \big\downarrow{\scriptstyle ev} & & \big\downarrow{\scriptstyle ev} & & \big\downarrow{\scriptstyle \mathrm{id}} \\ PU(\mathcal{H}) & \longrightarrow & P & \xrightarrow[\pi]{} & G \end{array}$$

Now [Sto96, Lemma 5.6] asserts that $ev : \mathcal{G}au^c(P) \to PU(\mathcal{H})$ is a (weak) homotopy equivalence. The long exact homotopy sequence and the Five Lemma then show that then $ev : \mathcal{S}tring_G^c \to P$ is also a homotopy equivalence. Hence it remains to show that $P \to G$ is a 3-connected cover. By definition of P its classifying map

$$p : G \longrightarrow BPU(\mathcal{H}) \simeq K(\mathbb{Z}, 3)$$

is a generator of $H^3(G, \mathbb{Z})$, hence it induces isomorphisms on the first three homotopy groups. Thus the pullback $P \cong p^* EPU(\mathcal{H})$ of the contractible space $EPU(\mathcal{H})$ kills exactly the first three homotopy groups, i.e. P is a 3-connected cover. □

In the rest of this section we want to prove the following modification and enhancement of the preceding proposition. For its formulation recall that an extension of Lie groups is a sequence of Lie groups $A \to B \to C$ such that B is a smooth locally trivial principal A-bundle over C [Nee07].

Theorem 4.3.6. *$\mathcal{S}tring_G$ is a smooth string group model according to Definition 4.3.1. Moreover, $\mathcal{S}tring_G$ is metrizable and there exists a Fréchet–Lie group structure on $\mathcal{S}tring_G$, unique up to isomorphism, such that*

$$\mathcal{G}au(P) \to \mathcal{S}tring_G \to G \tag{4.6}$$

is an extension of Lie groups.

Proof. We first show existence of the Lie group structure. To this end we recall that there exists an extension of Fréchet–Lie groups

$$\mathcal{G}au(P) \to \mathrm{Aut}(P)_0 \to \mathrm{Diff}(G)_0, \tag{4.7}$$

where $\mathrm{Aut}(P)_0$ is the inverse image $Q^{-1}(\mathrm{Diff}(G)_0)$ of the the identity component $\mathrm{Diff}(M)_0$ [Woc08, Theorem 2.14]. The embedding $G \hookrightarrow \mathrm{Diff}(G)_0$ given by left translation gives by the exponential law [GN11] a smooth homomorphism of Lie groups since the multiplication map $G \times G \to G$ is smooth. Pulling back (4.7) along this embedding then yields the extension (4.6). Moreover, $\mathcal{S}tring_G$ is metrizable by Lemma 4.2.3 and Lemma 4.2.6.

We now discuss the uniqueness assertion, so let $\mathcal{G}au(P) \to H_i \xrightarrow{q_i} G$ for $i = 1, 2$ be two extensions of Lie gropus. The requirement for it to be a locally trivial smooth principal bundle is equivalent to the existence of a smooth local section of q_i and we thus obtain a derived extension of Lie algebras

$$\mathfrak{gau}(P) \to L(H_i) \xrightarrow{L(q_i)} \mathfrak{g}.$$

The differential of the local smooth section implements a linear continuous section of $L(q_i)$ and thus we have a (non-abelian) extension of Lie algebras in the sense of [Nee06]. Now the equivalence classes of such extensions are parametrized by $H^2(\mathfrak{g}, \mathfrak{z}(\mathfrak{gau}(P)))$ [Nee06, Theorem II.7]. Since $\mathfrak{gau}(P) = C^\infty(P, \mathfrak{pu}(H))^K$ we clearly have $\mathfrak{z}(\mathfrak{gau}(P)) = C^\infty(P, \mathfrak{z}(\mathfrak{pu}(H)))^K$, which is trivial since $\mathfrak{z}(\mathfrak{pu}(H))$ is so. Consequently, we have a morphism

$$\begin{array}{ccccc} \mathfrak{gau}(P) & \longrightarrow & L(H_1) & \longrightarrow & \mathfrak{g} \\ \| & & \downarrow \varphi & & \| \\ \mathfrak{gau}(P) & \longrightarrow & L(H_2) & \longrightarrow & \mathfrak{g} \end{array}$$

of extensions of Lie algebras. The long exact homotopy sequence for the fibration $\mathcal{G}au(P) \to H_i \xrightarrow{q} G$ shows that H_i is 1-connected, and so φ integrates to a morphism

$$\begin{array}{ccccc} \mathcal{G}au(P) & \longrightarrow & H_1 & \longrightarrow & G \\ \| & & \downarrow \Phi & & \| \\ \mathcal{G}au(P) & \longrightarrow & H_2 & \longrightarrow & G \end{array}$$

of Lie groups. Since Φ makes this diagram commute it is automatically an isomorphism.

It remains to show that $\mathcal{S}tring_G$ is a smooth model for the String group. We have the following commuting diagram

$$\begin{array}{ccccc} \mathcal{G}au(P) & \longrightarrow & \mathcal{S}tring_G & \longrightarrow & G \\ \downarrow & & \downarrow & & \| \\ \mathcal{G}au^c(P) & \longrightarrow & \mathcal{S}tring_G^c & \longrightarrow & G \end{array}.$$

By Proposition 4.2.5 the inclusion $\mathcal{G}au(P) \hookrightarrow \mathcal{G}au^c(P)$ is a homotopy equivalence. Since, furthermore, $\mathcal{S}tring_G \to G$ and $\mathcal{S}tring_G^c \to G$ are bundles, they are in particular fibrations and we obtain long exact sequences of homotopy groups. Applying the Five Lemma we see that the maps $\pi_n(\mathcal{S}tring_G) \to \pi_n(\mathcal{S}tring_G^c)$ are isomorphisms for all n. By Proposition 4.3.5 we know that $\mathcal{S}tring_G^c$ is a 3-connected cover, hence also $\mathcal{S}tring_G$.

□

Remark 4.3.7. *Note that the proof of the uniqueness assertion only used the fact that the center of* $\mathfrak{gau}(P)$ *is trivial. In fact, this shows that for an arbitrary (regular) Lie group* H *which is a* $K(\mathbb{Z},2)$ *and has trivial* $\mathfrak{z}(L(H))$ *there exists, up to isomorphism, at most one Lie group* $\widehat{H}$, *together with smooth maps* $H \longrightarrow \widehat{H}$ *and* $\widehat{H} \longrightarrow G$ *turning*

$$H \longrightarrow \widehat{H} \longrightarrow G$$

into an extension of Lie groups. Moreover, the proof shows that the uniqueness is not only up to isomorphism of Lie groups, but even up to isomorphism of extensions.

4.4 2-groups and 2-group models

One of the main problems about string group models is that they are not very tightly determined. In fact, the underlying space is just determined up to weak homotopy equivalence. This implies that the group structure can only be determined up to A_∞-equivalence and the smooth structure is not determined at all. Part of this problem is that there is in general not a good control about the fiber of $\mathcal{String}_G \longrightarrow G$, only the underlying homotopy type is determined to be a $K(\mathbb{Z},2)$.

Some of the problems can be cured by using 2-group models. This setting allows to fix the fiber more tightly. In particular there is a nice model of $K(\mathbb{Z},2)$ as a 2-group, see Example 4.4.3 below and weak equivalences of 2-groups are more restrictive than homotopy equivalences of their geometric realizations. We first want to adapt the definition and basic properties of Lie 2-groups (as already treated in section 3.2.4) to the infinite dimensional world.

Definition 4.4.1. A (strict) Lie 2-group is a category $\mathcal{G}$ such that the set of objects $\mathcal{G}_0$ and the set of morphisms $\mathcal{G}_1$ are Lie groups, all structure maps

$$s,t:\mathcal{G}_1 \longrightarrow \mathcal{G}_0 \quad i:\mathcal{G}_0 \longrightarrow \mathcal{G}_1 \quad \text{and} \quad \circ : \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \longrightarrow \mathcal{G}_1$$

are Lie group homomorphisms and s,t are submersions[1]. In the case that $\mathcal{G}_0$ and $\mathcal{G}_1$ are metrizable, we call $\mathcal{G}$ a metrizable Lie 2-group. A morphism between 2-groups is a functor $f:\mathcal{G} \longrightarrow \mathcal{G}'$ that is a Lie group homomorphism on the level of objects and on the level of morphisms.

One reason to consider 2-groups here is that they can serve as models for topological spaces by virtue of the following construction.

Definition 4.4.2. Let $\mathcal{G}$ be a Lie 2-group. Then the nerve $N\mathcal{G}$ of the category $\mathcal{G}$ is a simplicial manifold by Proposition 4.7.3. Using this we define the *geometric*

[1]Submersion in the sense that it is locally a projection, see Appendix 4.7

realization of $\mathcal{G}$ to be the geometric realization of the simplicial space $N\mathcal{G}$, i.e., the coend

$$\int^{[n]\in\Delta} (N\mathcal{G})_n \times \Delta[n] = \bigsqcup_n (N\mathcal{G})_n \times \Delta[n] \ / \ \sim .$$

Note that the coend is taken in the category of compactly generated spaces.

Example 4.4.3. *1. Consider the category $\mathcal{B}U(1)$ with one object and automorphisms given by the group $U(1)$. This is clearly a Lie 2-group. The geometric realization $|\mathcal{B}U(1)|$ is the classifying space $BU(1)$, hence a $K(\mathbb{Z},2)$. The 2-group $\mathcal{B}A$ exists moreover for each abelian Lie group A.*

2. If G is an arbitrary Lie group, then it gives rise to a 2-group by considering it as category with only identity morphisms. More precisely, in this case $\mathcal{G}_0 = \mathcal{G}_1 = G$ and all structure maps are the identity.

3. Let $K \xrightarrow{\partial} L$ be a smooth crossed module of groups ([Nee07, Definition 3.1]). Then we can form a Lie 2-group $\mathcal{G}$ using the Lie groups $\mathcal{G}_0 := L$ and $\mathcal{G}_1 := K \rtimes L$ together with the smooth maps $s(k,l) = l$, $t(k,l) = \partial(k)l$, $i(l) = (1,l)$ and $(k,l)\circ(k',l') = (kk',l)$. Up to some technicalities, each Lie 2-group arises from a crossed module in this way.

Lemma 4.4.4. *If $\mathcal{G}$ is a metrizable Lie 2-group, then*

1. *all spaces $N\mathcal{G}_n$ have the homotopy type of a CW complex;*
2. *the nerve $N\mathcal{G}$ is good, i.e. all degeneracies are closed cofibrations;*
3. *the nerve $N\mathcal{G}$ is proper, i.e Reedy cofibrant as a simplicial space (with respect to the Strom model structure);*
4. *the canonical map from the fat geometric realization $\|N\mathcal{G}\|$ to the ordinary geometric realization $|\mathcal{G}|$ is a homotopy equivalence;*
5. *the geometric realization $|\mathcal{G}|$ has the homotopy type of a CW-complex.*

Proof. 1) First note that all the spaces $(N\mathcal{G})_n$ are subspaces of $(\mathcal{G}_1)^n$ and thus are metrizable. Hence by Theorem 4.7.5 they have the homotopy type of a CW-complex.

2) Again using the fact that all $(N\mathcal{G})_n$ are metrizable and [Pal66, Theorem 7] we see that they are well-pointed in the sense that the basepoint inclusion is a closed cofibration. A statement of Roberts and Stevenson [RS, Proposition 18] then shows that $N\mathcal{G}$ is good, i.e., degeneracy maps are closed cofibrations. We roughly sketch a variant of their argument here: By the fact that $\mathcal{G}$ is a 2-group we can write the nerve as

$$\cdots \ker(s)\times\ker(s)\times\mathcal{G}_0 \rightrightarrows \ker(s)\times\mathcal{G}_0 \rightrightarrows \mathcal{G}_0$$

where the decomposition is a decomposition on the level of topological spaces. Hence to show that the degeneracies are closed cofibrations it suffices to show that $\ker(s)$

is well-pointed. But it is a retract of $\mathcal{G}_1 = \mathcal{G}_0 \times \ker s$ hence well pointed by the fact that $\mathcal{G}_1$ is well pointed.

3) Now we know that $N\mathcal{G}$ is good and in this case [Lew82, Corollary 2.4(b)] implies that $N\mathcal{G}$ is also proper.

4) By [Seg74, Proposition A1] (resp [tD74, Proposition 1]) the fat and the ordinary geometric realizations are homotopy equivalent.

5) Since all the spaces $(N\mathcal{G})_n$ have the homotopy type of a CW-complex, also the fat geometric realization has the homotopy type of a CW complex [Seg74, Proposition A1]. Thus also the ordinary realization by 4). □

Proposition 4.4.5. *If $\mathcal{G}$ and $\mathcal{G}'$ are metrizable Lie 2-groups and $f : \mathcal{G} \longrightarrow \mathcal{G}'$ is a homomorphism that is a weak homotopy equivalence on objects and morphisms, then*

$$|f| : \quad |\mathcal{G}| \longrightarrow |\mathcal{G}'|$$

is a homotopy equivalence.

Proof. First note that $Nf : N\mathcal{G} \longrightarrow N\mathcal{G}'$ is a levelwise weak homotopy equivalence. For the first two layers this is the assumption and for the rest it follows again from the product structure of the nerves given in the proof of Lemma 4.4.4 and the fact that Nf is also a product map. Then using [May74, Proposition A4] and the fact that $N\mathcal{G}$ and $N\mathcal{G}'$ are proper we conclude that also $|f| : |\mathcal{G}| \longrightarrow |\mathcal{G}'|$ is a weak homotopy equivalence. But since the geometric realizations have the homotopy type of a CW-complex, Whitehead's theorem shows that $|f|$ is an honest homotopy equivalence. □

For smooth groupoids there is a notion of weak equivalence which is inspired by equivalence of the associated stacks, see e.g. [Met03, Definition 58 and Proposition 60] and definition 3.2.21. We adopt this for infinite dimensional 2-groups.

Definition 4.4.6. A morphism $f : \mathcal{G} \longrightarrow \mathcal{G}'$ of Lie 2-groups is called *smooth weak equivalence* if the following conditions are satisfied:

1. it is smoothly essentially surjective: the map

$$s \circ \mathrm{pr}_2 : \mathcal{G}_0 \ {}_{f_0}\!\times_t \mathcal{G}'_1 \longrightarrow \mathcal{G}'_0$$

is a surjective submersion.

2. it is smoothly fully faithful: the diagram

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{f_1} & \mathcal{G}'_1 \\ \downarrow{\scriptstyle s\times t} & & \downarrow{\scriptstyle s\times t} \\ \mathcal{G}_0 \times \mathcal{G}_0 & \xrightarrow[f_0\times f_0]{} & \mathcal{G}'_0 \times \mathcal{G}'_0 \end{array}$$

is a pullback diagram.

Proposition 4.4.7. *Let $f : \mathcal{G} \longrightarrow \mathcal{G}'$ be a smooth weak equivalence between metrizable 2-groups. Then $|f| : |\mathcal{G}| \longrightarrow |\mathcal{G}'|$ is a homotopy equivalence.*

Proof. A smooth weak equivalence between 2-groups is in particular a topological weak equivalence of the underlying topological groupoids. But then Theorem 6.3 and Theorem 8.2. of [Noo08] together imply that the induced morphism $\|f\| : \|\mathcal{G}\| \longrightarrow \|\mathcal{G}'\|$ between the fat geometric realizations is a weak equivalence. Again by the fact the the fat realizations are homotopy equivalent to the geometric realizations this completes the proof. □

Now we also have to repeat the definition of smoothly separable 2-groups (see section 3.4). Note that for finite dimensional Lie groups each closed subgroup is split.

Definition 4.4.8. If $\mathcal{G}$ is a Lie 2-group, then we denote by $\underline{\pi}_0\mathcal{G}$ the group of isomorphism classes of objects in $\mathcal{G}$ and by $\underline{\pi}_1\mathcal{G}$ the group of automorphisms of $1 \in \mathcal{G}_0$. Note that $\underline{\pi}_1\mathcal{G}$ is abelian. We call $\mathcal{G}$ *smoothly separable* if $\underline{\pi}_1\mathcal{G}$ is a split Lie subgroup[2] of $\mathcal{G}_1$ and $\underline{\pi}_0\mathcal{G}$ carries a Lie group structure such that $\mathcal{G}_0 \longrightarrow \underline{\pi}_0\mathcal{G}$ is a submersion.

Proposition 4.4.9.
1. *A morphism between smoothly seperable Lie 2-groups is a smooth weak equivalence if and only if it induces Lie group isomorphisms on $\underline{\pi}_0$ and $\underline{\pi}_1$.*
2. *For a metrizable, smoothly seperable Lie 2-group $\mathcal{G}$ the sequence*

$$|\mathcal{B}\underline{\pi}_1\mathcal{G}| \longrightarrow |\mathcal{G}| \longrightarrow \underline{\pi}_0\mathcal{G}$$

is a fiber sequence of topological groups. Moreover, the right hand map is a fiber bundle and the left map is a homotopy equivalence to its fiber.

Proof. The first claim will be proved in Appendix 4.8. We thus show the second. Let us first consider the morphism $q : \mathcal{G} \longrightarrow \underline{\pi}_0\mathcal{G}$ of 2-groups where $\underline{\pi}_0\mathcal{G}$ is considered as a 2-group with only identity morphisms. Let $\mathcal{K}$ be the levelwise kernel of this map, i.e., $\mathcal{K}_0 = \ker(q_0)$ and $\mathcal{K}_1 = \ker(q_1)$. Since $q_1 = q_0 \circ s$ it is a submersion, $\mathcal{K}_0$ and $\mathcal{K}_1$ are Lie subgroups and $\mathcal{K}$ is a metrizable Lie 2-group. Then $N\mathcal{K} \longrightarrow N\mathcal{G} \longrightarrow N\underline{\pi}_0\mathcal{G}$ is an exact sequence of simplicial groups. It is easy to see that the geometric realization of this sequence is also exact, e.g., by using the fact that geometric realization preserves pullbacks [May74, Corollary 11.6]. Hence we have an exact sequence of topological groups.

$$|\mathcal{K}| \longrightarrow |\mathcal{G}| \longrightarrow \underline{\pi}_0\mathcal{G}$$

Moreover the right hand map is a $|\mathcal{K}|$-bundle since by the definition of smooth separability it admits local sections. Thus it only remains to show that $|\mathcal{B}\underline{\pi}_1\mathcal{G}| \simeq |\mathcal{K}|$. Now the inclusion $\mathcal{B}\underline{\pi}_1\mathcal{G} \longrightarrow \mathcal{K}$ is a smooth weak equivalence, which we can either see using the first part of the Proposition or by a direct argument. Then Proposition 4.4.7 shows that the realization is a homotopy equivalence. □

[2]Split Lie subgroup in the sense of Definition 4.7.2

Definition 4.4.10. Let G be a compact simple and simply connected Lie group. A *smooth 2-group model* for the string group is a smooth 2-group $\mathcal{G}$ which is smoothly seperable together with isomophisms

$$\underline{\pi}_0\mathcal{G} \xrightarrow{\sim} G \qquad \text{and} \qquad \underline{\pi}_1\mathcal{G} \xrightarrow{\sim} U(1)$$

such that $|\mathcal{G}| \longrightarrow G$ is a 3-connected cover.

Remark 4.4.11.
- *Note that for a smooth 2-group model the geometric realization $|\mathcal{G}|$ with the canonical map $|\mathcal{G}| \longrightarrow G$ is automatically a topological group model for the string group.*
- *For a 2-group $\mathcal{G}$ with isomorphisms $\underline{\pi}_0\mathcal{G} \xrightarrow{\sim} G$ and $\underline{\pi}_1\mathcal{G} \xrightarrow{\sim} U(1)$ we already know from Proposition 4.4.9 that $|\mathcal{G}| \longrightarrow G$ is a fibration with fiber $|\mathcal{B}U(1)| \simeq K(\mathbb{Z}, 2)$. Hence the condition that $|\mathcal{G}| \longrightarrow G$ is a 3-connected cover only ensures that it has the right level, i.e. the connecting homomorphism in the long exact homotopy sequence*

  $$\mathbb{Z} = \pi_3(G) \longrightarrow \pi_2(K(\mathbb{Z}, 2)) = \mathbb{Z}$$

  is an isomorphism.
- *Considering $\mathcal{S}tring_G$ as a category with only identity morphisms we obtain a 2-group as in Example 4.4.3. However, in this case $\underline{\pi}_1\mathcal{S}tring_G$ is trivial. So it is not a 2-group model as defined above, although its geometric realization is a topological group model.*

4.5 The string group as a 2-group

The previous remark shows that Lie 2-group models have more structure than topological or Lie group models for the string group. In this section we promote our Lie group model from Section 4.3 to such a Lie 2-group model. Therefore the setting will be as in Section 4.3: G is a compact simple, simply-connected Lie group and $P \longrightarrow G$ is a smooth $PU(\mathcal{H})$ bundle that represents the generator $1 \in H^3(G, \mathbb{Z}) \cong \mathbb{Z}$.

Clearly we have the central extension $U(1) \longrightarrow U(\mathcal{H}) \longrightarrow PU(\mathcal{H})$. Furthermore $PU(\mathcal{H})$ acts by conjugation on $U(\mathcal{H})$. Using these maps we obtain a sequence

$$C^\infty(G, U(1)) \longrightarrow C^\infty(P, U(\mathcal{H}))^{PU(\mathcal{H})} \longrightarrow \mathcal{G}au(P), \tag{4.8}$$

which is a central extension of Fréchet–Lie groups by Proposition 4.2.7.

For the next proposition note that that each smooth function $f \in C^\infty(G, U(1))$ is a quotient of a smooth function $\hat{f} \in C^\infty(G, \mathbb{R})$ by the fact the G is simply connected. If we identify $U(1)$ with $\mathbb{R}/\mathbb{Z}$ we may thus identify $C^\infty(G, U(1))$ with $C^\infty(G, \mathbb{R})/\mathbb{Z}$.

Lemma 4.5.1. *If μ is the Haar measure on G, then the map*

$$I_G\colon C^\infty(G,U(1)) \longrightarrow U(1), \qquad I_G\left[\hat{f}\right] := \left[\int_G \hat{f}\ d\mu\right]$$

is a smooth group homomorphism. This map I_G is invariant under the right action of G on $C^\infty(G,U(1))$ which is given by left multiplication in the argument.

Proof. We denote by $dI_G : C^\infty(G,\mathbb{R})$ the map on Lie algebras that is given by $dI_G(f) := \int_G f\ d\mu$. First note that dI_G is linear and continuous in the topology of uniform convergence since we have $|\int_G f\ d\mu| \leq \int_G |f|\ d\mu$. It thus is also continuous in the finer C^∞-topology and in particular smooth. Furthermore it is invariant under left multiplication with G. Moreover, dI_G factors since it maps $\mathbb{Z} \subset C^\infty(G,\mathbb{R})$ to $\mathbb{Z} \subset \mathbb{R}$. □

Now we can use the group homomorphism I_G to turn the smooth extension (4.8) into a $U(1)$ extension:

Definition 4.5.2. We define

$$\widehat{\mathcal{G}au}(P) := C^\infty(P,U(\mathcal{H}))^{PU(\mathcal{H})} \times U(1)\ /\ \sim,$$

where we identify $(\varphi\cdot\mu,\lambda) \sim (\varphi, I_G(\mu)\cdot\lambda)$ for $\mu \in C^\infty(G,U(1))$.

Proposition 4.5.3. *The sequence*

$$U(1) \longrightarrow \widehat{\mathcal{G}au}(P) \longrightarrow \mathcal{G}au(P) \tag{4.9}$$

is a central extension of metrizable Fréchet Lie groups and the space $\widehat{\mathcal{G}au}(P)$ is contractible.

Proof. By definition $\widehat{\mathcal{G}au}(P)$ is just the association of the bundle $C^\infty(P,U(\mathcal{H}))^{PU(\mathcal{H})}$ over $\mathcal{G}au(P)$ along the homomorphism $I_G : C^\infty(G,U(1)) \longrightarrow U(1)$. Hence it is a smooth manifold and a central extension of $\mathcal{G}au(P)$. More precisely we may take a locally smooth $C^\infty(G,U(1))$-valued cocycle describing the central extension (4.8). Composing this with I_G yields then a locally smooth cocycle representing the central extension (4.9) (cf. [Nee02, Proposition 4.2]). Since the modeling space is the product of the modeling space of the fiber and the base it is in particular Fréchet. In addition, $\widehat{\mathcal{G}au}(P)$ is metrizable by Lemma 4.2.3 and Lemma 4.2.6.

Now we come to the second part of the claim. In order to show that $\widehat{\mathcal{G}au}(P)$ is weakly contractible we first define another space $\widetilde{\mathcal{G}au}(P)$ using the homomorphism $ev : C^\infty(G,U(1)) \longrightarrow U(1)$ instead of I_G. More precisely,

$$\widetilde{\mathcal{G}au}(P) := C^\infty(P,U(\mathcal{H}))^{PU(\mathcal{H})} \times U(1)\ /\ \sim_{ev}$$

where we identify $(\varphi \cdot \mu, \lambda) \sim_{ev} (\varphi, \mu(1) \cdot \lambda)$ for $\mu \in C^\infty(G, U(1))$. Note that ev is smooth since arbitrary point evaluations are so. Thus $\widetilde{\mathcal{G}au}(P)$ is a $U(1)$ central extension of $\mathcal{G}au(P)$ as well an also metrizable by Lemma 4.2.6.

We claim that the $\widehat{\mathcal{G}au}(P)$ and $\widetilde{\mathcal{G}au}(P)$ are homeomorphic as spaces (not as groups). Therefore we first show that the homomorphisms ev and I_G are homotopic as group homomorphisms, i.e. there is a homotopy

$$H : C^\infty(G, U(1)) \times [0,1] \longrightarrow U(1)$$

such that each $H_t := H(-,t)$ is a Lie group homomorphism, $H_0 = ev$ and $H_1 = I_G$. We first define the smooth map

$$dH \colon C^\infty(G, \mathbb{R}) \times [0,1] \longrightarrow \mathbb{R}, \quad (f,t) \longmapsto t \cdot f(1) + (1-t) \cdot \int_G f \, d\mu$$

Since each dH_t maps $\mathbb{Z}$ into $\mathbb{Z}$ it in particular induces a smooth group homomorphism H_t via the identification $C^\infty(G, U(1)) \cong C^\infty(G, R)/\mathbb{Z}$. Now we can use H_t to define a $U(1)$-bundle E over $\mathcal{G}au(P) \times [0,1]$ by

$$E := C^\infty(P, U(\mathcal{H}))^{PU(\mathcal{H})} \times U(1) \times [0,1] \;/\; \sim_H$$

where we identify $(\varphi \cdot \mu, \lambda, t) \sim_H (\varphi, H(\mu, t) \cdot \lambda, t)$. Obviously $E\big|_{\mathcal{G}au(P) \times 0} \cong \widetilde{\mathcal{G}au}(P)$ and $E\big|_{\mathcal{G}au(P) \times 1} \cong \widehat{\mathcal{G}au}(P)$. Thus $\widetilde{\mathcal{G}au}(P)$ and $\widehat{\mathcal{G}au}(P)$ are isomorphic as continuous bundles [tD08, Theorem 14.3.2].

Since we now know that $\widehat{\mathcal{G}au}(P) \cong \widetilde{\mathcal{G}au}(P)$, it is sufficient to show that $\widetilde{\mathcal{G}au}$ is contractible. To this end we first pick a point $p \in P$ in the fiber over $1 \in G$. Evaluation at p yields a group homomorphism

$$ev : \mathcal{G}au(P) = C^\infty(P, PU(\mathcal{H}))^{PU(\mathcal{H})} \longrightarrow PU(\mathcal{H}).$$

which is a weak homotopy equivalence by [Sto96, Lemma 5.6] and Proposition 4.2.5. We now define another Lie group homomorphism $\Phi : \widetilde{\mathcal{G}au}(P) \longrightarrow U(\mathcal{H})$ by $\Phi([\varphi, \lambda]) := \lambda \cdot \varphi(p)$. By definition of $\widetilde{\mathcal{G}au}(P)$ this is well defined and the diagram

$$\begin{array}{ccccc}
U(1) & \longrightarrow & \widetilde{\mathcal{G}au}(P) & \longrightarrow & \mathcal{G}au(P) \\
\| & & \downarrow \Phi & & \downarrow ev \\
U(1) & \longrightarrow & U(\mathcal{H}) & \longrightarrow & PU(\mathcal{H})
\end{array}$$

commutes. Since ev is a weak homotopy equivalence it follows from the long exact homotopy sequence and the Five Lemma that also Φ is a weak homotopy equivalence. Therefore the weak contractibility of $\widetilde{\mathcal{G}au}(P)$ is implied by the weak contractibility of $U(\mathcal{H})$. This also implies contractibility of $\widetilde{\mathcal{G}au}(P)$ by Theorem 4.7.5. □

Combining the two sequences (4.6) and (4.9) we obtain an exact sequence

$$1 \longrightarrow U(1) \longrightarrow \widehat{\mathcal{G}au}(P) \stackrel{\partial}{\longrightarrow} \mathcal{S}tring_G \longrightarrow G \longrightarrow 1 \tag{4.10}$$

of Fréchet Lie groups, where ∂ is the composition $\widehat{\mathcal{G}au}(P) \longrightarrow \mathcal{G}au(P) \longrightarrow \mathcal{S}tring_G$. We furthermore define a smooth right action of $\mathcal{S}tring_G$ on $\widehat{\mathcal{G}au}(P)$ by:

$$[\varphi, \lambda]^f := [\varphi \circ f, \lambda] \qquad \text{for } f \in \mathcal{S}tring_G \subset \mathrm{Aut}(P). \tag{4.11}$$

Proposition 4.5.4. *The action is well defined. Together with the morphism* $\partial : \widehat{\mathcal{G}au}(P) \longrightarrow \mathcal{S}tring_G$ *this forms a smooth crossed module.*

Proof. The action is well-defined since for $\varphi \in C^\infty(P, U(\mathcal{H}))^{PU(\mathcal{H})}$, $\mu \in C^\infty(G, U(1))$ and $f \in \mathcal{S}tring_G$ we have

$$[(\varphi \cdot \mu) \circ f, \lambda] = [(\varphi \circ f) \cdot (\mu \circ Q(f)), \lambda] = [\varphi \circ f, I_G(\mu \circ Q(f)) \cdot \lambda] = [\varphi \circ f, I_G(\mu) \cdot \lambda]$$

where the last equality holds by the fact that I_G is invariant under left multiplication as shown in Lemma 4.5.1.

The action of $\mathrm{Aut}(P)$ on $\mathcal{G}au(P) \cong C^\infty(P, PU(\mathcal{H}))^{PU(\mathcal{H})}$, given by $\varphi^f := \varphi \circ f$ is the conjugation action of $\mathcal{G}au(P)$ on itself [Woc08, Remark 2.8]. This shows that ∂ is equivariant and that (4.10) and (4.11) define indeed a crossed module. It thus remains to show that the action map $\widehat{\mathcal{G}au}(P) \times \mathcal{S}tring_G \longrightarrow \widehat{\mathcal{G}au}(P)$ is smooth. Since $\mathcal{S}tring_G$ acts by diffeomorphisms it suffices to show that the restriction of the action map $U \times \widehat{\mathcal{G}au}(P) \longrightarrow \widehat{\mathcal{G}au}(P)$ for U some identity neighborhood in $\mathcal{S}tring_G$ is smooth. By Theorem 4.3.6 we find some U which is diffeomorphic to $\mathcal{G}au(P) \times O$ for some open $O \subset G$ with $1_G \in O$. Writing out the induced map $\widehat{\mathcal{G}au}(P) \times \mathcal{G}au(P) \times O \longrightarrow \widehat{\mathcal{G}au}(P)$ in local coordinates one sees that the smoothness of this map is implied from the smoothness of the action of $\mathcal{G}au(P)$ on $C^\infty(P, U(\mathcal{H}))^{PU(\mathcal{H})}$ and the smoothness of the natural action $C^\infty(G, U(H)) \times \mathrm{Diff}(G) \longrightarrow C^\infty(G, U(H))$, $(\varphi, f) \longmapsto \varphi \circ f$ [GN11]. $\square$

Definition 4.5.5. Let G be a compact simple and simply connected Lie group. Then we define $\mathtt{STRING}_G$ to be the metrizable Fréchet Lie 2-group associated to the crossed module $\big(\widehat{\mathcal{G}au}(P) \stackrel{\partial}{\longrightarrow} \mathcal{S}tring_G\big)$ according to example 4.4.3.

In more detail we have

$$\big(\mathtt{STRING}_G\big)_0 := \mathcal{S}tring_G \qquad \text{and} \qquad \big(\mathtt{STRING}_G\big)_1 := \widehat{\mathcal{G}au}(P) \rtimes \mathcal{S}tring_G$$

with structure maps given by

$$s(g,f) = f \qquad t(g,f) = \partial(g)h \qquad i(f) = (1,f) \quad \text{and} \quad (g,f) \circ (g',f') = (gg',f).$$

From the sequence (4.10) we obtain isomorphisms

$$\underline{\pi}_0\, \mathtt{STRING}_G = \mathrm{coker}(\partial) \xrightarrow{\sim} G \quad \text{and} \quad \underline{\pi}_1\, \mathtt{STRING}_G = \ker(\partial) \xrightarrow{\sim} U(1). \tag{4.12}$$

Moreover we can consider the Lie group $\mathcal{S}tring_G$ from Definition 4.3.4 also as a 2-group which has only identity morphisms, see Example 4.4.3. Then there is clearly an inclusion $\mathcal{S}tring_G \to \mathtt{STRING}_G$ of 2-groups.

Theorem 4.5.6. *The 2-group* $\mathtt{STRING}_G$ *together with the isomorphisms* (4.12) *is a smooth 2-group model for the string group (in the sense of Definition 4.4.10). The inclusion* $\mathcal{S}tring_G \to \mathtt{STRING}_G$ *induces a homotopy equivalence*

$$\mathcal{S}tring_G \to |\,\mathtt{STRING}_G\,|$$

Proof. We first want to show that the map $\mathcal{S}tring_G = |\mathcal{S}tring_G| \to |\,\mathtt{STRING}_G\,|$ is a homotopy equivalence. Therefore note that the inclusion functor $\mathcal{S}tring_G \to \mathtt{STRING}_G$ is given by the identity on the level of objects and by the canonical inclusion

$$\mathcal{S}tring_G \to \widehat{\mathcal{G}au} \rtimes \mathcal{S}tring_G$$

on the level of morphisms. Both of these maps are homotopy equivalences, the identity for trivial reasons and the inclusion by the fact that $\widehat{\mathcal{G}au}$ is contractible as shown in Proposition 4.5.3. Since, furthermore, both Lie-2-groups are metrizable we can apply Proposition 4.4.5 and conclude that the geometric realization of the functor is a homotopy equivalence.

It only reamains to show that $|\,\mathtt{STRING}_G\,| \to G$ is a 3-connected cover. The homotopy equivalence $\mathcal{S}tring_G \simeq |\,\mathtt{STRING}_G\,|$ clearly commutes with the projection to G. Thus the claim is a consequence of the fact that $\mathcal{S}tring_G$ is a smooth String group model (in particular a 3-connected cover) as shown in Theorem 4.3.6. □

Remark 4.5.7. *We obtain a crossed module* $\widetilde{\mathcal{G}au(P_eG)} \to \mathcal{PS}tring_G$ *from Remark 4.2.8, where* $\mathcal{PS}tring_G$ *is the restriction of the Lie group extension*

$$\mathcal{G}au(P) \to \mathrm{Aut}(P)_0 \to \mathrm{Diff}(G)_0 \tag{4.13}$$

from [Woc08, Theorem 2.14] to $G \subset \mathrm{Diff}(G)_0$ *and* $\mathcal{PS}tring_G \subset \mathrm{Aut}(P_eG)$ *acts canonically* $\widetilde{\mathcal{G}au(P_eG)} := C^\infty(P_eG, \widehat{\Omega G})^{\Omega G}$*. As in Definition 4.5.2 we then define* $\widehat{\mathcal{G}au}(P_eG)$ *to be associated to* $\widetilde{\mathcal{G}au(P_eG)}$ *along the homomorphism* I_G*. This furnishes another crossed module*

$$\widehat{\mathcal{G}au}(P_eG) \to \mathcal{PS}tring_G,$$

where the action of $\mathcal{PS}tring_G \subset \mathrm{Aut}(P_eG)$ *is defined in the same way as in as in* (4.11).

4.6 Comparison of string structures

One reason for the importance of Lie 2-groups is that they allow for a bundle theory as investigated in chapter 3. As mentioned in the introduction, these 2-bundles play for example a role in mathematical physics. In particular in supersymmetric sigma models, which are used to describe fermionic string theories, they serve as target space background data [FM06, Wal09, Bun09]. For a precise definition of 2-bundles we refer the reader to section 3.6. Note that there we have not explicitly considered infinite dimensioanl Lie 2-groups, but all the proofs carry over to this setting. We mainly need the following facts about smooth 2-bundles here, which we repeat for the convienience of the reader:

1. For a Lie 2-group $\mathcal{G}$ and a finite dimensional manifold M all 2-bundles form a bicategory $2\text{-}\mathcal{B}un_{\mathcal{G}}(M)$ (Definition 3.6.5).

2. For a smoothly separable, metrizable Lie 2-group $\mathcal{G}$ isomorphism classes of $\mathcal{G}$-2-bundles are in bijection with non-abelian cohomology $\check{\mathrm{H}}^1(M,\mathcal{G})$ and with isomorphism classes of continuous $|\mathcal{G}|$-bundles (Theorem 3.4.6, Theorem 3.5.20 and Theorem 3.7.1.

3. For a Lie group G considered as a Lie 2-group (as in example 4.4.3) the definition of 2-bundles reduces to that of 1-bundles. More precisely we have an equivalence of bicategories $\mathcal{B}un_G(M) \to 2\text{-}\mathcal{B}un_G(M)$ where $\mathcal{B}un_G(M)$ is considered as a bicategory with only identity 2-morphisms (Example 3.5.9). Moreover non-abelian cohomology $\check{\mathrm{H}}^1(M,G)$ reduces in this case to the ordinary Čech-cohomology.

4. For a morphism of $\mathcal{G} \to \mathcal{G}'$ of Lie 2-groups we have an induced functor

   $$2\text{-}\mathcal{B}un_{\mathcal{G}}(M) \to 2\text{-}\mathcal{B}un_{\mathcal{G}'}(M)$$

   and an induced morphism $\check{\mathrm{H}}^1(M,\mathcal{G}) \to \check{\mathrm{H}}^1(M,\mathcal{G}')$. For a smooth weak equivalence between metrizable, smoothly separable 2-groups the induced functor is an equivalence of bicategories. Theorem 3.6.11.

Proposition 4.6.1. *The inclusion* $\mathcal{S}tring_G \to \mathtt{STRING}_G$ *induces a functor*

$$\mathcal{B}un_{\mathcal{S}tring_G}(M) \to 2\text{-}\mathcal{B}un_{\mathtt{STRING}_G}(M)$$

which on isomorphism classes is given by the induced map

$$\check{\mathrm{H}}^1\big(M,\mathcal{S}tring_G\big) \to \check{\mathrm{H}}^1\big(M,\mathtt{STRING}_G\big)$$

for each finite dimensional manifold M. This map is a bijection.

Proof. This follows essentially from the fact that the geometric realizations of the functor $\mathcal{S}tring_G \longrightarrow |\,\mathtt{STRING}_G\,|$ is a homotopy equivalence as shown in Theorem 4.5.6. Then one knows that the induced map between isomorphism classes of continuous $\mathcal{S}tring_G$-bundles and $|\,\mathtt{STRING}_G\,|$-bundles is an isomorphisms. Then the claim follows by the facts given above. □

The importance of the last proposition is that it allows to directly compare $\mathcal{S}tring_G$-structures and $\mathtt{STRING}_G$-structures. We mainly built the 2-group model $\mathtt{STRING}_G$ in order to have such a comparison available. Now one can use the $\mathtt{STRING}_G$ 2-group and compare it in the world of Lie 2-groups to other smooth 2-group models and so obtain an overall comparison. We will make precise what this means in detail:

Definition 4.6.2. A morphism between 2-group models $\mathcal{G}$ and $\mathcal{G}'$ is a smooth homomorphism $f : \mathcal{G} \longrightarrow \mathcal{G}'$ such that the diagrams

$$\begin{array}{ccc} \underline{\pi}_0\mathcal{G} & \xrightarrow{\underline{\pi}_0 f} & \underline{\pi}_0\mathcal{G}' \\ & {\scriptstyle\sim}\searrow \quad \swarrow{\scriptstyle\sim} & \\ & G & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \underline{\pi}_1\mathcal{G} & \xrightarrow{\underline{\pi}_1 f} & \underline{\pi}_1\mathcal{G}' \\ & {\scriptstyle\sim}\searrow \quad \swarrow{\scriptstyle\sim} & \\ & U(1) & \end{array}$$

commute.

Proposition 4.6.3. *Let $f : \mathcal{G} \longrightarrow \mathcal{G}'$ be a morphism between metrizable, smoothly separable smooth 2-group models.*

1. *Then f is automatically a smooth weak equivalence of 2-groups.*
2. *The geometric realization $|f| : |\mathcal{G}| \longrightarrow |\mathcal{G}'|$ is a homotopy equivalence of topological groups. Furthermore it commutes with the projection to G and the inclusion of $|\mathcal{B}U(1)|$ (see proposition 4.4.9).*
3. *For a manifold M the induced functor*

$$f_* : 2\text{-}\mathcal{B}un_{\mathcal{G}}(M) \longrightarrow 2\text{-}\mathcal{B}un_{\mathcal{G}'}(M).$$

is an equivalence of bicategories.

Proof. The first assertion follows from the characterization of weak equivalences given in Proposition 4.4.9 and the second from Proposition 4.4.7. The last statement is then implied by fact 4 mentioned above. □

This shows that from such a morphism between 2-group models we can directly derive comparisons between the bundle theories. Of course one should allow spans of such morphisms. An interesting thing would be to give directly such a span connecting our model $\mathtt{STRING}_G$ to the model given in [BCSS07]. There are cohomological reasons to expect that such a span should exist [WW11].

4.7 Appendix: Locally convex manifolds and Lie groups

In this section we provide the necessary information to clarify the differential geometric background. If X, Y are locally convex vector spaces and $U \subset X$ is open, then $f\colon U \longrightarrow Y$ is called *continuously differentiable* if for each $v \in X$ the limit

$$df(x).v := \lim_{h \longrightarrow 0} \frac{1}{h}(f(x+hv) - f(x)) \tag{4.14}$$

exists and the map $U \times X \longrightarrow Y$, $(x, v) \longmapsto df(x).v$ is continuous. It is called *smooth* if the iterated derivatives $d^n f\colon U \times X^n \longrightarrow Y$ exist and are also continuous. Concepts like manifolds and tangent bundles carry over to this setting of differential calculus, in particular the notion of Lie groups and their associated Lie algebras [GN11]. Moreover, manifolds in this sense are in particular topological manifolds in the sense of [Pal66].

If M, N are manifolds and $f\colon M \longrightarrow N$ is smooth, then we call f an *immersion* if for each $m \in M$ there exist charts around m and $f(m)$ such that the corresponding coordinate representation of f is an inclusion of the modeling space of M as a direct summand into the modeling space of N. Analogously, f is called *submersion* if for each $m \in M$ the corresponding coordinate representation is a projection onto a direct summand (cf. [Lan99, §II.2], [Ham82, Definition 4.4.8]).

If G is a Lie group, then a closed subgroup $H \subset G$ is called *Lie subgroup* if it is also a submanifold. This is not automatically the case in infinite dimensions (cf. [Bou98b, Exercise III.8.2]). Moreover, if H is a closed Lie subgroup, then it need not be immersed as the example of a non-complemented subspace in a Banach space shows.

Lemma 4.7.1. *If $H \subset G$ is a closed subgroup and G/H carries an arbitrary Lie group structure such that $G \longrightarrow G/H$ is smooth, then the following are equivalent.*

1. *$G \longrightarrow G/H$ admits smooth local sections around each point.*
2. *$G \longrightarrow G/H$ is a locally trivial bundle.*
3. *$G \longrightarrow G/H$ is a submersion.*

In any of these cases H is an immersed Lie subgroup and G/H carries the quotient topology.

Proof. If $G \longrightarrow G/H$ admits local sections, then

$$q^{-1}(U) \ni g \longmapsto (q(g), g \cdot \sigma(q(g))^{-1}) \in U \times H$$

defines a local trivialization of $G \longrightarrow G/H$. This shows equivalence of the first two statements and with this aid on sees also the equivalence with the last statement.

From the second it follows in particular that $H \hookrightarrow G$ is an immersion. Since submersions are open, and since surjective open maps are quotient maps, the topology on G/H has to be the quotient topology. □

Definition 4.7.2. (cf. [Nee07, Definition 2.1]) A *split Lie subgroup* of a Lie group is a closed subgroup that fulfills one of the three equivalent conditions of the preceding lemma.

Note that each immersed Lie subgroup of a Banach–Lie group is split by [Bou98b, Proposition III.1.10]. This implies in particular that each closed subgroup of a finite-dimensinoal Lie group is split by [Bou98b, Theorem III.8.2]. Also note that if H is closed and normal and G/H carries a Lie group structure such that $G \longrightarrow G/H$ is smooth, then a single local smooth section can be moved around with the group multiplication to yield a local smooth section around each point.

Proposition 4.7.3. *If X, Y, Z are manifolds, $f\colon X \longrightarrow Z$ is smooth and $g\colon Y \longrightarrow Z$ is a submersion then the fiber product $X \times_Z Y$ exists in the category of smooth manifolds and the projection*

$$X \times_Z Y \longrightarrow X$$

is a submersion. Moreover the identity is a submersion and the composition of submersions is again a submersion. That means submersions form a Grothendick pretopology (see [Met03, Definition 5]) on the category of smooth manifolds

Proof. This is a slight generalization of [Ham82, 4.4.10]. The proof of [Lan99, Proposition II.2.6], showing that the first statement is a local one and of [Lan99, Proposition II.2.7], showing this for a projection carry over literally to our more general setting. Moreover, the question of being a submersion is also local, so [Lan99, Proposition II.2.7] shows that $X \times_Z Y \longrightarrow X$ is one. □

Corollary 4.7.4. *The fibers of a submersion are submanifolds.*

A manifold is called metrizable if the underlying topology is so. Note that metrizable is equivalent to paracompact and locally metrizable [Pal66, Theorem 1]. Thus a Fréchet manifold is metrizable if and only if it is paracompact. Moreover, we have the following

Theorem 4.7.5. *A metrizable manifold has the homotopy type of a CW-complex. In particular, weak homotopy equivalences between metrizable manifolds are homotopy equivalences.*

Proof. By [Pal66, Theorem 14] a metrizable manifold is dominated by CW-complex. By a theorem of Whitehead this implies that it has the homotopy type of a CW-complex (cf. [Hat02, Prop. A.11]). □

4.8 Appendix: A characterization of smooth weak equivalences

In this section we will exclusively be concerned with smoothly separable Lie 2-groups. Recall that for a smoothly separable Lie 2-group $\mathcal{G}$ we require among other things that $\underline{\pi}_1\mathcal{G}$ is a split Lie subgroup. Our main goal here is to prove part 1 of Proposition 4.4.9. This will be done in several steps.

Lemma 4.8.1. *Let $\mathcal{G}$ be a smoothly separable Lie 2-group Then the map $s \times t : \mathcal{G}_1 \longrightarrow \mathcal{G}_0 \times_{\underline{\pi}_0\mathcal{G}} \mathcal{G}_0$ is a surjective submersion.*

Proof. By definition the map $s \times t$ is a surjective map onto the submanifold $\mathcal{G}_0 \times_{\underline{\pi}_0\mathcal{G}} \mathcal{G}_0$ of $\mathcal{G}_0 \times \mathcal{G}_0$

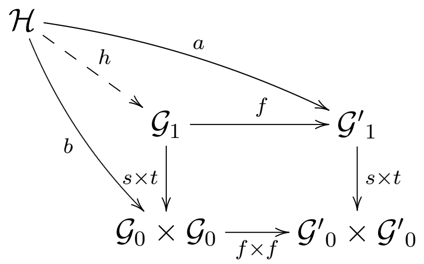
It admits local sections because its kernel $\underline{\pi}_1\mathcal{G}$ is a split Lie subgroup. By Lemma 4.7.1 this implies that it is a submersion. □

Proposition 4.8.2. *Let $f : \mathcal{G} \longrightarrow \mathcal{G}'$ be a morphism of smoothly separable Lie 2-groups inducing an isomorphism on $\underline{\pi}_1$. Then f is smoothly fully faithful, i.e.,*

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{f} & \mathcal{G}'_1 \\ \downarrow{\scriptstyle s\times t} & & \downarrow{\scriptstyle s\times t} \\ \mathcal{G}_0 \times \mathcal{G}_0 & \xrightarrow[f\times f]{} & \mathcal{G}'_0 \times \mathcal{G}'_0 \end{array}$$

is a pullback diagram of Lie groups.

Proof. It is clear that this is a pullback diagram of groups by the general theory of 2-groups. Let $\mathcal{H}$ be a Lie group and consider the diagram



where a, b are morphisms of Lie groups. We have to show that the unique map $h : \mathcal{H} \longrightarrow \mathcal{G}_1$ supplied by the pullback of groups is also smooth. By Lemma 4.8.1 there exists a smooth local section $\gamma\colon U \longrightarrow \mathcal{G}_1$ of $s \times t$, defined on an indentity neighbourhood $U \subset \mathcal{G}_0 \times_{\underline{\pi}_0} \mathcal{G}_0$. Since b maps to $\mathcal{G}_0 \times_{\underline{\pi}_0} \mathcal{G}_0$, $V := b^{-1}(U)$ is an open identity neighborhood in $\mathcal{H}$.

We now observe that

$$h'\colon V \longrightarrow \mathcal{G}_1, \quad x \longmapsto \gamma(b(x)) \cdot (\underline{\pi}_1 f_1)^{-1}(f_1(\gamma(b(x)))^{-1} \cdot a(x))$$

is smooth since $f_1(\gamma(b(x)))^{-1} \cdot a(x) \in \underline{\pi}_1\mathcal{G}'$ and f_1 restricts to a diffeomorphism $\underline{\pi}_1\mathcal{G} \longrightarrow \underline{\pi}_1\mathcal{G}'$. It satisfies $f_1 \circ h' = a|_V$, and we also have $(s \times t) \circ h' = b$ since γ is a section of $s \times t$. Thus h coincides with h' on V, showing that h is a smooth homomorphism of Lie groups. □

Proposition 4.8.3. *Let $f : \mathcal{G} \longrightarrow \mathcal{G}'$ be a morphism of smoothly separable Lie 2-groups inducing an isomorphism on $\underline{\pi}_0$. Then f is smoothly essentially surjective, i.e., the morphism*

$$s \circ \mathrm{pr}_2 : \mathcal{G}_0 \;{}_{f_0}\!\times_t \mathcal{G}'_1 \longrightarrow \mathcal{G}'_0$$

is a smooth submersion.

Proof. Surjectivity is clear because f is surjective on $\underline{\pi}_0$. To see that $s \circ \mathrm{pr}_2$ is a submersion we will construct a local smooth section. Since the map $p : \mathcal{G}_0 \longrightarrow \underline{\pi}_0\mathcal{G}$ is a submersion there exists a local section $\sigma\colon U \longrightarrow \mathcal{G}_0$ of p.

For brevity let us denote the "roundtrip" map, restricted to $V := p'^{-1}(\underline{\pi}_0 f(U))$ as $R = f_0 \circ \sigma \circ (\underline{\pi}_0 f)^{-1} \circ p'$. For $x \in V$ we then have $x \cong R(x)$ and thus $(x, R(x)) \in \mathcal{G}'_0 \times_{\underline{\pi}_0\mathcal{G}} \mathcal{G}'_0$. Now there exists a local smooth section $\tau\colon W \longrightarrow \mathcal{G}'_1$ of $s' \times t'$ for $W \subset V \times_{\underline{\pi}_0\mathcal{G}'} V$ open. Then

$$\begin{array}{rcl} S : (\mathrm{id}_{\mathcal{G}'_0}\big|_V \times R)^{-1}(W) & \longrightarrow & \mathcal{G}_0 \;{}_{f_0}\!\times_t \mathcal{G}'_1 \\ x & \longmapsto & (\sigma((\underline{\pi}_0 f)^{-1}(p'(x))), \tau(x, R(x))) \end{array}$$

is the required section since we have

$$f_0(\sigma((\underline{\pi}_0 f)^{-1}(p'(x)))) = R(x) = t(\tau(x, R(x)))$$

and $s(\tau(x)) = x$. □

Corollary 4.8.4. *If $f : \mathcal{G} \longrightarrow \mathcal{G}'$ is a morphism of smoothly separable Lie 2-groups inducing isomorphisms on $\underline{\pi}_0$ and $\underline{\pi}_1$ then f is a weak equivalence.*

The converse of the first part of Proposition 4.4.9 also holds:

Proposition 4.8.5. *A smooth weak equivalence $f : \mathcal{G} \longrightarrow \mathcal{G}'$ of smoothly separable Lie 2-groups induces isomorphisms on $\underline{\pi}_0$ and $\underline{\pi}_1$.*

Proof. Since f is in particular an equivalence of the underlying categories in the set-theoretic sense, it is clear that its induced morphisms $\underline{\pi}_0 f : \underline{\pi}_0\mathcal{G} \longrightarrow \underline{\pi}_0\mathcal{G}'$ and $\underline{\pi}_1 f : \underline{\pi}_1\mathcal{G} \longrightarrow \underline{\pi}_1\mathcal{G}'$ are group isomorphisms. From the diagram

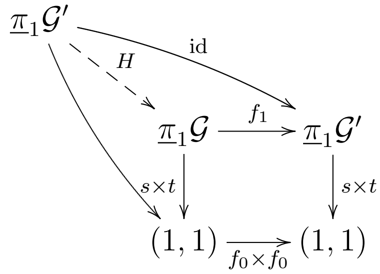
$$\begin{array}{ccc} \mathcal{G}_0 \;{}_{f_0}\!\times_t \mathcal{G}'_1 & \xrightarrow{\mathrm{pr}_2} & \mathcal{G}'_1 \\ \downarrow & & \downarrow{\scriptstyle s} \\ \mathcal{G}_0 & \xrightarrow{f_0} & \mathcal{G}'_0 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ \underline{\pi}_0\mathcal{G} & \xrightarrow[\underline{\pi}_0 f]{} & \underline{\pi}_0\mathcal{G}'_0 \end{array} \qquad (4.15)$$

we see that $\underline{\pi}_0 f$ is smooth since we can pick a local section $\sigma : \underline{\pi}_0\mathcal{G} \longrightarrow \mathcal{G}_0$ of the submersion $p : \mathcal{G}_0 \longrightarrow \underline{\pi}_0\mathcal{G}$, which shows that locally

$$\underline{\pi}_0 f = p' \circ f_0 \circ \sigma.$$

To see that $(\underline{\pi}_0 f)^{-1}$ is smooth as well we choose a local section $\sigma' : \underline{\pi}_0\mathcal{G}' \longrightarrow \mathcal{G}'_0$. Since we know that $s \circ \mathrm{pr}_2 : \mathcal{G}_0 \ {}_{f_0}\!\times_t \mathcal{G}'_1 \longrightarrow \mathcal{G}'_0$ is a submersion, we can also choose a section τ for that map, and composing $\tau \circ \sigma'$ with the projection to $\mathcal{G}_0$ and finally to $\underline{\pi}_0\mathcal{G}$ coincides with $(\underline{\pi}_0 f)^{-1}$ which is therefore smooth.

To see that $\underline{\pi}_1 f$ is a diffeomorphism we use the fact that the diagram of part 2 of the definition of a smooth weak equivalence is a pullback diagram. This implies in particular that the restriction of f_1 to the fiber over $(1,1)$, which is the submanifold $\underline{\pi}_1\mathcal{G}$, is a smooth bijective map. That its inverse is also smooth follows from the universal property of the pullback: there exists a unique smooth map $H : \underline{\pi}_1\mathcal{G}' \longrightarrow \underline{\pi}_1\mathcal{G}$ that makes the diagram



commute, so $f_1 \circ H = \mathrm{id}_{\underline{\pi}_1\mathcal{G}'}$ which means that H is the inverse of f_1 on $\underline{\pi}_1\mathcal{G}'$, which thus is smooth. □

This concludes the proof of the first part of Proposition 4.4.9.

Chapter 5

Equivariant Modular Categories via Dijkgraaf-Witten Theory

This last chapter of the thesis contains a study of higher categorical aspects of 3d topological field theory. The relation to 2d conformal theories that have been implicit in the previous four chapters has been explained in the introduction and section 1.3. In contrast to chapter 4, the model discussed in this chapter relies on discrete target space structures. This simplifies the situation drastically and allows to compute the topological field theory. This discrete setting also paves the way towards the more complicated target spaces and background fields from the preceding chapters.

Concretely, in this chapter we present a geometric construction of J-equivariant Dijkgraaf-Witten theory as an extended topological field theory based on a weak action of a finite group J on a finite group G. The construction yields an explicitly accessible class of equivariant modular tensor categories. For the action of a group J on a group G, the category is described as the representation category of a J-ribbon algebra that generalizes the Drinfel'd double of the finite group G.

5.1 Motivation

This chapter has two seemingly different motivations and, correspondingly, can be read from two different points of view, a more algebraic and a more geometric one. Both in the introduction and the main body of the chapter, we try to separate these two points of view as much as possible, in the hope to keep the chapter accessible for readers with specific interests.

5.1.1 Algebraic motivation: equivariant modular categories

Among tensor categories, modular tensor categories are of particular interest for representation theory and mathematical physics. The representation categories of several algebraic structures give examples of semisimple modular tensor categories:

1. Left modules over connected factorizable ribbon weak Hopf algebras with Haar integral over an algebraically closed field [NTV03].

2. Local sectors of a finite μ-index net of von Neumann algebras on $\mathbb{R}$, if the net is strongly additive and split [KLM01].

3. Representations of selfdual C_2-cofinite vertex algebras with an additional finiteness condition on the homogeneous components and which have semisimple representation categories [Hua05].

Despite this list and the rather different fields in which modular tensor categories arise, it is fair to say that modular tensor categories are rare mathematical objects. Arguably, the simplest incarnation of the first algebraic structure in the list is the Drinfel'd double $\mathcal{D}(G)$ of a finite group G. Bantay [Ban05] has suggested a more general source for modular tensor categories: a pair, consisting of a finite group H and a normal subgroup $G \triangleleft H$. (In fact, Bantay has suggested general finite crossed modules, but for this chapter, only the case of a normal subgroup is relevant.) In this situation, Bantay constructs a ribbon category which is, in a natural way, a representation category of a ribbon Hopf algebra $\mathcal{B}(G \triangleleft H)$. Unfortunately, it turns out that, for a proper subgroup inclusion, the category $\mathcal{B}(G \triangleleft H)$-mod is only premodular and not modular.

Still, the category $\mathcal{B}(G \triangleleft H)$-mod is modularizable in the sense of Bruguières [Bru00], and the next candidate for new modular tensor categories is the modularization of $\mathcal{B}(G \triangleleft H)$-mod. However, it has been shown [MS10] that this modularization is equivalent to the representation category of the Drinfel'd double $\mathcal{D}(G)$.

The modularization procedure of Bruguières is based on the observation that the violation of modularity of a modularizable tensor category $\mathcal{C}$ is captured in terms of a canonical Tannakian subcategory of $\mathcal{C}$. For the category $\mathcal{B}(G \triangleleft H)$-mod, this subcategory can be realized as the representation category of the the quotient group $J := H/G$ [MS10]. The modularization functor

$$\mathcal{B}(G \triangleleft H)\text{-mod} \longrightarrow \mathcal{D}(G)\text{-mod}$$

is induction along the commutative Frobenius algebra given by the regular representation of J. This has the important consequence that the modularized category $\mathcal{D}(G)$ is endowed with a J-action.

Experience with orbifold constructions, see [Kir04, Tur10] for a categorical formulation, raises the question of whether the category $\mathcal{D}(G)$-mod with this J-action can be seen in a natural way as the neutral sector of a J-modular tensor category.

We thus want to complete the following square of tensor categories

$$\begin{array}{ccc} \overset{J}{\circlearrowright} & & \overset{J}{\circlearrowright} \\ \mathcal{D}(G)\text{-mod} & \hookrightarrow & ??? \\ {\scriptstyle\text{modularization}}\ \big\uparrow\big\downarrow\ {\scriptstyle\text{orbifold}} & & \big\uparrow\big\downarrow\ {\scriptstyle\text{orbifold}} \\ \mathcal{B}(G \triangleleft H)\text{-mod} & \hookrightarrow & ??? \end{array} \tag{5.1}$$

Here vertical arrows pointing upwards stand for induction functors along the commutative algebra given by the regular representation of J, while downwards pointing arrows indicate orbifoldization. In the upper right corner, we wish to place a J-modular category, and in the lower right corner its J-orbifold which, on general grounds [Kir04], has to be a modular tensor category. Horizontal arrows indicate the inclusion of neutral sectors.

In general, such a completion need not exist. Even if it exists, there might be inequivalent choices of J-modular tensor categories of which a given modular tensor category with J-action is the neutral sector [ENO10].

5.1.2 Geometric motivation: equivariant extended TFT

Topological field theory is a mathematical structure that has been inspired by physical theories [Wit89] and which has developed into an important tool in low-dimensional topology. Recently, these theories have received increased attention due to the advent of *extended* topological field theories [Lur09b, SP09]. The present chapter focuses on three-dimensional topological field theory.

Dijkgraaf-Witten theories provide a class of extended topological field theories. They can be seen as discrete variants of Chern-Simons theories, which provide invariants of three-manifolds and play an important role in knot theory [Wit89]. Dijkgraaf-Witten theories have the advantage of being particularly tractable and admitting a very conceptual geometric construction.

A Dijkgraaf-Witten theory is based on a finite group G; in this case the 'field configurations' on a manifold M are given by G-bundles over M, denoted by $\mathcal{A}_G(M)$. Furthermore, one has to choose a suitable action functional $S : \mathcal{A}_G(M) \longrightarrow \mathbb{C}$ (which we choose here in fact to be trivial) on field configurations; this allows to make the structure suggested by formal path integration rigorous and to obtain a topological field theory. A conceptually very clear way to carry this construction out rigorously is described in [FQ93] and [Mor10], see section 5.2 of this chapter for a review.

Let us now assume that as a further input datum we have another finite group J which acts on G. In this situation, we get an action of J on the Dijkgraaf-Witten theory based on G. But it turns out that this topological field theory together with the J-action does not fully reflect the equivariance of the situation: it has been an important insight that the right notion is the one of equivariant topological field theories, which have been another point of recent interest [Kir04, Tur10]. Roughly speaking, equivariant topological field theories require that all geometric objects (i.e. manifolds of different dimensions) have to be decorated by a J-cover (see definitions 5.3.11 and 5.3.13 for details). Equivariant field theories also provide a conceptual setting for the orbifold construction, one of the standard tools for model building in conformal field theory and string theory.

Given the action of a finite group J on a finite group G, these considerations lead to the question of whether Dijkgraaf-Witten theory based on G can be enlarged to a J-equivariant topological field theory. Let us pose this question more in detail:

- What exactly is the right notion of an action of J on G that leads to interesting theories? To keep equivariant Dijkgraaf-Witten theory as explicit as the non-equivariant theory, one needs notions to keep control of this action as explicitly as possible.

- Ordinary Dijkgraaf-Witten theory is mainly determined by the choice of field configurations $\mathcal{A}_G(M)$ to be G-bundles. As mentioned before, for J-equivariant theories, we should replace manifolds by manifolds with J-covers. We thus need a geometric notion of a G-bundle that is 'twisted' by this J-cover in order to develop the theory parallel to the non-equivariant one.

Based on an answer to these two points, we wish to construct equivariant Dijkgraaf-Witten theory as explicitly as possible.

5.1.3 Summary of the results

This chapter solves both the algebraic and the geometric problem we have just described. In fact, the two problems turn out to be closely related. We first solve the problem of explicitly constructing equivariant Dijkgraaf-Witten and then use our solution to construct the relevant modular categories that complete the square (5.1).

Despite this strong mathematical interrelation, we have taken some effort to write the chapter in such a way that it is accessible to readers sharing only a geometric or algebraic interest. The geometrically minded reader might wish to restrict his attention to section 2 and 3, and only take notice of the result about J-modularity stated in theorem 5.4.35. An algebraically oriented reader, on the other hand, might simply accept the categories described in proposition 5.3.22 together with the structure described in propositions 5.3.23, 5.3.24 and 5.3.26 and then directly delve into section 4.

For the benefit of all readers, we present here an outline of all our findings. In section 2, we review the pertinent aspects of Dijkgraaf-Witten theory and in particular the specific construction given in [Mor10]. Section 3 is devoted to the equivariant case: we observe that the correct notion of J-action on G is what we call a weak action of the group J on the group G; this notion is introduced in definition 5.3.1. Based on this notion, we can very explicitly construct for every J-cover $P \twoheadrightarrow M$ a category $\mathcal{A}_G(P \twoheadrightarrow M)$ of P-twisted G-bundles. For the definition and elementary properties of twisted bundles, we refer to section 5.3.2 and for a local description to appendix 5.6.1. We are then ready to construct equivariant Dijkgraaf Witten theory along the lines of the construction described in [Mor10]. This is carried out in section 5.3.3 and 5.3.4. We obtain a construction of equivariant Dijkgraaf-Witten theory that is so explicit that we can read off the category $\mathcal{C}^J(G)$ it assigns to the circle $\mathbb{S}^1$. The equivariant topological field theory induces additional structure on this category, which can also be computed by geometric methods due to the explicit control of the theory, and part of which we compute in section 5.3.5. This finishes the geometric part of our work. It remains to show that the category $\mathcal{C}^J(G)$ is indeed J-modular.

To establish the J-modularity of the category $\mathcal{C}^J(G)$, we have to resort to algebraic tools. Our discussion is based on the appendix 6 of [Tur10] by A. Virélizier. At the same time, we explain the solution of the algebraic problems described in section 5.1.1. The Hopf algebraic notions we encounter in section 4, in particular Hopf algebras with a weak group action and their orbifold Hopf algebras might be of independent algebraic interest.

In section 4, we introduce the notion of a J-equivariant ribbon Hopf algebra. It turns out that it is natural to relax some strictness requirements on the J-action on such a Hopf algebra. Given a weak action of a finite group J on a finite group G, we describe in proposition 5.4.24 a specific ribbon Hopf algebra which we call the equivariant Drinfel'd double $\mathcal{D}^J(G)$. This ribbon Hopf algebra is designed in such a way that its representation category is equivalent to the geometric category $\mathcal{C}^J(G)$ constructed in section 3, compare proposition 5.4.25.

The J-modularity of $\mathcal{C}^J(G)$ is established via the modularity of its orbifold category. The corresponding notion of an orbifold algebra is introduced in subsection 4.4. In the case of the equivariant Drinfel'd double $\mathcal{D}^J(G)$, this orbifold algebra is shown to be isomorphic, as a ribbon Hopf algebra, to a Drinfel'd double. This implies modularity of the orbifold theory and, by a result of [Kir04], J-modularity of the category $\mathcal{C}^J(G)$, cf. theorem 5.4.35.

In the course of our construction, we develop several notions of independent interest. In fact, our chapter might be seen as a study of the geometry of chiral backgrounds. It allows for various generalizations, some of which are briefly sketched in the conclusions. These generalizations include in particular twists by 3-cocycles in group cohomology and, possibly, even the case of non-semi simple chiral backgrounds.

5.2 Dijkgraaf-Witten theory and Drinfel'd double

This section contains a short review of Dijkgraaf-Witten theory as an extended three-dimensional topological field theory, covering the contributions of many authors, including in particular the work of Dijkgraaf-Witten [DW90], of Freed-Quinn [FQ93] and of Morton [Mor10]. We explain how these extended 3d TFTs give rise to modular tensor categories. These specific modular tensor categories are the representation categories of a well-known class of quantum groups, the the Drinfel'd doubles of finite groups.

While this section does not contain original material, we present the ideas in such a way that equivariant generalizations of the theories can be conveniently discussed. In this section, we also introduce some categories and functors that we need for later sections.

5.2.1 Motivation for Dijkgraaf-Witten theory

We start with a brief motivation for Dijkgraaf-Witten theory from physical principles. A reader already familiar with Dijkgraaf-Witten theory might wish to take at least notice of the definition 5.2.2 and of proposition 5.2.3.

It is an old, yet successful idea to extract invariants of manifolds from quantum field theories, in particular from quantum field theories for which the fields are G-bundles with connection, where G is some group. In this chapter we mostly consider the case of a finite group and only occasionally make reference to the case of a compact Lie group.

Let M be a compact oriented manifold of dimension 1,2 or 3, possibly with boundary. As the *'space' of field configurations*, we choose G bundles with connection,

$$\mathcal{A}_G(M) := \mathcal{B}un_G^{\nabla}(M).$$

In this way, we really assign to a manifold a groupoid, rather than an actual space. The morphisms of the category take gauge transformations into account. We will nevertheless keep on calling it 'space' since the correct framework to handle $\mathcal{A}_G(M)$ is as a stack on the category of smooth manifolds.

Moreover, another piece of data specifying the model is a function defined on manifolds of a specific dimension,

$$S : \mathcal{A}_G(M) \longrightarrow \mathbb{C}$$

called the *action*. In the simplest case, when G is a finite group, a field configuration is given by a G-bundle, since all bundles are canonically flat and no connection data are involved. Then, the simplest action is given by $S[P] := 0$ for all P. In the case of a compact, simple, simply connected Lie group G, consider a 3-manifold M. In this situation, each G-bundle P over M is globally of the form $P \cong G \times M$, because $\pi_1(G) = \pi_2(G) = 0$. Hence a field configuration is given by a connection on the trivial bundle which is a 1-form $A \in \Omega^1(M, \mathfrak{g})$ with values in the Lie algebra of G. An example of an action yielding a topological field theory that can be defined in this situation is the Chern-Simons action

$$S[A] := \int_M \langle A \wedge dA \rangle - \frac{1}{6} \langle A \wedge A \wedge A \rangle$$

where $\langle\cdot,\cdot\rangle$ is the basic invariant inner product on the Lie algebra $\mathfrak{g}$.

The heuristic idea is then to introduce an invariant $Z(M)$ for a 3-manifold M by integration over all field configurations:

$$Z(M) := \text{“}\int_{\mathcal{A}_G(M)} d\phi\ e^{iS[\phi]}\text{ ”}.$$

Warning 5.2.1. *In general, this path integral has only a heuristic meaning. In the case of a finite group, however, one can choose a counting measure $d\phi$ and thereby reduce the integral to a well-defined finite sum. The definition of Dijkgraaf-Witten theory [DW90] is based on this idea.*

Instead of giving a well-defined meaning to the invariant $Z(M)$ as a path-integral, we exhibit some formal properties these invariants are expected to satisfy. To this end, it is crucial to allow for manifolds that are not closed, as well. This allows to cut a three-manifold into several simpler three-manifolds with boundaries so that the computation of the invariant can be reduced to the computation of the invariants of simpler pieces.

Hence, we consider a 3-manifold M with a 2-dimensional boundary ∂M. We fix boundary values $\phi_1 \in \mathcal{A}_G(\partial M)$ and consider the space $\mathcal{A}_G(M,\phi_1)$ of all fields ϕ on M that restrict to the given boundary values ϕ_1. We then introduce, again at a heuristic level, the quantity

$$Z(M)_{\phi_1} := \text{“}\int_{\mathcal{A}_G(M,\phi_1)} d\phi\ e^{iS[\phi]}\text{ ”}. \tag{5.2}$$

The assignment $\phi_1 \longmapsto Z(M)_{\phi_1}$ could be called a 'wave function' on the space $\mathcal{A}_G(\partial M)$ of boundary values of fields. These 'wave functions' form a vector space $\mathcal{H}_{\partial M}$, the *state space*

$$\mathcal{H}_{\partial M} := \text{“}L^2\big(\mathcal{A}_G(\partial M),\mathbb{C}\big)\text{ ”}$$

that we assign to the boundary ∂M. The transition to wave functions amounts to a linearization. The notation L^2 should be taken with a grain of salt and should indicate the choice of an appropriate vector space for the category $\mathcal{A}_G(\partial M)$; it should not suggest the existence of any distinguished measure on the category.

In the case of Dijkgraaf-Witten theory based on a finite group G, the space of states has a basis consisting of δ-functions on the set of isomorphism classes of field configurations on the boundary ∂M:

$$\mathcal{H}_{\partial M} = \mathbb{C}\big\langle \delta_{\phi_1} \mid \phi_1 \in Iso\mathcal{A}_G(\partial M)\big\rangle.$$

In this way, we associate finite dimensional vector spaces $\mathcal{H}_\Sigma$ to compact oriented 2-manifolds Σ. The heuristic path integral in equation (5.2) suggests to associate to a 3-manifold M with boundary ∂M an element

$$Z(M) \in \mathcal{H}_{\partial M}\ ,$$

or, equivalently, a linear map $\mathbb{C} \longrightarrow \mathcal{H}_{\partial M}$.

A natural generalization of this situation are cobordisms $M : \Sigma \longrightarrow \Sigma'$, where Σ and Σ' are compact oriented 2-manifolds. A cobordism is a compact oriented 3-manifold M with boundary $\partial M \cong \bar{\Sigma} \sqcup \Sigma'$ where $\bar{\Sigma}$ denotes Σ, with the opposite orientation. To a cobordism, we wish to associate a linear map

$$Z(M) : \mathcal{H}_\Sigma \longrightarrow \mathcal{H}_{\Sigma'}$$

by giving its matrix elements in terms of the path integral

$$Z(M)_{\phi_0,\phi_1} := \text{“} \int_{\mathcal{A}_G(M,\phi_0,\phi_1)} d\phi \; e^{iS[\phi]} \text{”}$$

with fixed boundary values $\phi_0 \in \mathcal{A}_G(\Sigma)$ and $\phi_1 \in \mathcal{A}_G(\Sigma')$. Here $\mathcal{A}_G(M, \phi_0, \phi_1)$ is the space of field configurations on M that restrict to the field configuration ϕ_0 on the ingoing boundary Σ and to the field configuration ϕ_1 on the outgoing boundary Σ'. One can now show that the linear maps $Z(M)$ are compatible with gluing of cobordisms along boundaries. (If the group G is not finite, additional subtleties arise; e.g. $Z(M)_{\phi_0,\phi_1}$ has to be interpreted as an integral kernel.)

Atiyah [Ati88] has given a definition of a topological field theory that formalizes these properties: it describes a topological field theory as a symmetric monoidal functor from a geometric tensor category to an algebraic category. To make this definition explicit, let $\mathfrak{Cob}(2, 3)$ be the category which has 2-dimensional compact oriented smooth manifolds as objects. Its morphisms $M : \Sigma \longrightarrow \Sigma'$ are given by (orientation preserving) diffeomorphism classes of 3-dimensional, compact oriented cobordism from Σ to Σ' which we write as

$$\Sigma \hookrightarrow M \hookleftarrow \Sigma'.$$

Composition of morphisms is given by gluing cobordisms together along the boundary. The disjoint union of 2-dimensional manifolds and cobordisms equips this category with the structure of a symmetric monoidal category. For the algebraic category, we choose the symmetric tensor category $\mathrm{Vect}_{\mathbb{K}}$ of finite dimensional vector spaces over an algebraically closed field $\mathbb{K}$ of characteristic zero.

Definition 5.2.2 (Atiyah). A *3d TFT* is a symmetric monoidal functor

$$Z : \mathfrak{Cob}(2, 3) \longrightarrow \mathrm{Vect}_{\mathbb{K}}.$$

Let us set up such a functor for Dijkgraaf-Witten theory, i.e. fix a finite group G and choose the trivial action $S : A_G(M) \longrightarrow \mathbb{C}$, i.e. $S[P] = 0$ for all G-bundles P on M. Then the path integrals reduce to finite sums over 1 hence simply count the number of elements in the category $\mathcal{A}_G$. Since we are counting objects in a category, the stabilizers have to be taken appropriately into account, for details see e.g. [Mor08, Section 4]. This is achieved by the *groupoid cardinality* (which is sometimes also called the Euler-characteristic of the groupoid Γ)

$$|\Gamma| := \sum_{[g]\in Iso(\Gamma)} \frac{1}{|Aut(g)|}.$$

A detailed discussion of groupoid cardinality can be found in [BD01] and [Lei08].

We summarize the discussion:

Proposition 5.2.3 ([DW90],[FQ93])**.** *Given a finite group G, the following assignment Z_G defines a 3d TFT: to a closed, oriented 2-manifold Σ, we assign the vector space freely generated by the isomorphism classes of G-bundles on Σ,*

$$\Sigma \quad \longmapsto \quad \mathcal{H}_\Sigma := \mathbb{K}\big\langle \delta_P \mid P \in Iso\mathcal{A}_G(\Sigma)\big\rangle \ .$$

To a 3 dimensional cobordism M, we associate the linear map

$$Z_G\Big(\Sigma \hookrightarrow M \hookleftarrow \Sigma'\Big): \quad \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$$

with matrix elements given by the groupoid cardinality of the categories $\mathcal{A}_G(M,P_0,P_1)$:

$$Z_G(M)_{P_0,P_1} := \big|\mathcal{A}_G(M,P_0,P_1)\big| \ .$$

Remark 5.2.4.

1. *In the original paper [DW90], a generalization of the trivial action $S[P]=0$, induced by an element η in the group cohomology $\mathrm{H}^3_{Gp}\big(G,\mathrm{U}(1)\big)$ with values in $\mathrm{U}(1)$, has been studied. We postpone the treatment of this generalization to a separate paper: in the present thesis, the term Dijkgraaf-Witten theory refers to the 3d TFT of proposition 5.2.3 or its extended version.*

2. *In the case of a compact, simple, simply-connected Lie group G, a definition of a 3d TFT by a path integral is not available. Instead, the combinatorial definition of Reshetikin-Turaev [RT91] can be used to set up a 3d TFT which has the properties expected for Chern-Simons theory.*

3. *The vector spaces $\mathcal{H}_\Sigma$ can be described rather explicitly. Since every compact, closed, oriented 2-manifold is given by a disjoint union of surfaces Σ_g of genus g, it suffices to compute the dimension of $\mathcal{H}_{\Sigma_g}$. This can be done using the well-known description of moduli spaces of flat G-bundles in terms of homomorphisms from the fundamental group $\pi_1(\Sigma_g)$ to the group G, modulo conjugation,*

$$Iso\mathcal{A}_G(\Sigma_g) \cong \mathrm{Hom}(\pi_1(\Sigma_g),G)/G$$

which can be combined with the usual description of the fundamental group $\pi_1(\Sigma_g)$ in terms of generators and relations. In this way, one finds that the space is one-dimensional for surfaces of genus 0. In the case of surfaces of genus 1*, it is generated by pairs of commuting group elements, modulo simultaneous conjugation.*

4. *Following the same line of argument, one can show that for a closed 3-manifold M, one has*

$$\left|\mathcal{A}_G(M)\right| = \left|\mathrm{Hom}(\pi_1(M), G)\right| \ / \ |G| \ .$$

This expresses the 3-manifold invariants in terms of the fundamental group of M.

5.2.2 Dijkgraaf-Witten theory as an extended TFT

Up to this point, we have considered a version of Dijkgraaf-Witten theory which assigns invariants to closed 3-manifolds $Z(M)$ and vector spaces to 2-dimensional manifolds Σ. Iterating the argument that has lead us to consider three-manifolds with boundaries, we might wish to cut the two-manifolds into smaller pieces as well, and thereby introduce two-manifolds with boundaries into the picture.

Hence, we drop the requirement on the 2-manifold Σ to be closed and allow Σ to be a compact, oriented 2-manifold with 1-dimensional boundary $\partial\Sigma$. Given a field configuration $\phi_1 \in \mathcal{A}_G(\partial\Sigma)$ on the boundary of the surface Σ, we consider the space of all field configurations $\mathcal{A}_G(\Sigma, \phi_1)$ on Σ that restrict to the given field configuration ϕ_1 on the boundary $\partial\Sigma$. Again, we linearize the situation and consider for each field configuration ϕ_1 on the 1-dimensional boundary $\partial\Sigma$ the vector space freely generated by the isomorphism classes of field configurations on Σ,

$$\mathcal{H}_{\Sigma,\phi_1} := \text{“}L^2\big(\mathcal{A}_G(\Sigma, \phi_1)\big)\text{”} = \mathbb{C}\big\langle \delta_\phi \mid \phi \in Iso\mathcal{A}_G(\Sigma, \phi_1)\big\rangle.$$

The object we associate to the 1-dimensional boundary $\partial\Sigma$ of a 2-manifold Σ is thus a map $\phi_1 \longmapsto \mathcal{H}_{\Sigma,\phi_1}$ of field configurations to vector spaces, i.e. a complex vector bundle over the space of all fields on the boundary. In the case of a finite group G, we prefer to see these vector bundles as objects of the functor category from the essentially small category $\mathcal{A}_G(\partial\Sigma)$ to the category $\mathrm{Vect}_{\mathbb{C}}$ of finite-dimensional complex vector spaces, i.e. as an element of

$$\mathrm{Vect}(\mathcal{A}_G(\partial\Sigma)) = \big[\mathcal{A}_G(\partial\Sigma), \mathrm{Vect}_{\mathbb{C}}\big].$$

Thus the extended version of the theory assigns the functor category $Z(S) = [\mathcal{A}_G(S), \mathrm{Vect}_{\mathbb{C}}]$ to a one dimensional, compact oriented manifold S. These categories possess certain additional properties which can be summarized by saying that they are 2-vector spaces in the sense of [KV94]:

Definition 5.2.5. 1. A *2-vector space* (over a field $\mathbb{K}$) is a $\mathbb{K}$-linear, abelian, finitely semi-simple category. Here finitely semi-simple means that the category has finitely many isomorphism classes of simple objects and each object is a finite direct sum of simple objects.

2. Morphisms between 2-vector spaces are $\mathbb{K}$-linear functors and 2-morphisms are natural transformations. We denote the 2-category of 2-vector spaces by $2\mathrm{Vect}_{\mathbb{K}}$

3. The *Deligne tensor product* $\boxtimes$ endows $2\mathrm{Vect}_{\mathbb{K}}$ with the structure of a symmetric monoidal 2-category.

For the Deligne tensor product, we refer to [Del90, Sec. 5] or [BK01, Def. 1.1.15]. The definition and the properties of symmetric monoidal bicategories (resp. 2-categories) can be found in [SP09, ch. 3].

In the spirit of definition 5.2.2, we formalize the properties of the extended theory Z by describing it as a functor from a cobordism 2-category to the algebraic category $2\mathrm{Vect}_{\mathbb{K}}$. It remains to state the formal definition of the relevant geometric category. Here, we ought to be a little bit more careful, since we expect a 2-category and hence can not identify diffeomorphic 2-manifolds. For precise statements on how to address the difficulties in gluing smooth manifolds with corners, we refer to [Mor09, 4.3]; here, we restrict ourselves to the following short definition:

Definition 5.2.6. $\mathfrak{Cob}(1,2,3)$ is the following symmetric monoidal bicategory:

- Objects are compact, closed, oriented 1-manifolds S.
- 1-Morphisms are 2-dimensional, compact, oriented collared cobordisms $S \times I \hookrightarrow \Sigma \hookleftarrow S' \times I$.
- 2-Morphisms are generated by diffeomorphisms of cobordisms fixing the collar and 3-dimensional collared, oriented cobordisms with corners M, up to diffeomorphisms preserving the orientation and boundary.
- Composition is by gluing along collars.
- The monoidal structure is given by disjoint union with the empty set $\emptyset$ as the monoidal unit.

Remark 5.2.7. *The 1-morphisms are defined as collared surfaces, since in the case of extended cobordism categories, we consider surfaces rather than diffeomorphism classes of surfaces. A choice of collar is always possible, but not unique. The choice of collars ensures that the glued surface has a well-defined smooth structure. Different choices for the collars yield equivalent 1-morphisms in* $\mathfrak{Cob}(1,2,3)$.

Obviously, extended cobordism categories can be defined in dimensions different from three as well. We are now ready to give the definition of an extended TFT which goes essentially back to Lawrence [Law93]:

Definition 5.2.8. An *extended 3d TFT* is a weak symmetric monoidal 2-functor

$$Z : \quad \mathfrak{Cob}(1,2,3) \longrightarrow 2\mathrm{Vect}_{\mathbb{K}} \ .$$

We pause to explain in which sense extended TFTs extend the TFTs defined in definition 5.2.2. To this end, we note that the monoidal 2-functor Z has to send the monoidal unit in $\mathfrak{Cob}(1,2,3)$ to the monoidal unit in $2\mathrm{Vect}_{\mathbb{K}}$. The monoidal unit in $\mathfrak{Cob}(1,2,3)$ is the empty set $\emptyset$, and the unit in $2\mathrm{Vect}_{\mathbb{K}}$ is the category $\mathrm{Vect}_{\mathbb{K}}$.

The functor Z restricts to a functor $Z|_\emptyset$ from the endomorphisms of $\emptyset$ in $\mathfrak{Cob}(1,2,3)$ to the endomorphisms of $\mathrm{Vect}_\mathbb{K}$ in $2\mathrm{Vect}_\mathbb{K}$. It follows directly from the definition that $\mathrm{End}_{\mathfrak{Cob}(1,2,3)}(\emptyset) \cong \mathfrak{Cob}(2,3)$. Using the fact that the morphisms in $2\mathrm{Vect}_\mathbb{K}$ are additive (which follows from $\mathbb{C}$-linearity of functors in the definition of 2-vector spaces), it is also easy to see that the equivalence of categories $\mathrm{End}_{2\mathrm{Vect}_\mathbb{K}}(\mathrm{Vect}_\mathbb{K}) \cong \mathrm{Vect}_\mathbb{K}$ holds. Hence we have deduced:

Lemma 5.2.9. *Let Z be an extended 3d TFT. Then $Z|_\emptyset$ is a 3d TFT in the sense of definition 5.2.2.*

At this point, the question arises whether a given (non-extended) 3d TFT can be extended. In general, there is no reason for this to be true. For Dijkgraaf-Witten theory, however, such an extension can be constructed based on ideas which we described at the beginning of this section. A very conceptual presentation of this this construction based on important ideas of [Fre95] and [FQ93] can be found in [Mor10]. Before we describe this construction in more detail in subsection 5.2.3, we first state the result:

Proposition 5.2.10. *[Mor10] Given a finite group G, there exists an extended 3d TFT Z_G which assigns the categories*

$$\left[\mathcal{A}_G(S), \mathrm{Vect}_\mathbb{K}\right]$$

to 1-dimensional, closed oriented manifolds S and whose restriction $Z_G|_\emptyset$ is (isomorphic to) the Dijkgraaf-Witten TFT described in proposition 5.2.3.

Remark 5.2.11. *One can iterate the procedure of extension and introduce the notion of a fully extended TFT which also assigns quantities to points rather than just 1-manifolds. It can be shown that Dijkgraaf-Witten theory can be turned into a fully extended TFT, see [FHLT10]. The full extension will not be needed in the present article.*

5.2.3 Construction via 2-linearization

In this subsection, we describe in detail the construction of the extended 3d TFT of proposition 5.2.10. An impatient reader may skip this subsection and should still be able to understand most of the chapter. He might, however, wish to take notice of the technique of 2-linearization in proposition 5.2.14 which is also an essential ingredient in our construction of equivariant Dijkgraaf-Witten theory in a sequel paper.

As emphasized in particular by Morton [Mor10], the construction of the extended TFT is naturally split into two steps, which have already been implicitly present in preceding sections. The first step is to assign to manifolds and cobordisms the configuration spaces $\mathcal{A}_G$ of G bundles. We now restrict ourselves to the case when G is a finite group. The following fact is standard:

- The assignment $M \longmapsto \mathcal{A}_G(M) := \mathcal{B}un_G$ is a contravariant 2-functor from the category of manifolds to the 2-category of groupoids. Smooth maps between manifolds are mapped to the corresponding pullback functors on categories of bundles.

A few comments are in order: for a connected manifold M, the category $\mathcal{A}_G(M)$ can be replaced by the equivalent category $\mathrm{Hom}\big(\pi_1(M), G\big)//G$ given by the action groupoid where G acts by conjugation. In particular, the category $\mathcal{A}_G(M)$ is essentially finite, if M is compact. It should be appreciated that at this stage no restriction is imposed on the dimension of the manifold M.

The functor $\mathcal{A}_G(-)$ can be evaluated on a 2-dimensional cobordism $S \hookrightarrow \Sigma \hookleftarrow S'$ or a 3-dimensional cobordism $\Sigma \hookrightarrow M \hookleftarrow \Sigma'$. It then yields diagrams of the form

$$\mathcal{A}_G(S) \longleftarrow \mathcal{A}_G(\Sigma) \longrightarrow \mathcal{A}_G(S')$$
$$\mathcal{A}_G(\Sigma) \longleftarrow \mathcal{A}_G(M) \longrightarrow \mathcal{A}_G(\Sigma').$$

Such diagrams are called spans. They are the morphisms of a symmetric monoidal bicategory $\mathfrak{Span}$ of spans of groupoids as follows (see e.g. [DPP04] or [Mor09]):

- Objects are (essentially finite) groupoids.
- Morphisms are spans of essentially finite groupoids.
- 2-Morphisms are isomorphism classes of spans of span-maps.
- Composition is given by forming weak fiber products.
- The monoidal structure is given by the cartesian product $\times$ of groupoids.

Proposition 5.2.12 ([Mor10])**.** *$\mathcal{A}_G$ induces a symmetric monoidal 2-functor*

$$\widetilde{\mathcal{A}_G} : \mathfrak{Cob}(1,2,3) \longrightarrow \mathfrak{Span}.$$

This functor assigns to a 1-dimensional manifold S the groupoid $\mathcal{A}_G(S)$, to a 2-dimensional cobordism $S \hookrightarrow \Sigma \hookleftarrow S'$ the span $\mathcal{A}_G(S) \longleftarrow \mathcal{A}_G(\Sigma) \longrightarrow \mathcal{A}_G(S')$ and to a 3-cobordism with corners a span of span-maps.

Proof. It only remains to be shown that composition of morphisms and the monoidal structure is respected. The first assertion is shown in [Mor10, theorem 2] and the second assertion follows immediately from the fact that bundles over disjoint unions are given by pairs of bundles over the components, i.e. $\mathcal{A}_G(M \sqcup M') = \mathcal{A}_G(M) \times \mathcal{A}_G(M')$. □

The second step in the construction of extended Dijkgraaf-Witten theory is the 2-linearization of [Mor08]. As we have explained in section 5.2.1, the idea is to associate to a groupoid Γ its category of vector bundles $\mathrm{Vect}_{\mathbb{K}}(\Gamma)$. If Γ is essentially finite, the category of vector bundles is conveniently defined as the functor category $\big[\Gamma, \mathrm{Vect}_{\mathbb{K}}\big]$. If $\mathbb{K}$ is algebraically closed of characteristic zero, this category is a 2-vector space, see [Mor08, Lemma 4.1.1].

- The assignment $\Gamma \longmapsto \mathrm{Vect}_{\mathbb{K}}(\Gamma) := [\Gamma, \mathrm{Vect}_{\mathbb{K}}]$ is a contravariant 2-functor from the bicategory of (essentially finite) groupoids to the 2-category of 2-vector spaces. Functors between groupoids are sent to pullback functors.

We next need to explain what 2-linearization assigns to spans of groupoids. To this end, we use the following lemma due to [Mor08, 4.2.1]:

Lemma 5.2.13. *Let $f : \Gamma \longrightarrow \Gamma'$ be a functor between essentially finite groupoids. Then the pullback functor $f^* : \mathrm{Vect}(\Gamma') \longrightarrow \mathrm{Vect}(\Gamma)$ admits a 2-sided adjoint $f_* : \mathrm{Vect}(\Gamma) \longrightarrow \mathrm{Vect}(\Gamma')$, called the pushforward.*

Two-sided adjoints are also called 'ambidextrous' adjoint, see [Bar09, ch. 5] for a discussion. We use this pushforward to associate to a span

$$\Gamma \xleftarrow{p_0} \Lambda \xrightarrow{p_1} \Gamma'$$

of (essentially finite) groupoids the *'pull-push'-functor*

$$(p_1)_* \circ (p_0)^* : \quad \mathrm{Vect}_{\mathbb{K}}(\Gamma) \longrightarrow \mathrm{Vect}_{\mathbb{K}}(\Gamma').$$

A similar construction [Mor08] associates to spans of span-morphisms a natural transformation. Altogether we have:

Proposition 5.2.14 ([Mor08])**.** *The functor $\Gamma \longmapsto \mathrm{Vect}_{\mathbb{K}}(\Gamma)$ can be extended to a symmetric monoidal 2-functor on the category of spans of groupoids*

$$\widetilde{\mathcal{V}_{\mathbb{K}}} : \mathfrak{S}\mathrm{pan} \longrightarrow 2\mathrm{Vect}_{\mathbb{K}}.$$

This 2-functor is called 2-linearization*.*

Proof. The proof that $\widetilde{\mathcal{V}_{\mathbb{K}}}$ is a 2-functor is in [Mor08]. The fact that $\widetilde{\mathcal{V}_{\mathbb{K}}}$ is monoidal follows from the fact that $\mathrm{Vect}_{\mathbb{K}}(\Gamma \times \Gamma') \cong \mathrm{Vect}_{\mathbb{K}}(\Gamma) \boxtimes \mathrm{Vect}_{\mathbb{K}}(\Gamma')$ for a product $\Gamma \times \Gamma'$ of essentially finite groupoids. □

Arguments similar to the ones in [DPP04, prop 1.10] which are based on the universal property of the span category can be used to show that such an extension is essentially unique.

We are now in a position to give the functor Z_G described in proposition 5.2.10 which is Dijkgraaf-Witten theory as an extended 3d TFT as the composition of functors

$$Z_G := \widetilde{\mathcal{V}_{\mathbb{K}}} \circ \widetilde{\mathcal{A}_G} : \quad \mathfrak{C}\mathrm{ob}(1,2,3) \longrightarrow 2\mathrm{Vect}_{\mathbb{K}}.$$

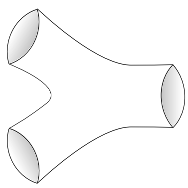
It follows from propositions 5.2.12 and 5.2.14 that Z_G is an extended 3d TFT in the sense of definition 5.2.8. For the proof of proposition 5.2.10, it remains to be shown that $Z_G|_{\emptyset}$ is the Dijkgraaf-Witten 3d TFT from proposition 5.2.3; this follows from a calculation which can be found in [Mor10, Section 5.2].

5.2.4 Evaluation on the circle

The goal of this subsection is a more detailed discussion of extended Dijkgraaf-Witten theory Z_G as described in proposition 5.2.10. Our focus is on the object assigned to the 1-manifold $\mathbb{S}^1$ given by the circle with its standard orientation. We start our discussion by evaluating an arbitrary extended 3d TFT Z as in definition 5.2.8 on certain manifolds of different dimensions:

1. To the circle $\mathbb{S}^1$, the extended TFT assigns a $\mathbb{K}$-linear, abelian finitely semisimple category $\mathcal{C}_Z := Z(\mathbb{S}^1)$.

2. To the two-dimensional sphere with three boundary components, two incoming and one outgoing, also known as the *pair of pants*,

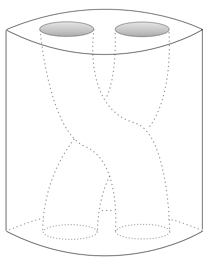


   the TFT associates a functor

$$\otimes : \mathcal{C}_Z \boxtimes \mathcal{C}_Z \longrightarrow \mathcal{C}_Z \quad ,$$

   which turns out to provide a tensor product on the category $\mathcal{C}_Z$.

3. The figure



   shows a 2-morphism between two three-punctured spheres, drawn as the upper and lower lid. The outgoing circle is drawn as the boundary of the big disk. To this cobordism, the TFT associates a natural transformation

$$\otimes \Longrightarrow \otimes^{\mathrm{opp}}$$

   which turns out to be a braiding.

Moreover, the TFT provides coherence cells, in particular associators and relations between the given structures. This endows the category $\mathcal{C}_Z$ with much additional structure. This structure can be summarized as follows:

Proposition 5.2.15. *For Z an extended 3d TFT, the category $\mathcal{C}_Z := Z(\mathbb{S}^1)$ is naturally endowed with the structure of a braided tensor category.*

For details, we refer to [Fre95] [Fre94] [Fre99] and [CY99]. This is not yet the complete structure that can be extracted: from the braiding-picture above it is intuitively clear that the braiding is not symmetric; in fact, the braiding is 'maximally non-symmetric' in a precise sense that is explained in definition 5.2.20. We discuss this in the next section for the category obtained from the Dijkgraaf-Witten extended TFT.

We now specialize to the case of extended Dijkgraaf-Witten TFT Z_G. We first determine the category $\mathcal{C}(G) := \mathcal{C}_Z$; it is by definition

$$\mathcal{C}(G) = \left[\mathcal{A}_G(\mathbb{S}^1), \mathrm{Vect}_{\mathbb{K}}\right].$$

It is a standard result in the theory of coverings that G-covers on $\mathbb{S}^1$ are described by group homomorphisms $\pi_1(\mathbb{S}^1) \longrightarrow G$ and their morphisms by group elements acting by conjugation. Thus the category $\mathcal{A}_G(\mathbb{S}^1)$ is equivalent to the action groupoid $G//G$ for the conjugation action. As a consequence, we obtain the abelian category $\mathcal{C}(G) \cong [G//G, \mathrm{Vect}_{\mathbb{K}}]$. We spell out this functor category explicitly:

Proposition 5.2.16. *For the extended Dijkgraaf-Witten 3d TFT, the category $\mathcal{C}(G)$ associated to the circle $\mathbb{S}^1$ is given by the category of G-graded vector spaces $V = \bigoplus_{g \in G} V_g$ together with a G-action on V such that for all $x, y \in G$*

$$x.V_g \subset V_{xgx^{-1}} \ .$$

As a next step we determine the tensor product on $\mathcal{C}(G)$. Since the fundamental group of the pair of pants is the free group on two generators, the relevant category of G-bundles is equivalent to the action groupoid $(G \times G)//G$ where G acts by simultaneous conjugation on the two copies of G. The 2-linearization $\widetilde{\mathcal{V}_{\mathbb{K}}}$ on the span

$$(G//G) \times (G//G) \leftarrow (G \times G)//G \longrightarrow G//G.$$

is treated in detail in [Mor10, rem. 5]; the result of this calculation yields the following tensor product:

Proposition 5.2.17. *The tensor product of V and W is given by the G-graded vector space*

$$(V \otimes W)_g = \bigoplus_{st=g} V_s \otimes W_t$$

together with the G-action $g.(v, w) = (gv, gw)$. The associators are the obvious ones induced by the tensor product in $\mathrm{Vect}_{\mathbb{K}}$.

In the same vein, the braiding can be calculated:

Proposition 5.2.18. *The braiding $V \otimes W \longrightarrow W \otimes V$ is for $v \in V_g$ and $w \in W$ given by*

$$v \otimes w \longmapsto gw \otimes v.$$

5.2.5 Drinfel'd double and modularity

The braided tensor category $\mathcal{C}(G)$ we just computed from the last section has a well-known description as the category of modules over a braided Hopf-algebra $\mathcal{D}(G)$, the Drinfel'd double $\mathcal{D}(G) := \mathcal{D}(\mathbb{K}[G])$ of the group algebra $\mathbb{K}[G]$ of G, see e.g. [Kas95, Chapter 9.4] The Hopf-algebra $\mathcal{D}(G)$ is defined as follows:

As a vector space, $\mathcal{D}(G)$ is the tensor product $\mathbb{K}(G) \otimes \mathbb{K}[G]$ of the algebra of functions on G and the group algebra of G, i.e. we have the canonical basis $(\delta_g \otimes h)_{g,h\in G}$. The algebra structure can be described as a smash product ([Mon93]), an analogue of the semi-direct product for groups: in the canonical basis, we have

$$(\delta_g \otimes h)(\delta_{g'} \otimes h') = \begin{cases} \delta_g \otimes hh' \text{ for } g = hg'h^{-1} \\ 0 \text{ else.} \end{cases}$$

where the unit is given by the tensor product of the two units: $\sum_{g\in G} \delta_g \otimes 1$. The coalgebra structure of $\mathcal{D}(G)$ is given by the tensor product of the coalgebras $\mathbb{K}(G)$ and $\mathbb{K}[G]$, i.e. the coproduct reads

$$\Delta(\delta_g \otimes h) = \sum_{g'g''=g} (\delta_{g'} \otimes h) \otimes (\delta_{g''} \otimes h)$$

and the counit is given by $\epsilon(\delta_1 \otimes h) = 1$ and $\epsilon(\delta_g \otimes h) = 0$ for $g \neq 1$ for all $h \in G$. It can easily be checked that this defines a bialgebra structure on $\mathbb{K}(G) \otimes \mathbb{K}[G]$ and that furthermore the linear map

$$S : (\delta_g \otimes h) \longmapsto (\delta_{h^{-1}g^{-1}h} \otimes h^{-1})$$

is an antipode for this bialgebra so that $\mathcal{D}(G)$ is a Hopf algebra. Furthermore, the element

$$R := \sum_{g,h\in G} (\delta_g \otimes 1) \otimes (\delta_h \otimes g) \in \mathcal{D}(G) \otimes \mathcal{D}(G)$$

is a universal R-matrix, which fulfills the defining identities of a braided bialgebra and corresponds to the braiding in proposition 5.2.18. At last, the element

$$\theta := \sum_{g\in G} (\delta_g \otimes g^{-1}) \in \mathcal{D}(G)$$

is a ribbon-element in $\mathcal{D}(G)$, which gives $\mathcal{D}(G)$ the structure of a ribbon Hopf-algebra (as defined in [Kas95, Definition 14.6.1]). Comparison with propositions 5.2.17 and 5.2.18 shows

Proposition 5.2.19. *The category $\mathcal{C}(G)$ is isomorphic, as a braided tensor category, to the category $\mathcal{D}(G)$-mod.*

The category $\mathcal{D}(G)$-mod is actually endowed with more structure than the one of a braided monoidal category. Since $\mathcal{D}(G)$ is a ribbon Hopf-algebra, the category of representations $\mathcal{D}(G)$-mod has also dualities and a compatible twist, i.e. has the structure of a ribbon category (see [Kas95, Proposition 16.6.2] or [BK01, Def. 2.2.1] for the notion of a ribbon category). Moreover, the category $\mathcal{D}(G)$-mod is a 2-vector space over $\mathbb{K}$ and thus, in particular, finitely semi-simple. We finally make explicit the non-degeneracy condition on the braiding that was mentioned in the last subsection.

Definition 5.2.20.

1. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. A *premodular tensor category* over $\mathbb{K}$ is $\mathbb{K}$-linear, abelian, finitely semisimple category $\mathcal{C}$ over $\mathbb{K}$ which has the structure of a ribbon category such that the tensor product is linear in each variable and the tensor unit is absolutely simple, i.e. $\mathrm{End}(\mathbf{1}) = \mathbb{K}$.

2. Denote by $\Lambda_{\mathcal{C}}$ a set of representatives for the isomorphism classes of simple objects. The braiding on $\mathcal{C}$ allows to define the *S-matrix* with entries in the field $\mathbb{K}$

$$s_{XY} := tr(R_{YX} \circ R_{XY}) \ ,$$

   where $X, Y \in \Lambda_{\mathcal{C}}$. A premodular category is called *modular*, if the S-matrix is invertible.

In the case of the Drinfel'd double, the S-matrix can be expressed explicitly in terms of characters of finite groups [BK01, Section 3.2]. Using orthogonality relations, one shows:

Proposition 5.2.21. *The category* $\mathcal{C}(G) \cong \mathcal{D}(G)$-mod *is modular.*

The notion of a modular tensor category first arose as a formalization of the Moore-Seiberg data of a two-dimensional rational conformal field theory. They are the input for the Turaev-Reshetikhin construction of three-dimensional topological field theories.

5.3 Equivariant Dijkgraaf-Witten theory

We are now ready to turn to the construction of equivariant generalization of the results of section 5.2. We denote again by G a finite group. Equivariance will be with respect to another finite group J that acts on G in a way we will have to explain. As usual, 'twisted sectors' [VW95] have to be taken into account for a consistent equivariant theory. A description of these twisted sectors in terms of bundles twisted by J-covers is one important result of this section.

5.3.1 Weak actions and extensions

Our first task is to identify the appropriate definition of a J-action. The first idea that comes to mind – a genuine action of the group J acting on G by group automorphisms – turns out to need a modification. For reasons that will become apparent in a moment, we only require an action up to inner automorphism.

Definition 5.3.1. 1. A *weak action* of a group J on a group G consists of a collection of group automorphisms $\rho_j : G \longrightarrow G$, one automorphism for each $j \in J$, and a collection of group elements $c_{i,j} \in G$, one group element for each pair of elements $i, j \in J$. These data are required to obey the relations:

$$\rho_i \circ \rho_j = \mathrm{Inn}_{c_{i,j}} \circ \rho_{ij} \qquad \rho_i(c_{j,k}) \cdot c_{i,jk} = c_{i,j} \cdot c_{ij,k} \quad \text{and} \quad c_{1,1} = 1$$

for all $i, j, k \in J$. Here Inn_g denotes the inner automorphism $G \longrightarrow G$ associated to an element $g \in G$. We will also use the short hand notation ${}^{j}g := \rho_j(g)$.

2. Two weak actions $(\rho_j, c_{i,j})$ and $(\rho'_j, c'_{i,j})$ of a group J on a group G are called isomorphic, if there is a collection of group elements $h_j \in G$, one group element for each $j \in J$, such that

$$\rho'_j = \mathrm{Inn}_{h_j} \circ \rho_j \quad \text{and } c'_{ij} \cdot h_{ij} = h_i \cdot \rho_i(h_j) \cdot c_{ij}$$

Remark 5.3.2. *1. If all group elements $c_{i,j}$ equal the neutral element, $c_{i,j} = 1$, the weak action reduces to a strict action of J on G by group automorphisms.*

2. A weak action induces a strict action of J on the outer automorphisms group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$.

3. In more abstract terms, a weak action amounts to a (weak) 2-group homomorphism $J \longrightarrow \mathrm{AUT}(G)$. Here $\mathrm{AUT}(G)$ denotes the automorphism 2-group of G. This automorphism 2-group can be described as the monoidal category of endofunctors of the one-object-category with morphisms G. The group J is considered as a discrete 2-group with only identities as morphisms. For more details on 2-groups, we refer to [BL04].

Weak actions are also known under the name *Dedecker cocycles*, due to the work [Ded60]. The correspondence between weak actions and extensions of groups is also termed *Schreier theory*, with reference to [Sch26]. Let us briefly sketch this correspondence:

- Let $(\rho_j, c_{i,j})$ be a weak action of J on G. On the set $H := G \times J$, we define a multiplication by

$$(g, i) \cdot (g', j) := \left(g \cdot {}^{i}(g') \cdot c_{i,j}\ ,\ ij\right). \tag{5.3}$$

  One can check that this turns H into a group in such a way that the sequence $G \longrightarrow H \longrightarrow J$ consisting of the inclusion $g \longmapsto (g, 1)$ and the projection $(g, j) \longmapsto j$ is exact.

- Conversely, let $G \longrightarrow H \xrightarrow{\pi} J$ be an extension of groups. Choose a set theoretic section $s : J \longrightarrow H$ of π with $s(1) = 1$. Conjugation with the group element $s(j) \in H$ leaves the normal subgroup G invariant. We thus obtain for $j \in J$ the automorphism $\rho_j(g) := s(j)\; g\; s(j)^{-1}$ of G. Furthermore, the element $c_{i,j} := s(i)s(j)s(ij)^{-1}$ is in the kernel of π and thus actually contained in the normal subgroup G. It is then straightforward to check that $\left(\rho_j, c_{i,j}\right)$ defines a weak action of J on G.

- Two different set-theoretic sections s and s' of the extension $G \longrightarrow H \longrightarrow J$ differ by a map $J \longrightarrow G$. This map defines an isomorphism of the induced weak actions in the sense of definition 5.3.1.2.

We have thus arrived at the

Proposition 5.3.3 (Dedecker, Schreier)**.** *There is a 1-1 correspondence between isomorphism classes of weak actions of J on G and isomorphism classes of group extensions $G \longrightarrow H \longrightarrow J$.*

Remark 5.3.4. *1. One can easily turn this statement into an equivalence of categories. Since we do not need such a statement here, we leave a precise formulation to the reader.*

2. Under this correspondence, strict actions *of J on G correspond to* split extensions. *This can be easily seen as follows: given a* split *extension $G \longrightarrow H \longrightarrow J$, one can choose the section $J \longrightarrow H$ as a group homomorphism and thus obtains a* strict *action of J on G. Conversely for a strict action of J on G it is easy to see that the group constructed in equation (5.3) is a semidirect product and thus the sequence of groups splits. To cover all extensions, we thus really need to consider weak actions.*

5.3.2 Twisted bundles

It is a common lesson from field theory that in an equivariant situation, one has to include “twisted sectors” to obtain a complete theory. Our next task is to construct the parameters labeling twisted sectors for a given weak action of a finite group J on G, with corresponding extension $G \longrightarrow H \longrightarrow J$ of groups and chosen set-theoretic section $J \longrightarrow H$. We will adhere to a two-step procedure as outlined after proposition 5.2.14. To this end, we will first construct for any smooth manifold a category of twisted bundles. Then, the linearization functor can be applied to spans of such categories.

We start our discussion of twisted G-bundles with the most familiar case of the circle, $M = \mathbb{S}^1$.

The isomorphism classes of G-bundles on $\mathbb{S}^1$ are in bijection to connected components of the free loop space $\mathcal{L}BG$ of the classifying space BG:

$$Iso\big(\mathcal{A}_G(\mathbb{S}^1)\big) = \mathrm{Hom}_{\mathrm{Ho(Top)}}(\mathbb{S}^1, BG) = \pi_0(\mathcal{L}BG).$$

Given a (weak) action of J on G, one can introduce twisted loop spaces. For any element $j \in J$, we have a group automorphism $j : G \longrightarrow G$ and thus a homeomorphism $j : BG \longrightarrow BG$. The j-twisted loop space is then defined to be

$$\mathcal{L}^j BG := \big\{ f : [0,1] \longrightarrow BG \mid f(0) = j \cdot f(1) \big\}.$$

Our goal is to introduce for every group element $j \in J$ a category $\mathcal{A}_G(\mathbb{S}^1, j)$ of j-twisted G-bundles on $\mathbb{S}^1$ such that

$$Iso\big(\mathcal{A}_G(\mathbb{S}^1, j)\big) = \pi_0(\mathcal{L}^j BG) \ .$$

In the case of the circle $\mathbb{S}^1$, the twist parameter was a group element $j \in J$. A more geometric description uses a family of J-covers P_j over $\mathbb{S}^1$, with $j \in J$. The cover P_j is uniquely determined by its monodromy j for the base point $1 \in \mathbb{S}^1$ and a fixed point in the fiber over 1. A concrete construction of the cover P_j is given by the quotient $P_j := [0,1] \times J/\sim$ where $(0,i) \sim (1, ji)$ for all $i \in J$. In terms of these J-covers, we can write

$$\mathcal{L}^j BG = \big\{ f : P_j \longrightarrow BG \mid f \text{ is } J\text{-equivariant} \big\}.$$

This description generalizes to an arbitrary smooth manifold M. The natural twist parameter in the general case is a J-cover $P \xrightarrow{J} M$.

Suppose, we have a weak J-action on G and construct the corresponding extension $G \longrightarrow H \xrightarrow{\pi} J$. The category of bundles we need are H-lifts of the given J-cover:

Definition 5.3.5. Let J act weakly on G. Let $P \xrightarrow{J} M$ be a J-cover over M.

- A P-twisted G-bundle over M is a pair (Q, φ), consisting of an H-bundle Q over M and a smooth map $\varphi : Q \longrightarrow P$ over M that is required to obey

  $$\varphi(q \cdot h) = \varphi(q) \cdot \pi(h)$$

  for all $q \in Q$ and $h \in H$. Put differently, a $P \xrightarrow{J} M$-twisted G-bundle is a lift of the J-cover P reduction along the group homomorphism $\pi : H \longrightarrow J$.

- A morphism of P-twisted bundles (Q, φ) and (Q', φ') is a morphism $f : Q \longrightarrow Q'$ of H-bundles such that $\varphi' \circ f = \varphi$.

- We denote the category of P-twisted G-bundles by $\mathcal{A}_G\big(P \longrightarrow M\big)$. For $M = \mathbb{S}^1$, we introduce the abbreviation $\mathcal{A}_G\big(\mathbb{S}^1, j\big) := \mathcal{A}_G\big(P_j \longrightarrow \mathbb{S}^1\big)$ for the standard covers of the circle.

Remark 5.3.6. *There is an alternative point of view on a P-twisted bundle (Q,φ): the subgroup $G \subset H$ acts on the total space Q in such a way that the map $\varphi : Q \longrightarrow P$ endows Q with the structure of a G-bundle on P. Both the structure group H of the bundle Q and the bundle P itself carry an action of G; for twisted bundles, an equivariance condition on this action has to be imposed. Unfortunately this equivariance property is relatively involved; therefore, we have opted for the definition in the form given above.*

A morphism $f : P \longrightarrow P'$ of J-covers over the same manifold induces a functor $f_* : \mathcal{A}_G\big(P \longrightarrow M\big) \longrightarrow \mathcal{A}_G\big(P' \longrightarrow M\big)$ by $f_*(Q,\varphi) := (Q, f\circ\varphi)$. Furthermore, for a smooth map $f : M \longrightarrow N$, we can pull back the twist data $P \longrightarrow M$ and get a *pullback functor* of twisted G-bundles:

$$f^* : \mathcal{A}_G\big(P \longrightarrow N\big) \longrightarrow \mathcal{A}_G\big(f^*P \longrightarrow M\big)$$

by $f^*(Q,\varphi) = (f^*Q, f^*\varphi)$. Before we discuss more sophisticated properties of twisted bundles, we have to make sure that our definition is consistent with 'untwisted' bundles:

Lemma 5.3.7. *Let the group J act weakly on the group G. For G-bundles twisted by the trivial J-cover $M\times J \longrightarrow M$, we have a canonical equivalence of categories*

$$\mathcal{A}_G\big(M\times J \longrightarrow M\big) \cong \mathcal{A}_G(M).$$

Proof. We have to show that for an element $(Q,\varphi) \in \mathcal{A}_G\big(M\times J \longrightarrow M\big)$ the H-bundle Q can be reduced to a G-bundle. Such a reduction is the same as a section of the associated fiber bundle $\pi_*(Q) \in \mathcal{B}un_J(M)$ see e.g. [Bau09, Satz 2.14]). Now $\varphi : Q \longrightarrow M\times J$ induces an isomorphism of J-covers $Q\times_H J \cong (M\times J)\times_H J \cong M\times J$ so that the bundle $Q \times_H J$ is trivial as a J-cover and in particular admits global sections.

Since morphisms of twisted bundles have to commute with these sections, we obtain in that way a functor $\mathcal{A}_G\big(M\times J \longrightarrow M\big) \longrightarrow \mathcal{A}_G(M)$. Its inverse is given by extension of G-bundles on M to H-bundles on M. □

We also give a description of twisted bundles using standard covering theory; for an alternative description using Čech-cohomology, we refer to appendix 5.6.1. We start by recalling the following standard fact from covering theory, see e.g. [Hat02, 1.3] that has already been used to prove proposition 5.2.16: for a finite group J, the category of J-covers is equivalent to the action groupoid $\mathrm{Hom}(\pi_1(M), J)//J$. (Note that this equivalence involves choices and is not canonical.)

To give a similar description of twisted bundles, fix a J-cover P. Next, we choose a basepoint $m \in M$ and a point p in the fiber P_m over m. These data determine a unique group morphism $\omega : \pi_1(M,m) \longrightarrow J$ representing P.

Proposition 5.3.8. *Let J act weakly on G. Let M be a connected manifold and P be a J-cover over M represented after the choices just indicated by the group homomorphism $\omega : \pi_1(M) \longrightarrow J$. Then there is a (non-canonical) equivalence of categories*

$$\mathcal{A}_G\big(P \longrightarrow M\big) \cong \mathrm{Hom}^{\omega}\big(\pi_1(M), H\big)//G$$

where we consider group homomorphisms

$$\mathrm{Hom}^{\omega}\big(\pi_1(M), H\big) := \big\{\mu : \pi_1(M) \longrightarrow H \mid \pi \circ \mu = \omega\big\}$$

whose composition restricts to the group homomorphism ω describing the J-cover P. The group G acts on $\mathrm{Hom}^{\omega}\big(\pi_1(M), H\big)$ via pointwise conjugation using the inclusion $G \longrightarrow H$.

Proof. Let $m \in M$ and $p \in P$ over m be the choices of base point in the J-cover $P \longrightarrow M$ that lead to the homomorphism ω. Consider a $(P \longrightarrow M)$ twisted bundle $Q \longrightarrow M$. Since $\varphi : Q \longrightarrow P$ is surjective, we can choose a base point q in the fiber of Q over m such that $\varphi(q) = p$. The group homomorphism $\pi_1(M) \longrightarrow H$ describing the H-bundle Q is obtained by lifting closed paths in M starting in m to paths in Q starting in q. They are mapped under φ to lifts of the same path to P starting at p, and these lifts are just described by the group homomorphism $\omega : \pi_1(M) \longrightarrow J$ describing the cover P. If the end point of the path in Q is qh for some $h \in H$, then by the defining property of φ, the lifted path in P has endpoint $\varphi(qh) = \varphi(q)\pi(h) = p\pi(h)$. Thus $\pi \circ \mu = \omega$.

□

Remark 5.3.9. *For non-connected manifolds, a description as in proposition 5.3.8 can be obtained for every component. Again the equivalence involves choices of base points on M and in the fibers over the base points. This could be fixed by working with pointed manifolds, but pointed manifolds cause problems when we consider cobordisms. Alternatively, we could use the fundamental groupoid instead of the fundamental group, see e.g. [May99].*

Example 5.3.10. *We now calculate the categories of twisted bundles over certain manifolds using proposition 5.3.8.*

1. *For the circle $\mathbb{S}^1$, $\omega \in \mathrm{Hom}(\pi_1(\mathbb{S}^1), J) = \mathrm{Hom}(\mathbb{Z}, J)$ is determined by an element $j \in J$ and the condition $\pi \circ \mu = \omega$ requires $\mu(1) \in H$ to be in the preimage $H_j := \pi^{-1}(j)$ of j. Thus, we have $\mathcal{A}_G(\mathbb{S}^1, j) \cong H_j//G$.*
2. *For the 3-Sphere $\mathbb{S}^3$, all twists P and all G-bundles are trivial. Thus, we have $\mathcal{A}_G(P \longrightarrow \mathbb{S}^3) \cong \mathcal{A}_G(\mathbb{S}^3) \cong pt//G$.*

5.3.3 Equivariant Dijkgraaf-Witten theory

The key idea in the construction of equivariant Dijkgraaf-Witten theory is to take twisted bundles $\mathcal{A}_G(P \longrightarrow M)$ as the field configurations, taking the place of G-bundles in section 5.2. We cannot expect to get then invariants of closed 3-manifolds M, but rather invariants of 3-manifolds M together with a twist datum, i.e. a J-cover P over M. Analogous statements apply to manifolds with boundary and cobordisms. Therefore we need to introduce extended cobordism-categories as $\mathfrak{Cob}(1,2,3)$ in definition 5.2.6, but endowed with the extra datum of a J-cover over each manifold.

Definition 5.3.11. $\mathfrak{Cob}^J(1,2,3)$ is the following symmetric monoidal bicategory:

- Objects are compact, closed, oriented 1-manifolds S, together with a J-cover $P_S \xrightarrow{J} S$.
- 1-Morphisms are collared cobordisms

$$S \times I \hookrightarrow \Sigma \hookleftarrow S' \times I$$

  where Σ is a 2-dimensional, compact, oriented cobordism, together with a J-cover $P_\Sigma \longrightarrow \Sigma$ and isomorphisms

$$P_\Sigma|_{(S\times I)} \xrightarrow{\sim} P_S \times I \qquad \text{and} \qquad P_\Sigma|_{(S'\times I)} \xrightarrow{\sim} P_{S'} \times I.$$

  over the collars.
- 2-Morphisms are generated by
  - orientation preserving diffeomorphisms $\varphi : \Sigma \longrightarrow \Sigma'$ of cobordisms fixing the collar together with an isomorphism $\widetilde{\varphi} : P_\Sigma \longrightarrow P_{\Sigma'}$ covering φ.
  - 3-dimensional collared, oriented cobordisms with corners M with cover $P_M \longrightarrow M$ together with covering isomorphisms over the collars (as before) up to diffeomorphisms preserving the orientation and boundary.
- Composition is by gluing cobordisms and covers along collars.
- The monoidal structure is given by disjoint union.

Remark 5.3.12. *In analogy to remark 5.2.7, we point out that the isomorphisms of covers are defined over the collars, rather than only over the the boundaries. This endows the glued cover with a well-defined smooth structure.*

Definition 5.3.13. An extended 3d J-TFT is a symmetric monoidal 2-functor

$$Z : \mathfrak{Cob}^J(1,2,3) \longrightarrow 2\mathrm{Vect}_{\mathbb{K}}.$$

Just for the sake of completeness, we will also give a definition of non-extended J-TFT. Therefore define the symmetric monoidal category $\mathfrak{Cob}^J(2,3)$ to be the endomorphism category of the monoidal unit $\emptyset$ in $\mathfrak{Cob}(1,2,3)$. More concretely, this category has as objects closed, oriented 2-manifolds with J-cover and as morphisms J-cobordisms between them.

Definition 5.3.14. A (non-extended) 3d J-TFT is a symmetric monoidal 2-functor

$$\mathfrak{Cob}^J(2,3) \longrightarrow \mathrm{Vect}_{\mathbb{K}}.$$

Similarly as in the non-equivariant case (lemma 5.2.9), we get

Lemma 5.3.15. *Let Z be an extended 3d J-TFT. Then $Z|_\emptyset$ is a (non-extended) 3d J-TFT.*

Now we can state the main result of this section:

Theorem 5.3.16. *For a finite group G and a weak J-action on G, there is an extended 3d J-TFT called Z_G^J which assigns the categories*

$$\mathrm{Vect}_{\mathbb{K}}\big(\mathcal{A}_G(P \longrightarrow S)\big) = \big[\mathcal{A}_G(P \longrightarrow S), \mathrm{Vect}_{\mathbb{K}}\big]$$

to 1-dimensional, closed oriented manifolds S with J-cover $P \longrightarrow S$.

We will give a proof of this theorem in the next sections. Having twisted bundles at our disposal, the main ingredient will again be the 2-linearization described in section 5.2.3.

5.3.4 Construction via spans

As in the case of ordinary Dijkgraaf-Witten theory, cf. section 5.2.3, equivariant Dijkgraaf-Witten Z_G^J theory is constructed as the composition of the symmetric monoidal 2-functors

$$\widetilde{\mathcal{A}_G} : \mathfrak{Cob}^J(1,2,3) \longrightarrow \mathfrak{Span} \quad \text{and} \quad \widetilde{\mathcal{V}_{\mathbb{K}}} : \mathfrak{Span} \longrightarrow 2\mathrm{Vect}_{\mathbb{K}}.$$

The second functor will be exactly the 2-linearization functor of proposition 5.2.14. Hence we can limit our discussion to the construction of the first functor $\widetilde{\mathcal{A}_G}$. As it will turn out, our definition of twisted bundles is set up precisely in such a way that the construction of the corresponding functor in proposition 5.2.12 can be generalized.

Our starting point is the following observation:

- The assignment $(P_M \xrightarrow{J} M) \longmapsto \mathcal{A}_G(P_M \xrightarrow{J} M)$ of twisted bundles to a twist datum $P_M \longrightarrow M$ constitutes a contravariant 2-functor from the category of manifolds with J-cover to the 2-category of groupoids. Maps between manifolds with cover are mapped to the corresponding pullback functors of bundles.

From this functor which is defined on manifolds of any dimension, we construct a functor $\widetilde{\mathcal{A}_G}$ on J-cobordisms with values in the 2-category $\mathfrak{Span}$ of spans of groupoids, where the category $\mathfrak{Span}$ is defined in section 5.2.3. To an object in $\mathfrak{Cob}^J(1,2,3)$, i.e. to a J-cover $P_S \longrightarrow M$, we assign the category $\mathcal{A}_G(P_S \longrightarrow S)$ of J-covers. To a 1-morphism $P_S \hookrightarrow P_\Sigma \hookleftarrow P'_S$ in $\mathfrak{Cob}^J(1,2,3)$, we associate the span

$$\mathcal{A}_G(P_S \longrightarrow S) \leftarrow \mathcal{A}_G(P_\Sigma \longrightarrow \Sigma) \longrightarrow \mathcal{A}_G(P_{S'} \longrightarrow S') \tag{5.4}$$

and to a 2-morphism of the type $P_\Sigma \hookrightarrow P_M \hookleftarrow P_{\Sigma'}$ the span

$$\mathcal{A}_G(P_\Sigma \longrightarrow \Sigma) \leftarrow \mathcal{A}_G(P_M \longrightarrow M) \longrightarrow \mathcal{A}_G(P_{\Sigma'} \longrightarrow \Sigma'). \tag{5.5}$$

We have to show that the assignment $\widetilde{\mathcal{A}_G} : \mathfrak{Cob}^J(1,2,3) \longrightarrow \mathfrak{Span}$. is a symmetric monoidal functor. In particular, we have to show that the composition of morphisms is respected.

Lemma 5.3.17. *Let $P_\Sigma \longrightarrow \Sigma$ and $P_{\Sigma'} \longrightarrow \Sigma'$ be two 1-morphisms in $\mathfrak{Cob}^J(1,2,3)$ which can be composed at the object $P_S \longrightarrow S$ to get the 1-morphism*

$$P_\Sigma \circ P_{\Sigma'} := \left(P_\Sigma \sqcup_{P_S \times I} P_{\Sigma'} \longrightarrow \Sigma \sqcup_{S\times I} \Sigma'\right) ,$$

where $I = [0,1]$ is the standard interval. (Recall that we are gluing over collars.) Then the category $\mathcal{A}_G\big(P_\Sigma \circ P_{\Sigma'}\big)$ is the weak pullback of the groupoids $\mathcal{A}_G(P_\Sigma \longrightarrow \Sigma)$ and $\mathcal{A}_G(P_{\Sigma'} \longrightarrow \Sigma')$ over $\mathcal{A}_G(P_S \longrightarrow S)$.

Proof. By definition the category

$$\mathcal{A}_G\big(P_\Sigma \circ P_{\Sigma'}\big)$$

has as objects twisted G-bundles over the 2-manifold $\Sigma \sqcup_{S\times I} \Sigma' =: N$. The manifold N admits an open covering $N = U_0 \cup U_1$ with $U_0 = \Sigma \setminus S$ and $U_1 = \Sigma' \setminus S$ where the intersection is the cylinder $U_0 \cap U_1 = S \times (0,1)$. By construction, the restrictions of the glued bundle $P_N \longrightarrow N$ to U_0 and U_1 are given by $P_\Sigma \setminus P_S$ and $P_{\Sigma'} \setminus P_S$.

The natural inclusions $U_0 \longrightarrow \Sigma$ and $U_1 \longrightarrow \Sigma'$ induce equivalences

$$\mathcal{A}_G(P_\Sigma \longrightarrow \Sigma) \xrightarrow{\sim} \mathcal{A}_G(P_N|_{U_0} \longrightarrow U_0)$$
$$\mathcal{A}_G(P_{\Sigma'} \longrightarrow \Sigma') \xrightarrow{\sim} \mathcal{A}_G(P_N|_{U_1} \longrightarrow U_1)$$

Analogously, we have an equivalence

$$\mathcal{A}_G\big(P_N|_{U_0\cap U_1} \longrightarrow U_0 \cap U_1\big) \xrightarrow{\sim} \mathcal{A}_G(P_S \longrightarrow S) \ .$$

At this point, we have reduced the claim to an assertion about descent of twisted bundles which we will prove in corollary 5.3.20. This corollary implies that $\mathcal{A}_G(P_N \longrightarrow N)$ is the weak pullback of $\mathcal{A}_G(P_N|_{U_0} \longrightarrow U_0)$ and $\mathcal{A}_G(P_N|_{U_1} \longrightarrow U_1)$ over $\mathcal{A}_G\big(P_N|_{U_0\cap U_1}\big)$. Since weak pullbacks are invariant under equivalence of groupoids, this shows the claim. □

We now turn to the promised results about descent of twisted bundles. Let $P \longrightarrow M$ be a J-cover over a manifold M and $\{U_\alpha\}$ be an open covering of M, where for the sake of generality we allow for arbitrary open coverings. We want to show that twisted bundles can be glued together like ordinary bundles; while the precise meaning of this statement is straightforward, we briefly summarize the relevant definitions for the sake of completeness:

Definition 5.3.18. Let $P \longrightarrow M$ be a J-cover over a manifold M and $\{U_\alpha\}$ be an open covering of M. The descent category $\mathcal{D}esc(U_\alpha, P)$ has

- Objects: families of $P|_{U_\alpha}$-twisted bundles Q_α over U_α, together with isomorphisms of twisted bundles $\varphi_{\alpha\beta} : Q_\alpha|_{U_\alpha \cap U_\beta} \xrightarrow{\sim} Q_\beta|_{U_\alpha \cap U_\beta}$ satisfying the cocycle condition $\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$.
- Morphisms: families of morphisms $f_\alpha : Q_\alpha \longrightarrow Q'_\alpha$ of twisted bundles such that over $U_{\alpha\beta}$ we have $\varphi'_{\alpha\beta} \circ (f_\alpha)|_{U_{\alpha\beta}} = (f_\beta)|_{U_{\alpha\beta}} \circ \varphi_{\alpha\beta}$.

Proposition 5.3.19 (Descent for twisted bundles)**.** *Let $P \longrightarrow M$ be a J-cover over a manifold M and $\{U_\alpha\}$ be an open covering of M. Then the groupoid $\mathcal{A}_G(P \longrightarrow M)$ is equivalent to the descent category $\mathcal{D}esc(U_\alpha, P)$.*

Proof. Note that the corresponding statements are true for H-bundles and for J-covers. Then the description in definition 5.3.5 of a twisted bundle as an H-bundle together with a morphism of the associated J-cover immediately implies the claim. □

Corollary 5.3.20. *For an open covering of M by two open sets U_0 and U_1 the category $\mathcal{A}_G(P \longrightarrow M)$ is the weak pullback of $\mathcal{A}_G(P|_{U_0} \longrightarrow U_0)$ and $\mathcal{A}_G(P|_{U_1} \longrightarrow U_1)$ over $\mathcal{A}_G(P|_{U_0 \cap U_1} \longrightarrow U_0 \cap U_1)$.*

In order to prove that the assignment (5.4) and (5.5) really promotes $\mathcal{A}_G$ to a symmetric monoidal functor $\widetilde{\mathcal{A}_G} : \mathfrak{Cob}^J(1,2,3) \longrightarrow \mathfrak{Span}$, it remains to show that $\mathcal{A}_G$ preserves the monoidal structure.

Now a bundle over a disjoint union is given by a pair of bundles over each component. Thus, for a disjoint union of J-manifolds $P \longrightarrow M = (P_1 \sqcup P_2) \longrightarrow (M_1 \sqcup M_2)$, we have $\mathcal{A}_G(P \longrightarrow M) \cong \mathcal{A}_G(P_1 \longrightarrow M_1) \times \mathcal{A}_G(P_2 \longrightarrow M_2)$. Note that the manifolds M, M_1 and M_2 can also be cobordisms. The isomorphism of categories is clearly associative and preserves the symmetric structure. Together with lemma 5.3.17, this proves the next proposition.

Proposition 5.3.21. *$\mathcal{A}_G$ induces a symmetric monoidal functor*

$$\widetilde{\mathcal{A}_G} : \mathfrak{Cob}^J(1,2,3) \longrightarrow \mathfrak{Span}$$

which assigns the spans (5.4) *and* (5.5) *to 2 and 3-dimensional cobordisms with J-cover.*

5.3.5 Twisted sectors and fusion

We next proceed to evaluate the J-equivariant TFT Z_G^J constructed in the last section on the circle, as we did in section 5.2.4 for the non-equivariant TFT. We recall from section 5.3.2 the fact that over the circle $\mathbb{S}^1$ we have for each $j \in J$ a standard cover P_j. The associated category

$$\mathcal{C}(G)_j := Z_G^J\big(P_j \longrightarrow \mathbb{S}^1\big)$$

is called the j-twisted sector of the theory; the sector $\mathcal{C}(G)_1$ is called the neutral sector. By lemma 5.3.7, we have an equivalence $\mathcal{A}_G(P_1 \longrightarrow \mathbb{S}^1) \cong \mathcal{A}_G(\mathbb{S}^1)$; hence we get an equivalence of categories $\mathcal{C}(G)_1 \cong \mathcal{C}(G)$, where $\mathcal{C}(G)$ is the category arising in the non-equivariant Dijkgraaf-Witten model, we discussed in section 5.2.4. We have already computed the twisted sectors as abelian categories in example 5.3.10 and note the result for future reference:

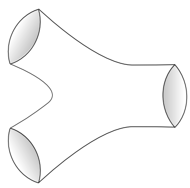
Proposition 5.3.22. *For the j-twisted sector of equivariant Dijkgraaf-Witten theory, we have an equivalence of abelian categories*

$$\mathcal{C}(G)_j \cong [H_j//G, \mathrm{Vect}_{\mathbb{K}}] \ ,$$

where $H_j//G$ is the action groupoid given by the conjugation action of G on $H_j := \pi^{-1}(j)$. More concretely, the category $\mathcal{C}(G)_j$ is equivalent to the category of H_j-graded vector spaces $V = \bigoplus_{h \in H_j} V_h$ together with a G-action on V such that

$$g.V_h \subset V_{ghg^{-1}} .$$

As a next step, we want to make explicit additional structure on the categories $\mathcal{C}(G)_j$ coming from certain cobordisms. Therefore, consider the pair of pants $\Sigma(2,1)$:



The fundamental group of $\Sigma(2,1)$ is the free group on two generators. Thus, given a pair of group elements $j, k \in J$, there is a J-cover $P_{j,k}^{\Sigma(2,1)} \longrightarrow \Sigma(2,1)$ which restricts to the standard covers P_j and P_k on the two ingoing boundaries and to the standard cover P_{jk} on the outgoing boundary circle. (To find a concrete construction, one should fix a parametrization of the pair of pants $\Sigma(2,1)$.) The cobordism $P_{j,k}^{\Sigma(2,1)}$ is a morphism

$$P_{j,k}^{\Sigma(2,1)} : \big(P_j \longrightarrow \mathbb{S}^1\big) \sqcup \big(P_k \longrightarrow \mathbb{S}^1\big) \longrightarrow \big(P_{jk} \longrightarrow \mathbb{S}^1\big) \tag{5.6}$$

in the category $\mathfrak{Cob}^J(1,2,3)$. Applying the equivariant TFT-functor Z_G^J yields a functor

$$\otimes_{jk} : \quad \mathcal{C}(G)_j \boxtimes \mathcal{C}(G)_k \longrightarrow \mathcal{C}(G)_{jk}.$$

We describe this functor in terms of the equivalent categories of graded vector spaces as a functor

$$H_j//G\text{-mod} \times H_k//G\text{-mod} \longrightarrow H_{jk}//G\text{-mod} \ .$$

Proposition 5.3.23. *For objects* $V = \bigoplus_{h \in H_j} V_h$ *in* $H_j//G$-mod *and* $W = \bigoplus W_h$ *in* $H_k//G$-mod *the product* $V \otimes_{jk} W \in H_{jk}//G$-mod *is given by*

$$(V \otimes_{jk} W)_h = \bigoplus_{st=h} V_s \otimes W_t$$

together with the action $g.(v \otimes w) = g.v \otimes g.w$.

Proof. As a first step we have to compute the span $\widetilde{\mathcal{A}_G}(P^{\Sigma(2,1)}_{j,k})$ associated to the cobordism $P^H_{j,k}$. From the description of twisted bundles in proposition 5.3.8 and the fact that the fundamental group of $\Sigma(2,1)$ is the free group on two generators, we derive the following equivalence of categories:

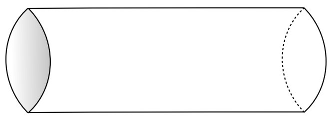
$$\mathcal{A}_G\big(P^{\Sigma(2,1)}_{jk} \longrightarrow \Sigma(2,1)\big) \cong (H_j \times H_k)//G \ .$$

Here we have $H_j \times H_k = \{(h,h') \in H \times H \mid \pi(h) = j, \ \pi(h') = k\}$, on which G acts by simultaneous conjugation. This leads to the span of action groupoids

$$H_j//G \times H_k//G \longleftarrow (H_j \times H_k)//G \longrightarrow H_{jk}//G$$

where the left map is given be projection to the factors and the right hand map by multiplication. Applying the 2-linearization functor $\widetilde{\mathcal{V}_{\mathbb{K}}}$ from proposition 5.2.14 amounts to computing the corresponding pull-push functor. This yields the result. $\square$

Next, we consider the 2-manifold $\Sigma(1,1)$ given by the cylinder over $\mathbb{S}^1$, i.e. $\Sigma(1,1) = \mathbb{S}^1 \times I$:



There exists a cover $P^{\Sigma(1,1)}_{j,x} \longrightarrow \Sigma(1,1)$ for $j, x \in J$ that restricts to P_j on the ingoing circle and to $P_{xjx^{-1}}$ on the outgoing circle. The simplest way to construct this cover is to consider the cylinder $P_j \times I \longrightarrow \mathbb{S}^1 \times I$ and to use the identification of $P^{\Sigma(1,1)}_{j,x}$ over (a collaring neighborhood of) the ingoing circle by the identity and over the outgoing circle the identification by the morphism $P_{\Sigma(1,1)}|_{\mathbb{S}^1 \times 1} = P_j \longrightarrow P_{xjx^{-1}}$ given by conjugation with x. In this way, we obtain a cobordism that is a 1-morphism

$$P^{\Sigma(1,1)}_{j,x} : (P_j \longrightarrow \mathbb{S}^1) \longrightarrow (P_{xjx^{-1}} \longrightarrow \mathbb{S}^1) \tag{5.7}$$

in the category $\mathfrak{Cob}^J(1,2,3)$ and hence induces a functor

$$\phi_x : \mathcal{C}(G)_j \longrightarrow \mathcal{C}(G)_{xjx^{-1}}.$$

We compute the functor on the equivalent action groupoids explicitly:

Proposition 5.3.24. *The image under ϕ_x of an object $V = \bigoplus V_h \in H_j//G$-mod is the graded vector space with homogeneous component*

$$\phi_x(V)_h = V_{s(x)hs(x)^{-1}}$$

for $h \in H_j$ and with G-action on $v \in V_h$ given by $s(x)gs(x)^{-1} \cdot v$.

Proof. As before we compute the span $\widetilde{\mathcal{A}_G}(P_{j,x}^{\Sigma(1,1)})$. Using explicitly the equivalence given in the proof of proposition 5.3.8, we obtain the span of action groupoids

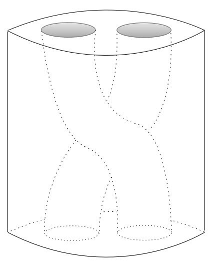
$$H_j//G \leftarrow H_j//G \longrightarrow H_{xjx^{-1}}//G$$

where the left-hand map is the identity and the right map is given by

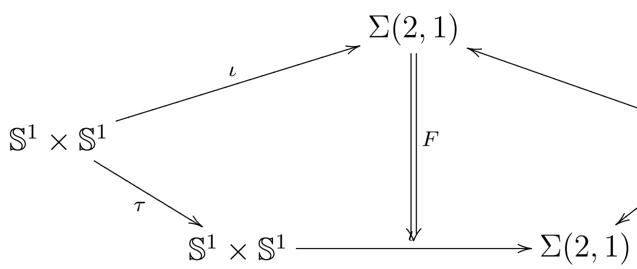
$$(h, g) \longmapsto \left(s(x)hs(x)^{-1}, s(x)gs(x)^{-1}\right) .$$

Computing the corresponding pull-push functor shows the claim. □

Finally we come to the structure corresponding to the braiding of section 5.2.4. Note that the cobordism that interchanges the two ingoing circles of the pair of pants $\Sigma(2,1)$, as in the following picture,



can also be realized as the diffeomorphism $F : \Sigma(2,1) \longrightarrow \Sigma(2,1)$ of the pair of pants that rotates the ingoing circles counterclockwise around each other and leaves the outgoing circle fixed. In this picture, we think of the cobordism as the cylinder $\Sigma(2,1) \times I$ where the identification with $\Sigma(2,1)$ on the top is the identity and on the bottom is given by the diffeomorphism F. More explicitly, denote by $\tau : \mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1$ the map that interchanges the two copies. We then consider the following diagram in the two-category $\mathfrak{Cob}(1,2,3)$:



where $\iota : \mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow \Sigma(2,1)$ is the standard inclusion of the two ingoing boundary circles into the trinion $\Sigma(2,1)$.

Our next task is to lift this situation to manifolds with J-covers. On the ingoing trinion, we take the J cover $P_{jk}^{\Sigma(2,1)}$. We denote the symmetry isomorphism in $\mathfrak{Cob}^J(1,2,3)$ by τ as well. Applying the diffeomorphism of the trinion explicitly, one sees that the outgoing trinion will have monodromies jkj^{-1} and j on the ingoing circles. Hence we have to apply a J-cover $P_{j,k}^{\Sigma(1,1)}$ of the cylinder $\Sigma(1,1)$ first to one insertion. The next lemma asserts that then the 2-morphism in $\mathfrak{Cob}^J(1,2,3)$ is fixed:

Lemma 5.3.25. *In the 2-category* $\mathfrak{Cob}^J(1,2,3)$*, there is a unique 2-morphism*

$$\hat{F} : \quad P_{j,k}^{\Sigma(2,1)} \Longrightarrow \big(P_{jkj^{-1},j}^{\Sigma(2,1)}\big) \circ \tau \circ \big(\mathrm{id} \sqcup P_{j,k}^{\Sigma(1,1)}\big)$$

that covers the 2-morphism F in $\mathfrak{Cob}(1,2,3)$.

Proof. First we show that a morphism $\tilde{F} : P_{jkj^{-1},j}^{\Sigma(2,1)} \longrightarrow P_{j,k}^{\Sigma(2,1)}$ can be found that covers the diffeomorphism $F : \Sigma(2,1) \longrightarrow \Sigma(2,1)$. This morphism is most easily described using the action of F on the fundamental group $\pi_1(\Sigma(2,1))$ of the pair of pants. The latter is a free group with two generators which can be chosen as the paths a, b around the two ingoing circles, $\pi_1(\Sigma(2,1)) = \mathbb{Z} * \mathbb{Z} = \langle a, b\rangle$. Then the induced action of F on the generators is $\pi_1(F)(a) = aba^{-1}$ and $\pi_1(F)(b) = a$. Hence, we find on the covers $F^* P_{j,k} \cong P_{jkj^{-1},j}$. This implies that we have a diffeomorphism $\tilde{F} : P_{jkj^{-1},j} \longrightarrow P_{j,k}$ covering F.

To extend $\tilde{F}$ to a 2-morphism in $\mathfrak{Cob}^J(1,2,3)$, we have to be a bit careful about how we consider the cover $P_{jkj^{-1},j}^{\Sigma(2,1)} \longrightarrow \Sigma(2,1)$ of the trinion as a 1-morphism. In fact, it has to be considered as a morphism $(P_j \longrightarrow \mathbb{S}^1) \sqcup (P_k \longrightarrow \mathbb{S}^1) \longrightarrow P_{jk} \longrightarrow \mathbb{S}^1$ where the ingoing components are first exchanged and then the identification of $P_k \longrightarrow \mathbb{S}^1$ and $P_{jkj^{-1}} \longrightarrow \mathbb{S}^1$ via the conjugation isomorphisms $P_{j,k}^{\Sigma(1,1)}$ induced by covers of the cylinders is used first, compare the lower arrows in the preceding commuting diagram. This yields the composition $\big(P_{jkj^{-1},j}^{\Sigma(2,1)}\big) \circ \tau \circ \big(\mathrm{id} \sqcup P_{j,k}^{\Sigma(1,1)}\big)$ on the right hand side of the diagram. □

The next step is to apply the TFT functor Z_G^J to the 2-morphism $\hat{F}$. The target 1-morphism of $\hat{F}$ can be computed using the fact that Z_G^J is a symmetric monoidal 2-functor; we find the following functor $\mathcal{C}(G)_j \otimes \mathcal{C}(G)_k \longrightarrow \mathcal{C}(G)_{jk}$:

$$Z_G^J\Big(\big(P_{jkj^{-1},j}^{\Sigma(2,1)}\big) \circ \tau \circ \big(\mathrm{id} \sqcup P_{j,k}^{\Sigma(1,1)}\big)\Big) = (-)^j \otimes_{kjk^{-1},k}^{op} (-)$$

We thus have the functor which acts on objects as $(V,W) \longmapsto \phi_j(W) \otimes V$ for $V \in \mathcal{C}(G)_j$ and $W \in \mathcal{C}(G)_k$.

Then $c := Z_G^J(\hat{F})$ is a natural transformation $(-) \otimes_{j,k} (-) \Longrightarrow (-)^j \otimes_{jkj^{-1},j}^{op} (-)$ i.e. a family of isomorphisms

$$c_{V,W} : V \otimes_{j,k} W \xrightarrow{\sim} \quad \phi_j(W) \otimes_{jkj^{-1},j} V \tag{5.8}$$

in $\mathcal{C}(G)_{jk}$ for $V \in \mathcal{C}(G)_j$ and $W \in \mathcal{C}(G)_k$.

We next show how this natural transformation is expressed when we use the equivalent description of the categories $\mathcal{C}(G)_j$ as vector bundles on action groupoids:

Proposition 5.3.26. *For* $V = \bigoplus V_h \in H_j//G$-mod *and* $W = \bigoplus W_h \in H_k//G$-mod *the natural isomorphism* $c_{V,W} : V \otimes W \longrightarrow \phi_j(W) \otimes V$ *is given by*

$$v \otimes w \longmapsto (h \cdot s(j)^{-1}).w \otimes v$$

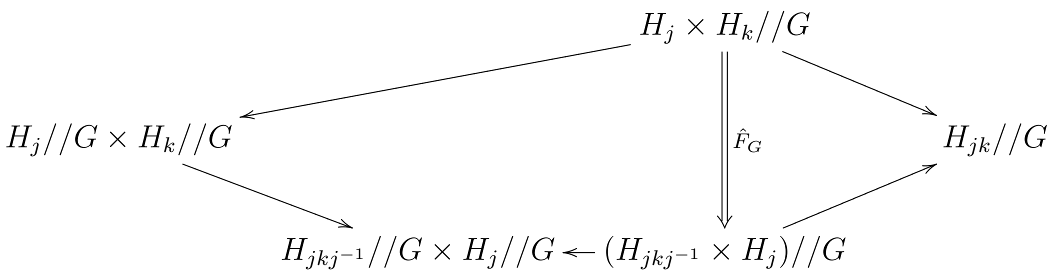
for $v \in V_h$ *with* $h \in H_j$ *and* $w \in W$.

Proof. We first compute the 1-morphism in the category $\mathfrak{Span}$ of spans of finite groupoids that corresponds to the target 1-morphisms $\left(P^{\Sigma(2,1)}_{jkj^{-1},j}\right) \circ \tau \circ \left(id \sqcup P^{\Sigma(1,1)}_{j,k}\right)$. From the previous proposition, we obtain the following zig-zag diagram:

$$H_j//G \times H_k//G \longrightarrow H_{jkj^{-1}}//G \times H_j//G \leftarrow (H_{jkj^{-1}} \times H_k)//G \longrightarrow H_{jk}//G \ .$$

The first morphism is given by the morphisms implementing the J-action that has been computed in the proof of proposition 5.3.24, composed with the exchange of factors. The second 1-morphism is obtained from the two projections and the last 1-morphism is the product in the group H.

Thus, the 2-morphism $\hat{F}$ yields a 2-morphism $\hat{F}_G$ in the diagram



where $\hat{F}_G$ is induced by the equivariant map $(h, h') \longmapsto (hh'h^{-1}, h)$. Once the situation is presented in this way, one can carry our explicitly the calculation along the lines described in [Mor10, Section 4.3] and obtain the desired result. □

A similar discussion can in principle be carried out to compute the associators. More generally, structural morphisms on $H//G$-mod can be derived from suitable 3-cobordisms. The relevant computations become rather involved. On the other hand, the category $H//G$-mod also inherits structural morphisms from the underlying category of vector spaces. We will use in the sequel the latter type of structural morphism.

5.4 Equivariant Drinfel'd double

The goal of this section is to show that the category $\mathcal{C}^J(G) := \bigoplus_{j\in J} \mathcal{C}(G)_j$ comprising the categories we have constructed in proposition 5.3.22 has a natural structure of a J-modular category.

Very much like ordinary modularity, J-modularity is a completeness requirement for the relevant tensor category that is suggested by principles of field theory. Indeed, it ensures that one can construct a J-equivariant topological field theory, see [Tur10]. For the definition of J-modularity we refer to [Kir04, Definition 10.1].

To establish the structure of a modular tensor category on the category found in the previous sections, we realize this category as the representation category of a finite-dimensional algebra, more precisely of a J-Hopf algebra. This section is organized as follows: we first recall the notions of equivariant fusion categories and of equivariant ribbon algebras, taking into account a suitable form of weak actions. In section 4.3, we then present the appropriate generalization of the Drinfel'd double that describes the category $\mathcal{C}^J(G)$. We then describe its orbifold category as the category of representations of a braided Hopf algebra, which allows us to establish the modularity of the orbifold category. We then apply a result of [Kir04] to deduce that the structure with which we have endowed $\mathcal{C}^J(G)$ is the one of a J-modular tensor category.

The Hopf algebraic structures endowed with weak actions we introduce in this section might be of independent interest.

5.4.1 Equivariant fusion categories.

Let $1 \longrightarrow G \longrightarrow H \stackrel{\pi}{\longrightarrow} J \longrightarrow 1$ be an exact sequence of finite groups. The normal subgroup G acts on H by conjugation; denote by $H//G$ the corresponding action groupoid. We consider the functor category $H//G\text{-mod} := [H//G, \mathrm{Vect}_{\mathbb{K}}]$, where $\mathbb{K}$ is an algebraically closed field of characteristic zero. The category $H//G$-mod is the category of H-graded vector spaces, endowed with an action of the subgroup G such that $g.V_h \subset V_{ghg^{-1}}$ for all $g \in G, h \in H$.

An immediate corollary of proposition 5.3.22 is the following description of the category $\mathcal{C}^J(G) := \bigoplus_{j\in J} \mathcal{C}(G)_j$ as an abelian category:

Proposition 5.4.1. *The category $\mathcal{C}^J(G)$ is equivalent, as an abelian category, to the category $H//G$-mod. In particular, the category $\mathcal{C}^J(G)$ is a 2-vector space in the sense of definition 5.2.5.*

Proof. Proposition 5.3.22 gives the equivalence $\mathcal{C}(G)_j \cong H_j//G$-mod of abelian categories, where $H_j := \pi^{-1}(j)$. The equivalence of categories $\mathcal{C}^J(G) \cong H//G$-mod now follows from the decomposition $H = \bigsqcup_{j\in J} H_j$. By [Mor08, Lemma 4.1.1], the representation category of a finite groupoid is a 2-vector space. □

Representation categories of finite groupoids are very close in structure to representation categories of finite groups. In particular, there is a complete character theory that describes the simple objects, see appendix 5.6.2.

We next introduce equivariant categories.

Definition 5.4.2. Let J be a finite group and $\mathcal{C}$ a category.

1. A *categorical action* of the group J on the category $\mathcal{C}$ consists of the following data:

   - A functor $\phi_j : \mathcal{C} \longrightarrow \mathcal{C}$ for every group element $j \in J$.
   - A functorial isomorphism $\alpha_{i,j} : \phi_i \circ \phi_j \xrightarrow{\sim} \phi_{ij}$ for every pair of group elements $i, j \in J$

   such that the coherence conditions

   $$\alpha_{ij,k} \circ \alpha_{i,j} = \alpha_{i,jk} \circ \phi_i(\alpha_{j,k}) \quad \text{and} \quad \phi_1 \cong \mathrm{id}$$

   hold.

2. If $\mathcal{C}$ is a monoidal category, we only consider actions by monoidal functors ϕ_j and require the natural transformations to be monoidal natural transformations. In particular, for each group element $j \in J$, we have the additional datum of a natural isomorphism

   $$\gamma_j(U, V) : \phi_j(U) \otimes \phi_j(V) \xrightarrow{\sim} \phi_j(U \otimes V)$$

   for each pair of objects U, V of $\mathcal{C}$ such that the following diagrams commute:

   $$\begin{array}{ccc} {}^{jk}X \otimes {}^{jk}Y & \xrightarrow{\gamma_{jk}(X,Y)} & {}^{jk}(X \otimes Y) \\ \Big\downarrow{\scriptstyle \alpha_{jk}(X)\otimes\alpha_{jk}(Y)} & & \Big\downarrow{\scriptstyle \alpha_{jk}(X\otimes Y)} \\ {}^{j}({}^{k}(X)) \otimes {}^{j}({}^{k}(Y)) & \xrightarrow[{}^{j}\gamma_k(X,Y)\circ\gamma_j({}^{k}X,{}^{k}Y)]{} & {}^{j}({}^{k}(X \otimes Y)) \end{array}$$

3. A J*-equivariant* category $\mathcal{C}$ is a category with a decomposition $\mathcal{C} = \bigoplus_{j \in J} \mathcal{C}_j$ and a categorical action of J, subject to the compatibility requirement

   $$\phi_i \mathcal{C}_j \subset \mathcal{C}_{iji^{-1}}$$

   with the grading.

   (Moreover, an isomorphism $\phi_j(1) \longrightarrow 1$ has to be chosen; we will suppress this isomorphism in our discussion.) We use the notation ${}^{j}U := \phi_j(U)$ for the image of an object $U \in \mathcal{C}$ under the functor ϕ_j.

4. A *J-equivariant tensor category* is a J-equivariant monoidal category $\mathcal{C}$, subject to the compatibility requirement that the tensor product of two homogeneous elements $U \in \mathcal{C}_i, V \in \mathcal{C}_j$ is again homogeneous, $U \otimes V \in \mathcal{C}_{ij}$.

Remark 5.4.3. *For any category $\mathcal{C}$, consider the category $\mathrm{AUT}(\mathcal{C})$ whose objects are automorphisms of $\mathcal{C}$ and whose morphisms are natural isomorphisms. The composition of functors and natural transformations endow $\mathrm{AUT}(\mathcal{C})$ with the natural structure of a strict tensor category. A categorical action of a finite group J on a category $\mathcal{C}$ then amounts to a tensor functor $\phi : J \longrightarrow \mathrm{AUT}(\mathcal{C})$, where J is seen as a tensor category with only identity morphisms, compare also remark 5.3.2.3.*

Similarly, we consider for a monoidal category $\mathcal{C}$ the category $\mathrm{AUT}mon(\mathcal{C})$ whose objects are monoidal automorphisms of $\mathcal{C}$ and whose morphisms are monoidal natural automorphisms. The categorical actions we consider for monoidal categories are then tensor functors $\phi : J \longrightarrow \mathrm{AUT}mon(\mathcal{C})$ For more details, we refer to [Tur10] Appendix 5.

The category $H//G$-mod has a natural structure of a monoidal category: the tensor product of two objects $V = \oplus_{h\in H} V_h$ and $W = \oplus_{h\in H} W_h$ is the vector space $V \otimes W$ with H grading given by $(V \otimes W)_h := \oplus_{h_1 h_2 = h} V_{h_1} \otimes W_{h_2}$ and G action given by $g.(v \otimes w) = g.v \otimes g.w$. The associators are inherited from the underlying category of vector spaces.

Proposition 5.4.4. *Consider an exact sequence of groups $1 \longrightarrow G \longrightarrow H \longrightarrow J \longrightarrow 1$. Any choice of a a set-theoretic section $s : J \longrightarrow H$ allows us to endow the abelian category $H//G$-mod with the structure of a J-equivariant tensor category as follows: the functor ϕ_j is given by shifting the grading from h to $s(j)hs(j)^{-1}$ and replacing the action by g by the action of $s(j)gs(j)^{-1}$. The isomorphism $\alpha_{i,j} : \phi_i \circ \phi_k \longrightarrow \phi_{ij}$ is given by the left action action of the element*

$$\alpha_{i,j} = s(i)s(j)s(ij)^{-1} \ .$$

The fact that the action is only a weak action thus accounts for the failure of s to be a section in the category of groups.

Proof. Only the coherence conditions $\alpha_{ij,k} \circ \alpha_{i,j} = \alpha_{i,jk} \circ \phi_i(\alpha_{j,k})$ remain to be checked. By the results of Dedecker and Schreier, cf. proposition 5.3.3, the group elements $s(i)s(j)s(ij)^{-1} \in G$ are the coherence cells of a weak group action of J on H. By definition 5.3.1, this implies the coherence identities, once one takes into account that that composition of functors is written in different order than group multiplication. □

We have derived in section 5.3.5 from the geometry of extended cobordism categories more structure on the geometric category $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j$. In particular, we collect the functors $\otimes_{jk} : \mathcal{C}(G)_j \boxtimes \mathcal{C}(G)_k \longrightarrow \mathcal{C}(G)_{jk}$ from proposition 5.3.23 into a functor

$$\otimes : \mathcal{C} \boxtimes \mathcal{C} \longrightarrow \mathcal{C}. \tag{5.9}$$

Another structure are the isomorphisms $V \otimes W \longrightarrow \phi_j(W) \otimes V$ for $V \in \mathcal{C}(G)_j$, described in proposition 5.3.26. Together with the associators, this suggests to endow the category $\mathcal{C}^J(G)$ with a structure of a braided J-equivariant tensor category:

Definition 5.4.5. A *braiding* on a J-equivariant tensor category is a family

$$c_{U,V} : U \otimes V \longrightarrow {}^jV \otimes U$$

of isomorphisms, one for every pair of objects $U \in \mathcal{C}_i, V \in \mathcal{C}_j$, which are natural in the sense that for any pair $f : U \longrightarrow U'$, $g : V \longrightarrow V'$ of morphisms, the identity

$$c_{U',V'}(f \otimes g) = ({}^jg \otimes f)c_{U,V},$$

holds. Moreover, a braiding is required to satisfy an analogue of the hexagon axioms (see [Tur10, appendix A5]) and to be preserved under the action of J, i.e. the following diagram commutes for all objects U, V with $U \in \mathcal{C}_j$ and $i \in J$

$$\begin{array}{ccccc}
{}^i(U \otimes V) & \xrightarrow{{}^i(c_{U,V})} & {}^i({}^jV \otimes U) & \xrightarrow{\gamma_i} & {}^i({}^jV) \otimes {}^iU \\
\downarrow{\scriptstyle \gamma_i} & & & & \downarrow{\scriptstyle \alpha_{ij}(V)\otimes \mathrm{id}} \\
{}^iU \otimes {}^iV & \xrightarrow[c_{{}^iU,{}^iV}]{} & {}^{ij^{-1}}({}^iV) \otimes {}^iU & \xrightarrow[\alpha_{iji^{-1},i(V)}\otimes \mathrm{id}]{} & {}^{ij}V \otimes {}^iU
\end{array}$$

Remark 5.4.6.

1. *It should be appreciated that a braided J-equivariant category is not, in general, a braided category. Its* neutral component $\mathcal{C}_1$ *with* $1 \in J$ *the neutral element, is a braided tensor category.*

2. *By replacing the underlying category by an equivalent category, one can replace a weak action by a strict action, compare [Tur10, Appendix A5]. In our case, weak actions actually lead to simpler algebraic structures.*

3. *The J-equivariant monoidal category $H//G$*-mod *has a natural braiding isomorphism that has been described in proposition 5.3.26*

We use the equivalence of abelian categories between $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j$ and $H//G$-mod to endow the category $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j$ with associators. The category has now enough structure that we can state our next result:

Proposition 5.4.7. *The category $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j$, with the tensor product functor from (5.9), can be endowed with the structure of a braided J-equivariant tensor category such that the isomorphism $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j \cong H//G$*-mod *becomes an isomorphism of braided J-equivariant tensor categories.*

Proof. The compatibility with the grading is implemented by definition via the graded components $\otimes_{jk}$ of $\otimes$ and the graded components of $c_{V,W}$. It remains to check that the action is by tensor functors and that the braiding satisfies the hexagon axiom. The second boils down to a simple calculation and the first is seen by noting that the action is essentially an index shift which is preserved by tensoring together the respective components. □

5.4.2 Equivariant ribbon algebras

In the following, let J again be a finite group. To identify the structure of a J-modular tensor category on the geometric category $\mathcal{C}^J(G) = \bigoplus_{j\in J}\mathcal{C}(G)_j$, we need dualities. This will lead us to the discussion of (equivariant) ribbon algebras. Apart from strictness issues, our discussion closely follows [Tur10]. We start our discussion with the relevant category-theoretic structures.

Definition 5.4.8. 1. A *J-equivariant ribbon category* is a J-braided category with dualities and a family of isomorphisms $\theta_V : V \longrightarrow {}^jV$ for all $j \in J, V \in \mathcal{C}_j$, such that θ is compatible with duality and the action of J (see [Tur10, VI.2.3] for the identities). In contrast to [Tur10], we allow weak J-actions and thus require the diagram

$$\begin{array}{ccccccc}
U\otimes V & & & \xrightarrow{\quad\theta_{U\otimes V}\quad} & & & {}^{ji}(U\otimes V) \\
\downarrow{\scriptstyle \theta_U\otimes\theta_V} & & & & & & \uparrow{\scriptstyle \alpha_{ji}\circ{}^j(\gamma_i)} \\
{}^jU\otimes{}^iV & & & & & & {}^j({}^iU\otimes{}^iV) \\
& \searrow{\scriptstyle R_{{}^jU,{}^iV}} & & & & \nearrow{\scriptstyle {}^j(R_{{}^iV,U})} & \\
& & {}^j({}^iV)\otimes{}^jU & \xrightarrow{\quad\gamma_j\quad} & {}^j({}^iV\otimes U) & &
\end{array}$$

to commute for $U \in \mathcal{C}_j$ and $V \in \mathcal{C}_i$.

2. A *J-equivariant fusion category* is an abelian semi-simple J-equivariant ribbon category.

Remark 5.4.9. *The following facts directly follow from the definition of J-equivariant ribbon category: the neutral component $\mathcal{C}_1$ is itself a braided tensor category. In particular, it contains the tensor unit of the J-equivariant tensor category. The dual object of an object $V \in \mathcal{C}_j$ is in the category $\mathcal{C}_{j^{-1}}$.*

We will not be able to directly endow the geometric category $\mathcal{C}^J(G) = \bigoplus_{j\in J}\mathcal{C}(G)_j$ with the structure of a J-equivariant fusion category. Rather, we will realize an equivalent category as the category of modules over a suitable algebra. To this end, we introduce in several steps the notions of a J-ribbon algebra and analyze the extra structure induced on its representation category.

Definition 5.4.10. Let A be an (associative, unital) algebra over a field $\mathbb{K}$. A *weak J-action* on A consists of an algebra automorphism $\varphi_j \in \mathrm{Aut}(A)$, one for every element $j \in J$, and an invertible element $c_{ij} \in A$, one for every pair of elements $i, j \in J$, such that for all $i, j, k \in J$ the following conditions hold:

$$\varphi_i \circ \varphi_j = \mathrm{Inn}_{c_{i,j}} \circ \varphi_{ij} \qquad \varphi_i(c_{j,k}) \cdot c_{i,jk} = c_{i,j} \cdot c_{ij,k} \quad \text{and} \quad c_{1,1} = 1$$

Here Inn_x with x an invertible element of A denotes the algebra automorphism $a \longmapsto xax^{-1}$. A weak action of a group J is called strict, if $c_{i,j} = 1$ for all pairs $i, j \in J$.

Remark 5.4.11. *As discussed for weak actions on groups in remark 5.3.2, a weak action on a $\mathbb{K}$-algebra A can be seen as a categorical action on the category which has one object and the elements of A as endomorphisms.*

We now want to relate a weak action $(\varphi_j, c_{i,j})$ of a group J on an algebra A to categorical actions on the representation category A-mod. To this end, we define for each element $j \in J$ a functor on objects by

$${}^j(M, \rho) := (M, \rho \circ (\varphi_j \otimes \mathrm{id}_M))$$

and on morphisms by ${}^j f = f$. For the functorial isomorphisms, we take $\alpha_{i,j}(M, \rho) := \rho(c_{ij} \otimes \mathrm{id}_M)$. The next lemma is immediate from the definitions:

Lemma 5.4.12. *Given a weak action of J on a $\mathbb{K}$-algebra A, these data define a categorical action on the category* A-mod.

We next turn to an algebraic structure that yields J-equivariant tensor categories.

Definition 5.4.13. A *J-Hopf algebra* over $\mathbb{K}$ is a Hopf algebra A with a J-grading $A = \bigoplus_{j \in J} A_j$ and a weak J-action such that:

- The algebra structure of A restricts to the structure of an associative algebra on each homogeneous component so that A is the direct sum of the components A_j as an algebra.
- J acts by homomorphisms of Hopf algebras.
- The action of J is compatible with the grading, i.e. $\varphi_i(A_j) \subset A_{iji^{-1}}$
- The coproduct $\Delta : A \longrightarrow A \otimes A$ respects the grading, i.e.
$$\Delta(A_j) \subset \bigoplus_{p,q \in J, pq = j} A_p \otimes A_q \ .$$
- The elements $(c_{i,j})_{i,j \in J}$ are group-like, i.e $\Delta(c_{i,j}) = c_{i,j} \otimes c_{i,j}$.

Remark 5.4.14. *1. For the counit ϵ and the antipode S of a J-Hopf algebra, the compatibility relations with the grading $\epsilon(A_j) = 0$ for $j \neq 1$ and $S(A_j) \subset A_{j^{-1}}$ are immediate consequences of the definitions.*

2. The restrictions of the structure maps endow the homogeneous component A_1 of A with the structure of a Hopf algebra with a weak J-action.

3. J-Hopf algebras with strict J-action have been considered under the name "J-crossed Hopf coalgebra" in [Tur10, Chapter VII.1.2].

4. The invertible elements c_{ij} of a J-Hopf algebra that are part of the definition of the weak J-action fulfill the identity $\epsilon(c_{i,j})\epsilon((c_{i,j})^{-1}) = 1$.

We will normalize them in such a way that the identity $\epsilon(c_{i,j}) = 1$ for all $i, j \in J$ holds.

The category A-mod of finite-dimensional modules over a J-Hopf algebra inherits a natural duality from the duality of the underlying category of $\mathbb{K}$-vector spaces. The weak action described in Lemma 5.4.12 is even a monoidal action, since J acts by Hopf algebra morphisms. A grading on A-mod can be given by taking $(A\text{-mod})_j = A_j\text{-mod}$ as the j-homogeneous component. From the properties of a J-Hopf algebra one can finally deduce that the tensor product, duality and grading are compatible with the J-action. We have thus arrived at the following statement:

Lemma 5.4.15. *The category of representations of a J-Hopf algebra has a natural structure of a $\mathbb{K}$-linear, abelian J-equivariant tensor category with compatible duality as introduced in definition 5.4.5.*

The representation category of a braided Hopf algebra is a braided tensor category. If the Hopf algebra has, moreover, a twist element, its representation category is even a ribbon category. We now present J-equivariant generalizations of these structures. To this end, we introduce for a J-Hopf algebra A a linear endomorphism τ^J of $A \otimes A$ that acts on $a \otimes b \in A_i \otimes A_j$ as

$$\tau^J(a \otimes b) = \varphi_i^{-1}(b) \otimes a \tag{5.10}$$

We call this linear map the J-flip on A.

Definition 5.4.16. Let A be a J-Hopf algebra. An *R-matrix* in A is an invertible element $R = \sum_{i,j \in J} R_{i,j} \in A \otimes A$ with $R_{ij} \in A_i \otimes_{\mathbb{K}} A_j$ which satisfies the following conditions:

- $$R\Delta(a) = \tau^J \Delta(a) R$$

- For any triple $i, j, k \in J$, we have the following equivariant version of the Yang-Baxter relations:

$$(\mathrm{id}_{A_i} \otimes \Delta_{j,k})(R_{i,jk}) = (R_{i,k})_{1[j]3}(R_{i,j})_{12[k]}$$

$$(\Delta_{i,j} \otimes \mathrm{id}_{A_k})(R_{ij,k}) = ((\varphi_j \otimes \mathrm{id}_{A_k})(R_{j^{-1}ij,k})_{1[j]3}(R_{j,k})_{[i]23}$$

where $\Delta_{i,j} : A_{ij} \longrightarrow A_i \otimes A_j$ are the components of the coproduct and for $r = \sum_r r' \otimes r'' \in A \otimes A$ we denote

  - $r_{12[k]} = r \otimes 1_k \in A^{\otimes 3}$
  - $r_{[i]23} = 1_i \otimes r$
  - $r_{1[j]3} = \sum_r r' \otimes 1_j \otimes r''$

with $1 = \sum_{j \in J} 1_j$.

A J-Hopf algebra with an R-matrix is called a *J-braided Hopf algebra.*

Remark 5.4.17.

- *A J-braided Hopf algebra is not, in general, a braided Hopf algebra.*
- *The component A_1 is a braided Hopf algebra.*

For the twist, we proceed similarly:

Definition 5.4.18. Let A be a braided J-Hopf algebra with R-matrix R. A *twist element* in A is an invertible element $\theta = \sum_{j \in J} \theta_j \in A$ with $\theta_j \in A_j$ that obeys the following conditions:

- For all $j \in J, a \in A_j$:

$$\varphi_j(a) = \theta_j^{-1} a \theta_j$$

- The elements $(\theta_j)_{j \in J}$ are invariant under the antipode and the action of J, and furthermore compatible with the R-matrix, i.e.

$$\Delta_{ji}(\theta_{ji}) = (\theta_j \otimes \theta_i)[\Delta_{ji}(c_{ji})(\tau(\mathrm{id}_{H_i} \otimes \varphi_j)R_{i,j})R_{j,i}]^{-1} \qquad \text{for all } i, j \in J \ .$$

A J-braided Hopf algebra with a twist is called *J-ribbon algebra.*

Let A be a J-ribbon algebra. By lemma 5.4.15, its representation category A-mod has a natural structure of a J-equivariant tensor category with compatible dualities. To find the structure of a J-ribbon category, we have to find a J-equivariant braiding and twist.

To this end, we consider for objects $V \in A_i$-mod and $W \in A_j$-mod the morphism $R_{VW} := \tau^J \circ R._{-} : V \otimes W \longrightarrow {}^iW \otimes V$ constructed from the the left-action $R._{-}$ of R on the $A \otimes A$-module $V \otimes W$ and the J-flip τ^J introduced in (5.10). The twist endomorphism θ_V for $V \in A$-mod is defined by the left action of the twist element as well, $\theta_V := \theta^{-1}$.

The morphisms R and θ can be checked to endow the J-equivariant category A-mod with a J-equivariant braiding and twist. We have thus derived:

Proposition 5.4.19. *The representation category of a J-ribbon algebra is a J-ribbon category.*

Remark 5.4.20. *In [Tur10], Hopf algebras and ribbon Hopf algebras with strict J-action have been considered. The next subsection will give an illustrative example where the natural action is not strict.*

5.4.3 Equivariant Drinfel'd Double

The goal of this subsection is to construct a J-crossed ribbon algebra, given a finite group G with a weak J-action. As explained in subsection 5.3.1, such a weak J-action amounts to a group extension

$$1 \longrightarrow G \longrightarrow H \xrightarrow{\pi} J \longrightarrow 1 \tag{5.11}$$

with a set-theoretical splitting $s : J \longrightarrow H$.

We start from the well-known fact reviewed in subsection 5.2.5 that the Drinfel'd double $\mathcal{D}(H)$ of the finite group H is a ribbon Hopf algebra. The double $\mathcal{D}(H)$ has a canonical basis $\delta_{h_1} \otimes h_2$ indexed by pairs h_1, h_2 of elements of H. Let $G \subset H$ be a subgroup. We are interested in the vector subspace $\mathcal{D}^J(G)$ spanned by the basis vectors $\delta_h \otimes g$ with $h \in H$ and $g \in G$.

Lemma 5.4.21. *The structure maps of the Hopf algebra $\mathcal{D}(H)$ restrict to the vector subspace $\mathcal{D}^J(G)$ in such a way that the latter is endowed with the structure of a Hopf subalgebra.*

Remark 5.4.22. *The induced algebra structure on $\mathcal{D}^J(G)$ is the one of the groupoid algebra of the action groupoid $H//G$.*

The Drinfel'd double $\mathcal{D}(H)$ of a group H has also the structure of a ribbon algebra. However, neither the R-matrix nor the the ribbon element yield an R-matrix or a ribbon element of $\mathcal{D}^J(G) \subset \mathcal{D}(H)$. Rather, this Hopf subalgebra can be endowed with the structure of a J-ribbon Hopf algebra as in definition 5.4.18.

To this end, consider the partition of the group H into the disjoint subsets $H_j := \pi^{-1}(j) \subset H$. It gives a J-grading of the algebra A as a direct sum of subalgebras:

$$A_j := \langle \delta_h \otimes g \rangle_{h \in H_j, g \in G} \ .$$

The set-theoretical section s gives a weak action of J on A that can be described by its action on the canonical basis of A_j:

$$\varphi_j(\delta_h \otimes g) := \ (\delta_{s(j)hs(j)^{-1}} \otimes s(j)gs(j)^{-1}) \ ;$$

the coherence elements are

$$c_{ij} := \sum_{h \in H} \delta_h \otimes s(i)s(j)s(ij)^{-1} \ .$$

Now the compatibility relations of grading and weak J-action with the Hopf algebra structure that have been formulated in definition 5.4.13 can be checked by straightforward calculations. We summarize our finding:

Proposition 5.4.23. *The Hopf algebra $\mathcal{D}^J(G)$, together with the grading and weak J-action derived from the weak J-action on the group G, has the structure of a J-Hopf algebra.*

We now turn to the last piece of structure, an R-matrix and twist element in $\mathcal{D}^J(G)$. Consider the element $R = \sum_{ij} R_{i,j} \in \mathcal{D}^J(G) \otimes \mathcal{D}^J(G)$ with homogeneous elements R_{ij} defined as

$$R_{i,j} := \sum_{h_1 \in H_i, h_2 \in H_j} (\delta_{h_1} \otimes 1) \otimes (\delta_{h_2} \otimes h_1 s(j)^{-1}). \tag{5.12}$$

The element R_{ij} is invertible with inverse

$$R_{i,j}^{-1} = \sum_{h_1 \in H_i, h_2 \in H_j} (\delta_{h_1} \otimes 1) \otimes (\delta_{h_2} \otimes s(j) h_1^{-1}).$$

We also introduce a twist element $\theta = \sum_{j \in J} \theta_j \in \mathcal{D}^J(G)$ with $\theta_j := \sum_{h \in H_j} (\delta_h \otimes hs(j)^{-1}) \in A_j$ for every element $j \in J$. Again, a straightforward computation yields

Proposition 5.4.24. *The elements R and θ endow the J-Hopf algebra $\mathcal{D}^J(G)$ with the structure of a J-ribbon algebra that we call the J-Drinfel'd double of G.*

We are now ready to come back to the J-equivariant tensor category $\mathcal{C}^J(G) = \bigoplus_{j \in J} \mathcal{C}(G)_j$ described in proposition 5.4.7. From this proposition, we know that the category $\mathcal{C}^J(G)$ is equivalent to $H//G$-mod $\cong \mathcal{D}^J(G)$-mod as a J-equivariant tensor category. Also J-action and tensor product coincide with the ones on $\mathcal{D}^J(G)$-mod. Moreover, the equivariant braiding of $\mathcal{C}^J(G)$ computed in proposition 5.3.26 is just the J-flip composed with action of the R-matrix of $\mathcal{D}^J(G)$ given in (5.12) which is the equivariant braiding in $\mathcal{D}^J(G)$-mod.

This allows us to transfer also the other structure on representation category of the J-Drinfel'd double $\mathcal{D}^J(G)$ described in proposition 5.4.24 to the category $\mathcal{C}^J(G)$:

Proposition 5.4.25. *The J-equivariant tensor category $\mathcal{C}^J(G) = \bigoplus_{j \in J} \mathcal{C}(G)_j$ described in proposition 5.4.7 can be endowed with the structure of a braided J-equivariant fusion category such that it is equivalent, as a J-equivariant fusion category, to the category $\mathcal{D}^J(G)$*-mod.

Remark 5.4.26. *At this point, we have constructed a J-equivariant fusion category $\mathcal{C}^J(G) = \bigoplus_{j \in J} \mathcal{C}(G)_j$ with neutral component $\mathcal{C}(G)_1 \cong \mathcal{D}(G)$-mod from a weak action of the group J on the group G, or in different words, from a 2-group homomorphisms $J \longrightarrow \mathrm{AUT}(G)$ with $\mathrm{AUT}(G)$ the automorphism 2-group of G.*

In this remark, we very briefly sketch the relation to the description of J-equivariant fusion categories with given neutral sector $\mathcal{B}$ in terms of 3-group homomorphisms $J \longrightarrow \underline{Pic}(\mathcal{B})$ given in [ENO10]. Here $\underline{Pic}(\mathcal{B})$ denotes the so called Picard 3-group whose objects are invertible module-categories of the fusion category $\mathcal{B}$. The group structure comes from the tensor product of module categories which can be defined since the braiding on $\mathcal{B}$ allows to turn module categories into bimodule categories.

Using this setting, we give a description of our J-equivariant fusion category $\mathcal{D}^J(G)$-mod in terms of a functor $\Xi : J \longrightarrow \underline{Pic}(\mathcal{D}(G))$. To this end, we construct a 3-group homomorphism $\mathrm{AUT}(G) \longrightarrow \underline{Pic}(\mathcal{D}(G))$ and write Ξ as the composition of this functor and the functor $J \longrightarrow \mathrm{AUT}(G)$ defining the weak J-action.

The 3-group homomorphism $\mathrm{AUT}(G) \longrightarrow \underline{Pic}(\mathcal{D}(G))$ is given as follows: to an object $\varphi \in \mathrm{AUT}(G)$ we associate the twisted conjugation groupoid $G//^{\varphi}G$, where G acts on itself by twisted conjugation, $g.x := gx\varphi(g)^{-1}$. This yields the category $G//^{\varphi}G$-mod $:= [G//^{\varphi}G, \mathrm{Vect}_{\mathbb{K}}]$ which is naturally a module category over $\mathcal{D}(G)$-mod. Morphisms $\varphi \longrightarrow \psi$ in $\mathrm{AUT}(G)$ are given by group elements $g \in G$ with $g\varphi g^{-1} = \psi$; to such a morphism we associate the functor $L_g : G//^{\varphi}G \longrightarrow G//^{\psi}G$ given by conjugating with $g \in G$ on objects and morphisms. This induces functors of module categories $G//^{\varphi}G$-mod $\longrightarrow G//^{\psi}G$-mod. Natural coherence data exist; one then shows that this really establishes the desired 3-group homomorphism.

5.4.4 Orbifold category and orbifold algebra

It remains to show that the J-equivariant ribbon category $\mathcal{C}^J(G) = \bigoplus_{j\in J}\mathcal{C}(G)_j$ described in 5.4.24 is J-modular. To this end, we will use the orbifold category of the J-equivariant category:

Definition 5.4.27. Let $\mathcal{C}$ be a J-equivariant category. The *orbifold category $\mathcal{C}^J$* of $\mathcal{C}$ has:

- as objects pairs $(V, (\psi_j)_{j\in J})$ consisting of an object $V \in \mathcal{C}$ and a family of isomorphisms $\psi_j : {}^jV \longrightarrow V$ with $j \in J$ such that $\psi_i \circ {}^i\psi_j = \psi_{ij}$.
- as morphisms $f : (V, \psi_j^V) \longrightarrow (W, \psi_j^W)$ those morphisms $f : V \longrightarrow W$ in $\mathcal{C}$ for which $\psi_j \circ {}^j(f) = f \circ \psi_j$ holds for all $j \in J$.

In [Kir04], it has been shown that the orbifold category of a J-ribbon category is an ordinary, non-equivariant ribbon category:

Proposition 5.4.28. *1. Let $\mathcal{C}$ be a J-ribbon category. Then the orbifold category $\mathcal{C}^J$ is naturally endowed with the structure of a ribbon category by the following data:*

- *The tensor product of the objects $(V, (\psi_j^V))$ and $(W, (\psi_j^W))$ is defined as the object $(V \otimes W, (\psi_j^V \otimes \psi_j^W))$.*

- *The tensor unit for this tensor product is* $\mathbf{1} = (\mathbf{1}, (\mathrm{id}))$
- *The dual object of* $(V, (\psi_j))$ *is the object* $(V^*, (\psi_j^*)^{-1})$*, where* V^* *denotes the dual object in* $\mathcal{C}$*.*
- *The braiding of the two objects* $(V, (\psi_j^V))$ *and* $(W, (\psi_j^W))$ *with* $V \in \mathcal{C}_j$ *is given by the isomorphism* $(\psi_j \otimes \mathrm{id}_V) \circ c_{V,W}$*, where* $c_{V,W} : V \otimes W \longrightarrow {}^jW \otimes V$ *is the* J*-braiding in* $\mathcal{C}$*.*
- *The twist on an object* $(V, (\psi_j))$ *is* $\psi_j \circ \theta$*, where* $\theta : V \longrightarrow {}^jV$ *is the twist in* $\mathcal{C}$*.*

2. *If* $\mathcal{C}$ *is a* J*-equivariant fusion category, then the orbifold category* $\mathcal{C}^J$ *is even a fusion category.*

It has been shown in [Kir04] that the J-modularity of a J-equivariant fusion category is equivalent to the modularity as in definition 5.2.20 of its orbifold category. Our problem is thus reduced to showing modularity of the orbifold category of $\mathcal{D}^J(G)$-mod.

To this end, we describe orbifoldization on the level of (Hopf-)algebras: given a J-equivariant algebra A, we introduce an orbifold algebra $\widehat{A}^J$ such that its representation category $\widehat{A}^J$-mod is isomorphic to the orbifold category of A-mod.

Definition 5.4.29. Let A be an algebra with a weak J-action (φ_j, c_{ij}). We define on the vector space $\widehat{A}^J := A \otimes \mathbb{K}[J]$ a unital associative multiplication which is defined on an element of the form $(a \otimes j)$ with $a \in A$ and $j \in J$ by

$$(a \otimes i)(b \otimes j) := a\varphi_i(b)c_{ij} \otimes ij \ .$$

This algebra is called the *orbifold algebra* $\widehat{A}^J$ of the J-equivariant algebra A with respect to the weak J-action.

If A is even a J-Hopf algebra, it is possible to endow the orbifold algebra with even more structure. To define the coalgebra structure on the orbifold algebra, we use the standard coalgebra structure on the group algebra $\mathbb{K}[J]$ with coproduct $\Delta_J(j) = j \otimes j$ and counit $\epsilon_J(j) = 1$ on the canonical basis $(j)_{j\in J}$. The tensor product coalgebra on $A \otimes \mathbb{K}[J]$ has the coproduct and counit

$$\Delta(a \otimes j) = (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_{\mathbb{K}[J]})(\Delta_A(a) \otimes j \otimes j), \quad \text{and} \quad \epsilon(a \otimes j) = \epsilon_A(a) \qquad (5.13)$$

which is clearly coassociative and counital.

To show that this endows the orbifold algebra with the structure of a bialgebra, we have first to show that the coproduct Δ is a unital algebra morphism. This follows from the fact, that Δ_A is already an algebra morphism and that the action of J is by coalgebra morphisms. Next, we have to show that the counit ϵ is a unital algebra morphism as well. This follows from the fact that the action of J commutes with the counit and from the fact that we take normalized elements $c_{i,j}$, see remark 5.4.14 3.). The compatibility of ϵ with the unit is obvious.

In a final step, one verifies that the endomorphism

$$S(a \otimes j) = (c_{j^{-1},j})^{-1} \varphi_{j^{-1}}(S_A(a)) \otimes j^{-1}$$

is an antipode. Altogether, one arrives at

Proposition 5.4.30. *If A is a J-Hopf-algebra, then the orbifold algebra $\widehat{A}^J$ has a natural structure of a Hopf algebra.*

Remark 5.4.31. 1. *The algebra $\widehat{A}^J$ is not the fixed point subalgebra A^J of A; in general, the categories A^J-mod and $\widehat{A}^J$-mod are inequivalent.*

2. *Given any Hopf algebra A with weak J-action, we have an exact sequence of Hopf algebras*

$$A \longrightarrow \widehat{A}^J \longrightarrow \mathbb{K}[J] \ . \tag{5.14}$$

In particular, A is a sub-Hopf algebra of $\widehat{A}^J$. In general, there is no inclusion of $\mathbb{K}[J]$ into $\widehat{A}^J$ as a Hopf algebra.

3. *If the action of J on the algebra A is strict, then the algebra A is a module algebra over the Hopf algebra $\mathbb{K}[J]$ (i.e. an algebra in the tensor category $\mathbb{K}[J]$-mod). Then the orbifold algebra is the smash product $A\#\mathbb{K}[J]$ (see [Mon93, Section 4] for the definitions). The situation described occurs, if and only if the exact sequence (5.14) splits.*

The next proposition justifies the name "orbifold algebra" for $\widehat{A}^J$:

Proposition 5.4.32. *Let A be a J-Hopf algebra. Then there is an equivalence of tensor categories*

$$\widehat{A}^J\text{-mod} \cong (A\text{-mod})^J \ .$$

Proof. • An object of $(A\text{-mod})^J$ consists of a $\mathbb{K}$-vector space M, an A-action $\rho : A \longrightarrow \mathrm{End}(M)$ and a family of A-module morphisms $(\psi_j)_{j\in J}$. We define on the same $\mathbb{K}$-vector space M the structure of an $\widehat{A}^J$ module by $\tilde{\rho} : \widehat{A}^J \longrightarrow \mathrm{End}(M)$ with $\tilde{\rho}(a \otimes j) := \rho(a) \circ \psi_j$. One next checks that, given two objects (M, ρ, ψ) and (M', ρ', ψ') in $(A\text{-mod})^J$, a $\mathbb{K}$-linear map $f \in \mathrm{Hom}_{\mathbb{K}}(M, M')$ is in the subspace $\mathrm{Hom}_{(A\text{-mod})^J}(M, M')$ if and only if it is in the subspace $\mathrm{Hom}_{\widehat{A}^J\text{-mod}}(M, \tilde{\rho}), (M', \tilde{\rho}'))$.

We can thus consider a $\mathbb{K}$-linear functor

$$F : (A\text{-mod})^J \longrightarrow \widehat{A}^J\text{-mod} \tag{5.15}$$

which maps on objects by $(M, \rho, \psi) \longmapsto (M, \tilde{\rho})$ and on morphisms as the identity. This functor is clearly fully faithful.

To show that the functor is also essentially surjective, we note that for any object $(M, \tilde{\rho})$ in $\widehat{A}^J$-mod, an object in $(A\text{-mod})^J$ can be obtained as follows: on the underlying vector space, we have the structure of an A-module by restriction, $\rho(a) := \tilde{\rho}(a \otimes 1_J)$. A family of equivariant morphisms is given by $\psi_j := \tilde{\rho}(\mathbf{1} \otimes j)$. Clearly its image under F is isomorphic to $(M, \tilde{\rho})$. This shows that the functor F is an equivalence of categories, indeed even an isomorphism of categories.

- The functor F is also a strict tensor functor: consider two objects (M_1, ρ_1, ψ_1) and (M_2, ρ_2, ψ_2) in $(A\text{-mod})^J$. The functor F yields the following action of the orbifold Hopf algebra $\widehat{A}^J$ on the $\mathbb{K}$-vector space $M_1 \otimes_{\mathbb{K}} M_2$:

$$\tilde{\rho}_{M_1 \otimes M_2}(a \otimes j) = \rho_1 \otimes \rho_2(\Delta(a)) \circ (\psi_1(j) \otimes \psi_2(j)) \ .$$

Since the coproduct on $\widehat{A}^J$ was just given by the tensor product of coproducts on A and $\mathbb{K}[G]$, this coincides with the tensor product of $F(M_1, \rho_1, \psi_1)$ and $F(M_2, \rho_2, \psi_2)$ in $\widehat{A}^J$-mod.

□

In a final step, we assume that the J-equivariant algebra A has the additional structure of a J-ribbon algebra. Then, by proposition 5.4.19, the category A-mod is a J-ribbon category and by proposition 5.4.28 the orbifold category $(A\text{-mod})^J$ is a ribbon category. The strict isomorphism (5.15) of tensor categories allows us to transport both the braiding and the ribbon structure to the representation category of the orbifold Hopf algebra $\widehat{A}^J$. General results [Kas95, Proposition 16.6.2] assert that this amounts to a natural structure of a ribbon algebra on $\widehat{A}^J$. In fact, we directly read off the R-matrix and the ribbon element. For example, the R-matrix $\hat{R}$ of $\hat{A}^J$ equals

$$\hat{R} = \hat{\tau}\ c_{\widehat{A}^J, \widehat{A}^J}(1_{\widehat{A}^J} \otimes 1_{\widehat{A}^J}) \in \widehat{A}^J \otimes \widehat{A}^J \ \ ,$$

where the linear map $\hat{\tau}$ flips the two components of the tensor product $\widehat{A}^J \otimes \widehat{A}^J$. This expression can be explicitly evaluated, using the fact that $A \otimes \mathbb{K}[J]$ is an object in $(A\text{-mod})^J$ with A-module structure given by left action on the first component and that the morphisms ψ_j are given by left multiplication on the second component. We find for the R-matrix of $\widehat{A}^J$

$$\begin{aligned}\hat{R} &= \sum_{i,j \in J} (\mathrm{id} \otimes \psi_j)(\rho \otimes \rho)(R)((1_A \otimes 1_J) \otimes 1_A \otimes 1_J) \\ &= \sum_{i,j \in J} ((R_{i,j})_1 \otimes 1_J) \otimes ((R_{i,j})_2 \otimes j))\end{aligned}$$

where R is the R-matrix of A. The twist element of $\widehat{A}^J$ can be computed similarly; one finds

$$\theta = \left(\sum_{j \in J} \psi_j \circ \rho(\theta^{-1})(1_{A_j} \otimes 1) \right)^{-1} = \left(\sum_{j \in J} (\theta_j)^{-1} \otimes j \right)^{-1}$$

We summarize our findings:

Corollary 5.4.33. *If A is a J-ribbon algebra, then the orbifold algebra $\widehat{A}^J$ inherits a natural structure of a ribbon algebra such that the equivalence of tensor categories in proposition 5.4.32 is an equivalence of ribbon categories.*

5.4.5 Equivariant modular categories

In this subsection, we show that the orbifold category of the J-equivariant ribbon category $\mathcal{C}^J(G)$-mod is J-modular. A theorem of Kirillov [Kir04, Theorem 10.5] then immediately implies that the category $\mathcal{C}^J(G)$-mod is J-modular.

Since we have already seen in corollary 5.4.33 that the orbifold category is equivalent, as a ribbon category, to the representation category of the orbifold Hopf algebra, it suffices the compute this Hopf algebra explicitly. Our final result asserts that this Hopf algebra is an ordinary Drinfel'd double:

Proposition 5.4.34. *The $\mathbb{K}$-linear map*

$$\begin{array}{rccc} \Psi : & \widehat{\mathcal{D}^J(G)}^J & \longrightarrow & \mathcal{D}(H) \\ & (\delta_h \otimes g \otimes j) & \longmapsto & (\delta_h \otimes gs(j)) \end{array} \tag{5.16}$$

is an isomorphism of ribbon algebras, where the Drinfel'd double $\mathcal{D}(H)$ is taken with the standard ribbon structure introduced in subsection 2.5.

This result immediately implies the equivalence

$$(\widehat{\mathcal{D}^J(G)}\text{-mod})^J \cong \mathcal{D}(H)\text{-mod}$$

of ribbon categories and thus, by proposition 5.2.21, the modularity of the orbifold category, so that we have finally proven:

Theorem 5.4.35. *The category $\mathcal{C}^J(G) = \bigoplus_{j \in J} \mathcal{C}(G)_j$ has a natural structure of a J-modular tensor category.*

Proof of proposition 5.4.34. We show by direct computations that the linear map Ψ preserves product, coproduct, R-matrix and twist element:

- Compatibility with the product:

$$\Psi((\delta_h \otimes g \otimes j)(\delta'_h \otimes g' \otimes j')) = \Psi((\delta_h \otimes g) \cdot {}^j(\delta'_h \otimes g')c_{jj'} \otimes jj')$$
$$= \Psi\left((\delta_h \otimes g) \cdot (\delta_{s(j)h's(j)^{-1}} \otimes s(j)g's(j)^{-1}) \cdot \sum_{h'' \in H} (\delta_{h''} \otimes s(j)s(j')s(jj')^{-1}) \otimes jj'\right)$$
$$= \Psi\left(\delta(h, gs(j)hs(j)^{-1}g^{-1})(\delta_h \otimes gs(j)g's(j')s(jj')^{-1}) \otimes jj'\right)$$
$$= \delta(h, gs(j)hs(j)^{-1}g^{-1})(\delta_h \otimes gs(j)g's(j'))$$
$$= (\delta_h \otimes gs(j)) \cdot (\delta_{h'} \otimes g's(j'))$$
$$= \Psi(\delta_h \otimes g \otimes j)\Psi(\delta_{h'} \otimes g' \otimes j')$$

- Compatibility with the coproduct:

$$\begin{aligned}(\Psi \otimes \Psi)\Delta(\delta_h \otimes g \otimes j) &= \sum_{h'h''=h} \Psi(\delta_{h'} \otimes g \otimes j) \otimes \Psi(\delta_{h''} \otimes g \otimes j) \\ &= \sum_{h'h''=h} (\delta_{h'} \otimes gs(j)) \otimes (\delta_{h''} \otimes gs(j)) \\ &= \Delta\left(\Psi(\delta_h \otimes g \otimes j)\right)\end{aligned}$$

- The R-matrix of $\widehat{\mathcal{D}^J(G)}^J$ has been determined in the lines preceding corollary 5.4.33:
$$R = \sum_{j,j' \in J} \sum_{h \in H_j, h' \in H_{j'}} (\delta_h \otimes 1_G \otimes 1_J) \otimes (\delta_{h'} \otimes hs(j)^{-1} \otimes j)$$
This implies
$$\begin{aligned}(\Psi \otimes \Psi)(R) &= \sum_{j,j' \in J} \sum_{h \in H_j, h' \in H_{j'}} \Psi(\delta_h \otimes 1_G \otimes 1_J) \otimes \Psi(\delta_{h'} \otimes hs(j)^{-1} \otimes j) \\ &= \sum_{h\ h' \in H} (\delta_h \otimes 1) \otimes (\delta_{h'} \otimes h),\end{aligned}$$
which is the standard R-matrix of the Drinfel'd double $\mathcal{D}(H)$.

- The twist in $\widehat{\mathcal{D}^J(G)}^J$ is by corollary 5.4.33 equal to

$$\theta = \left(\sum_{j \in J} \sum_{h \in H_j} (\delta_h \otimes hs(j)^{-1} \otimes j)\right)^{-1}$$

and thus it gets mapped to the element

$$\Psi(\theta) = \left(\sum_{j \in J} \sum_{h \in H_j} \Psi(\delta_h \otimes hs(j)^{-1} \otimes j) \right)^{-1}$$
$$= \left(\sum_{h \in H} (\delta_h \otimes h) \right)^{-1} = \sum_{h \in H} (\delta_h \otimes h^{-1})$$

which is the twist element in $\mathcal{D}(H)$.

□

5.4.6 Summary of all tensor categories involved

We summarize our findings by discussing again the four tensor categories mentioned in the introduction, in the square of equation (5.1), thereby presenting the explicit solution of the algebraic problem described in section 5.1.1. Given a finite group G with a weak action of a finite group J, we get an extension $1 \longrightarrow G \longrightarrow H \longrightarrow J \longrightarrow 1$ of finite groups, together with a set-theoretic section $s : J \longrightarrow H$.

Proposition 5.4.36.
We have the following natural realizations of the categories in question in terms of categories of finite-dimensional representations over finite-dimensional ribbon algebras:

1. *The premodular category introduced in [Ban05] is* $\mathcal{B}(G \triangleleft H)$-mod*. As an abelian category, it is equivalent to the representation category* $G//H$-mod *of the action groupoid* $G//H$*, i.e. to the category of* G*-graded* $\mathbb{K}$*-vector spaces with compatible action of* H*.*

2. *The modular category obtained by modularization is* $\mathcal{D}(G)$-mod*. As an abelian category, it is equivalent to* $G//G$-mod*.*

3. *The* J*-modular category constructed in this chapter is* $\mathcal{D}^J(G)$-mod*. As an abelian category, it is equivalent to* $H//G$-mod*.*

4. *The modular category obtained by orbifoldization from the* J*-modular category* $\mathcal{D}(G)$-mod *is equivalent to* $\mathcal{D}(H)$-mod*. As an abelian category, it is equivalent to* $H//H$-mod*.*

Equivalently, the diagram in equation (5.1), has the explicit realization:

$$\begin{array}{ccc} \overset{J}{\circlearrowright} & & \overset{J}{\circlearrowright} \\ \mathcal{D}(G)\text{-mod} & \hookrightarrow & \mathcal{D}^J(G)\text{-mod} \\ \scriptstyle{modularization}\uparrow\downarrow\scriptstyle{orbifold} & & \uparrow\downarrow\scriptstyle{orbifold} \\ \mathcal{B}(G \lhd H)\text{-mod} & \hookrightarrow & \mathcal{D}(H)\text{-mod} \end{array} \tag{5.17}$$

We could have chosen the inclusion in the lower line as an alternative starting point for the solution of the algebraic problem presented in introduction 5.1.1. Recall from the introduction that the category $\mathcal{B}(G \lhd H)$-mod contains a Tannakian subcategory that can be identified with the category of representations of the quotient group $J = H/G$. The Tannakian subcategory and thus the category $\mathcal{B}(G \lhd H)$-mod contain a commutative Frobenius algebra given by the algebra of functions on J; recall that the modularization function was just induction along this algebra. The image of this algebra under the inclusion in the lower line yields a commutative Frobenius algebra in the category $\mathcal{D}(H)$-mod. In a next step, one can consider induction along this algebra to obtain another tensor category which, by general results [Kir04, Theorem 4.2] is a J-modular category.

In this approach, it remains to show that this J-modular tensor category is equivalent, as a J-modular tensor category, to $\mathcal{D}^J(G)$-mod and, in a next step that the modularization $\mathcal{D}(G)$-mod can be naturally identified with the neutral sector of the J-modular category. This line of thought has been discussed in [Kir01, Lemma 2.2] including the square (5.17) of Hopf algebras. Our results directly lead to a natural Hopf algebra $\mathcal{D}^J(G)$ and additionally show how the various categories arise from extended topological field theories which are built on clear geometric principles and through which all additional structure of the algebraic categories become explicitly computable.

5.5 Outlook

Our results very explicitly provide an interesting class J-modular tensor categories. All data of these theories, including in particular the representations of the modular group $SL(2, \mathbb{Z})$ on the vector spaces assigned to the torus, are directly accessible in terms of representations of finite groups. Also series of examples exist in which closed formulae for all quantities can be derived, e.g. for the inclusion of the alternating group in the symmetric group.

Our results admit generalizations in various directions. In fact, in this thesis, we have only studied a subclass of Dijkgraaf-Witten theories. The general case requires, apart from the choice of a finite group G, the choice of an element of

$$H^3_{Gp}(G, U(1)) = H^4(\mathcal{A}_G, \mathbb{Z})\,.$$

This element can be interpreted [Wil08] geometrically as a 2-gerbe on $\mathcal{A}_G$. It is known that in this case a quasi-triangular Hopf algebra can be extracted that is exactly the one discussed in [DPR90]. Indeed, our results can also be generalized by including the additional choice of a non-trivial element

$$\omega \in H^4_J(\mathcal{A}_G, \mathbb{Z}) \equiv H^4(\mathcal{A}_G//J, \mathbb{Z}) \ .$$

Only all these data together allow to investigate in a similar manner the categories constructed by Bantay [Ban05] for crossed modules with a boundary map that is not necessarily injective any longer. We plan to explain this general case in a subsequent publication.

5.6 Appendix

5.6.1 Appendix: Cohomological description of twisted bundles

In this appendix, we give a description of P-twisted bundles as introduced in definition 5.3.5 in terms of local data. This local description will also serve as a motivation for the term 'twisted' in twisted bundles. Recall the relevant situation: $1 \longrightarrow G \longrightarrow H \xrightarrow{\pi} J \longrightarrow 1$ is an exact sequence of groups. Let $P \xrightarrow{J} M$ be a J-cover. A P-twisted bundle on a smooth manifold M is an H-bundle $Q \longrightarrow M$, together with a smooth map $\varphi : Q \longrightarrow P$ such that $\varphi(qh) = \varphi(q)\pi(h)$ for all $q \in Q$ and $h \in H$.

We start with the choice of a contractible open covering $\{U_\alpha\}$ of M, i.e. a covering for which all open sets U_α are contractible. Then the J-cover P admits local sections over U_α. By choosing local sections s_α, we obtain the cocycle

$$j_{\alpha\beta} := s_\alpha^{-1} \cdot s_\beta : U_\alpha \cap U_\beta \longrightarrow J$$

describing P.

Let (Q, φ) be a P-twisted G-bundle over M. We claim that we can find local sections

$$t_\alpha : U_\alpha \longrightarrow Q$$

of the H-bundle Q which are compatible with the local section of the J-cover P in the sense that $\varphi \circ t_\alpha = s_\alpha$ holds for all α.

To see this, consider the map $\varphi : Q \longrightarrow P$; restricting the H-action on Q along the inclusion $G \longrightarrow H$, we get a G-action on Q that covers the identity on P. Hence Q has the structure of a G-bundle over P. Note that the image of s_α is contractible, since U_α is contractible. Thus the G-bundle $Q \longrightarrow P$ admits a section s'_α over the image of s_α. Then $t_\alpha := s'_\alpha \circ s_\alpha$ is a section of the H-bundle $Q \longrightarrow M$ that does the job.

With these sections $t_\alpha : U_\alpha \longrightarrow Q$, we obtain the cocycle description

$$h_{\alpha\beta} := t_\alpha^{-1} \cdot t_\beta : U_\alpha \cap U_\beta \longrightarrow H$$

of Q.

The set underlying the group H is isomorphic to the set $G \times J$. The relevant multiplication on this set depends on the choice of a section $J \longrightarrow H$; it has been described in equation (5.3):

$$(g, i) \cdot (g', j) := \big(g \cdot {}^{i}(g') \cdot c_{i,j} \ , \ ij\big) \ .$$

This allows us to express the H-valued cocycles $h_{\alpha\beta}$ in terms of J-valued and G-valued functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G \ .$$

By the condition $\varphi \circ t_\alpha = s_\alpha$, the J-valued functions are determined to be the J-valued cocycles $j_{\alpha\beta}$. Using the multiplication on the set $G \times J$, the cocycle condition $h_{\alpha\beta} \cdot h_{\beta\gamma} = h_{\alpha\gamma}$ can be translated into the following condition for $g_{\alpha\beta}$

$$g_{\alpha\beta} \cdot {}^{j_{\beta\gamma}}\big(g_{\beta\gamma}\big) \cdot c_{j_{\alpha\beta}, j_{\beta\gamma}} = g_{\alpha\gamma} \tag{5.18}$$

over $U_\alpha \cap U_\beta \cap U_\gamma$. This local expression can serve as a justification of the term P-twisted G-bundle.

We next turn to morphisms. A morphism f between P-twisted bundles (Q, φ) and (Q', ψ) which are represented by twisted cocycles $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ is represented by a coboundary

$$l_\alpha := (t'_\alpha)^{-1} \cdot f(t_\alpha) : U_\alpha \longrightarrow H$$

between the H-valued cocycles $h_{\alpha\beta}$ and $h'_{\alpha\beta}$. Since f satisfies $\psi \circ f = \varphi$, the J-component $\pi \circ l_\alpha : U_\alpha \longrightarrow H \longrightarrow J$ is given by the constant function to $e \in J$. Hence the local data describing the morphism f reduce to a family of functions

$$k_\alpha : U_\alpha \longrightarrow G.$$

Under the multiplication (5.3), the coboundary relation $l_\alpha \cdot h_{\alpha\beta} = h'_{\alpha\beta} \cdot l_\beta$ translates into

$$k_\alpha \cdot {}^{e}\big(g_{\alpha\beta}\big) \cdot c_{e, j_{\alpha\beta}} = g'_{\alpha\beta} \cdot {}^{j_{\alpha\beta}}\big(k_\beta\big) \cdot c_{j_{\alpha\beta}, e}$$

One can easily conclude from the definition 5.3.1 of a weak action that ${}^{e}g = g$ and $c_{e,g} = c_{g,e} = e$ for all $g \in G$. Hence this condition reduces to the condition

$$k_\alpha \cdot g_{\alpha\beta} = g'_{\alpha\beta} \cdot {}^{j_{\alpha\beta}}\big(k_\beta\big) \ . \tag{5.19}$$

We are now ready to present a classification of P-twisted bundles in terms of Čech-cohomology.

Therefore we define the relevant cohomology set:

Definition 5.6.1. Let $\{U_\alpha\}$ be a contractible cover of M and $(j_{\alpha\beta})$ be a Čech-cocycle with values in J.

- A $(j_{\alpha\beta})$-twisted Čech-cocycle is given by a family
$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$$
satisfying relation (5.18).
- Two such cocycles $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ are cobordant if there exists a coboundary, that is a family of functions $k_a : U_\alpha \longrightarrow G$ satisfying relation (5.19).
- The twisted Čech-cohomology set $\check{\mathrm{H}}^1_{j_{\alpha\beta}}(M,G)$ is defined as the quotient of twisted cocycles modulo coboundaries.

Warning 5.6.2. *It might be natural to guess that the so defined twisted Čech-cohomology $\check{\mathrm{H}}^1_{j_{\alpha\beta}}(M,G)$ agrees with the preimage of the class $[j_{\alpha\beta}]$ under the map $\pi_* : \check{\mathrm{H}}^1(M,H) \longrightarrow \check{\mathrm{H}}^1(M,J)$. This turns out to be wrong: The natural map*

$$\begin{array}{rcl} \check{\mathrm{H}}^1_{j_{\alpha\beta}}(M,G) & \longrightarrow & \check{\mathrm{H}}^1(M,H) \\ [g_{\alpha\beta}] & \longmapsto & [(g_{\alpha,\beta}, j_{\alpha\beta})] \end{array} ,$$

is, in general, not injective. The image of this map is always the fiber ${\pi_*}^{-1}[j_{\alpha\beta}]$.

We summarize our findings:

Proposition 5.6.3. *Let P be a J-cover of M, described by the cocycle $j_{\alpha\beta}$ over the contractible open cover $\{U_\alpha\}$. Then there is a canonical bijection*

$$\check{\mathrm{H}}^1_{j_{\alpha\beta}}(M,G) \cong \left\{ \begin{array}{c} \textit{Isomorphism classes of } P\textit{-twisted} \\ G\textit{-bundles over } M \end{array} \right\}.$$

5.6.2 Appendix: Character theory for action groupoids

In this subsection, we explicitly work out a character theory for finite action groupoids $M//G$; in the case of $M = pt$, this theory specializes to the character theory of a finite group (cf. [Isa94] and [Ser77]). In the special case of a finite action groupoid coming from a finite crossed module, a character theory including orthogonality relation has been presented in [Ban05]. In the sequel, let $\mathbb{K}$ be a field and denote by $\mathrm{Vect}_{\mathbb{K}}(M//G)$ the category of $\mathbb{K}$-linear representations of $M//G$.

Definition 5.6.4. Let $((V_m)_{m\in M}, (\rho(g))_{g\in G})$ be a $\mathbb{K}$- linear representation of the action groupoid $M//G$ and denote by $P(m)$ the projection of $V = \bigoplus_{n\in M} V_n$ to the homogeneous component V_m. We call the function

$$\begin{aligned} \chi : M \times G &\longrightarrow \mathbb{K} \\ \chi(m,g) &:= \mathrm{Tr}_V(\rho(g)P(m)) \end{aligned}$$

the *character of the representation.*

Example 5.6.5. *On the* $\mathbb{K}$*-vector space* $H := \mathbb{K}(M) \otimes \mathbb{K}[G]$ *with canonical basis* $(\delta_m \otimes g)_{m\in M, g\in G}$*, we define a grading by* $H_m = \bigoplus_g \mathbb{K}(\delta_{g.m} \otimes g)$ *and a group action by* $\rho(g)(\delta_m \otimes h) = \delta_m \otimes gh$*. This defines an object in* $\mathrm{Vect}_{\mathbb{K}}(M//G)$*, called the regular representation. The character is easily calculated in the canonical basis and found to be*

$$\chi_H(m, g) = \sum_{(n,h)\in M\times G} \delta(g, 1)\delta(h.m, n) = \delta(g, 1)|G|$$

Definition 5.6.6. We call a function

$$f : M \times G \longrightarrow \mathbb{K}$$

an *action groupoid class function* on $M//G$, if it satisfies

$$f(m, g) = 0 \text{ if } g.m \neq m \quad \text{ and } \quad f(h.m, hgh^{-1}) = f(m, g) \ .$$

The character of any finite dimensional representation is a class function.

From now on, we assume that the characteristic of $\mathbb{K}$ does not divide the order $|G|$ of the group G. This assumption allows us to consider the following normalized non-degenerate symmetric bilinear form

$$\langle f, f'\rangle := \frac{1}{|G|} \sum_{g\in G, m\in M} f(m, g^{-1})f'(m, g). \tag{5.20}$$

In the case of *complex* representations, one can show, precisely as in the case of groups, the equality $\chi(m, g^{-1}) = \overline{\chi(m, g)}$ which allows introduce the hermitian scalar product

$$(\chi, \chi') := \frac{1}{|G|} \sum_{g\in G, m\in M} \overline{\chi(m, g)}\chi'(m, g) \ . \tag{5.21}$$

Lemma 5.6.7. *Let* $\mathbb{K}$ *be algebraically closed. The characters of irreducible* $M//G$*-representations are orthogonal and of unit length with respect to the bilinear form (5.20).*

Proof. The proof proceeds as in the case of finite groups: for a linear map $f : V \longrightarrow W$ on the vector spaces underlying two irreducible representations, one considers the intertwiner

$$f^0 = \frac{1}{|G|} \sum_{g\in G, m\in M} \rho_W(g^{-1})P_W(m)fP_V(m)\rho_V(g). \tag{5.22}$$

and applies Schur's lemma. □

A second orthogonality relation

$$\sum_{i \in I} \chi_i(m,g)\chi_i(n,h^{-1}) = \sum_{z \in G} \delta(n, z.m)\delta(h, zgz^{-1})$$

can be derived as in the case of finite groups, as well.

Combining the orthogonality relations with the explicit form for the character of the regular representation, we derive in the case of an algebraically closed field whose characteristic does not divide the order $|G|$ use a standard reasoning:

Lemma 5.6.8. *Every irreducible representation V_i is contained in the regular representation with multiplicity $d_i := \dim_{\mathbb{K}} V_i$.*

As a consequence, the following generalization of Burnside's Theorem holds:

Proposition 5.6.9. *Denote by $(V_i)_{i \in I}$ a set of representatives for the isomorphism classes of simple representations of the action groupoid and by $d_i := \dim_{\mathbb{K}} V_i$ the dimension of the simple object. Then*

$$\sum_{i \in I} |d_i|^2 = |M||G|$$

Proof. One combines the relation $\dim H = \sum_{i \in I} d_i \dim V_i$ from Lemma 5.6.8 with the relation $\dim H = |M||G|$. □

In complete analogy to the case of finite groups, one then shows:

Proposition 5.6.10. *The irreducible characters of $M//G$ form an orthogonal basis of the space of class functions with respect to the scalar product (5.20).*

The above proposition allows us to count the number of irreducible representations. On the set

$$A := \{(m,g) | g.m = m\} \subset M \times G$$

the group G naturally acts by $h.(m,g) := (h.m, hgh^{-1})$. A class function of $M//G$ is constant on G-orbits of A; it vanishes on the complement of A in $M \times G$. We conclude that the number of irreducible characters equals the number of G-orbits of A.

This can be rephrased as follows: the set A is equal to the set of objects of the inertia groupoid $\Lambda(M//G) := [\bullet//\mathbb{Z}, M//G]$. Thus the number of G-orbits of A equals the number of isomorphism classes of objects in $\Lambda(M//G)$, thus $|I| = |Iso(\Lambda(M//G))|$.

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Thomas Nikolaus
Higher Categorical Structures in Geometry - General Theory and Applications to Quantum Field Theory
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Zusammenfassung

In dieser Arbeit untersuchen wir mathematische Strukturen die in der Quantenfeldtheorie eine Rolle spielen. Insbesondere konzentrieren wir uns dabei auf die Beschreibung von Hintergrundsdaten für Sigma-Modelle und die Beschreibung von gewissen topologischen Feldtheorien. In der formalen Beschreibung und Klassifikation der zugehörigen geometrischen Objekte spielen höhere Kategorien, insbesondere Bikategorien, eine wichtige Rolle.

Der zentrale Beitrag des ersten Kapitels ist die 'Abstiegsperspektive' auf die Definition von Bündelgerben und Jandl-Strukturen. Dies ist die Basis für die Theorie von 2-Stacks, die wir in Kapitel 2 entwickeln. Insbesondere erweitern wir 2-Stacks, die auf der Kategorie der glatten Mannigfaltigkeiten definiert sind, zu 2-Stacks auf der Kategorie der Lie-Gruppoide. Ein fundamentales technische Resultat ist nun, dass diese Fortsetzung eines Stacks invariant unter Morita-Äquivalenz von Lie Gruppoiden ist. Unter Verwendung dieses Resultats können wir eine allgemeine 'Stackifizierungs-Vorschrift' für beliebige 2-Prästacks angeben und Bündelgerben sowie Jandl-Gerben als Spezialfälle dieser allgemeinen Konstruktion identifizieren.

In Kapitel 3 entwickeln wir einen präzisen formalen Rahmen für vier verschiedene Versionen von nicht-abelschen Gerben. Dabei handelt es sich um die in der Literatur verschiedentlich untersuchten Čech-Kozykel, klassifizierenden Abbildungen, nicht-abelschen Bündelgerben und prinzipalen 2-Bündel. Zusätzlich zu einer konsistenten und vollständigen Definition behandeln wir Strukturaussagen und Vergleichsresultate, die zeigen, dass die vier Versionen äquivalent sind.

In Kapitel 4 geben wir eine neue, konkrete Konstruktion der String-Gruppe an. Diese Gruppe spielt unter Anderem eine Rolle in supersymmetrischen Sigma-Modellen zur Anomalie-Kürzung. Genauer konstruieren wir zunächst ein unendlich-dimensionales glattes Modell für die String-Gruppe. Dieses Modell erweitern wir dann zu einer 2-Gruppe. Die so konstruierte 2-Gruppe kann als Strukturgruppe für die allgemeine Bündeltheorie, die wir in Kapitel 3 entwickelt haben, dienen.

Im letzten Kapitel behandeln wir schließlich eine äquivariante Verallgemeinerung der sogenannten erweiterten Dijkgraaf-Witten Theorie, einer dreidimensionalen topologischen Feldtheorie. Unsere Erweiterung basiert auf der Wahl einer endlichen Gruppe J, die auf einer anderen endlichen Gruppe G wirkt. Wir verwenden geometrischen Methoden zur Konstruktion des vollen TFT 2-Funktors. Aus diesem können wir anschließend die Daten einer äquivarianten modularen Tensorkategorie gewinnen und die Theorie algebraisch, mittels einer Hopf-Algebra, verstehen.

Lebenslauf

Persönliche Daten

Name:	Thomas Nikolaus
Adresse:	Von-Essen-Straße 7 22089 Hamburg
E-Mail:	Thomas.Nikolaus@uni-hamburg.de
Geburtstag:	06.06.1984
Geburtsort:	Esslingen am Neckar

Bildungsgang

2003	Abitur am Otto-Hahn-Gymnasium Gifhorn
2009	Diplom in Mathematik an der Universität Hamburg
2009–2011	Doktorand an der Universität Hamburg

Beschäftigungen

2006 - 2009	Stipendium der Studienstiftung des deutschen Volkes
2009 - 2010	wissenschatlicher Mitarbeiter am DFG Sonderforschungsbereich 676 "Particles, Strings, and the Early Universe"
2010 - 2011	wissenschatlicher Mitarbeiter am Landesexzellenzcluster "Connecting particles with the cosmos"