On Exceptional Geometry and Supergravity

Hermann Nicolai
MPI für Gravitationsphysik, Potsdam
(Albert-Einstein-Institut)

Based on joint work with B. deWit, and H. and M. Godazgar:

[dWN:NPB274(1986),1302.6219; GGN:1303.1013,1307.8295,1309.0266,1312.1061]

as well as ongoing work with O. Hohm and H. Samtleben

[GGNHS: hopefully to appear soon]

Motivation

There are many indications of exceptional geometrical structures in maximal supergravity and M theory:

- **Ubiquity of exceptional groups: $\mathbf{E}_{6(6)}$, $\mathbf{E}_{7(7)}$, $\mathbf{E}_{8(8)}, \ldots$** [Cremmer,Julia(1979)]
- **Presence of form fields beyond standard geometry**
- **Extra (central charge) coordinates beyond $D = 11$?**

has led to several attempts to generalise geometry

- **Double Field Theory** [Siegel(1992);Hull(2005);Hohm,Hull,Zwiebach(2010),...]
- **Generalised geometry (and 'non-geometry')** [Berman,Cederwall, Kleinschmidt,Thompson(2013); Coimbra,Strickland-Constable,Waldram(2014);...]
- **Exceptional geometry** [dWN(1986,2001);HN(1987);KNS(2000);Hillmann(2009); Berman,Goadazgar,Perry,West(2011);Coimbra,Strickland-Constable,Waldram(2011); GGN(2013); Hohm,Samtleben(2013)]

Generalised Geometry

Idea: 'lift' exceptional structures found in lower dimensions back up to $D = 11$ (or $D = 10$).

- **Extend tangent space in accordance with R symmetries** [dWN(1986);HN(1987)]
- **Extend tangent space to include p-forms** [Hitchin(2003);Gualtieri(2004)]
- **Include windings of M2,M5, and KK branes** [Hull(2007);Pacheco,Waldram(2008)]
- **Extra (central charge) coordinates** [...,Siegel(1993);dWN(2001);West(2003);Hillmann(2009);Berman,Perry(2011)]

Exceptional duality symmetries necessitate new geometric structures (vielbeine, connections,...) and (perhaps) extra dimensions beyond $D = 11$ $\rightarrow$ two options:

- ***Postulate* new structures *ad hoc* ('top-down approach').**
- ***Derive* them by re-writing original theory ('bottom-up').**
- **In either case must ascertain full consistency, either intrinsically or by comparison with original theory.**

Cartan's Theorem (1909)

... states that the most general algebra of vector fields on a manifold consists (essentially) of the following three: diffeomeorphisms, volume preserving diffeomorphisms, or symplectomorphisms. Or: there are no exceptional algebras of vector fields! Thus, if a generalised vielbein $\mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}$ transforms according to

$$\mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}(y) \;\rightarrow\; \mathcal{V}'^{\mathcal{M}}{}_{\mathcal{A}}(y') = \frac{\partial y'^{\mathcal{M}}}{\partial y^{\mathcal{N}}} \mathcal{V}^{\mathcal{N}}{}_{\mathcal{A}}(y)$$

we can never arrange things such that

$$\frac{\partial y'^{\mathcal{M}}(y)}{\partial y^{\mathcal{N}}} \in \mathrm{E}_{7(7)} \subset \mathrm{GL}(56, \mathbb{R}) \qquad \textbf{for all } y$$

$\Rightarrow$ extra coordinates are not for real!

... as was to be expected since there appear to exist no consistent supergravity theories beyond $D = 11$ dimensions (at least, no one has found any so far...)!

More Motivation

What is to be gained from re-writing a known theory ($D = 11$ supergravity [CJS(1978)]) into a form that is (or is not??) on-shell equivalent to the original theory?

- **Derivation of non-linear Kaluza-Klein ansätze**
  - **Consistency of S^7 compactification** [dWN(1987),Pilch,HN(2012),GGN(2013)]
  - **Scherk-Schwarz compactifications** [Samtleben(2008);GGN(2013)]
- **Understanding origin of embedding tensor from higher dimensions and compactification.**
- **... and perhaps: *new* maximal supergravities?** [Dall'Agata,Inverso,Trigiante(2012);dWN(2013)]

Also, crucial new insights for (a long term project!)

- **Infinite dimensional extensions: $\mathbf{E}_{10}$** [Julia(1983);DHN(2002),...] **or $\mathbf{E}_{11}$** [West(2001)] **and emergent space-time?**

Reminder: $\mathrm{E}_{7(7)}$ from dimensional reduction

Starting from $D = 11$ supergravity [Cremmer,Julia,Scherk (1978)] **split coordinates as $z^M = (x^\mu, y^m)$ and perform 4+7 split of bosonic fields G_{MN} and A_{MNP}:**

$$
\begin{aligned}
G_{MN}: &\quad G_{mn}(28) \oplus G_{m\mu}(7) \oplus G_{\mu\nu}(1) \\
A_{MNP}: &\quad A_{mnp}(35) \oplus A_{\mu mn}(21) \oplus A_{\mu\nu m}(7) \oplus A_{\mu\nu\rho}(1)
\end{aligned}
$$

To get proper count of scalar degrees of freedom $\rightarrow$ dualize seven 2-form fields $A_{\mu\nu m}$ [Cremmer,Julia (1979)]

$$28 + 35 + 7 = 70 \rightarrow \mathcal{V}(x) \in \mathrm{E}_{7(7)}/\mathrm{SU}(8)$$

Key Question: is this structure peculiar to torus reduction, or can it be lifted back up to $D = 11$?

And: is there a way to reformulate $D = 11$ (or IIA, IIB,...) supergravity that makes these hidden symmetries manifest? [$\rightarrow$ B.deWit and HN,NPB274(1986)363; HN,PLB187(1987)316]

Dualities in eleven dimensions

3-form/6-form duality

$$F_{M_1\cdots M_7} \;=\; 7! D_{[M_1} A_{M_2\cdots M_7]} + 7! \frac{\sqrt{2}}{2} A_{[M_1 M_2 M_3} D_{M_4} A_{M_5 M_6 M_7]}$$
$$-\frac{\sqrt{2}}{192} i \epsilon_{M_1\cdots M_{11}} \left(\overline{\Psi}_R \tilde{\Gamma}^{M_8\cdots M_{11} RS} \Psi_S + 12 \overline{\Psi}^{M_8} \tilde{\Gamma}^{M_9 M_{10}} \Psi^{M_{11}} \right)$$

defines dual 6-form $A^{(6)} \equiv A_{MNPQRS}$, with

$$\delta A_{MNPQRS} = -\frac{3}{6!\sqrt{2}} \bar{\varepsilon} \Gamma_{MNPQR} \Psi_{S]} + \frac{1}{8} \bar{\varepsilon} \Gamma_{[MN} \Psi_P A_{QRS]}$$

Relations are valid on-shell and at full non-linear level.

By contrast, dualisation of gravity works only at linear level, and without matter sources:

$$G_{MN} = \eta_{MN} + h_{MN} : \qquad h_{MN} \longleftrightarrow h_{M_1\cdots M_8|N}$$

In particular, 'dual supergravity' does not even exist at linear level. [Bergshoeff,deRoo,Kerstan,Kleinschmidt,Riccioni(2008)]

Existing no go theorems suggest that $D = 11$ Lorentz covariance must be abandoned if interactions are to be included consistently! [Bekaert,Boulanger,Henneaux(2003)]

$\Rightarrow$ more 4+7 decompositions:

$$A_{MNPQRS}: \quad A_{mnpqrs}(7) \oplus A_{\mu mnpqr}(21) \oplus A_{\mu\nu mnpq}(35) \oplus A_{\mu\nu\rho mnp}(35) \oplus \cdots$$

$$h_{M_1\cdots M_8|N}: \quad \emptyset \oplus h_{\mu mnpqrst|u}(7) \oplus h_{\mu\nu mnpqrs|t}(49) \oplus h_{\mu\nu\rho mnpqr|s}(147) \oplus \cdots$$

Now we see that also fields other than scalars can be re-packaged into $\mathrm{E}_{7(7)}$ multiplets in eleven dimensions:

$$\textbf{Vectors}: \quad 7 \oplus 21 \oplus \bar{21} \oplus \bar{7} = \mathbf{56} \qquad \text{(electromagnetic duality)}$$

$$\textbf{2-forms}: \quad 7 \oplus 35 \oplus 49 \oplus \cdots = \mathbf{133} \qquad (\mathrm{E}_{7(7)} \text{ Noether current})$$

$$\textbf{3-forms}: \quad 1 \oplus 35 \oplus 147 \oplus \cdots = \mathbf{912} \qquad \text{(embedding tensor)}$$

$\rightarrow$ Beyond kinematics main challenge is to show that full $D = 11$ theory (supersymmetry variations and field equations) can be rewritten in an $\mathrm{E}_{7(7)}\times$SU(8) covariant way!

NPB274(1986)363 in short

4+7 decomposition of elfbein (in triangular gauge)

$$E_M{}^A(x,y) \;=\; \begin{pmatrix} \Delta^{-1/2} e'_\mu{}^\alpha & B_\mu{}^m e_m{}^a \\ 0 & e_m{}^a \end{pmatrix} \;\;, \qquad \Delta \equiv \det e_m{}^a$$

Similar redefinitions of fermions $\rightarrow$ chiral $\mathrm{SU}(8)$

$$\varphi'_\mu = \Delta^{-1/4}(i\gamma_5)^{-1/2} e'_\mu{}^\alpha (\Psi_\alpha - \tfrac{1}{2}\gamma_5\gamma_\alpha\Gamma^a\Psi_a) \;\;, \quad \varphi_\mu{}^A \;\textbf{ or }\; \varphi_{\mu A} \equiv \tfrac{1}{2}(1\pm\gamma_5)\,\varphi'_{\mu A}$$

$$\chi'_{ABC} = \tfrac{3}{4}\sqrt{2} i \Delta^{-1/4}(i\gamma_5)^{-1/2}\Psi_{a[A}\Gamma^a_{BC]} \;\;, \quad \chi^{ABC} \;\textbf{ or }\; \chi_{ABC} \equiv (1\pm\gamma_5)\chi'_{ABC}$$

$$\Rightarrow \quad \delta B_\mu{}^m = \frac{\sqrt{2}}{8} e^m_{AB}\left[2\sqrt{2}\bar{\varepsilon}^A \varphi^B_\mu + \bar{\varepsilon}_C \gamma'_\mu \chi^{ABC}\right] + \textbf{h.c.}$$

with generalised vielbein $\equiv$ GV

$$e^m_{AB} = i\Delta^{-1/2}(\Phi^T\Gamma^m\Phi)_{AB} \;\;, \quad \Phi(x,y) \in \mathrm{SU}(8)$$

whence e^m_{AB} becomes an $\mathrm{SU}(8)$ tensor!

Tangent space symmetry: $SO(1,10) \;\rightarrow\; SO(1,3)\times \mathrm{SU}(8)$

Generalization to remaining $\mathbf{21+21+7=49}$ vectors: [dWN,GGN(2013)]

$$\mathcal{B}_\mu{}^m = -\frac{1}{2} B_\mu{}^m, \qquad \mathcal{B}_{\mu\, mn} = -3\sqrt{2}\left(A_{\mu mn} - B_\mu{}^p A_{pmn}\right),$$

$$\mathcal{B}_\mu{}^{mn} = -3\sqrt{2}\,\eta^{mnp_1\ldots p_5}\left(A_{\mu p_1\cdots p_5} - B_\mu{}^q A_{qp_1\cdots p_5} - \frac{\sqrt{2}}{4}\left(A_{\mu p_1 p_2} - B_\mu{}^q A_{qp_1p_2}\right)A_{p_3p_4p_5}\right)$$

$$\mathcal{B}_{\mu\, m} = -18\,\eta^{n_1\ldots n_7}\Big(A_{\mu n_1\ldots n_7,m} + (3\tilde{c}-1)\left(A_{\mu n_1\ldots n_5} - B_\mu{}^p A_{pn_1\ldots n_5}\right)A_{n_6n_7m}$$

$$+\,\tilde{c}A_{n_1\ldots n_6}\left(A_{\mu n_7 m} - B_\mu{}^p A_{pn_7m}\right) + \frac{\sqrt{2}}{12}\left(A_{\mu n_1n_2} - B_\mu{}^p A_{pn_1n_2}\right)A_{n_3n_4n_5}A_{n_6n_7m}\Big)$$

where $\mathcal{B}_{\mu m}$ = dual (magnetic) graviphoton. Requiring

$$\delta B_{\mu\, mn} = \frac{\sqrt{2}}{8} e_{mn\, AB}\left[2\sqrt{2}\bar{\varepsilon}^A\varphi_\mu^B + \bar{\varepsilon}_C\gamma'_\mu\chi^{ABC}\right] + \textbf{h.c.}$$

leads to more generalised vielbein components $\Rightarrow$ extend e^m_{AB} to full 56-plet $(e^m_{AB}, e_{mnAB}, e^{mn}_{AB}, e_{mAB}) \equiv$ 56-bein in eleven dimensions!

56-bein in eleven dimensions

$$\mathcal{V}^m{}_{AB} = \frac{\sqrt{2}i}{8} e^m_{AB} = -\frac{\sqrt{2}}{8}\Delta^{-1/2}\Gamma^m_{AB} \equiv \mathcal{V}^{m8}{}_{AB} \equiv -\mathcal{V}^{8m}{}_{AB},$$

$$\mathcal{V}_{mnAB} = -\frac{\sqrt{2}}{8}\Delta^{-1/2}\left(\Gamma_{mnAB} + 6\sqrt{2}A_{mnp}\Gamma^p_{AB}\right),$$

$$\mathcal{V}^{mn}{}_{AB} = -\frac{\sqrt{2}}{8}\cdot\frac{1}{5!}\,\eta^{mnp_1\cdots p_5}\Delta^{-1/2}\Bigg[\Gamma_{p_1\cdots p_5AB} + 60\sqrt{2}A_{p_1p_2p_3}\Gamma_{p_4p_5AB}$$
$$- 6!\sqrt{2}\left(A_{qp_1\cdots p_5} - \frac{\sqrt{2}}{4}A_{qp_1p_2}A_{p_3p_4p_5}\right)\Gamma^q_{AB}\Bigg],$$

$$\mathcal{V}_{m\,AB} = -\frac{\sqrt{2}}{8}\cdot\frac{1}{7!}\,\eta^{p_1\cdots p_7}\Delta^{-1/2}\Bigg[(\Gamma_{p_1\cdots p_7}\Gamma_m)_{AB} + 126\sqrt{2}\ A_{mp_1p_2}\Gamma_{p_3\cdots p_7AB}$$
$$+ 3\sqrt{2}\times 7!\left(A_{mp_1\cdots p_5} + \frac{\sqrt{2}}{4}A_{mp_1p_2}A_{p_3p_4p_5}\right)\Gamma_{p_6p_7AB}$$
$$+ \frac{9!}{2}\left(A_{mp_1\cdots p_5} + \frac{\sqrt{2}}{12}A_{mp_1p_2}A_{p_3p_4p_5}\right)A_{p_6p_7q}\Gamma^q{}_{AB}\Bigg]$$

$\mathcal{V}(e, A^{(3)}, A^{(6)})$ **has all the requisite properties of an** $\mathrm{E}_{7(7)}$ **matrix:**

$$\mathcal{V}_{\mathtt{MN}}{}^{AB} \equiv (\mathcal{V}_{\mathtt{MN}\,AB})^* \quad , \quad \mathcal{V}^{\mathtt{MN}AB} \equiv (\mathcal{V}^{\mathtt{MN}}{}_{AB})^*$$

where we have combined the GL(7) indices into SL(8) indices

$$\mathcal{V}_{\mathtt{MN}} \equiv \big(\mathcal{V}_{mn}, \mathcal{V}_{m8}\big) \quad , \quad \mathcal{V}^{\mathtt{MN}} \equiv \big(\mathcal{V}^{mn}, \mathcal{V}^{m8}\big)$$

With proper $\mathbf{E}_{7(7)}$ **indices** $\mathcal{M}, \mathcal{N}, \dots$ **in 56 representation**

$$\mathcal{V}_{\mathcal{M}} \equiv \big(\mathcal{V}_{\mathtt{MN}}, \mathcal{V}^{\mathtt{MN}}\big) \quad , \quad \mathcal{V}^{\mathcal{M}} = \Omega^{\mathcal{M}\mathcal{N}}\mathcal{V}_{\mathcal{N}} \equiv \big(\mathcal{V}^{\mathtt{MN}}, -\mathcal{V}_{\mathtt{MN}}\big)$$

and symplectic form $\Omega^{\mathcal{M}\mathcal{N}}$

$$\begin{aligned}
\mathcal{V}_{\mathcal{M}}{}^{AB}\mathcal{V}_{\mathcal{N}\,AB} - \mathcal{V}_{\mathcal{M}\,AB}\mathcal{V}_{\mathcal{N}}{}^{AB} &= i\,\Omega_{\mathcal{M}\mathcal{N}}, \\
\Omega^{\mathcal{M}\mathcal{N}}\mathcal{V}_{\mathcal{M}}{}^{AB}\mathcal{V}_{\mathcal{N}\,CD} &= i\,\delta^{AB}_{CD}, \\
\Omega^{\mathcal{M}\mathcal{N}}\mathcal{V}_{\mathcal{M}}{}^{AB}\mathcal{V}_{\mathcal{N}}{}^{CD} &= 0 \quad \Rightarrow \quad \in Sp(56, \mathbb{R})
\end{aligned}$$

(for $\mathbf{E}_{7(7)}$ **have to work a little harder...)**

$\Rightarrow$ **$\mathbf{E}_{7(7)}$ covariant form of vector transformation in** $D = 11$**:**

$$\delta\mathcal{B}^{\mathcal{M}}_{\mu} = i\mathcal{V}^{\mathcal{M}}{}_{AB}\left(\bar{\varepsilon}_C\gamma_\mu\chi^{ABC} + 2\sqrt{2}\bar{\varepsilon}^A\psi^B_\mu\right) + \text{h.c.}$$

Extending general covariance

Standard behaviour under internal diffeomorphisms $\xi^m = \xi^m(x,y)$:

$$\delta\mathcal{V}^m{}_{AB} = \xi^p\partial_p\mathcal{V}^m{}_{AB} - \partial_p\xi^m\mathcal{V}^p{}_{AB} - \frac{1}{2}\partial_p\xi^p\mathcal{V}^m{}_{AB}$$

$$\delta\mathcal{V}_{mn\,AB} = \xi^p\partial_p\mathcal{V}_{mn\,AB} - 2\,\partial_{[m}\xi^p\mathcal{V}_{n]p\,AB} - \frac{1}{2}\partial_p\xi^p\mathcal{V}_{mn\,AB}$$

$$\delta\mathcal{V}^{mn}{}_{AB} = \xi^p\partial_p\mathcal{V}^{mn}{}_{AB} + 2\,\partial_p\xi^{[m}\mathcal{V}^{n]p}{}_{AB} + \frac{1}{2}\partial_p\xi^p\mathcal{V}^{mn}{}_{AB}$$

$$\delta\mathcal{V}_{m\,AB} = \xi^p\partial_p\mathcal{V}_{m\,AB} + \partial_m\xi^p\mathcal{V}_{p\,AB} + \frac{1}{2}\partial_p\xi^p\mathcal{V}_{m\,AB}$$

Due to its explicit dependence on $A^{(3)}$ and $A^{(6)}$ $\mathcal{V}$ also transforms under 2-form gauge transformations with parameter $\xi_{mn}(x,y)$:

$$\delta A_{mnp} = 3!\,\partial_{[m}\xi_{np]} \quad , \quad \delta A_{mnpqrs} = 3\sqrt{2}\,\partial_{[m}\xi_{np}A_{qrs]} \quad \Rightarrow$$

$$\delta\mathcal{V}^m{}_{AB} = 0, \qquad \delta\mathcal{V}_{mn\,AB} = 36\sqrt{2}\,\partial_{[m}\xi_{np]}\,\mathcal{V}^p{}_{AB},$$

$$\delta\mathcal{V}^{mn}{}_{AB} = 3\sqrt{2}\,\eta^{mnpqrst}\partial_p\xi_{qr}\,\mathcal{V}_{st\,AB}, \qquad \delta\mathcal{V}_{m\,AB} = 18\sqrt{2}\,\partial_{[m}\xi_{np]}\,\mathcal{V}^{np}{}_{AB}$$

Idem for 5-form gauge transformations

$$\delta A_{mnp} = 0 \ , \quad \delta A_{mnpqrs} = 6!\, \partial_{[m}\xi_{npqrs]} \quad \Rightarrow$$

$$\delta\mathcal{V}^{m}{}_{AB} = \delta\mathcal{V}_{mn\,AB} = 0, \qquad \delta\mathcal{V}^{mn}{}_{AB} = 6\cdot 6!\sqrt{2}\,\eta^{mnp_1\cdots p_5}\partial_{[q}\xi_{p_1\cdots p_5]}\mathcal{V}^{q}{}_{AB},$$

$$\delta\mathcal{V}_{m\,AB} = 3\cdot 6!\sqrt{2}\,\eta^{n_1\cdots n_7}\partial_{[m}\xi_{n_1\cdots n_5]}\mathcal{V}_{n_6 n_7 AB}$$

These formulas can be neatly summarised as

$$\delta_\Lambda \mathcal{V}_{\mathcal{M}\,AB} = \hat{\mathcal{L}}_\Lambda \mathcal{V}_{\mathcal{M}\,AB}$$

with $\Lambda^{\mathcal{M}} \equiv (\xi^m, \xi_{mn}, \xi^{mn}, \xi_m)$ **and generalised Lie derivative:**

$$\hat{\mathcal{L}}_\Lambda X_{\mathcal{M}} = \frac{1}{2}\Lambda^{\mathcal{N}}\partial_{\mathcal{N}} X_{\mathcal{M}} + 6(t^\alpha)_{\mathcal{M}}{}^{\mathcal{N}}(t_\alpha)_{\mathcal{P}}{}^{\mathcal{Q}}\partial_{\mathcal{Q}}\Lambda^{\mathcal{P}} X_{\mathcal{N}} + \frac{1}{2}w\,\partial_{\mathcal{N}}\Lambda^{\mathcal{N}} X^{\mathcal{M}}$$

$\Rightarrow$ **unifies internal diffeomorphisms and tensor gauge transformations and suggests extra coordinates: 4+56 instead of 4+7?**

But only consistent with Section Constraint:

$$t_\alpha^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{M}}\otimes\partial_{\mathcal{N}} = \Omega^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{M}}\otimes\partial_{\mathcal{N}} = 0 \ \Leftrightarrow \ \partial_{\mathcal{M}} = 0 \text{ for } \mathcal{M} \neq m$$

[Coimbra,Strickland-Constable,Waldram(2012);Berman,Cederwall,Kleinschmidt,Thompson(2013)]

Back to seven (or six) internal coordinates!

Generalised Vielbein Postulate = GVP

56-bein obeys a generalisation of the usual GVP, both for external and internal dimensions. For external dimensions, we have

$$\partial_\mu \mathcal{V}_{\mathcal{M}\,AB} + 2\hat{\mathcal{L}}_{\mathcal{B}_\mu} \mathcal{V}_{\mathcal{M}\,AB} + \mathcal{Q}^{C}_{\mu\,[A} \mathcal{V}_{\mathcal{M}\,B]C} = \mathcal{P}_{\mu\,ABCD} \mathcal{V}_{\mathcal{M}}{}^{CD}$$

where $\hat{\mathcal{L}}_\Lambda$ was defined above. To be compared with $D = 4$ relation

$$\partial_\mu \mathcal{V}_{\mathcal{M}\,ij} - g\mathcal{B}_\mu{}^{\mathcal{P}} X_{\mathcal{P}\mathcal{M}}{}^{\mathcal{N}} + \mathcal{Q}^{k}_{\mu[i} \mathcal{V}_{\mathcal{M}\,j]k} \mathcal{V}_{\mathcal{N}\,ij} = \mathcal{P}_{\mu\,ijkl} \mathcal{V}_{\mathcal{M}}{}^{kl}$$

where $X_{\mathcal{M}}$ generate the gauge algebra $\Rightarrow$ furnishes higher dimensional origin of embedding tensor $\Theta_{\mathcal{M}}{}^{\alpha}$ via

$$X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} \equiv \Theta_{\mathcal{M}}{}^{\alpha} (t_\alpha)_{\mathcal{N}}{}^{\mathcal{P}}$$

This correspondence has been checked for S^7 compactification (where gauging is purely electric) [GGN:1309.0266] **and Scherk-Schwarz compactifications** [GGN:1312.1061] **(where gauge fields are usually both electric and magnetic).**

$\rightarrow$ may thus explain new SO(8) gaugings [Dall'Agata,Inverso,Trigiante, PRL109(2012)201301] **via U(1) duality rotation in $D = 11$!**

Internal GVP *à la* dWN and GGN

$$\partial_m \mathcal{V}_{\mathcal{M}\,AB} - \boldsymbol{\Gamma}_{m\mathcal{M}}{}^{\mathcal{N}} \mathcal{V}_{\mathcal{N}\,AB} + Q^C_{m[A} \mathcal{V}_{\mathcal{M}\,B]C} = P_{m\,ABCD} \mathcal{V}_{\mathcal{M}}{}^{CD}$$

with SU(8) connection

$$Q_{mA}{}^B = -\frac{1}{2}\omega_{m\,ab}\,\Gamma^{ab}_{AB} + \frac{\sqrt{2}}{48} F_{mabc}\,\Gamma^{abc}_{AB} + \frac{\sqrt{2}}{14\cdot 6!} F_{mabcdef}\,\Gamma^{abcdef}_{AB},$$

and 'non-metricity'

$$P_{mABCD} = \frac{\sqrt{2}}{32} F_{mabc} \Gamma^a_{[AB} \Gamma^{bc}_{CD]} - \frac{\sqrt{2}}{56\cdot 5!} F_{mabcdef}\,\Gamma^a_{[AB}\Gamma^{bcdef}_{CD]}$$

$\mathbf{E}_{7(7)}$-valued generalised 'affine' connection $\boldsymbol{\Gamma}_{m\mathcal{M}}{}^{\mathcal{N}} = \boldsymbol{\Gamma}_m{}^\alpha (t_\alpha)_{\mathcal{M}}{}^{\mathcal{N}}$:

$$(\boldsymbol{\Gamma}_m)_n{}^p \equiv -\Gamma^p_{mn} + \tfrac{1}{4}\delta^p_n \Gamma^q_{mq}, \qquad (\boldsymbol{\Gamma}_m)_8{}^8 = -\tfrac{3}{4}\,\Gamma^n_{mn},$$

$$(\boldsymbol{\Gamma}_m)_8{}^n = \sqrt{2}\eta^{np_1\cdots p_6}\,\Xi_{m|p_1\cdots p_6}, \qquad (\boldsymbol{\Gamma}_m)^{n_1\cdots n_4} = \tfrac{1}{\sqrt{2}}\eta^{n_1\cdots n_4 p_1p_2p_3}\,\Xi_{m|p_1p_2p_3}$$

where

$$\Xi_{p|mnq} \equiv D_p A_{mnq} - \frac{1}{4!} F_{pmnq} \qquad \Rightarrow \qquad \Xi_{[m|npq]} = 0$$

$$\Xi_{p|m_1\cdots m_6} \equiv D_p A_{m_1\cdots m_6} - \frac{1}{7!} F_{pm_1\ldots m_6} + \ldots \qquad \Rightarrow \qquad \Xi_{[p|m_1\cdots m_6]} = 0$$

- **These connections (as determined from $D = 11$ supergravity) satisfy all covariance properties!**
- **but have non-vanishing components only along seven dimensions, vanish along all other directions.**

So what about connection coefficients for $\mathcal{M} \neq m$

$$\Rightarrow \ \partial_{\mathcal{M}} \mathcal{V}_{\mathcal{N}\,AB} - \mathbf{\Gamma}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} \mathcal{V}_{\mathcal{P}\,AB} + Q^{C}_{\mathcal{M}[A} \mathcal{V}_{\mathcal{N}\,B]C} \ = \ P_{\mathcal{M}\,ABCD} \mathcal{V}_{\mathcal{N}}{}^{CD} \ ??$$

Possible (and even required, see below), but:

- **Connections become highly ambiguous, and are not fixed by requiring absence of (generalised) torsion.**
- **Full (generalised) covariance incompatible with expressibility in terms of $\mathcal{V}$ and $\partial\mathcal{V}$ only.**
- **Remarkably, supersymmetric theory is insensitive to these ambiguities and other difficulties!**

Torsion

Definition from generalised geometry [CSW(2014);Cederwall,Edlund,Karlsson(2013)]

$$\mathcal{T}_{NK}{}^{M} \;=\; \mathbf{\Gamma}_{NK}{}^{M} - 12\,\mathbb{P}^{M}{}_{K}{}^{P}{}_{Q}\,\mathbf{\Gamma}_{PN}{}^{Q} + 4\,\mathbb{P}^{M}{}_{K}{}^{P}{}_{N}\mathbf{\Gamma}_{QP}{}^{Q}$$

This is the 912 representation in 56 $\times$ 133 $\rightarrow$ 56 $\oplus$ 912 $\oplus$ 6480.

A simple component-wise calculation using the components of $\mathbf{\Gamma}$ shows that the generalised torsion does indeed vanish, *e.g.*

$$\begin{aligned}\mathcal{T}_{m8\,n8}{}^{p8} \;&=\; \mathbf{\Gamma}_{m8\,n8}{}^{p8} - 48\,\mathbb{P}^{p8}{}_{n8}{}^{q8}{}_{r8}\,\mathbf{\Gamma}_{q8\,m8}{}^{r8} + 16\,\mathbb{P}^{p8}{}_{n8}{}^{q8}{}_{m8}\mathbf{\Gamma}_{r8\,q8}{}^{r8}\\ &=\; \mathbf{\Gamma}_{[mn]}{}^{p} - \frac{2}{3}\mathbf{\Gamma}_{r[m}{}^{r}\delta^{p}_{n]} \;=\; 0\end{aligned}$$

if ordinary torsion $\Gamma_{[mn]}{}^{p} = 0$. Similarly (using $\mathbb{P}^{pq}{}_{n8}{}^{r8}{}_{st} = -\frac{1}{12}\delta^{pq}_{n[s}\delta^{r}_{t]}$)

$$\begin{aligned}\mathcal{T}_{m8\,n8}{}^{pq} &= \mathbf{\Gamma}_{m8\,n8}{}^{pq} + 2\mathbf{\Gamma}_{r8\,m8}{}^{r[p}\delta^{q]}_{n}\\ &= 3\sqrt{2}\eta^{pqt_1\ldots t_5}\left(\Xi_{m|nt_1\ldots t_5} - \Xi_{n|mt_1\ldots t_5} + 5\Xi_{t_1|mnt_2\ldots t_5}\right)\\ &= 21\sqrt{2}\eta^{pqt_1\ldots t_5}\Xi_{[m|nt_1\ldots t_5]} \;=\; 0 \qquad \textbf{\textit{etc.}}\end{aligned}$$

$\Rightarrow$ irreducibility properties of $\mathbf{\Gamma}_{\mathcal{MN}}{}^{\mathcal{P}}$ are crucial for $\mathcal{T}_{\mathcal{MN}}{}^{\mathcal{P}} = 0$!

[GGNHS, to appear]

Absorbing non-metricity

[Hehl,VonDerHeyde,Kerlick,Nester(1976); M.Perry, private communication]

Cf. GVP of ordinary differential geometry

$$\partial_m e_n{}^a + \omega_m{}^a{}_b e_n{}^b - \Gamma^p_{mn} e_p{}^a = 0$$

But there is a more general expression

$$\partial_m e_n{}^a + \omega_m{}^a{}_b e_n{}^b - \Gamma^p_{mn} e_p{}^a = T_{mn}{}^p e_p{}^a + P_m{}^a{}_b \, e_n{}^b$$

with torsion $T_{mn}{}^p$ and non-metricity $P_{m\,np} \equiv \frac{1}{2} D_m g_{np}$, which can be absorbed by redefinitions

$$\Gamma^p_{mn} \longrightarrow \Gamma^p_{mn} - P_{(m}{}^c{}_{|d|} \, e_{n)}{}^d e^p{}_c,$$
$$T_{mn}{}^p \longrightarrow T_{mn}{}^p - P_{[m}{}^c{}_{|d|} \, e_{n]}{}^d e^p{}_c$$

Idem for exceptional geometry:

$$\mathbf{\Gamma}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} \longrightarrow \tilde{\mathbf{\Gamma}}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} = \mathbf{\Gamma}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} - i\Big(\mathcal{V}_{\mathcal{N}}{}^{AB} P_{\mathcal{M}\,ABCD} \mathcal{V}^{\mathcal{P}\,CD} - \mathcal{V}_{\mathcal{N}\,AB} P_{\mathcal{M}}{}^{ABCD} \mathcal{V}^{\mathcal{P}}{}_{CD}\Big)$$

so that the internal GVP becomes

$$\partial_{\mathcal{M}} \mathcal{V}_{\mathcal{N}\,AB} - \tilde{\mathbf{\Gamma}}_{\mathcal{M}\mathcal{N}}{}^{P} \mathcal{V}_{\mathcal{P}\,AB} + Q^C_{\mathcal{M}[A} \mathcal{V}_{\mathcal{N}\,B]C} = 0$$

Supersymmetric theory

Supersymmetry variations of bosonic fields

$$
\begin{aligned}
\delta e_\mu{}^\alpha &= \bar{\varepsilon}^A \gamma^\alpha \psi_{\mu A} + \bar{\varepsilon}_A \gamma^\alpha \psi_\mu^a \\
\delta \mathcal{B}_\mu^{\mathcal{M}} &= i \mathcal{V}^{\mathcal{M}}{}_{AB} \left(\bar{\varepsilon}_C \gamma_\mu \chi^{ABC} + 2\sqrt{2} \bar{\varepsilon}^A \psi_\mu^B \right) + \text{h.c.} \\
\delta \mathcal{V}^{\mathcal{M}}{}_{AB} &= 2\sqrt{2} \mathcal{V}^{\mathcal{M}CD} \left(\bar{\varepsilon}_{[A} \chi_{BCD]} + \frac{1}{24} \epsilon_{ABCDEFGH} \bar{\varepsilon}^E \chi^{FGH} \right)
\end{aligned}
$$

are *derived* from $D = 11$ SUGRA in [dWN,GGN], while *postulated* in recent approaches to exceptional geometry.

To establish agreement for the supersymmetry variations of fermions is more tricky! Recall [dWN(1986)]

$$
\begin{aligned}
\delta \psi_\mu^A &\propto \cdots + e^{mAB} \partial_m (\gamma_\mu \varepsilon_B) + \frac{1}{2} e^{mAB} Q_{mB}{}^C \gamma_\mu \varepsilon_C - \frac{1}{2} e^m_{CD} P_m^{ABCD} \gamma_\mu \varepsilon_D \\
\delta \chi^{ABC} &\propto \cdots + e^{m[AB} \partial_m \varepsilon^{C]} - \frac{1}{2} e^{m[AB} Q_{mD}{}^{C]} \varepsilon^D - \\
&\qquad - \frac{1}{2} e^m_{DE} P_m^{DE[AB} \varepsilon^{C]} - \frac{2}{3} e^m_{DE} P_m^{ABCD} \varepsilon^E
\end{aligned}
$$

To absorb non-metricity P_m^{ABCD} in these variations, must redefine SU(8) connection [GGNHS, to appear]

$$Q_{mA}{}^{B} \;\to\; \mathcal{Q}_{\mathcal{M}A}{}^{B} \equiv Q_{\mathcal{M}A}{}^{B} + \mathbb{Q}_{\mathcal{M}A}{}^{B}$$

where

$$\mathbb{Q}_{\mathcal{M}A}{}^{B} \;=\; R_{\mathcal{M}A}{}^{B} + \mathcal{U}_{\mathcal{M}A}{}^{B}$$

with

$$\begin{aligned}
R_{\mathcal{M}\,A}{}^{B} \;&\equiv\; \frac{4i}{3}\left(\mathcal{V}^{nBC}\mathcal{V}_{\mathcal{M}}{}^{DE}\,P_{nACDE} + \mathcal{V}^{n}{}_{AC}\mathcal{V}_{\mathcal{M}DE}\,P_{n}{}^{BCDE}\right) \\
&\quad + \frac{20i}{27}\left(\mathcal{V}^{nDE}\mathcal{V}_{\mathcal{M}}{}^{BC}\,P_{nACDE} + \mathcal{V}^{n}{}_{DE}\mathcal{V}_{\mathcal{M}AC}\,P_{n}{}^{BCDE}\right) \\
&\quad - \frac{7i}{27}\,\delta_{A}{}^{B}\left(\mathcal{V}^{nCD}\mathcal{V}_{\mathcal{M}}{}^{EF}\,P_{nCDEF} + \mathcal{V}^{n}{}_{CD}\mathcal{V}_{\mathcal{M}EF}\,P_{n}{}^{CDEF}\right) \\
\mathcal{U}_{\mathcal{M}A}{}^{B} \;&=\; \mathcal{V}_{\mathcal{M}\,CD}\,u^{CD,B}{}_{A} - \mathcal{V}_{\mathcal{M}}{}^{CD}\,u_{CD,A}{}^{B}
\end{aligned}$$

where $u^{[CD,B]}{}_{A} \equiv 0\,,\; u^{CA,B}{}_{C} \equiv 0$ **in 1280 of SU(8).**

Redefinition requires SU(8) connection components along $\mathcal{M} \neq m$!

Leads to very compact expressions:

$$\begin{aligned} \delta\psi_\mu^A \; &\propto \; \cdots + \mathcal{V}^{\mathcal{M}\,AB}\mathcal{D}_\mathcal{M}(\mathcal{Q})_B{}^C(\gamma_\mu\varepsilon_C) \\ \delta\chi^{ABC} \; &\propto \; \cdots + \mathcal{V}^{\mathcal{M}\,[AB}\mathcal{D}_\mathcal{M}(\mathcal{Q})\varepsilon^{C]} \end{aligned}$$

Also: requires extra components $\mathcal{Q}_\mathcal{M}$ for $\mathcal{M} \neq m$ and

$$\mathbf{\Gamma}_{\mathcal{M}\mathcal{N}}{}^\mathcal{P} \; \to \; \widehat{\mathbf{\Gamma}}_{\mathcal{M}\mathcal{N}}{}^\mathcal{P} \equiv \tilde{\mathbf{\Gamma}}_{\mathcal{M}\mathcal{N}}{}^\mathcal{P} + i\left(\mathcal{V}^\mathcal{P}{}_{AB}\mathbb{Q}_\mathcal{M}{}^A{}_C\mathcal{V}_\mathcal{N}{}^{BC} - \mathcal{V}^{\mathcal{P}AB}\mathbb{Q}_{\mathcal{M}A}{}^C\mathcal{V}_{\mathcal{N}BC}\right)$$

After all these operations we are left with fully covariant and torsion-free connections and a standard GVP

$$\partial_\mathcal{M}\mathcal{V}_{\mathcal{N}\,AB} \, - \, \widehat{\mathbf{\Gamma}}_{\mathcal{M}\mathcal{N}}{}^\mathcal{P}\mathcal{V}_{\mathcal{P}\,AB} \, + \, \mathcal{Q}^C_{\mathcal{M}[A}\mathcal{V}_{\mathcal{N}\,B]C} \, = \, 0$$

NB: absence of torsion does *not* fix affine connection uniquely, irremovable ambiguity is in 1280 of SU(8).

[Coimbra,Strickland-Constable,Waldram(2012);Cederwall,Edlund,Karlsson(2013);GGNHS(2014)]

Conclusions

- **Starting from 'old' results [dWN(1986);GGN(2013)] one can construct generalised SU(8) and affine connections that satisfy all required covariance properties.**
- **These *cannot* be written in terms of just $\mathcal{V}$ and $\partial\mathcal{V}$, unlike in General Relativity, even with zero torsion.**
- **SUSY theory smartly picks just the right combinations which are insensitive to ambiguities/difficulties encountered in generalised geometry constructions.**
- **Only in this supersymmetric context 'old' results agree with more recent constructions! [GGNHS(2014)]**
- **New theories by ω-deformations?**