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Lecture Notes for Felix Klein Lectures

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Abstract: These are lecture notes for a series of talks at the Hausdorff Mathematical Institute, Bonn, Oct. 1-11, 2012
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Contents

"This whole book is but a draught nay, but the draught of a draught." - Herman Melville

1. Prologue

The following are some notes adapted from a set of 8 lectures I gave in Bonn, October 1-11, 2012. As will become immediately apparent if you look at them, they are very rough and are a work in progress. Constructive criticism and suggestions are welcome.

The most up-to-date version will be available at
www.physics.rutgers.edu/∼ gmoore/FelixKleinLectureNotes.pdf

2. Lecture 1, Monday Oct. 1: Introduction: (2,0) Theory and Physical Mathematics

When Felix Klein came to study here at the University of Bonn in 1865 he wanted to become a physicist! In those days there was a single chair of Mathematics and Experimental Physics, and the occupant, Julius Plücker, was interested in geometry, not physics. So Felix Klein became a geometer instead. Nevertheless, later on when he was well established at Göttingen, Felix Klein turned back to mathematical physics. And indeed, the 20th century has shown that he really was a physicist all along, because physics and geometry are so deeply interconnected.

♣Yuji says Plücker was interested in physics. Check it out. ♣

I mention all this because I am a physicist, using physical heuristics, such as interacting quantum field theories and string theories – which are not rigorously defined – to learn about mathematics, and using mathematics to sharpen our understanding of quantum field theory and string theory. I call this intellectual endeavor "physical mathematics."

In these lectures I will undoubtedly err in being too loose and heuristic. Most of what I say is not settled mathematics, but I will try to separate what is physical heuristics from clearly established mathematics.

I will be aiming to explain some new mathematics which is motivated by the physical voodoo. This is largely, but not exclusively, the content of a project with D. Gaiotto and A. Neitzke [91, 94, 96, 97, 100]. I will also be explaining some work from various projects with Dmitry Belov, Frederik Denef, Emanuel Diaconescu, Dan Freed, and Yuji Tachikawa.

While these papers are not rigorous mathematics, they are perhaps precise enough to be turned into rigorous mathematics. Here is a list of some of the mathematical themes which are relevant to these lectures:

1. A new (2008) construction of hyperkähler metrics on certain manifolds (including moduli spaces of solutions to Hitchin's equations on a Riemann surface C) using generalized Donaldson-Thomas invariants. Curiously, this uses an integral equation formally identical to Zamolodchikov's TBA.

2. The work of GMN, especially [91, 94] motivated T. Bridgeland and I. Smith to formulate (rigorously) stability conditions on certain 3-Calabi-Yau categories in terms of spaces of quadratic differentials with singularities.

3. A new construction of certain hyperholomorphic connections (e.g. on the universal bundle over the Hitchin moduli space). This can be used to give a method to construct explicit solutions to Hitchin's equations on a Riemann surface C. Again, a key step uses an integral equation generalizing the kind seen in inverse scattering theory.

4. New methods of constructing coordinate systems (in some cases cluster coordinates, possibly cluster coordinates in general) on moduli spaces of (twisted) flat local systems on Riemann surfaces with prescribed singularities. Here there are extensive connections to the work of Fock and Goncharov [68]-[75].

5. New methods of constructing the generalized DT invariants discussed by Kontsevich and Soibelman. This involves a new geometrical construction (generalizing some facts about foliations of surfaces by quadratic differentials) which GMN called *spectral networks* [94, 102, 100].

6. The theories we will be discussing push the boundary of what we usually understand by "quantum field theory," and requires some extension of those notions. First, as explained in §**** we must generalize to the notation of an "n-dimensional field theory valued in an $(n+1)$-dimensional field theory." More radically, the Gaiotto factorization property of theories of class S suggests the notion of a "conformal field theory valued in higher dimensional field theories" This latter idea has been made somewhat precise in some restricted sense in [144, 83] and is probably the key to understanding the AGT phenomena [4].

7. The framework I will discuss also has some implications for the theory of WKB asymptotics of sections of flat connections on higher-rank local systems. This should have some application to the theory of λ-connections. ♣Proper refs? Arinkin? Simpson? ♣

There are other potential mathematical applications which I will mention at various points. But the larger point here, which I will dwell on for the remainder of this lecture, is that there are some quantum field theories whose existence was only suspected in the mid 1990's but whose existence seems to predict and unify a large array of deep mathematical facts.

First I must say some brief words on how we'll be thinking about quantum field theory.

2.1 Quantum Field Theory

It is not my purpose here to give some kind of foundational approach to quantum field theory, even though we will use it. There are two viewpoints of quantum field theory I will draw upon:

2.1.1 Extended QFT, defects, and bordism categories

There is a viewpoint which began with the Atiyah-Segal axioms for topological field theory and has evolved, under the influence of numerous mathematicians, [1] into a concept of extended field theory. This is an extension of the "three tiered theories" discussed by G. Segal in his Felix Klein lectures of two years ago.

The extended field theories are based on n-categories, where, for our very limited purposes, we need only say that an n-category is a category whose morphism spaces are $(n-1)$-categories. In this way of counting, a -1-category is a complex number, a 0-category is a vector space, and a 1-category is what one usually means by "category."[2]

A good example of an n-category is the category $BORD(n)$ of topological n-manifolds with corners:

1. Objects ("zero-morphisms") are 0-manifolds.
2. Morphisms are 1-manifolds defining bordisms between 0-manifolds
3. 2-morphisms are 2-manifolds defining bordisms between the 1-manifolds

etc. up to n-manifolds.

FIGURE ILLUSTRATING BORD-2.

According to the extended field theory viewpoint, an n-dimensional field theory $\mathcal{F}_n$ is some kind of functor[3] from a geometric bordism category to an n-category

$$\mathcal{F}_n : Bord^{\rm structure}(n) \to \mathcal{C} \tag{2.1}$$

where the domain $Bord^{\rm structure}(n)$ is an n-category like $BORD(n)$ but where the manifolds might be endowed with various topological and/or geometric structures. The codomain can be, rather generally, some n-category. Codomain categories involving linear spaces and maps between them is a very important class of examples. Categories generalize algebras, and in this sense, $\mathcal{F}_n$ should be viewed as a generalization of a homomorphism.

Thus the field theory associates to a closed n-manifold M_n (perhaps with topological and/or geometrical structure) a complex number (i.e. a "-1-category") - the partition function - usually denoted

$$\mathcal{F}_n[M_n] = Z(M_n) \in \mathbb{C} \tag{2.2}$$

and to a closed $(n-1)$-manifold M_{n-1} a vector space (i.e. a "0-category") - the space of states - usually denoted

$$\mathcal{F}_n[M_{n-1}] = \mathcal{H}(M_{n-1}) \in {\rm VECT}_{\mathbb{C}} \tag{2.3}$$

The "monoidal functor" property ensures the usual physical properties under disjoint union and gluing, respectively, which follow from *locality*. But now in the extended case we can keep going, and to a closed $(n-k-1)$-manifold M_{n-k-1} we associate a k-category

[1] among them J. Baez, L. Crane, J. Dolan, D. Freed, M. Hopkins, M. Kapranov, D. Kazhdan, R. Lawrence, J. Lurie, N. Reshetikhin, G. Segal, C. Teleman, V. Turaev, Voevodsky, Yetter. I learned these things primarily from Freed and Segal.

[2] Note that a -1 category is an object in a certain 0-category, namely, the vector space of complex numbers. A 0-category is an object in the 1-category of vector spaces, and a 1-category is an object in the 2-category of categories...

[3] symmetric monoidal functor of weak n-categories

$\mathcal{F}_n[M_{n-k-1}]$, with associated coherence relations reflecting various gluing properties. In a sense, the extended field theory is simply taking the notion of locality in quantum field theory to its logical conclusion.

We can clarify the physical meaning of the k-categories $\mathcal{F}_n[M_{n-k-1}]$ for $k \geq 1$ by considering the physics of *defects*. Recall that in field theory there is supposed to be a *state-operator correspondence*: In a field theory on an n-manifold, M_n, if we cut out a small n-dimensional ball $B_n(r, P)$ around a point $P \in M_n$ then the standard axioms imply that there is a vector space $\mathcal{H}(\partial B_n(r, P))$. Consider the amplitudes where we have a state $\psi \in \mathcal{H}(\partial B_n(r, P))$ inserted at this boundary. The "limit" as $r \to 0$ defines a disturbance localized at a point. Taking this "limit" of the whole statespace is therefore supposed to define the linear space of local operators of the theory at the point P.

A local operator is an example of a 0-dimensional *defect* in a quantum field theory. One important lesson of the past few years is that it is important to include more general *defects*. In a quantum field theory *defects* are, roughly speaking, local modifications of physical quantities (correlators, path integrals, Hilbert spaces...) and are associated to positive codimension submanifolds. [4] The inclusion of this data in the definition of a QFT goes beyond the traditional approaches to quantum field theory such as the Whiteman axioms of constructive QFT or the Haag-Kastler axioms of algebraic QFT.

For example, while a point defect is a local operator a theory might well also include line defects. Famous examples include Wilson operators in gauge theories and 't Hooft line operators in four-dimensional gauge theories. Sometimes two different theories cannot be distinguished by their local operators, but can be distinguished by their collection of line defects. Now, one can go on to higher dimensions: Codimension 1 defects are "domain walls" (of which boundary conditions are a special case). In the following chapters defects of both dimension and codimension two will play an important role.

We now relate the physical idea of defects to the extended category viewpoint. (Here we are following a discussion from A. Kapustin's ICM talk [124].) Consider defects localized on a k-dimensional submanifold $P_k \subset M_n$, again possibly endowed with topological and/or geometric structures. We consider a tubular neighborhood of P_k bounded by a bundle of linking spheres S_r^{n-k-1} of radius r. Then the boundary conditions of the field theory, in the "limit" as $r \to 0$ is supposed to describe k-dimensional defects. According to the definition (2.1) we should associate a k-category $\mathcal{F}_n(S_r^{n-k-1})$ to the linking sphere and the limit as $r \to 0$ should describe the k-category of k-dimensional defects localized on P_k. This is a generalization of the state-operator correspondence.

Why should k-dimensional defects form a k-category? The main physics point is that one can embed lower dimensional defects within higher dimensional defects. For example, Instead of considering a Wilson line in gauge theory along a closed path ℓ

$$\mathrm{Tr}_R \mathrm{Pexp} \oint_{\ell} A \tag{2.4}$$

[4] We say *defect* instead of *operator* because they are not operators on any Hilbert space in general. Moreover, they might have internal degrees of freedom not present in the "ambient" or "bulk" QFT.

one could insert a suitable local operator $\mathcal{O}(P)$ at a point $P \in \ell$ and consider instead the "operator"

$$\mathrm{Tr}_R \mathcal{O}(P) \mathrm{Pexp} \oint_{\ell(P)} A \tag{2.5}$$

where $\ell(P)$ is the "open path" which begins and ends at P. For k dimensional defects with $k > 1$ we can have defects within defects within defects ...

Thus, the 0-morphisms of the k-category $\lim_{r\to 0} \mathcal{F}_n[S_r^{n-k-1}]$ are the labels of the various k-dimensional defects which can live on P_k. The 1-morphisms of the k-category $\lim_{r\to 0} \mathcal{F}_n[S_r^{n-k-1}]$ are labels of the $(k-1)$-dimensional defects that can live within the k-dimensional defects on P_k, and more generally:

ℓ-morphisms of the k-category $\lim_{r\to 0} \mathcal{F}_n[S_r^{n-k-1}]$ are the labels of $(k-\ell)$-dimensional defects that can live within the k-dimensional defect on P_k.

DRAW DEFECTS WITHIN DEFECTS AS ILLUSTRATION OF MORPHISMS IN A HIGHER CATEGORY CATEGORY

A useful example to bear in mind is the case where $k = (n-1)$. These are domain walls, linked by the zero-sphere S^0, a disjoint union of two points. In general domain walls can even separate different theories. In particular, if the "empty theory" is on one side of the domain wall then we are describing boundary conditions for $\mathcal{F}_n$. The boundary conditions on a bounding $(n-1)$-fold P_{n-1} are 0-morphisms in an $(n-1)$-category $\mathcal{F}_n[P_{n-1}]$, the $(n-2)$-dimensional defects lying within this $(n-1)$-dimensional boundary are the 1-morphisms of the $(n-1)$-category, and so forth.

Let us specialize to $n = 2$. In this case we have a familiar picture of boundaries labeled by boundary conditions $a, b, ...$ which are 0-morphisms in a category, and boundary-condition-changing operators $\mathcal{O}_{ab}(P) \in \mathrm{Hom}(a, b)$ as local operators inserted on the boundary:

PICTURE

Using the state-operator correspondence on a semicircle, this picture can be changed to

PICTURE WITH SEMIDISK CUT OUT

which finally gives the link to the original discussion from [143] where a simple argument was presented explaining why boundary conditions in a 2D TFT should be thought of as objects in a category. We recall the argument here: The interval with boundary conditions a, b maps the the "space of open strings" $\mathcal{O}_{ab}$,

FIGURE

the basic open string interaction

FIGURE

means that there is a bilinear map $\mathcal{O}_{ab} \times \mathcal{O}_{bc} \to \mathcal{O}_{ac}$ and the different ways of cutting and gluing the diagram

FIGURE 3 INGOING AND ONE OUTGOING INTERVAL

imply that that bilinear map is associative. But these are just the defining axioms for a $\mathbb{C}$-linear category. Hence, the boundary conditions are objects in a category.

2.1.2 Traditional Wilsonian Viewpoint

There is a rather different point of view of QFT, which is the one usually adopted by physicists. It is based on the traditional Wilsonian view that quantum field theories are constructed from fixed points of the renormalization group - known as scale invariant field theories - and are obtained by deformations from these. [5]

The picture is - very roughly - that there is a space $\mathcal{S}_\Lambda$ of *cutoff* quantum field theories. These are defined with a positive finite parameter Λ, which has dimensions of mass, that is, under a scaling of lengths by a factor s, $\Lambda \to \Lambda/s$. The parameter Λ cuts off the divergences of quantum field theory. The cutoff is highly noncanonical and can be introduced in many different ways (not reflected in the notation). The idea is that with - say - a particular method of cutting off divergences the spaces of cutoff theories $\mathcal{S}_\Lambda$ for different Λ are all diffeomorphic to a common fixed (infinite-dimensional) space of coupling constants for all local operators in the Lagrangian. Changing the cutoff defines a diffeomorphism

$$R_{\Lambda,\Lambda'} : \mathcal{S}_\Lambda \to \mathcal{S}_{\Lambda'} \tag{2.6}$$

(which only depends on the ratio Λ/Λ'.) This map is defined by requiring that the coupling constants change so that with the different cutoff physical correlators are equal. Now we have the diagram:

$$\begin{array}{ccc} \mathcal{S}_\Lambda & \xrightarrow{R_{\Lambda,\Lambda'}} & \mathcal{S}_{\Lambda'} \\ & {\scriptstyle f_\Lambda} \searrow \quad \swarrow {\scriptstyle f_{\Lambda'}} & \\ & \mathcal{S} & \end{array} \tag{2.7}$$

The composition $f_{\Lambda'} R_{\Lambda',\Lambda} f_\Lambda^{-1}$ is a one-parameter family of diffeomorphisms $\mathcal{S} \to \mathcal{S}$ and is generated by a vector field known as the "*beta function*" (for historical reasons).

There is a (typically finite dimensional) subvariety of conformal field theories $\mathcal{C} \subset \mathcal{S}$. The tangent space of the infinite-dimensional space of all theories is identified with the space of local operators and, when restricted to $\mathcal{C}$,

$$T\mathcal{S}|_{\mathcal{C}} \tag{2.8}$$

this space of local operators breaks into three summands according to the eigenvalue of the scaling operator ♣Clarify this ♣

1. Relevant perturbations flow away from the CFT locus. $\Delta > 1$.
2. Marginal perturbations are tangent to the CFT locus.
3. Irrelevant perturbations flow back toward the CFT locus.

There are typically only finitely many relevant and marginal directions. ♣Figure ♣

Non-conformal theories are defined by simultaneously taking $\Lambda \to \infty$ while scaling a point in $\mathcal{S} - \mathcal{C}$ back to $\mathcal{C}$ along a relevant trajectory.

For more on this see, e.g., [24, 149][Wetterich review?].

[5] In these notes we will assume that scale invariance implies conformal invariance, and henceforth refer to these as "conformal field theories." Whether or not scale invariance really does imply conformal invariance is a topic still debated by physicists.

Thus, in this point of view, conformal field theories play a central role in the study of quantum field theory. All other local quantum field theories are "merely" perturbations of conformal field theories. The intuition is that "at short distances" or "at high energies" the quantum field theory "looks like" the conformal field theory of which it is a perturbation.

Conformal field theories have been rigorously constructed in two-dimensions. For some time it was almost universally accepted by physicists that interacting quantum field theories (and in particular conformal field theories) could only exist in spacetime dimensions $n \leq 4$.

I will be explaining reasons - albeit physical and heuristic - why in the mid-1990's *many string theorists came to believe that the long-accepted standard lore is wrong and that there exist nontrivial interacting conformal field theories in five and six dimensions.*

The reason for explaining this is that the existence of six-dimensional conformal field theories with maximal supersymmetry seems to lie at the heart of a number of remarkable discoveries in physical mathematics which have been made over the past decade. If they could be rigorously, or even semi-rigorously constructed this would provide a beautiful unifying framework for understanding a broad array of deep results in mathematics.

2.2 Compactification, Low Energy Limit, and Reduction

In order to make the above claim plausible we should first review some standard procedures by which physicists produce lower-dimensional field theories from higher-dimensional field theories.

Given an n-dimensional quantum field theory there are three closely related constructions that produce field theories in lower dimension:

VerKleinung:

If K_j is a compact closed manifold of dimension $j \leq n$ then we can consider the compactification of the theory $\mathcal{F}_n$ on K_j. This is the $(n-j)$-dimensional theory, denoted $\mathcal{F}/K$, which assigns to an s-dimensional manifold M_s the k-category

$$(\mathcal{F}/K)[M_s] := \mathcal{F}_n[M_s \times K_j] \tag{2.9}$$

The RHS is a k-category for $k = n - j - s - 1$.

If $\mathcal{F}$ is defined on a bordism category with geometric data then this definition only makes sense if we assume the geometric data on the RHS is a "direct product" in the appropriate sense. For example, if the theory depends on a metric then we choose a metric on K and on the RHS we have a direct sum metric. If the theory depends on spin structure then we must choose a product spin structure, etc. [6]

Note that, trivially,

$$(\mathcal{F}/K)/K' = (\mathcal{F}/K')/K = \mathcal{F}/(K \times K') \tag{2.10}$$

For example, if $\mathcal{F}$ is a Lagrangian field theory with a finite number of fields then $\mathcal{F}/K$ will typically involve an infinite number of fields. A very simple example is provided by a

[6] In physics warped metrics are of great interest and these would be excluded by this simple-minded definition.

free complex scalar field (massive or massless) in n-dimensions. (We take ϕ to be complex for simplicity.) It is a Lagrangian field theory with action:

$$S = \int_{M_n} \mathrm{vol}\,(g) \left\{ \frac{1}{2}\sum_{\mu=1}^{n} \partial_\mu\phi\partial^\mu\phi^* + \frac{1}{2}m^2\phi\phi^* \right\} \tag{2.11}$$

If we consider manifolds of the type $M_{n-1}\times S^1_R$ with product metric $g = \bar{g}\oplus R^2(d\theta)^2$ where $\theta\sim\theta+2\pi$ then we can of course make a Fourier decomposition of the general[7] field configuration

$$\phi = \sum_{k\in\mathbb{Z}} \phi_k(x)e^{\mathrm{i}k\theta} \tag{2.12}$$

and substitute into the Lagrangian to obtain an action for a theory of infinitely many complex scalar fields in $(n-1)$ dimensions:

$$S = 2\pi R\int_{M_{n-1}} \mathrm{vol}\,(\bar{g}) \left\{ \frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{\mu=1}^{n-1} \partial_\mu\phi_k\partial^\mu\phi_k^* + \frac{1}{2}\left(m^2+\left(\frac{k}{R}\right)^2\right)\phi_k\phi_k^* \right\} \tag{2.13}$$

Remarks

1. This is sometimes called "Kaluza-Klein compactification" or "Kaluza-Klein reduction" in the literature but the terms are used inconsistently. (That's a different Klein, Oscar Klein.) *VerKleinung* seems like a good name since in German "eine Verkleidung" is "a disguise," and in this construction we really do look at the theory in disguise. On the other hand, by restricting j-dimensions to be of the form K_j we are restricting the theory, and hence making it smaller.

2. If we consider the verKleinung of a Yang-Mills gauge theory with compact gauge group G along $K = S^1$ we obtain a Yang-Mills-Higgs theory with gauge group given by the loop group.

Compactification: In theories that depend on a metric (or even on a conformal class of a metric) we can take an "IR limit" of the compactified theory $\mathcal{F}/K$ by considering the limit in which all distance scales in K become small. We will denote this theory by $\mathcal{F}//K$, although the notation is *not* intended to suggest any relation to symplectic reduction. In the above example of a scalar field, in n dimensions then the low energy compactified theory in $(n-1)$ dimensions consists of a single scalar field at energy scales $m\le E\ll 1/R$. At energies $E\ll m$ the theory is trivial.

In general if $\mathcal{F}$ is a Lagrangian field theory with a finite number of fields then $\mathcal{F}//K$ will typically again involve a finite number of fields.

Once again we have

$$(\mathcal{F}//K)//K' = (\mathcal{F}//K')//K \tag{2.14}$$

which is somewhat less trivial, because we can have relative scales between K and K' and hence the lower dimensional theory can have "new" dimensionless parameters. For example, a special case of this, applied to the six-dimensional conformal field theories, implies S-duality of $d = 4, \mathcal{N} = 4$ supersymmetric Yang-Mills, a highly nontrivial statement.

♣Distinguish between cases where $\mathcal{F}//K$ is a well-defined quantum field theory and just an effective field theory? ♣

Remarks

1. This procedure is also sometimes called "Kaluza-Klein compactification" or "Kaluza-Klein reduction" in the literature but the terms are used inconsistently.

2. The low energy effective theory might or might not exist as a well-defined *quantum* field theory.

3. In general, the compactification on any closed $(n-2)$ dimensional manifold M_{n-2} defines a two-dimensional theory $\mathcal{F}_n//M_{n-2}$ which therefore has a category of boundary conditions. This is one nice interpretation of the category $\lim_{r\to 0} \mathcal{F}_n[M^r_{n-2}]$.

There is a third formal procedure to obtain one Lagrangian field theory from another:

Reduction: Suppose we consider manifolds of the form $M_{n-j} \times K$ where K is of dimension j and there is a Lie group G acting transitively on K. We may restrict field space to the fields which are invariant under G. In a Lagrangian field theory we obtain a new action principle by substituting invariant fields and "dividing by the volume of the group G". [8] This is called the reduced field theory, or more properly the *reduction with respect to the G-action.*

Example: If we consider n-manifolds of the form $M_{n-j} \times \mathbb{R}^j$ and we consider the group of translations by $\mathbb{R}^j$ then we can reduce. This is often called "dimensional reduction" in the literature. If instead we took $M_{n-j} \times (S^1)^j$ and used translation we would get the same set of fields.

2.3 Relations between theories

As I have already said, the mere existence of the six-dimensional (2,0) theory implies many nontrivial relations between aspects of physical mathematics associated with lower dimensional objects.

A schematic view of just SOME of the relations is shown in 1.

The key to the acronyms in the figure is:

1. 5DSYM(G): 5-dimensional susys Yang-Mills. Note the gauge group G and not just the Lie algebra

2. Class S: d=4 $\mathcal{N} = 2$ theories of class S. We will discuss these in detail in §7. Here we can define interesting BPS states and BPS invariants, which turn out to be closely related to the (generalized, motivic) Donaldson-Thomas invariants of Kontsevich and Soibelman. Moreover there is a beautiful realization of the S-duality groupoid as the modular groupoid of a Riemann surface.

[7] Fourier summable!

[8] The volume of G in its natural measure might well be infinite. In this case we should regularize and then take a limit.

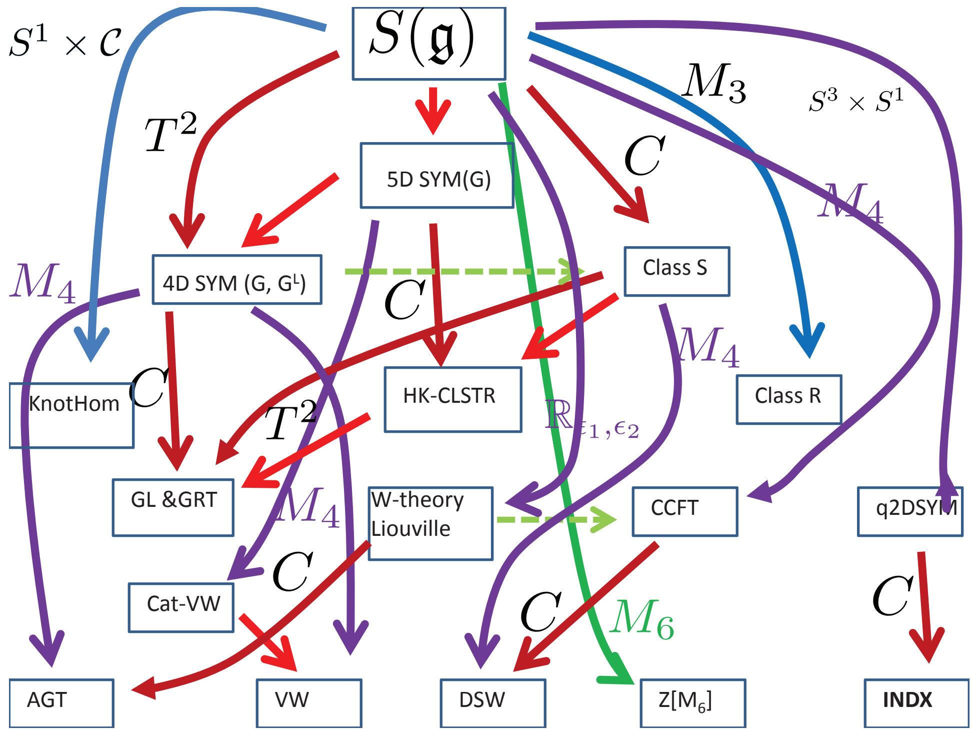

Figure 1: Some of the relations between theories and physical quantities predicted by the existence of the six-dimensional superconformal fixed points. Each row refers to theories in successively lower dimensions. Thus, from the viewpoint of extended field theory, each row refers to a (higher) category associated to the six-dimensional theory. The green dashed arrows indicate that one set of theories is a subset of the other. The red arrows indicate compactification plus low energy limit on a circle of radius R: $\mathcal{F} \to \mathcal{F}//S^1$. Brown arrows represent compactification on 2-manifolds. Green arrows on 3-manifolds. Purple arrows on 4-manifolds.

3. A special case is for $C = T^2$: $\mathcal{N} = 4$, $d = 4$ SYM. Note there are two groups G and its Langlands-GNO dual ${}^LG$. S-duality is thus implicit in the picture.

4. KnotHom: Witten's approach to knot homology [178] is based on studying the $(2,0)$ theory on six-manifolds of the form $M_3 \times \mathbb{R} \times \mathcal{C}$, where $\mathcal{C}$ is a so-called "cigar geometry," a disk with a metric approaching a semiinfinite cylinder near the boundary. See Sec. 5 of [178].

5. HK-CLSTR: Rozansky-Witten theory for hyperkähler manifold $\mathcal{M}$. In class S this is a Hitchin moduli space. There is a construction of cluster coordinates on this space, very similar to the Fock-Goncharov coordinates, which we will discuss in detail.

6. Class R: Invented by Dimofte-Gaiotto-Gukov [58, 59], this is a class of 3d superconformal theories from compactification on 3-manifolds M_3 (typically knot complements in S^3). They enjoy the symplectic action by $Sp(2g, \mathbb{Z})$ defined by Witten and are

closely related to 3d Chern-Simons-Witten theories for complex gauge group. Mathematically they are closely related to the Bloch group. [59] [also cite Dimofte SCGP talk].

7. GL& GRT: Geometric Langlands program and Geometric Representation Theory (Geometric Satake theorem, DAHA, Kazhdan-Lusztig theory etc.)[9]

8. W-Theory/Liouville: The famous AGT [REFS] relation between compactification on "Ω-deformed $\mathbb{R}^4$ (working equivariantly on $\mathbb{R}^4$ wrt to $so(4)$ actions) and the conformal blocks of CFT's with W-algebra symmetry. In particular for the case of $\mathfrak{g} = A_1$ we get Liouville conformal blocks.

9. CCFT: Chiral conformal field theory. Examples include chiral theories of bosons and fermions. The theories are related to an interesting construction in string theory used to compute black hole entropy and and black hole counting functions. (This is the MSW (0,4) sigma model obtained by wrapping a 5-brane on a divisor in a CY 3-fold.) These partition fucntions, count BH entropy for Calabi-Yau compactification typically involve mock modular forms.[REFS].

10. q2DYM: Work of Rastelli et. al. [83, 84, 85, 86] shows that there is a partially defined topological field theory associated with this compactification which is closely related to 2d Yang-Mills theory at zero area. In fact, a special case is the q-deformed Yang-Mills theory, but it involves interesting further deformations thereof. The superconformal index, denoted INDX in the $d = 0$ row involves integrals over elliptic gamma functions and S-duality implies nontrivial identities on these integrals. In a beautiful development it turned out that the subject of elliptic hypergeometric functions had previously (but recently) been investigated for completely independent reasons by Spiridonov et.al. [163], where some of the required identities had already been discovered.

    In general, for mathematicians that like curious special functions and the amazing identities they satisfy, this subject of $(2,0)$ superconformal theories should be a gold mine.

11. Cat-VW: Categorification of the Vafa-Witten invariant of four-manifolds. Cohomology of instanton moduli space

12. AGT: This is the AGT relation proper, which identifies the Liouville conformal blocks with Nekrasov (or Instanton) partition functions for the $\mathcal{N} = 2$ theories of class S. There are natural generalizations involving W-algebras.

13. VW: The Vafa-Witten invariants computing modular generating functions for the Euler characters of instanton moduli spaces.

14. DSW: Donaldson/Seiberg-Witten invariants. An interesting question is whether the new discoveries about N=2 d=4 theories will lead to new results in this area.

[9] I learned this latter connection from D. Ben-Zvi.

15. $Z[M_6]$: In the abelian theories, on a general 6-manifold (with appropriate geometric and topological structures, discussed below) there is the "five-brane partition function" discussed by Witten, Dolan-Nappi, Henningson, Hopkins-Singer, Belov-Moore, Monnier, and others. It is a natural generalization of the Dedekind eta function. On certain six-manifolds such as $E \times \mathbb{C}P^2$, where E is an elliptic curve it has a limit which is literally $\vartheta(\tau)/\eta(\tau)$ where τ represents the complex structure of E and $\vartheta(\tau)$ is one of the "thetanullwerthe" for one of the spin structures on E.

Of course, closed paths on this diagram typically lead to interesting results in physical mathematics.

There is far too much in this diagram to cover in any short series of lectures. We will be focusing on just Class S and HK-CLSTR. This is the part with which I am best acquainted.

3. Background material on superconformal and super-Poincaré algebras

It is very appropriate to begin with supersymmetry algebras in this lecture series: Felix Klein's Erlangen program defines geometry as the study of quantities invariant under symmetry. This applies very well to the automorphisms of (super) space time, and we even will be concerned with the relation between projective geometry (conformal symmetry) and affine geometry (poincare symmetry) which was one of Klein's main points.

3.1 Why study these?

Our more immediate reason for studying the algebras is that :

1. Conformal symmetry forms the basis of our understanding of quantum field theory as we discussed briefly in §????

2. Supersymmetry gives control and leads to theories where exact computations are possible. This has led to the rich connections to mathematics.

3.2 Poincaré and conformal symmetry

Consider affine Minkowski space $\mathbb{M}^{1,d-1}$: It is the d-dimensional real affine space whose tangent space carries a Lorentz metric of signature $-1, +1^{d-1}$.

The automorphism group of affine Minkowski space is the Poincaré group and it's Lie algebra will be denoted $iso(1, d-1)$. Generators are translations P and "rotations" M.

Similarly, we can consider the group which preserves the causal structure of spacetime - which takes light-cones to lightcones. This is the conformal group. Its Lie algebra is denoted $conf(1, d-1)$. Note that dilations D centered on some point preserve lightcones. Also, choosing an origin we can define $I : x^\mu \to x^\mu/x^2$. Conjugation by I is an inner automorphism of the conformal group which takes $IPI^{-1} = K$, the special conformal transformations. The generators of $conf(1, d-1)$ are D, P, K, M. Stereographic projection from a sphere in $\mathbb{M}^{1,d-1} \times \mathbb{M}^{1,1}$ shows that

$$conf(1, d-1) \cong so(2, d) \tag{3.1}$$

The dilation operator D is the boost in $\mathbb{M}^{1,1}$.

3.3 Super-Poincaré algebras

By the Coleman-Mandula/O'Raifeartaigh theorem [10] the symmetries of the S-matrix of a Poincaré invariant quantum field theory must have Lie algebra of the form:

$$\mathfrak{SP}^0 = iso(1, d-1) \oplus \mathfrak{k} \tag{3.2}$$

where $\mathfrak{k}$ is a compact real Lie algebra.

Some theories can be enriched by having a $\mathbb{Z}_2$-graded Lie algebra as a symmetry of the S-matrix.

Recall that a $\mathbb{Z}_2$-graded Lie algebra is a $\mathbb{Z}_2$-graded vector space

$$\mathfrak{S} = \mathfrak{S}^0 \oplus \mathfrak{S}^1 \tag{3.3}$$

with a graded-bracket $[,] : \mathfrak{S} \otimes \mathfrak{S} \to \mathfrak{S}$ such that

$$[X, Y] = (-1)^{1+|X||Y|}[Y, X] \tag{3.4}$$

on homogeneous elements and such that $[,]$ satisfies the graded Jacobi identity.

The graded Jacobi identity can be broken into 4 cases:

BBB: $\mathfrak{S}^0$ is an ordinary Lie algebra

BBF: $\mathfrak{S}^1$ is a module for $\mathfrak{S}^0$

FFB: There is an $\mathfrak{S}^0$ equivariant map $\mathrm{Sym}^2\mathfrak{S}^1 \to \mathfrak{S}^0$

FFF: $[F_1, [F_2, F_3]] + cyclic = 0$. (No signs with this order.)

There is a generalization of CM-O'R due to Haag-Lopuszanski-Sohnius [REF] for graded Lie algebra symmetries of the S-matrix of a relativistic quantum field theory on Minkowski space: If the theory contains massless particles and satisfies some physically reasonable criteria then the graded Lie algebra symmetry of the S-matrix must be a *super-Poincaré symmetries*:

Definition A d-dimensional superPoincaré symmetry $\mathfrak{SP}$ is a super Lie algebra such that

1. $\mathfrak{SP}^0 = iso(1, d-1) \oplus \mathfrak{k}$ where $\mathfrak{k}$ is a real reductive Lie algebra with compact semisimple summand.

2. $\mathfrak{SP}^1$ is a real spinorial representation of $iso(1, d-1)$.

3. $\mathrm{Sym}^2\mathfrak{SP}^1 \to \mathfrak{SP}^0$ is nonvanishing

♣Should we say nondegenerate? ♣

We will denote superPoincaré algebras of super-Minkowski space $\mathbb{M}^{1,d-1|s}$ by $\mathfrak{SP}(\mathbb{M}^{1,d-1|s})$.

♣Again, a la Klein, this should be viewed as a superalgebra preserving some geometric structures on the supermanifold $\mathbb{M}^{1,d-1|s}$. Should be something about preserving Minkowski metric and an odd leftinvariant distribution. ♣

For a super-Poincaré algebra elements of $\mathfrak{SP}^1$ are called supersymmetry operators. The global symmetries in $\mathfrak{k}$ which do not commute with the supersymmetries are known as *R-symmetries.*

Poincaré supersymmetry algebras exist in all dimensions. Choose a real spinorial representation S with a Poincare invariant symmetric pairing

$$S \otimes S \to V^* \tag{3.5}$$

♣Say more about when such pairings exist. ♣

where $V = T_p\mathbb{M}^{1,d-1}$. (Such representations always exist.) Then taking $\mathfrak{SP}^1 = S$ we use

[10] For a nice discussion, see [171], appendix B

this to define the odd bracket. Thus we have

$$[Q, Q'] \sim P \tag{3.6}$$

The FFF superJacobi identity is trivially satisfied.

In supersymmetric theories, the Hamiltonian is related to the square of a Hermitian supersymmetry operator, a fact with momentous consequences.

The contrast with superconformal symmetry will be quite striking.

3.4 Superconformal algebras

In a theory with only massless particles the CM-OR/HLS theorem allows a generalization of $iso(1, d-1)$ to $so(2, d)$.

A superconformal algebra is a super-Lie algebra which contains the conformal algebra of Minkowski space as a subalgebra and whose odd part is a spinorial representation of the conformal algebra. More precisely we will make a

Definition A d-dimensional *superconformal Lie algebra* is a super Lie algebra $\mathfrak{SC} = \mathfrak{SC}^0 \oplus \mathfrak{SC}^1$ over $\mathbb{R}$ such that

1. $\mathfrak{SC}^0 = so(2, d) \oplus \mathfrak{k}$ for some $d \geq 1$ with $\mathfrak{k}$ a compact real Lie algebra.

2. $\mathfrak{SC}^1$ is a real spinorial representation of $so(2, d)$

3. $\mathrm{Sym}^2 \mathfrak{SC}^1 \to \mathfrak{SC}^0$ is nonvanishing

♣Again, should we demand nondegenerate or positive pairing? ♣

Nahm's theorem [146] gives a clean classification. He proves a Lemma:

Lemma. Suppose $\mathfrak{SC}$ is a superconformal algebra. Then, $\mathfrak{SC} = \mathfrak{g}_s \oplus \mathfrak{z}$ where $\mathfrak{g}_s$ is a *simple* super Lie algebra and $\mathfrak{z}$ is central.

♣Clarify what Nahm means by "up to possible extension by an algebra of outer automorphisms. ♣

The idea of the proof is to use the fact that the commutator gives a positive definite form on $\mathfrak{SC}^1$. Then we study the center and note that

a.) $Z(\mathfrak{SC}) \subset \mathfrak{SC}^0$

b.) If $W \subset \mathfrak{SC}^1$ is invariant under $so(2, d)$ then $[so(2, d), [W, W]] = so(2, d)$.

♣Need to clarify his argument for this. ♣

c.) Therefore the maximal soluble ideal C of $\mathfrak{SC}$ must be even and therefore $[C, \mathfrak{SC}^1] = 0$.

d.) Next $\mathfrak{k} = \mathfrak{r} \oplus \mathfrak{a}$ where $\mathfrak{r}$ is semisimple and $\mathfrak{a}$ is abelian. So $C \subset \mathfrak{a}$ and hence $C = Z(\mathfrak{SC})$.

Now, one uses the important classification of Kac of simple super Lie algebras. See [117, 118, 119] and [152, 153]. From that classification we see that the condition that $\mathfrak{g}^1$ be spinorial is very constraining. In the classical superalgebras $\mathfrak{g}^1$ is a representation whose dimension grows linearly with the rank, but the spinor representations have a dimension which grows exponentially with the rank. Now a case-by-case analysis proves Nahm's theorem:

Theorem: The complete list of superconformal algebras is:

1. 7 types for $d = 2$, one type has continuous deformations.

2. $d = 3$: $osp(N|4) \cong [so(N) \oplus so(2, 3)]^0 \oplus (N, 4)$, $N \geq 1$

3. $d = 4$: $u(2,2|N) \cong [so(2,4) \oplus u(N)]^0 \oplus [(4,N) \oplus (\bar{4},\bar{N})]_{\mathbb{R}}$, $N \geq 1$,

4. $d = 4$: $psu(2,2|4) \cong [so(2,4) \oplus su(4)]^0 \oplus [(4,4) \oplus (\bar{4},\bar{4})]_{\mathbb{R}}$,

5. $d = 5$: $osp(2,5|2) = [so(2,5) \oplus usp(2)]^0 \oplus (8,2)_{\mathbb{R}}$,

6. $d = 6$: $osp(2,6|2k) = [so(2,6) \oplus usp(2k)]^0 \oplus (8,2k)_{\mathbb{R}}$, $k = 1,2,3,\ldots$

Remarks:

1. The notation $usp(2k)$ means the real Lie algebra of the compact symplectic group of $k \times k$ unitary matrices over the quaternions. It is often denoted $sp(k)$.

2. We are also taking some liberties with the notation. The correct superalgebra with the correct real structure is in fact $osp(8^*|2k)$. Note that the vector of $so(2,6)$ is isomorphic to $\mathbb{R}^8$ but we use the spinor $\Delta_\pm \cong \mathbb{H}^4$, so the odd generators of $osp(2,6|2k)$ are in the wrong representation. Traditionally people write $osp(2,6|2k)$, however.

3. The algebras in $d = 6$ are called $(k,0)$ superalgebras. The odd generators are in a chiral spinor representation of $so(2,6)$ and therefore we can distinguish $(k,0)$ superalgebras from $(0,k)$ superalgebras. Note there is no $(1,1)$ superconformal algebra.

4. The existence of these superconformal algebras all rely on special properties of Lie algebras giving isomorphisms between vector-like representations of sl or so and spinorial representations. The magic stops in 8 dimensions with $so(6,2)$ with the triality automorphism.

5. In physics there is an important distinction between Q and S supersymmetries. We diagonalize the scaling operator D acting on the odd part. The eigenspace $[D,Q] = -\frac{1}{2}Q$ are the Q supersymmetries and the eigenspace $[D,S] = +\frac{1}{2}S$ are the S supersymmetries. Since $[Q,Q] \sim P$ the Q supersymmetries are called *Poincaré* supersymmetries. Note that $[K,Q] \sim S$ and $[S,S] \sim K$.

3.5 Six-dimensional superconformal algebras

We are focusing on six dimensions, and hence we should consider the Lie superalgebra $osp(2,6|2k)$. It turns out that the representations of this algebra require particles of spin ≥ 2 for $k > 2$. ("Spin" is defined by the weights of the representations of $Spin(2)$ subgroups of the rotation subgroup $Spin(4)$.) It is thought that interacting field theories of higher spin particles with a finite number of fields cannot exist and hence there is a distinguished, largest, superconformal algebra $osp(2,6|4)$. This is known as the "$(2,0)$ algebra" in physics.

In this section we will describe that Lie superalgebra in a bit more detail.

3.5.1 Some group theory

Let us prepare with a bit of group theory first.

We denote real Clifford algebras by $Cl(t_+, s_-)$. The irreducible spinor representation is denoted by $\Delta(t_+, s_-)$ if nonchiral and $\Delta(t_+, s_-)_\pm$ where the subscript refers to the chirality, that is, the sign of the volume operator. [11]

We begin with the $\mathbb{Z}_2$-graded Clifford algebras over the real numbers:

$$Cl(2_+, 6_-) \cong Cl(2_-, 6_+) \cong Cl(2_+, 2_-) \otimes Cl(4_\pm) \cong \mathrm{End}(\mathbb{R}^{4|4}) \otimes \mathbb{H} \tag{3.7}$$

where all tensor products are $\mathbb{Z}_2$=graded tensor products of algebras, $\mathbb{H}$ is the even algebra of quaternions, and $\mathrm{End}(\mathbb{R}^{m|n})$ is the $\mathbb{Z}_2$-graded algebra of endomorphisms of the graded vector space $\mathbb{R}^{m|n}$. [12] It is immediate that the even subalgebra is $\mathbb{H}(4) \oplus \mathbb{H}(4)$ and hence there are two spinor representations $\Delta_\pm \cong \mathbb{H}^4$. The sign refers to the value of the Clifford volume element $\omega_{2,6}$ acting on the representation. In physics this is called the *chirality.*

Viewing $\Delta(2,6)_+$ as a complex 8-dimensional representation of $so(2,6)$ we have the Clebsch-Gordon decompositions:

$$\mathrm{Sym}^2(\Delta(2,6)_+) \cong \mathbb{C} + (\Lambda^4\mathbb{C}^8)^+ \cong 1 + 35 \tag{3.8}$$

$$\Lambda^2(\Delta(2,6)_+) \cong \Lambda^2\mathbb{C}^8 \cong so(6,2) \otimes \mathbb{C} \tag{3.9}$$

When we reduce to super-Poincaré subalgebras we will need the Clifford algebras

$$Cl(1_+, 5_-) \cong Cl(1_-, 5_+) \cong Cl(1_+, 1_-) \otimes Cl(4_\pm) \cong \mathrm{End}(\mathbb{R}^{2|2}) \otimes \mathbb{H} \tag{3.10}$$

and hence the even subalgebra is $\mathbb{H}(2) \oplus \mathbb{H}(2)$. It follows that the irreducible spinor representations of $so(1,5)$ are $\Delta(1,5)_\pm \cong \mathbb{H}^2$.

We will also need to know how the spinor representations $\Delta_\pm$ of $so(2,6)$ pull back to spinors of $so(1,5)$ under the obvious embedding. The volume elements are related by $\omega_{2,6} = \omega_{1,5}\omega_{1,1}$ and $\omega_{1,1}^2 = 1$ so under

$$so(2,6) \hookleftarrow so(1,5) \oplus so(1,1) \tag{3.11}$$

the chiral spinors pull back as

$$\Delta(2,6)_\pm \to \Delta(1,5)_{+,-\frac{1}{2}} \oplus \Delta(1,5)_{-,+\frac{1}{2}} \tag{3.12}$$

Thus the Q supersymmetries have one chirality and the S-supersymmetries have the other chirality.

We will need to know the Clebsch-Gordon decompositions:

$$\Lambda^2\Delta(1,5)_\pm \cong 6 \tag{3.13}$$

$$\mathrm{Sym}^2\Delta(1,5)_\pm \cong (\Lambda^3\mathbb{C}^6)^\pm = 10 \tag{3.14}$$

[11] This requires a choice of orientation, and when $r - s = 2\mathrm{mod}4$ an arbitrary choice of normalization of ω.

[12] Do note confuse $\mathbb{R}^{1,d-1}$, a vector space with Lorentz-signature metric with the superspace $\mathbb{R}^{1|d-1}$ with a one-dimensional even subspace and a $(d-1)$-dimensional odd subspace!

Finally, we will use the special isomorphism for the R-symmetry $usp(4) \cong so(5)$. Accordingly we should know that

$$Cl(5_\pm) \cong Cl(3_\mp) \otimes \mathrm{End}(\mathbb{R}^{1|1}) \tag{3.15}$$

from which one finds

$$Cl(5_\pm)^0 \cong \mathbb{H}(2) \tag{3.16}$$

so the irreducible spinor of $so(5)$ (isomorphic to the vector of $usp(4)$) is $\Delta(5) \cong \mathbb{H}^2$.

We will need to know the Clebsch-Gordon decompositions:

$$\mathrm{Sym}^2(\Delta(5)) = so(5) \tag{3.17}$$

$$\Lambda^2(\Delta(5)) = 1 + 5 \tag{3.18}$$

3.5.2 The $(2,0)$ superconformal algebra $\mathfrak{SC}(\mathbb{M}^{1,5|32})$

The superconformal algebra $\mathfrak{SC}(\mathbb{M}^{1,5|32})$ is usually referred to in the physics literature as the $(d = 6)$ $(2,0)$ algebra. [13] To lighten the notation slightly we will just write $\mathfrak{SC}$ in this section.

We have

$$\mathfrak{SC}^0 = so(2,6) \oplus usp(4) = so(2,6) \oplus so(5) \tag{3.19}$$

$$\mathfrak{SC}^1 = \begin{cases} (\Delta_+(2,6) \otimes_{\mathbb{C}} \Delta(5))_{\mathbb{R}} & (2,0) \quad \text{algebra} \\ (\Delta_-(2,6) \otimes_{\mathbb{C}} \Delta(5))_{\mathbb{R}} & (0,2) \quad \text{algebra} \end{cases} \tag{3.20}$$

N.B. $\Delta_\pm(2,6)$ are pseudo-real: As a complex vector space it is 8-dimensional, but since it is actually quaternionic there is a multiplication by the quaternion $\mathfrak{j}$ which acts as an anti-linear operator J squaring to -1. Δ' is also pseudo-real, and as a complex vector space it is 4-dimensional. We take the tensor product over $\mathbb{C}$ to get a 32-dimensional complex vector space but then the product of pseudoreal structures $J_1 \otimes J_2$ is a *real structure* on the tensor product and we use this projection to get 32 real spinors. In physics the reality conditions are expressed by a symplectic Majorana-Weyl condition.

$$(\mathcal{Q}_{\alpha i})^\dagger = J^{\alpha\beta} J^{ij} \mathcal{Q}_{\beta j} \qquad \alpha, \beta = 1, \ldots, 8; \quad i, j = 1, \ldots, 4 \tag{3.21}$$

From (3.8), (3.9) and (3.17), (3.18) the symmetric product of $\mathrm{Sym}^2(\Delta(2,6)_+ \otimes_{\mathbb{R}} \Delta(5))_{\mathbb{R}}$ has an equivariant projection to $\mathfrak{SC}^0$:

$$\begin{aligned} \mathrm{Sym}^2(\Delta(2,6)_+ \otimes_{\mathbb{R}} \Delta(5)) &\cong \mathrm{Sym}^2(\Delta(2,6)_+) \otimes \mathrm{Sym}^2(\Delta(5)) \oplus \Lambda^2(\Delta(2,6)_+) \otimes \Lambda^2(\Delta(5)) \\ &\cong (1 + 35) \otimes so(5)_R \oplus so(2,6) \otimes (1 + 5) \\ &\to 1 \otimes so(5)_R \oplus so(2,6) \otimes 1 \end{aligned} \tag{3.22}$$

where in the last line we project onto the singlet representation as indicated.

Remarks:

[13] The terminology is ambiguous, because there is a good deal of interesting literature on the $d = 2$ $(2,0)$ superconformal algebra also.

1. As far as I know there is no simple conceptual explanation that the FFF identity is satisfied other than that $osp(2,6|4)$ exists.

2. Since the spinor is chiral we can define the $(2,0)$ algebra with chirality Δ_+ and the $(0,2)$ algebra with chirality Δ_-. There is no $(2,2)$ superconformal algebra, but there is a $(2,2)$ super-Poincaré algebra.

3.6 $d = 6$ $(2,0)$ super-Poincaré $\mathfrak{SP}(\mathbb{M}^{1,5|16})$ and the central charge extensions

We will perturb in various ways away from the superconformal fixed points.

If the perturbation (e.g. moving in the "Coulomb branch" below) preserves Poincaré symmetry then a super-Poincaré supersymmetry is preserved. The generators D and K are broken and the even subalgebra will be

$$\mathfrak{SP}^0 = iso(1,5) \oplus so(5) \tag{3.23}$$

and the odd subalgebra

$$\mathfrak{SP}^1 = (\Delta(1,5)_+ \otimes_{\mathbb{C}} \Delta(5))_{\mathbb{R}} \tag{3.24}$$

Heeding the warning below (3.20) we see that $\mathfrak{g}_p^1$ is a 16-dimensional complex vector space with a reality condition (symplectic Majorana-Weyl condition) leading to 16 real supercharges.

Note that $[K,Q] \sim S$, so if conformal symmetry is broken to Poincaré symmetry then under the decomposition (3.12) the S-supersymmetries are also broken.

The superPoincaré subalgebra can be enlarged by central charges. That is because, by (3.13),(3.14),(3.17),(3.18) we have

$$\mathrm{Sym}^2 \mathfrak{SP}^1 = \mathbb{C}^6 \oplus \Lambda^2\mathbb{C}^6 \oplus (\Lambda^3\mathbb{C}^6)^+ \tag{3.25}$$

In physics notation, the super Poincaré algebra can be extended by "central charges". We can simultaneously symmetrize or anti-symmetrize the spin and R-symmetry indices, and there is an antisymmetric invariant tensor for $spin(5)$, C^{ij} so we can write

$$\{Q_\alpha^i, Q_\beta^j\} = J^{ij} P_{\alpha\beta} + Z_{[\alpha\beta]}^{[ij]} + \tilde{Z}_{(\alpha\beta)}^{(ij)} \tag{3.26}$$

or, putting in Lorentz indices:

$$\{Q_\alpha^i, Q_\beta^j\} = J^{ij}\gamma^\mu_{\alpha\beta} P_\mu + \gamma^\mu_{\alpha\beta}\gamma^{ij}_a Z^a_\mu + \gamma^{\mu\nu\rho}_{\alpha\beta}\gamma^{ij}_{ab}\tilde{Z}^{ab}_{\mu\nu\rho} \tag{3.27}$$

The extending operators Z and $\tilde{Z}$ are in the representations $\mathbb{R}^6 \otimes \mathbb{R}^5$ of $so(1,5) \oplus so(5)$ and $(\Lambda^3\mathbb{R}^6)^+ \otimes \Lambda^2\mathbb{R}^5$ respectively, and hence definitely not central! This is a misnomer preserved in the physics literature for historical reasons.

♣Say what Z commutators with things are ♣

Physically, the existence of these "central charges" is of great significance. The existence of Z^a_μ suggests that theories with $(2,0)$ super Poincaré symmetry can admit (BPS) string-like (1-brane) objects (or defects), and the existence of $Z^{ab}_{\mu\nu\rho}$ shows that such theories can also admit four-dimensional, i.e. codimension two, (3-brane) BPS objects (or defects). The former possibility was emphasized in [Seiberg-Witten, strings] and the latter was stressed in [114]. Both the 1-branes (and 2-dimensional defects) and 3-branes (and 4-dimensional defects) will be very important in our story. The reason for this is that in supersymmetric theories with p-branes described by a "Green-Schwarz action" one finds indeed p-form "central charges" extending the superPoincaré algebra [21].

3.7 Compactification and preserved supersymmetries

The above discussion relies on the automorphisms of Minkowski space $\mathbb{M}^{1,5}$ or, better, its supermanifold extension.

In these notes we will want to study theories on

$$\mathbb{M}^{1,s} \times K \tag{3.28}$$

for various compact (or even just cocompact) manifolds K of dimension s. Here we would like to make some comments on preserved subalgebras of the supersymmetry algebras.

In order to define a theory $\mathcal{F}/K$ or $\mathcal{F}//K$ we need to choose (in general) a product Riemannian structure on (3.28). In addition, for all the global symmetries of the theory - which include the R-symmetries of the superalgebra - we need to choose a bundle with connection. Denoting the global symmetry by H we need to choose (P_H, ∇_H). Fields and operators will be sections of associated bundles to P_H in various representations.

Now, when we compactify with these structures we generally break symmetries in the theory. The preserved symmetries are associated to covariantly constant quantities. It is clear that the isometries of $iso(\mathbb{M}^{1,s})$ are unbroken, but it is also clear that for example, in a superconformal theory the dilations and all superconformal generators K are broken by the compactification. It follows that the S-supersymmetries are broken. Moreover, generically, rotations in the structure group of TK are also broken, and translations along K are broken unless there are suitably covariantly constant vectors on K.

What about the Poincaré Q-supersymmetries? As we have said above, part of the data of the theory requires us to choose a bundle P_H for the global symmetry group H with connection ∇_H. Restricting attention to the R-symmetry group G_R we have P_R and ∇_R. The Poincaré supercharges are in an associated bundle to

$$P_{spin}(X_6) \times_{X_6} P_R, \tag{3.29}$$

where P_R is the R-symmetry bundle. We can view $\mathfrak{SP}^1$ (or rather its dual) as defining the representation for an associated bundle. The covariantly constant spinors in this bundle lead to preserved supersymmetries

$$\epsilon \cdot Q \qquad \nabla\epsilon = 0 \tag{3.30}$$

Example 1: $s = 5$, $K = S^1$; P_R, ∇_R trivial. Now the preserved part of the Poincaré supersymmetry has even part

$$\mathfrak{SP}^0 = iso(1,4) \oplus \mathbb{R} \oplus so(5)_R \tag{3.31}$$

The summand $\mathbb{R}$ is for translations along K. This becomes an *internal* symmetry of the theory, whose characters can be interpreted as a charge.

There is clearly a covariantly constant spinor along the circle if we choose the periodic (a.k.a. nonbounding, nontrivial, Ramond) spin structure. Under the inclusion

$$iso(1,5) \hookleftarrow iso(1,4) \tag{3.32}$$

the chiral spinors $\Delta_\pm(1,5)$ both pull back to the (nonchiral) irreducible spinor $\Delta(1,4)$ of $iso(1,4)$:

$$\Delta_\pm(1,5) \to \Delta(1,4) \tag{3.33}$$

and hence the odd part of the superalgebra will be

$$\mathfrak{S}\mathfrak{P}^1 = (\Delta(1,4) \otimes \Delta(5))_{\mathbb{R}} \tag{3.34}$$

Again $\Delta(1,4) \cong \mathbb{H}^2$ is pseudoreal so there are 16 real supercharges.

The resulting superalgebra appears in 5-dimensional super-Yang-Mills theory which will be important below.

Example 2: $s = 4$, $K = S^1 \times S^1$; P_R, ∇_R trivial. We choose RR spin structure. The discussion here is quite similar to the above. Now we have

$$\mathfrak{S}\mathfrak{P}^0 = iso(1,3) \oplus \mathbb{R}^2 \oplus so(5)_R \tag{3.35}$$

and the Q-susy's pull back as

$$iso(1,4) \hookleftarrow iso(1,3) \tag{3.36}$$

$$\Delta_\pm(1,5) \to \Delta(1,3)_+ \oplus \Delta(1,3)_- \tag{3.37}$$

where $\Delta(1,3)_\pm \cong \mathbb{C}^2$ are the spinor reps of $so(1,3) \cong sl(2,\mathbb{C})$ and $\Delta(1,3)_- \cong \Delta(1,3)_+^*$. Therefore we have

$$\mathfrak{S}\mathfrak{P}^1 = \big(\Delta(1,3)_+ \otimes \Delta(5) \oplus \Delta(1,3)_+^* \otimes \Delta(5)\big)_{\mathbb{R}} \tag{3.38}$$

giving a Poincaré superalgebra with 16 supersymmetries.

3.7.1 Emergent symmetries

At this point we must make an important remark. The four-dimensional superPoincaré algebra we obtained above is *not* the famous superalgebra $\mathfrak{S}\mathfrak{P}(\mathbb{M}^{1,3|16})$. The latter has an $so(6) = su(4)$ R symmetry. The enhancement of $so(5) \to so(6)$ is an example of an *emergent symmetry* in the IR theory. Moreover, the famous $d = 4, \mathcal{N} = 4$ supersymmetric Yang-Mills theory, which is obtained by low energy compactification of the d=6 (2,0) theory has *superconformal* symmetry. The S-supersymmetries in the IR are also emergent.

♣Can we view $so(5) \oplus \mathbb{R}^2$ as a Wigner contraction of $so(6)$? ♣

3.7.2 Partial Topological Twisting

In general, if K is a curved manifold there will be no covariantly constant spinors in the associated $\mathfrak{S}\mathfrak{P}^1$ bundle on K and hence there will be no (guaranteed) preserved supersymmetries in the theory $\mathcal{F}//K$ (there might be emergent supersymmetries).

However, there is a beautiful procedure, introduced by Edward Witten in [172] called *topological twisting*. Choose an embedding of the holonomy group of X in the R-symmetry group,

$$\mu : Hol(X) \to G_R \tag{3.39}$$

(in general such an embedding does not exist, but suppose that one does). Next we use it to define a principal G_R bundle - which we will call the *R-symmetry bundle* P_R. The principal bundle $P_{spin}(X)$ (i.e. the double-cover of the oriented frame bundle) has reduction of structure group to $Hol(X)$. We use μ to define a G_R principal bundle with reduction of structure group to $Hol(X)$. Thus we have a bundle

$$P_{spin}(X) \times P_R \tag{3.40}$$

with reduction of structure group to $Hol(X)$. Moreover, we endow P_R with a connection ∇_R so that

$$\nabla^{LC} = \mu^*(\nabla_R) \tag{3.41}$$

In this case, there might be covariantly constant spinors in $\mathfrak{S}\mathfrak{P}^1$. Note that the R-symmetry connection is an external gauge field which is part of the data for the problem. It is not integrated over in the path integral.

In practical terms then, to see what kinds of topological twistings are possible we choose a subgroup $H \subset Spin(1, d-1)$ and choose embeddings

$$H \hookrightarrow Spin(1, d-1) \times G_R \tag{3.42}$$

by taking $g \in H \mapsto (g, \mu(g))$. Next we consider $\mathfrak{S}\mathfrak{P}^1$ as a representation of H. The H-invariant subspace will be the space of preserved supersymmetries when we compactify on manifolds with holonomy group $Hol(X) = H$ if we also couple to an R-symmetry connection as above.

Example: An important example for the following chapters is the following. We will consider the $(2,0)$ theory on $X_6 = \mathbb{M}^{1,3} \times C$ where C is a Riemannian surface, possibly with punctures. The metric reduces the structure algebra to

$$so(1,5) \hookleftarrow so(1,3) \oplus so(2)_{st} \tag{3.43}$$

where $so(2)_{st}$ is the Lie algebra of the structure group $SO(2)_{st}$ of the tangent bundle TC.

Now *choose* an embedding

$$so(5)_R \hookleftarrow so(3)_R \oplus so(2)_R \tag{3.44}$$

We choose a principal $Spin(2)$-bundle $P_R \to C$ so that we can reduce the structure algebra of $P_{spin}(C) \times P_R$ to the diagonal subalgebra

$$so(2)_D \hookrightarrow so(2)_{st} \oplus so(2)_R \tag{3.45}$$

and we choose ∇_R to be equal to the Levi-Civita spin connection on C.

The preserved supersymmetries will have even part:

$$\mathfrak{SP}^0 = iso(1,3) \oplus \oplus so(3)_R \tag{3.46}$$

where the $so(2)_D \oplus so(3)_R$ will be considered to be global R-symmetries. To find the preserved odd generators we look for the $so(2)_D$ invariants in the decomposition

$$[\Big(\Delta(1,3)_{+,\frac{1}{2}} \oplus \Delta(1,3)_{-,-\frac{1}{2}}\Big) \otimes \Big(\Delta(3)_{\frac{1}{2}} \oplus \Delta(3)_{-\frac{1}{2}}\Big)]_{\mathbb{R}} \tag{3.47}$$

to give

$$\mathfrak{SP}^1 = [\Delta(1,3)_+ \otimes \Delta(3) \oplus \Delta(1,3)_- \otimes \Delta(3)]_{\mathbb{R}} \tag{3.48}$$

The super algebra given by (3.46) and (3.48) is the standard $d=4$ $\mathcal{N}=2$ superPoincaré algebra.

3.7.3 Embedded Four-dimensional $\mathcal{N}=2$ algebras and defects

There is an extension of the topological twisting explained above which is even valid at the superconformal level. This is important for the existence of codimension two defects which preserve superconformal symmetry.

As before we choose an embedding

$$so(2,4) \oplus so(2)_{st} \hookrightarrow so(2,6) \tag{3.49}$$

by singling out two directions, say x_4, x_5 in $\mathbb{M}^{1,5}$. These are the directions transverse to the codimension two defect.

As before we *choose* an embedding

$$so(3)_R \oplus so(2)_R \hookrightarrow so(5)_R \tag{3.50}$$

and then consider the *diagonal* embedding

$$so(2)_D \hookrightarrow so(2)_{st} \oplus so(2)_R \tag{3.51}$$

Now we can define an embedding of a four-dimensional superconformal algebra into the six dimensional $(2,0)$ superconformal algebra

$$su(2,2|2) \hookrightarrow osp(2,6|4) \tag{3.52}$$

by taking

$$\mathfrak{SC}^0 = so(2,4) \oplus so(2)_D \oplus so(3)_R \tag{3.53}$$

while the odd part are the invariants under $so(2)_D$:

$$\mathfrak{SC}^1 = [\Delta_+(2,4) \otimes \Delta(3) \oplus \Delta_-(2,4) \otimes \Delta(3)]_{\mathbb{R}} \tag{3.54}$$

Using the group theory reviewed above and the specific projection (3.22) one can show that the symmetric pairing used in the six-dimensional superconformal algebra indeed restricts to the one appropriate for the four-dimensional $\mathcal{N}=2$ superconformal algebra.

♣What about K-transformations in the presence of defects. Aren't these broken? ♣

3.8 Unitary Representations of the $(2,0)$ algebra

This is where we could explain about short representations and their characters.

Important for discussion of local operators of the theory.

TM of (2,0)

Say something about the definition of chiral operators.

3.9 List of topological twists of the $(2,0)$ algebra

There are a finite number of possible holonomy algebras in $so(6)$.

There are a finite number of embeddings of these algebras in $so(5)_R$.

So it is possible to list them:

1. For a general six-manifold there is no embedding of the $so(6)$ holonomy algebra into the $so(5)$ R-symmetry algebra: There is to topological field theory on a general Riemannian six-manifold.

2. For $\mathbb{M}^{1,0} \times M_5$ with M_5 a general five-manifold there is one embedding where we consider the diagonal $so(5)_D \hookrightarrow so(5)_{st} \oplus so(5)_R$. Pulling back the spinor representations and looking for singlets we find exactly one topological supersymmetry.

3. For a $\mathbb{M}^{1,1} \times M_4$ with M_4 a general 4-manifold the holonomy group has algebra $\mathfrak{h} = so(4) \cong su(2)_L \oplus su(2)_R$. Under

$$so(1,1) \oplus so(4) \hookrightarrow so(1,5) \tag{3.55}$$

we have

$$(2,1)_{+\frac{1}{2}} \oplus (1,2)_{-\frac{1}{2}} \leftarrow 4_+ \cong \Delta(1,4)_+ \tag{3.56}$$

where the subscript indicates the weight under $so(1,1)$, the Lorentz group of $\mathbb{M}^{1,1}$. Note that this is *not* part of the holonomy group we use for topological twisting, but we keep track of it to keep track of the chirality of the preserved supersymmetries.

Now, for the embedding $\mu : so(4) \to so(5)$ there is only one choice and it is the obvious one. The 5 pulls back to $4+1$ and the spinor pulls back as

$$(2,1) \oplus (1,2) \leftarrow 4 \cong \Delta(5) \tag{3.57}$$

The invariants in the tensor product are $(1,1)_{+\frac{1}{2}} \oplus (1,1)_{-\frac{1}{2}}$ giving $(1,1)$ supersymmetry in $\mathbb{M}^{1,1}$.

If we specialize to Kähler manifolds the holonomy is

$$\mathfrak{h} = u(2) \cong su(2)_L \oplus u(1)_R \subset su(2)_L \oplus su(2)_R \subset so(4) \tag{3.58}$$

Now (3.56) specializes to

$$2^0_{+\frac{1}{2}} \oplus 1^{\frac{1}{2}}_{-\frac{1}{2}} \oplus 1^{-\frac{1}{2}}_{-\frac{1}{2}} \leftarrow 4_+ \cong \Delta(1,4)_+ \tag{3.59}$$

and (3.60) specializes to

$$2^0 \oplus 1^{+\frac{1}{2}} \oplus 1^{-\frac{1}{2}} \leftarrow 4 \cong \Delta(5) \tag{3.60}$$

taking the tensor product and looking for $\mathfrak{h}$ singlets we find

$$1^0_{+\frac{1}{2}} \oplus 1^0_{-\frac{1}{2}} \oplus 1^0_{-\frac{1}{2}} \tag{3.61}$$

giving $(1,2)$ supersymmetry.

If we specialize to hyperkähler manifolds then the holonomy algebra is just $\mathfrak{h} = su(2)_L$. Now we have

$$2_{+\frac{1}{2}} \oplus 1_{-\frac{1}{2}} \oplus 1_{-\frac{1}{2}} \leftarrow 4_+ \cong \Delta(1,4)_+ \tag{3.62}$$

and

$$2 \oplus 1 \oplus 1 \leftarrow 4 \cong \Delta(5) \tag{3.63}$$

and the space of $\mathfrak{h}$-invariants in the tensor product of (3.62) and (3.63) gives $(1,4)$ supersymmetry.

♣There might be other embeddings of the holonomies of kahler and hk into the R-symmetry group ♣

4. For $\mathbb{M}^{1,2} \times M_3$ where M_3 has general $so(3)$ holonomy we take the standard embedding

$$so(1,2) \oplus so(3) \hookrightarrow so(1,5) \tag{3.64}$$

$$(2,2) \leftarrow 4 \cong \Delta(1,5)_+ \tag{3.65}$$

and choosing

$$so(3) \hookrightarrow so(3) \oplus so(2) \hookrightarrow so(5) \tag{3.66}$$

$$2 \oplus 2 \leftarrow 4 \cong \Delta(5) \tag{3.67}$$

giving holonomy singlets $(1,2) \oplus (1,2)$, that is, four preserved supersymmetries.

♣other embeddings? ♣

4. For $\mathbb{M}^{1,3} \times M_2$: Discussed above.

5. In addition we could have $M_2 \times M_4$ - Donaldson twist in class S. etc.

4. Four-dimensional BPS States and (primitive) Wall-Crossing

4.1 The $d = 4, \mathcal{N} = 2$ super-Poincaré algebra $\mathfrak{SP}(\mathbb{M}^{1,3|8})$.

We now write out the $\mathcal{N} = 2$ superalgebra much more explicitly. We mostly follow the conventions of Bagger and Wess [22] for $d = 4, \mathcal{N} = 1$ supersymmetry. In particular $SU(2)$ indices are raised/lowered with $\epsilon^{12} = \epsilon_{21} = 1$. Components of tensors in the irreducible spin representations of $so(1,3)$ are denoted by $\alpha, \dot{\alpha}$ running over $1,2$. The rules for conjugation are that $(\mathcal{O}_1\mathcal{O}_2)^\dagger = \mathcal{O}_2^\dagger \mathcal{O}_1^\dagger$ and $(\psi_\alpha)^\dagger = \bar{\psi}_{\dot{\alpha}}$.

The $d = 4, \mathcal{N} = 2$ supersymmetry algebra will be denoted by $\mathfrak{s}$. It has even and odd parts:

$$\mathfrak{s} = \mathfrak{s}^0 \oplus \mathfrak{s}^1 \tag{4.1}$$

where the even subalgebra is

$$\mathfrak{s}^0 = iso(1,3) \oplus su(2)_R \oplus u(1)_R \oplus \mathbb{C} \tag{4.2}$$

and the odd subalgebra, as a representation of $\mathfrak{s}^0$ is

$$\mathfrak{s}^1 = \left[(2,1;2)_{+1} \oplus (1,2;2)_{-1}\right] \tag{4.3}$$

A basis for the odd superalgebra is usually denoted:
Q_α^A, $\bar{Q}_{\dot\alpha}^A$,
The reality constraint is:

$$(Q_\alpha{}^A)^\dagger = \bar{Q}_{\dot\alpha A} := \varepsilon_{AB}\bar{Q}_{\dot\alpha}^B \tag{4.4}$$

The commutators of the odd generators are:

$$\begin{aligned} \{Q_\alpha{}^A, \bar{Q}_{\dot\beta B}\} &= 2\sigma^m_{\alpha\dot\beta}P_m\delta^A_B \\ \{Q_\alpha{}^A, Q_\beta{}^B\} &= 2\epsilon_{\alpha\beta}\epsilon^{AB}\bar{Z} \\ \{\bar{Q}_{\dot\alpha A}, \bar{Q}_{\dot\beta B}\} &= -2\epsilon_{\dot\alpha\dot\beta}\epsilon_{AB}Z \end{aligned} \tag{4.5}$$

Remarks:

1. The last summand in $\mathfrak{s}^0$ is the central charge Z.
2. P_m is the Hermitian energy-momentum vector with $P^0 \geq 0$.
3. The commutators of the even generators with the odd generators are indicated by the indices. In particular, $SU(2)_R$ rotates the index A.
4. Under the $u(1)_R$ symmetry Q_α^A has charge $+1$ and hence $\bar{Q}_{\dot\alpha}^A$ has charge -1. The $u(1)_R$ symmetry can be broken explicitly by couplings in the Lagrangian, spontaneously by vevs, or it can be anomalous.
5. In supergravity one sometimes does not have $su(2)_R$ symmetry.

4.2 BPS particle representations of four-dimensional $\mathcal{N} = 2$ superpoincaré algebras

It is not our purpose in these notes to give a systematic catalogue of the representations of all the relevant super algebras, but for the $d = 4$ $\mathcal{N} = 2$ super-Poincaréalgebra some representation theory is absolutely essential for what follows.

4.2.1 Particle representations

The $\mathcal{N} = 2$ supersymmetry algebra acts unitarily on the Hilbert space of our physical theory. Therefore we should understand well the unitary irreps of this algebra.

We will be particularly interested in single-particle representations. We will construct these using the time-honored method of induction from a little superalgebra, going back to Wigner's construction of the unitary irreps of the Poincaré group.

A particle representation is characterized in part by the Casimir $P^2 = M^2$. We will only discuss massive representations with $M > 0$. A massive particle can be brought to rest. It defines a state such that

$$P^m|\psi\rangle = M\delta_0^m|\psi\rangle \tag{4.6}$$

where $M > 0$ is the mass. The little superalgebra is then

$$\mathfrak{s}^0_\ell \oplus \mathfrak{s}^1 \tag{4.7}$$

with $\mathfrak{s}^0_\ell = so(3) \oplus su(2)_R \oplus u(1)_R$. (We will sometimes drop the $u(1)_R$ summand.)

The states satisfying (4.6) form a finite dimensional representation ρ of the little superalgebra. The algebra of the odd generators acting on ρ is that of a *Clifford algebra* and therefore we try to represent that.

To make an irreducible representation of the $Q, \bar{Q}$'s we need to diagonalize the quadratic form on the RHS. This can be done as follows (we will find the following a convenient computation in §8 on line operators):

Let us assume that $Z \neq 0$. (The case $Z = 0$ can be found in Chapter II of Bagger-Wess [22].

A particle at rest at the origin $x^i = 0$ of spatial coordinates is invariant under spatial involution. This suggests we consider the *involution* of the the superalgebra given by parity together with $U(1)_R$ symmetry rotation by a phase: Denote this by $I(\zeta)$. The involution decomposes the supersymmetries into

$$\mathfrak{s}^1 = \mathfrak{s}^{1,+} \oplus \mathfrak{s}^{1,-} \tag{4.8}$$

Define:

$$\mathcal{R}_\alpha{}^A = \xi^{-1} Q_\alpha{}^A + \xi \sigma^0_{\alpha\dot{\beta}} \bar{Q}^{\dot{\beta}A} \tag{4.9}$$

$$\mathcal{T}_\alpha{}^A = \xi^{-1} Q_\alpha{}^A - \xi \sigma^0_{\alpha\dot{\beta}} \bar{Q}^{\dot{\beta}A} \tag{4.10}$$

for the supersymmetries transforming as ± 1 under the involution, respectively. Here ξ is a phase: $|\xi| = 1$ and the R-symmetry rotation is by $\zeta = \xi^{-2}$.

These operators satisfy the Hermiticity conditions of the quaternions:

$$(\mathcal{R}_\alpha{}^A)^\dagger = \epsilon^{\alpha\beta} \epsilon_{AB} \mathcal{R}_\beta{}^B \tag{4.11}$$

or, explicitly:

$$\begin{aligned} (\mathcal{R}_1{}^1)^\dagger &= -\mathcal{R}_2{}^2 \\ (\mathcal{R}_1{}^2)^\dagger &= \mathcal{R}_2{}^1 \end{aligned} \tag{4.12}$$

Then, on V we compute:

$$\{\mathcal{R}_\alpha{}^A, \mathcal{R}_\beta{}^B\} = 4\left(M + \mathrm{Re}(Z/\zeta)\right) \epsilon_{\alpha\beta} \epsilon^{AB} \tag{4.13}$$

$$\{\mathcal{T}_\alpha{}^A, \mathcal{T}_\beta{}^B\} = 4\left(-M + \mathrm{Re}(Z/\zeta)\right) \epsilon_{\alpha\beta} \epsilon^{AB} \tag{4.14}$$

Together with the Hermiticity conditions we now see that

$$\left(\mathcal{R}_1{}^1 + (\mathcal{R}_1{}^1)^\dagger\right)^2 = \left(\mathcal{R}_1{}^2 + (\mathcal{R}_1{}^2)^\dagger\right)^2 = 4(M + \mathrm{Re}(Z/\zeta)) \tag{4.15}$$

Now consider a unitary representation of the superalgebra on which the central charge operator acts as a scalar $Z \in \mathbb{C}$. Since the square of an Hermitian operator must be positive semidefinite we obtain the *BPS bound*:

$$M + \mathrm{Re}(Z/\zeta) \geq 0. \tag{4.16}$$

This bound holds for any ζ, and therefore we can get the *strongest* bound by taking $\zeta = -e^{i\alpha}$, where we define a phase α by

$$Z := e^{i\alpha}|Z| \tag{4.17}$$

With this choice of ζ we get the famous BPS bound:

$$M \geq |Z| \tag{4.18}$$

Moreover, when we make the choice $\zeta = -e^{i\alpha}$, a little computation shows that

$$\begin{aligned}
\{\mathcal{R}_\alpha{}^A, \mathcal{R}_\beta{}^B\} &= 4\,(M - |Z|)\,\epsilon_{\alpha\beta}\epsilon^{AB} \\
\{\mathcal{T}_\alpha{}^A, \mathcal{T}_\beta{}^B\} &= -4\,(M + |Z|)\,\epsilon_{\alpha\beta}\epsilon^{AB} \\
\{\mathcal{R}_\alpha^A, \mathcal{T}_\beta^B\} &= 0
\end{aligned} \tag{4.19}$$

It is the third line which results from this particular choice of ζ. For general phases ζ the third graded commutator would be nonzero.

How shall we construct representations of (4.7) ?

When $M > |Z|$ the representations in this case are known as "non-BPS" or "long" representations of the superalgebra. When $M = |Z|$ they are *BPS* or *short* representations. We will first discuss the case $M > |Z|$ and then indicate the modification for $M = |Z|$.

When $M - |Z| \neq 0$ we can make a suitable positive rescaling to define generators $\hat{\mathcal{R}}$ and $\hat{\mathcal{T}}$ such that

$$\begin{aligned}
\{\hat{\mathcal{R}}_\alpha{}^A, \hat{\mathcal{R}}_\beta{}^B\} &= \epsilon_{\alpha\beta}\epsilon^{AB} \\
\{\hat{\mathcal{T}}_\alpha{}^A, \hat{\mathcal{T}}_\beta{}^B\} &= -\epsilon_{\alpha\beta}\epsilon^{AB} \\
\{\hat{\mathcal{R}}_\alpha^A, \hat{\mathcal{T}}_\beta^B\} &= 0
\end{aligned} \tag{4.20}$$

Then we have two (graded) commuting Clifford algebras $\mathfrak{s}^{1,\pm}$ The representation theory of the $\hat{\mathcal{R}}'s$ and $\hat{\mathcal{T}}'s$ can be considered separately

4.2.2 The half-hypermultiplet

Let us consider the irreducible Clifford module generated by the $\hat{\mathcal{R}}_\alpha^A$. The resulting Clifford module will also be a representation of $so(3) \oplus su(2)_R$. To see this note that we may form

$$T_{(\alpha\beta)} = \frac{1}{2}\mathcal{R}_\alpha{}^A\mathcal{R}_\beta{}^B\epsilon_{AB} \tag{4.21}$$

which generates a copy of $so(3)$ and

$$T^{(AB)} = \frac{1}{2}\mathcal{R}_\alpha{}^A\mathcal{R}_\beta{}^B\epsilon^{\alpha\beta} \tag{4.22}$$

which generates a commuting copy of $su(2)$. The operators (4.21) and (4.22) clearly act on the Clifford module so the Clifford module is an $so(3) \oplus su(2)_R$ module. Furthermore, the operators $\mathcal{R}_\alpha{}^A$ transform under these exactly the same way as they do under $\mathfrak{s}^0_\ell = so(3) \oplus su(2)_R$, namely the $(\frac{1}{2};\frac{1}{2})$, so as a representation of $\mathfrak{s}^0_\ell$ the Clifford module is the same as that given by the action of (4.21) and (4.22).

Since the Clifford generators transform in the $(\frac{1}{2};\frac{1}{2})$ of $su(2) \oplus su(2)$, which is just the vector of $so(4)$, the irreducible Clifford module must be the Dirac spinor which is the $(\frac{1}{2};0) \oplus (0;\frac{1}{2})$. This important representation is known as the *half-hypermultiplet*:

$$\rho_{hh} \cong (0;\frac{1}{2}) \oplus (\frac{1}{2};0). \tag{4.23}$$

Note that it is $\mathbb{Z}_2$ graded with $\rho^0_{hh} \cong (0;\frac{1}{2})$ and $\rho^1_{hh} \cong (\frac{1}{2};0)$. See Figure 2.

A more explicit demonstration goes as follows:

Note that $\mathfrak{s}^{1,+}$ is itself a sum of two (graded) commuting Clifford algebras on two generators. For example $\mathfrak{s}^{1,+}$ is the graded tensor product of the Clilfford algebra

$$(\hat{\mathcal{R}}^1_1)^2 = (\hat{\mathcal{R}}^2_2)^2 = 0 \qquad \& \qquad \{\hat{\mathcal{R}}_1{}^1, \hat{\mathcal{R}}_2{}^2\} = -1 \tag{4.24}$$

with the Clifford algebra

$$(\hat{\mathcal{R}}^2_1)^2 = (\hat{\mathcal{R}}^1_2)^2 = 0 \qquad \& \qquad \{\hat{\mathcal{R}}_1{}^2, \hat{\mathcal{R}}_2{}^1\} = 1 \tag{4.25}$$

and the algebra generated by (4.24) graded commutes with that generated by (4.25).

To construct an explicit module we should choose a Clifford vacuum. Clearly it is natural to regard either $\hat{\mathcal{R}}^1_1$ or $\hat{\mathcal{R}}^2_2$ as a creation operator, and then the other serves as an annihilation operator. A similar remark holds for $\hat{\mathcal{R}}^2_1$ and $\hat{\mathcal{R}}^1_2$. Now consider a Clifford vacuum

$$\mathcal{R}_1{}^A|\Omega\rangle = 0 \qquad A = 1,2 \tag{4.26}$$

The irreducible Clifford representation of (4.24) generated by $|\Omega\rangle$ is the span of

$$\rho_{hh} = Span\{|\Omega\rangle, \mathcal{R}_2{}^1|\Omega\rangle, \mathcal{R}_2{}^2|\Omega\rangle, \mathcal{R}_2{}^1\mathcal{R}_2{}^2|\Omega\rangle\} \tag{4.27}$$

We can compute the representation of $so(3) \oplus su(2)_R$ since $|\Omega\rangle$ is the unique highest weight state of $J_3 \in so(3)_{st}$ and a singlet under $su(2)_R$. But now it is straightforward to see that the representation is $(\frac{1}{2};0) \oplus (0;\frac{1}{2})$.

Of course, the same remarks apply to the irreducible Clifford module representing the algebra of the $\hat{\mathcal{T}}_\alpha{}^A$.

4.2.3 Long representations

It is shown in [22] that the general representation of $[so(3) \oplus su(2)_R] \oplus \mathfrak{s}^{1,+}$ is of the form

$$\rho_{hh} \otimes \mathfrak{h} \tag{4.28}$$

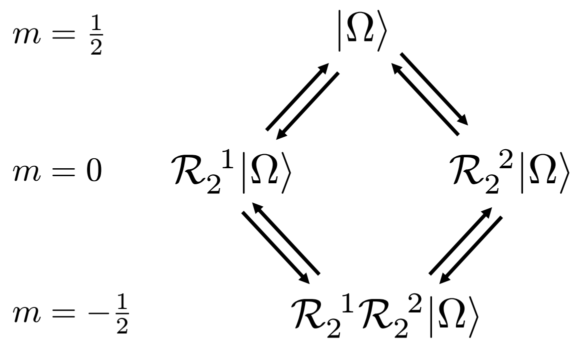

Figure 2: Showing the action of the supersymmetries in the basic half-hypermultiplet representation.

where $\mathfrak{h}$ is an arbitrary representation of $\mathfrak{s}_\ell^0 \cong so(3) \oplus su(2)_R$. [14]

Now, to get representations of the full superalgebra we apply this construction to the Clifford algebras generated by the $\mathcal{R}_\alpha^A$ and the $\mathcal{T}_\alpha^A$, and the general representation of the little superalgebra is of the form

$$\text{LONG} \quad \text{REP}: \qquad \rho_{hh} \otimes \rho_{hh} \otimes \mathfrak{h} \tag{4.29}$$

where $\mathfrak{h}$ is an arbitrary finite dimensional representation of $so(3) \oplus su(2)_R$, and the first and second factors are the half-hypermultiplet representations for $\mathcal{R}$ and $\mathcal{T}$, respectively.

Example: The smallest (long) representation is obtained by taking $\mathfrak{h}$ to be the trivial one-dimensional representation:

$$\rho_{hh} \otimes \rho_{hh} = 2(0;0) \oplus (0;1) \oplus 2(\frac{1}{2};\frac{1}{2}) \oplus (1;0). \tag{4.30}$$

Physically this means the multiplet has 5 scalar fields, 2 of which are singlets under $su(2)_R$ and 3 of which are in a triplet. In addition there are 4 Dirac fermions and finally there is a (massive) spin one field.

[14] An alternative and elegant proof was explained to me by Dan Freed and Andy Neitzke: Suppose A is an algebra with a unique irreducible module V and a group G acts as automorphisms of A. We claim that the most general representation of the semidirect product $\mathbb{C}[G] \times A$ is $V \otimes W$ where A acts on V and W is a representation of a central extension $\tilde{G}$ of G. To prove this note that for every g we can use the automorphism action on A to obtain a new module V^g of A. Shur's lemma then says that $L_g = Hom_A(V, V^g)$ is a one-dimensional complex vector space, i.e. a complex line. Moreover there is a coherent product $L_{g_1} \otimes L_{g_2} \to L_{g_1 g_2}$. This defines a central extension $\tilde{G}$ of G by $\mathbb{C}^*$. Now take any module M for $\mathbb{C}[G] \times A$. As an A module it is of the form $V \otimes W$ where W is a vector space and A acts on V. Now we can once again twist by g to form $M^g := V^g \otimes W$, and the statement that $\tilde{G}$ acts on M is the statement that $g \to Hom_A(W, W^g)$ gives a representation of $\tilde{G}$. On the other hand, again by Schur's lemma $Hom_A(W, W^g) = L_g \otimes \text{End}(W)$ and this shows that W is a representation of $\tilde{G}$.

4.2.4 Short representations of $\mathcal{N}=2$

When the bound (4.18) is saturated something special happens: The quadratic form in the Clifford algebra of the ${\mathcal{R}_\alpha}^A$ degenerates and becomes zero. In a unitary representation these operators must therefore be represented as zero. Such representations are called "short" or *BPS representations*.

Definition: We refer to the ${\mathcal{R}_\alpha}^A$ as *preserved supersymmetries* and the ${\mathcal{T}_\alpha}^A$ as *broken supersymmetries.*

The representations are now "shorter" – since we need only represent the clifford algebra $\mathfrak{s}^{1,-}$ generated by the $\mathcal{T}$'s. Thus we have the BPS or short representations:

$$\text{SHORT} \quad \text{REP}: \qquad \rho_{hh} \otimes \mathfrak{h} \tag{4.31}$$

where $\mathfrak{h}$ is an arbitrary finite-dimensional unitary representation of $so(3) \oplus su(2)_R$.

The two main examples are

1. The simplest representation is the half-hypermultiplet with $\mathfrak{h}$ equals the one-dimensional trivial rep and is just ρ_{hh}. It consists of a pair of scalars in a doublet of $su(2)_R$ and a Dirac fermion.

2. Another important representation is the *vectormultiplet* obtained by taking $\mathfrak{h} = (\frac{1}{2}; 0)$. As a representation of $so(3) \oplus su(2)_R$ it is

$$\rho_{vm} \cong (0;0) \oplus (\frac{1}{2};\frac{1}{2}) \oplus (1;0) \tag{4.32}$$

4.3 Field Representations

The particle representations described above have corresponding free field multiplets. The two most important are:

1. Vectormultiplet: Let $\mathfrak{g}$ be a compact Lie algebra. An $\mathcal{N}=2$ vectormultiplet has a scalar φ, fermions $\psi_{\alpha A}$, in the $(\mathbf{2};\mathbf{1}) \otimes 2$ of $so(1,3) \oplus su(2)_R$, (and their complex conjugates $\bar{\psi}_{\dot{\alpha}A} := ({\psi_\alpha}^A)^\dagger$), an Hermitian gauge field A_m and an auxiliary field $D_{AB} = D_{BA}$ satisfying the reality condition $(D_{AB})^* = -D^{AB}$. After multiplication by i all these fields are valued in the adjoint of $\mathfrak{g}$. Taking the case $\mathfrak{g} = u(1)$ we recognize the particle content of the vm described above.[15]

   USUAL QUARTET PICTURE OF VM FIELDS. SU(2)R ACTS HORIZONTALLY

2. Hypermultiplet: Consists of 2 complex scalar fields, in the spin $\frac{1}{2}$ representation of $SU(2)_R$ and a pair of Dirac fermions which are singlets under $SU(2)_R$. [16]

   USUAL QUARTET PICTURE OF HM FIELDS. SU(2)R ACTS HORIZONTALLY

[15] A possible source of confusion here: These fields correspond to massless particle representations, which we have not discussed.

[16] The half-hypermultiplet has $\dim_c \rho^0 = 2$. That corresponds to two *real* scalar fields. The full hypermultiplet field representation has four *real* scalar fields, and hence the corresponding particle representation has $\dim_c \rho^0 = 4$.

3. In $N = 2$ supergravity, there is in addition, a supermultiplet with the graviton and 2 gravitinos. We will not need the explicit form in these lectures.

In these lectures we are mostly focussing on vectormultiplets. The supersymmetry transformations of the vectormultiplet are:

$$
\begin{aligned}
[Q_{\alpha A}, \varphi] &= -2\psi_{\alpha A} \\
[\bar{Q}_{\dot{\alpha} A}, \varphi] &= 0 \\
[Q_{\alpha A}, A_m] &= i\bar{\psi}_{\dot{\beta} A}(\bar{\sigma}_m)^{\dot{\beta}}{}_{\alpha} \\
[\bar{Q}_{\dot{\alpha} A}, A_m] &= -i(\bar{\sigma}_m)_{\dot{\alpha}}{}^{\beta}\psi_{\beta A} \\
[Q_{\alpha A}, \psi_{\beta B}] &= \sigma^{mn}_{\beta\alpha}F_{mn}\epsilon_{AB} + iD_{AB}\epsilon_{\beta\alpha} + \frac{i}{2}g\epsilon_{\beta\alpha}\epsilon_{AB}[\varphi^\dagger, \varphi] \\
[\bar{Q}_{\dot{\alpha} A}, \psi_{\beta B}] &= -i\epsilon_{AB}\sigma^m_{\beta\dot{\alpha}}D_m\varphi \\
[Q_{\alpha A}, D_{BC}] &= \left(\epsilon_{AB}\sigma^{m\dot{\beta}}_{\alpha}D_m\bar{\psi}_{\dot{\beta} C} + B \leftrightarrow C\right) \\
&\quad + g\left(\epsilon_{AB}[\varphi^\dagger, \psi_{\alpha C}] + B \leftrightarrow C\right)
\end{aligned}
\tag{4.33}
$$

4.4 Families of Quantum Vacua

In d=4 N=2 theories a very important phenomenon is that there is not a unique relativistic vacuum, rather there is a space of vacua, and the properties of the theory change.

This can be rather easily demonstrated, at the classical level, in some of the simplest N=2 theories.

Example: For example, in pure $SU(K)$ $\mathcal{N} = 2$ gauge theory the classical potential energy is just

$$\frac{1}{e^2}\int d^3\vec{x}\mathrm{Tr}\left\{\vec{E}^2 + \vec{B}^2 + \vec{\Pi}^2 + (\vec{D}\varphi)^2 + ([\varphi^\dagger, \varphi])^2\right\} \tag{4.34}$$

and hence the energy is minimized for flat gauge fields (which are therefore gauge equivalent to zero on $\mathbb{R}^3$ with φ a space-time independent normal matrix. Gauge transformations act by conjugation by a unitary matrix and so φ may be diagonalized:

$$\varphi_{vac} = Diag\{a^1, \dots, a^K\} \qquad \sum a^i = 0 \tag{4.35}$$

This leaves unbroken a Weyl group symmetry so the gauge invariant parameters are the $(K-1)$ Casimirs $\mathrm{Tr}\varphi^i = u_i$ for $i = 2, \dots, K$.

In general for $\mathfrak{g}$-gauge theory we have r Casimirs u_i, i running over the degrees of the generators of the ring of invariants. We have classical vacua:

$$\mathcal{B} = \mathfrak{t} \otimes \mathbb{C}/W \tag{4.36}$$

which itself may be identified with a vector space.

These vacua define boundary conditions at infinity for path integrals:

$$u_i = \lim_{\vec{x}\to\infty} \mathrm{Tr}(\varphi(\vec{x}))^i \tag{4.37}$$

for $i = 2, \ldots, K$. The moduli space thus looks like a copy of $\mathbb{C}^{K-1}$.

The very surprising claim - due to Seiberg and Seiberg and Witten - is that, in fact, in the exact quantum theory *this family of vacua is not lifted.* That is there is a family of quantum vacua $|\Omega(u)\rangle$ $u \in \mathcal{B}$ with

$$\langle \Omega(u) | \mathrm{Tr} \varphi^i(x) | \Omega(u) \rangle = u_i \qquad i = 2, \ldots, K \tag{4.38}$$

The argument is based on the strong constraints on N=2 supersymmetric Lagrangians [REF: B. DeWitt and A. van Proeyen; Ferrara et.al.]

1. The N=2 SYM is a symmetry of the quantum theory. It can only be broken if there is an anomaly or it can be broken spontaneously. There is no (perturbative) evidence for anomalies. Therefore we can assume that the N=2 SYM is a symmetry of the IR theory. We assume the IR theory has an action principle. Therefore the LEEA is N=2 supersymmetric.

2. The LEEA does not allow a superpotential which could lift the classical vacua, because that would be inconsistent with the structure of N=2 Lagrangians.

3. We cannot have quantum effects induce FI terms, again because that would be inconsistent with the N=2 structure of the LEEA. The argument here promotes a coupling constant to a vev of a scalar in a weakly gauged vectormultiplet. But the structure of N=2 Lagrangians again forbids the kind of couplings which would lead to such quantum corrections.

4. Therefore there is a family of quantum Coulomb vacua in 1-1 correspondence with the classical vacua.

In fact, again appealing to the general structure of N=2 Lagrangians the exact space of all quantum vacua is a rather complicated stratified space:

$$\mathcal{M} = \amalg_\alpha \left(\mathcal{H}_\alpha \times \mathcal{SK}_\alpha \right) / \sim \tag{4.39}$$

FIRST DESCRIBE THE STRATA AT FIXED ALPHA: Locally a product space.

Here $\mathcal{H}_\alpha$ are called *Higgs branches* where the rank of some simple factor of the gauge group is broken down to a group of smaller rank. In the maximal Higgs branch the gauge group is broken completely. Again by constraints of supersymmetry the Higgs branches are hyperkähler manifolds. In Lagrangian field theory they are hyperkähler quotients. The $\mathcal{SK}_\alpha$ are *special Kähler manifolds.* For a mathematical definition of a special Kahler manifold see [STROMINGER] [76].

For a physicist the term *Coulomb branch* refers to a family of quantum vacua where at low energies there is an unbroken *abelian* gauge theory. The maximal Coulomb branch, $\mathcal{B}$ is the set of vacua in which the rank of the gauge group is preserved: The low energy abelian gauge theory has full rank.

Using the kind of reasoning above it is argued in Section 3 of [17] that in Lagrangian field theories, the hyperkähler metric on the Higgs branch does not receive quantum corrections. The Seiberg-Witten papers show that although the Coulomb branch is the same space quantum mechanically as classically, the special Kähler metric on the branch does receive (important!) quantum corrections.

Indeed, the LEEA can change at a complex codimension one locus

$$\mathcal{B}^{sing} \subset \mathcal{B} \tag{4.40}$$

We define

$$\mathcal{B}^{*} = \mathcal{B} - \mathcal{B}^{sing} \tag{4.41}$$

For us $\mathcal{B}$ is going to be the base of a Hitchin system.

In equation (4.39) the various branches are glued together in complicated ways to make a singular space. For an example of how this arises in $SU(K)$ N=2 QCD see [17].

Remarks: Also we can talk about families of theories by varying coupling constants and masses.

4.5 Families of Hilbert Spaces

Let us now restrict attention to the maximal Coulomb branch $\mathcal{B}$.

The Hilbert spaces will be continuous families of unitary representations of the $d = 4$ $\mathcal{N} = 2$ Poincaré algebra. They will typically be highly reducible, including single-particle representations as well as multi-particle states. We will be focusing on the single-particle spectrum. It defines a family of Hilbert spaces over the family. Write the fiber over $u \in \mathcal{B}$ as $\mathcal{H}_u$.

The particle representations are characterized by a mass, a central charge Z, and a representation of the little superalgebra. The BPS or short representations are much more rigid, since $M = |Z|$ and, it turns out, they are something we can control mathematically.

Over $\mathcal{B}^*$ the Hilbert space admits a grading by an integral lattice Γ:

$$\mathcal{H}_u = \oplus_{\gamma\in\Gamma}\mathcal{H}_{u,\gamma} \tag{4.42}$$

The lattice is typically a lattice of charges associated with the low energy abelian gauge theory. In fact, it will turn out that for the family of Hilbert spaces $\mathcal{H}$ the grading lattice in fact defines a *local system of lattices over* $\mathcal{B}^*$. That is, there is nontrivial monodromy of the Gauss-Manin connection.

Moreover, as a representation of $\mathcal{N} = 2$, it turns out that:

a.) The operator Z is in fact a scalar Z_γ on each summand $\mathcal{H}_\gamma$.

b.) The value of Z_γ it is linear:

$$Z_{\gamma_1+\gamma_2} = Z_{\gamma_1} + Z_{\gamma_2} \tag{4.43}$$

I do not know a fundamental reason why this had to be true. It is true in all examples I know.

Moreover, the value of Z_γ depends on the point $u \in \mathcal{B}$, so we write $Z_\gamma(u)$. This is defined to be the *central charge function*:

$$Z \in \mathrm{Hom}(\Gamma, \mathbb{C}). \tag{4.44}$$

The Coulomb branch $\mathcal{B}$ is a special Kähler manifold, in particular it is a complex manifold and, it turns out (by supersymmetry) that Z is a holomorphic function of u.

4.6 The BPS Index and the Protected Spin Character

In trying to enumerate the BPS spectrum of a theory one encounters an important difficulty. It can happen that a non-BPS particle representation has a mass $M(u,\dots)$, which depends on u, as well as other parameters, generically satisfies $M(u,\dots) > |Z(u)|$ but for special values of u, (or the other parameters) it satisfies $M(u,\dots) = |Z(u)|$. When this happens, the non-BPS representation becomes a sum of BPS representations: This is clear since in the non-BPS representation $\rho_{hh}\otimes\rho_{hh}\otimes\mathfrak{h}$ the $\mathcal{R}$-supersymmetries act as zero on the first factor and hence we obtain a BPS representation $\rho_{hh}\otimes\mathfrak{h}'$ with

$$\mathfrak{h}' = \mathfrak{h}\otimes\left[(\frac{1}{2};0)\oplus(0;\frac{1}{2})\right] \tag{4.45}$$

as a representation of $\mathfrak{s}^0_\ell$.

The problem is, that while "true" BPS representations which do not mix with non-BPS representations are - to some extent - independent of parameters and constitute a more solvable sector of the theory, the "fake" BPS representations of the above type are more difficult to control. In particular they can appear and disappear as a function of other parameters (such as hypermultiplet moduli).

We need a way to separate the "fake" BPS representations from the "true" BPS representations. One way to do this is to consider an *index*:

Definition An *index* is a function of $\mathcal{N}=2$ representations which is continuous on continuous families of representations and which vanishes on long representations.

When the theory has $SU(2)_R$ symmetry we can introduce a nice index known as the *protected spin character.* [17]

A representation ρ of the massive little superalgebra $\mathfrak{s}^0_\ell\oplus\mathfrak{s}^1$ has a character, defined by:

$$\mathrm{ch}(\rho) = \mathrm{Tr}_\rho x_1^{2J_3}x_2^{2I_3} \tag{4.46}$$

where J_3 is a generator of $so(3)$ and I_3 is a generator of $su(2)_R$.

Note that

$$\mathrm{ch}(\rho_{hh}) = x_1 + x_1^{-1} + x_2 + x_2^{-1} \tag{4.47}$$

and therefore, for the general long representation (4.29) of $\mathcal{N}=2$ we get:

$$\mathrm{ch}(LONG) = (x_1 + x_1^{-1} + x_2 + x_2^{-1})^2\mathrm{ch}(\mathfrak{h}) \tag{4.48}$$

while for the general short representation (4.31) of $\mathcal{N}=2$ we get:

$$\mathrm{ch}(SHORT) = (x_1 + x_1^{-1} + x_2 + x_2^{-1})\mathrm{ch}(\mathfrak{h}) \tag{4.49}$$

The difference is by a factor of $(x_1 + x_1^{-1} + x_2 + x_2^{-1})$.

This suggests that we should consider the specialization:

$$x_1\frac{\partial}{\partial x_1}\left(\mathrm{Tr}x_1^{2J_3}x_2^{2I_3}\right)|_{x_1=-x_2=y} := \mathrm{Tr}(2J_3)(-1)^{2J_3}(-y)^{2\mathcal{J}_3} \tag{4.50}$$

[17] This was suggested to me by Juan Maldacena.

where $\mathcal{J}_3 = J_3 + I_3$.

It is clear that this specialization vanishes on long representations, and therefore it vanishes on "fake" BPS representations. On the other hand from the character of the BPS representation (4.49) we get instead

$$\mathrm{Tr}(2J_3)(-1)^{2J_3}(-y)^{2\mathcal{J}_3} = (y - y^{-1})\mathrm{ch}(\mathfrak{h})|_{x_1=-x_2=y} \tag{4.51}$$

Now, when we combine the Γ-grading (4.42) of $\mathcal{H}$ with the BPS condition we get a grading of the BPS subspace:

$$\mathcal{H}_u^{BPS} = \oplus_{\gamma\in\Gamma}\mathcal{H}_{u,\gamma}^{BPS} \tag{4.52}$$

and on $\mathcal{H}_{u,\gamma}^{BPS}$ the energy is $|Z(\gamma;u)|$.

In all known examples it turns out that $\mathcal{H}_{u,\gamma}^{BPS}$ are finite dimensional, and therefore we can form the traces

$$\mathrm{Tr}_{\mathcal{H}_{u,\gamma}^{BPS}}(2J_3)(-1)^{2J_3}(-y)^{2\mathcal{J}_3}. \tag{4.53}$$

We define the *Protected Spin Character* by the equation

$$(y - y^{-1})\Omega(\gamma;u;y) := \mathrm{Tr}_{\mathcal{H}_{u,\gamma}^{BPS}}(2J_3)(-1)^{2J_3}(-y)^{2\mathcal{J}_3} \tag{4.54}$$

that is:

$$\Omega(\gamma;u;y) = \mathrm{ch}(\mathcal{H}_{u,\gamma}^{\rm int})|_{x_1=-x_2=y} = \mathrm{Tr}_{\mathcal{H}_{u,\gamma}^{\rm int}}(-1)^{2J_3}(-y)^{2\mathcal{J}_3} \tag{4.55}$$

where

$$\mathcal{H}_{u,\gamma}^{BPS} = \rho_{hh}\otimes\mathcal{H}_{u,\gamma}^{\rm int} \tag{4.56}$$

Example/Exercise: Show that the contribution of a half-hypermultiplet representation in $\mathfrak{h}_\gamma$ to Ω is just $\Omega(\gamma;u;y) = 1$ and the contribution of a vectormultiplet in $\mathfrak{h}_\gamma$ is $\Omega(\gamma;u;y) = y + y^{-1}$.

Remarks:

1. If we specialize to $y = -1$ then we get the *BPS index* $\Omega(\gamma;u)$.

2. Exercise: Show that we could have defined the BPS index directly via

$$\Omega(\gamma;u) = -\frac{1}{2}\mathrm{Tr}_{\mathcal{H}_{\gamma,u}^{BPS}}(2J_3)^2(-1)^{2J_3} = \mathrm{Tr}_{\mathfrak{h}}(-1)^{2J_3} \tag{4.57}$$

   This quantity is known as the *second helicity supertrace.*

3. The BPS index (4.57) can be defined even if the $\mathcal{N} = 2$ superalgebra does not have an unbroken $su(2)_R$ symmetry. This is important since in supergravity we generally do not have that symmetry and should only work with BPS indices, and not protected spin characters.

4. Now, these BPS indices are piecewise constant functions of u, but they can still jump discontinuously. This is the phenomenon of *wall-crossing*, which we will discuss in detail in §**??**.

4.7 An Open Problem: Compute the BPS Spectrum

Sketch what is and is not known:

1. *The Seiberg-Witten paradigm*: In a celebrated pair of papers, Seiberg and Witten [155, 156] found a basic paradigm for computing the central charge functions $Z_\gamma(u)$ for the case of SQCD with $SU(2)$ gauge group. The $Z_\gamma(u)$ are periods of a meromorphic one-form (the Seiberg-Witten differential) on a holomorphic family of Riemann surfaces (the Seiberg-Witten curve) over the Coulomb branch.

FIGURE OF SW CURVE OVER B.

From the Seiberg-Witten solution we expect two things to be exactly solvable:

a.) Z and LEEA

b.) BPS spectrum.

Following the SW solution there was an enormous amount of work determining the SW curve and differential for a large class of theories. We will describe an infinite class of theories (class S) for which they can be straightforwardly written down.

However, to this day, there is no algorithm, given an N=2 d=4 theory with a Coulomb branch to write down the SW curve family over that branch. Indeed, I do not even know a general principle why there should exist a SW curve. All that N=2 guarantees is a holomorphic family of abelian varieties. There is no fundamental reason they have to be Jacobians of Riemann surfaces.

2. However, this does not determine the BPS spectrum. It does not tell us which charges γ are realized by BPS particles.

3. Mid 90's to early 2000's: Isolated examples of BPS spectra. $SU(2)$ $\mathcal{N}_f = 0, 1, 2, 3, 4$ at special masses [35][33] $SU(2)$ + adjoint [Ferrari?]

4. Changed dramatically in the past few years. Starting with a deeper understanding of the Wall-crossing phenomenon.

5. Now N=2 BPS spectra are known for a compact Calabi-Yau.

4.8 Positivity Conjectures and geometric quantities

The BPS spectrum is largely unknown, but we can state a general conjecture about it:

Reference [96] stated a pair of conjectures concerning the protected spin character, known as the *positivity conjecture* and the *no-exotics conjecture*. These are meant to apply only to field-theoretic (and not string-theoretic) BPS states. The positivity conjecture asserts that $\Omega(\gamma; u, y)$, regarded as a function of y, can be written as a positive integral linear combination of $SU(2)$ characters. That is:

$$\Omega(\gamma; u, y) = \sum_{n \geq 1} d(\gamma; u; n) \chi_n(y) \tag{4.58}$$

where

$$\chi_n(y) := \mathrm{Tr}_n y^{2J} = \frac{y^n - y^{-n}}{y - y^{-1}} \tag{4.59}$$

is the character in the n-dimensional representation of $SU(2)$ and the $d(\gamma; u; n)$ are piecewise constant functions of u. The positivity conjecture states that $d(\gamma; u; n) \geq 0$ for all γ

and all points u on the Coulomb branch. It would follow if the center of $SU(2)_R$ acts trivially on $\mathcal{H}_{int}$, i.e., that $\mathcal{H}_{int}$ contains only integral spins. We will call this the *integral spin property*. It is stronger than the positivity conjecture. The even stronger *no-exotics* conjecture posits that in fact only states with trivial $SU(2)_R$ quantum numbers contribute to the protected spin character. When there are no exotics the naive spin character coincides with the protected spin character.

Reference [56] discusses how these conjectures are related to statements about motivic DT invariants defined by Kontsevich and Soibelman [128, 129]. It often happens that we can model BPS states by BPS field configurations described by a moduli space $\mathcal{M}(\gamma; u)$ of some objects:

– objects in a derived category of sheaves of some fixed Chern classes

– objects in a derived category of representations of a quiver

The working hypothesis in connecting math and physics is that, when the moduli space of BPS states is smooth we can identify

$$\mathcal{H}_{int}(\gamma; u) \cong \oplus_{p,q} H^{p,q}(\mathcal{M}(\gamma; u)) \tag{4.60}$$

Moreover, under this isomorphism the action of the spin group $SU(2)_{spin}$ should be identified with the standard Lefschetz action on cohomology. Thus, $2J_{spin}$ acts on the (p, q)-graded piece as $p + q - m$, where $m = \dim_{\mathbb{C}} \mathcal{M}$. Furthermore, $2J_R$ acts with weight $p - q$ on the (p, q)-graded piece. Granting these identifications the protected spin character (**??**) becomes

$$\Omega(\gamma; u; y) = \sum_{p,q\in\mathbb{Z}} (-1)^{p-q} y^{2p-m} h^{p,q}(\mathcal{M}(\gamma; u)) \tag{4.61}$$

for compact and smooth moduli spaces.

What is the mathematical import of (4.61) ? Recall that the $\chi_{\tilde{y}}$-genus of a smooth projective variety V is defined by

$$\chi_{\tilde{y}}(V) := \sum_{p,q\in\mathbb{Z}} (-1)^{p+q} \tilde{y}^p h^{p,q}(V). \tag{4.62}$$

Therefore

$$\Omega(\gamma; u; y) = y^{-m} \chi_{\tilde{y}}(\mathcal{M}(\gamma; u))\big|_{\tilde{y}=y^2} \tag{4.63}$$

Reference [56] further conjectures that

$$\Omega(\gamma; u; y) = \sum_{r,s\in\frac{1}{2}\mathbb{Z}} (-1)^{r-s} y^{2r} \mathfrak{h}^{r,s}(\gamma; u) \tag{4.64}$$

for *any* charge γ and point u on the Coulomb branch, even when the moduli spaces of BPS states are singular. [DEFINED VIA THEORY OF MOTIVES].

The absence of exotics conjecture translates into the condition $\mathfrak{h}^{r,s}(\gamma; u) = 0$ for all $r \neq s$. If this holds,

$$\Omega(\gamma; u; y) = \sum_{r\in\frac{1}{2}\mathbb{Z}} y^{2r} \mathfrak{h}^{r,r}(\gamma; u) \tag{4.65}$$

If the moduli space is smooth we can further write:

$$\Omega(\gamma; u; y) = y^{-m} P(\mathcal{M}(\gamma; u); y^2) \tag{4.66}$$

where P is the Poincaré polynomial.

SAY MORE ABOUT MOTIVES????

Remarks:

1. Reference [56] investigates the BPS states in an important set of examples: The field-theoretic BPS states arising in the approach via geometric engineering (type IIA string theory on a noncompact CY constructed as a family of resolved ADE singularities over a projective line). The BPS states can be related to cohomology classes on moduli spaces of quiver representations. In this context one can show in that the relevant motives are all rational functions of the motive of the affine line, and hence the no-exotics conjecture is true for BPS states in pure $SU(K)$ gauge theory for all K.

2. Historical remarks about spin vs. pcs?

3. SAY SOMETHING ABOUT THE SEMICLASSICAL PICTURE USING MONOPOLE MODULI SPACES?

4.9 The wall-crossing phenomenon

Now we come to a fundamental point. Although Ω is an index, nevertheless, it can change as we vary vacuum parameters.

The essential physical point here is that *it can happen that BPS particles can form new boundstates which are themselves BPS!*.

This was discovered in the context of $d = 2$ $\mathcal{N} = (2, 2)$ supersymmetric field theories by Cecotti, Fendley, Intriligator, and Vafa [42]. It played a crucial role for Seiberg and Witten in understanding that they BPS spectrum they proposed for pure $SU(2)$ $\mathcal{N} = 2$ SYM was consistent [156]. We will explore the physics of this a bit more later. For now, let us just accept that it can happen.

Given that two BPS particles can make a BPS boundstate we can ask if that boundstate can decay. For example, could there be a tunneling process where the particles fly apart to infinity?

The answer is - in general - NO!

If BPS particles of charges γ_1 and γ_2 form a BPS boundstate of charge $\gamma_1 + \gamma_2$ then we can compute the binding energy:

$$|Z(\gamma_1 + \gamma_2; u)| - |Z(\gamma_1; u)| - |Z(\gamma_2; u)| \tag{4.67}$$

Because Z is linear in γ we can use the triangle inequality to conclude that this is nonpositive. Moreover, this binding energy is negative, and therefore the particles cannot separate to infinity *unless* $Z(\gamma_1; u)$ and $Z(\gamma_2; u)$ are parallel complex numbers.

This special locus, where the boundstate might become unstable is known as the *wall of marginal stability*:

$$MS(\gamma_1, \gamma_2) := \{u | 0 < Z(\gamma_1; u)/Z(\gamma_2; u) < \infty \qquad \text{and} \qquad \Omega(\gamma_1; u; y)\Omega(\gamma_2; u; y) \neq 0\} \tag{4.68}$$

Along such walls boundstates can decay. This raises a natural question: Can we describe quantitatively how the BPS spectrum changes across a wall of marginal stability? Such a formula is called a wall-crossing formula.

We will give a simple physical argument for a wall-crossing formula in §4.11 below. But first we need to introduce one more physical idea.

Remarks

1. In general an index of a family of operators can change when at some locus in the family the operator ceases to be Fredholm. It would be nice to interpret the N=2 d=4 wall-crossing this way. In string theory BPS wall crossing it can be so interpreted by relating BPS indices to an index of a the Dirac-Ramond operator in a sigma model with supersymmetry.

4.10 The charge lattice Γ

4.10.1 Dirac quantization of dyonic charge in 4d Maxwell theory

By definition the Coulomb branch $\mathcal{B}$ is the space of quantum vacua so that the low-energy effective theory contains an unbroken abelian gauge theory. Let us suppose the abelian gauge group is of rank r, so there are r Maxwell fieldstrengths $F^I \in \Omega^2(\mathbb{M}^{1,3})$ (we will say this more invariantly later).

Now in such theories particles can carry electric and magnetic charges, so to a particle we can assign a collection $\gamma_g = (p^I, q_I)_{I=1,\dots,r}$. Particles which carry both electric and magnetic charge are typically called *dyons*.

Dirac showed that in quantum theory there is a quantization law on the charges of dyons. One easy argument for this proceeds by considering the spin in the electromagnetic field in the presence of two dyons.

Consider pair of an electron and a monopole in $\mathbb{R}^3$. Draw field lines and see that there is nontrivial spin in the electromagnetic field from $\vec{E} \times \vec{B}$. Get Dirac quantization.

Simple generalization: Two dyons of (magnetic, electric) charges (q_{m_i}, q_{e_i}) $i = 1, 2$ in $\mathbb{R}^{1,3}$. Computing the angular momentum of the electromagnetic field around the midpoint separating them

$$\vec{J}_{ij} = \int_{\mathbb{R}^3} d^3\vec{x} \left(x_i(\vec{E} \times \vec{B})_j - x_j(\vec{E} \times \vec{B})_i \right) \tag{4.69}$$

Find

$$\vec{J} = \frac{1}{c}(q_{m_1} q_{e_2} - q_{m_2} q_{e_1})\hat{r} \tag{4.70}$$

and therefore, by quantization of angular momentum

$$q_{m_1} q_{e_2} - q_{m_2} q_{e_1} = \hbar c \frac{s}{2} \tag{4.71}$$

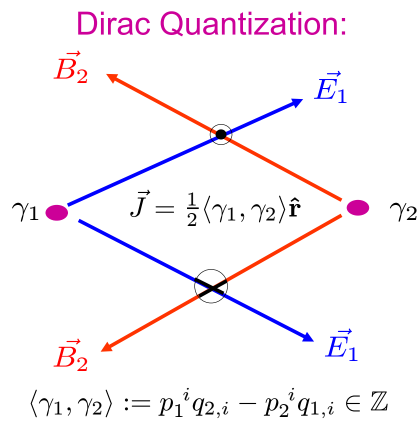

Figure 3: A pair of dyons in four-dimensions produces an electromagnetic field with spin around their axis.

where s is an integer.

We conclude that the electric and magnetic gauge charges should form a *symplectic lattice* Γ_g of rank $2r$ if there are r independent abelian gauge fields.

4.10.2 General Picture: Global Symmetry Charges

The abelian gauge theory has a global symmetry group $\mathcal{G}_{\text{gauge}}$ which (in a self-dual formalism) is a torus of rank $2r$. Its group of characters is Γ_g.

In general, however, there will be a larger group of global symmetries. The connected component of the subgroup of abelian global symmetries will contain the group $\mathcal{G}_{\text{gauge}}$ canonically, but might be larger:

$$0 \to \mathcal{G}_{\text{gauge}} \to \mathcal{G} \to \mathcal{G}_{\text{flavor}} \to 0 \tag{4.72}$$

The set of global gauge currents are is canonically determined because these couple to gauge fields in the action. The "flavor symmetry currents" are not canonically defined: We can shift

$$J_\mu^{flavor} \to J_\mu^{flavor} + J_\mu^{gauge} \tag{4.73}$$

In other words, (4.72) does not split canonically.

The lattice Γ is defined as the lattice of characters of the connected component of the abelian subgroup of global symmetries. It has an integral-valued antisymmetric form:

$$\langle\cdot,\cdot\rangle : \Lambda^2\Gamma \to \mathbb{Z} \tag{4.74}$$

This can be justified using Seiberg's trick of regarding the flavor symmetries as gauge symmetries for a weakly gauged group [154]. Only the electrically charged particles under this flavor group survive and hence we have

♣Actually, this trick is pretty important and deserves a separate discussion. It is used extensively in SW, and in [17]. ♣

$$\Gamma_f = Ann(\langle\cdot,\cdot\rangle) \tag{4.75}$$

The Pontryagin dual statement to (4.72) is

$$0 \to \Gamma_f \to \Gamma \to \Gamma_g \to 0 \tag{4.76}$$

This sequence can be split, but not naturally. What that means is that the local system can have monodromy on closed paths $\wp$ in $\mathcal{B}^*$ of the form

$$\gamma_f \oplus \gamma_g \to (\gamma_f + N(\wp)\cdot\gamma_g) \oplus (M(\wp)\cdot\gamma_g) \tag{4.77}$$

where $M(\wp)$ is a monodromy transformation valued in $Sp(2r;\mathbb{Z})$ and $N(\wp) \in \mathrm{Hom}(\Gamma_g, \Gamma_f)$.

♣ GIVE AN EXAMPLE HERE OR REFER TO ONE LATER ♣

4.11 Primitive Wall Crossing Formula

4.11.1 Denef's boundstate radius formula

In ordinary Maxwell theory dyons will either attract or repel each other. However, in $\mathcal{N}=2$ supersymmetric field theory and supergravity there are other forces. There are forces due to scalar fields and (in supergravity) gravitational fields.

In [49, 48, 50] Frederik Denef found certain exact solutions of BPS particles (in $\mathcal{N}=2$ supergravity) which can be interpreted as boundstates of individual BPS particles. He computed the boundstate radius for these solutions and found a very interesting and general result:

If two dyonic BPS particles or black holes of electromagnetic charges γ_1, γ_2 in a vacuum u form a BPS boundstate then that boundstate has total electromagnetic charge $\gamma_1+\gamma_2$ and boundstate radius :

$$R_{12} = \frac{1}{2}\langle\gamma_1,\gamma_2\rangle \frac{|Z(\gamma_1;u)+Z(\gamma_2;u)|}{\mathrm{Im}Z(\gamma_1;u)\overline{Z(\gamma_2;u)}} \tag{4.78}$$

Remarks:

1. Note that (4.78) can equivalently be written as

$$R_{12} = \frac{1}{2}\langle\gamma_1,\gamma_2\rangle \frac{1}{\mathrm{Im}e^{-i\alpha}Z(\gamma_1;u)} \tag{4.79}$$

   where α is the phase of $Z(\gamma_1;u)+Z(\gamma_2;u)$, so in the limit $|Z(\gamma_2;u)| \gg |Z(\gamma_1;u)|$ (something called the probe approximation limit) equation (4.79) reduces to

$$R_{12} = \frac{1}{2}\langle\gamma_1,\gamma_2\rangle \frac{1}{\mathrm{Im}Z_{\gamma_1}e^{-i\alpha_2}} \tag{4.80}$$

   a formula we will come back to later when we talk about line defects.

2. Since the boundstate radius must be positive, a crucial corollary of the above result is the *Denef stability condition*: If BPS particles of charges γ_1 and γ_2 form a BPS boundstate in the vacuum u then it must necessarily be that

$$\langle\gamma_1,\gamma_2\rangle \mathrm{Im}Z(\gamma_1;u)\overline{Z(\gamma_2;u)} > 0. \tag{4.81}$$

4.11.2 A simple physical derivation of the primitive wcf

Suppose there is a BPS boundstate of BPS particles of charges γ_1, γ_2 and an experimentalist dials the vacuum moduli u, at infinity, so as to approach a wall of marginal stability, $MS(\gamma_1, \gamma_2)$, from the stable side (4.81), crossing at u_{ms}.

It follows from Denef's boundstate radius formula that $R_{12} \to \infty$: We literally see the states leaving the Hilbert space. This confirms our suspicion about the wall.

But now we can be quantitative: How many states do we lose?

The Hilbert space of states of the boundstate is a tensor product of the space of states of the constituents of the particle of charge γ_1, the space of states of the constituents of the particle of charge γ_2 and the states associated to the electromagnetic field of the pair of dyons. Therefore, the PSC changes by:

$$\Delta\Omega(\gamma; u; y) = \pm\chi_{|\langle\gamma_1,\gamma_2\rangle|}(y)\Omega(\gamma_1; u_{ms}; y)\Omega(\gamma_2; u_{ms}; y) \tag{4.82}$$

where $\chi_n = \mathrm{ch}_{\rho_n}$ is the character of an $SU(2)$ representation of dimension n. The rationale for the first factor is that the electromagnetic field carries a representation of $so(3)$ of dimension $|\langle\gamma_1, \gamma_2\rangle|$, that is, of spin [18]

$$J_{\gamma_1,\gamma_2} := \frac{1}{2}(|\langle\gamma_1, \gamma_2\rangle| - 1) \tag{4.83}$$

The + sign occurs when we move from a region of instability to stability.

Remarks:

1. If $\langle\gamma_1, \gamma_2\rangle = 0$ then $J_{12} = -1/2$ and we have the representation of $so(3)$ given by the zero vector space. It has character = 0.

2. Note that the quantity in equation (4.67) is always nonpositive, and is in fact negative even in the Denef-stable region. Thus, negativity of (4.67) is a necessary condition for having a true boundstate, but not a sufficient condition.

3. Similarly, the Denef stability condition is a *necessary* condition, but not a sufficient condition for the existence of a boundstate. Indeed, we can also define an *anti-marginal stability wall* to be a wall where the complex numbers $Z(\gamma_1; u)$ and $Z(\gamma_2; u)$ anti-align (i.e. Z_1 and $-Z_2$ are parallel complex numbers). In this case the anti-marginal wall separates a Denef-stable region from an unstable region. Suppose a boundstate existed in the stable region near a wall of anti-marginal stability. Note that the boundstate radius in Denef's formula also diverges across such a wall, but it is impossible to have a boundstate decay in this case, since that would violate energy conservation! (Show this!). We conclude that such boundstates cannot exist, even in a region of Denef stability. This would appear to pose a paradox if, as does indeed happen, a marginal stability wall can be connected to an anti-marginal stability wall

[18] The classical computation of **** ABOVE gives $J = \frac{1}{2}|\langle\gamma_1, \gamma_2\rangle|$ but in fact there is a quantum correction and the correct result is $J = \frac{1}{2}(|\langle\gamma_1, \gamma_2\rangle| - 1)$. This quantum correction is best seen by studying the quantum mechanics of a probe particle. It can also be seen in the quiver quantum mechanics.

through a region of Denef stability. The resolution of the paradox can be found in [14, 13].

4. Now it is important here that we take γ_1 and γ_2 to be *primitive vectors*, since otherwise there can be more complicated decays and boundstates, as we will see. [19] In particular, a single BPS boundstate of charge $N\gamma_1$ where $N > 1$ is an integer is necessarily a boundstate at threshhold (and therefore very subtle). It can split up adiabatically into N-particle states and these can form more complicated configurations than we have taken into account.

5. The first quantitative wall-crossing formulae for BPS states were given in the context of $d = 2$ $\mathcal{N} = (2, 2)$ theories by Cecotti and Vafa in [41]. We will call it the CVWCF and describe it below. The above simple argument was first stressed in [51], and it can be generalized to a so-called semi-primitive formula. However, as just noted, it does not apply to the most general boundstates. The formula that describes the most general boundstate wall crossing is the KSWCF.

6. Note that $\mathrm{Im} Z_1 \bar{Z}_2 > 0$ means that the complex numbers Z_1 and Z_2 are oriented so that Z_1 is counterclockwise to Z_2 at an angle less than π. As u crosses a wall of marginal stability the vectors Z_1, Z_2 rotate to become parallel and then exchange order.

5. The abelian tensormultiplet

5.1 Quick Reminder on the Hodge $*$

Let X_n be an oriented Riemannian n-manifold. It can be of Euclidean or Minkowskian signature. It can be compact or not. When these distinctions are important we will mention them.

In local coordinates, if $\alpha, \beta \in \Omega^\ell(X)$ then we can define the local inner product of forms:

$$(\alpha, \beta) = \frac{1}{\ell!} g^{\mu_1\nu_1} \cdots g^{\mu_\ell\nu_\ell} \alpha_{\mu_1\cdots\mu_\ell} \beta_{\nu_1\cdots\nu_\ell} \tag{5.1}$$

Note that $(\alpha, \beta) = (\beta, \alpha)$, always.

In this chapter we will make extensive use of the Hodge $*$ operator

$$* : \Omega^\ell(X_n) \to \Omega^{n-\ell}(X_n) \tag{5.2}$$

One way to characterize it is that

$$(\alpha, \beta) := \frac{\alpha \wedge *\beta}{\mathrm{vol}\,(g)} \tag{5.3}$$

Note that

1.

$$*^2 = (-1)^{\ell(n-\ell)} \mathrm{sign}\det g_{\mu\nu} \tag{5.4}$$

2. On middle-dimensional forms $*$ only depends on $g_{\mu\nu}$ through its conformal class.

[19] A vector γ in a lattice is said to be *primitive* if it is not an nontrivial integral multiple of another lattice vector. That is if $\frac{1}{N}\gamma$ is not in the lattice for any integer $N > 1$.

5.2 The $(2,0)$ tensormultiplet

The $(2,0)$ superPoincaré algebra has a basic field multiplet known as the *tensormultiplet*. Letting $V_R \cong \mathbb{R}^5$ be the fundamental representation of the $Spin(5)$ R-symmetry the fields are (B, ψ, Y) where

$$B \in \Omega^2(\mathbb{M}^{1,5}) \tag{5.5}$$

$$\psi \in \Gamma(\mathbb{M}^{1,5}; (\Delta(1,5)_+ \otimes \Delta(5))_{\mathbb{R}}) \tag{5.6}$$

(note the chirality of the spinors is the same as that of the supersymmetries) and

$$Y \in \Gamma(\mathbb{M}^{1,5}; V_R) \tag{5.7}$$

The supersymmetry operators act, schematically, as

$$\begin{aligned} [Q, Y] &= \psi \\ [Q, \psi] &= \Gamma \cdot dY + \Gamma \cdot H \\ [Q, H] &= \Gamma \cdot d\psi \end{aligned} \tag{5.8}$$

where $\Gamma\cdot$ stands for various Clifford contractions and we have defined

$$H := dB \in \Omega^3_d(\mathbb{M}^{1,5}) \tag{5.9}$$

that is, H is a closed (and exact) 3-form. For the precise form of the transformations see [131] [OTHER REFS]. (Our normalization is spelled out in Appendix E of [96].)

A very interesting exercise shows that if we try to verify that this is a representation of $\mathfrak{SP}(\mathbb{M}^{1,5|16})$ we find that we must restrict the fields to be *on-shell*, that is the fields must satisfy

$$\begin{aligned} \partial^2 Y &= 0 \\ \Gamma \cdot d\psi &= 0 \\ H &= *H \end{aligned} \tag{5.10}$$

The first equation says that the five scalar fields Y are free and massless. The second equation says that ψ satisfies the free massless chiral Dirac equation. The third is the most interesting. Coupled with $dH = 0$ it gives a *self-dual* equation of motion.

Remarks

1. The tensormultiplet was first written down in [HoweSierraTownsend]. The demonstration that it is in a superconformal multiplet is in [46, 30].

2. There are two choices of convention we make here. First, we choose an orientation of spacetime. This determines a preferred Clifford volume element Γ^{012345}. Next, we choose a chirality for our supersymmetries and hence for the spinors in the chiral multiplet. Here we have chosen the eigenvalue $+1$ for the action of Γ^{012345}. Having chosen that, the chirality of the fieldstrength H is determined. With the above choices it is $H = + * H$. ♣Check sign. ♣

5.3 A lightning review of generalized Maxwell theory

Let us review a few basics about the classicial field theory of a generalized Maxwell field.

As above: Let X_n be an oriented Riemannian n-manifold. It can be of Euclidean or Minkowskian signature. It can be compact or not. When these distinctions are important we will mention them.

Let V be a real vector space with a positive symmetric bilinear form λ. Then we can define the classical generalized Maxwell theory by taking the fieldstrength to be

$$F \in \Omega^{\ell}(X;V) \tag{5.11}$$

(*smooth* differential forms valued in V) and the generalized Maxwell equations to be

$$\begin{aligned} dF &= 0 \\ d * F &= 0 \end{aligned} \tag{5.12}$$

and the energy-momentum tensor to be the symmetric tensor whose value on a vector field v is

$$T_F(v) = \lambda(\iota_v F, \iota_v F) - \frac{1}{2}(v,v)\lambda(F,F). \tag{5.13}$$

Here combined the inner product λ on V with the local Hodge inner product (5.3) above.

The above equations define classical generalized Maxwell theory. The solutions of the equations of motion are harmonic forms. They form a finite dimensional vector space if X is compact Euclidean and an infinite dimesional space of wave-solutions moving at the speed of light if X is noncompact Lorentzian. We compute the energy-momentum of the waves from T_F.

The theory has an electric-magnetic duality symmetry. We could equally well use $\tilde{F} \in \Omega^{n-\ell}(X)$ and the relation between the two is $\tilde{F} = *F$. The equations of motion are obviously exchanged and with a little work one can check that $T_F = T_{\tilde{F}}$.

If we want an action principle we break the electric-magnetic duality symmetry and choose one equation as preferred and to be solved first. Say $dF = 0$. Then $F \in \Omega^{\ell}_d(X)$ is a closed ℓ-form. If we vary *within the space of closed ℓ-forms* then we can take the action principle

$$S = \pi \int_X \lambda(F,F) \tag{5.14}$$

Varying $F \to F + d\delta a$ within the subspace with fixed deRham cohomology class we find the equation of motion $d * F = 0$. Varying with respect to the metric gives T_F. ♣eliminate that restriction? ♣

We can introduce external electric and magnetic currents into the classical equations:

$$\begin{aligned} dF &= J_m \\ d * F &= J_e \end{aligned} \tag{5.15}$$

Note that these currents are necessarily closed:

$$\begin{aligned} J_m &\in \Omega^{\ell+1}_d(X) \\ J_e &\in \Omega^{n-\ell+1}_d(X) \end{aligned} \tag{5.16}$$

The presence of the currents changes the geometric nature of the field.

For example, for $\ell = 2$ we can consider the Maxwell field, in the absence of magnetic current, to be a connection on a complex line bundle over X. In the presence of J_m this interpretation no longer applies.

If we allow J_e, J_m to be delta-function supported currents then we can allow p-brane analogs of Dirac monopoles into the theory. There are electric $p = (\ell - 2)$ and magnetic $p = (n - \ell - 2)$-branes in the theory. For example, for Maxwell theory $n = 4$ and $\ell = 2$ so there are electric and magnetic particles. For $n = 6$ and $\ell = 3$ there are electric and magnetic strings.

5.4 Differential cohomology

In the quantum theory we interpret the Maxwell fieldsrength as the curvature of a connection on a principal $U(1)$ bundle over X. The curvature is then quantized. When we do this we realize that there is more gauge invariant information in the Maxwell field than just the fieldstrength.

The correct way to describe the gauge invariant information in the Maxwell field (and its ℓ-form generalizations) is via *differential cohomology* also known as *Deligne-Cheeger-Simons cohomology* . Here we will be telegraphic. For a more extended discussion go see the talks at the SCGP Jan. 2011 conference where you can see a nice talk by J. Simons himself on the subject [151].

The differential cohomology group famously satisfies two compatible exact sequences:

$$0 \to \overbrace{H^{\ell-1}(M;\mathbb{R}/\mathbb{Z})}^{\text{flat}} \to \check{H}^{\ell}(M) \stackrel{\text{fieldstrength}}{\longrightarrow} \Omega^{\ell}_{\mathbb{Z}}(M) \to 0$$

$$0 \to \underbrace{\Omega^{\ell-1}(M)/\Omega^{\ell-1}_{\mathbb{Z}}(M)}_{\text{Topologically trivial}} \to \check{H}^{\ell}(M) \stackrel{\text{char.class}}{\longrightarrow} \underbrace{H^{\ell}(M;\mathbb{Z})}_{\text{Topological sector}} \to 0$$

The flat fields form a compact abelian group so we also have the useful exact sequence:

$$0 \to \overbrace{H^{\ell-1}(M;\mathbb{R})/H^{\ell-1}(M;\mathbb{Z})}^{\text{Wilson lines}} \to H^{\ell-1}(M;\mathbb{R}/\mathbb{Z}) \longrightarrow \underbrace{\text{Tors}(H^{\ell}(M;\mathbb{Z}))}_{\text{Discrete Wilson lines}} \to 0$$

As indicated above, in physics we identify

1. $c \in H^{\ell}(M_n;\mathbb{Z})$: Dirac's quantization. We will refer to it as the *characteristic class* or *topological class*.

2. $F \in \Omega^{\ell}(M_n)$: Maxwell's fieldstrength Our normalization is that F will have *integral* periods, so that

$$c_{\mathbb{R}} = [F] \tag{5.17}$$

under the isomorphism $H^{\ell}(M;\mathbb{R}) \cong H^{\ell}_{dR}(M)$.

3. $H^\ell(M_n;\mathbb{R})/H^\ell(M_n;\mathbb{Z})$: Wilson lines.

4. Noncanonically $\check{H}^\ell(M)$ is a product of abelian groups of the form

$$T \times \Gamma \times V \tag{5.18}$$

where T is a connected torus, (the torus of Wilson lines), Γ is a discrete group, (the group of topological sectors $H^\ell(M;\mathbb{Z})$) and V is an infinite dimensional vector space. Physically, it corresponds to the "oscillator modes" of the field. It can be taken to be isomorphic to $\mathrm{Im}d^\dagger$.

5. One thing which is not often appreciated by physicists that $\check{H}^\ell(M)$ is an abelian group. We will exploit this group structure in our discussion of the Hilbert space below in Section **??**.

6. Now that fluxes are quantized the value of λ is meaningful even in the rank 1 case.

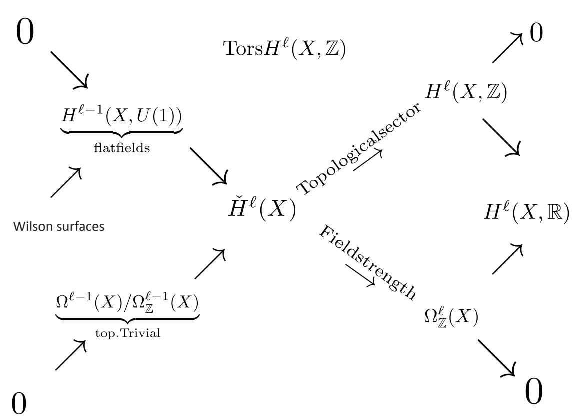

Figure 4: A Cheeger-Simons diagram summarizing how various groups fit together in differential cohomology, together with their physical interpretations. The symbol ???? indicates that the square commutes. ♣ FINISH DIAGRAM!!!♣

5.4.1 Examples

We now look at three simple examples.

Periodic scalar

For $\ell = 1$, $\check{H}^1(M) = Map(M, U(1))$ is the space of identified with a periodic scalar fields on M: $\varphi : M \to U(1)$. $F = \frac{1}{2\pi i}\varphi^{-1}d\varphi$ is the fieldstrength. The integral periods are the winding numbers of φ around 1-cycles and are measured by the characteristic class

$c \in H^1(M;\mathbb{Z})$. Flat field is the constant field φ, a constant phase. To be even more explicit, if we take $M = S^1$ then $\check{H}^1(S^1) = LU(1)$ is the famous loop group. Then

$$\varphi(\sigma) = \exp\left[2\pi i\phi_0 + 2\pi i w\sigma + \sum_{n\neq 0}\frac{\phi_n}{n}e^{2\pi i n\sigma}\right] \tag{5.19}$$

where $\sigma \sim \sigma + 1$ is a coordinate on the circle, illustrating very explicitly the meaning of the decomposition $T \times \Gamma \times V$ and identifying V with oscillator modes. In this case we have a field mapping M to a circle of radius R and $\lambda = R^2$ is the radius of a circle.

$\ell = 2$: Quantum Maxwell

In this case $\check{H}^2(M)$ is the set of isomorphism classes of principal $U(1)$ bundles over M with connection. c is the first chern class, $F = F(\nabla)$ is the curvature form of the connection etc. In this case $\lambda = e^{-2}$ is the electromagnetic coupling.

$\ell = 3$: Gerbe connections

Appears in the WZW theory. The field theory of these is important in $n = 6, 10, 26$ because of the B field of string theory. Electric sources are strings.

$\ell = 4$: M-theory C-field

See [52, 53, 54].

5.4.2 Ring Structure and Pairing

There are some important and nontrivial structures on the differential cohomology groups:

1. The (rather subtle) graded ring structure

$$\check{H}^{\ell_1} \times \check{H}^{\ell_2} \to \check{H}^{\ell_1+\ell_2} \tag{5.20}$$

The fieldstrength and characteristic class multiply in the expected way:

$$F(\check{\chi}_1 \cdot \check{\chi}_2) = F(\check{\chi}_1) \wedge F(\check{\chi}_2) \qquad\qquad c(\check{\chi}_1 \cdot \check{\chi}_2) = c(\check{\chi}_1) \cup c(\check{\chi}_2) \tag{5.21}$$

The formula for the product of the holonomy is more subtle.

The product is graded symmetric:

$$\check{\chi}_1 \cdot \check{\chi}_2 = (-1)^{\ell_1\ell_2}\check{\chi}_2 \cdot \check{\chi}_1 \tag{5.22}$$

♣ GIVE SOME EXAMPLES. There is a nice description of the multiplication $\check{H}^1 \times \check{H}^2 \to \check{H}^2$ where we pull back a standard bundle with connection on $S^1 \times S^1$. It should have first Chern class = 1 and translation invariant fieldstrength and holonomy – ???? Are there similar nice models for low degree examples?? ♣

2. There is a theory of integration. If X is a compact oriented n-manifold then, unlike de Rham theory, $\check{H}^{n+1}(X)$ is nonzero. In fact, $\check{H}^1(pt) \cong \mathbb{R}/\mathbb{Z}$. We can define an integral

$$\int^{\check{H}} : \check{H}^{n+1}(X) \to \mathbb{R}/\mathbb{Z} \tag{5.23}$$

by noting that since $H^{n+1}(X,\mathbb{Z}) = 0$ a differential character of degree $(n+1)$ must be topologically trivial so can be represented by some globally well defined n-form A_n (modulo gauge transformations) and we take the value of the integral to be $\int_{X_n} A_n \mathrm{mod}\mathbb{Z}$. In general if $\mathcal{X} \to \mathcal{P}$ is a family of compact oriented manifolds M of dimension n then we can define:

$$\int^{\check{H}}_{\mathcal{X}/\mathcal{P}} : \check{H}^s(\mathcal{X}) \to \check{H}^{s-n}(\mathcal{P}) \tag{5.24}$$

3. If $X_n = \partial B_{n+1}$ is the boundary of an $(n+1)$-manifold and $[\check{A}] \in \check{H}^{n+1}(X_n)$ can be extended *as a differential character* to $\check{H}^{n+1}(B_{n+1})$ then we can say

$$\int^{\check{H}}_X [\check{A}] = \int_{B_{n+1}} F \mathrm{mod}\mathbb{Z} \tag{5.25}$$

where F is the fieldstrength of the extending character. *Warning: It is wrong simply to extend the fieldstrength. The definition would then be ambiguous. This is a mistake frequently made in the literature!*

4. Now, using the fact that $\check{H}^1(pt) = \mathbb{R}/\mathbb{Z}$ we have the canonical pairing

$$\check{H}^\ell \times \check{H}^{n+1-\ell} \to \mathbb{R}/\mathbb{Z} \tag{5.26}$$

defined by

$$\langle [\check{A}_1], [\check{A}_2] \rangle := \int^{\check{H}} [\check{A}_1] \cdot [\check{A}_2] \tag{5.27}$$

5. An important special case of the pairing: If $[\check{A}_1]$ is topologically trivial then we may represent it by some $A_1 \in \Omega^{\ell_1 - 1}$ and then the pairing only depends on the fieldstrength of $[\check{A}_2]$: so pairing is

$$\langle [\check{A}_1], [\check{A}_2] \rangle = \int_M A_1 F_2 \mathrm{mod}\mathbb{Z} \tag{5.28}$$

so, in particular, if $[\check{A}_2]$ is also topologically trivial then

$$\langle [\check{A}_1], [\check{A}_2] \rangle = \int_M A_1 dA_2 \mathrm{mod}\mathbb{Z} \tag{5.29}$$

6. Another useful special case: $[\check{A}_1]$ is flat then it is represented by a class $\alpha_1 \in H^{\ell_1 - 1}(M, \mathbb{R}/\mathbb{Z})$ and then the pairing only depends on the characteristic class a_2 of $[\check{A}_2]$ and is given by

$$\langle [\check{A}_1], [\check{A}_2] \rangle = \int_M \alpha_1 a_2 \in \mathbb{R}/\mathbb{Z} \tag{5.30}$$

7. An absolutely crucial aspect of this pairing is that if X_n is compact and orientable then the pairing

$$\check{H}^{\ell}(X_n) \times \check{H}^{n-\ell+1}(X_n) \to \check{H}^1(pt) = \mathbb{R}/\mathbb{Z} \tag{5.31}$$

is a perfect pairing. [REFS!] This is the notion of Poincaré duality for differential cohomology. It says that $\check{H}^{n-\ell+1}(X_n)$ is the Pontryagin dual group to the abelian group $\check{H}^{\ell}(X_n)$. That will be important later.

5.5 Choice of Dirac Quantization: The torus theories

When we consider the quantum theory of generalized abelian gauge fields we must make a *choice* of Dirac quantization. This ultimately boils down to a choice of a generalized cohomology theory. As Hopkins and Singer showed [112], given a generalized cohomology theory one can produce an associated differential theory.

If we begin with $F \in \Omega_d^{\ell}(X;V)$ and ask for a corresponding quantum theory then there might be several different corresponding quantum theories corresponding to different choices of Dirac quantization. The differential cohomology theory described in 5.4 above corresponds to Eilenberg-MacLane cohomology with

$$H^{\ell}(X) = [X, K(\mathbb{Z}, \ell)] \tag{5.32}$$

But we could choose other cohomology theories. For example, in type II string theory it is thought that the proper generalized cohomology is not a product of Eilenberg-MacLane theories but rather K-theory. [REFS].

Here we will simply replace the Eilenberg-MacLane spaces $K(\mathbb{Z}, \ell)$ by $K(\Pi, \ell)$ where Π is a free abelian group of rank r.

The terms in Figure 4 change as follows:

1. The fieldstrengths are valued in

$$\Omega_{\Pi}^{\ell}(X, V) \tag{5.33}$$

where $V = \Pi \otimes \mathbb{R}$. Thus we have r independent fieldstrengths, (as we began in the classical theory). Now the subscript means the periods are valued in the image of Π in V (and hence are clsoed). This is the flux quantization.

2. The topological sectors are given by

$$H\Pi^{\ell}(X) := [X, K(\Pi, \ell)] \tag{5.34}$$

3. The topologically trivial gauge fields are

$$\Omega^{\ell-1}(X)/\Omega_{\Pi}^{\ell-1}(X) \tag{5.35}$$

4. The group of flat gauge fields is

$$H^{\ell-1}(X; T) \tag{5.36}$$

where $T = V/\Pi$ is a torus.

These are called "torus theories" because for $\ell = 2$ we have a generalized Maxwell gauge theory with compact gauge group the torus $T = V/\Pi$. In general, differential cohomology is a groupoid and the group of automorphisms of any object, which corresponds physically to global gauge transformations, is $H^{\ell-2}(X;T)$. ♣Careful. Check it is not the dual torus. ♣

5.6 A Reminder on Heisenberg Groups

In the following material we will be mentioning Heisenberg groups frequently. Here is a brief reminder of the central facts:

Theorem 1: Let G be an abelian group. Isomorphism classes of central extensions of G by $U(1)$:

$$1 \to U(1) \to \tilde{G} \to G \to 1 \tag{5.37}$$

are in 1-1 correspondence with skew, alternating bihomomorphisms:

$$s : G \times G \to U(1) \tag{5.38}$$

which means

skew: $s(x, y) = 1/s(y, x)$

alternating: $s(x, x) = 1$

bihomomorphism: $s(x + x', y) = s(x, y)s(x', y)$.

The basic idea of the proof is that if we have a cocycle defining the central extension then $s(x, y) = c(x, y)/c(y, x)$. The nontrivial point is that given an s we can find a c up to a coboundary.

Theorem 2: If $\tilde{G}$ is a central extension of a locally compact abelian group G then the unitary irreducible representations (up to isomorphims) of $\tilde{G}$ in which $U(1)$ acts by scalar multiplication are in 1-1 correspondence with the unitary irreducible representations of $Z(\tilde{G})$ in which $U(1)$ acts by scalar multiplication.

A special case of this is the Stone-von Neumann theorem.

Example Recall that for an abelian group A with a translationally invariant measure, $L^2(A)$ can be viewed as the unique irrep of the Heisenberg group $Heis(A \times \widehat{A})$, where $\widehat{A}$ is the Pontryagin dual group. A acts by translation and $\widehat{A}$ acts by multiplication:

$$(T_{a_0}\psi)(a) := \psi(a + a_0).$$
$$(M_\chi\psi)(a) := \chi(a)\psi(a)$$

then

$$T_{a_0}M_\chi = \chi(a_0)M_\chi T_{a_0}.$$

from which we obtain the cocycle:

$$c((a_1, \chi_1), (a_2, \chi_2)) = \frac{1}{\chi_1(a_2)} \tag{5.39}$$

as a further specialization of this we could take $A = \mathbb{Z}_n$ giving the famous finite Heisenberg group ♣check sign on RHS ♣

$$\langle q, u, v | u^n = 1, v^n = 1, uq = qu, vq = qv, uv = qvu \tag{5.40}$$

or we could take $A = \mathbb{R}$ and $\hat{A} = \mathbb{R}^*$ giving the usual Heisenberg group of quantum mechanics.

Remark: Note that skew implies $s(x,x)^2 = 1$ so $s(x,x) = \pm 1$. There is an analog of Theorems 1 and 2 where we drop the condition that s is alternating and speak instead of $\mathbb{Z}_2$-graded Heisenberg groups and representations.

5.7 Quantization of the Torus Theories

We comment briefly on some aspects of the quantum theory with gauge equivalence classes given by $\check{H}\Pi^\ell$.

We must choose a bilinear form b on V, and we take it to be induced by a bilinar form on Π.

5.7.1 Partition function

The partition function is straightfoward in the absence of either electric or magnetic current. It has the form

$$Z = \int_{\check{H}\Pi^\ell(X)} \mu e^{-\pi \int \lambda(F,F)} = Z_{osc} Z_{flux} \tag{5.41}$$

where μ is a translation invariant measure on $\check{H}\Pi^\ell(X)$ induced by the Riemannian metric.

When we generalize this to include the presence of electric current the generalization is straightforward. We can view the electric current as the "fieldstrength" of a differential class

$$[\check{j}_e] \in (\check{H}\Pi^*)^{n-\ell+1}(X) \tag{5.42}$$

and then we insert the electric coupling

$$\exp[2\pi i \langle [\check{A}], [\check{j}_e] \rangle] \tag{5.43}$$

into the path integral. If $[\check{A}]$ is topologically trivial this reduces to the standard insertion $\exp[2\pi i \int A^I_{\ell-1} J_{e,I}]$.

On the other hand, if $J_m \neq 0$ then even the classical coupling (5.43) is problematical. Nevertheless it can still be defined [77]. The idea is that we must pass to a cochain formulation of differential cohomology and at that level $[\check{j}_m]$ is represented by a differential cocycle $\check{j}_m$ which must be trivializable by a cochain $\delta \check{A} = \check{j}_m$. Then since $\check{j}_e$ is also a differential cocycle the expression

$$\exp[2\pi i \int^{\check{C}} \check{A} \cdot \check{j}_e] \tag{5.44}$$

still makes sense. However, it must now be regarded as a section of a line bundle with connection over the space $\mathcal{P}$ of parameters of the theory. That space can be taken to include the space of topologically trivial currents themselves

$$\mathcal{P} = \check{H}\Pi^{\ell+1}(X)_0 \times (\check{H}\Pi^*)^{n-\ell+1}(X)_0 \tag{5.45}$$

where the subscript 0 indicates that we restrict to the topologically trivial sector. Indeed, the line bundle is given by

$$\int_{\mathcal{X}/\mathcal{P}}^{\check{H}} [\check{j}_e] \cdot [\check{j}_m] \in \check{H}^2(\mathcal{P}). \tag{5.46}$$

Recall that $\check{H}^2(\mathcal{P})$ is the space of isomorphism classes of line bundle with connection over $\mathcal{P}$. In particular, the curvature of the line bundle is the two form $\int_{\mathcal{X}/\mathcal{P}}^{H} J_e \wedge J_m$.

The partition function $Z(\check{j}_e, \check{j}_m)$ would now be interpreted as a functional integral over the trivializations of $\check{j}_m$ (which is a torsor for $\check{H}\Pi^\ell(X)$) with the usual action and the coupling (5.44) inserted. This makes sense since the line bundle with connection $\int \check{j}_e \cdot \check{j}_m$ is translation invariant over this torsor.

The main lesson we should draw from this discussion is that *in the presence of simultaneous electric and magnetic current the partition function $Z(\check{j}_e, \check{j}_m)$ cannot be regarded as a function but is rather a section of a line bundle.*

5.7.2 Hilbert space

Here for simplicity we restrict attention to a single generalized maxwell field $[\check{A}] \in \check{H}^\ell(X)$.

♣Need to generalize to torus theories ♣

In the Hamiltonian formulation we take $X = \mathbb{R} \times S$ where S is a spatial slice. For simplicity we take S to be compact.

From the action principle we derive the relation between the classical field and the canonical momentum:

$$\Pi = 2\pi\lambda(*F)|_S \tag{5.47}$$

the phase space is

$$T^*\check{H}^\ell(S) = \check{H}^\ell(S) \times \Omega_d^{n-\ell}(S) \tag{5.48}$$

and standard quantization should give $L^2(\check{H}^\ell(S))$ with respect to some measure on $\check{H}^\ell(S)$. [20]

The Heisenberg relations are as follows: Π and F become operator-valued $(n-\ell)$ and ℓ-forms, respectively and we have the relations

$$[\int_S \omega_1 F, \int_S \omega_2 \Pi] = i\hbar \int_S \omega_1 d\omega_2 \tag{5.49}$$

for differential forms ω_1, ω_2 of the appropriate degrees.

While straightforward, this raises two issues:

[20] We are going to be cavalier about issues of functional analysis here, believing that in this Gaussian theory such points can be dealt with completely rigorously. Roughly speaking our wavefunctionals should have Gaussian falloff for large fieldstrengths: $\psi(\check{A}) \sim \exp[-\int_S \kappa F * F]$ where κ is some positive constant. We choose a basic such falloff for the groundstate and construct the whole Hilbert space by action of the operators $F(\check{A})$ and Π on that state. Then we construct view wavefunctions as L^2 half-densities. The story should be similar to G. Segal's discussion of the Hilbert space of a massive scalar field. An important related issue we are not discussing here is the issue of polarization. We are studying representations of the Heisenberg algebra with energy bounded from below.

- Of course, we have broken manifest EM duality. Quantum EM duality suggests that we should have an isomorphic description in terms of $L^2(\check{H}^{n-\ell}(S))$. How does this work?

- There are no general rules in quantum physics for quantizing disconnected phase spaces.

We can solve both problems by exploiting the fact that $\check{H}^\ell(S)$ is an abelian group.

We can apply this in our case because of the beautiful fact that if S is compact and oriented then we have Poincaré-Pontryagin duality: The canonical pairing

$$\check{H}^\ell(S) \times \check{H}^{n-\ell}(S) \to \mathbb{R}/\mathbb{Z} \tag{5.50}$$

is in fact a perfect pairing. Therefore $\check{H}^{n-\ell}(S)$ is the Pontryagin dual group and we can apply the construction of §5.6.

Next, the Stone-von Neumann theorem guarantees that we have a unique irrep of this group where the central $U(1)$ acts by scalars. Therefore

The Hilbert space $\mathcal{H}(S)$ of the theory on a spatial slice S is, up to isomorphism, the unique SvN irrep of

$$\mathrm{Heis}(\check{H}^\ell(S) \times \check{H}^{n-\ell}(S))$$

with the cocycle given by canonical pairing. This is a manifestly EM dual formulation of Hilbert space!

5.7.3 Vertex operators

If $v \in \Pi^*$ and $\Sigma_{\ell-1}$ is a closed oriented $(\ell-1)$-cycle we have "vertex operators" which we may schematically write

$$\mathcal{V}(v, \Sigma_{\ell-1}) := \exp[2\pi i \int_{\Sigma_{\ell-1}} \langle v, [\check{A}] \rangle] \tag{5.51}$$

If $[\check{A}]$ is topologically trivial then it can be represented (up to gauge transformation) by a globally well-defined ℓ-form A and we can simply write

$$\mathcal{V}(v, \Sigma_{\ell-1}) := \exp[2\pi i \int_{\Sigma_{\ell-1}} \langle v, A \rangle] \tag{5.52}$$

Note that the "large gauge transformations" shift A by a closed $(\ell-1)$-form with periods in Π, so the vertex operator is gauge invariant. If $\Sigma_{\ell-1} = \partial B_\ell$ is a boundary it can be written as

$$\exp[2\pi i \int_{B_\ell} \langle v, F \rangle] \tag{5.53}$$

These expressions can be inserted into the path integral and thus define $(\ell-1)$-dimensional defects in the theory. For example they will be the basis for constructing the surface defects in the $(2,0)$ theory on the Coulomb branch. They are often referred to in the physical literature as "Wilson surface operators."

The usual vertex operator case is obtained by putting $n = 2$ and $\ell = 1$.

♣ Need to put in conditions for mutual locality of the operators ♣

♣What about monodromy defects? Cod 2 defects? ♣

5.8 Chiral theories

5.8.1 Classical chiral theory

In $n = 4s+2$ dimensions with Lorentzian signature $*^2 = 1$ and we can impose a self-duality condition on solutions to the Maxwell equations for $\ell = 2s+1$.

In the higher rank case we take $F \in \Omega^\ell(M; V)$. In order to write the self-duality equations we require an extra structure:

1. $n = 0\text{mod}4$: $I^2 = -1$, complex structure on V allows us to write the self-duality equation: $F = \pm(* \otimes I)F$. (e.g. Seiberg-Witten theory)

2. $n = 2\text{mod}4$: $I^2 = +1$, an involution, or equivalently, a projection operator on V allows us to write: $F = \pm(* \otimes I)F$. (e.g. Narain theory)

For a physical theory one needs to write an energy-momentum tensor. Recall that previously this required us to endow V with a positive symmetric form. Now, we demand that this is compatible with the extra structure of I:

1. For $n = 0\text{mod}4$ we require compatibility between the positive quadratic form $g(v, w)$ and the complex structure: $b(Iv, Iw) = b(v, w)$. This allows us to define an symplectic form $\omega(v, w) := b(v, Iw)$. The space of coupling constants is then the Seigel upper half-plane: $Sp(V)/U(V)$.

2. For $n = 2\text{mod}4$ we require an orthogonal structure compatible with the involution so that $g(v) = \langle v, Iv\rangle$ is a positive definite metric. The space of coupling constants is then the Grassmannian: $O(p, q))/O(p) \times O(q)$.

After making choices can write actions for $n = 0\text{mod}4$ and also for $n = 2\text{mod}4$ if $p = q$.

5.8.2 Challenges for Quantum Theory

Let us return to the case of a single self-dual $\ell = 2s+1$ form in $4s+2$ dimensions. We want now to discuss the quantum theory. There are three problems which immediately arise:

1. The obvious action is zero $\int F * F = \int FF = 0$. Nevertheless, there is an action, but it involves further choices, as described below.

2. $[F] = [*F]$: How can flux be quantized !?

3. NB. Now $J_e = J_m$ (more generally, there is an isomorphism between differential cohomology groups where electric and magnetic currents are defined.) Therefore we have simultaneous presence of electric and magnetic current and the theory will be anomalous.

5.8.3 Approach via chiral factorization

Physically we expect the theory of a nonchiral $\ell = 2s+1$ form in $4s+2$ dimensions to "factorize" as a self-dual and anti-self-dual theory.

To make this intuition plain consider for example the theory on $M = \mathbb{R} \times S$. If we work with topologically trivial fields $F = da$ then a satisfies the wave equation:

$$d_M^\dagger d_M a = 0 \tag{5.54}$$

but in this dimension and degree the waveoperator $d_M^\dagger d_M$ splits:

$$\partial_t^2 + d^\dagger d = (\partial_t - *d)(\partial_t + *d) \tag{5.55}$$

where d and Hodge $*$ refer to S. In particular, on a $2s$ form $*d : \Omega^{2s}(S) \to \Omega^{2s}(S)$ so it makes sense to define chiral and antichiral waves:

$$\begin{aligned} (\partial_t - *d)a_L &= 0 \\ (\partial_t + *d)a_R &= 0 \end{aligned} \tag{5.56}$$

The general solution will be a sum of chiral and anti-chiral solutions

Closely related to this, in first quantization the (reductive part of the) little group is $SO(n-2) = SO(4s)$. The usual representation Λ^{2s} now splits into irreducibles:

$$\Lambda^{2s} \cong \Lambda_+^{2s} \oplus \Lambda_-^{2s} \tag{5.57}$$

because $*^2 = +1$ in Euclidean signature on $\mathbb{R}^{4s}$.

Also, if we decompose a nonchiral fieldstrength $F = F^+ + F^-$ into its chiral and anti-chiral parts:

$$F^\pm = \frac{1}{2}(F \pm *F) \tag{5.58}$$

then (somewhat nontrivially)

$$T[F] = T[F^+] + T[F^-], \tag{5.59}$$

so, at least naively, we expect the dynamics of the modes to decouple.

For all these reasons it becomes interesting to try to "split" the Hilbert space of the non-self-dual field and also the partition function of the non-self-dual field coupled to external electric and magnetic currents.

We take the action of the nonchiral theory to be

$$S = \int_X \pi \lambda F * F \tag{5.60}$$

In the case of $n = 2, \ell = 1$ this is the famous Gaussian model with a periodic boson target. Then $\lambda = R^2$ where R is the radius of the circle target space. When R^2 is rational we have the rational Gaussian model, which is a simple example of an RCFT.

♣Need to generalize to the torus case. ♣

Partition function. We consider the partition function of the nonchiral theory where we couple to an external source $J \in \Omega^3(X_6)$. Note that,

$$\int_X F^+ J^- = \int_X F^- J^+ = 0 \tag{5.61}$$

where $F^{\pm}$ is the self-dual/anti-self-dual projection. In Euclidean space $*^2 = -1$ so we define it to be

$$F^{\pm} = \frac{1}{2}(F \pm i * F) \tag{5.62}$$

Now we simply study the partition function $Z(J)$ of the nonchiral theory coupled to J.

$$Z(J) = Z_{osc} \sum_{f \in \mathcal{H}^{2s+1}(X;\mathbb{Z})} e^{-\pi \int_X \lambda(f,f) + \int_X fJ} \tag{5.63}$$

Choosing a Lagrangian decomposition $\mathcal{H}^{2s+1}(X;\mathbb{Z}) = L \oplus L^{\perp}$ we do a Poisson transformation on the sum over $L^{\perp}$. When $\lambda = p/q$ is *rational* we arrive at an expression in the form of a finite sum:

$$Z(J) = Z_{osc} \sum_{\alpha,\beta} N_{\alpha,\beta} \Theta_{\alpha}(J^+) \Theta_{\beta}(J^-) \tag{5.64}$$

where α, β run over a basis for an irreducible representation of a Heisenberg group extension of $H^{2s+1}(X_{4s+2}; \Pi^*/\Pi)$.

♣Describe the group precisely and how it acts on the theta functions. ♣

We interpret $\Theta_{\alpha}(J^+)$ and $\Theta_{\beta}(J^-)$ as "conformal blocks" – at least in so far as concerns the dependence on $J^{\pm}$. The Theta functions are theta functions of level $2pq$. See [28] for the detailed computation.

♣♣ Remark on the inclusion of the quadratic refinement in the partition function and how it modifies the factorization into conformal blocks. This was done first in the case of the chiral boson in [10, 11]. See Appendix E of [28]. ♣♣

Remark that the chiral factorization of Z_{osc} involves an interesting generalization of the η-function and the gravitational anomaly an interesting generalization of the phases coming from modular transformations.

Hilbert space. In [80] Section 4.1 it is explained that the Heisenberg algebra representation on $L^2(\check{H}^{2s+1}(X_{4s+1}))$ has two commuting subalgebras of "chiral" and "antichiral" vertex operators. These have irreducible representations labeled by characters of the group of $2pq$ torsion points in $H^{2s}(Y;\mathbb{R}/\mathbb{Z})$:

$$\mathcal{H} = \oplus_{\alpha,\beta} N_{\alpha,\beta} \mathcal{H}_{\alpha} \otimes \bar{\mathcal{H}}_{\beta} \tag{5.65}$$

EXPLAIN VERTEX OPERATOR ALGEBRA OF THE GAUSSIAN MODEL AND FACTORIZATION IN THAT CASE.

(with p, q relatively prime) This generalizes the famous rational Gaussian model. It was discussed in [80]

♣Say something about the extended algebra and the category of finite reps with $2pq$ simple objects. ♣

5.9 Torus Chern-Simons Theories

Another use of the differential cohomology theory is to define torus Chern-Simons theories.

In 3mod4 dimensions, i.e. $n = 4s + 3$ if Π has an integral symmetric form κ then we can define the spin Chern-Simons action

$$S_{CS} = \pi \langle [\check{A}], [\check{A}] \rangle \tag{5.66}$$

where $[\check{A}] \in \check{H}\Pi^{2s+2}(X_n)$ and the pairing uses the form b. Note that with our normalization F has integral periods with minimal period one. If the integral symmetric form is even the expression is well-defined modulo 2π and can be used in an action, but if the integral symmetric form is odd then extra topological data is required.

An important example is the case $n = 3$. For $[\check{A}]$ topologically trivial we have action

$$S = \int_{X_3} k_{IJ} A^I dA^J \tag{5.67}$$

Here A^I are one-forms indexed by $I = 1, \dots, r$ and k_{IJ} is a symmetric integral matrix. This action has been much studied (especially in the condensed matter literature) and was studied from the viewpoint of these notes in [27]. If for all I, k_{II} are even, then we have a topological field theory. If, for some I, k_{II} is odd then in order to define the action (5.66) we need a spin structure.

♣What is the analog topological condition for $s > 0$? HS say it is an integral Wu structure. How does this specialize to spin cs? ♣

Now there is a conjectural statement [21] that these theories only depend on

a.) The finite group $D = \Pi^*/\Pi$. Note that this finite group comes equipped with a bilinear form $\bar{\kappa} : D \times D \to \mathbb{R}/\mathbb{Z}$ defined by

$$\bar{\kappa}(\bar{x}, \bar{y}) = \kappa(x, y) \mathrm{mod} \mathbb{Z} \tag{5.68}$$

where x, y are lifts of $\bar{x}, \bar{y}$ to Π^*.

b.) A quadratic refinement $q : D \to \mathbb{R}/\mathbb{Z}$ of the bilinear form $\bar{\kappa}$ that, is

$$q(x + y) - q(x) - q(y) + q(0) = \bar{\kappa}(x, y) \tag{5.69}$$

Note that by the Gauss-Milgram formula

$$\frac{1}{\sqrt{|D|}} \sum_{x \in D} e^{2\pi i q(x)} = e^{2\pi i \sigma/8} \tag{5.70}$$

q defines an integer modulo 8, $\sigma \mathrm{mod} 8$.

c.) A lift of $\sigma \mathrm{mod} 8$ to $\sigma \mathrm{mod} 24$.

♣Not so obvious this is the right data for all $4s + 3$. ♣

For $n = 3$ this has been shown for the partition function and Hilbert space in [27]. It has been extended to 0-1-2-3 extended topological field theory in [81][125]. It appears that there is no obstruction to generalizing this to higher dimensions, although this has not been done, and we will just assume it here for the case of $n = 7$.

What is clear is that the topological field theory has a Hilbert space on 6-manifolds which is an irreducible representation of a "Heisenberg group extension" of

$$H^{2s+1}(X_{4s+2}, D) \tag{5.71}$$

where the skew bihomomorphism defining the extension is given by combining the symmetric bilinear form 5.68 on D with the natural pairing on forms from multiplying and integrating. Then we use Theorem 1 of 5.6 to obtain a Heisenberg group *up to isomorphism*.

♣Is this a nondegenerate pairing? Otherwise there is not a unique irrep ♣

In the 3-2-1 case we have

dim 3: CS invariant

[21] Unpublished work in progress by D. Freed and G. Moore.

dim 2: Hilbert space of conformal blocks of the chiral WZW theory with torus group and level in $H^4(BT;\mathbb{Z})$ determines by Π. This is an irreducible representation of $Heis(H^1(\Sigma_2; D))$.

dim 1: Category of highest weight representations of a Heisenberg extension of the loop group $\check{H}\Pi^1(S^1)$.

We expect that in the 7-6-5 case

dim 7: CS invariant

dim 6: Hilbert space of conformal blocks is an irrep of $Heis(H^3(X_5; D))$.

dim 5: Category of highest weight representations of the infinite-dimensional abelian group $Heis(\check{H}\Pi^3(X_5))$.

Remarks:

1. There is an analog in 1mod4 dimensions where we choose an anti-symmetric integral form on Π. We will not need this in these notes.

2. Comment on case $F = L \times \hat{L}$. Equivalent to Chern-Simons gauge theory for finite gauge group L [25, 81, 125].

5.10 Generalizing the notion of "field theory"

5.10.1 Anomalous Field Theories

Anomalous field theories are characterized by the property that their partition functions and correlation functions are not numbers but rather sections of line bundles.

A familiar example of this from 2d CFT follows from the conformal anomaly. In a two-dimensional conformal field theory the partition function $Z(g)$, where g is the Riemannian metric on a two-dimensional surface Σ is *not* invariant under Weyl rescalings of the metric. Rather, as is very well-known

$$Z(e^{2\sigma}g) = e^{cS_L(\sigma;g)}Z(g) \tag{5.72}$$

where $S_L(\sigma; g)$ is the famous Liouville action and c is the central charge. The central charge is generally nonzero and always nonzero in nontrivial unitary theories.

Now, one would expect that a "conformal field theory" should only depend on the conformal class of the metric. Equation (5.72) says that this is almost true. Indeed we can view the statement as literally true if we recall that the the space of conformal structures $Conf(\Sigma)$ is the base of a principal fibration:

$$Weyl \ltimes Diff^+ \to Met(\Sigma) \to Conf(\Sigma) \tag{5.73}$$

Now, the Liouville action in (5.72) defines a one-dimensional representation of $Weyl$ and equation (5.72) says that $Z(g)$ is a section of an associated line bundle over $Conf(\Sigma)$.

♣Not really. Liouville is a cocycle. Ref. Shatashvili. ♣

Now, we can invent a *three-dimensional* invertible field theory whose Hilbert space on (Σ, g) is just a one-dimensional line. We could then say that $Z(g)$ is valued in the Hilbert space of that 3d theory.

♣Say more precisely what that 3d theory is. Something like η invariant of 3d Dirac. ♣

5.10.2 n-dimensional field theory valued in an $(n+1)$-dimensional field theory

Chiral bosons and chiral torus Chern-Simons:

On a surface Σ_2 CSW gives a finite-dimensional vector space. In the chiral theory the partition function is a vector in that vector space. To specify which vector we must give a conformal block, i.e. a vector in $\mathcal{H}(\Sigma_2)$.

On a circle Σ_1 CSW gives a category of representations. So the chiral CFT "Hilbert space" is valued in a category of representations. If we choose an object in the category then that specifies a particular Hilbert space.

We call this "a 2-dimensional field theory valued in the 3-dimensional topological field theory of Chern-Simons-Witten."

We can summarize the discussion more formally we a series of definitions:

Definition 1: An n-dimensional field theory $\tilde{\mathcal{F}}^{(n)}$ valued in an (ordinary) $(n+1)$-dimensional field theory $\mathcal{F}^{(n+1)}$ would be one where

$$\tilde{\mathcal{F}}^{(n)}(X) \in \mathcal{F}^{(n+1)}(X) \tag{5.74}$$

for all X. In particular, the partition function of an n-dimensional field theory is a vector in the Hilbert space of an $(n+1)$-dimensional field theory. So there is a vector space of partition functions. In $(n-1)$ dimensions the Hilbert space is an object in a category, etc.

Definition 2: An *invertible field theory* is one where the partition function is always nonvanishing and the Hilbert space is a one-dimensional vector space. At the next level the category of vector spaces is invertible in the category of (linear) categories, so The notion of invertibility can be given for a full extended TFT, but we will not need it. ♣Ref to Freed-Teleman? ♣

Definition 3: Finally we can define an *anomalous n-dimensional field theory* is an n-dimensional field theory valued in a invertible $(n+1)$-dimensional

Remarks:

1. In the 2d case this notion was suggested long ago by Segal as a "Weak conformal field theory" [G.Segal, Notes on CFT. In 60th birthday volume] It was implicit in much of the work of physicists on RCFT.
2. In [177] Witten is suggesting some version of this idea for the six-dimensional (2,0) theory. He calls them "vector valued theories." Witten's discussion was followed up in [107].
3. The point of view given here was developed in discussion with Dan Freed. It can be made much more precise.

5.11 The issue of an action

It is often said that there is no action for the self-dual field. This is not really true. There is a classical action, provided one is willing to make some extra choices.

Let us consider the simplest case where there is no nontrivial topology. For example we could consider the theory on $\mathbb{R}^{1,4s+1}$. The space $V := \Omega^\ell(M_n)$ with $\ell = 2s+1$ has a symplectic structure

$$\omega(\phi_1, \phi_2) := \int_M \phi_1 \wedge \phi_2 \tag{5.75}$$

1. First, as in the ordinary nonself-dual theory, to formulate an action we restrict attention to fields $\mathcal{R} \in V_{cl} := \Omega^{\ell}_{d}(M)$. We will vary within this space. Note that it is a Lagrangian subspace of V. We call the fieldstrength $\mathcal{R}$ so that it will not be confused with the classical self-dual field we see in the semiclassical physics of this theory. We will do the path integral over closed fields $\mathcal{R}$ modulo gauge transformations by

2. Now, we *choose* another Lagrangian subspace $V_m \subset V$, assumed to be maximal Lagrangian and transversal to V_{cl} (i.e. $V_{cl} \cap V_m = \{0\}$) and moreover

$$V = V_m \oplus *V_m \tag{5.76}$$

is a decomposition into maximal Lagrangian subspaces. We will demand that V_m and $*V_m$ are transverse. (This condition can be slightly relaxed.)

3. Now, given (5.76) there is a unique decomposition of any $\mathcal{R} \in V_{cl}$ as

$$\mathcal{R} = \mathcal{R}_m + \mathcal{R}_e \tag{5.77}$$

with $\mathcal{R}_m \in V_m$ and $\mathcal{R}_e \in V_e := *V_m$.

4. The Lorentzian signature action for the ϵ-self-dual field is then

$$S = \pi \int (\mathcal{R}_e * \mathcal{R}_e + \epsilon \mathcal{R}_e \mathcal{R}_m) \tag{5.78}$$

There are two nice features of this action:

5. First, the action is stationary iff the ϵ-self-dual field $\mathcal{F} := \mathcal{R}_e - \epsilon * \mathcal{R}_e$ is closed:

$$d\mathcal{F} = 0 \tag{5.79}$$

Thus, the set of stationary points of the action is the set of solutions of the self-dual equations of motion for $\mathcal{F}$.

6. The second nice feature is that if we consider the action as a functional of both the metric and the field $\mathcal{R}$ then, varying the metric holding $\mathcal{R}$ fixed the action varies into

$$\delta S = \frac{\pi}{2} \int \mathrm{vol}\,(g) \delta g^{\mu\nu} T(\mathcal{F})_{\mu\nu} \tag{5.80}$$

where $T(\mathcal{F})_{\mu\nu}$ is the standard energy-momentum tensor for the ϵ-self-dual field $\mathcal{F} := \mathcal{R}_e + \epsilon * \mathcal{R}_e$.

7. The proof that the variation of the action gives (5.79) goes as follows:

$$\begin{aligned} \delta S &= \pi \int 2\delta\mathcal{R}_e * \mathcal{R}_e + \epsilon \delta\mathcal{R}_e \mathcal{R}_m + \epsilon \mathcal{R}_e \delta\mathcal{R}_m \\ &= \pi \int 2\delta\mathcal{R}_e * \mathcal{R}_e + 2\epsilon \delta\mathcal{R}_e \mathcal{R}_m \\ &= 2\pi \int \delta\mathcal{R}(*\mathcal{R}_e + \epsilon \mathcal{R}_m) \\ &= 2\pi \int d(\delta c)(*\mathcal{R}_e + \epsilon \mathcal{R}_m) \end{aligned} \tag{5.81}$$

Where in the second line we used the fact that both $\mathcal{R}$ and $\delta\mathcal{R}$ are in V_{cl}, which is Lagrangian. In the third line we notice that $*\mathcal{R}_e + \epsilon\mathcal{R}_m \in V_m$, and hence we can replace $\delta\mathcal{R}_e$ by $\delta\mathcal{R}$. In the fourth line we use the fact that variations of $\mathcal{R}$ in V_{cl} are exact. Now integration by parts gives $d(*\mathcal{R}_e + \epsilon\mathcal{R}_m) = 0$. Finally, $d\mathcal{R} = d\mathcal{R}_e + d\mathcal{R}_m = 0$, so $d(*\mathcal{R}_e + \epsilon\mathcal{R}_m) = 0$ is equivalent to $d(*\mathcal{R}_e - \epsilon\mathcal{R}_e) = 0$ which is equivalent to $d(\mathcal{R}_e - \epsilon * \mathcal{R}_e) = 0$.

♣Check that you reproduce the correct symplectic form on the solutions of the equations of motion. ♣

Remarks

1. We would conjecture that there is no *local* Lorentz invariant choice of Lagrangian subspace V_m, V_e. This this has not been carefully investigated. It would be nice to have a rigorous statement.

2. There are other action principles: PST, Henneaux et. al. COMMENT ON RELATION

3. MENTION Monnier claim that quadratic fluctuation determinant comes out wrong for one chiral boson but ok for two...

5.12 Compactification to 5 dimensions: 5D Abelian SYM

If $X_6 = X_5 \times S^1$. We choose a coordinate $\theta \sim \theta + 2\pi$ on S^1 and a direct sum metric $ds^2_{X_5} + R^2(d\theta)^2$.

Then there is a natural Lagrangian decomposition of the space of fields

$$V_m = \ker \iota(\partial_\theta) \subset \Omega^3(X_6) \tag{5.82}$$

$$V_e = *V_m \tag{5.83}$$

So elements of V_m are $h = \frac{1}{3!}h_{mnp}(x,\theta)d^{mnp}$ where x^m are coordinates on X_5 and elements of V_e are of the form $F = \frac{1}{2!}F_{mn}(x,\theta)dx^{mn}$. We can decompose

$$\mathcal{R} = F \wedge d\theta + h \tag{5.84}$$

Note that $dR = 0$ implies

$$\begin{aligned} dF &= \partial_\theta h \\ dh &= 0 \end{aligned} \tag{5.85}$$

where d is the exterior derivative on X_5.

Then the action (5.78) becomes

$$\int_{X_5} \oint d\theta \left(\frac{1}{R} F * F + F \wedge h\right) \tag{5.86}$$

where $*$ is the 5-dimensional Hodge star.

In the low energy limit we keep only the zeroth Fourier mode on the circle and then $dF = 0$ and $\int_{X_5} F \wedge h$ does not contribute to the equation of motion, so that we recover - at least classically - 5D Maxwell theory with $g^2_{YM} = R$:

$$\frac{1}{R} \int F * F \tag{5.87}$$

Note that we have derived $g^2_{YM} \sim R$, as indeed had to be the case just from dimensional analysis. This will be much more nontrivial in the nonabelian case, where there is no field multiplet and no action.

5.13 Compactification to four dimensions

Let us consider the low energy reduction $\mathcal{F}//\Sigma$ where Σ is a two-dimensional surface with (Euc. sig.) Riemannian metric.

In the the low energy compactification H can be written as a sum of terms

$$H = F \wedge \alpha \tag{5.88}$$

where α is a harmonic form on Σ and F is a 2-form on $\mathbb{M}^{1,3}$. Thus, we can say that the low energy fieldstrength is a 2-form $\mathbb{F}$ valued in $V = \mathcal{H}^1(\Sigma)$, the space of harmonic one-forms on Σ.

Note that

1. V is a symplectic vector space

$$\langle \alpha, \beta \rangle := \int_\Sigma \alpha \wedge \beta \tag{5.89}$$

2. $*^2 = -1$ on one-forms on Σ. So V has a complex structure.

3. We have a positive compatible complex structure.

Thus, the low energy limit is a *self-dual abelian gauge theory in* $\mathbb{M}^{1,3}$. Let us define that formally:

5.13.1 Self-dual abelian gauge theory in 4d

Let V be a real symplectic vector space. Our fieldstrength will be $\mathbb{F} \in \Omega^2(\mathbb{M}^{1,3} \otimes V)$.

In order to define the theory we add the data of a *positive compatible complex structure.*

This means

1. *Complex structure*: There is an $\mathbb{R}$-linear operator $\mathcal{I} : V \to V$ such that $\mathcal{I}^2 = -1$.

2. *Compatible* Moreover $\langle \mathcal{I}(v), \mathcal{I}(v') \rangle = \langle v, v' \rangle$ for all $v, v' \in V$.

3. *Positive* Since $\mathcal{I}$ is compatible with the symplectic product we can introduce the *symmetric* bilinear form
$$g(v, v') := \langle v, \mathcal{I}(v') \rangle \tag{5.90}$$
We will assume that g is positive definite. We will often denote the metric simply by (v, v').

Now we have

$$\mathbb{F} \in \Omega^2(M_4; V) \tag{5.91}$$

Now $*^2 = -1$ on $\Omega^2(M_4)$ for M_4 of Lorentzian signature, so $s := * \otimes \mathcal{I}$ squares to 1 and as in 5.8.1 we can impose the ϵ-self-duality constraint

$$s\mathbb{F} = \epsilon \mathbb{F} \tag{5.92}$$

where $\epsilon = +1$ for a *self-dual field* and $\epsilon = -1$ for an *anti-self-dual field.*

The dynamics of the field is simply the flatness equation:

$$d\mathbb{F} = 0 \tag{5.93}$$

but in order to recognize this as a generalization of Maxwell's theory we need to do some linear algebra.

5.13.2 Lagrangian decomposition of a symplectic vector space with compatible complex structure

In our physical considerations we will be choosing "duality frames." This will amount to choosing a Darboux basis for Γ denoted $\{\alpha_I, \beta^I\}$ where $I = 1, \dots, r$. Our convention is that

$$\begin{aligned} \langle \alpha_I, \alpha_J \rangle &= 0 \\ \langle \beta^I, \beta^J \rangle &= 0 \\ \langle \alpha_I, \beta^J \rangle &= \delta_I{}^J \end{aligned} \tag{5.94}$$

The $\mathbb{Z}$-linear span of α_I is a maximal Lagrangian sublattice L_1 while that for β^I is another L_2 and we have a Lagrangian decomposition:

$$\Gamma \cong L_1 \oplus L_2. \tag{5.95}$$

Upon complexification we have $V \otimes_{\mathbb{R}} \mathbb{C} \cong V^{0,1} \oplus V^{1,0}$. The $\mathbb{C}$-linear extension of $\mathbb{C}$ is $-i$ on $V^{0,1}$ and $+i$ on $V^{1,0}$.

Given a Darboux basis we can define a basis for $V^{0,1}$:

$$f_I := \alpha_I + \tau_{IJ}\beta^J \qquad I = 1, \dots, r \tag{5.96}$$

while $V^{1,0}$ is spanned by

$$\bar{f}_I := \alpha_I + \bar{\tau}_{IJ}\beta^J \qquad I = 1, \dots, r \tag{5.97}$$

So we have:

$$\mathcal{I}(f_I) = -i f_I \qquad \mathcal{I}(\bar{f}_I) = +i \bar{f}_I \tag{5.98}$$

Compatibility if $\mathcal{I}$ with the symplectic structure now implies $\langle f_I, f_J \rangle = 0$ and hence $\tau_{IJ} = \tau_{JI}$. It is useful to write τ_{IJ} in its real and imaginary parts

$$\tau_{IJ} = X_{IJ} + iY_{IJ} \tag{5.99}$$

Positive definiteness of g implies Y_{IJ} is positive definite. It will be convenient to denote the matrix elements of the inverse by Y^{IJ} so

$$Y^{IJ}Y_{JK} = \delta^I{}_K \tag{5.100}$$

Using the inverse transformations to (5.96) (5.97):

$$\begin{aligned} \beta^I &= -\frac{i}{2}Y^{IJ}(f_J - \bar{f}_J) \\ \alpha_I &= \frac{i}{2}\bar{\tau}_{IJ}Y^{JK}f_K - \frac{i}{2}\tau_{IJ}Y^{JK}\bar{f}_K \end{aligned} \tag{5.101}$$

We compute the action of $\mathcal{I}$ in the Darboux basis:

$$\begin{aligned} \mathcal{I}(\alpha_I) &= \alpha_K(Y^{-1}X)^K{}_I + \beta^K(Y + XY^{-1}X)_{KI} \\ \mathcal{I}(\beta^I) &= -\alpha_K Y^{KI} - \beta^K(XY^{-1})_K{}^I \end{aligned} \tag{5.102}$$

5.13.3 Lagrangian formulation

Equations (5.91), (5.92), (5.93) summarize the entire theory in a manifestly invariant way.

Usually physicists choose a Darboux basis or "duality frame" for V. We can then define components of $\mathbb{F}$:

$$\mathbb{F} = \alpha_I F^I - \beta^I G_I \tag{5.103}$$

If we impose the ϵSD equations (5.92) then, using (5.102) we can solve for G_I in terms of F^I and the complex structure:

$$G_J = -\epsilon Y_{JK} * F^K - X_{JK}F^K \tag{5.104}$$

Now the equation (5.93) splits naturally into two

$$dF^I = 0 \tag{5.105}$$

$$dG_J = -\epsilon d\left(Y_{JK} * F^K + \epsilon X_{JK}F^K\right) = 0 \tag{5.106}$$

(5.105) is the Bianchi identity and (5.106) is the equation of motion of a generalization of Maxwell theory.

Because of (5.105) we can locally solve $F^I = dA^I$ and thereby write an action principle with action proportional to

$$\int_{M_4} \left(\mathrm{Im}\tau_{JK}F^J * F^K + \epsilon \mathrm{Re}\tau_{JK}F^J F^K\right) \tag{5.107}$$

This is part of the low energy action in the Seiberg-Witten theory.

1. From this Lagrangian we learn the physical reason for demanding that the bilinear form $g(v, v')$ be positive definite. If Y_{IJ} were diagonal then $Y_{IJ} = \delta_{IJ}\frac{1}{e_I^2}$ would be the matrix of coupling constants, and X_{IJ} would be a matrix of theta-angles. Thus, the data of the complex structure on V summarizes the complexified gauge coupling of the theory.

2. Note that the difference between the self-dual and anti-self-dual case is the relative sign of the parity-odd term, as expected.

3. The self-dual equations (5.91), (5.92), (5.93) do not follow from a relativistically invariant action principle. This is a famous surprising property of self-dual field theories. However, once one chooses a duality frame (which induces a Lagrangian splitting in the space of fields) one can indeed write an action principle, as above. A very similar remark applies to the self-dual theory of (abelian) tensormultiplets in six-dimensional supergravity. Once one chooses a Lagrangian decomposition of fieldspace one can write a covariant action principle for this self-dual theory [28]. This is not unrelated to the above examples: If we compactify (with a partial topological twist described in [94]) the six-dimensional abelian tensormultiplet theory on the Seiberg-Witten curve to get an abelian gauge theory in four dimensions we obtain precisely the Seiberg-Witten effective Lagrangian!

5.13.4 Duality Transformations

There is of course no unique choice of duality frame for V, although in different physical regimes one duality frame can be preferred. The change of description between different duality frames is given by an integral symplectic transformation $(\alpha_I, \beta^I) \to (\tilde{\alpha}_I, \tilde{\beta}^I)$ and leads to standard formulae for strong-weak electromagnetic duality transformations in abelian gauge theories.

An important special case of this is the torus $\Sigma = S^1 \times S^1$ where successive compactifications on the two S's distinguishes two duality frames, related by $\tau \to -1/\tau$.

Remark: The link between the self-dual 3-form in six dimensions and electric-magnetic duality after toroidal compactification was first pointed out in [Verlinde; Witten]. Note in particular that it means the S-duality of abelian d=4 N=4 SYM has been interpreted *geometrically*.

5.14 Compactification to Three Dimensions

Let us consider the compactification of a self-dual 3-form on an oriented 3-manifold X_3. At long distances the IR theory will be described by a topological field theory. We claim that it is in the class of abelian Chern-Simons topological field theories and therefore can be characterized by the data (D, q, σ).

The finite group D is $D = \mathrm{Tors}(H^2(X_3; \mathbb{Z}))$, and the bilinear form $\bar{\kappa}$ is defined by the torsion pairing:

$$(a_1, a_2)_T = \int a_1 \alpha_2 \tag{5.108}$$

where $\alpha_2 \in H^1(X_3; U(1))$ is any preimage of a_2 under the Bockstein map.

q is given by ...

σmod24 is given by ...

One way to derive the result is to consider the Hilbert space of the 6d theory on 5-manifolds of the form $\Sigma \times X$ where Σ is a compact oriented surface and applying $(\mathcal{F}//X)//\Sigma = (\mathcal{F}//\Sigma)//X$.

As we have seen, compactifying along Σ gives a self-dual abelian gauge theory where the gauge group has rank $\frac{1}{2}b_1(\Sigma) = g$. On the other hand, from the results of [80], we the Hilbert space of the theory is given by the unique irreducible representation of

$$\mathrm{Heis}(\check{H}^2_{\Pi}(X_3)) \tag{5.109}$$

where $\Pi = \mathcal{H}^2(\Sigma;\mathbb{Z})$ can be thought of as the lattice of harmonic one-forms with integral periods. The commutator function is given by combining the canonical pairing on $\check{H}^2(X_3)$ (which is symmetric) with the intersection pairing on Π (which is antisymmetric). As in the discussion in [80] the groundstates will then be in an irreducible representation of the Heisenberg group

$$\mathrm{Heis}(\mathrm{Tors}(H^2(X_3;\Pi))) \tag{5.110}$$

where the group commutator is determined by

$$s(\alpha_1\otimes\lambda_1,\alpha_2\otimes\lambda_2) = (\alpha_1,\alpha_2)_T\langle\lambda_1,\lambda_2\rangle \tag{5.111}$$

On the other hand, now from the viewpoint of $(\mathcal{F}//X)//\Sigma$, comparing with equations (4.7) and (4.8) of [27] we see that the finite group of the abelian Chern-Simons theory should indeed be D.

Remarks: 1. Compare Cecotti, Cordova, Vafa. 2. Reference [27] determines the quadratic refinement and σmod24 can be determined from the representation of the modular group for $\mathcal{H}(T^2)$. This in turn follows from the gravitational anomalies of the six-dimensional self-dual form. This determines the quadratic refinement and σmod24.

5.15 Compactification to Two Dimensions

Get Narain theory, chiral and anti-chiral bosons according to self-dual and anti-self-dual forms on M_4.

6. Physical Heuristics: The Interacting $(2,0)$ theories

"It is all true, or it ought to be; and more and better besides. - Winston Churchill

6.1 Due warning to the reader

In what follows we try to explain the reasons many string theorists believe in the existence of the remarkable $(2,0)$ superconformal field theories. In §??? we attempt to write down with some precision the ground rules physicists use when speculating about these theories.

Mathematicians will find this discussion extremely frustrating. A mathematician could well ask: "Is this mathematics?" The answer is "No." It is not even physical mathematics. The relevant question is "Can it be turned into mathematics?"

6.2 Lightning review: Type II strings and D-branes

1. Dp-branes: Extended p-dimensional spatial objects (extending in time to make $p+1$-dimensional worldvolumes).

2. They carry bundles with connection. There is thus a Yang-Mills gauge theory (possibly together with other fields) describing low energy dynamics on the brane. The "basic" D-brane carries a $U(1)$ bundle with connection. N "coincident" branes means that there is a $U(N)$ bundle with connection (the origin of the terminology is explained below). In the description via conformal field theory one takes a tensor product of a conformal field theory operator algebra with $\mathrm{End}(\mathbb{C}^N)$. At low energies on the brane, the physics is described by $U(N)$ SYM as obtained by dimensional reduction from $d = 10$ SYM.

3. *Symmetry enhancement*: We consider a system of N_1+N_2 Dp-branes in $\mathbb{M}^{1,9}$ localized in the dimensions parametrized by $(X^9, \dots, X^{p+1})$. The branes feel no mutual forces, so long as they are parallel. Therefore, $N_1 + N_2$ basic branes can be put at arbitrary points in $\mathbb{R}^{9-p}$.

   Let us say that N_1 coincident branes are at $(X^9, \dots, X^{9-p}) = 0$ and N_2 are at $(X^9, \dots, X^{9-p}) = (d, 0, \dots, 0)$. Among the massive excitations of this system are open strings stretched from one brane to the other. They have masses d/ℓ_s^2 and are charged, in the representation $(N_1, \bar{N}_2)$ (with one orientation) and in the $(\bar{N}_1, N_2)$ (with the other orientation).

   At energies
   $$\frac{d}{\ell_s^2} \ll E \ll \frac{1}{\ell_s} \tag{6.1}$$
   there is a good description of the brane physics in terms of low energy effective field theory. Moreover, for $d \to 0$ the gauge symmetry is enhanced from $U(N_1) \times U(N_2)$ to $U(N_1 + N_2)$. This reflects the decomposition
   $$u(N_1 + N_2) \cong u(N_1) \oplus u(N_2) \oplus (N_1, \bar{N}_2) \oplus (\bar{N}_1, N_2) \tag{6.2}$$

4. *Geometrization of the Higgs mechanism*: The above symmetry enhancement is exactly the kind of enhancement one finds in spontaneous symmetry breaking by adjoint Higgs fields. There is a purely gauge theoretic description of this in terms of the $(p+1)$-dimensional SYM on the brane.

   Because the gauge theory on the D-branes is actually a dimensional reduction of 10D SYM. The 10d gauge field A_M gives a gauge field A_μ on the Dp brane, $\mu = 0, \dots, p$ and $(d-p)$ scalar fields X_m, $m = 9, \dots, 9-p$. The gauge theory on the Dp brane has potential energy
   $$V = \sum_{i \neq j} \mathrm{Tr}([X_i, X_j])^2 \tag{6.3}$$
   and the (supersymmetric) vacua are the simultaneously diagonalizable matrices. The gauge invariant information is a point in $\mathrm{Sym}^N(\mathbb{R}^{9-p})$. This may be interpreted as a space parametrizing the positions of N basic branes in the transverse $9-p$ dimensions.

Thus, the transverse scalar fields are simultaneously Higgs fields and (nonabelian) coordinates in space. The formula for the mass of the broken gauge fields

$$M \sim |\Delta X|/\ell_s^2 \tag{6.4}$$

now takes on a significant new interpretation when we vies ΔX as the expectation value of the Higgs field: We have a *geometrization of the Higgs mechanism.*

5. Finally, for the discussion below, we note that Dp-branes can "wrap cycles." What this means is that the worldvolume can be taken to be of the form $\mathbb{M}^{1,s} \times K_{p-s}$ where K_{p-s} is some compact submanifold of spacetime. There are conditions K_{p-s} must satisfy if the configuration is to be stable and/or supersymmetric. At low energies the dynamics is described by an $s+1$-dimensional gauge theory (possibly + other fields).

6.3 Physical Construction 1: Type IIB strings on a singular $K3$ surface

6.3.1 Hyperkahler resolution of surface singularities

Let us first recall a well-known but remarkable fact from geometry.

If $\Gamma \subset SU(2)$ is a discrete subgroup there is a well-known 1-1 correspondence with the simply laced simple Lie algebras of ADE type: $\Gamma \leftrightarrow \mathfrak{g}(\Gamma)$.

Now there is a resolution of singularities

$$X_\Gamma \to \mathbb{C}^2 \tag{6.5}$$

and moreover, thanks to Kronheimer, we can in fact write a family of hyperkähler metrics on X_Γ.

There is a basis for homology $\Sigma_i \in H_2(X_\Gamma;\mathbb{Z})$ such that

$$\Sigma_i \cdot \Sigma_j = -C_{ij} \tag{6.6}$$

so we have a natural identification of $H_2(X_\Gamma;\mathbb{Z})$ with the root lattice $\Lambda_{rt}(\mathfrak{g}(\Gamma))$.

6.3.2 IIA string theory on a hyperkähler resolution of singularities

6.3.3 IIB string theory on a hyperkähler resolution of singularities

Witten's construction [173]:

We consider Type IIB string theory. In the classical approximation to describe groundstates of this theory we have to choose:

1. A 10-dimensional oriented spin manifold equipped with certain fields satisfying the IIB equations of motion.

2. String theorists claim there is a family of exact solutions to the fully quantum problem where the 10-manifold is $\mathbb{R}^{1,5} \times K3$ and we have:

1. A constant real valued scalar field (the dilaton). Its value (vev) is $g_s^2 = e^{2\phi}$.

2. A metric $ds^2 = ds^2_{1,3} \oplus ds^2_{K3}$ which is the direct sum of a Minkowski and hyperkähler metric. [22]

3. The theory also has a *lengthscale*, called the string-length.

The moduli space of these fields is $\mathbb{R}_+ \times \mathbb{R}_+ \times \Gamma\backslash O(3,19)/O(3) \times O(19)$, where $\Gamma \subset O(3,19;\mathbb{Z})$ is finite index.

Now, string theorists consider a one-parameter family of these solutions $g^2(\epsilon)$ and $ds^2_{K3}(\epsilon)$, $\epsilon \to 0$ so that the K3 surface develops an ADE singularity. So, near some point on the K3 there is a neighborhood that "looks like" a resolution of $\mathbb{C}^2/\Gamma'$ and $\Gamma' \subset SL(2,\mathbb{C})$ is a discrete group. The exceptional curves have size ϵ. At the same time $g^2(\epsilon) \sim \epsilon^2$.

♣Not quite right. the strings should become tensionless for $\epsilon = 0$. ♣

The remarkable claim here is that – even though the coupling constant $g^2(\epsilon) \to 0$ because the K3 surface is developing a singularity there are compensating strong interaction effects and a nontrivial interacting six-dimensional theory results. [23]

The basic phenomenon comes from the local modes near the singularity. Thus, although there is a finite list of possible ADE singularities that can occur in K3 manifolds we can instead consider IIB theory on a hyperkähler resolution of the $\mathbb{C}^2/\Gamma$ singularity, in the limit that the exceptional divisors have area going to zero like ϵ. Thus, we expect the existence of six-dimensional theories $S[\mathfrak{g}]$ labeled by simple and simply-laced algebras. Since the string coupling has gone to zero they should be non-gravitational.

♣so why are they superconformal from this point of view? ♣

6.4 Physical Construction 2: Parallel $M5$-branes

In this section we explain an observation of Strominger [164] which was roughly contemporaneous with Witten's, and constructs the theories from a different viewpoint.

6.4.1 Lightning review of M-theory

The essential facts about M-theory we need are:

1. M-theory is a hypothetical quantum theory which has 11-dimensional superPoincaré symmetry $\mathfrak{sp}(11|32)$ on 11-dimensional super-Minkowski space. It is not a conformal field theory (in accordance with Nahm's theorem) and has a length scale, called the 11-dimensional Planck length ℓ_m.

2. At energies $E \ll \ell_m^{-1}$ there is an effective theory – not UV complete – given by 11-dimensional supergravity. It has a supergravity field multiplet of $(g_{MN}, C_{MNP}, \psi_{M\alpha})$.

3. M-theory has two types of branes: they have 2+1 and 5+1-dimensional worldvolumes.

[22] In fact, it is easy to make a larger family with moduli space $\mathbb{R}_+ \times \Gamma'\backslash O(5,21;\mathbb{R})/O(5) \times O(21)$, but we will not need this.

[23] The essential fact is that the $(2,2)$ supersymmetric sigma model with target space $\mathcal{R}_\Gamma$ has a singularity in the complexified Kähler moduli space. This occurs when the Kähler parameters of the exceptional curves shrinks to a point and the holonomy of the (NS) B-field on the exceptional divisors is tuned to a special value. It is very similar to the "conifold" singularity of string theory on Calabi-Yau 3-folds.

4. The M2 brane has tension $1/\ell_m^3$. The worldvolume theory of the basic brane is an $\mathcal{N} = 8$ supersymmetric theory with 8 scalar fields. There is *no obvious generalization of the nonabelian theory.* In the past few years, thanks to the breakthroughs of Bagger-Lambert-Gustavsson and ABJM, there has been a great deal of progress in understanding the theories describing the IR physics of N coincident M2 branes.

5. The M5 brane has tension $1/\ell_m^6$. The basic M5 brane carries the $u(1)$ abelian tensor-multiplet. Again, *there is no obvious nonabelian generalization.* The IR dynamics of N coincident M5 branes is the $\mathfrak{g} = u(N)$ interacting $(2,0)$ theory, and is the subject of these notes. It remains largely mysterious.

6. ♣FILL IN: M-theory/IIA duality on circle bundles. ♣

Remark: The low energy degrees of freedom on the M2 and M5 brane were derived from the soliton solutions of supergravity via the collective coordinate method in the theory of solitons. For the crucial case of the M5 brane see [36].

♣Other references? ♣

6.4.2 Stacks of M5-branes

Stack of M5 branes, analogous to stacks of Dp-branes. There is an analogous construction of tensionfull strings.

d=distance between M5's.

$d = \ell_m \epsilon$ and $T_{M2} = 1/\ell_m^3$ so $T_{string} = \epsilon/\ell_m^2$.

Here ℓ_m is the 11-dimensional Planck length.

We send $\epsilon \to 0$ and study correlation functions and scattering at energies and momentum in the range:

$$\frac{d}{\ell_m^3} \ll E^2 \ll \frac{1}{\ell_m^2} \tag{6.7}$$

The claim is that scattering of the string-like excitations will be governed by a local quantum field theory and moreover there exist local operators whose correlation functions behave like those of a local quantum field theory.

♣What the hell does that mean for a mathematician? ♣

Remarks:

1. Note that there is no dimensionful parameter in the construction, so if we really believe the gravitons decouple and a field-theoretic object results then we should expect it to be scale invariant, and preserving 16 supersymmetries. "Hence" $(2,0)$ superconformal.

2. Comparing with the discussion of the symmetry enhancement for Dp branes, and recalling the fact that the basic M5 brane has a $(2,0)$ tensor multiplet on it, which includes a gerbe connection, one is tempted to think that the nonabelian theory obtained by the stack of M5 branes in the low energy limit is a "theory of nonabelian gerbes." This is reinforced by the close relation between the M5 brane and the D4 brane: In the duality $IIAStringTheory/X_{10} = M/X_{10} \times S^1$ the stack of N M5 branes wrapped on the S^1 is claimed to be equivalent (even in the nonabelian case) to the stack of N D4 branes. This has led to many efforts to define a theory of

nonabelian gerbes and nonabelian 7D Chern-Simons theory. At present there is no generally accepted version of what such a theory should be. For some examples of attempts see [110, 111, 67]. Other authors, notably Witten, [CITE GRAEME BIRTHDAY VOLUME] have expressed strong doubts that a sensible classical theory of nonabelian gerbes can be defined.

6.5 Duality relation between the two pictures

1. Type IIA(B) on $X_6 \times \mathbb{R}^3 \times S^1_R$ with NS5-branes at positions θ^i_9 is T-dual to Type IIB(A) on $X^6 \times TN$ with positions of the NS5-branes giving the TN centers and the periodic coordinate giving the (NS) B-field holonomy.

2. When NS5's approach each other the B-field goes to zero. Need to explain that the supersymmetric sigma model on the resolution of $\mathbb{C}^2/\mathbb{Z}_N$ has hyperkahler parameters in $\mathbb{R}^3$ for each exceptional divisor and a B-field holonomy.

6.6 Ground rules for working with interacting $(2,0)$ theories

From these heuristic physical pictures we can infer a number of ground rules, which we might try to dignify with the name of "axioms." In practice, when physicists work with $(2,0)$ theories they assume these axioms and deduce consequences from them. This is what we will do in subsequent lectures. The formulation of the axioms should be viewed as a work in progress. We will comment on justifications and limitations of the axioms below. [24]

Basic data for $(2,0)$ Theories

1. A real reductive Lie algebra $\mathfrak{g}$, so $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_{ss}$, where $\mathfrak{z}$ is the center, and the semisimple part $\mathfrak{g}_{ss} = [\mathfrak{g}, \mathfrak{g}]$ is compact.
2. A nondegenerate Ad-invariant bilinear form b such that all roots have length two.
3. A full lattice $\Pi = \Gamma \oplus \Gamma'$ with $\Gamma \subset \mathfrak{z}$ and $\Gamma' \subset [\mathfrak{g}, \mathfrak{g}]$ given by $\Gamma' = \mathrm{Hom}(U(1), \tilde{T})$ where $\tilde{T}$ is the maximal torus of the simply connected Lie group with Lie algebra $\mathfrak{g}_{ss}$.

♣Do we need the bilinear form to be integral, or even on Γ? Might be needed for mutual locality of surface defects. Note arguments of Seiberg-Taylor. ♣

The string theorists claim that to the basic data $(\mathfrak{g}, b, \Pi)$ is associated a local quantum field theory $S(\mathfrak{g}, b, \Pi)$ such that:

Working Axioms for $(2,0)$ Theories

[24] My first attempt to write out the axioms was greatly improved by comments from the "Theory X" group, especially by remarks of D. Freed.

1. (A.) *Domain of definition*: The domain of definition is supermanifolds (or Lorentzian or Euclidean signature) $M^{6|32}$ where the reduced space is a spin 6-manifold with a conformal metric and an R-symmetry bundle with connection. (B.) *Codomain*: It is valued in a 7D field theory $\mathcal{F}_7$ which encodes the conformal, gravitational, and R-symmetry anomalies. That theory is an invertible field theory times a 7d abelian Chern-Simons topological field theory based on the finite group $D = \Pi^*/\Pi$.

♣Defined on (2,0) superconformal manifolds? ♣

♣Might need more topological data like lift of w_4 ♣

♣Might not need to be spin. Need spin structure on tangent bundle + R-symmetry bundle. ♣

♣Need to say something about reflection positivity/Wightman axioms? ♣

2. As in any conformal field theory, the vector space of local operators in the theory is isomorphic, as a representation of the superconformal algebra to the space of states on S^5 in radial quantization. (Note the relevant Heisenberg group is trivial in this case, so there is a unique space of states.) This should be a unitary representation of the $(2,0)$ algebra. Some unitary representations are "known" to be present from the relation to 5D SYM and to M-theory. See below for details. s

3. If we consider the theory on $X_5 \times S^1$ with S^1 of radius R with periodic spin structure and study the effective action at length scales $\gg R$, that is, at energies $\ll 1/R$ then the theory is described by the supersymmetric (with 16 real supersymmetries) Yang-Mills theory with a compact gauge group G_{adj} but whose principal bundle has a possibly nontrivial characteristic class in $H^2(X_5; D)$ and whose coupling constant satisfies $\lambda_5^2 \propto R$.

♣It is actually possible to give the precise constant of proportionality. ♣

4. If $\mathfrak{g}$ is simple, then on $\mathbb{M}^{1,5}$ there is a moduli space of vacua given by

$$\mathcal{M} \cong (\mathbb{R}^5 \otimes \mathfrak{t})/\mathcal{W} \tag{6.8}$$

where $\mathfrak{t}$ is a Cartan subalgebra and $\mathcal{W}$ is the Weyl group.

5. The low energy dynamics around smooth points on this moduli space are described by abelian tensor multiplets valued in $\mathfrak{t}$.

♣Say this more precisely. Actually, here need to specify group structure for the toral self-dual theory. ♣

6. In addition, the Hilbert space of the theory includes BPS quantum states which are described, semiclassically, by finite tension strings, charged under the abelian tensormultiplets, with "charges given by roots of $\mathfrak{g}$."

♣say what that means ♣

7. There are supersymmetric surface defects labeled by $(\mathcal{R}, \Sigma, \vec{n})$, where Σ is an oriented surface in X_6, $\mathcal{R}$ is a representation of $\mathfrak{g}$, and $\vec{n} : \Sigma \to \mathbb{R}^5$ is a map into the fundamental representation of $Spin(5)$ R-symmetry. In the IR, far out on the moduli space of vacua [25] these are well-approximated by a sum over the weights of $\mathcal{R}$ of the abelian B-field holonomy + supersymmetrizing terms. There is a restriction on the embedding of the surface and the map $\vec{n}$ which determines how many susy's are preserved.

8. There exists a set of codimension two superconformal defects. (Corresponding to the 3-brane central charges.) They are labeled by conjugacy classes of homomorphisms $\rho : sl(2, \mathbb{C}) \to \mathfrak{g}_c$ and embeddings of $so(2) \to so(5)$. Insertion of such defects

♣Careful! This might just be for $SU(K)$! Check CDT. ♣

[25] This means, in the limit where the vacua in (6.8) go to infinity, staying far from the fixed point loci of $\mathcal{W}$.

leads to a new theory with a global symmetry whose Lie algebra is the maximal Lie subalgebra $\mathfrak{g}_{global}$ of $\mathfrak{g}$ centralizing $\mathrm{Im}(\rho)$. The defects preserve a superconformal symmetry isomorphic to $su(2,2|2) \subset osp(2,6|4)$. They have "mass deformations" labeled by $\mathfrak{t}_{global,c}$.

♣What kind of manifolds are these defined on? In $\mathbb{M}^{1,5}$ they are defined on hyperplanes. What is the generalization? ♣

6.7 Remarks on and justification of the axioms

In this section we explain how the above axioms are "derived" from the physical pictures and make some further comments:

6.7.1 Basic Data

Note we specify a Lie algebra, not a Lie group. The abelian part $\mathfrak{z}$ corresponds to the noninteracting theory of §5 and $\mathfrak{g}_{ss}$ corresponds to the interacting part. The theories associated to the summands are locally but not necessarily globally independent of each other, as in the case of $\mathfrak{g} = u(K)$.

The semisimple part is compact so that 5d SYM in Axiom 3 has correct sign kinetic term.

The restriction on the roots comes about because on the Coulomb branch, (see below) and in the presence of a defect or string of charge $\alpha \in \mathfrak{t}^*$, the fieldstrength $H \in \Omega^3(\mathbb{M}^{1,5}; \mathfrak{t})$ must satisfy

$$dH = 2\pi\alpha\delta^4(W_2) \tag{6.9}$$

where $\delta^4(W_2)$ is a current (i.e. delta-function supported differential form) Poincaré dual to the worldvolume W_2. In order for this equation to make sense we must have an identification $\mathfrak{t} \cong \mathfrak{t}^*$. Put differently, the electric and magnetic currents must be isomorphic, and this requires an isomorphism $\mathfrak{t} \cong \mathfrak{t}^*$. The specific normalization condition on the roots (which implies that the simple summands in $\mathfrak{g}_{ss}$ is of ADE type) is argued in [106] to follow from the cancelation of worldsheet anomalies of the strings mentioned in "axiom" 6.

♣Should give this argument! ♣

Note the absence of a level in the nonabelian theories. This is very different from the chiral nonabelian WZW theories.

6.7.2 Axiom 1: Domain and codomain of definition

The choice of the domain and codomain is strongly influenced by past investigation of the abelian tensormultiplets as reviewed in §5.

Domain

The spacetime must be oriented in order to formulate the self-duality condition on the fieldstrength H. In addition we have fermions in the abelian tensormultiplet theory and therefore we require a spin structure on the direct sum $TX_6 \oplus \mathcal{R}$ where $\mathcal{R}$ is the rank 5 "R-symmetry" bundle with $SO(5)$ structure group. More subtle topological conditions come about from:

1. The partition function [Witten, Hopkins-Singer, Belov-Moore, Henningson, Monnier]. HS found the need for an integral lift of w_4 to formulate the Euclidean partition function. Maybe this is properly encoded in the 7d TFT.

2. The Hamiltonian formulation [79, 80].

3. Investigations of surface operator correlators [161].

A final remark on the domain: We will make use of the freedom to formulate the theory when there is a nontrivial $Spin(5)$ bundle $P \to X_6$ with connection when we consider the topologically twisted theories. However, the best formulation surely makes use of a "(2,0) superconformal structure" on a super-manifold $M^{6|32}$. The $(2,0)$ energy-momentum supermultiplet is thought to be [Bergsheoff-van Proeyen]:

$$(T_{\mu\nu}, J^i_{\mu\alpha}, R^{ab}_{\mu}, \dots) \tag{6.10}$$

As far as I know the structure of such supermanifolds has not been investigated.

Codomain

It is not at all obvious that the 7D theory of the abelian tensor multiplets should be extended as claimed to the interacting theories. This is suggested by

1. Investigations of singletons in the AdS/CFT correspondence [142]. (See §6.8 below.)

♣There should be a corresponding type IIB argument. Some analog of singleton modes from IIB supergravity multiplet should fail to decouple. This is related to the embedding of Γ into the unimodular self-dual $II^{5,21}$ lattice. ♣

2. Reduction to theories of chiral bosons on 4-manifolds.
3. ARGUE BY MOVING ONTO THE COULOMB BRANCH?

For example, for the $su(K)$ theory the appropriate 7D TFT has Chern-Simons action $S = \pi K(\check{C}, \check{C})$ where $\check{C} \in \check{H}^4(X_7)$ is a differential character. This was discussed in §5.9. The space of conformal blocks will be an irreducible representation of $\mathrm{Heis}(H^3(X_6; \mathbb{Z}/K\mathbb{Z})$.

Furthermore the 7D is tensored with an invertible field theory representing the Weyl and gravitational anomalies:

♣Could $\check{C}$ live in a torsor for differential characters? ♣

For the case of $\mathfrak{g} = su(N)$, and at large N the Weyl anomaly was computed in [105, 26]:

$$\delta S = \kappa N^3 \int \mathrm{vol}\,(g)\delta\sigma \left(E_6 + \sum_i c_i I_i(g) \right) \tag{6.11}$$

where E_6 is the Euler density and I_i are three independent Weyl-invariant combinations of curvatures and covariant derivatives.

The integrated form of the anomaly (i.e. the analog of the Liouville action) for this theory has not be written down although a partial result (corresponding to the contribution of the Euler density to the anomaly) has written in equation (B.17) of [64]. As far as I know the result has not been discussed for the D and E theories and even for A-theories the above is only leading large N result. For the D and E theories the coefficients c_i above could well be different.

There are also gravitational and R-symmetry anomalies for the (2,0) theory. The local gravitational anomalies are summarized by the anomaly line bundle [112]:

$$\int_{\mathcal{X}/\mathcal{P}} \frac{1}{8}(\check{L}_8 - \check{\lambda}^2) \tag{6.12}$$

where we integrate over a family of 6-manifolds, $\check{L}_8$ is a differential character lift of the Hirzebruch genus and $\check{\lambda}$ is a differential lift of an integral lift of λ of the Wu class w_4. This formula encodes the local gravitational anomalies computed in [CITE ALVAREZ-GAUME-WITTEN]. It also agrees with results on global gravitational anomalies has been recently confirmed by [WITTEN,MONNIER].

The R-symmetry anomalies are subtle and have been discussed from the viewpoint of M-theory in [Freed, Harvey, Minasian, Moore]. ♣Say this better. ♣

In more physical terms, the statement that the theory is valued in the invertible Weyl anomalous theory encodes the conformal Ward identities of physical quantities such as the partition function and the local correlation functions.

6.7.3 Axiom 2: Hilbert space as a representation of the superconformal algebra

In radial quantization there is a state-operator correspondence.

Local operators can be organized into irreducible unitary representations of $osp(2,6|4)$. These in turn are built by constructing Verma modules on irreducible representations of the maximal compact even subalgebra $so(2) \oplus so(6) \oplus so(5)$ and dividing by null vectors. The Verma modules are characterized by weights $(\epsilon; j; j')$ where j is a dominant weight of $so(6)$ and j' a dominant weight of $so(5)$. The weight ϵ of $so(2)$ is the scaling dimension. Unitarity implies it is positive. The possible null vectors were investigated in [32].

This reference found various kinds of "short representations" where there are nontrivial null vectors.

The "shortest" representations have $j = 0$ and j' has Dynkin indices $j' = (k, 0)$ and then $\epsilon = 4k$. These are examples of "chiral primary fields." They are annihilated by supersymmetries ♣ SAY MORE PRECISELY WHICH SUSY's♣

If we wish to count states in a representation annihilated by a supercharge Q and $S = Q^\dagger$ then we may form the superconformal index, defined by

$$I(\mu) = \mathrm{Tr}(-1)^{2J} e^{-\beta\{Q,S\}+\mu} \tag{6.13}$$

where J is any generator in $so(6)$ and μ is in the weightspace of the bosonic subalgebra commuting with Q and S.

Reference [32] used the AdS/CFT correspondence to derive a formula for

$$I(\mu) = \mathrm{Tr}_{\mathcal{H}(S^5)}(-1)^{2J} e^{-\beta\{Q,S\}+\mu} \tag{6.14}$$

for the A_K theory in the limit $K \to \infty$. Their method of computation was to consider a Fock space of gravitons for 11-dimensional supergravity on $AdS_7 \times S^4$. This Fock space is a unitary representation of the isometry superalgebra $osp(2,6|4)$. Part of the claim of the AdS/CFT conjecture is that, as a representation of $osp(2,6|4)$ this Fock space should be isomorphic to $\mathcal{H}(S^5)$, at least for $K \to \infty$ ♣and ignoring singletons and also ....? ♣

Most importantly for us is the consequence that there are chiral primary representations (the "shortest" ones, described above) which in the 5D $su(K)$ SYM theory become $\mathrm{Tr}\Phi^{(I_1} \cdots \Phi^{I_n)}$, $n \geq 2$. The main claim is that they lift to local operators in the 6d theory. This is supported by arguments in [1]. Heuristically they are $\mathrm{Tr}Y^{(I_1} \cdots Y^{I_n)}$, $n \geq 2$, where Y^I are 5 scalars in $\mathfrak{g}$ of dimension two. (Except, there is no nonabelian field multiplet, so the last sentence is "very heuristic.")

More generally, for any invariant polynomial $\mathcal{P}_d$ on the Lie algebra $\mathfrak{g}$ of degree d we postulate the existence of a chiral primary multiplet of scalar fields $\mathcal{P}_d^{(I_1,\ldots,I_d)}$ which transform in the $\mathrm{Sym}^d(\mathbb{R}^5)$ of $Spin(5)_R$ and have scaling dimension $\Delta = 2d$. In the related 5D SYM theory they are

$$\mathcal{P}_d(\Phi^{I_1}, \dots, \Phi^{I_d}) \tag{6.15}$$

with $j = 0$, and $j' = (d, 0)$ and $\epsilon = 2d$. We will need these to characterize cod two defects.

6.7.4 Axiom 3: Relation to Super-Yang-Mills

1. This absolutely crucial axiom can be deduced from the IIB string perspective by T-duality. If we consider type II string theory on $\mathbb{M}^{1,4} \times S^1 \times K3$ then

$$\frac{1}{g_s(IIA)} = \frac{R_B}{g_s(IIB)} \tag{6.16}$$

   but IIA on a K3 singularity leads to nonabelian ADE gauge theory.

2. From the M5-brane viewpoint the reduction of a basic M5 brane on a circle leads to a D4 brane with abelian gauge group. Combined with the Coulomb branch axiom 3 and the geometric Higgs mechanism explained in §**??** we again arrive at the picture that multiple basic M5-branes wrapped on a circle are described by SYM with a compact gauge group with Lie algebra $\mathfrak{g}$.

3. We should stress that the action of the low energy effective SYM has the form

$$\frac{1}{R}\int \mathrm{Tr} F * F + D\Phi * D\Phi + \sum_{i\neq j}[\Phi_i, \Phi_j]^2 + \cdots \tag{6.17}$$

   The R in the denominator follows from conformal invariance of the 6-dimensional theory together with dimensional analysis. As stressed by Witten this is extremely unusual. Compactification of a six-dimensional Yang-Mills theory would give a factor of R in the numerator in (6.17): Normal KK reduction from $(d+1)$ to d dimensions of YM on a circle of radius R gives Yang-Mills-Higgs with

$$\frac{R}{\lambda_{d+1}^2} = \frac{1}{\lambda_d^2} \tag{6.18}$$

   We have seen how the inverse power of R can arise from an action principle in six-dimensions in the *abelian* case, but the nonabelian case is a different matter altogether. There is no field multiplet - let alone an action principle in the interacting (2,0) theories.

4. We should also explain about the nature of the gauge group. The conformal blocks transform in a Heisenberg representation of $Heis(H^3(X_6; D))$ in general. However, for $X_6 = X_5 \times S^1$ there is a natural Lagrangian decomposition

$$H^3(X_6; D) \cong H^2(X_5; D) \oplus H^3(X_5; D) \tag{6.19}$$

   There is therefore a natural basis of conformal blocks given by elements of the finite group $H^2(X_5; D)$. On the other hand, $D \cong Z(\tilde{G})$ is isomorphic to the center of the simply connected compact Lie group with Lie algebra $\mathfrak{g}$. Therefore an element of $H^2(X_5; D)$ can be viewed as a characteristic class of a principal G_{adj} bundle over X_5. [26] Thus, when reducing the partition function we should get the "partition function" [27] of 5D SYM with gauge group G_{adj} and where we sum over principal bundles with fixed characteristic class in $H^2(X_5; D)$.

5. In the 5D SYM the scalar fields Φ^I have dimension one. On the Coulomb branch the $(2,0)$ theory has scalar fields Y^I of dimension 2. The relation between these is

$$\Phi^I = RY^I \tag{6.20}$$

♣We have a notation problem here.... ♣

6.7.5 Axiom 4: Coulomb branch in Minkowski space

a. Classical moduli of 5D SYM.
b. Strominger picture for A_r theories
c. IIB viewpoint?

6.7.6 Axiom 5: Low energy dynamics on the Coulomb branch

a. IIB - self-dual F5 on exceptional divisors give self-dual abelian B-fields. periods of RR and NSNS B-fields, and HK moment maps combine into so(5) multiplet of scalars. Strings charged under these the self-dual 2-form as roots.

♣But 2 of the five are periodic scalars? Maybe the scaling limit kills the periodicity. ♣

b. M5 - self-dual multiplet are the low energy fluctuations of the basic M5 brane. [36]. From both viewpoints: $\mathcal{M}$ is the space of vevs $\langle Y_s^I \rangle$, making contact with Axiom 3.
Remarks:

1. It should be noted that the DBI action of the M5 brane suggests that even in the abelian case there is a nonlinear version of the $(2,0)$ tensormultiplet [115]. Such nonlinearities would only show up when the fieldstrength is on the order of the 11-dimensional Planck scale and hence will disappear in the low-energy limit used to define the $(2,0)$ theory.

6.7.7 Axiom 6: String Excitations

a. Wrapped D3 branes on exceptional divisors
b. Open M2 branes stretched between parallel M5's.
SAY WHAT CHARGED UNDER ALPHA MEANS.

6.7.8 Axiom 7: Surface Defects

a. We have seen that the abelian theory has surface defects in Section
b. 5D SYM has "monopole surface defects" labeled by representations $\mathcal{R}$ of $\mathfrak{g}$. Look at these on the Coulomb branch. Natural uplift to 6d.

---

[26] This class measures the obstruction to lifting the bundle to a principal $\tilde{G}$ bundle.
[27] in quotes because there is no UV complete theory

c. Witten's paper on knot homology [178], Section 5.1.4 has an extensive discussion of this axiom. He does not regard it as obvious that the surface defects exist for all representations.

d. Infinite length open M2's ending on M5: Note this only gives surface defects in the fundamental representations. These are labeled by a choice of direction in $\mathbb{R}^5$, i.e. a vector on S^5.

e. Far out on the Coulomb branch the surface defect should be well-described by a sum of surface defects for the abelian theory, namely

$$\sum_{v\in WT(\mathcal{R})} \exp[2\pi i \int_\Sigma (v,B) + \cdots] \tag{6.21}$$

where v runs over the weights of the representation $\mathcal{R}$.

$$\exp\left[2\pi i \int_\Sigma B + \kappa n^I Y^I \mathrm{vol}\,(\Sigma)\right] \tag{6.22}$$

where $\mathrm{vol}\,(\Sigma)$ is the volume form on Σ from the induced metric, κ is a constant, and we can assume without loss of generality that $\vec{n}$ is a unit vector in Euclidean $\mathbb{R}^5$ with components n^I. Let ξ^α, $\alpha = 1,2$ be a local coordinate system on Σ. Then supersymmetries $\epsilon_i^r Q_r^i$ will annihilate this operator provided

$$\epsilon_i^r \left(\frac{d\xi^\alpha \wedge d\xi^\beta \partial_\alpha X^M \partial_\beta X^N}{\mathrm{vol}\,(\Sigma)} (\gamma_{MN})_r{}^s \delta^i{}_j + \kappa (n^I \Gamma^I)^i{}_j \delta_r{}^s \right) = 0 \tag{6.23}$$

where $X^M(\xi)$ denote the embedding of the surface into M_6. In order to preserve supersymmetry this equation must be satisfied for *constant* unbroken supersymmetries $\epsilon_i^r Q_r^i$. For a flat surface and constant n^I half the supersymmetries will be preserved with $\kappa = \pm 1$. More generally, (analogously to super Yang-Mills) Σ can be a curved surface and n^I can vary. An example which will be important below arises when the surface is $\mathbb{R} \times \wp$ where $\wp$ is a curve in, say, the 12 the plane. If we decompose the R-symmetry space $\mathbb{R}^5 = \mathbb{R}^2 \oplus \mathbb{R}^3$, identify the $\mathbb{R}^2$ summand with the 12 plane, and take n^I to be the unit tangent vector to $\wp$ then one-quarter of the supersymmetry is preserved. In general, the R symmetry is broken to $so(4)$ by the direction n^I.

ALSO GET SURFACE DEFECTS FROM INTERSECTION WITH M5: SEE ALDAY-TACHIKAWA.

6.7.9 Axiom 8: Half-BPS Codimension Two Defects

Points to make

1. The existence of such objects is allowed by the "central extension" of the superalgebra (3.27).
2. We will take the defects to have worldvolume $\mathbb{M}^{1,3}$ or $\mathbb{M}^{1,2} \times S^1$ and the transverse space will be $\mathbb{R}^2$ or $\mathbb{R} \times S^1$. We will also consider both the Euclidean and the cigar metrics on the transverse $\mathbb{R}^2$. More general geometries do not seem to have been discussed.

3. One "definition" of the defect in the context of the M5-brane construction is that it corresponds to a basic M5-brane intersecting the "stack" of K M5-branes in codimension two, and preserving supersymmetry. If the stack of branes is located at $X^{6,7,8,9,10} = 0$ in the transverse $\mathbb{R}^5$ and fills the $\mathbb{M}^{1,5}$ parametrized by $X^{0,1,2,3,4,5}$ then the transverse brane will intersect the stack at $X^{4,5} = 0$, filling a worldvolume parametrized by $X^{0,1,2,3,4}$ and spanning a real 2-plane in the transverse $\mathbb{R}^5$. Any plane will do, and the six-dimensional Grassmannian of 2-planes in $\mathbb{R}^5$, $SO(5)/SO(3) \times SO(2)$, parametrizes the embeddings $so(2) \oplus so(3) \to so(5)$ which characterize the topological twist enabling us to define embeddings of $su(2,2|2) \hookrightarrow osp(2,6|4)$. Conditions on intersecting branes in M-theory which preserve (♣ Poincare? ♣) supersymmetry were investigated in [169].

♣How do we construct the defects from type IIB on a hyperkahler resolution? ♣

4. The defects can be defined, in part, by the singularity of local operators as they approach the defect. For example, in the $SU(K)$ case, characterized by $\rho : sl(2,\mathbb{C}) \to su(K)$ deformed by mass parameters, according to axiom 2 there are local chiral operators $\mathcal{P}_d$ transforming in the d^{th} symmetric power $\mathrm{Sym}^d(V_R)$ of $Spin(5)_R$. These have singularity

$$\mathcal{P}_d(\Phi^{4+i5})(z)D \sim \frac{\mathcal{P}_d(m)}{z^j} D' + \cdots \tag{6.24}$$

♣Check this! What happens when $m \to 0$? Subleading singularity? ♣

5. Because the defect has global symmetry G_{glob} we can couple it to four-dimensional N=2 theories. This is the connection to the 3-brane picture of [114].

6. Another property is that when the defect is put on a circle, so it wraps $\mathbb{M}^{1,2} \times S^1$, then the 6d theory at long distances is 5d SYM. The claim is that in this case the long distance theory is 5d SYM coupled (via the global symmetry currents) to the superconformal theory of type $T^\rho[G]$ of Gaiotto-Witten. The latter can be viewed as long distance limits of quiver theories.

♣Careful! ρ or ρ^D here? ♣

7. A separate property is that if we consider the transverse space to be $\mathbb{R}^2$ but with a cigar metric with asymptotic radius R then at long distances (compared to R) the theory should be described by 5D SYM on a half-space. The defect then induces boundary conditions on the 5 scalar fields. They consist of the Nahm pole:

♣define cigar metric. clarify when it is an embedded submanifold in a hk TN manifold. ♣

$$Y^i \sim \frac{\rho(\mathfrak{t}^i)}{y} + regular \qquad i = 1,2,3 \tag{6.25}$$

$$Y^{4+i5} \sim m \tag{6.26}$$

with $[m, \rho(\mathfrak{t}^i)] = 0$. This is part of the Gaiotto-Witten theory of supersymmetric boundary conditions of d=4 N=4 SYM. (Lifted up to 5d.)

♣Say what happens with the 5D SYM gauge fields ♣

8. The previous two pictures are beautifully compatible by consider a defect wrapping $\mathbb{M}^{1,2} \times S^1_{R_1}$ with transverse space a $\mathbb{R}^2$ with cigar metric R_2. Generalizing the standard S-duality discussion, we can compactify on the $S^1_{R_1}$ first and then look at the IR theory at distances $\gg R_2$ or first "compactify" on the cigar and then compactify on the R_1.

The S-duality between d=4 N=4 with gauge group G Nahm boundary conditions and d=4 N=4 with gauge group G^L coupled to $T^\rho(G)$ is part of the Gaiotto-Witten theory. See Figure 5. ♣Say the right scfml theory here!! ♣

9. Discuss taxonomy and geometry of complex orbits and nilpotent orbits.

10. Which compact group should we take for the unbroken global symmetry? This is a physical question. We have just characterized its Lie algebra. Gaiotto-Maldaenca construct local operators in theories of class S by wrapping 2-dimensional defects on C. These transform in representations of the global symmetry. From that discussion for $\mathfrak{g} = su(K)$ full punctures the global symmetry should be the simply connected group $SU(K)$.

11. ♣ Say something about anomaly cancellation conditions ♣

12. There are important sublteties for the D, and E cases discussed by Distler, Chacaltana, and Tachikawa [45]. If the defect induces a coupling to $T_\rho[G]$ in the 5D SYM then [45] argues that the corresponding singularity in the Hitchin system is

$$\varphi(z) \sim \frac{\tilde{\rho}(\sigma^+)}{z} dz + \cdots \tag{6.27}$$

where $\tilde{\rho}$ is another homomorphism $\tilde{\rho} : sl(2, \mathbb{C}) \to \mathfrak{g}$. Both ρ and $\tilde{\rho}$ are in 1-1 correspondence with nilpotent orbits in $\mathfrak{g}$ (when the mass parameters are zero) and the claim is that they are related by the Spaltenstein map, together with some subtleties involving discrete groups.

♣There should be a better way to say this in terms of sigma models for the flag variety... ♣

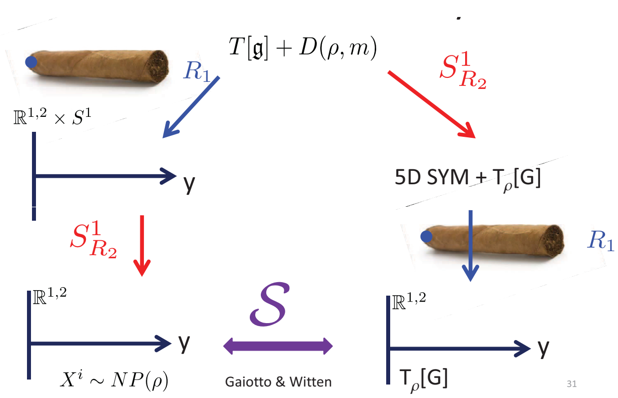

Figure 5: Showing the relation of surface defects to the Gaiotto-Witten theory of boundary conditions for 4(5)D SYM and S-duality.

6.8 Information from the AdS/CFT correspondence

The A_K and D_K theories are supposed to have well-defined large K limits which can be described - to leading order in K - by 11-dimensional supergravity on $AdS_7 \times S^4$ and $AdS_7 \times \mathbb{R}P^4$, respectively.

The metric and G-flux are given by taking the "near horizon limit" of the solution to the 11-dimensional sugra equations representing the stack of K M5 branes:

$$ds^2 = f^{-1/3}(dx^\mu dx_\mu) + f^{2/3} dy^a dy^a \tag{6.28}$$

$$f = 1 + \frac{\pi K \ell_m^3}{|y|^3} \tag{6.29}$$

$$G = K \mathrm{vol}\,(S^4) \tag{6.30}$$

The result is a Freund-Rubin solution on $AdS_7 \times S^4$ with radii of curvature

$$R_{AdS_7} = 2R_{S^4} = 2(\pi K)^{1/3}\ell_m \tag{6.31}$$

The (mysterious and largely unknown)M-theory quantum corrections to supergravity on this space are expected to be proportional to $\ell_m/R_{AdS_7} \sim K^{-1/3}$, suggesting that the boundary theory has a good $1/K$ expansion. That boundary theory is supposed to be the large K limit of the $su(K)$ $(2,0)$ theory.

In addition, M-theory has a parity symmetry taking $G \to -G$. Dividing by this symmetry gives a solution whose near horizon limit is $AdS_7 \times \mathbb{R}P^4$ with G a twisted differential form. Supergravity on this space is supposed to be dual (at large K) to be the $(2,0)$ D_K theories.

This sheds light on

1. The relevant 7d TFT, through the coupling to singletons [142].

2. The operator spectrum of the theory, and in particular the superconformal index, as described above.

6.9 Relation to N=4 SYM

Compactification on a torus: Geometrization of S-duality of N=4 SYM.

$$(S(\mathfrak{g})//S^1_{R_1})//S^1_{R_2} = (S(\mathfrak{g})//S^1_{R_2})//S^1_{R_1} \tag{6.32}$$

6.10 Attempts at a more rigorous construction

1. In [1] Aharony, Berkooz, and Seiberg studied the DLCQ of the the A_{k-1} theory and formulated the theory in the momentum N sector in terms of the quantum mechanics of the "instanton particle" (see below). Thus they were led to formulate the theory in terms of the large N limit of supersymmetric quantum mechanics on the moduli space of $SU(k)$ instantons of instanton number N on $\mathbb{R}^4$, $\mathcal{M}_{N,k}$. They encountered difficulties with the singularities of moduli space, but in principle this approach is sound, and should not have been dropped.

2. Recently, in a very interesting paper [131] Lambert and Papageorgakis attempted to find a six-dimensional analog of the BLG theory [104, 23].

3. Some recent explorations of possible UV completions of 5D SYM (which would define the six-dimensional theory) appear in [60, 132, 110] More work is needed here.

4. "Deconstruction": Take a certain limit of four-dimensional $d = 4, \mathcal{N} = 2$ theories, or $d = 4, \mathcal{N} = 1$ theories [20]. We discuss it in more detail in §6.11 below.

6.11 The "lattice gauge theory" or "deconstruction" approach

6.11.1 An elementary computation

An extremely elementary but instructive prototype of the arguments used here is given by a theory of N complex scalar fields φ_j, $j = 1, \dots, N$ in D dimensions with a common mass μ, but which have certain quadratic interactions. We extend j to a periodic variable so $\varphi_{j+N} := \varphi_j$ and take the action to be (we take Lorentz-signature $(-, +^{D-1})$.

$$S = -\int_{X_D} \sqrt{g} \sum_{j=1}^{N} \left\{ \frac{1}{2} \partial_\mu \varphi_j \partial^\mu \varphi_j^* + \frac{1}{2} \left(\frac{N}{2\pi R} \right)^2 |\varphi_{j+1} - \varphi_j|^2 + \frac{1}{2} \mu^2 |\varphi_j|^2 \right\} \tag{6.33}$$

Here R is a parameter with dimensions of length.

What is the spectrum of this theory? The "quadratic interactions" are easily diagonalized with a finite Fourier transform:

$$\hat{\phi}_j := \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i \frac{jk}{N}} \varphi_k \tag{6.34}$$

Making this change of variables the action becomes

$$S = -\int_{X_D} \sqrt{g} \sum_{j=1}^{N} \left\{ \frac{1}{2} \partial_\mu \hat{\phi}_j \partial^\mu \hat{\phi}_j^* + \frac{1}{2} \left(\mu^2 + \left(\frac{N}{\pi R} \right)^2 \sin^2 \left(\frac{\pi k}{N} \right) \right) |\hat{\phi}_j|^2 \right\} \tag{6.35}$$

From this we can easily deduce that there is a set of N particles of masses

$$\mu_k^2 = \mu^2 + \left(\frac{N}{\pi R} \right)^2 \sin^2 \left(\frac{\pi k}{N} \right) \tag{6.36}$$

with $k = 0, \dots, N-1$. For $k \ll N$ the masses are approximately $\mu^2 + \left(\frac{k}{N}\right)^2$, which resembles a Kaluza-Klein tower of particles from compactification of a scalar field of mass μ on a circle, at least in a certain range of values of k and for large N.

Indeed, we claim that the quadratic interactions suppress various fluctuations of the N independent scalar fields φ_j so much so that if we define:

$$\Phi(x^\mu, 2\pi R \frac{j}{N}) := \sqrt{\frac{N}{2\pi R}} \varphi_j(x^\mu) \tag{6.37}$$

then, for $j \to \infty$, and $N \to \infty$ holding j/N fixed the field Φ has a smooth limit as a function on $X_D \times S^1$, $\Phi(x^\mu, x^{D+1})$ and moreover

$$\frac{\partial}{\partial x^{D+1}}\Phi(x^\mu, x^{D+1}) = \lim_{j,N\to\infty} \frac{\Phi(x^\mu, 2\pi R\frac{j+1}{N}) - \Phi(x^\mu, 2\pi R\frac{j}{N})}{2\pi R/N} \tag{6.38}$$

With this understood, the series of field theories labeled by N has as a limit the field theory of a free $(D+1)$-dimensional scalar field Φ of mass μ.

6.11.2 The quiver gauge theory

We now consider a $d = 4$ $\mathcal{N} = 2$ quiver gauge theory with the extended $\hat{A}_{N-1}$ Dynkin diagram. That is, the gauge group is $SU(K)^N$ and there are hypermultiplets $Q_{i,i+1}, \tilde{Q}_{i+1,i}$ associated with each link in the $(K, \bar{K}) \oplus (\bar{K}, K)$ representations.

This is a theory of class S associated to the N-punctured torus.

♣Where does this remark go? Needs much elaboration. Witten's paper. ♣

For the UV theory we restrict to the case where the coupling constant is the same for each $SU(K)$ gauge factor and equal to G.

This theory has a complicated manifold of vacua. The Higgs branch is the hyperkähler quotient of

$$\oplus_{arr}\left(\mathrm{End}(\mathbb{C}^K) \oplus \mathrm{End}(\mathbb{C}^K)^*\right) \tag{6.39}$$

by $\prod_{vert} SU(K)$. We confine our attention to a one-parameter subspace of the Higgs branch of the theory. Identifying $(K, \bar{K}) \cong \mathrm{End}(\mathbb{C}^K)$ we take

♣No FI parameters allowed... ♣

$$\langle Q_j \rangle = \Phi 1_{K\times K} \tag{6.40}$$

$$\langle \tilde{Q}_j \rangle = 0 \tag{6.41}$$

where Φ is a real number.

♣How do we break the R-symmetry and split the HM into 1 + 3? ♣

The vacuum expectation value breaks the gauge symmetry $SU(K)^N \to SU(K)_{diag}$.

Now reference [20] proposes that we consider a triple scaling limit of theories:

1. $N \to \infty$: Large N limit of theories
2. $G \to \infty$ Large coupling limit.
3. $\Phi \to \infty$ Large vev limit.

More precisely, we define:

$$2\pi R_5 := \frac{N}{G\Phi} \tag{6.42}$$

$$2\pi R_6 := \frac{G}{\Phi} \tag{6.43}$$

and take the large N limit $N \to \infty$ and simulatenously the large coupling limit $G \to \infty$ and the large vev limit $\Phi \to \infty$ so that R_5 and R_6 are held fixed.

♣You should only take limits with dimensionless quantities. Say this right. ♣

The claim is that this "reproduces" the $(2,0)$ theory with all its KK and nonperturbative modes, compactified on a torus with radii R_5, R_6.

1. Demonstrate that there is a naive limit which is 5D SU(K) SYM with

$$g_5^2 = \frac{G}{\Phi} \tag{6.44}$$

2. Explain that this is almost the quiver gauge theory of D3 branes at a $\mathbb{C}^2/\mathbb{Z}_N$ singularity. Actually the D3 branes have a $U(K)^N$ gauge theory. Explain heuristic relation to the Strominger picture through the use of some dualities.

3. Main evidence is the "Near BPS spectrum." But what does this mean on the Higgs branch?

6.12 Field theory vs. little string theory

Finally, we address an extremely tricky issue.

Given the "derivations" from type II string theory, M-theory, and the large N limit of a sequence of $d = 4, \mathcal{N} = 2$ theories it is by no means obvious that the resulting object - assuming it even exists - is a conventional quantum field theory.

In fact, many early papers on the subject asserted the contrary. Claiming that the object is a non-critical theory of strongly interacting strings.

Little string theory is "defined" by the theory on the NS5 brane where $g_s \to 0$ holding ℓ_s fixed. Therefore, $\ell_p \to 0$, $M_p \to \infty$. Therefore gravity decouples. Still the T-duality takes the NS5 brane to the NS5 brane. So the theory should have T-duality.

We can also view this as the theory of M5-branes with a *transverse* M-theory circle in the limit that the circle shrinks to zero radius.

Little string theory is NOT a low energy limit. So it need not be a field theory. The low energy limit of little string theory is supposed to be $(2, 0)$ theory.

There are two types:

There is type iia little string theory from the IIA NS5 brane: It has (2,0) supersymmetry.

There is type iib little string theory from the IIB NS5 brane: It has (1,1) supersymmetry.

The DLCQ formulation involves now a *sigma model* with target space the moduli space of instantons.

A good review of LST is the one by Aharony [2]. There are also several excellent papers by D. Kutasov and collaborators.

1. LST has strings and T-duality, but no gravity.
2. LST has local operators, but the do not satisfy LSZ reduction.
3. LST has a Hagedorn degeneracy of states.

The subject is closely related to M(atrix) theory [150, 31].

7. Theories of class S

7.1 Definition

7.1.1 Data

For simplicity take $\mathfrak{g}$ to be a simple Lie algebra.

1. An interacting $(2, 0)$ theory $S[\mathfrak{g}]$.

2. A punctured surface C with a finite number of punctures $\mathfrak{s}_n$.

3. A choice of half-BPS cod two defects $D_n = D(\rho_n, [V], \mathfrak{m}^{(n)})$ at the punctures $\mathfrak{s}_n$. They must all preserve the same supersymmetry and hence are associated with the same point $[V] \in Gr_2(\mathbb{R}^5)$. For this reason also the theory must be partially topologically twisted as described in §3.7.2.

7.1.2 Definition

We consider the partially topologically twisted theory using the construction of §3.7.2 with $SO(2)_{st}$ interpreted as the structure group of the tangent bundle of C.

With this twisting we take area to zero limit: [28]

$$S(\mathfrak{g}, C, D) := S(\mathfrak{g})//(C, D) \tag{7.1}$$

Since C is noncompact we require boundary conditions at the punctures, this is indicated in the notation on the RHS.

Remark: Say what the punctures do to vevs of chiral operators (as in GMN II - sec. 3).

7.2 Decoupling the Weyl modes: Gaiotto gluing, modular groupoid and S-duality

Now there is an important claim that the topologically twisted sector of the theory is independent of the Weyl modes of the metric on C. That is, for correlators which are invariant under some of the preserved supersymmetries, the correlation functions are invariant under a Weyl-rescaling of the metric on C.

There are two arguments for this: First, there is a very beautiful argument of Anderson et. al. [12] based on holographic renormalization group flow. But, strictly speaking, this only applies to $su(N)$ theories for large N. [SAY MORE ABOUT WHAT THEY SHOW]

Second, there is an argument based on the relation to 5D SYM [98] that the only dependence of certain quantities such as the moduli space metric on the HIggs branch enters through the total area of C, and then only as an overall scale. The Coulomb branch metric is independent of A.

Since

$$METRIC(C)/WEYL(C) \ltimes DIFF^+(C) = \mathcal{M}(C) \tag{7.2}$$

is the moduli space of complex structures we conclude that the theory $S(\mathfrak{g}, C, D)$ only depends on the moduli space of complex structures.

Now, a deep observation of Gaiotto's [92] is that there is the following close relationship between gluing of Riemann surfaces and gauging of global symmetries of the defects: Consider a pair of Riemann surfaces C_L and C_R with collections of defects D_L and D_R. To these surfaces and defects we assign 4d $\mathcal{N} = 2$ field theories $S(\mathfrak{g}, C_L, D_L)$ and $S(\mathfrak{g}, C_R, D_R)$. Let us focus on one puncture from each surface $\mathfrak{s}_L$ and $\mathfrak{s}_R$. We assume that they are "full punctures" meaning that there is a global G symmetry, where G is a compact Lie group with global symmetry $\mathfrak{g}$. The product field theory $S(\mathfrak{g}, C_L, D_L) \times S(\mathfrak{g}, C_R, D_R)$ has a global $G \times G$ symmetry associated with the punctures $\mathfrak{s}_L$ and $\mathfrak{s}_R$. We can gauge the diagonal subgroup $G \hookrightarrow G \times G$ with coupling constant

♣WHICH GLOBAL FORM? ♣

♣Say how θ is normalized. ♣

[28] There are some special cases where there are subtleties. These are discussed in [98]. [OTHER REFS????]

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi \mathrm{i}}{g^2} \tag{7.3}$$

to produce a new quantum field theory which we denote:

$$S(\mathfrak{g}, C_L, D_L) \times_{G,\tau} S(\mathfrak{g}, C_R, D_R) \tag{7.4}$$

On the other hand, we could also choose local coordinates z_L and z_R near the punctures $\mathfrak{s}_L$ and $\mathfrak{s}_R$ (such that z_L and z_R vanish at the punctures) and construct the glued Riemann surface with the standard plumbing fixture identification:

$$z_L z_R = q \tag{7.5}$$

We can, of course, construct a class S theory associated to this new Riemann surface which we will denote

$$S(\mathfrak{g}, C_L \times_q C_R, D_{LR}) \tag{7.6}$$

where D_{LR} denotes the remaining defects.

Gaiotto's conjecture is that *if we identify* $q = e^{2\pi i\tau}$ *then the two four-dimensional field theories are the same:*

$$S(\mathfrak{g}, C_L \times_q C_R, D_{LR}) = S(\mathfrak{g}, C_L, D_L) \times_{G,\tau} S(\mathfrak{g}, C_R, D_R) \tag{7.7}$$

Remark: Analytic coordinate redefinition and finite renormalization of τ.

7.2.1 Gaiotto decomposition and S-duality

a.) Go to boundary of Teichmuller space. Characterize by a pants decomposition, or trivalent graph.

b.) Claim that there is a superconformal trinion theory associated with three full punctures. For $\mathfrak{g} = A_1$ is is just a theory of free hypermultiplets.

c.) The theories associated to different boundaries of Teichmüller space projecting to the same point in moduli space (i.e. related by the modular group) must be equivalent: This is a generalization of the S-duality of $\mathcal{N} = 4$. It was first mentioned in [176] for the case of a sphere with punctures and generalized in [92] to general class S.

d.) We have a weak coupling description for each pants decomposition since this corresponds to long thin tubes, and hence weak couplings.

e.) On the other hand the pants decompositions label boundary regions of Teichmuller space. Consider two different boundary regions projecting to the same region in moduli space. The theory $S(\mathfrak{g}, C, D)$ only depends on the complex structure, not on its pants decomposition, and hence the weak coupling descriptions must be two different presentations of the same theory: This is a generalized S-duality [92].

f.) In describing the monodromy and "gluing" of conformal blocks of rational conformal field theory it turned out to be useful to introduce the "modular groupoid" whose objects are pants decompositions and whose morphisms are composed of sequences of elementary moves representing braiding, fusing, and a genus-one S-transformation [140, 139, 141]. (For an in-depth mathematical discussion see [Kirillov].

g.) This appearance of a mathematical structure central to RCFT is not an accident. It is deeply related to the AGT conjecture [4] and other factorization phenomena in the theory of class S. Some of which include behavior of Higgs branches [144, 98] and the behavior of the superconformal indices [83, 84, 85, 86, 99, 101].

7.2.2 A conformal field theory valued in four-dimensional theories

Mathematical challenge: A novel idea: Two-dimensional CFT valued in four-dimensional field theories. Make it precise.

This is the key insight that leads to, for example, AGT [4].

7.3 The Higgs and Coulomb branches

Different branches: Separations in $\mathbb{R}^2_{6,7} \times \mathbb{R}^3_{8,9,10}$

♣N.B. Different from Witten and GMN-II labeling of coordinates ♣

The maximal Higgs branch:

A TFT valued in holomorphic symplectic manifolds [Moore-Tachikawa]. Gives and rigorous and precise manifestation of the Gaiotto gluing phenomenon.

7.4 Relation to the Hitchin system

Now consider the maximal Coulomb branch: All branes in $\mathbb{R}^3_{8,9,10}$ at zero.

Compactification on $S^1 \times C$. Usual commutative diagram argument.

BPS equations for 5D SYM = Hitchin equations.

Effect of defects: Poles in φ.

IR and UV: The single M5 (tensormultiplet) on the spectral curve.

Relation of sheets of the covering to the Coulomb branch.

7.5 Recovery of the Seiberg-Witten Paradigm

Derive part of SW action from DBI action as in PaperII. Conclusion:

SW curve = spectral curve. Natural SW differential.

Similarly, express the local system of charges $\Gamma \to \mathcal{B}$ in terms of $H_1(\Sigma;\mathbb{Z})$.

7.6 BPS states in class S

Geometrical picture of "BPS states":

Goes back to [127, 137, 138], and is based on the relation to open M2 branes. [COMMENT ON RELEVANT CALIBRATED MANIFOLDS HERE?]

The separation of the sheets reflects the Coulomb branch. Strominger's open M2 branes stretching between them will end on closed one-cycles on Σ and project to string webs on C:

Label (locally) the sheets by $i, j, ... = 1, \dots, K$.

Definition: A *WKB path of phase* ϑ is an integral path on C such that for *some* pair of sheets

$$\langle \lambda_i - \lambda_j, \partial_t \rangle = e^{i\vartheta} \tag{7.8}$$

A1 case: Trajectories of a quadratic differential. Thus, we are interested in generalizations of the theory of trajectories of quadratic differentials.

INCORPORATE PICTURES FROM CALTECH TALK

Generic WKB paths have both endpoints on singular points $\mathfrak{s}_n$.

Separating WKB paths have one endpoint on a branch point $\mathfrak{b}$ and one endpoint on a singular point $\mathfrak{s}_n$. For simple branchpoints in the neighborhood of $\mathfrak{b}$ we get a trivalent graph.

At generic ϑ these are the only kinds of WKB paths we find. We get a foliation of C.

But at critical values of ϑ we get string webs.

a.) *Saddle trajectories*: Connect two branch points.

♣if connecting $\mathfrak{b}$ to itself do we give it a different name? ♣

b.) *closed trajectories*

c.) *string junctions* Only happens when $K > 3$.

Definition: A *string web* is a union of WKB paths with endpoints on branchpoints or string junctions.

Examples: (WITH PICTURES)

HYPER

VECTOR:

TRIVALENT JUNCTION:

Moduli and number of loops: Relation to spin.

These webs lift to closed cycles on Σ. The central charge of the corresponding state is the period of λ on the homology class γ of the cycle:

$$Z_\gamma = \frac{1}{\pi}\oint_\gamma \lambda = e^{\mathrm{i}\vartheta_c}|Z_\gamma| \tag{7.9}$$

7.7 The Witten construction

Relation to Witten construction of N=2,d=4 theories from M-theory.

D4's stretch along x^6 between NS5's.

x^6 is promoted to a cylinder coordinate $t = \exp[-(x^6 + ix^{10})]$. The D4's lift to $M5$ and we have a reduced curve

$$v^K \prod_{\alpha=0}^{n}(t - t_\alpha) \tag{7.10}$$

On the Coulomb branch this splits into the spectral curve.

The modern interpretation is: This is the case of class S with $C = \mathbb{C}^*$ with "full defects" at $0, \infty$ and "simple defects" at the t_α.

Remark: Brane bending and geometrization of the renormalization group!

Mathematical application: Novel isomorphisms of Hitchin systems (with different rank, for example). Includes some of the examples of Boalch.

7.8 Mirror picture

Mirror picture: CY from resolved family of ADE singularities over a curve.

Paper of KLMVW

7.9 Some novel isomorphisms of Hitchin systems

Requires introduction of D6's which get moved and the Hanany-Witten phenomenon. We move the D6's and get nontrivial isomorphisms of Hitchin systems with different ranks.

Because we need D6's this goes beyond the class S construction.

Special cases by Boalch.

Related to nontrivial Deligne isomorphisms?

8. Line defects, framed BPS states and wall-crossing

8.1 Definition of susy line defects

Point defects in space, extend along Eucl/Mink. time direction.

Kapustin definition: Superconformal boundary conditions on $AdS_2 \times S^2$ preserving the subalgebra of $su(2,2|2)$ fixed by the involution $I(\zeta)$ of §4.2.1

Preserved ζ-susys: Recall the $\mathcal{R}^A_\alpha$.

Examples.

Susy Wilson line

't Hooft operator singular conditions on A, φ.

8.2 Framed BPS states

Hilbert space is modified.

Grading by torsor for Γ.

Modified BPS bound $E \geq -Re(Z/\zeta)$.

Physical interpretation:

$$\lim_{M\to+\infty} (|\zeta M - Z_\gamma| - M) = -\mathrm{Re}(Z_\gamma/\zeta) \tag{8.1}$$

Sketch of the spectrum.

Definition of framed BPS states and framed protected spin character $\overline{\underline{\Omega}}(L_\zeta, \gamma; y; \zeta; u)$.

8.3 Wall-crossing of framed BPS states

The Denef radius formula (for framed states)

BPS Walls and their physical meaning

$$W(\gamma) = \{(u,\zeta) | Z_\gamma(u)/\zeta < 0\} \tag{8.2}$$

EXPLAIN WHY IT IS < 0 and not just real.

Halo particles mutually local: Fock spaces of halo states.

Generating function

$$F(L) = \sum_\gamma \overline{\underline{\Omega}}(L,\gamma) X_\gamma \tag{8.3}$$

X_γ satisfy Heisenberg algebra. HOW TO EXPLAIN THAT?

Change of the generating function across BPS wall:

Example: $F(L,c_-) = X_{\gamma_c}$. Suppose $\langle\gamma_c,\gamma_h\rangle = +1$ and $\Omega(\gamma_h) = +1$. Then

$$X_{\gamma_c} X_{\gamma_h} = y X_{\gamma_c+\gamma_h} = y^2 X_{\gamma_h} X_{\gamma_c} \tag{8.4}$$

So we compute

$$\begin{aligned}\Phi(X_{\gamma_h}) X_{\gamma_c} \Phi(X_{\gamma_h})^{-1} &= X_{\gamma_c}(1 + y^{-1} X_{\gamma_h}) \\ &= X_{\gamma_c} + X_{\gamma_c+\gamma_h}\end{aligned} \tag{8.5}$$

♣SHOW MORE STEPS! ♣

Motivic KS transformation. DEMONSTRATE WITH AN EXAMPLE THE FOCK SPACE COMBINATORICS.

Remark on Positivity and No Exotics???

8.4 Derivation of the motivic KSWCF

As we have seen in §4.11 the vanilla BPS states also undergo wall-crossing.

It turns out that *consistency* of the framed BPS states in the neighborhood of a marginal stability wall implies the "motivic KSWCF".

Usual diagram of BPS walls intersecting at point on MS and two paths.

Go through the fermionic Fock space combinatorics, as in paper III.

Remarks

1. When combined with the no-exotics conjecture (or even its weaker strong positivity counterpart) there is a surprising feature of framed BPS wall-crossing. A priori the KS/cluster transformations would be expected to produce $\overline{\underline{\Omega}}(L,\gamma;y)$ which involve characters of general elements of the representation ring. But this appears not to happen. We believe this is related to similar "positivity conjectures" in the cluster algebra literature.

   ♣ Be more precise [universal Laurent phenomena of FG??] ♣

8.5 Special Cases of the KSWCF

Take $y \to -1$ to get differential operators.

$$Y_\gamma Y_{\gamma'} = (-1)^{\langle\gamma,\gamma'\rangle} Y_{\gamma+\gamma'} \tag{8.6}$$

Then K_γ is the automorphism of the algebra generated by Y_γ defined by

$$K_\gamma(Y_{\gamma'}) := (1 - Y_\gamma)^{\langle\gamma',\gamma\rangle} Y_{\gamma'} \tag{8.7}$$

Specialize $\gamma = a\gamma_1 + b\gamma_2$ to get the form:
$(x,y) \to (x(1-(-1)^{ab} x^a y^b)^a, y ...)$
Rearrangement of $K_{\gamma_2} K_{\gamma_1}$ for $\langle\gamma_1,\gamma_2\rangle = m$.

$$K_{\gamma_2} K_{\gamma_1} =: \prod_{a/b \searrow} K_{a\gamma_1+b\gamma_2}^{\Omega_{a,b}(m)} \tag{8.8}$$

The product on the RHS is taken over all nonnegative integers a, b so that as we read from left to right the quantity a/b is nonincreasing. In particular $0/1 = 0$ corresponding to γ_2 is on the right and $1/0 = \infty$ corresponding to γ_1 is on the left.

Remark: Active vs. passive convention. We are defining K_γ as an automorphism of the algebra of twisted functions. It could also be considered as a diffeomorphism of the algebraic torus. Then the identities should have the ordering of the K's reversed, because the functor for Diff to Aut of functions is contravariant.

Examples:

$a = b = 1$: Recover the primitive WCF.

$a = 1$, b arbitrary: Recover the semiprimitive WCF

$m = 1$: Pentagon identity:

$$K_{\gamma_2} K_{\gamma_1} = K_{\gamma_1} K_{\gamma_1+\gamma_2} K_{\gamma_2} \tag{8.9}$$

as can easily be checked with a few lines of computation.

$m = 2$: In this case we have:

$$K_{\gamma_2} K_{\gamma_1} = \Pi_L K^{-2}_{\gamma_1+\gamma_2} \Pi_R \tag{8.10}$$

$$\begin{aligned} \Pi_L &= \prod_{n=0 \nearrow \infty} K_{(n+1)\gamma_1 + n\gamma_2} = K_{\gamma_1} K_{2\gamma_1+\gamma_2} \cdots \\ \Pi_R &= \prod_{n=\infty \searrow 0} K_{n\gamma_1 + (n+1)\gamma_2} = \cdots K_{\gamma_1+2\gamma_2} K_{\gamma_2} \end{aligned} \tag{8.11}$$

♣Indicate the trick to prove this. ♣

PHYSICAL REALIZATIONS:

$m = 1$: AD3

$m = 2$: SU(2) juggle: Demonstrate spectrum of SU(2) Nf=0.

Comment on wild wall crossing $m \neq 0, 1, 2$?

8.6 The spectrum generator

The KS formula strongly suggests that a useful quantity to compute is the *spectrum generator*. To define it, we choose a phase ϑ, or equivalently a half-plane in the complex plane (with phases between ϑ and $\varphi + \pi$. We then form the product:

$$\mathbb{S}(\vartheta; u) := \prod_{\gamma : \vartheta \leq \arg -Z_\gamma < \vartheta + \pi} K_\gamma^{\Omega(\gamma; u)}, \tag{8.12}$$

where the product is taken in order of increasing $\arg -Z_\gamma$ as we read from left to right.

The spectrum generator $\mathbb{S}(\vartheta; u)$ is a symplectic (or Poisson) transformation acting on the functions on the algebraic torus $\Gamma^* \otimes \mathbb{C}^*$. Given a central charge function, and hence an ordering of the phases of Z_γ there is a *unique* factorization of $\mathbb{S}(\vartheta; u)$ into a product of KS-transformations ordered as in (15.11). This is proved in [94], Section 2.2 using a filtration on the algebra given by defining a Euclidean metric on Γ. $\mathbb{S}(\vartheta; u)$ thus captures all the BPS degeneracies of the theory, since BPS states with central charges in the other half-plane are just the anti-particles of the ones counted by the SG. We assume that ϑ is sufficiently generic that no BPS particle has central charge of phase ϑ.

♣SAY MORE HERE? ♣

The SG can be in principle a compact way of summarizing a very complicated BPS spectrum. It is invariant under wall-crossing, so long as no BPS ray enters or leaves the half-space $\mathbb{H}_{\vartheta+\pi/2}$. As we will see in §**** and §**** below it can be explicitly computed in certain theories without any *a priori* knowledge of the BPS spectrum. In such cases it thus serves to derive the BPS spectrum - at least in principle. Given a symplectic transformation and an ordering of the phases of Z_γ it is in practice rather difficult to find a factorization of the form (15.11).

♣ REMARK ON "MONODROMY". Note that $K_{-\gamma}K_\gamma \neq 1$. Rather is a monodromy transformation. ♣

8.7 Line defects in theories of class S

Wrapping the (2,0) surface defects on C:

a.) Point defect (Local operator)

b.) Line defect: $L_{\wp,\zeta}$.

♣Need to explain how and why the phase ζ appears. ♣

c.) Surface defect: $\mathbb{S}_z$.

Classification of simple line defects: DMO for A1. Higher rank: Open.

Line defect vevs: Trace holonomy of flat connection.

Traffic Rules: Can really compute!! [Strings 2010]. Example of $SU(2)$ Wilson line.

9. Darboux Functions and Hyperkähler metrics

9.1 Line defect vevs

Wrap line defect on Euclidean circle:

$$\langle L_\zeta \rangle_m \tag{9.1}$$

depends on the vacuum $m \in \mathcal{M}$.

Susy of L_ζ implies this is a *holomorphic* function on $\mathcal{M}$.

So: $F(L)$ with $\mathcal{Y}_\gamma \to X_\gamma$ defines a noncommutative deformation of the algebra of holomorphic functions on $\mathcal{M}$. Return to this in Cluster Variety section below.

9.2 The Darboux expansion

We write the vev as a trace:

$$\langle L_\zeta \rangle = \mathrm{Tr}_{\mathcal{H}_{u,L_\zeta}}(-1)^F e^{-2\pi R H} e^{i\theta\cdot\mathcal{Q}}\sigma(\mathcal{Q}). \tag{9.2}$$

Expression in terms of trace implies the Darboux expansion:

$$\langle L_\zeta \rangle_m = \sum_\gamma \underline{\overline{\Omega}}(L_\zeta, \gamma)\mathcal{Y}_\gamma \tag{9.3}$$

This defines the $\mathcal{Y}_\gamma$ functions (if there are "enough" line defects).

Note that the large R limit defines nice the "semi-flat twistor functions":

$$\mathcal{Y}_\gamma^{\rm sf} := \exp\left(\frac{\pi R Z_\gamma}{\zeta} + i\theta_\gamma + \pi R\zeta\bar{Z}_\gamma\right) \tag{9.4}$$

9.3 The TBA construction of Darboux functions

Construct functions with basic 6 properties is a Riemann-Hilbert problem.

Solution: A TBA.

Iterating the TBA: interpret as semiclassical + sigma model instanton (from BPS particle going around the circle).

State result on convergence of the series expansion for sufficiently tame BPS spectrum (As in file ExplicitMetrics.tex)

Riemann-Hilbert problem is related to Differential equations: Interpreting KS transformations as Stokes matrices. Generalization of tt* geometry to 4 dimensions.

9.4 3D Compactification and HK geometry

Explain how you get Seiberg-Witten moduli space $\mathcal{M}$ as a fibration of tori over the Coulomb branch $\mathcal{B}$.

Do the explicit abelian duality. [GGI lectures]

A complex algebraic completely integrable system.

$d = 3, \mathcal{N} = 4$ supersymmetry: $\mathcal{M}$ must be hyperkähler .

The semi-flat metric: Will be quantum corrected.

Example: PTN: Get the Gibbons-Hawking form.

9.5 Twistors and Hitchin's theorem

Hitchin's theorem: Rephrasing hyperkähler metric in terms of family of holomorphic symplectic manifolds.

9.6 Twistor sections for $\mathcal{M}$

Turns out: The twistor functions for the semiflat metric are precisely the functions $\mathcal{Y}_\gamma^{\rm sf}$ we found above. [Neitzke-Pioline, unpublished].

Torus fibrations implies $(\mathbb{C}^*)^r$ coordinate patches: New interpretation of the *same* Darboux functions.

Pulling back ϖ_ζ from algebraic torus:

$$\{\mathcal{Y}_{\gamma_1}, \mathcal{Y}_{\gamma_2}\}_{\omega_\zeta} = \langle\gamma_1, \gamma_2\rangle \mathcal{Y}_{\gamma_1}\mathcal{Y}_{\gamma_2} \tag{9.5}$$

Implies Construction of HK metric from BPS degeneracies.

Interpretation of KSWCF: Continuity of the metric across $MS(\gamma_1, \gamma_2)$.

EXAMPLE: Full twistor functions for the (generalized) PTN case.

9.7 Relation to Fock-Goncharov coordinates for the A_1 case

Recall definition of cluster variety.

The $(\mathbb{C}^*)^r$ coordinate charts are cluster coordinate charts for $\mathcal{M}$. [???]

For the case of A_1 theories of class S

a.) Define Fock-Goncharov coordinates.

b.) Show that our $\mathcal{Y}_\gamma$ (perhaps modified by a quadratic refinement) are the FG coordinates for the moduli of flat $SL(2, \mathbb{C})$ local systems with flag data at punctures.

9.7.1 KS transformations from morphisms of decorated ideal triangulations

Flip, juggle, and pop here?? Or put this together in the spectral networks chapter?

10. Coupled 2d-4d systems

10.1 Defining a class of susy surface defects

UV Definition: Superconformal boundary conditions on $AdS_3 \times S^1$.

We consider defects which are straight lines in $\mathbb{R}^3$ preserving a $d = 2$ $\mathcal{N} = (2,2)$ superPoincaré symmetry.

Twisted chiral multiplet from the VM.

Two examples:

a.) Couple 2d theory to 4d theory

b.) Monodromy (Gukov-Witten) defects. Reduction of structure group.

10.1.1 IR effective theory

Solenoid picture. (Nonperiodic) Gukov-Witten parameters. Large gauge transformations.

Prepotential + superpotential

Subtleties of the superpotential: Gauge dependence/Monodromy.

Defines a Γ_i-torsor for Γ.

10.2 Important Example: The canonical surface defects $\mathbb{S}_z$

M2 description

Chiral ring: Equation for an SW curve.

Massive vacua = sheets of the covering.

Difference of superpotentials is relative H_1.

10.3 New BPS degeneracies

μ and ω:

Solitons on $\mathbb{S}_z$: Geometric picture in terms of open string webs. PICTURES OF THESE FROM SPECTRAL NETWORKS PAPER.

In the class S case μ counts with signs open string webs.

We will give a very precise way to compute μ using spectral networks technique below.

Similarly, $\omega(\gamma; \gamma_i)$.

Affine linear rule:

Again a precise result comes from spectral networks.

10.4 Supersymmetric Interfaces

Domain walls and Janus.

Defects within defects as in extended CFT: line defect L_ζ within surface defects $\mathbb{S}$ and $\mathbb{S}'$.

Framed BPS states. They again have wc: S-factors.

C: Moduli of surface defects [BAD NOTATION ?????]

Wall crossing for $\underline{\overline{\Omega}}$ Physical reason: 2d soliton binds to the domain wall.

So again we have a WCF.

This gives the 2d/4d WCF.

10.4.1 Class S supersymmetric interfaces

$\Gamma_{ij'}(z, z')$ is relative homology. Grades the framed BPS states.

$L_{\wp(z,z'),\zeta}$ only depends on *homotopy class* of $\wp$.

Wall-crossing in z: $\underline{\overline{\Omega}}$ jumps across the WKB paths of phase ζ:

10.5 2d/4d WCF

10.6 Reduction to 3D/1D systems: hyperkähler geometry

SUSY implies: Must be Hyperholomorphic vector bundles over $\mathcal{M}$.

Hyperholomorphic connection on hyperkähler manifold: F is type $(1,1)$ in all complex structures.

Semiflat Limit: HH line bundles.

Integral equations: Draw analogy to integral equations of inverse scattering theory.

Quantum corrections from 2D solitons running around the surface defect (wrapped on compactifying circle): Only the vector bundle $V_{\mathbb{S}} \to \mathcal{M}$ is well defined.

Interpretation of susy interfaces in terms of parallel transport. The fact that $\langle L_{\wp,\zeta} \rangle$ only depends on the homotopy class of $\wp$ is a reflection of the flatness of the connection $\mathcal{A}$.

10.6.1 Example: HH Line bundle over (generalized) PTN

10.6.2 The case of class S

1. Identify $V_{\mathbb{S}}$ with the universal (twisted) bundle over Hitchin moduli space.

2. Implication: In principle a systematic approach to solving the Hitchin equations themselves.

10.7 Example: The $\mathbb{C}P^1$ model coupled to the $d = 4$ $\mathcal{N} = 2$ SU(2) theory

Draw the WMS for the solitons for weak and strong 2d and 4d couplings.

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HERE ARE THE ACTUAL LECTURES AS GIVEN IN BONN

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11. Lecture 2, Tuesday Oct.2 : Theories of class S and string webs

11.1 Motivation

There are two motivations for this work:

a.) For physical reasons we are interested in the BPS degeneracies of d=4 $\mathcal{N} = 2$ theories. There are two things we can expect to find exactly in an $\mathcal{N} = 2$ theory: The low energy effective action and the BPS spectrum. The LEEA is fairly well understood. Less is known about the BPS spectrum. The technique of spectral networks gives one approach to deriving this spectrum for the theories of class S.

b.) BPS degeneracies are useful for other applications. As we have seen, knowledge of BPS degeneracies allows us to produce explicit hyperkähler metrics on some interesting spaces, such as the Seiberg-Witten/Hitchin moduli space.

11.2 Recap for the impatient reader who has skipped Sections 2-9

Let us summarize the structures we have on the maximal Coulomb branch for theories of class S:

Physics tells us that there is a six-dimensional theory $S(\mathfrak{g})$ associated with a simple, simply laced, compact real Lie algebra $\mathfrak{g}$.

Here we will take $\mathfrak{g} = su(K)$. (At some points, in the interest of pedagogical simplicity we will relax the mathematical condition of simplicity and take $\mathfrak{g} = u(K)$.) The generalization of the constructions below to other Lie algebras is work in progress.

As explained in §7, when we are given a compact Riemann surface C with punctures $\mathfrak{s}_n$, and some data D_n at the punctures we can produce a four-dimensional quantum field theory with $\mathcal{N} = 2$ Poincaré supersymmetry. We denote it by $S(\mathfrak{g}, C, D)$. Moreover, as explained in §??? when this theory is further compactified on a circle of radius R the resulting theory is a 3-dimensional sigma model. If we compactify on a circle with nonbounding spin structure (periodic boundary conditions for fermions) then the resulting three-dimensional sigma model has 8 supercharges. In this case the target space $\mathcal{M}$ - known as the Seiberg-Witten moduli space - has a fibration

$$\mathcal{M} \to \mathcal{B} \tag{11.1}$$

where $\mathcal{B}$ is a special Kähler manifold - physically $\mathcal{B}$ is the maximal Coulomb branch of the four-dimensional theory and the generic fibers are compact tori. It follows from theorems in supersymmetry that since the sigma model has Poincaré symmetry $\mathfrak{SP}(\mathbb{M}^{1,2|8})$ the target space $\mathcal{M}$ is hyperkähler .

Now, for theories of class S the moduli space $\mathcal{M}$ can be identified with a Hitchin moduli space. [29] This conclusion is an example of the nontrivial conclusions one can obtain by considering a closed loop in the diagram of relations in Figure 1. The compactification of $S(\mathfrak{g})$ on a circle gives a d=5 SYM theory with gauge group $SU(K)$ (or $PSU(K)$) and the space of BPS field configurations for compactification of this theory on C is easily found from the supersymmetry transformations of the d=5 SYM vectormultiplet. Those equations are the Hitchin equations. See Figure 6.

So, to summarize, we have:

1. A Riemann surface C with punctures $\mathfrak{s}_n$.

2. A principal $U(K)$ bundle $P \to C$ with unitary connection ∇ (locally written as $\nabla = d + A$) and "Higgs field" $\varphi \in \Gamma(C; \mathcal{K}_C \otimes \mathrm{ad}P)$.

[29] This might in fact be a characterizing property of class S.

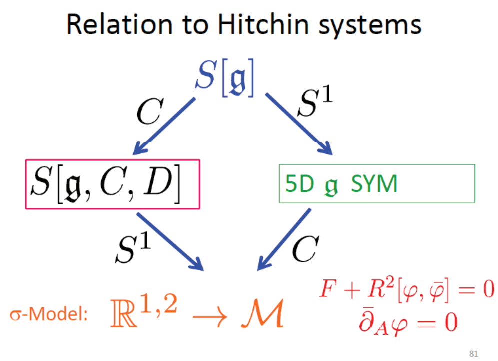

Figure 6: Two ways of viewing the compactification to three dimensions.

3. Boundary conditions: The simplest boundary conditions we can have are:

$$\varphi \sim (\mathfrak{r}_n + \delta_0 + \delta_1 z + \cdots)\frac{dz}{z} + reg. \tag{11.2}$$

where z is a local coordinate near $\mathfrak{s}_n$ so that $z = 0$ corresponds to the puncture. For simplicity we will take here $\mathfrak{r}_n$ to be a regular semisimple element of $\mathfrak{g}$. We can fix a gauge near $z = z_n$ and consider

$$\mathfrak{r}_n = Diag\{\mathfrak{m}_1^{(n)}, \ldots, \mathfrak{m}_K^{(n)}\} \tag{11.3}$$

In this gauge we also have gauge field

$$A \sim \frac{\alpha}{2i}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right) + reg \tag{11.4}$$

where $\alpha \in u(K)$ is a generic element of the Cartan:

$$\alpha = \mathrm{Diag}\{\alpha_1^{(n)}, \ldots, \alpha_K^{(n)}\} \tag{11.5}$$

4. It is important to stress that physics demands that we also consider other boundary conditions where the Higgs field has a higher order pole (asymptotically free theories) and/or where the coefficient of the pole of the Higgs field is in a nilpotent orbit of the Lie algebra. The detailed form of the boundary conditions for these other cases can be found in Sec. 3 and 4 of [94].

The moduli space $\mathcal{M}$ of solutions to Hitchin's equations:

$$F + R^2[\varphi, \bar{\varphi}] = 0, \tag{11.6}$$
$$\bar{\partial}_A \varphi := d\bar{z}\,(\partial_{\bar{z}}\varphi + [A_{\bar{z}}, \varphi]) = 0, \tag{11.7}$$
$$\partial_A \bar{\varphi} := dz\,(\partial_z \bar{\varphi} + [A_z, \bar{\varphi}]) = 0. \tag{11.8}$$

with the above boundary conditions can be identified with the SW moduli space of vacua of the theory $S(su(K), C, D)$ compactified on S^1 with radius R. [30]

The description from compactification of $S(\mathfrak{g}, C, D)$ on a circle makes it clear that $\mathcal{M}$ must be hyperkähler . (Of course this is a well-known mathematical result.) Therefore there is a twistor sphere of complex structures. We will let $\zeta \in \mathbb{C}^*$ denote a point in the stereographically projected twistor sphere so that $\zeta = 0$ corresponds to complex structure I, (and $\zeta = \infty$ corresponds to the complex structure $-I$) in which there is a holomorphic fibration:

$$\mathcal{M} \to \mathcal{B} \tag{11.9}$$

We can understand the nature of the base $\mathcal{B}$ better if we consider the spectral curve. The equation for the spectral curve

$$\Sigma := \{\lambda | \det(\lambda - \varphi) = 0\} \subset T^*C \tag{11.10}$$

(where we take the determinant in the fundamental representation of $su(K)$) can be written as

$$\lambda^K + \lambda^{K-1}\phi_1 + \lambda^{K-2}\phi_2 + \cdots + \phi_K = 0 \tag{11.11}$$

where ϕ_i is a meromorphic section of $\mathcal{K}_C^{\otimes i}$ with singularities at the $\mathfrak{s}_n$ implied by the boundary condition (11.2). [31]

The base of the fibration (11.9) parametrizes the gauge invariant data in the Higgs field φ. The fiber can be thought of - roughly speaking - as the set of flat connections A compatible with these gauge invariant data. The base is therefore parametrized by tuples $(\phi_1, \phi_2, \dots, \phi_K)$ of meromorphic differentials. The singularities of the differentials follow from the boundary condition. In the easiest case of simple poles with regular semisimple residue the leading singularities of these differentials are determined by the boundary condition (11.2):

$$\phi_j \sim \frac{e_j(-\mathfrak{m})}{z^j}(dz)^j + \cdots \tag{11.12}$$

where e_j is the elementary symmetric function. The subleading singularities depend on the subleading terms δ_0, δ_1 in the expansion of the Higgs field (11.2). Therefore, the Hitchin base is a torsor for the vector space

♣Explain this important point more. Need to explain the difference between normalizable and nonnormalizable parameters and that you can change the subleading singularities with normalizable parameters. ♣

$$\oplus_{j=1}^K H^0\left(\bar{C}; \mathcal{K}_{\bar{C}}^{\otimes j} \otimes \mathcal{O}\left(-\sum_n (j-1)\mathfrak{s}_n\right)\right). \tag{11.13}$$

We will often denote a generic point in $\mathcal{B}$ by u.

A very important point for what follows is that the equation (11.11) defines a $K:1$ branched covering

$$\pi : \Sigma \to C \tag{11.14}$$

[30] The parameter R is usually not written in the Hitchin equations but we include it since it has a well-defined physical meaning, and cannot be rescaled out of the problem if we also specify the "masses" $\mathfrak{m}_i$.

[31] We stress that φ is *not* holomorphic or meromorphic. It depends on z and $\bar{z}$ in local holomorphic coordinates on C. Nevertheless, it follows from the Hitchin equations that the ϕ_i are meromorphic.

Connection to physics: In §7 it is shown that Σ has the physical interpretation of being the Seiberg-Witten curve of the d=4 theory $S(\mathfrak{g}, C, D)$. Because of the physical interpretations we will sometimes refer to C as the "ultraviolet curve" and Σ as the "infrared" or "Seiberg-Witten" curve. Note that Σ canonically has a meromorphic one-form defined on it, namely λ itself. In physics this is known as the "Seiberg-Witten differential." [32]

For generic data $u \in \mathcal{B}$ the branch points of $\Sigma \to C$ will be simple branch points. However, on a special sublocus $\mathcal{B}^{\text{sing}}$ two different branchpoints will collide.

Example 1: Consider $\mathfrak{g} = su(2)$, $C = \mathbb{C}P^1 - \{\mathfrak{s}_n\}$, with regular singular points. We consider the stereographic projection of C to $\mathbb{C}$, and assume $z \to \infty$ is not a singular point.

In this case the general meromorphic quadratic differential with only second order poles must look like

$$\lambda^2 = \frac{Q_{2n+2}(z)}{(D_{n+3}(z))^2}(dz)^2. \tag{11.15}$$

where $Q_{2n+2}(z)$ is a polynomial of degree $2n+2$ and $D_{n+2}(z)$ is a polynomial of degree $n+2$ whose roots are at the singular points $\mathfrak{s}_n$. We assume that the zeroes of $Q_{2n+2}(z)$ are not located at the $\mathfrak{s}_n$. Using the boundary conditions and Liouville's theorem we know that we can write:

$$\lambda^2 = \sum_{a=1}^{n+3} \left(\frac{\mathfrak{m}_a^2}{(z-z_a)^2} + \frac{c_a}{z-z_a} \right) dz^2. \tag{11.16}$$

for some parameters c_a. These parameters label the possible quadratic differentials. Requiring that λ^2 is regular at $z = \infty$ gives three conditions on the c_a,

$$\begin{aligned} &\sum_{a=1}^{n+3} c_a = 0, \\ &\sum_{a=1}^{n+3} z_a c_a = -\sum_{a=1}^{n+3} \mathfrak{m}_a^2, \\ &\sum_{a=1}^{n+3} z_a^2 c_a = -2\sum_{a=1}^{n+3} \mathfrak{m}_a^2 z_a. \end{aligned} \tag{11.17}$$

The Coulomb branch $\mathcal{B}$ is the space of c_a solving (11.17). Because (11.17) is an inhomogeneous linear equation for the c_a, it is an affine space of dimension n. The singular locus is the place where zeroes of $Q_{n+3}(z)$ coincide.

Example 2: By taking a scaling limit where zeroes of Q_{n+3} collide we can obtain a series of theories with irregular singular points. See Section 9.2 of [94] as well as Section 9.2 of [96] for the detailed procedure.

In the case where we have one singular point at infinity we have a series of theories known as the "Argyres-Douglas" theories, and labeled AD_N, $N = 1, 2, \ldots$. The reason for the name is that they can be considered as perturbations of a series of superconformal field

[32] In the original work of Seiberg and Witten the meromorphic differential was defined up to some ambiguities. In the theories of class S there is a canonical such differential.

theories in $d = 4$ the first example of which (AD_3) was discovered in [15]. We also highly recommend [16], which clarified many obscure points.

The SW curve for this sequence of theories is

$$\lambda^2 = P_N(z)(dz)^2 \tag{11.18}$$

where $P_N(z)$ is an order N polynomial. The boundary conditions for (A, φ) are infinity are expressed in terms of the quantity $\Delta(z)$ defined by defined by the leading terms in the large z expansion:

$$\sqrt{P_N(z)} = \Delta(z) + o(z^{-1}) \tag{11.19}$$

When N is even, $\Delta(z)$ contains a smallest term m/z and we have

$$A_0 = \begin{pmatrix} -m^{(3)} & 0 \\ 0 & m^{(3)} \end{pmatrix} \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) \tag{11.20}$$

and

$$\varphi_0 = \begin{pmatrix} \Delta(z) & 0 \\ 0 & -\Delta(z) \end{pmatrix}, \tag{11.21}$$

When N is odd $\Delta(z)$ we have (**??**),

$$\varphi_0 = \Delta(z) \begin{pmatrix} 0 & (\bar{z}/z)^{1/4} \\ (z/\bar{z})^{1/4} & 0 \end{pmatrix}, \tag{11.22}$$

$$A_0 = \frac{1}{8}\sigma^3 \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right). \tag{11.23}$$

Example 3: One very basic example is the double-covering of $C = \mathbb{C}^*$ in T^*C given by

$$\lambda^2 = \left(\frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z} \right) (dz)^2 \tag{11.24}$$

This is the Seiberg-Witten curve for pure $SU(2)$ gauge theory. In this case there is an irregular singularity at $z = 0, \infty$. Indeed, note that the leading singularity is $(dz)^2/z^3$ rather than the expected $(dz)^2/z^4$ for a regular singular point. The boundary conditions for the Higgs field are now

$$\varphi \sim \frac{\Lambda}{\sqrt{2|z|}} \begin{pmatrix} 0 & 1 \\ \frac{\bar{z}}{|z|} & 0 \end{pmatrix} \frac{dz}{z} + reg. \tag{11.25}$$

$$A = -\frac{1}{8}\sigma^3 \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) + Reg \tag{11.26}$$

Note that

$$\mathrm{Tr}\varphi^2 = \frac{\Lambda^2}{z^3}(dz)^2 + \cdots \tag{11.27}$$

The singular points are also branchpoints of the covering so there are four branch points in all, the other two being at the zeroes of the RHS of (12.7):

$$z_\pm = -\frac{u}{\Lambda^2} \pm \sqrt{\left(\frac{u}{\Lambda^2}\right)^2 - 1} \tag{11.28}$$

In this case $\mathcal{B}$ is identified with the complex plane, and the singular locus $\mathcal{B}^{\text{sing}}$ arises where the two branch points $z_\pm$ collide, namely at $u = \pm\Lambda^2$.

In general we define $\mathcal{B}^{\text{sing}}$ to be the sublocus where branchpoints of the covering $\Sigma \to C$ collide. Here there are new massless particles in the theory. We let

$$\mathcal{B}^* = \mathcal{B} - \mathcal{B}^{\text{sing}} \tag{11.29}$$

Now, associated with the K-fold branched cover $\pi : \Sigma \to C$ we can form a local system of lattices over $\mathcal{B}^*$:

$$\Gamma := \ker \pi_* \subset H_{1,cpt}(\Sigma; \mathbb{Z}) \tag{11.30}$$

♣check! ♣

Both C and Σ are noncompact because of the punctures. There is a well-defined intersection form on the compactly supported 1-cycles on Σ. Let

$$\Gamma_f := \text{Ann}(\langle \cdot, \cdot \rangle) \tag{11.31}$$

the quotient is then a symplectic lattice and we have

$$0 \to \Gamma_f \to \Gamma \to \Gamma_g \to 0 \tag{11.32}$$

Let $\bar{C}$ and $\bar{\Sigma}$ be the normalizations of C and Σ (fill in points). We can identify Γ_g with $H_1(\bar{\Sigma}; \mathbb{Z})$.

The physical interpretation shows that Γ_g is a symplectic lattice of electric and magnetic charges in the low energy d=4 abelian gauge theory, while Γ_f is a lattice of charges of an abelian group of global symmetries.

In these terms, the fiber of the Hitchin fibration over $u \in \mathcal{B}^*$ includes the compact torus $\Gamma^*_{u,g} \otimes \mathbb{R}/\mathbb{Z}$. In complex structure I this torus inherits a complex structure and is an abelian variety.

♣Clarify the issue of components of the Hitchin fibration ♣

Also, we can define the central charge function $Z \in \text{Hom}(\Gamma, \mathbb{C})$ by

♣convention for angles. divide by 2pi Z ? ♣

$$Z_\gamma := \frac{1}{\pi} \oint_\gamma \lambda \tag{11.33}$$

In the physical interpretation, these are the central charges of an $\mathcal{N} = 2$ superPoincaré algebra.

11.3 WKB paths and string webs

As described in §7.6, the description of $S(\mathfrak{g})$ in terms of the membranes of M-theory leads to a nice geometrical description of *BPS states.*

11.3.1 WKB paths

Definition A *WKB path of phase* ϑ is a solution of the differential equation

$$\langle \lambda_i - \lambda_j, \partial_t \rangle = e^{\mathrm{i}\vartheta} \tag{11.34}$$

for *some* ordered pair of sheets i, j of the covering 11.14.

There are a number of remarks to make about this definition:

1. This is a differential equation: Locally $\lambda_i = f_i(z)dz$ so we have

$$(f_i(z(t)) - f_j(z(t)))\frac{dz}{dt} = e^{\mathrm{i}\vartheta} \tag{11.35}$$

   the solutions locally foliate the surface. We can think of introducing a local coordinate

$$w^{ij} = \int^z (\lambda_i - \lambda_j) \tag{11.36}$$

   and then the foliation is by straight lines in the w^{ij}-plane:

$$\mathrm{Im}\left(w^{ij}e^{-\mathrm{i}\vartheta}\right) = 0 \tag{11.37}$$

   FIGURE OF STRAIGHT LINES ROTATED FROM LINES PARALLEL TO x-AXIS BY $-\vartheta$

2. This is a local definition. The branched covering $\Sigma \to C$ has nontrivial monodromy so there is no global labeling of the sheets (i.e. solutions to (11.11)) over C. Nevertheless, as long as the solutions do not run into a branch point we can continue the sheets and the differential equation. We will in fact be quite interested in the long-time evolution of these equations.

3. In the A_1 case the two sheets are given by $\lambda = \pm\sqrt{\phi_2}$. In this case the equation defines the trajectory of $\pm\sqrt{\phi_2}$ and if we do not try to orient them then we get the foliations of C associated with a quadratic differential. These have been studied in depth in [Strebel-book; Teichmuller ref.]. In a sense what we are going to discuss involves generalization of that work.

4. The reason for the name "WKB path" will be evident later.

11.3.2 Local Behavior

Now let us study the local behavior of some of these WKB trajectories: There are two easy things we can say:

1. Near a singular point $\mathfrak{s}_n$ we have

$$\lambda_i \sim \mathfrak{m}_i^{(n)}\frac{dz}{z} + reg. \tag{11.38}$$

in a local coordinate with $z = 0$ at $\mathfrak{s}_n$. Therefore the ij trajectories asymptote to:

$$z(t) = z_0 \exp(\xi^{(ij)}t), \tag{11.39}$$

where we defined

$$\xi^{(ij)} = \frac{e^{i\vartheta}}{\mathfrak{m}_i - \mathfrak{m}_j}. \tag{11.40}$$

For generic ϑ and generic masses $\mathfrak{m}_i$ the solution is a spiral. There is now a natural ordering on the sheets in the neighborhood of $\mathfrak{s}_n$: We say that $i < j$ if, as $t \to \infty$, we have $z(t)$ spiraling into $z = 0$ as in

FIGURE OF AN ij TRAJECTORY SPIRALING INTO 0

So according to this definition,

$$i < j \qquad \text{iff} \qquad \mathrm{Re}\,\xi^{(ij)} < 0 \qquad \text{iff} \qquad \mathrm{Re}\,e^{-i\vartheta}\mathfrak{m}_i < \mathrm{Re}\,e^{-i\vartheta}\mathfrak{m}_j. \tag{11.41}$$

Note particularly that the region near the singularity functions as a basin of attraction for all the trajectories

DRAW FIGURE OF C WITH SOME BASINS OF ATTRACTION INDICATED: FIG *

2. We should also discuss that happens at an irregular singular point, where

$$\lambda \sim \frac{dz}{z^r} + \cdots \tag{11.42}$$

with $r > 1$. These still define attractive basins for WKB paths but the nature of the paths in the neighborhood of the point is somewhat different. As an example, consider $\mathfrak{g} = su(2)$ and the AD_N theories once again:

$$\lambda^2 = P_N(z)(dz)^2 \tag{11.43}$$

Then there is a single irregular singular point at $z = \infty$. For large z we have

$$\int^z \lambda \sim const.z^{\frac{(N+2)}{2}} + \cdots \tag{11.44}$$

so that there are $(N+2)$ rays at infinity and all WKB Paths will asymptote to one of those rays. In a local coordinate near the irregular singularity the pattern of WKB paths will look like

If we imagine drawing a small circle around the irregular singular point then we see that there are four marked points on that circle. In general, around irregular singular points we should think that there is a small circle with some number (one or larger) of marked points.

3. For generic $u \in \mathcal{B}^*$ all the branch points are simple branch points. These are simple zeroes of the discriminant $\prod_{i<j}(\lambda_i - \lambda_j)^2$. Near an (ij) branch-point $\mathfrak{b}$ choose a local coordinate z with $z = 0$ at $\mathfrak{b}$. Then we can write:

$$\lambda_i = (\kappa + \sqrt{z} + \cdots)dz \qquad \lambda_j = (\kappa - \sqrt{z} + \cdots)dz \tag{11.45}$$

where κ is some constant which is in general nonzero. Therefore

$$\int_{\mathfrak{b}}^z (\lambda_i - \lambda_j) \sim \frac{4}{3} z^{3/2} = e^{i\vartheta} t \tag{11.46}$$

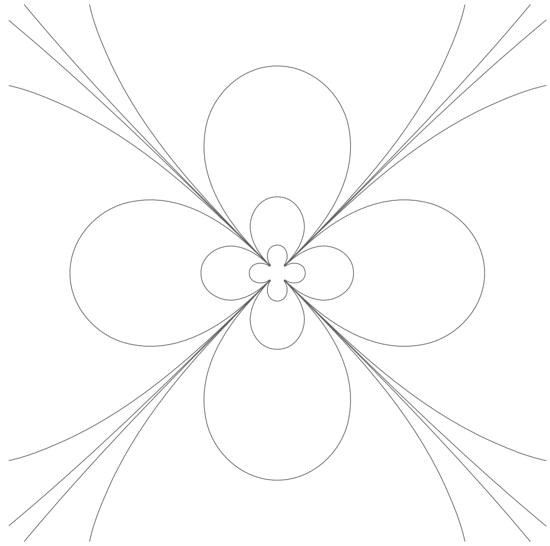

Figure 7: WKB paths in the neighborhood of an irregular singular point with four Stokes sectors.

Solving for $z(t)$, with $t \geq 0$ and redefining the positive time evolution parameter as $\tau = t^{2/3}$ we get:

$$z(\tau) = \tau e^{\frac{2i\vartheta}{3}} \tag{11.47}$$

Since ϑ is only defined mod $2\pi\mathbb{Z}$ there are three outward oriented trajectories emanating from $\mathfrak{b}$. If we choose a cut then one is a ji trajectory and the other two are ij trajectories as in Figure 47,

♣Explain better about the two sheets being interchanged and the need for a cut to distinguish them. ♣

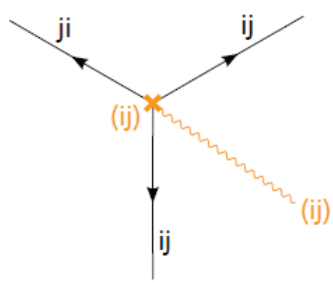

Figure 8: WKB paths in the neighborhood of a simple branchpoint exchanging sheets ij.

11.3.3 Global behavior

Now, what can we say more globally?

First, consider *generic* ϑ. Then, we expect, there are only two kinds of trajectories:

1. *Generic WKB paths* begin and end on singular points, because the neighborhoods of these points serve as basins of attraction.

GO BACK TO FIG * AND FILL IN A GENERIC TRAJECTORY

2. There are *separating WKB paths* which begin on branch points but end on singular points.

GO BACK TO FIG * AND FILL IN A SEPARATING TRAJECTORY FALLING INTO A BASIN OF ATTRACTION

For $K = 2$ these are theorems about trajectories of quadratic differentials. For $K > 2$ they seem like reasonable statements, but are ultimately conjectures.

3. However, for special angles ϑ_c we find trajectories such that *neither* end is on a singular point. There are two basic ways this can happen

a.) A trajectory can begin on a branch point and end on a branch point.

b.) A trajectory can be closed as in Figure 9

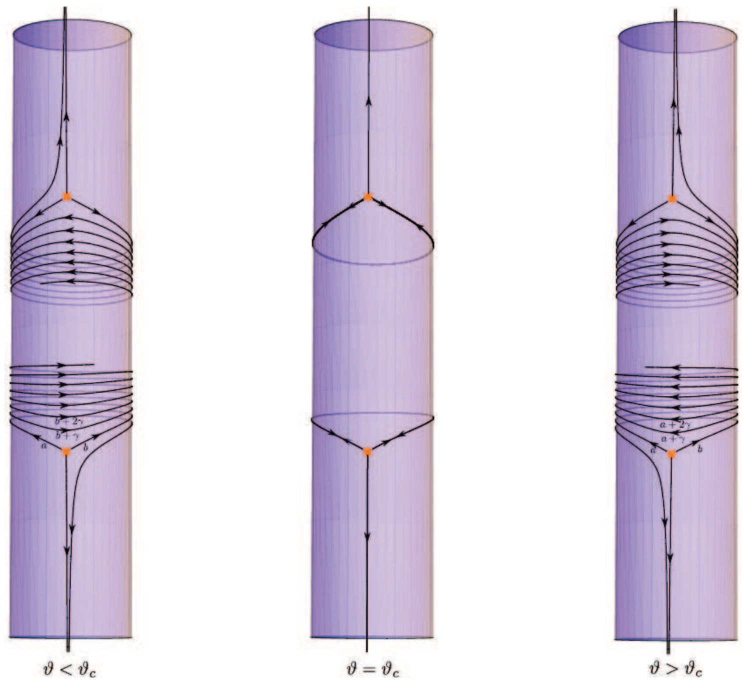

Figure 9: In the middle picture, at a critical angle ϑ_c there is a trajectory joining a branch point to itself. There are also closed trajectories foliating part of the cylinder.

Remark: The case when C has no punctures is rather different. In this case there are no basins of attraction and generic trajectories never end. The physics of such theories is qualitatively different and has not been much explored.

11.3.4 Definition of string webs

When $K = 2$ string webs are the special WKB paths discussed above which arise at special critical values of ϑ.

However, when $K > 2$ there is a new phenomenon which can occur:

Let us introduce *3-string junctions*: These are configurations of WKB paths of phase ϑ which look like Figure 10. Here we have $K > 2$ and three distinct sheets i, j, k. A WKB path of type ij meets a WKB path of type jk. We stop drawing those paths at the intersection point and continue with a WKB path of type ik.

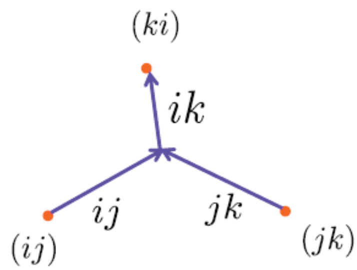

Figure 10: A trivalent junction used in constructing string webs.

For critical values of ϑ we can form graphs whose vertices are string junctions and whose endpoints are branchpoints.

Definition A *string web* is a connected graph whose segments consist of WKB paths and vertices consist of string junctions such that the endpoints of the graph (if any) lie on branch points.

When we have a string web then there is a lift of the web to a *closed* cycle in Σ. This closed cycle has a homology class $\gamma \in \Gamma$ which we call the "charge" of the web.

Example 1: The saddle connection. See Figure 11

Example 2: For the closed cycle see Figure 12

Example 3: For the 3-string junction see Figure 13.

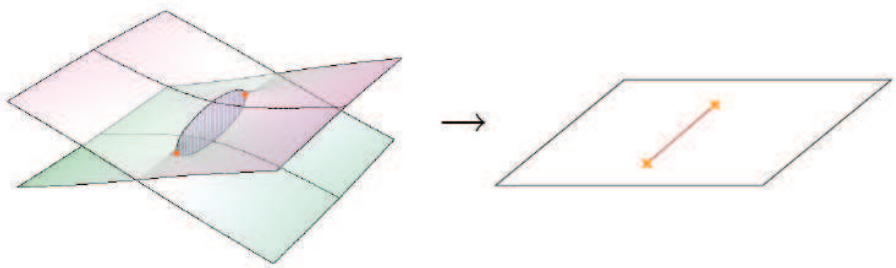

Figure 11: The lift of a string web saddle connection to Σ.

Remarks:

1. A string web has an associated central charge. Indeed, along the lift to Σ the one form $e^{-i\vartheta}\lambda$ is real. So
$$Z_\gamma = e^{i\vartheta_c}|Z_\gamma| \tag{11.48}$$
where ϑ_c is the phase of the web.

2. An experimental fact seems to be that when there are only string webs with no moduli then the set of critical phases ϑ_c is finite. However, when string webs with moduli occur then there are infinitely many critical phases which accumulate at the critical phases of the webs with moduli.

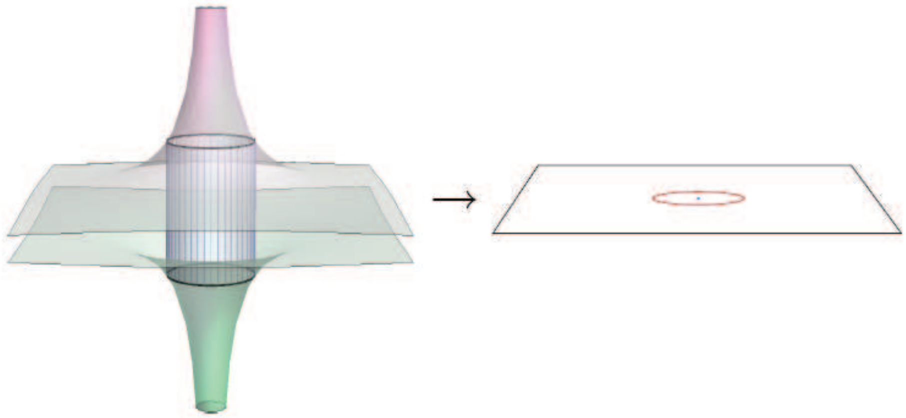

Figure 12: The lift of a string web closed cycle to Σ.

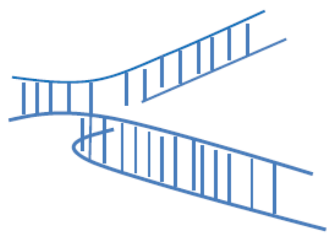

Figure 13: The lift of a string junction to Σ.

3. The M-theory viewpoint which gave rise to this definition of string webs suggests an interesting theorem in geometry. Suppose that T^*C admits a hyperkähler metric. (This is not always the case, but it is true in many examples.) We study minimal area surfaces in T^*C which end on 1-dimensional closed curves in Σ. The homology classes of the closed curves should be the same as the homology classes of string webs.

11.3.5 Counting string webs

Physically string webs represent BPS states with $\mathcal{N} = 2$ central charge Z_γ. As such they have BPS degeneracies $\Omega(\gamma; u)$ which are expected to satisfy the KSWCF. We will show, using spectral networks, how to define such a set of piecewise continuous integers. (For the $su(2)$ case this was shown a few years ago in [94].)

Roughly speaking, the BPS degeneracies $\Omega(\gamma; u)$ should be thought of as counting, with signs, the number of string webs.

In general, the webs come in moduli spaces. For example, for the closed curves in the case $\mathfrak{g} = su(2)$ the modulus arises because there is an annulus foliated by string webs. See Figure 9. The moduli space in this case is an interval. The boundaries of the interval represent a critical WKB path that begins and ends and the same branchpoint.

In general we expect that if the web has ℓ loops there will be a moduli space which is an ℓ-dimensional manifold with corners. Naively the formula for $\Omega(\gamma; u)$ for such a web would be $(-1)^\ell(\ell+1)$, but examples show that the situation is more complicated than that.

Unfortunately, there is no known direct definition of $\Omega(\gamma;u)$ in terms of enumerative invariants of the moduli space of string webs.

What we can say at the moment is

1. For $\mathfrak{g}=su(2)$ other considerations show that: it is known that

   Saddle connection (hypermultiplet): $\Omega(\gamma;u)=1$

   Closed curve (vectormultiplet): $\Omega(\gamma;u)=-2$

   So the count is similar to, but not exactly, the Euler character of the moduli of the webs. We will give a precise geometrical formulation of $\Omega(\gamma;u)$ later using spectral networks.

2. If one considers the DBI action for M2 branes then, in principle, one should be able to compute a supersymmetric quantum mechanics from the collective coordinate expansion around minimal area surfaces in T^*C ending on cycles in Σ in homology class γ. These calibrated surfaces form a moduli space which is an ℓ-dimensional manifold with corners (where ℓ is the number of loops). We compute $Tr(-1)^F$ in that susy quantum mechanics of collective coordinates, and that should be $\Omega(\gamma)$.

3. We will nevertheless give a precise algorithm for computing $\Omega(\gamma;u)$ using spectral networks. See equation (13.14).

12. Lecture 3, Thursday Oct. 4: Surface Defects and Spectral Networks

How can we determine the BPS degeneracies $\Omega(\gamma;u)$? How can we count the string webs?

The key new idea comes from physics: We introduce certain surface defects into the four-dimensional theory. Together with supersymmetric interfaces this will introduce two new kinds of BPS degeneracies: the 2d soliton degeneracies $\mu(a)$ and the framed BPS degeneracies $\underline{\overline{\Omega}}(\wp,\vartheta,a)$. While this might seem to complicate the story even further, it will turn out that it is actually quite helpful: Constructing these new BPS degeneracies first will give us the necessary handle to control the four-dimensional BPS degeneracies $\Omega(\gamma;u)$.

12.1 The canonical surface defect

In theories of class S, to each point $z\in C$ we associate a $(1+1)$-dimensional quantum field theory with $d=2$, $\mathcal{N}=(2,2)$ Poincaré supersymmetry. This dimension 2 "surface defect" will be denoted $\mathbb{S}_z$.

Physically we should view the surface defect as living in the x^3-t plane in Minkowski spacetime $\mathbb{M}^{1,3}$ located on the plane $x^1=x^2=0$

FIGURE OF THE x3-t PLANE

We have a 2d-4d coupled system: A d=2 QFT on the plane at $x^1=x^2=0$ is coupled to a d=4 field theory on $\mathbb{M}^{1,3}$. In the original $d=6$ $(2,0)$ theory there is a surface defect living at the surface $x^1 = x^2 = 0$ and $z \in C$ in $\mathbb{M}^{1,3} \times C$. In the original M-theory description we have an 11-dimensional Lorentzian signature spacetime of the form

$$\mathbb{M}^{1,3} \times T^*C \times \mathbb{R}^3 \tag{12.1}$$

where T^*C is equipped with a hyperkähler metric[33] There are K $M5$-branes in this spacetime located on the zero-section of T^*C and at the origin of $\mathbb{R}^3$. In these terms $\mathbb{S}_z$ is defined by a semi-infinite M2 brane whose worldvolume is $\mathbb{R}^2 \times \mathbb{R}_+$. The boundary is the surface $x^1 = x^2 = 0$ in $\mathbb{M}^{1,3}$, the point $(z, 0) \in T^*C$, and the origin of $\mathbb{R}^3$. The three-dimensional worldvolume extends along a ray in the $\mathbb{R}^3$ factor. (This ray breaks the $SU(2)_R$ symmetry down to $SO(2)$.)

♣So shouldn't we include an extra direction in the data of the surface defect? ♣

Example 1: A good example is the Hitchin system on $C = \mathbb{C}P^1$ with a minimal irregular singularity at infinity. After projecting C to the complex plane we have an irregular singularity at $z = \infty$ and no other singularities. This is the AD_1 theory discussed in (11.18) with $N = 1$. In particular its SW curve is

$$\lambda^2 = z(dz)^2 \tag{12.2}$$

In this case it turns out there are no d=4 BPS states. Indeed the WKB paths are easily plotted

DRAW WKB PATHS, SEPARATING AND GENERAL

In general, in order to search for string webs we need only vary ϑ from 0 to π. In this case the lines simply rotate by $2\pi/3$. There are no string webs. Thus, the $d = 4$ theory is empty, but the $d = 2$ theory is nontrivial. In fact, it is just a LG model with superpotential

♣put this general remark earlier? ♣

$$W = \frac{1}{3}\Phi^3 - z\Phi \tag{12.3}$$

Note that

1. The Riemann surface C can be viewed as a parameter space for the superpotential.
2. The equation for the critical points of this LG theory is the same as the equation for the spectral curve.

These will turn out to be general features.

Example 2: We can also consider the $\mathbb{C}P^1$ sigma model coupled to the pure $SU(2)$ gauge theory. The chiral ring is well-known to be

$$x^2 = \Lambda_{2d}^2 e^t \tag{12.4}$$

It is also well-known that if one adds a twisted mass the chiral ring is modified to

$$x^2 = \Lambda_{2d}^2 e^t + 2u \tag{12.5}$$

[33] This might not exist in some cases. Since we are primarily interested in the case with punctures a hyperkähler metric should exist.

and it turns out that the effect of 4d instantons gives

$$x^2 = \Lambda_{2d}^2 e^t + 2u + \frac{\Lambda_{4d}^4}{\Lambda_{2d}^2 e^t} \tag{12.6}$$

Now with the change of variable $z = e^t$ and $\lambda = x(dt)$ we recognize the SU(2) SW curve (taking $\Lambda_{2d} = \Lambda_{4d} = \Lambda$, for simplicity):

$$\lambda^2 = \left(\frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z} \right) (dz)^2 \tag{12.7}$$

In general, it is much harder to say what QFT the field theory $\mathbb{S}_z$ is in conventional Lagrangian terms. But we *can* say something about the massive vacua and the superpotential. A simple argument based on the M-theory picture leads to the crucial claim:

The equation for the critical points of the superpotential - that is, the equation determining the vacua of the theory - is *identical* to the equation determining the SW curve (11.11):

$$\lambda^K + \lambda^{K-1}\phi_1 + \lambda^{K-2}\phi_2 + \cdots + \phi_K = 0 \tag{12.8}$$

Thus, we should identify the preimages of the branched cover $\pi : \Sigma \to C$ over $z \in C$ with the massive vacua of the 1+1 dimensional field theory $\mathbb{S}_z$.

FIGURE OF SHEETS OF COVER SIGMA OVER C.

12.2 Solitons on the canonical surface defect $\mathbb{S}_z$

The $d = 2$ $\mathcal{N} = (2,2)$ theory $\mathbb{S}_z$ can have BPS solitons interpolating between different vacua. Recall that these different vacua are identified with the preimages $z^{(i)}$ of $\pi : \Sigma \to C$.

As with 4d BPS states, the 2d BPS solitons have a representation in terms of "open string webs."

Definition: An *open string web* of phase ϑ for $\mathbb{S}_z$ is a connected graph of WKB paths and three string junctions with one end at z and all others (if they exist) on branch points of the covering $\pi : \Sigma \to C$.

For examples see Figure 14

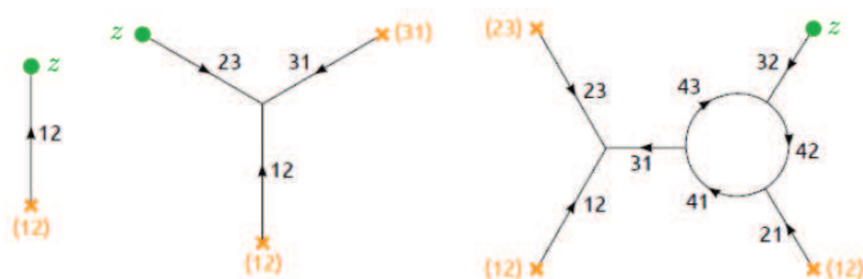

Figure 14: A trivalent junction used in constructing string webs.

♣Should say something about the M2-brane interpretation. ♣

As with the string webs, the open string webs have a lift to Σ. Instead of being a closed path we now get an open path whose endpoints are the vacua $z^{(i)}$ and $z^{(j)}$ above z. This path defines a relative homology path, and we express the "charge" of the soliton in terms of relative homology of Σ. See Figure 15.

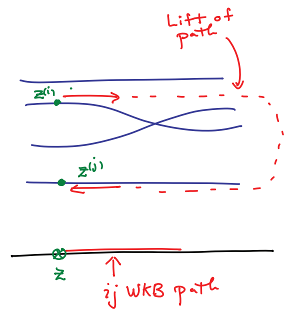

Figure 15: The charge of a soliton is a chain with two endpoints above z.

To say this a little more precisely, let us introduce a slightly more general notion, which we will need later:

For $z_1, z_2 \in C$ let $z_1^{(i)}$ denote the lifts to Σ (in some local trivialization of the branched covering) and similarly for $z_2^{(j)}$. Then we consider the set of 1-chains on Σ so that

$$\partial \mathfrak{c} = z_2^{(j)} - z_1^{(i)} \tag{12.9}$$

Modding out by boundaries defines a torsor $\Gamma_{ij}(z_1, z_2)$ for Γ.

For simplicity we will take $\mathfrak{g} = gl(K)$ in this talk so that $\Gamma = H_1(\Sigma; \mathbb{Z})$.

It will also be convenient to define:

$$\Gamma(z_1, z_2) = \cup_{i,j} \Gamma_{ij}(z_1, z_2) \tag{12.10}$$

The lift of an open string web defines a homology class in $\Gamma(z, z)$. We will refer to this as a "charge" because in the ambient four-dimensional gauge theory it indeed sources the low energy abelian gauge fields.

We can now define a soliton degeneracy for solitons of charge $a \in \Gamma(z, z)$. Roughly speaking, we once again count with signs the open string webs at z. Once again, we will be a little vague about the precise definition since we will give a more precise construction later. We will denote the soliton degeneracies by $\mu(a)$. Physically we expect that

♣Give precise forward ref Formal Parallel Transport Theorem ♣

$$\mu(a) = \mathrm{Tr}_{\mathcal{H}_a^{BPS}} F(-1)^F \tag{12.11}$$

where $\mathcal{H}_a^{BPS}$ is a space of BPS solitons of charge a and F is a fermion number. The soliton has a $d = 2$ $\mathcal{N} = (2, 2)$ central charge, and it is given by

$$Z_a = \frac{1}{\pi} \oint_a \lambda \tag{12.12}$$

the phase of Z_a is $e^{\mathrm{i}\vartheta}$.

Remarks:

1. As opposed to the closed string webs, for *any* phase ϑ there is some z which supports an open string web. For this we need only look in the neighborhood of a branch point $\mathfrak{b}$. On the other hand, if we fix a 2d theory, that is, if we fix z then as before only for special critical angles ϑ_c will there be open string webs with an endpoint at z. These are the phases of the BPS solitons in the theory $\mathbb{S}_z$.

2. It turns out that there is a tricky sign ambiguity in the definition $\mu(a)$ which stems from ambiguities in defining the fermion number F. Really one needs to work on the unit circle bundle in the tangent bundle of C and Σ. Roughly speaking, when counting paths there are signs corresponding to whether these lifted paths wind around the circle fiber and even or odd number of times. In the interest of simplicity we are going to suppress that subtlety in these notes. For the complete story see [100].

12.3 Physical Definition of a spectral network

Now we can define a spectral network:

Definition: Given a branched covering $\pi : \Sigma \to C$ equipped with meromorphic differential λ, and a phase ϑ we define $\mathcal{W}_\vartheta$ to be the set of points $z \in C$ so that $\mathbb{S}_z$ has solitons of charge $a \in \Gamma(z, z)$ such that

$$Z_a = e^{\mathrm{i}\vartheta} |Z_a| \tag{12.13}$$

and $\mu(a) \neq 0$.

This is not a very practical definition because we have not yet said how to compute the $\mu(a)$ precisely. Nevertheless, it provides an important physical intuition. Moreover, there are two comments we can make immediately:

1. We note that it is easy to determine $\mathcal{W}_\vartheta$ in the neighborhood of a branch point $\mathfrak{b}$ of type (ij). As we have noted, in this neighborhood we have

$$\int_{\mathfrak{b}}^z (\lambda_i - \lambda_j) \sim \frac{4}{3} z^{3/2} \tag{12.14}$$

and hence we have the trivalent vertex of WKB paths as noted above. If z lies on any one of these paths we can easily construct an open string web simply by following the WKB path back to the branch point. The lift of this path is a path on Σ which begins on $z^{(i)}$ moves along a path in sheet i covering the WKB path on C back to the ramification point, and then moves back on sheet j along a path covering the WKB path ending on $z^{(j)}$. We will call it the simple soliton, or *simpleton* (of type ij) for short. Note that the mass of such a soliton is $\sim |z^{3/2}|$.

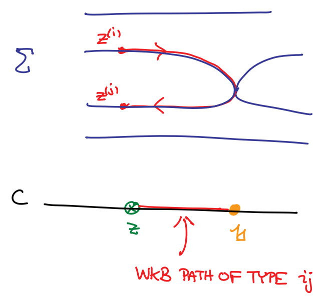

Figure 16: The simple soliton, or simpleton.

2. The network $\mathcal{W}$ is in general a complicated graph in C. See, for example Figure 17. The graph is made of segments and these segments must be WKB paths of type ij and phase ϑ. and we will call them *S-walls of type ij* (if the corresponding WKB path is of type ij). The reason for the name "wall" will soon be apparent.

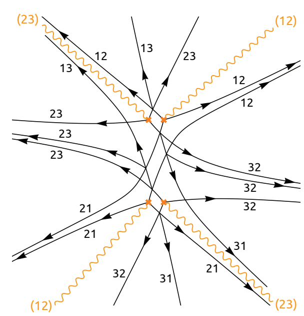

Figure 17: An example of a spectral network. It is a complicated graph and the segments, generically denoted by p, are WKB paths of phase ϑ.

The lattices $\Gamma(z, z)$ form a local system over C so that continuous variation of z along a segment as a lift to the fibers. That is, a homology class $a_z \in \Gamma(z, z)$ has a natural parallel transport as z is moved along the segment:

$$a_z \to a_{z'} \tag{12.15}$$

as z moves to z' along the segment. This continuation will sometimes be understood in formulae to follow. Indeed, with this understanding, if there is a soliton of charge $a \in \Gamma_{ij}(z, z)$, for z on an S-wall of type ij then $\mu(a)$ remains constant as z is continued along the segment. Thus, it makes sense to write $\mu(a, p)$ where p represents the segment. The soliton degeneracies in general *will* change when WKB paths in $\mathcal{W}_\vartheta$ intersect, as we will soon see.

12.4 Supersymmetric Interfaces and framed BPS states

Before we say how the soliton degeneracies change it is very convenient to introduce a second set of BPS degeneracies - the framed BPS degeneracies.

Now we consider domain walls, or supersymmetric interfaces between the $1+1$ dimensional theories represented by $\mathbb{S}_{z_1}$ and $\mathbb{S}_{z_2}$. If we think of C as parametrizing superpotentials then a way of making a domain wall, or janus, is to let the superpotential, i.e. to let $z(x)$ vary along some path as x ranges over an interval of the x^3-axis, as shown in Figure 18. That is, in the $\mathcal{N}=2$ superpotential $W(\Phi; z)$ depending on LG superfields Φ and depending parametrically on $z \in C$, we now substitute $W(\Phi; z(x))$ into the Lagrangian.

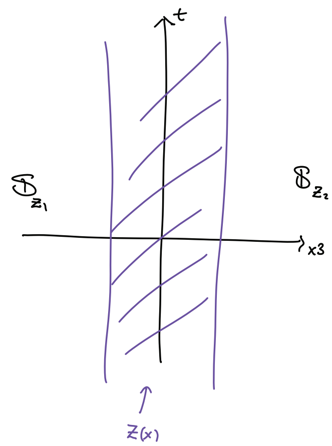

Figure 18: The superpotential changes in the shaded region. This describes a continuous path $z(x)$ on C, which we denote by $\wp(z_1, z_2)$.

Thus, these domain walls are represented, geometrically, by continuous paths in C, denoted $\wp(z_1, z_2)$. N.B. these paths need not be WKB paths. They are just continuous paths and specifying the path is part of the specification of the interface. Note that z_1, z_2 are in C and hence never coincide with singular points $\mathfrak{s}_n$. Moreover, for the questions we will be dealing with, they are unparametrized paths. We could, in principle collapse the domain in which $z(x)$ vaires in Figure 18 to be arbitrarily small.

We will be interested in *supersymmetric interfaces* which preserve 2 of the 4 supersymmetries preserved by the defects $\mathbb{S}_z$. The supersymmetries preserved by the interface will have the form

$$Q + e^{\mathrm{i}\vartheta}\bar{Q} \tag{12.16}$$

A susy interface is thus determined by two independent pieces of data:

1. A continuous path $\wp(z_1, z_2)$
2. A phase ϑ

We denote these defects by $L_{\wp,\vartheta}$ because they should be thought of as line defects embedded within surface defects.

♣Should put these within the context of lecture one on extended field theories. ♣

Once again supersymmetric interfaces have an associated set of BPS states known as "framed BPS states." Again there is a geometrical interpretation in terms of "millipedes. Here we attach WKB webs ending on the path $\wp$ so that there is a lift of the resulting connected set to a relative cycle defining a homology class $a \in \Gamma(z_1, z_2)$. The nontrivial point here is that there are no other boundaries on Σ. That is, the lift of the millipede to Σ is a chain with

$$\partial \mathfrak{c} = z_2^{(j)} - z_1^{(i)} \tag{12.17}$$

In analogy to what we had before, the chain $\mathfrak{c}$ determines a homology class $a \in \Gamma_{ij}(z_1, z_2)$ which is called the "charge" of the framed BPS state.

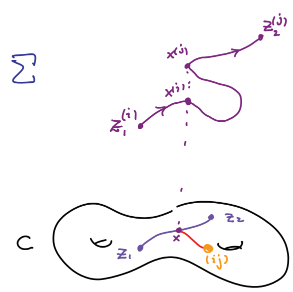

Figure 19: The lift of a millipede joining onto $\wp(z_1, z_2)$ has only two boundary points.

Once again we imagine counting these millipedes with signs to define a framed BPS invariant

$$\underline{\overline{\Omega}}(L_{\wp,\vartheta}, a) \tag{12.18}$$

Again we will not determine these directly from this definition.

Remark An important aspect of framed BPS states is that their energy is not de-

♣This remark is not used in this Lecture. Move it. ♣

termined just by the central charge Z_a, as with solitons and 4d BPS states. Rather the energy of these states is

$$E = -\mathrm{Re}(e^{-\mathrm{i}\vartheta} Z_a) \tag{12.19}$$

12.5 The Formal Parallel Transport Theorem

We have now discussed several BPS invariants:

$$\Omega(\gamma) \qquad \gamma \in \Gamma \qquad Z_\gamma = \frac{1}{\pi} \oint_\gamma \lambda \tag{12.20}$$

$$\mu(a) \qquad a \in \Gamma(z, z) \qquad Z_a = \frac{1}{\pi} \oint_a \lambda \tag{12.21}$$

$$\underline{\overline{\Omega}}(\wp, \vartheta, a) \qquad a \in \Gamma(z_1, z_2) \qquad E = -\mathrm{Re}(e^{-\mathrm{i}\vartheta} Z_a) \tag{12.22}$$

The key to understanding all these BPS degeneracies is to introduce a generating function for the framed BPS states and to formulate its properties:

$$F(\wp, \vartheta) := \sum_{a \in \Gamma(z_1, z_2)} \underline{\overline{\Omega}}(\wp, \vartheta, a) X_a \tag{12.23}$$

We will be taking the X_a in the *homology path algebra*. Let us explain this:

Recall that to any category $\mathcal{C}$ we may associate a ring $R(\mathcal{C})$. As an abelian group, $R(\mathcal{C})$ is the free group on the space of morphisms:

$$R(\mathcal{C}) = \bigoplus_{f \in \mathrm{Mor}(\mathcal{C})} \mathbb{Z} \cdot \ell_f. \tag{12.24}$$

The ring structure in $R(\mathcal{C})$ is defined by

$$\ell_{f_1} \cdot \ell_{f_2} := \begin{cases} 0 & \text{if } f_1 \text{ and } f_2 \text{ are not composable,} \\ \ell_{f_1 f_2} & \text{if } f_1 \text{ and } f_2 \text{ are composable.} \end{cases} \tag{12.25}$$

We can apply this to the fundamental groupoid $\pi_{\leq 1}(C)$ of C to obtain the *homotopy path algebra*. There is a natural homomorphism of $R(\pi_{\leq 1}(C))$ to the homology path algebra of C. Concretely, the homology path algebra has generators X_a for any relative homology class $a \in \Gamma(z_1, z_2)$ and relations

$$X_{a_1} X_{a_2} = \begin{cases} X_{a_1 + a_2} & \mathrm{end}(a_1) = \mathrm{beg}(a_2) \\ 0 & \text{else} \end{cases} \tag{12.26}$$

where $a_1 + a_2$ is the homology of any concatenated path. In what follows we will restrict attention to the homology path algebra, although it is important - both physically and mathematically - that the considerations in fact can be applied to the homotopy path algebra.

Now we call the formal generating function

$$F(\wp, \vartheta) = \sum_a \underline{\overline{\Omega}}(\wp, \vartheta, a) X_a. \tag{12.27}$$

the *formal parallel transport* for reasons which will become apparent below. It is useful because of the following theorem:

Theorem: Then there exists a unique set of BPS degeneracies

1. $\overline{\underline{\Omega}}(L_{\wp,\vartheta}, a)$, $\forall \wp, a \in \Gamma(z_1, z_2), z_1, z_2 \in C - \mathcal{W}_\vartheta$
2. $\mu(a)$, $a \in \Gamma(z, z)$, $\forall z \in C$

such that $F(\wp, \vartheta$ satisfies the following four properties:

1. *Homotopy invariance*: $F(\wp_1, \vartheta) = F(\wp_2, \vartheta)$ if $\wp_1 \sim \wp_2$ are homotopic with fixed endpoints in C.
2. *Homomorphism property*: $F(\wp_1, \vartheta)F(\wp_2, \vartheta) = F(\wp_1 \circ \wp_2, \vartheta)$ when $\mathrm{end}(\wp_1) = \mathrm{beg}(\wp_2)$. This applies when the endpoints of $\wp_1, \wp_2$ do not lie on $\mathcal{W}_\vartheta$.
3. *Local triviality*: If $\wp \cap \mathcal{W}_\vartheta = \emptyset$ then
$$F(\wp, \vartheta) = \sum_{i=1}^{K} X_{\wp^{(i)}} := D(\wp) \tag{12.28}$$
Here $\wp^{(i)}$ denote the K lifts of $\wp$ to Σ.
4. *Detour rule*: When $\wp \cap \mathcal{W}_\vartheta \neq \emptyset$ F satisfies the detour rule. Because of item 2 above it suffices to say what happens when a small path $\wp$ crosses one segment of type ij in $\mathcal{W}_\vartheta$ as in

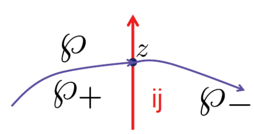

Figure 20: A small path $\wp$ crosses a single S-wall of type ij. Because of the homomorphism property it suffices to give the Detour rule for this case. ♣ Fix figure so intersection point is z_* ♣

Then we have

$$\begin{aligned} F(\wp, \vartheta) &= D(\wp) + D(\wp_+) \left(\sum_{a \in \Gamma_{ij}(z_*, z_*)} \mu(a) X_a \right) D(\wp_-) \\ &= D(\wp) + X_{\wp_+^{(i)}} \left(\sum_{a \in \Gamma_{ij}(z_*, z_*)} \mu(a) X_a \right) X_{\wp_-^{(j)}} \\ &= D(\wp_+) \left(\prod_{a \in \Gamma_{ij}(z_*, z_*)} (1 + \mu(a) X_a) \right) D(\wp_-) \end{aligned} \tag{12.29}$$

where the different ways of writing the RHS each has different advantages.

Remarks:

1. Typically there is only one soliton to worry about in the detour rule.

2. The detour rule is really a wall-crossing formula for the framed bps degeneracies $\underline{\overline{\Omega}}(\wp, \vartheta, a)$. Imagine a family of supersymmetric interfaces where the endpoint z moves along the path $\wp(z_1, z_2)$ and z crosses an S-wall of type ij. As z crosses the wall a new millipede appears. Physically it says that a susy interface or domain wall can, as we vary its parameters z, emit or absorb a BPS soliton.

   FIGURE OF x-SPACE WITH SOLITON BEING EMITTED

3. In the proof of the theorem we sketch below we suppose that ϑ is generic. That is, suppose ϑ does not support any string webs. The soliton degeneracies can still be determined when ϑ supports string webs, but the result is more difficult. The details are in [100].

Now let us sketch the idea of the proof:

One key idea is to introduce a *mass filtration*. We put a cutoff on the spectral network $\mathcal{W}_\vartheta$ to include only the solitons with mass below the cutoff:

Definition: $\mathcal{W}_\vartheta[\Lambda] \subset \mathcal{W}_\vartheta$ is the subset consisting of z so that all solitons of charge $a \in \Gamma(z, z)$ have $|Z_a| \leq \Lambda$.

Of course, when $\Lambda \to \infty$ we recover the full spectral network.

On the other hand, we claim that when $\Lambda \to 0$ only the simpletons contribute. Recall these have $M(z) \sim |z|^{3/2}$ where z is a coordinate in the neighborhood of a branch point.

As we have seen, it is easy to draw the spectral network in the neighborhood of branch points.

Now we apply homotopy invariance together with the detour rule and compare paths $\wp_1$ and $\wp_2$ in Figure 21

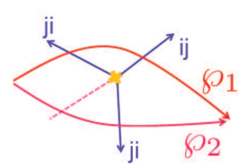

Figure 21: The spectral network in the neighborhood of a branch point. The homotopy axiom plus the detour axiom applied to the two paths $\wp_1$ and $\wp_2$ shows that $\mu(a) = 1$ for the simpletons with z on the WKB paths near the branch point.

From a careful analysis with signs (done in detail in [100]) we learn the important result that for $a \in \Gamma(z, z)$ where z lies on the spectral network near the branch point $\mathfrak{b}$ we must have

$$\mu(a) = 1. \tag{12.30}$$

Now that we know the "initial conditions" for the spectral network, we can start growing it: We simply evolve the differential equations for the WKB paths.

This will be fine until the paths start intersecting. Then new things can happen.

To see what happens, we impose the homotopy constraints to learn some important consistency rules for how the soliton degeneracies $\mu(a,p)$ must change as z is moved along WKB trajectories. As we have noted, if z evolves along a segment and a is continuously evolved in the obvious way then $\mu(a,p)$ remains constant along the segment p.

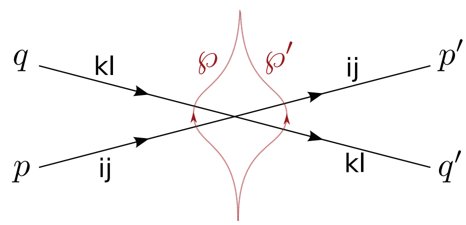

Figure 22: When ij and kl trajectories meet with i,j disjoint from k,l no new solitons are created in the theory $\mathbb{S}_z$ as z is continued along a trajectory through the intersection point. This is shown by applying the homotopy axiom plus the detour axiom to the two paths shown here.

A.) When an S-wall of type ij intersects and S-wall of type kl and i,j is disjoint from k,l then homotopy equivalence of $F(\wp,\vartheta)$ for the two paths in Figure 22 plus the detour rule shows that

$$\mu(a,p) = \mu(a',p'). \tag{12.31}$$

B.) However, when and ij S-wall intersects a jk S-wall then, because $(\lambda_i - \lambda_j) + (\lambda_j - \lambda_k) = (\lambda_i - \lambda_k)$ necessarily there is an intersecting ik WKB path as in

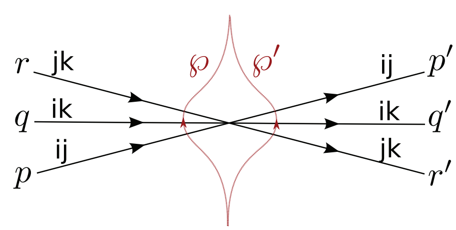

Figure 23: When an ij WKB segment in a spectral network collides with a jk path in a spectral network there is in general also an ik path in the spectral network which simultaneously collides. The collision point is called a *joint*. In this case the soliton degneracies of $\mathbb{S}_z$ change, as z moves along the ik trajectory across the joint. The change in soliton number satisfies the CVWCF.

We refer to such an intersection point z_* as a *joint*.

Now, homotopy equivalence of $F(\wp,\vartheta)$ for the two paths plus the detour rule shows that

$$\begin{aligned} \mu(a',p') &= \mu(a,p) \\ \mu(c',r') &= \mu(c,r) \\ \mu(b',q') &= \mu(b,q) + \mu(a_*,p)\mu(b_*,r) \end{aligned} \tag{12.32}$$

Here a_* is a path that joins $z_*^{(i)}$ to $z_*^{(j)}$ and b_* is a path that joins $z_*^{(j)}$ to $z_*^{(k)}$. Note that we can compose these with simple paths along the ik segment to produce a new soliton for $c' \in \Gamma(z', z')$ with $z' \in r'$.

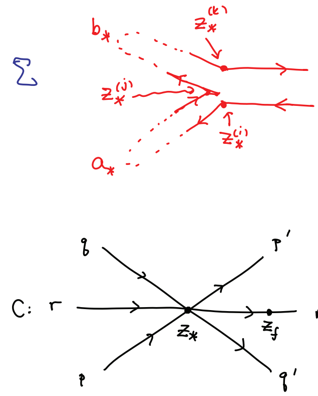

Figure 24: The new solitons which are created after crossing through a joint z_*.

Now we know how to evolve the spectral network indefinitely.

In fact, this can be put on a computer. Go to

http://www.ma.utexas.edu/users/neitzke/spectral-network-movies/

(or just google on "Spectral Network Movies") to see beautiful animations based on a Mathematica code written by A. Neitzke.

As we evolve the network forward in time the WKB paths will eventually get trapped in the basins of attraction. From a local analysis near the singular points we expect that the joints will accumulate at the singular points, but this is a completely controlled (and relatively uninteresting) phenomenon:

Claim: For any $\epsilon > 0$, On $C - \cup_n D(\epsilon, \mathfrak{s}_n)$ there are only finitely many joints. This follows because the velocities of the paths are finite away from the singular points.

Remarks

1. For $K = 2$ there are no joints. Separating WKB paths simply end at singular points. The resulting spectral network is dual to an *ideal triangulation* of C. An ideal triangulation is a triangulation with all vertices at the punctures. This is important in making contact with the work of Fock and Goncharov. (Their work is based on choosing triangulations.) The sense in which it is dual is that the spectral network divides C up into cells. We then choose a generic WKB path within each cell. It connects singular points to singular points. These are the edges of a triangulation of C, with one branch point within each triangle. See Figure 25.

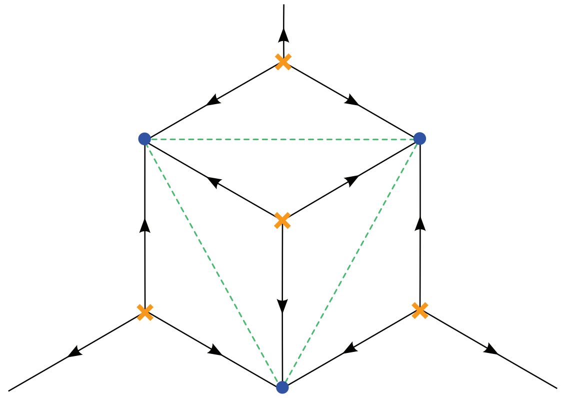

Figure 25: The ideal triangulation dual to a spectral network with $K = 2$.

2. The rule (12.32) for evolving the soliton degeneracies through a joint is identical to the Cecotti-Vafa wall-crossing formula.

3. We have now shown how to construct the spectral network $\mathcal{W}_\vartheta$, together with the soliton degeneracies $\mu(a)$ and the framed BPS degeneracies $\overline{\underline{\Omega}}(\wp, \vartheta, a)$. An important assumption we have been making is that ϑ is *generic*. In particular that means that no S-wall of type ij collides with an ij branch point or with an S-wall of type ji going the other way.

13. Lecture 4, Friday October 5: Morphisms of Spectral Networks and the 2d-4d Wall-Crossing Formula

13.1 Morphisms of $\mathcal{W}_\vartheta$

We now consider what happens to the spectral networks and the formal parallel transport $F(\wp, \vartheta)$ as we vary ϑ.

If ϑ is generic then sufficiently small variations $\vartheta \to \vartheta + \delta\vartheta$ will simply isotope the spectral network slightly.

Around the branch points the spectral network simply rotates a little as in Figure 26:

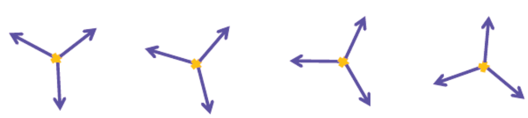

Figure 26: The change in a spectral network near a branch point as the angle ϑ is increased.

However, at critical values of ϑ something more drastic can happen: An ij path can run into a ji path with the opposite orientation.

Note that when this happens the path will change course drastically as in Figure 27:

At the critical value ϑ_c we will have a *degenerate spectral network* by which we mean that some segmenets will involve *two-way streets*. These are segments which where an ij wall coincides with a ji wall (necessarily of opposite orientation).

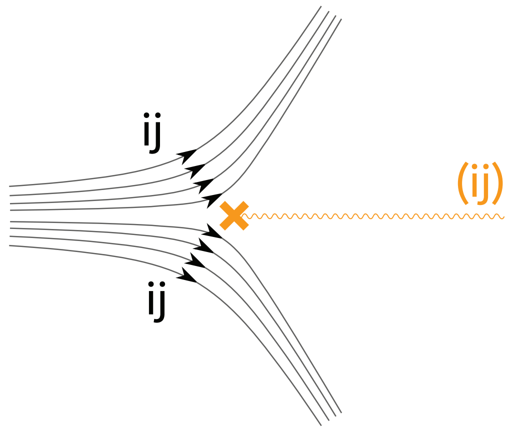

Figure 27: Here ϑ varies through a critical value at which an ij S-wall crashes on an ij branch point. Note that there is a large change in the spectral network as ϑ varies from $\vartheta^- = \vartheta_c - \epsilon$ to $\vartheta^+ = \vartheta + \epsilon$.

For $K = 2$ there are 3 things which can happen which we call *flip, juggle, and pop*:

Example 1: *flip*: A good example of this is the basic flip shown in Figure 28. This can be considered as a local picture within a spectral network or as the spectral network for the AD_2 theory $\lambda^2 = (z^2 - m^2)(dz)^2$.

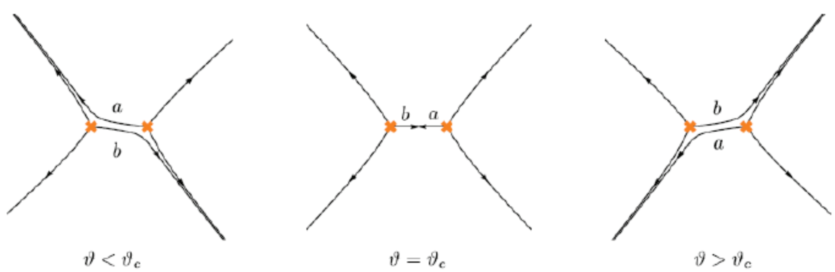

Figure 28: The basic flip associated with two branch points of the same type (ij).

The reason for the name "flip" is that in the dual triangulation we are flipping the diagonal in a rectangle as in Figure 29

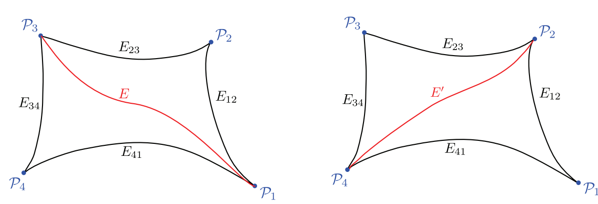

Figure 29: The triangulation dual to the spectral network flips.

Example 2 *juggle*: A second example is the *juggle*, already shown in Figure 9 above. Note that infinitely many ij streets of charge $b + n\gamma$, $n \geq 0$ are colliding with the ji street

a at each of the two branch points.

♣Explain the name "juggle" ♣

Example 3 *pop*: When $(\mathfrak{m}_1^{(n)} - \mathfrak{m}_2^{(n)})e^{-i\vartheta}$ becomes pure imaginary we have a pop transition llustrated in Figure 30:

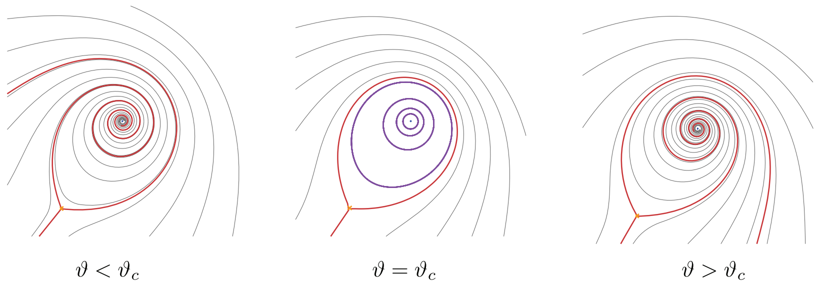

Figure 30: A pop transition.

Example 4: When $K > 2$ we can have a new phenomenon. A 3-string junction develops

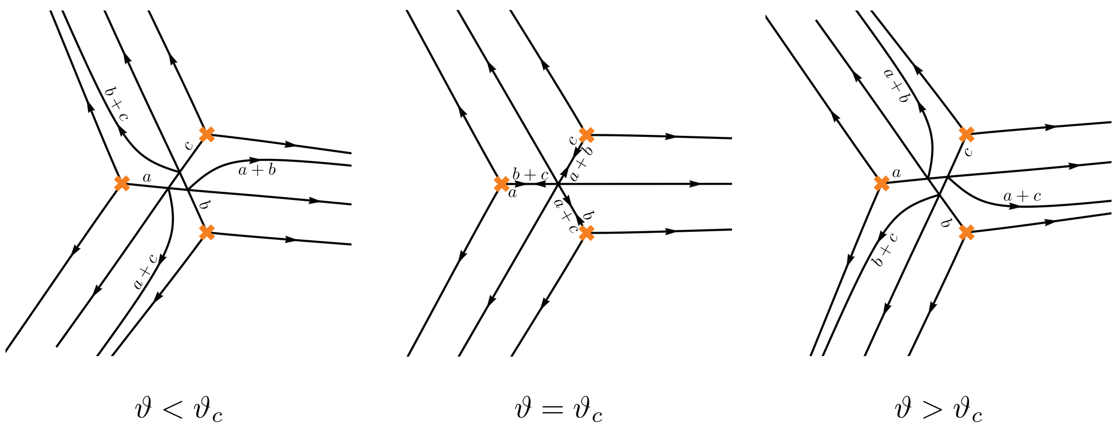

Figure 31: When $K > 2$ there is a new kind of degeneration of a spectral network associated with a string junction.

Again, some beautiful movies of these morphisms of spectral networks can be seen at A. Neitzke's homepage quoted above.

In general, our two-way streets can end at branch points as in Figure 32:

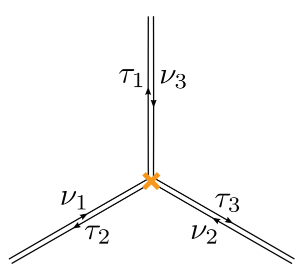

Figure 32: Two-way streets ending on a branch point.

Or the two-way streets can end at a joint. The most general situation is shown in Figure 33:

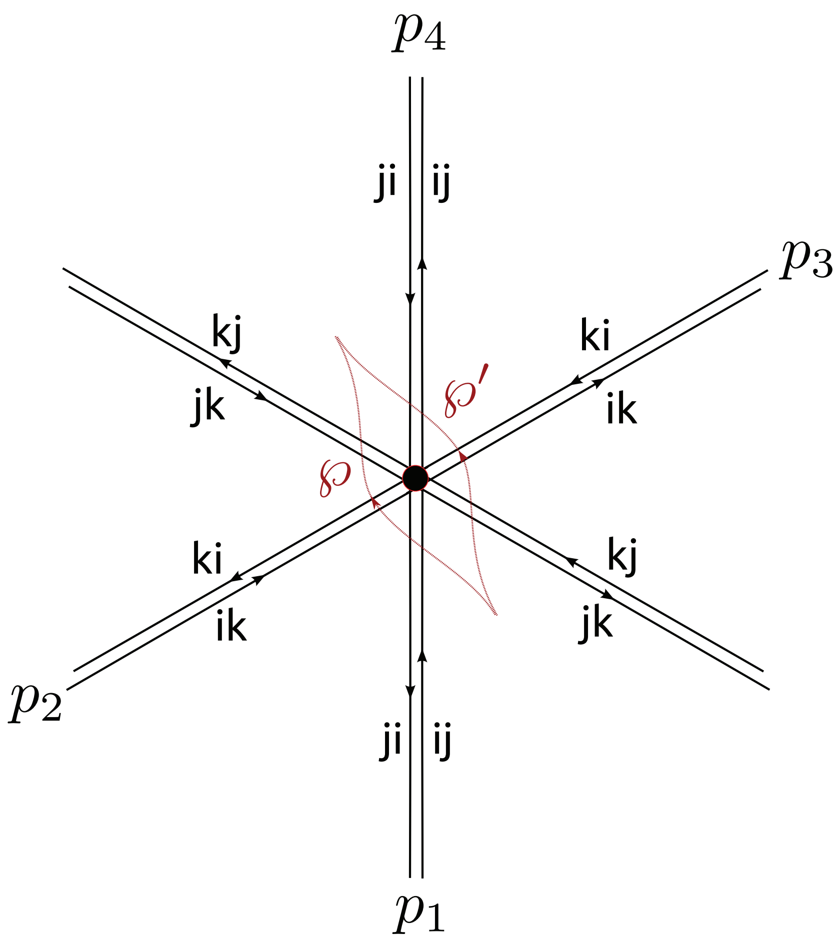

Figure 33: The basic flip associated with two branch points of the same type (ij).

Note that the WKB paths falling into a singular point are *ordered* by $i < j$. Therefore, two-way streets cannot end on singular points. Therefore, the two-way streets must construct *4d string webs*!

An important point is that at the critical values ϑ_c where there are two-way streets the degenerate spectral network $\mathcal{W}_\vartheta$ contains string webs.

Thus, we identify the critical values at which the spectral network becomes degenerate with the critical values at which there are closed string webs, that is, 4d BPS states.

13.2 How the formal parallel transport changes with ϑ

Let us now study how the formal parallel transports change as we vary ϑ holding $\wp$ fixed. That is, we now study wall-crossing in the framed BPS states as we vary ϑ.

The first thing that can happen is that an S-wall of type ij can move across the initial point or the final point of a path $\wp(z_1, z_2)$.

FIGURE OF SEVERAL S-WALLS OF TYPE ij MOVING INTO A PATH $\wp$

At a critical angle ϑ_c the path will intersect the initial point z_1. Then we can make a new millipede - equivalently, we should apply the detour rule -

$$F(\wp, \vartheta_c^+) = (1 + \mu(a) X_a) F(\wp, \vartheta_c^-) \tag{13.1}$$

where $a \in \Gamma(z_1, z_1)$, and again, in general we must sum over all such solitons.

There is a similar formula for what happens if an S-wall crosses into $\wp$ from the endpoint.

$$F(\wp, \vartheta_c^+) = F(\wp, \vartheta_c^-)(1 + \mu(a) X_a)^{-1} \tag{13.2}$$

For later purposes of the 2d-4d wall-crossing formula note that this can be written:

$$F(\wp, \vartheta_c^+) = (1 + \mu(z) X_a) F(\wp, \vartheta_c^-)(1 + \mu(z) X_a)^{-1} \tag{13.3}$$

♣Note the sign is not obvious here, but it must work this way. ♣

Let us now consider how the formal parallel transport jumps when ϑ crosses a critical value at which there are string webs (equivalently, at which there are two-way streets within the web).

We will make a genericity assumption about the point $u \in \mathcal{B}$ in the Coulomb branch at which we are working:

At the critical value ϑ_c at which a web exists we assume that all string webs have charge $\gamma = N\gamma_0$ where γ_0 is some primitive vector in Γ and N is a positive integer. That is

$$e^{-\mathrm{i}\vartheta_c} Z_\gamma > 0 \qquad \Rightarrow \qquad \gamma = N\gamma_0 \tag{13.4}$$

We will denote $\Gamma_0 = \mathbb{Z}\gamma_0$.

Now we have a

Conjecture: There exists a set of signs $\xi_\gamma = \pm 1$ and closed cycles $L(\gamma)$ for $\gamma \in \Gamma_0$ so that the change of the framed BPS degeneracies is given by

$$F(\wp, \vartheta_c^+) = \mathcal{K}_{\Gamma_0} F(\wp, \vartheta_c^-) \tag{13.5}$$

where $\mathcal{K}_{\Gamma_0}$ is a linear operator on the homology path algebra of Σ of the form

$$\mathcal{K}_{\Gamma_0} X_a = \prod_{\gamma \in \Gamma_0} (1 + \xi_\gamma X_\gamma)^{\langle a, L(\gamma) \rangle} X_a \tag{13.6}$$

Note that since Γ_0 is a rank one lattice this can be written as an (a priori infinite) product:

$$\prod_{\gamma \in \Gamma_0} (1 + \xi_\gamma X_\gamma)^{\langle a, L(\gamma) \rangle} = \prod_{N=1}^{\infty} (1 + \xi_{N\gamma_0} X_{\gamma_0}^N)^{\langle a, L(N\gamma_0) \rangle} \tag{13.7}$$

Let us explain how these signs ξ_γ and closed cycles $L(\gamma)$ are determined.

Consider any two-way street $\hat{p}$ in the degenerate spectral network $\mathcal{W}_{\vartheta_c}$. Suppose $z \in \hat{p}$ then there are two kinds of solitons of type ij and ji:

$a \in \Gamma_{ij}(z, z)$ with $\mu(a) \neq 0$

$b \in \Gamma_{ji}(z, z)$ with $\mu(b) \neq 0$

Of course, they both have phase $e^{\mathrm{i}\vartheta_c}$.

We can put these together to form a *closed* path on Σ, which we denote $cl(a+b)$. By our genericity assumption $cl(a+b) = N\gamma_0$ for some integer N.

Example: Consider a simple two-way street between two branch points of type (ij).

DRAW A FIGURE OF THE LIFT TO SIGMA

Now, for each two-way street $\hat{p}$ choose some point (any point) $z \in \hat{p}$ and write the generating function:

$$Q(\hat{p}) = 1 + \sum_{a \in \Gamma_{ij}(z,z); b \in \Gamma_{ji}(z,z)} \mu(a)\mu(b) X_{cl(a+b)} \tag{13.8}$$

Now, we *conjecture*, based on many examples, that there is a unique way of writing $Q(\hat{p})$ as a *finite* product of the form

$$Q(\hat{p}) = \prod_{\gamma \in \Gamma_0} (1 + \xi_\gamma X_\gamma)^{\alpha_\gamma(\hat{p})} \tag{13.9}$$

where the ξ_γ are signs.

Remarks

1. The kind of ambiguity we are trying to eliminate with the signs is the formal identity

$$\frac{1}{1-x} = (1+x)(1+x^2)(1+x^4)(1+x^8)\cdots \tag{13.10}$$

2. This conjecture is well-supported by numerous examples, and is an established fact for $K = 2$.

3. The conjecture is also strongly suggested by the physical *halo picture* in which various particles bind to the line defect. [SEE SECTION *** ABOVE FOR THE HALO DESCRIPTION OF WALL-CROSSING]. The factors here are simply Fock space partition functions of various free bosons and free fermions. Recall that the partition function of a free boson is just

$$Z^{\text{freeboson}} = \frac{1}{1-q^\omega} \tag{13.11}$$

and of a free fermion is just

$$Z^{\text{freeboson}} = 1 + q^\omega \tag{13.12}$$

In a field theory we only expect a finite number of types of halo particles. Note that this viewpoint also neatly explains why $Q(\hat{p})$, considered as a function of the variable X_{γ_0} has all its zeroes and singularities on the unit circle.

Now, assuming such a factorization exists we can define $L(\gamma)$ by summing over all the two-way streets:

$$L(\gamma) := \sum_{\hat{p}} \alpha_\gamma(\hat{p}) \hat{p}_\Sigma \tag{13.13}$$

Here $\hat{p}_\Sigma$ is the lift of $\hat{p}$ to Σ. Such a lift is uniquely determined by the orientation and labels ij.

DRAW FIGURE OF LIFT OF $\hat{p}_\Sigma$.

It is not obvious that $L(\gamma)$ is a closed cycle. When the two way street ends at a branch point the lift meets the ramification point in Σ above $\mathfrak{b}$ and is clearly closed. However, when the end of the two way street is at a joint as in Figure 33 above then $\partial\hat{p}_\Sigma$ is nonzero. However, once again using the homotopy invariance of $F(\wp, \vartheta)$ (say, comparing the two paths in Figure 33) across the joint it is possible to show that in the sum over $\hat{p}_\Sigma$ all the boundaries cancel and $L(\gamma)$ is a closed cycle.

Now, for reasons explained in the next section we can finally give our precise mathematical definition of $\Omega(\gamma; u)$ for theories of class S:

$$L(\gamma) := \Omega(\gamma; u)\gamma \qquad\qquad \gamma \in \Gamma_0 \tag{13.14}$$

WRITE OUT "KIRCHOFF LAWS?"

13.3 Examples

Example 1: Consider again the simple flip associated with two branch points of the same type (ij). Choose any point z on the two-way street joining these two branch points. Then there is $a \in \Gamma_{ij}(z, z)$ going to the one branch point and $b \in \Gamma_{ji}(z, z)$ going to the other. There are no other ij walls colliding with this one. So $\mu(a) = \mu(b) = 1$ (because these are both simpletons) and

$$Q(\hat{p}) = 1 + X_\gamma \tag{13.15}$$

where $\gamma := cl(a + b)$. Then $L(\gamma) = \hat{p}_\Sigma$ is closed and is indeed γ. So $\Omega(\gamma) = +1$.

Example 2: Consider again the case of the juggle, shown in Figure 9. Focus first on the lower trajectories. At the critical phase. There is a two-way street $\hat{p}^1$ beginning and ending at the branch point. If we choose a point z on this street then there is one path a of type ij winding around the cylinder clockwise (looking down from the top) and infinitely paths $b + n\gamma$, $n \geq 0$ of type ji winding around the cylinder counterclockwise.

These all collide at the critical angle and become a single circle, beginning and ending at the branch point. We have $cl(a + b) = \gamma$ if we choose any z on the circle. These were all simpletons so

$$\mu(a) = 1 \qquad \& \qquad \mu(b + n\gamma) = 1 \qquad n \geq 0 \tag{13.16}$$

so for the lower branch point

$$Q(\hat{p}) = 1 + X_\gamma + X_{2\gamma} + \cdots = \frac{1}{1 - X_\gamma} \tag{13.17}$$

so that $\alpha_\gamma(\hat{p}^1) = -1$ for this two-way street. Now there is an identical story for the upper branch point and hence

$$L(\gamma) = -\hat{p}^1_\Sigma - \hat{p}^2_\Sigma = -2\gamma \tag{13.18}$$

and hence

$$\Omega(\gamma) = -2 \tag{13.19}$$

Example 3: *Triangle state*. Referring to Figure 31 we see that $\gamma = cl(a + b + c)$, $Q(\hat{p}^i) = 1 + X_\gamma$ for $i = 1, 2, 3$ so $\alpha_\gamma(\hat{p}^i) = 1$ and $L(\gamma) = \hat{p}^1_\Sigma + \hat{p}^2_\Sigma + \hat{p}^3_\Sigma = \gamma$. So $\Omega(\gamma) = 1$ and we have a hypermultiplet.

13.4 2d-4d WCF

One reason for defining Ω as in (13.14) is that these degeneracies behave just like the BPS degeneracies of the d=4 N=2 BPS states and indeed satisfy the Kontsevich-Soibelman wall-crossing-formula as u is varied.

Let us explain the basic reason for this, and then write out the 2d-4d wall-crossing formula in some more detail.

Let us consider paths $\wp(z,z)$ which begin and end at a point $z \in C$ and study the subalgebra of the homology path algebra generated by $F(\wp(z,z),\vartheta)$. Of course, everything also depends on $u \in \mathcal{B}$ since the branched covering $\pi : \Sigma \to C$ depended on u, so we will write for the moment $F(\wp,\vartheta,u)$.

Now consider the space $LC \times \mathcal{B}^* \times S^1$, where LC is the loop space of C, $\mathcal{B}^*$ is the space of nonsingular points on the Coulomb branch, and the S^1 is for $e^{\mathrm{i}\vartheta}$.

Consider a contractible closed loop ℓ in this space:

$$\ell(t) = (\wp_t(z(t),z(t)), u(t), \vartheta(t))$$

with $0 \leq t \leq 1$. Let

$$F(t) := F(\wp_t(z(t),z(t)), u(t), \vartheta(t)) \tag{13.20}$$

On the one hand, since the loop is contractible $F(1) = F(0)$.

On the other hand, $F(t)$ changes by the ordered product of the transformations $J(t_c)$ which occur as described above. For example, if $\vartheta(t_c)$ is a critical phase for $u(t_c)$ then there is a $\mathcal{K}_{\Gamma_0}$-type transformation, etc. The total transformation

$$\mathfrak{M} = \prod_{t_c} J(t_c) \tag{13.21}$$

takes $F(0)$ to $F(1)$:

$$\mathfrak{M}F(0) = F(1) \tag{13.22}$$

but as we said, $F(1) = F(0)$ so

$$\mathfrak{M}F(\wp,\vartheta,u) = F(\wp,\vartheta,u) \tag{13.23}$$

We would like to claim that this implies $\mathfrak{M} = 1$, from which we deduce the 2d-4d WCF. For we must be careful because in general the objects $F(\wp,\vartheta)$ only forms a subalgebra of the homology path algebra. Nevertheless in [100] it is argued that this subalgebra is sufficiently large to conclude that $\mathfrak{M} = 1$. This in turn implies the 2d-4d wall-crossing-formula.

It is worth writing out the 2d4d wall-crossing formula a bit more explicitly:

As we vary u or ϑ or z the formal parallel transport operators change by various automorphisms of the homology path algebra. There are two basic kinds of transformations:

$$S_a : X_b \to (1+\mu(a)X_a)X_b(1+\mu(a)X_a)^{-1} \tag{13.24}$$

$$\mathcal{K}_{\Gamma_0} : X_b \to \prod_{\gamma\in\Gamma_0} (1+\xi_\gamma X_\gamma)^{\langle b, L(\gamma)\rangle} X_b \tag{13.25}$$

It is convenient to write this second transformation

$$\mathcal{K}_{\Gamma_0} = \prod_{\gamma \in \Gamma_0} \mathcal{K}_\gamma^{\omega(\gamma;b)} \tag{13.26}$$

where we define:

$$\omega(\gamma; b) := \langle b, L(\gamma) \rangle \tag{13.27}$$

These integer functions are known as *2d-4d degeneracies.* Note that they satisfy an affine-linearity property: [34]

$$\omega(\gamma; b + \gamma') = \omega(\gamma; b) + \langle \gamma, \gamma' \rangle \Omega(\gamma) \tag{13.28}$$

14. Summary of material from week 1

We consider the Hitchin equations on a Riemann surface with marked points $\mathfrak{s}_n$.

We construct the spectral curve $\pi : \Sigma \to C$ with equation

$$\lambda^K + \lambda^{K-1}\phi_1 + \cdots + \phi_k = 0 \tag{14.1}$$

and interpret the Hitchin base as interpreting tuples of meromorphic differentials $u = (\phi_1, \ldots, \phi_K)$ with prescribed singularities at sf_n.

Def: A *WKB path of phase ϑ* is a path where, for some ordered pair of sheets i, j of the spectral cover $\langle \lambda_i - \lambda_j, \partial_t \rangle = e^{\mathrm{i}\vartheta}$.

4d BPS states are represented by closed string webs: Connected graphs of WKB paths of phase ϑ with no ends on $\mathfrak{s}_n$. Three string junctions are allowed.

2d BPS states in $\mathbb{S}_z$ are open string webs: These are like closed string webs but one path must end at z.

Closed string webs lift to closed cycles and have a charge $\gamma \in \Gamma = H_1(\Sigma, \mathbb{Z})$. They have an $\mathcal{N} = 2$ central charge $Z_\gamma = \frac{1}{\pi} \oint_\gamma \lambda$. Closed string webs of charge γ are counted by $\Omega(\gamma; u)$

Open string webs lift to chains with $\partial \mathfrak{c} = z^{(j)} - z^{(i)}$ for two sheets $i \neq j$ and have a charge $a \in \Gamma_{ij}(z, z) := \{\mathfrak{c} | \partial \mathfrak{c} = z^{(j)} - z^{(i)}\}/\partial(*)$. They have an $\mathcal{N} = 2$ central charge $Z_a = \frac{1}{\pi} \oint_a \lambda$. Open string webs of charge a are counted by $\mu(a; u)$.

The spectral network $\mathcal{W}_\vartheta$ is defined to be the set of $z \in C$ such that there is some $a \in \Gamma(z, z)$ with $Z_a \in e^{\mathrm{i}\vartheta}\mathbb{R}_-$ and $\mu(a) \neq 0$.

The formal parallel transport along a continuous path $\wp(z_1, z_2) \in C$ for a line defect $L_{\wp,\vartheta}$ is defined by $F(L_{\wp,\vartheta}) := \sum_{a \in \Gamma(z_1, z_2)} \overline{\underline{\Omega}}(L_{\wp,\vartheta}, a) X_a$ where X_a are in the homology path algebra of Σ and the framed BPS degeneracies are determined by homotopy, homomorphism, local triviality and the detour rule (= wall-crossing for framed BPS degeneracies). This uniquely determines both the framed BPS degeneracies and the soliton degeneracies $\mu(a; u)$.

[34]There is a beautiful physical interpretation of the affine-linearity property (13.28). It again relies on the halo picture alluded to above. The shift is due to the existence of orbits or BPS particles around the domain wall on the surface defect and the factor $\langle \gamma, \gamma' \rangle$ is just a Landau-level degeneracy. See §4.7.2 and Appendix of [97] for a full explanation.

One can then grow the spectral network from simpletons near the branch points by evolving the differential equations and using the intersection rules to continue μ past joints.

As we vary ϑ there are two changes of $F(L_{\wp,\vartheta})$

S-morphisms: $F \to (1+\mu(a)X_a)F(1+\mu(a)X_a)^{-1}$ when the network moves into or out of the path $\wp$

K-morphisms: At ϑ_0 so that the spectral network has two-way streets there exist string webs of charge $\gamma \in \gamma_0\mathbb{Z} := \Gamma_0$. ϑ_0 is then the phase of the 4d BPS state. Then $F^+ = \mathcal{K}_{\Gamma_0}F^-$ changes by

$$X_a \to K_{\Gamma_0}X_a = \prod_{\gamma\in\Gamma_0} K_\gamma^{\omega(\gamma,\cdot)}X_a = \prod_{\gamma\in\Gamma_0}(1+\xi_\gamma X_\gamma)^{\langle a,L(\gamma)\rangle}X_a \tag{14.2}$$

where ξ_γ are signs and $L(\gamma)$ is a closed cycle on Σ determined by the combinatorics of the degenerate spectral network.

We can obtain the 4d BPS degeneracies from $L(\gamma) = \Omega(\gamma;u)\gamma)$ and we define the *2d4d degeneracies* by

$$\omega(\gamma;a) := \langle a, L(\gamma)\rangle \tag{14.3}$$

note that it is affine linear:

$$\omega(\gamma;a+\gamma') = \omega(\gamma;a) + \langle\gamma,\gamma'\rangle\Omega(\gamma) \tag{14.4}$$

Finally, we argued that there is a 2d4d WCF for the degeneracies $\mu(a)$ and $\omega(\gamma;a)$ which is modeled closely on the conventional 4d KSWCF.

15. Lecture 5, Monday, October 8: Wall-Crossing Formulae

By studying the morphisms of spectral networks we were led to study two kinds of transformations of variables in the homology path algebra of Σ and to state a 2d4d wall-crossing formula.

In this chapter we will look at that formula a little more carefully and study some examples and generalizations.

15.1 Formal statement of the 2d4d WCF

To state the 2d4d wall-crossing formula we have four pieces of data:

1. *Vacuum Groupoid*: Let $\mathbb{V}[z]$ be the groupoid with objects $z^{(i)}$ and morphisms

$$Hom(z^{(i)}, z^{(j)}) = \Gamma_{ij} \tag{15.1}$$

   where Γ_{ij} is a Γ-torsor and we identify $\Gamma \cong \Gamma_{ii}(z,z) = Aut(z^{(i)})$. Here Γ is an integral lattice with integral antisymmetric form.

2. *Central charge*: $Z : \Gamma(z,z) \to \mathbb{C}$ is linear $Z_{a+b} = Z_a + Z_b$ when $a+b$ is defined (composable morphisms).

3. *BPS data*: $\Omega : \Gamma \to \mathbb{Z}$, $\mu : \cup_{i\neq j}\Gamma_{ij}(z,z) \to \mathbb{Z}$, $\omega : \Gamma \times \cup_{i,j}\Gamma_{ij} \to \mathbb{Z}$, the latter satisfies the affine linearity relation

$$\omega(\gamma, a+\gamma') = \omega(\gamma, a) + \langle\gamma,\gamma'\rangle\Omega(\gamma) \tag{15.2}$$

4. *Sign cocycle* $\sigma(a,b) \in \{\pm 1\}$ satisfies the cocycle relation

$$\sigma(a,b)\sigma(a+b,c) = \sigma(a,b+c)\sigma(b,c) \tag{15.3}$$

when the morphisms are composable.

Then we make three definitions:

1. A *BPS ray* is a ray in the complex plane of the form

$$\ell_\gamma := Z(\gamma)\mathbb{R}_- \qquad \omega(\gamma,\cdot) \neq 0 \tag{15.4}$$

$$\ell_a := Z(a)\mathbb{R}_- \qquad \mu(a) \neq 0 \tag{15.5}$$

2. The *twisted vacuum groupoid algebra* $\mathbb{C}[\mathbb{V}]$:

$$X_a X_b = \begin{cases} \sigma(a,b)X_{a+b} & a+b \quad \textit{composable} \\ 0 & \textit{else} \end{cases} \tag{15.6}$$

3. Two automorphisms of $\mathbb{C}[\mathbb{V}]$:

$$S_a^\mu : X_b \to (1+\mu(a)X_a)X_b(1+\mu(a)X_a)^{-1} \tag{15.7}$$

$$\mathcal{K}_\gamma^{\omega(\gamma,\dot{})} : X_a \to (1+X_\gamma)^{\omega(\gamma,a)}X_a \tag{15.8}$$

We will apply this to our case where $\Gamma_{ij} = \Gamma_{i,j}(z,z)$, and the stability data Z is given by the integral of λ and the BPS data μ and ω are piecewise constant functions of $(z,u) \in C \times \mathcal{B}^*$.

Now the statement of the 2d-4d wcf is essentially identical to the KSWCF:

Consider point $(z,u) \in C \times \mathcal{B}^*$. For an angular sector $\sphericalangle$ in the complex plane we define an automorphism of the groupoid algebra,

$$A(\sphericalangle) = : \prod_{\gamma:\ell_\gamma\subset\sphericalangle} \mathcal{K}_\gamma^\omega \prod_{a:\ell_a\subset\sphericalangle} \mathcal{S}_a^\mu : \tag{15.9}$$

where the normal ordering symbol indicates that the factors — be they of type $\mathcal{K}$ or $\mathcal{S}$ — are ordered so that reading from left to right we encounter factors associated with rays successively in the counterclockwise direction. Then the 2d4d wall-crossing formula says: *$A(\sphericalangle)$ is constant, as a function of (u,z), as long as no BPS rays cross the boundary of $\sphericalangle$.* This is equivalent to saying that there is no monodromy around contractible loops in $C \times \mathcal{B}^*$.

The nontrivial point here is that as (u,z) change the central charges Z_γ and Z_a, $a \in \Gamma(z,z)$ also change. The BPS rays within the angular sector $\sphericalangle$ can very well change phase order. However, the automorphisms $\mathcal{K}_\gamma$ and $\mathcal{S}^{\mu}_{\gamma_{ij}}$ do *not* commute. The only way the automorphism $A(\sphericalangle)$ can remain the same if the ordering changes is if the dengeracies μ and ω also change. Now, given an automorphism $A(\sphericalangle)$ of $\mathbb{C}[\mathbb{V}]$ and given an ordering of the central charges Z_γ, Z_a there is a *unique* way to factor $A(\sphericalangle)$ as in (15.9). This is proved in [94], Section 2.2 using a filtration on the algebra given by defining a Euclidean metric on Γ. Therefore, given μ, ω for one ordering and μ', ω' for the other ordering, we can, in principle, solve for μ', ω' in terms of μ, ω. This is therefore a wall-crossing-formula.

♣SAY MORE HERE? Must also be in KS. Give ref. ♣

Remark The 2d/4d wcf can be viewed as a special case of the general KSWCF for a graded Lie algebra. We take A to be the (twisted) algebra of functions on the Poisson torus $\Gamma^* \otimes \mathbb{C}^*$. Then we take

$$\mathfrak{g} = Mat_{K\times K}(A) \oplus SympVect(T) \tag{15.10}$$

in the KS formalism. However, the "standard" wall-crossing formula, just based on $SympVect(T)$ (i.e. just using the transformations K_γ) has a "motivic generalization" which we discuss briefly in §15.4 below. As far as I know, the motivic generalization of the 2d4d formula is not known. Nevertheless, physical reasoning strongly suggests that there is such a generalization. See §15.4 below.

♣Say more here. Give generators and relations. See email exchange with Andy ♣

15.1.1 The spectrum generator

The KS formula strongly suggests that a useful quantity to compute is the *spectrum generator*. To define it, we choose a phase ϑ, or equivalently a half-plane in the complex plane (with phases between ϑ and $\vartheta + \pi$. We then form the product:

♣Written for 4d case. Write it for the 2d4d case. ♣

$$\mathbb{S}(\vartheta; u) := \prod_{\gamma:\vartheta \le \arg -Z_\gamma < \vartheta+\pi} K_\gamma^{\Omega(\gamma;u)}, \tag{15.11}$$

where the product is taken in order of increasing $\arg -Z_\gamma$ as we read from left to right.

The spectrum generator $\mathbb{S}(\vartheta; u)$ is a symplectic (or Poisson) transformation acting on the functions on the algebraic torus $\Gamma^* \otimes \mathbb{C}^*$. Given a central charge function, and hence an ordering of the phases of Z_γ there is a *unique* factorization of $\mathbb{S}(\vartheta; u)$ into a product of KS-transformations ordered as in (15.11). $\mathbb{S}(\vartheta; u)$ thus captures all the BPS degeneracies of the theory, since BPS states with central charges in the other half-plane are just the anti-particles of the ones counted by the SG. We assume that ϑ is sufficiently generic that no BPS particle has central charge of phase ϑ.

The SG can be in principle a compact way of summarizing a very complicated BPS spectrum. It is invariant under wall-crossing, so long as no BPS ray enters or leaves the half-space $\mathbb{H}_{\vartheta+\pi/2}$. As we will see in §**** and §**** below it can be explicitly computed in certain theories without any *a priori* knowledge of the BPS spectrum. In such cases it thus serves to derive the BPS spectrum - at least in principle. Given a symplectic transformation and an ordering of the phases of Z_γ it is in practice rather difficult to find a factorization of the form (15.11). This can be done by using a filtration on the algebra given by a height function on Γ, but even implementing this on a computer one runs out of power rather rapidly.

Open problem: Find an efficient algorithm to factorize - preferably exactly - elements of the automorphism algebra of $\mathbb{C}[\mathbb{V}]$ *in terms of* $\mathcal{K}$ *and* $\mathcal{S}$ *transformations.*

♣ Comment on what happens with mutations. ♣

15.2 Walls of Marginal Stability and the Four types of 2d4d wall crossing

The BPS rays will exchange orders as *walls of marginal stability*. These are defined by the generalization of (4.68):

$$MS(\gamma_1,\gamma_2) := \{(u,z)|Z(\gamma_1;u) \parallel Z(\gamma_2;u) \quad \text{and} \quad \exists a,b \in \Gamma(z,z) \quad s.t. \quad \omega(\gamma_1;a;u)\omega(\gamma_2;b;u) \neq 0\} \tag{15.12}$$

Moreover, there are purely 2d walls of marginal stability:

$$MS(a,b) := \{(u,z)|Z_a \parallel Z_b \quad \text{and} \quad \mu(a;u)\mu(b;u) \neq 0\} \tag{15.13}$$

although we are only interested in the cases where S^{μ}_{a} and S^{μ}_{b} do not commute so if $a \in \Gamma_{ij}(z,z)$ and $b \in \Gamma_{kl}(z,z)$ then there should be only three distinct indices among i,j,k,l.

There can also be mixed walls where $Z_a \parallel Z_\gamma$:

$$M(a,\gamma) := ETCETC. \tag{15.14}$$

This can be usefully broken into two subcases according to whether, if $a \in \Gamma_{ij}(z,z)$ there is or is not an occupied ray $b \in \Gamma_{ji}(z,z)$ with $a+b=\gamma$.

Thus, altogether, there are four interesting cases to consider:

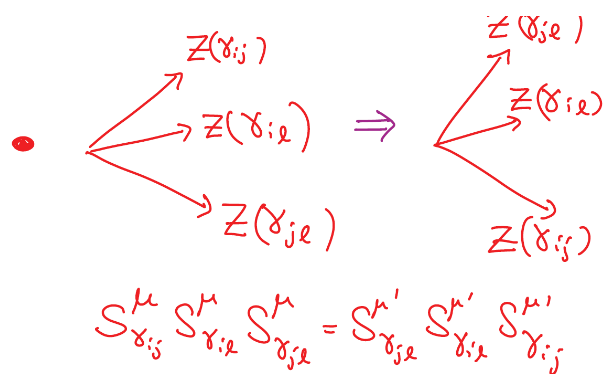

Figure 34: One of four kinds of wall-crossing in the 2d-4d case. This is the case corresponding to the CVWCF. Here γ_{ij} denotes some chain in $\Gamma_{ij}(z,z)$ etc. We have $\gamma_{i\ell} = \gamma_{ij} + \gamma_{jk}$.

In all cases except the KSWCF it is possible to solve explicitly for the new degeneracies in terms of the old ones.

For the CVWCF, we have already shown how it follows. In the WCF it is essentially just a fact about multiplication of 3×3 matrices (if we think of the X_a for simpletons $a \in \Gamma_{ij}$ as upper triangular matrices:

$$(1+\mu_{12}e_{12})(1+\mu_{13}e_{13})(1+\mu_{23}e_{23}) = (1+\mu'_{23}e_{23})(1+\mu'_{13}e_{13})(1+\mu'_{12}e_{12}) \tag{15.15}$$

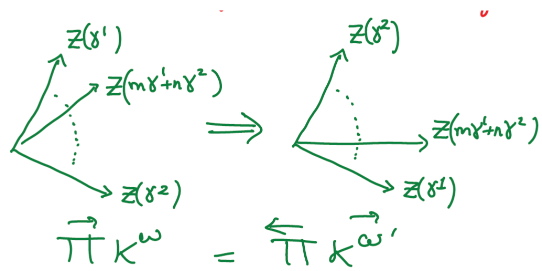

Figure 35: One of four kinds of wall-crossing in the 2d-4d case. This is the case corresponding to the KSWCF

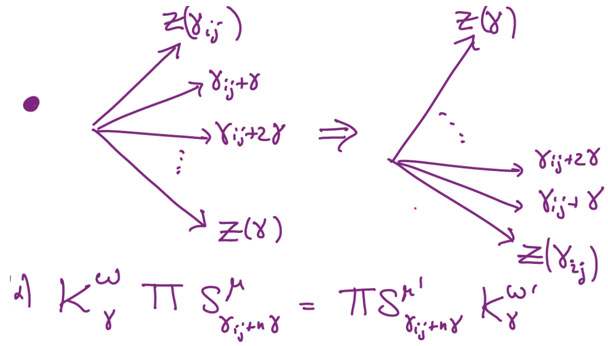

Figure 36: One of four kinds of wall-crossing in the 2d-4d case. This is a mixed case.

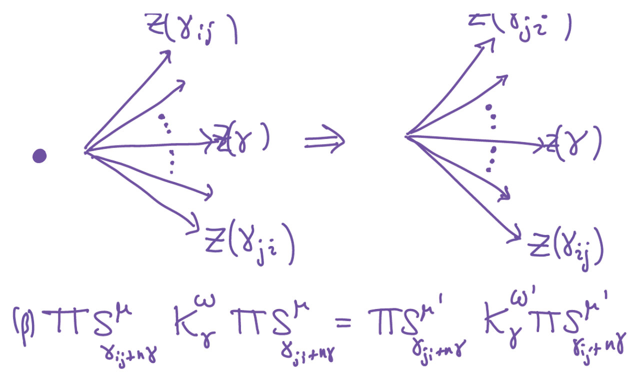

Figure 37: One of four kinds of wall-crossing in the 2d-4d case.This is a mixed case.

The two "new" formulae of Fig. 36 and 37 can also be solved explicitly. Suppressing signs and defining

$$\Sigma_a := \sum_{n=0}^{\infty} \mu(a+n\gamma) X_{\gamma}^{n} \tag{15.16}$$

$$\Pi_a := \prod_{n=1}^{\infty} (1+X_{\gamma})^{\omega(n\gamma;a)} \tag{15.17}$$

the formula from Figure 36 is solved by

$$\begin{aligned} \Sigma'_{ij} &= (1+X_\gamma)^{-\omega(\gamma,\gamma_{ij})}\Sigma_{ij} \\ \Pi'_{ij} &= \Pi_{ij} \end{aligned} \tag{15.18}$$

and the formula from Figure 37 is solved by

$$\begin{aligned} \Pi'_{ji} &= \Delta^{-2}\Pi_{ij}, \\ \Sigma'_{ij} &= \Delta^{-1}\Sigma_{ij}, \\ \Sigma'_{ji} &= \Delta^{-1}\Sigma_{ji}, \end{aligned} \tag{15.19}$$

where $\Delta := \Pi_{ji} + \sigma(\gamma_{ij},\gamma_{ji})\Sigma_{ij}\Sigma_{ji}X_\gamma$.

15.3 Special Cases of the 4d KSWCF

In our discussion of the 2d4d wcf have been a bit cavalier about the signs. If we take just the standard 4d KSWCF then the signs are best handled by writing the twisted group algebra of an algebraic torus:

$$Y_\gamma Y_{\gamma'} = (-1)^{\langle\gamma,\gamma'\rangle} Y_{\gamma+\gamma'} \tag{15.20}$$

Then K_γ is the automorphism of the algebra generated by Y_γ defined by

$$K_\gamma(Y_{\gamma'}) := (1-Y_\gamma)^{\langle\gamma',\gamma\rangle} Y_{\gamma'} \tag{15.21}$$

Rearrangement of $K_{\gamma_2}K_{\gamma_1}$ for $\langle\gamma_1,\gamma_2\rangle = m$.

$$K_{\gamma_2}K_{\gamma_1} =: \prod_{a/b \searrow} K_{a\gamma_1+b\gamma_2}^{\Omega_{a,b}(m)} \tag{15.22}$$

♣Specialize $\gamma = a\gamma_1 + b\gamma_2$ to get the form: $(x,y) \to (x(1-(-1)^{ab}x^a y^b)^a, y\ldots)$ etc. ♣

The product on the RHS is taken over all nonnegative integers a, b so that as we read from left to right the quantity a/b is nonincreasing. In particular $0/1 = 0$ corresponding to γ_2 is on the right and $1/0 = \infty$ corresponding to γ_1 is on the left.

Remark: Active vs. passive convention. We are defining K_γ as an automorphism of the algebra of twisted functions. It could also be considered as a diffeomorphism of the algebraic torus. Then the identities should have the ordering of the K's reversed, because the functor for Diff to Aut of functions is contravariant.

There are two central examples related to these identities:

15.3.1 Example 1: Pentagon identity

The first example is $m = 1$:

$$K_{\gamma_2}K_{\gamma_1} = K_{\gamma_1}K_{\gamma_1+\gamma_2}K_{\gamma_2} \tag{15.23}$$

as can easily be checked with a few lines of computation.

The simplest realization of this identity in our story is obtained by taking the AD_3 theory:

$$\lambda^2 = z^3 - 3\Lambda^2 z + u \tag{15.24}$$

There are four branch points, including $z = \infty$ and hence $\Sigma \to C$ is an elliptic curve with a puncture. The base $\mathcal{B}$ is just the u-plane and $\mathcal{B}^{\text{sing}} = \{u = \pm 2\Lambda^2\}$. There is a single irregular singularity at $z = \infty$ and in this case 5 marked points on the circle at infinity. The typical dual triangulation looks like Figure 38:

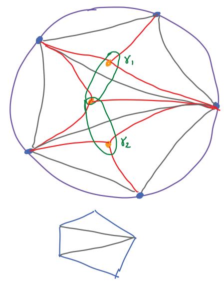

Figure 38: The ideal triangulation associated with the AD_3 theory with small u. The grey curves are generic WKB paths in the cells determined by the separating WKB paths. The two green curves lift to cycles γ_1, γ_2 which form a basis for the charge lattice. They have intersection product $\langle \gamma_1, \gamma_2 \rangle = +1$.

The local system Γ_g is defined by the anti-invariant curves under the Deck transformation of $\Sigma \to C$. It has rank 2 and we can choose a basis from cycles as shown in Figure 38.

At small values of u one finds only two critical phases when varying ϑ through an interval of length π. These have charges γ_1 and γ_2 as indicated. So there is a wall of marginal stability

$$Z_{\gamma_1}(u) \parallel Z_{\gamma_2}(u), \tag{15.25}$$

This wall passing through $\mathcal{B}^{\text{sing}}$ (where one or the other period vanishes). The u-plane is divided into two regions by the wall of marginal stability The inner region is called the "strong coupling region" and the outer region is called the "weak coupling region." [35] The figure looks very much like that for the $SU(2)$ example Figure 43 below.

If we consider how the spectral network evolves for u in the two different regions then we find there are either two or three critical phases as shown in Figure 39:

Comparing the transformations of $F(L_{\wp,\vartheta})$ for the two evolutions of ϑ gives the pentagon identity on KS transformations. In this case the closed path is illustrated in Figure 40:

15.3.2 AD_N theories

The generalization of this story to N^{th} order polynomials

$$\lambda^2 = P_N(z)(dz)^2 \tag{15.26}$$

[35] The reason for the terminology is based on the similarity to the $SU(2)$ u-plane. Physically we are perturbing a superconformal point and there is no gauge coupling.

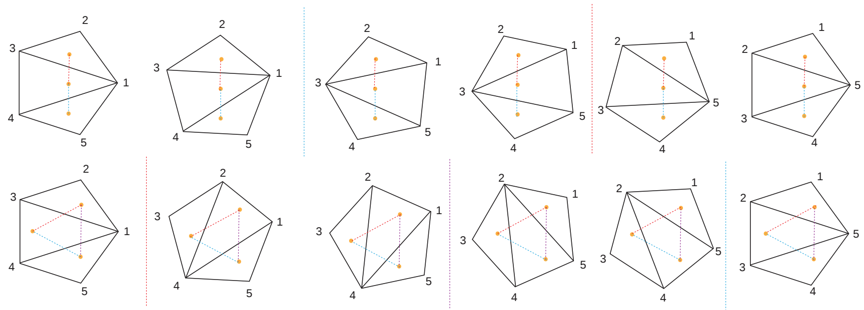

Figure 39: For u in the "strong coupling" and "weak coupling" regions there are different numbers of flips of the triangulation dual to the spectral network $\mathcal{W}_\vartheta$ as we vary ϑ through a range of π.

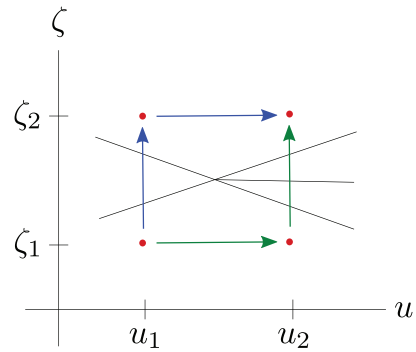

Figure 40: Let $\zeta = e^{i\vartheta}$. Then the closed path in (u, ϑ) space is illustrated here, along with walls of marginal stability. ♣ IT IS CONFUSING TO USE THE VARIABLE ζ. CHANGE THE PICTURE SO THAT VERTICAL AXIS IS ϑ ♣

is well-understood [162, 94]:

1. The Hitchin base is parametrized by the coefficients of $P_N(z)$ of degree less than $\frac{1}{2}(N-2)$. Recall that the boundary conditions at the singularity at infinity are specified by $\Delta(z)$ defined by

$$\sqrt{P_N(z)} = \Delta(z) + o(1/z) \tag{15.27}$$

   These are determined by the coefficients of z^k for $k \geq (N-2)/2$.

2. There are $(N+2)$ marked points on the circle at infinity. The WKB triangulations correspond to triangulations of the $(N+2)$-gon.

3. The BPS spectrum in the various chambers of the u-plane is finite and ranges from $(N-1)$ to $\frac{1}{2}N(N-1)$ states. These BPS states all correspond to simple saddle connections with $\Omega(\gamma; u) = 1$. (In physics they all correspond to hypermultiplets.)

4. Moreover, the triangulations can be considered to be vertices of the associahedron. Wall-crossings correspond to edges of the associahedron. The consistency of all wall-crossings is guaranteed by the pentagon identity. This is just a version of the MacLane coherence theorem of category theory.

15.3.3 The Juggle/Vectormultiplet

For $m = 2$ we have the juggle identity:

$$K_{\gamma_2} K_{\gamma_1} = \Pi_L K_{\gamma_1+\gamma_2}^{-2} \Pi_R \tag{15.28}$$

$$\begin{aligned} \Pi_L &= \prod_{n=0\nearrow\infty} K_{(n+1)\gamma_1+n\gamma_2} = K_{\gamma_1} K_{2\gamma_1+\gamma_2} \cdots \\ \Pi_R &= \prod_{n=\infty\searrow 0} K_{n\gamma_1+(n+1)\gamma_2} = \cdots K_{\gamma_1+2\gamma_2} K_{\gamma_2} \end{aligned} \tag{15.29}$$

This identity is rather more challenging to prove directly as an identity of transformations K_γ.

♣Indicate the trick to prove this. ♣

This is related to the BPS spectrum of the pure $SU(2)$ $N_f = 0$ theory. As we have said, the spectral curve equation is

$$\lambda^2 = \left(\frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z}\right)(dz)^2 \tag{15.30}$$

Again there is a rank two lattice Γ_g. For $u = 0$ the degenerate spectral networks are shown in Figure 41:

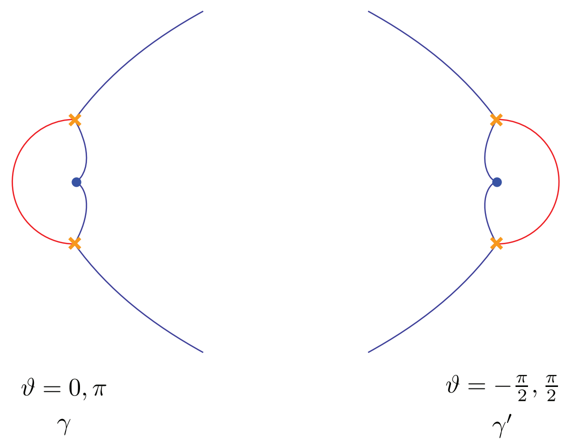

Figure 41: For $u = 0$ there are only two critical phases in a range of π. They lead to the two spectral networks shown here. The red shows the saddle connections with charges γ_1 and γ_2.

The cycles which support BPS states are shown in Figure 42. In this case we have $\langle \gamma_1, \gamma_2 \rangle = +2$.

Once again there is a single wall of MS as shown in Figure 43

For u in the strong coupling region we have just two BPS states. Now the wall crossing formula predicts that in the weak coupling region there are infinitely many BPS states.

We can check this geometrically:

For u in the weak coupling region we have infinitely many flips as $\vartheta \to \vartheta_c$, where ϑ_c corresponds to the phase of the vectormultiplet. These correspond to the closed string webs which go from one branch point to another but wrap n times around the annulus.

DRAW THE BPS RAYS ACCUMULATING ON THE VM BPS RAY. OVERLAY THIS ON THE u-PLANE TO SHOW THE DIFFERENT BEHAVIORS OF THE BPS RAYS IN THE TWO REGIONS.

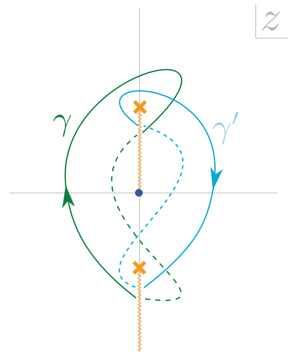

Figure 42: Cycles γ_1 and γ_2 supporting BPS states at $u = 0$.

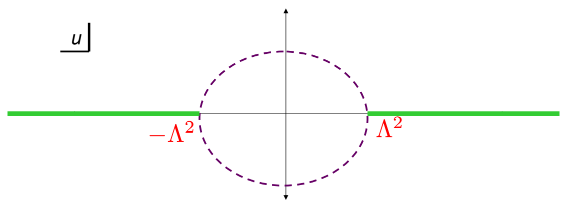

Figure 43: The u-plane for the pure $SU(2)$ gauge theory. The SW curve becomes singular at $u = \pm\Lambda^2$. As a result there is monodromy in the charge lattice Γ around these two points. We have chosen cuts shown in green to trivialize the corresponding local system. The marginal stability curve is shown in dashed purple and separates a strong coupling region near $u = 0$ from a weak coupling region near $u = \infty$. ♣ OVERLAY ON THIS TWO PICTURES OF THE CENTRAL CHARGE PLANE WITH THE DISPOSITION OF BPS RAYS♣

15.3.4 2d4d Wall crossing for the $SU(2)$ model

When we consider the 2d4d wall-crossing for the $SU(2)$ model of Example 2 things start to become very intricate. We now consider the 2-complex dimensional space (z, u), $z \in \mathbb{C}^*$, $u \in \mathcal{B}^*$. The full chamber structure has not been worked out, although the example has been extensively discussed in Section 8 of [97]. Roughly speaking there are 3 separate cases.

1. For small u ("strong 4d coupling") there are only two 4d BPS states. The corresponding degenerate networks are shown in Figure 41. Combining these two gives the walls in the z-plane at which ω and μ will jump. See Figure 44. In each of these these regions there are 2 or 3 solitons.

2. For large u ("weak coupling region") There are infinitely many 4d BPS states, as we have just observed. In particular, the "vectormultiplet" ray, which is the accu-

mulation point of the infinitely many hypermultiplet rays corresponds to the critical angle ϑ_{vm} in Figure 9. The degenerate spectral network at ϑ_{vm} divides the z-plane into three regions shown in Figure 45. To obtain the full set of walls for ω and μ we should overlay the spectral networks for all the phases of the hypermultiplets. These look like Figure 46. Outside the annular region defined by the vectormultiplet there are countably infinitely many chambers. Each chamber is an open set and supports a finite number of solitons. The chambers shrink in size as we approach the boundary of the annular region. SImulatneously the BPS spectrum grows. It is finite in each chamber but grows without bound. For z inside the VM annulus things are considerably more intricate. There are an infinite number of 2d solitons. (See Section 8.3.4 of [97].) There are infinitely many walls. There is also an uncountably infinite number of chambers (=complement of the walls) which consist of single points and form a set of full measure.

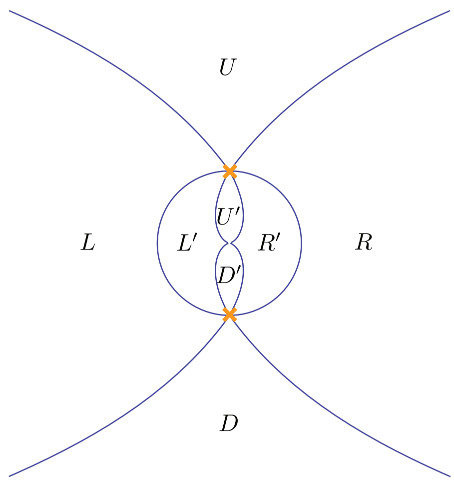

Figure 44: For $u = 0$ the 8 regions of the z-plane shown here have different soliton degeneracies, with the jumps predicted by the 2d4d wall-crossing formula.

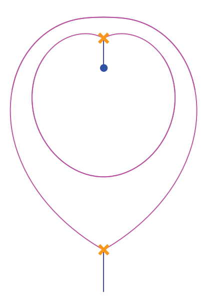

Figure 45: The degnerate spectral network in the z-plane for the critical phase corresponding to the vectormultiplet.

15.3.5 Wild wall crossing

What happens at higher rank? Are there examples with $m \neq 0, 1, 2$?

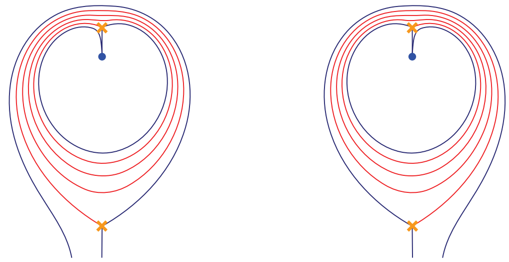

Figure 46: Some sample degenerate spectral networks for the critical phases corresponding to two hypermultiplets of type $n\gamma_1 + (n+1)\gamma_2$ and $(n+1)\gamma_1 + n\gamma_2$.

Certainly there are some qualitative differences at rank $K > 2$.

For example, already for $\mathfrak{g} = su(3)$ one has BPS states of arbitrarily high spin. By contrast, for $\mathfrak{g} = su(2)$ only $\Omega = +1$ and $\Omega = -2$ degeneracies appear.

There are further surprises in store for us and these are topics of intense current research.

A useful paper by Weist, http://arxiv.org/pdf/0903.5442v2.pdf investigates Euler characters of the m-Kronecker quiver $K(m)$. Then Weist conjectures the asymptotics for $n \to \infty$:

$$\log \Omega_{na+a_0,nb+b_0}(m) \sim nC(m)\sqrt{mab - a^2 - b^2} \tag{15.31}$$

with

$$C(m) = \frac{1}{\sqrt{m-2}}\left((m-1)^2 \log(m-1)^2 - (m^2-2m)\log(m^2-2m)\right) \tag{15.32}$$

Weist proves this is true for the case $(na + a_0, nb + b_0) = (d + 1, d)$.

♣See also papers of Gross-Pandharipande, Reineck-Stoppa-Weist, and Oda. ♣

The important point for us is that the $\Omega_{a,b}(m)$ grow exponentially with charge! Note that we only get a real result for

$$\frac{m - \sqrt{m^2-4}}{2} < \frac{a}{b}, \frac{b}{a} < \frac{m + \sqrt{m^2-4}}{2} \tag{15.33}$$

We should interpret this to mean that the degeneracies have only power law behavior when a, b do not satisfy these inequalities.

There are physical arguments which suggest that one should not have exponentially growing degeneracies of states in quantum field theory. The basic statement is that the density of states in a d-dimensional quantum field theory should be sub-exponential in the energy for large energy.

At large E the density of states should be that of the UV conformal fixed point used to define the theory.

Consider a d-dimensional CFT in finite spatial volume V at temperature T. Both the energy and the entropy should be extensive in the volume. Therefore, since there are no dimensionful parameters in a CFT by dimensional analysis we must have energy

$$E = \kappa_1 V T^d \tag{15.34}$$

and entropy

$$S(E) = \kappa_1 V T^{d-1} \tag{15.35}$$

Here V is the $(d-1)$-dimensional spatial volume and κ_1, κ_2 are some dimensionless constants associated with the CFT. (We are assuming the CFT has a finite number of fields.) Eliminating T gives the relation:

$$S(E) = \kappa_3 V^{1/d} E^{(d-1)/d} \tag{15.36}$$

so the entropy should be sub-exponential in the energy.

Since the BPS degeneracies are signed sums over states at an energy $|Z_\gamma|$ and since Z_γ is linear in the charge γ it follows that $\Omega(\gamma; u)$ is expected to grow subexponentially in the charge γ.

On the other hand, A. Neitzke and T. Mainiero appear to have found examples of spectral networks which do indeed predict BPS degeneracies growing exponentially with charge. It is not entirely clear what is going on and this is under investigation.

15.4 The "motivic" generalization

The 4d KSWCF has an important "motivic" generalization. It has a natural place in the physical approach to the subject. The image of the "motive of the affine line" in the ring $\mathbb{C}((q))$ is just the variable which couples to the spin of BPS states [55, 57, 96, 13].

From our present viewpoint we can approach the subject as follows:

For paths $\wp(z, z)$ which are based loops we can take a "trace":

$$\mathrm{Tr} F(\wp, \vartheta) := \sum_{i=1}^{K} \sum_{a \in \Gamma_{ii}(z,z)} \underline{\overline{\Omega}}(L_{\wp,\vartheta}, a) X_a \tag{15.37}$$

Note that there is no longer any z dependence, so we have lost all dependence on $\mathbb{S}_z$.

The way to interpret this is that what we are discussing is a line defect in the absence of surface defects.

Once again, M-theory provides a very intuitive picture. We consider M-theory on the 11-dimensional Lorentzian signature manifold

$$\mathbb{R}_t \times \mathbb{R}^3 \times T^*C \times \mathbb{R}^3 \tag{15.38}$$

with a background metric which solves Einstein's equations. So, we take T^*C has a hyperkähler metric (and assume we are in a situation where it admits such a metric).

Now, within this 11-manifold we have K basic M5-branes localized on

$$\mathbb{M}^{1,3} \times C \times \vec{0} \tag{15.39}$$

where C is embedded in T^*C via the zero section.

The line defects come from semi-infinite M2 branes which have boundary

$$\partial(M2) = (\mathbb{R}_t \times \{\vec{0}\}) \times \{\wp\} \times \{\vec{0}\} \tag{15.40}$$

The M2 worldvolume stretches along the fiber directions of $T * C$ with $\langle \lambda, \partial_n \rangle = e^{i\vartheta}$ for the normal direction n. This is the geometric origin of the angle ϑ in $L_{\wp,\vartheta}$. It is well-known that intersecting branes at angles can preserve some subset of supersymmetry [?].

In the three-dimensional Euclidean space the line defect just sits at a point, say $\vec{x} = 0$. Therefore there is an $SO(3)_{space} \times SO(3)_R$ symmetry in the problem. The second $SO(3)$ factor comes from rotations in the final $\mathbb{R}^3$ factor of the spacetime.

In the presence of the line defect the 4d Hilbert space is modified to $\mathcal{H}_{L_{\wp,\vartheta}}$, this space is still graded by Γ: [36]

$$\mathcal{H}_{L_{\wp,\vartheta}} = \oplus \mathcal{H}_{L_{\wp,\vartheta},\gamma} \tag{15.41}$$

♣How do we see from this viewpoint that γ lives in a Γ-torsor? ♣

and it is possible to put a BPS condition on states in the presence of $L_{\wp,\vartheta}$ so that we also have

$$\mathcal{H}^{BPS}_{L_{\wp,\vartheta}} = \oplus \mathcal{H}^{BPS}_{L_{\wp,\vartheta},\gamma} \tag{15.42}$$

We can now introduce a "refined index" or "protected spin character"

$$\overline{\underline{\Omega}}(L_{\wp,\vartheta}, \gamma; u; y) = \mathrm{Tr}_{\mathcal{H}^{BPS}_{L_{\wp,\vartheta},\gamma}} y^{2J_3}(-y)^{2I_3} \tag{15.43}$$

The "halo picture" of wall-crossing tells us how these degeneracies change:

Near a wall

$$W(\gamma_0) := \{(u,\vartheta) | Z_{\gamma_0}(u)/e^{i\vartheta} \in \mathbb{R}_-\} \tag{15.44}$$

we find a collection of states in the Hilbert space $\mathcal{H}^{BPS}_{L_{\wp,\vartheta}}$ which are very well-described by a Fock space. (The approximation becomes exact as (u,ϑ) approach the wall.) This is based on a semiclassical description of boundstates of "halo particles" BPS states of charges γ_h with $\gamma_h = \ell\gamma_0$, with $\ell \in \mathbb{Z}_+$ which are constrained to move in a sphere around some "core particle" of charge γ_c at a radius

$$R \sim \langle \gamma_c, \gamma_h \rangle \frac{1}{2\mathrm{Im}(Z_{\gamma_0}/e^{i\vartheta})} \tag{15.45}$$

Note that this radius goes to infinity as the wall is crossed, so the states in the Fock space leave the spectrum.

From the physical derivation we find the important facts that the creation oscillators which build the $\mathbb{Z}_2$-graded Fock space are not based on $\mathcal{H}^{BPS}_{\gamma_h}$ but rather on

$$(J_{\gamma_c,\gamma_h}) \otimes \mathcal{H}^{BPS}_{\gamma_h} \tag{15.46}$$

where (J_{γ_c,γ_h}) is the spin representation of $SO(3)_{space}$ of dimension $|\langle \gamma_c, \gamma_h \rangle|$. Introducing a variable q conjugate to the grading by the "Γ_0-charge" we can say that

$$\oplus_{N \geq 0} q^N \mathcal{H}^{\mathrm{halo}}_{\gamma_c + N\gamma_0} = \mathcal{H}^{BPS}_{\gamma_c} \otimes \otimes_{\ell=1}^{\infty} \mathcal{F}[q^\ell (J_{\gamma_c,\ell\gamma_0}) \otimes \mathcal{H}^{BPS}_{\gamma_h}] \tag{15.47}$$

With this physical picture in mind suppose we consider now

$$F(L_{\wp,\vartheta}, y) \sim \sum_\gamma \overline{\underline{\Omega}}(L_{\wp,\vartheta}; \gamma; u; y) X_\gamma \tag{15.48}$$

[36] Actually, by a torsor for Γ

with X_γ in the homology path algebra. For simplicity let us first consider the case where there is just one kind of halo particle with charge $\gamma_h = \gamma_0$ and moreover it is spinless

$$\mathrm{Tr}_{\mathcal{H}^{BPS}_{L_{\wp,\vartheta},\gamma}} y^{2J_3}(-y)^{2I_3} = 1 \tag{15.49}$$

Then the halo configurations make a contribution to $F(L_{\wp,\vartheta}, y)$ of the form

$$X_{\gamma_c}(1 + y^{n-1}X_{\gamma_h})(1 + y^{n-3}X_{\gamma_h}) \cdots (1 + y^{3-n}X_{\gamma_h})(1 + y^{1-n}X_{\gamma_h}) \tag{15.50}$$

Here $n = \langle\gamma_c, \gamma_h\rangle$ is an integer. If we expand this out and use (**??**) we get

$$x_{\gamma_c}(1+y^{n-1}x_{\gamma_h})(1+y^{n-3}x_{\gamma_h}) \cdots (1+y^{3-n}x_{\gamma_h})(1+y^{1-n}x_{\gamma_h}) = \sum_{j=0}^{|n|} P^{(n)}_j(y) x_{\gamma_c+j\gamma_h} \tag{15.51}$$

where $P^{(n)}_j(y)$ are symmetric integral Laurent polynomials in y. In fact, as is clear from the Fermionic Fock space interpretation $P^{(n)}_j(y)$ is just the character of the j^{th} antisymmetric product $\Lambda^j \rho_{|n|}$ where ρ_N is the N-dimensional irreducible representation of $SU(2)$.

Now we use a trick. Introduce *non-commuting* variables satisfying a relation generalizing (**??**): [37]

$$X_\gamma X_{\gamma'} = y^{\langle\gamma,\gamma'\rangle} X_{\gamma+\gamma'}. \tag{15.52}$$

We then claim that the same Laurent polynomials $P^{(n)}_j = ch\Lambda^j\rho_n$ appear when we expand out the product

$$X_{\gamma_c}\Phi_n(X_{\gamma_h}) = \sum_{j=0}^{|n|} P^{(n)}_j(y) X_{\gamma_c+j\gamma_h} \tag{15.53}$$

where

$$\Phi_n(\xi) := \begin{cases} \prod_{s=1}^{n}(1 + y^{-(2s-1)}\xi) & n > 0 \\ 1 & n = 0 \\ \prod_{s=1}^{|n|}(1 + y^{(2s-1)}\xi) & n < 0 \end{cases} \tag{15.54}$$

We prove this by first expanding out $\Phi_n(X_{\gamma_h})$ with coefficients $X_{j\gamma_h}$. For this purpose the X_{γ_h} can be taken as commutative variables satisfying **??**. It is clear from the "fermionic combinatorics" that the coefficient of $X_{j\gamma_h}$ is (say, for $n > 0$) a polynomial in y^{-1} which is the character $\mathrm{ch}\Lambda^j\rho_n$ up to an overall multiplication by a power of y^{-1}. By comparing the lowest power of y^{-1} with that for the character we see that that power is precisely canceled by

$$X_{\gamma_c}X_{j\gamma_h} = y^{jn}X_{\gamma_c+j\gamma_h} \tag{15.55}$$

[37]Here the noncommutative torus is introduced as a convenient mathematical device. It surely has a compelling physical meaning and motivation, and it would be good to find one. The introduction of these variables has an interesting interpretation in terms of Wilson lines operators in a Chern-Simons theory in the work of Cecotti and Vafa.

Now we can summarize the framed BPS wall-crossing more elegantly by introducing a "formal parallel transport" valued in the noncommutative algebra:

$$F(L_{\wp,\vartheta}) = \sum_{\gamma\in\Gamma} \overline{\underline{\Omega}}(L_{\wp,\vartheta},\gamma;y)\hat{X}_\gamma \tag{15.56}$$

and the wall-crossing across $W(\gamma_0)$ is summarized by

$$\hat{X}_\gamma \to \hat{X}_\gamma \Phi_n(\hat{X}_{\gamma_0}) \tag{15.57}$$

where $n = \langle\gamma,\gamma_0\rangle$. This is still not the best formulation because the substitution for $\hat{X}_\gamma$ depends on γ itself. A better way to express this is to introduce the quantum dilogarithm:

$$\Phi(X) := \prod_{k=1}^{\infty}(1+y^{2k-1}X)^{-1} \tag{15.58}$$

and then we note that the substitution is equivalent to

♣Say something about $|y|<1$ here? ♣

$$\hat{F} \to \Phi(\hat{X}_{\gamma_0})\hat{F}\Phi(\hat{X}_{\gamma_0})^{-1} \tag{15.59}$$

♣Show a little more of the computation here? ♣

We just derived equation (15.59) under the simplifying assumption (15.49), but the same reasoning leads to the following general formula:

The 4d protected spin characters themselves are sums of $SU(2)$ characters and can be expanded as finite sums:

$$\Omega(\gamma;y;u) = \sum_m a_m(\gamma;u)(-y)^m \tag{15.60}$$

Then across $W(\gamma_0)$ we find that $\hat{F}$ is conjugated by

$$S_{\Gamma_0} = \prod_{\gamma\in\Gamma_0}\prod_m(\Phi((-y)^m X_\gamma))^{a_m(\gamma;u)} \tag{15.61}$$

It is shown in [96] how this conjugation precisely captures the Fock space combinatorics of the halo particles which appear/disappear upon crossing a wall.

Now, the "motivic" version of the Kontsevich-Soibelman wall crossing formula is the obvious generalization with an ordered product of S_{Γ_0}. For example, if $\langle\gamma_1,\gamma_2\rangle = +1$ then the analog of (15.23) is the identity on dilogarithm functions:

$$\Phi(X_{\gamma_2})\Phi(X_{\gamma_1}) = \Phi(X_{\gamma_1})\Phi(X_{\gamma_1+\gamma_2})\Phi(X_{\gamma_2}) \tag{15.62}$$

♣Maybe put all this in a separated chapter on the "halo picture" ? ♣

Remarks

1. Consistency of the wall crossing of $\hat{F}$ around closed loops proves the "motivict KSWCF": The ordered product $\prod S_{\Gamma_0}$ is unchanged. The argument relies on the claim that there are "sufficiently many" line defects.

2. The $\hat{F}$'s have interesting positivity properties under wall-crossing...

3. If we now include 2d4d degeneracies by considering an supersymmetric interface $L_{\wp(z_1,z_2),\vartheta}$ then in the geometrical M-theory interpretation there is still an $SO(2)$ symmetry. Therefore we believe that there is a "motivic" version of the 2d4d wall-crossing formula. Some aspects of what it would look like were discussed in [97], but some important details need to be understood better.

4. We put "motivic" in quotation marks because we do not actually discuss the true motivic invariants of Kontsevich and Soibelman, but only their image in a ring of Laurent polynomials.

16. Lecture 6: Tuesday, Oct. 9: Coordinates for Moduli Spaces of Flat Connections

In this chapter we are going to start investigating some interesting functions defined (in patches) on the moduli space $\mathcal{M}$ of flat connections. In the next chapter we will use this to find ways of constructing explicitly the parallel transport operators associated with the flat connection $\mathcal{A}$ associated with a solution (φ, A) of the Hitchin system and a complex structure ζ.

16.1 Review: Moduli spaces of complex flat gauge fields and moduli spaces of local systems

A flat connection is characterized, up to gauge equivalence, by its monodromy. Therefore, the moduli space of flat G_c connections on C is

$$\mathrm{Hom}(\pi_1(C,*), G_c)/G_c \tag{16.1}$$

where we quotient by the conjugation action.

Let us make this more concrete for $G_c = SL(K,\mathbb{C})$. If we choose a basepoint $*$ and a standard set of cycles $\alpha_k, \beta_k,\ k = 1,\dots,g_{\bar{C}}$, generating $\pi_1(\bar{C},*)$ as well as cycles δ_n surrounding the singular points $\mathfrak{s}_n$, with $n = 1,\dots,s$. The monodromy around these cycles is given by $SL(K,\mathbb{C})$ matrices A_k, B_k and C_n satisfying the monodromy constraint

$$\prod_k [A_k, B_k] \prod_{n=1}^{s} C_n = 1 \tag{16.2}$$

modulo overall conjugation.

Thus we can compute the dimension of the moduli space:

$$\begin{aligned}\dim_c \mathcal{M}(C, SL(K,\mathbb{C})) &= \dim_c G_c(2g_C + s) - \dim_c G_c - \dim_c G_c - \sum_n (\dim_c G_c - \dim_c \mathcal{O}_n)\\ &= \dim_c G_c(2g_C - 2) + \sum_n \dim_c \mathcal{O}_n\end{aligned} \tag{16.3}$$

where $\mathcal{O}_n$ corresponding to the fixed conjugacy class C_n of the monodromy around $\mathfrak{s}_n$.

There are exceptional cases where equation (16.3) does not hold. This can happen if the conjugacy classes are nongeneric. Certainly when the dimension comes out negative one is in such a situation.

Later in this chapter we will need the case $G_c = GL(K,\mathbb{C})$. In the case of s singular points with conjugacy classes given by orbits of regular semisimple elements $\mathcal{O}_n = \mathcal{O}(\mu^{(n)})$, where

$$\mu^{(n)} = Diag\{\mu_1^{(n)}, \dots, \mu_K^{(n)}\} \tag{16.4}$$

with all eigenvalues distinct, the dimension is

$$\begin{aligned} \dim_c \mathcal{M}^{flat}(C, GL(K,\mathbb{C}), \{\mathcal{O}(\mu^{(n)})\}) &= (2g_C + s)K^2 - 2(K^2-1) - Ks \\ &= (2g_C - 2)K^2 + s(K^2 - K) + 2 \end{aligned} \tag{16.5}$$

The reason for the "extra 2" can be seen in the first line. The determinant of the equation can only be satisfied for a certain constraint on the orbits, and when it is satisfied it is not an independent equation. So there are only K^2-1 equations. Also, when dividing by conjugation only the K^2-1 dimensional $SL(K,\mathbb{C})$ subgroup acts effectively.

In the case of irregular singular points for $G_c = SL(K,\mathbb{C})$ or $G_c = GL(K,\mathbb{C})$ instead of a semisimple monodromy matrix we will have unipotent *Stokes matrices* $S_n^{(\alpha)}$, $\alpha = 1, \dots, L_n$. Around an irregular singular point with N Stokes lines the Stokes matrices will be, in an appropriate gauge, alternating upper and lower triangular unipotent matrices and the product around the singularity

$$\prod_{\alpha=1}^{L_n} S_n^{(\alpha)} \tag{16.6}$$

acts as a *formal* monodromy which should be inserted into the usual relation imposed on representations of $\pi_1(C)$ in $SL(K,\mathbb{C})$. Then the moduli space of local systems is the space of solutions (modulo conjugation) to

$$\prod_k [A_k, B_k] \prod_n C_n \prod_{n'} \prod_{\alpha=1}^{L_{n'}} S_{n'}^{(\alpha)} = 1 \tag{16.7}$$

where the product on n is over regular singular points and the product over n' is over the irregular singular points.

♣Is this correct? Perhaps also need to include factor of "formal monodromy" for the Stokes sectors. ♣

In addition, we will be prescribing *flag data* at the punctures. This was an important innovation of Fock and Goncharov, and arises very naturally in the physics, as we will see. Thus, we assume given a monodromy invariant flag at the marked points. For $SL(K,\mathbb{C})$ or $GL(K,\mathbb{C})$ there are $K!$ invariant flags, so the space of local systems with flag data is a $(K!)^s$-fold cover of the moduli space of local systems.

♣Write an appendix on Stokes phenomenon and Stokes rays ♣

♣Say what we do for irregular singular points. ♣

16.1.1 Relation to Hitchin systems

♣This section doesn't belong here! ♣

The moduli space of flat connections is closely related to the moduli space of solutions to Hitchin's equations.

In complex structure $\zeta \neq 0, \infty$ the moduli space $\mathcal{M}$ can be identified with the moduli space of $SL(K,\mathbb{C})$ (or $GL(K,\mathbb{C})$) local systems, i.e., with complex flat connections as follows. The flat connection $\mathcal{A}$ is related to the solution (A,φ) of the Hitchin system through the relation

$$\mathcal{A} = \frac{R}{\zeta}\varphi + A + R\zeta\bar{\varphi} \tag{16.8}$$

One easily checks that if (A,ϕ) solve the Hitchin equations then

$$d\mathcal{A} + \mathcal{A}^2 = 0 \tag{16.9}$$

and conversely if $\mathcal{A}$ is flat for a family of ζ then (A,φ) solve the Hitchin equations.

The boundary conditions in the Hitchin system correspond to prescribing conjugacy classes of monodromy of the local system. In the case of regular singularities the conjugacy class of the monodromies around the singularities $\mathfrak{s}_n$ can be specified by

♣Be clear about gauge equivalences here and in the Higgs case. In the later case we fix a gauge near $\mathfrak{s}_n$, but here we only specify the monodromy up to conjugacy. ♣

$$\mu^{(n)} = Diag\{\mu_1^{(n)}, \dots, \mu_K^{(n)}\} \tag{16.10}$$

From (16.8) we see that the relation to the boundary data in the Hitchin system is

$$\mu_i^{(n)} = \exp[2\pi i \left(\frac{R\mathfrak{m}_i^{(n)}}{\zeta} - \mathrm{i}\alpha_i^{(n)} - R\zeta\bar{\mathfrak{m}}_i^{(n)} \right)] \tag{16.11}$$

where $\mathfrak{m}_i^{(n)}$ and $\alpha_i^{(n)}$ are defined in equation (11.2) et. seq.

16.2 Higgs bundles and the abelianization map

A *Higgs bundle* is a holomorphic vector bundle on $E \to C$, which we will regard as a complex vector bundle with a $\bar{\partial}$ operator, with a meromorphic section $\varphi \in \Gamma(\mathcal{K}_C \otimes \mathrm{End}(E))$.

To a Higgs bundle one can associate an *abelianization map*. This is simply the association of the Higgs bundle with the pair of the spectral curve Σ together with the eigenline bundle

$$\mathcal{L} := \ker(\varphi - \lambda) \subset \pi^* E \tag{16.12}$$

over the spectral curve. One can recover the Higgs bundle from the line bundle over the spectral curve simply by taking $E = \pi_*(\mathcal{L})$, which has fibers:

$$E_z = \oplus_i \mathcal{L}_{z^{(i)}}. \tag{16.13}$$

Similarly, one can recover the Higgs field from the pushforward of the canonical multiplication map by $\lambda : \mathcal{L} \to \mathcal{L} \otimes \mathcal{K}_\Sigma$. Thus, one can identify the moduli space of Higgs bundles with the space of pairs $(\Sigma, \mathcal{L})$. Since a line bundle has an abelian structure grouip this is called the abelianization map.

Remark: One can also identify the Hitchin moduli space with the moduli space of Higgs bundles if we restrict to line bundles $\mathcal{L}$ of appropriate degree. The Hitchin fibration, in these terms, is then just the forgetful map $(\Sigma, \mathcal{L}) \to \Sigma$.

16.3 Generalized Spectral Networks

For some purposes, especially for the discussion of coordinates on moduli spaces of flat connections it is convenient to have a more flexible notion of a "spectral network." Here we consider again a graph, but dispense with the requirement that the edges be WKB paths of phase ϑ for some spectral curve.

Rather, we define a purely topological notion. We begin again with a Riemann surface C possibly with boundaries and with marked points $\mathfrak{s}_n$. Each boundary component has at least one marked point, and marked points in the interior of C are called "punctures." We are also given a $K:1$ branched cover $\pi : \Sigma \to C$, but no mention is made of how we got this branched cover. In particular there is no meromorphic differential λ. We assume that all branch points are simple and that the cover is unramified (although possibly nontrivial) over the boundaries. Let C' be C minus the branch points.

Definition: A *spectral network subordinate to the covering* Σ is a collection

$$\mathcal{W} = (o(\mathfrak{s}_n), \{z_\mu\}, \{p_c\}) \tag{16.14}$$

where the symbols refer to the following data:

D1. For each singular point $\mathfrak{s}_n$, $o(\mathfrak{s}_n)$ is a *partially ordered subset* of the set of sheets of Σ over a neighborhood of $\mathfrak{s}_n$. $o(\mathfrak{s}_n)$ must contain at least two elements, and if $\mathfrak{s}_n$ is a puncture, $o(\mathfrak{s}_n)$ must contain *all* of the sheets over a neighborhood of $\mathfrak{s}_n$.

D2. $\{z_\mu\}$ is a locally finite collection of points on C', called *joints*.

D3. $\{p_c\}$ is a finite or countable collection of closed segments (i.e. images of *embeddings* of $[0,1]$ into C), called *walls* or *streets* (depending which metaphor is more useful in a given context. For each orientation o of the street p_c, p_c is labeled with an ordered pair of distinct sheets of the covering $\Sigma \to C$ over p_c. Reversing the orientation reverses this ordered pair of sheets. So p_c comes with two labels which we could write as (o, ij) and $(-o, ji)$.

The data must satisfy the following conditions:

C1. The segments p_c cannot cross one another (but they are allowed to have common tangents). Each p_c must begin on a branch point or a joint, and must end on a joint or a singular point. Any compact subset of C' intersects only finitely many segments.

C2. Around each branch point $\mathfrak{b}$ there is a neighborhood where $\mathcal{W}$ looks like Figure **??**. That is, each branch point of type (ij) is an endpoint of three streets which carry labels (o, ij) or (o, ji), and the streets encountered consecutively traveling around a loop around $\mathfrak{b}$ have oppositely ordered sheets.

C3. Around each joint z_μ there is a neighborhood where $\mathcal{W}$ looks like Figure 22 or Figure 23.

C4. If a segment with label ij ends at a singular point $\mathfrak{s}_n$, then i and j lie in the ordered subset $o(\mathfrak{s}_n)$, and with respect to the ordering of $o(\mathfrak{s}_n)$ we have $i < j$.

The main point is that for a generalized spectral network we can once again determine degeneracies $\overline{\underline{\Omega}}(\wp, \mathcal{W}, a)$ and μ using the same four axioms as in the formal parallel transport theorem of §12.5

16.4 Nonabelianization map

We are now going to describe a kind of inverse of the abelianization map, but it will apply in a more general setting.

Suppose we give ourselves the data

1. A branched cover $\pi : \Sigma \to C$ with only simple branch points.
2. A complex line bundle $L \to \Sigma$ and flat $GL(1, \mathbb{C})$ connection ∇^{ab} on Σ.

We would like to push this forward to C to make a nonabelian flat connection on a rank K complex vector bundle over C.

The natural thing to do is to take the pushforward of the line bundle:

$$E = \pi_*(L) \tag{16.15}$$

whose fibers at a generic point $z \in C$ are

$$E|_z = \oplus_{i=1}^{K} L_{z^{(i)}}. \tag{16.16}$$

However, we cannot just push down $\pi_*(\nabla^{\mathrm{ab}})$ because of the monodromy around branch points: Around a branch point the lines L_i get permuted because the preimages $z^{(i)}$ in the spectral curve equation get permuted. But the pushed-forward connection preserves the lines.

Therefore, $\pi_*(\nabla^{\mathrm{ab}})$ makes sense on $C' := C - \{\textit{branch points}\}$, but it does not extend over the branch points.

We will now show that a (generalized) spectral network $\mathcal{W}$ gives just the right data needed to push forward an abelian connection to get a flat nonabelian connection downstairs which does extend over the branch points:

Theorem: Given $(\Sigma, L, \nabla^{\mathrm{ab}})$ and a spectral network $\mathcal{W}$ there is a complex rank K bundle $E_{\mathcal{W}} \to C$ with flat connection $\nabla_{\mathcal{W}}$ such that

1. On $C - \mathcal{W}$ we have $E_{\mathcal{W}}|_{C-\mathcal{W}} \cong \pi_*(L)|_{C-\mathcal{W}}$
2. For $z_1, z_2 \in C - \mathcal{W}$ the parallel transport along a path $\wp(z_1, z_2)$, from $E_{\mathcal{W}}|_{z_1}$ to $E_{\mathcal{W}}|_{z_2}$ is given by
$$\mathbb{F}(\wp) = \sum_{a \in \Gamma(z_1, z_2)} \overline{\underline{\Omega}}(\wp, \mathcal{W}, a) \mathcal{Y}_a \tag{16.17}$$
where
$$\mathcal{Y}_a := \exp \int_a \nabla^{\mathrm{ab}} \tag{16.18}$$

is the element of $\mathrm{Hom}(\oplus_i L_{z_1^{(i)}}, \oplus_j L_{z_2^{(j)}})$ defined as follows: If $a \in \Gamma_{ij}(z_1, z_2)$ then $\mathcal{Y}_a$ is zero on $L_{z_1^{(k)}}$ for $k \neq i$, and for $k = i$, it is in

$$\mathrm{Hom}(L_{z_1^{(i)}}, L_{z_2^{(j)}}) \tag{16.19}$$

and is just defined by parallel transport along the chain a on Σ with ∇^{ab}. N.B. The path $\wp$ is allowed to intersect the spectral network.

Sketch of proofs: There are two approaches to proving this theorem

Approach 1: We define $E_{\mathcal{W}}$ so that this is tautologically true. We let the equation [38]

$$\psi(z_2) = \psi(z_1) F(\wp) \tag{16.20}$$

define a sheaf of flat sections. That is, given an open set $\mathcal{U} \subset C$ we let $\mathcal{F}[\mathcal{U}]$ denote all sections ψ of E over $\mathcal{U} \cap (C - \mathcal{W})$ such that for any path $\wp \subset \mathcal{U}$ with endpoints in $C - \mathcal{W}$ we have (16.20).

♣SAY MORE HERE! NONTRIVIAL POINT IS THAT IT EXTENDS OVER $\mathcal{W}$ ♣

Approach 2: We use the spectral network $\mathcal{W}$ to define an explicit set of transition functions as follows:

Denote the connected components of the complement of $\mathcal{W}$ by $\mathcal{R}^\alpha$. Trivialize the bundle on each of these regions:

$$E_{\mathcal{W}}|_z \cong \oplus_{i=1}^K L_{z^{(i)}} \tag{16.21}$$

For a path $\wp$ in a region $\mathcal{R}^\alpha$ which does not cross $\mathcal{W}$ the formula

$$\nabla_{\mathcal{W}} = \pi_*(\nabla^{\mathrm{ab}}) \tag{16.22}$$

makes sense. Now consider an S-wall of type ij separating $\mathcal{R}^\alpha$ from $\mathcal{R}^\beta$.

FIGURE OF S-WALL OF TYPE ij SEPARATING REGIONS

We now define the transition function

$$T^{\alpha\beta} = 1 + \sum_{a \in \Gamma_{ij}(z,z)} \mu(a) \mathcal{Y}_a \in \mathrm{End}(E_z) \tag{16.23}$$

where if $a \in \Gamma_{ij}(z,z)$ then we define $\mathcal{Y}_a \in \mathrm{End}(E_z)$ as in the statement of the theorem above (that is, it is zero on all lines except $L_{z^{(i)}}$ etc.)

Now we take

$$E_{\mathcal{W}} := (\amalg_\alpha E|_{\bar{\mathcal{R}}^\alpha}) / \sim \tag{16.24}$$

where the equivalence relation identifies

$$(\psi, z) \in (E|_{\mathcal{R}^\alpha})_z \sim (T^{\alpha\beta}\psi, z) \in (E|_{\mathcal{R}^\beta})_z \tag{16.25}$$

Now, by construction of the detour rule, $F(\wp, \mathcal{W})$ is indeed the parallel transport of a well-defined connection on $E_{\mathcal{W}}$ across the S-walls of type ij.

Note that in our local trivialization the $\mathcal{Y}_a$ are proportional to matrix units e_{ij}.

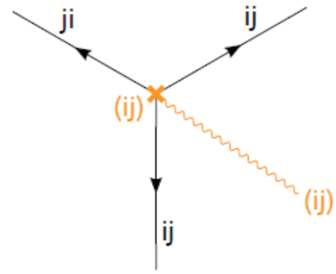

Figure 47: WKB paths in the neighborhood of a simple branchpoint exchanging sheets ij.

The main computation now is to check that the bundle $E_{\mathcal{W}}$ and connection $\nabla_{\mathcal{W}}$ really do extend over the branch points. So we return to the basic picture of Figure 47

But this computation is essentially the same as that used in the formal parallel transport theorem to check the homotopy invariance across a branch point. This concludes the proof.

Remarks:

1. Ultimately the computation boils down to a simple matrix identity

$$\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix} \tag{16.26}$$

   which is valid for $\mu^2 = 1$. The matrix on the RHS expresses the monodromy which exchanges sheets. As promised, there are some tricky signs. One cannot write such an identity with all $+1$ signs.

2. In order for this theorem to work at all there are some very important sign considerations, which we alluded to before. It is very important that we actually discuss *twisted connections*. These are flat connections on the unit circle bundles $\tilde{\Sigma}$ of $T\Sigma$ and $\tilde{C}$ of TC constrained to have holonomy -1 around the circle fibers.

3. Given a spin structure on C, Σ we can identify the twisted flat connections with ordinary flat connections. A spin structure gives us a $2:1$ cover $p : C^{spin} \to \tilde{C}$ which over each fiber over $z \in C$ is just the nontrivial double-cover of the circle by the circle. Then $p^*\nabla$ has holonomy $+1$ around the circle fibers of $C^{spin} \to C$ and therefore the connection can be descended to a flat connection on C.

4. As we mentioned before, there is a generalization of the formal parallel transport theorem to the case where X_a is in the *homotopy* path algebra of Σ. There is then an analogous form of nonabelianization which allows us to push forward nonabelian flat rank κ bundles on Σ to flat rank κK bundles on C. This is related to some interesting physics in M theory. The sheets of the branched cover $\pi : \Sigma \to C$

[38] Our convention will be that successive linear operators are written successively to the right.

represent the separation of a stack of K M5-branes corresponding to a point on the Coulomb branch of vacua. However, there can be subspaces of the Coulomb branch where nonabelian gauge symmetry survives in the IR. This would happen if we begin with NK M5 branes on C and there are N M5 branes on each of the sheets. ♣Say this more coherently. ♣

5. An important point about the connection $\nabla_{\mathcal{W}}$ is that it preserves a *flag structure* at the singular points. For example, at a puncture with a regular singular point we have the picture

   SPIRAL WITH ij PATH GOING IN

   and the orientation going into the puncture defines an ordering of the sheets $i < j$. Now, with this ordering, the transition matrices $T^{\alpha\beta}$ are *upper triangular* in the decomposition $\oplus L_i$. Therefore, they preserve the flag structure:

$$F^i := \oplus_{j \geq i} L_{z^{(j)}} \tag{16.27}$$

   Note that the upper index reflects the *codimension* so that our flags are

$$0 = F^K \subset F^{K-1} \subset \cdots \subset F^1 \subset F^0 = E_{\mathcal{W}} \tag{16.28}$$

   The data of a spectral network picks out one of the $K!$ invariant filtrations at each puncture.

16.5 Mapping moduli spaces

The theorem of the previous section defines a mapping of moduli spaces. In this section we examine some of its properties.

Suppose that the abelian $GL(1)$ connection has prescribed monodromies around the lifts of the punctures $\mathfrak{s}_n^{(i)}$:

$$\mu_{n,i} = \exp\left(\oint_{\mathfrak{s}_n^{(i)}} \nabla^{\rm ab}\right) \tag{16.29}$$

Then we have constructed a map of moduli spaces

$$\Psi_{\mathcal{W}} : \mathcal{M}(\Sigma, GL(1), \{\mu_{n,i}\}) \to \mathcal{M}_F(C, GL(K), \{\mathcal{O}_n\}) \tag{16.30}$$

Here the subscript F on the RHS indicates that we have prescribed a flag structure at each puncture. Moreover, the monodromy around the puncture $\mathfrak{s}_n$ is in the conjugacy class $\mathcal{O}_n$ defined in equation (16.4) above.

Now, each of the moduli spaces above is holomorphic symplectic.

For $\mathcal{M}(\Sigma, ...)$ it is

$$\varpi_\Sigma = \int_\Sigma \delta\alpha \wedge \delta\alpha \tag{16.31}$$

where α denotes the abelian connection 1-form.[39]

[39] Here and below, we use the freedom to fix a gauge so that we consider only variations $\delta\alpha$ which vanish near the punctures. This makes the integral (16.31) convergent, despite the fact that the connections we consider are singular.

For $\mathcal{M}_F(C, ...)$ it is

$$\varpi_C = \int_C \mathrm{Tr}\, \delta\mathcal{A} \wedge \delta\mathcal{A} \tag{16.32}$$

with $\mathcal{A}$ the nonabelian connection 1-form.

Now, by the construction we have given, it is clear that when $\delta\alpha$ has support away from $\mathcal{W}$ we have

$$\Psi_{\mathcal{W}}^*(\varpi_C) = \varpi_\Sigma. \tag{16.33}$$

and a short argument shows that this is also true even when $\delta\alpha$ has support intersecting $\mathcal{W}$.

Thus, the map $\Psi_{\mathcal{W}}$ *is holomorphic symplectic.*

Now let us consider the dimensions of these moduli spaces. As we have noted above

$$\dim_{\mathbb{C}} \mathcal{M}(C, GL(K); \{\mathcal{O}_n\}) = (2g_C + s - 2)K^2 - Ks + 2. \tag{16.34}$$

On the other hand, using the Riemann-Hurwitz relation

$$\dim_{\mathbb{C}} \mathcal{M}(\Sigma, GL(1); \{\mu_{n,i}\}) = 2g_\Sigma = K(2g_C - 2) + B + 2, \tag{16.35}$$

where B is the branching number (which is simply the number of branch points, since we assume all branch points are simple.)

It follows that

$$\dim_{\mathbb{C}} \mathcal{M}_F(C, GL(K); \{\mathcal{O}\}) - \dim_{\mathbb{C}} \mathcal{M}(\Sigma, GL(1); \{\mu_{n,i}\}) = (K^2 - K)(2g_C + s - 2) - B. \tag{16.36}$$

Since $\Psi_{\mathcal{W}}$ is symplectic it is in particular locally injective, so the quantity in (16.36) must be nonnegative. We thus get a topological restriction on the branching number of a branched cover $\Sigma \to C$ of degree K admitting a spectral network:

$$B \leq (K^2 - K)(2g_C + s - 2). \tag{16.37}$$

This is a curious inequality on the purely topological notion of spectral network and it would be interesting to see how to prove it directly from the topological definition of a generalized spectral network.

For the case of a WKB spectral network we have

$$B = \deg(\mathrm{Div}_0(\Delta)) \tag{16.38}$$

where $\Delta = \prod_{i<j}(\lambda_i - \lambda_j)^2$ is the discriminant of the spectral curve. This is a meromorphic differential of order $K^2 - K$ and it has s poles of order $2 \times \frac{1}{2}K(K-1)$ and hence for this case

$$B = (K^2 - K)(2g_C - 2 + s) \tag{16.39}$$

and the bound (16.37) is saturated.

In this case, where the bound is saturated the dimensions of the moduli spaces are equal and since $\Psi_{\mathcal{W}}$ is symplectic we have *local coordinates.* They are valued in certain patches $\mathcal{U}_{\mathcal{W}}$. We don't know much about these patches in general. They depend on the spectral network. In some examples, (including some below) these patches are Zariski open sets in the moduli space.

16.6 Darboux functions

The gauge equivalence class of a $GL(1,\mathbb{C})$ connection is determined uniquely by its holonomies:

$$\mathcal{Y}_\gamma := \exp \oint_\gamma \nabla^{\rm ab} \tag{16.40}$$

By our map $\Psi_{\mathcal{W}}$ we can regard them as locally defined (on $\mathcal{U}_{\mathcal{W}}$) functions on the moduli space $\mathcal{M}_F(C, GL(K); \{\mathcal{O}\})$.

Some properties of these functions:

1. $\mathcal{Y}_\gamma$ are locally holomorphic on $\mathcal{M}_F(C, GL(K,\mathbb{C}), \{\mathcal{O}_n\})$. (That is, they are holomorphic in patches $\mathcal{U}_{\mathcal{W}}$.)

2. $\mathcal{Y}_\gamma \mathcal{Y}_{\gamma'} = (-1)^{\langle\gamma,\gamma'\rangle}\mathcal{Y}_{\gamma+\gamma'}$ (The sign arises because we should be working with twisted local systems.) If we choose a basis for the annihilator of $\langle\cdot,\cdot\rangle$, that is, for the flavor lattice Γ_f then we just have

$$\mu_{n,i} = P\exp \oint_{\mathfrak{s}_n^{(i)}} \nabla^{\rm ab} = \mathcal{Y}_{\gamma_{n,i}} \tag{16.41}$$

   Choosing a basis $\{\bar{\gamma}_t\}$ for the symplectic quotient $\Gamma/\Gamma_f \cong \Gamma_g$ therefore gives a set of coordinates on $\mathcal{M}(\Sigma, GL(1), \mathfrak{m})$.

3. An easy computation shows that $\{\mathcal{Y}_\gamma, \mathcal{Y}_{\gamma'}\} = \langle\gamma,\gamma'\rangle\mathcal{Y}_\gamma\mathcal{Y}_{\gamma'}$ For this reason we refer to the $\mathcal{Y}_\gamma$ as *Darboux functions.* If we choose a basis $\bar{\gamma}_t$ for the symplectic quotient lattice Γ_g and let $C_{ts} = \langle\bar{\gamma}_t, \bar{\gamma}_s\rangle$ then we can write

$$\tilde{\omega}_\Sigma = C^{st} d\log\mathcal{Y}_{\gamma_s} \wedge d\log\mathcal{Y}_{\gamma_s} \tag{16.42}$$

4. If we compare coordinates from two different WKB spectral networks $\mathcal{W}_+$ and $\mathcal{W}-$ at $\vartheta^\pm$ across a critical WKB phase ϑ_0 at which we have a morphism associated with a BPS state of charge $\gamma \in \Gamma_0 = \mathbb{Z}\gamma_0$ then

$$\Psi_{\mathcal{W}^+}(\nabla^{\rm ab}) = \Psi_{\mathcal{W}^-}(\mathcal{K}_{\Gamma_0}(\nabla^{\rm ab})) \tag{16.43}$$

$$\mathcal{K}_{\Gamma_0}\mathcal{Y}_\gamma = \prod_{\gamma'\in\Gamma_0} (1-\mathcal{Y}_{\gamma'})^{\langle\gamma,\gamma'\rangle\Omega(\gamma')}\mathcal{Y}_\gamma \tag{16.44}$$

   This follows from the WCF for the framed BPS degeneracies we found before.

5. Note that these are cluster-like transformations. Sometimes we can find a redefinition of coordinates by signs $\mathcal{Y}_\gamma \to \sigma(\gamma)\mathcal{Y}_\gamma$ where $\sigma(\gamma)$ is a quadratic refinement of the intersection form mod 2 so that all the transformations have positive signs. This kind of positivity property is very important in the theory of Fock and Goncharov. For A_1 theories of class S, there is a canonical quadratic refinement that makes all transformations with hypermultiplets positive in this sense.

♣Can one always get only + signs? I think not with vm's. Clarify this point. ♣

16.7 The spectrum generator

Recall that the spectrum generator is the KS product over a half-plane (15.11)

$$\mathbb{S}(\vartheta; u) := \prod_{\gamma:\vartheta\leq \arg -Z_\gamma<\vartheta+\pi} K_\gamma^{\Omega(\gamma;u)}, \tag{16.45}$$

There is a nice interpretation of this in terms of the nonabelianization maps $\Psi_{\mathcal{W}_\vartheta}$.

If we continuously change $\vartheta \to \vartheta + \pi$ then on the one hand, the spectral networks undergo the morphsims associated with the BPS states contributing to the product in (15.11).

Let $\mathcal{W}_\vartheta^* := \mathcal{W}_{\vartheta+\pi}$. Then $\mathbb{S}(\vartheta; u)$ is the transformation determined by

$$\Psi_{\mathcal{W}^*}(L, \tilde{\nabla}^{\rm ab}) = \Psi_{\mathcal{W}}(L, \nabla^{\rm ab}) \tag{16.46}$$

and hence

$$\tilde{\mathcal{Y}}_{\gamma'} = \mathbb{S}(\vartheta; u)\mathcal{Y}_{\gamma'} \qquad \forall \gamma' \in \Gamma \tag{16.47}$$

where $\tilde{\mathcal{Y}}_{\gamma'}$ are the holonomies of $\tilde{\nabla}^{\rm ab}$.

On the other hand, $\mathcal{W}_\vartheta^* := \mathcal{W}_{\vartheta+\pi}$ is almost the same spectral network as $\mathcal{W}_\vartheta$! Recall that the segments are made of WKB paths

$$\langle \lambda_i - \lambda_j, \partial_t\rangle = e^{i\vartheta} \tag{16.48}$$

All we have done is exchange ij walls for ji walls and reversed the ordering of sheets at the punctures!

Indeed, from the construction of the bundle we have given, it is also clear that

$$\Psi_{\mathcal{W}^*}(L, \nabla^{\rm ab}) = \Big(\Psi_{\mathcal{W}}(L^\vee, \nabla^{{\rm ab},\vee})\Big)^\vee \tag{16.49}$$

where $\vee$ refers to the dual bundle. The reason is that $\mathcal{W} \to \mathcal{W}^*$ just takes flags to dual flags and takes a transpose of the transition functions. But this means that

$$\Psi_{\mathcal{W}}(L, \nabla^{\rm ab})^\vee = \Psi_{\mathcal{W}}(L^\vee, \tilde{\nabla}^{{\rm ab},\vee}) \tag{16.50}$$

so taking the dual bundle should give coordinates $1/\tilde{\mathcal{Y}}_\gamma$. Therefore, if one can compute independently the relation between coordinates for a bundle and its dual one has computed the BPS spectrum generator!

17. Lecture 7, Wednesday, October 10: Relation to Fock-Goncharov Coordinates

In this lecture we discuss the relation of the Darboux coordinates with the coordinate systems on moduli spaces of local systems introduced by Fock and Goncharov. This relation opens the door to connections to other nice aspects of mathematics and physics, for examples, cluster algebra theory and higher Teichmüller theory.

Fock and Goncharov associate coordinates to $\mathcal{M}_F(C, GL(K), \{\mathcal{O}_n\})$ using the data of an ideal triangulation, for $K = 2$, and an m-triangulation, for $K > 2$.

17.1 Line decompositions of the space of flat sections

♣This section might be out of place ♣

Let $D \subset C$ be a domain on which the covering $\Sigma \to C$ has been trivialized.

Let $\mathcal{E}(D)$ be the vector space of flat sections of the flat $\mathfrak{g} = sl(K, \mathbb{C})$ connection ∇, that is, solutions to

$$\nabla\psi = 0 \tag{17.1}$$

on the domain $D \subset C$. This is a K-dimensional vector space. Indeed, for any $z_0 \in D$, the initial condition $\psi(z_0)$ determines the section ψ uniquely.

Suppose we have a spectral network $\mathcal{W} \subset C$. Then we let $\mathcal{R}^\alpha$ be the connected components of the complement $C - \mathcal{W}$:

$$C - \mathcal{W} = \amalg_\alpha \mathcal{R}^\alpha \tag{17.2}$$

In each region $\mathcal{R}^\alpha$ we have an isomorphism

$$E|_{\mathcal{R}} \cong \oplus_{i=1}^K L_i \tag{17.3}$$

and a corresponding decomposition of the space of flat sections:

$$\mathfrak{L}_i^{\mathcal{R}} := \{\psi \in \mathcal{E} \mid \psi(z) \in L_i|_z \ \forall z \in \mathcal{R}\} \subset \mathcal{E}. \tag{17.4}$$

Note that for $\psi \in \mathfrak{L}_i^\alpha$ if $z \in \mathcal{R}^\beta$ with $\beta \neq \alpha$ in general $\psi(z) \notin L_j|_z$.

17.2 Fock-Goncharov coordinates for $K = 2$

For $K = 2$ Fock and Goncharov define a coordinate chart and coordinate system on $\mathcal{M}_F(C, SL(2, \mathbb{C}), \mathcal{O}_n)$ associated to an ideal triangulation of C as follows.

Each edge of the triangulation sits in a rectangle, as shown in Figure

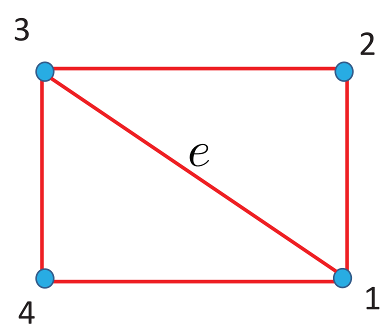

Figure 48: A rectangle in an ideal triangulation of C

Now, let ψ_i be distinguished flat sections defined by the lines at the singular points $\mathfrak{s}_1, \ldots, \mathfrak{s}_4$. [40] Or course the ψ_i are only defined up to scale so they should be thought of as

[40] In degenerate situations these points are sometimes identified.

defining *lines* in the two-dimensional vector space $\mathcal{E}(\mathcal{R})$ of flat sections over the rectangle - cuts.

Four lines in a two-dimensional vector space, i.e. four points on $\mathbb{C}P^1$, have a cross ratio. This is the FG coordinate associated to the edge e. Call it r_e.

In order to write out a concrete formula for this coordinate let us trivialize

$$\Lambda^2 E \cong C \times \mathbb{C} \tag{17.5}$$

by choosing a volume form on the fibers vol(E). Then define

$$(\psi_1(z), \psi_2(z)) := \frac{\psi_1(z) \wedge \psi_2(z)}{\mathrm{vol}\,(E)(z)} \tag{17.6}$$

For ψ_1, ψ_2 flat sections this will be constant in z so we just denote it by (ψ_1, ψ_2). Of course, these are only defined up to scale.

To be precise, for the edge e in Figure 48 we have

$$r_e^T := -\frac{(\psi_1, \psi_2)(\psi_3, \psi_4)}{\psi_4, \psi_1)(\psi_2, \psi_3)} \tag{17.7}$$

Now, when we consider a flipped triangulation, as in 49.

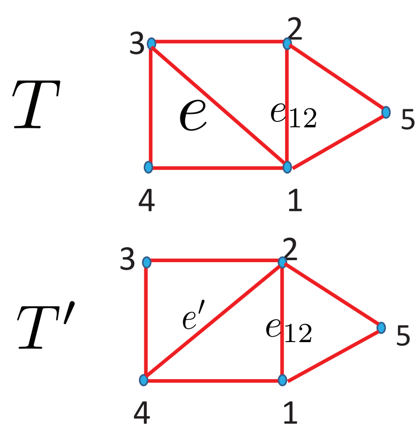

Figure 49: Flipping a triangulation.

Now, 5 points in $\mathbb{C}P^1$ have only two independent cross ratios. So there must be a linear relation. Now that

$$r_{e_{12}}^{T'} = -\frac{(15)(24)}{(52)(41)} \qquad r_{e_{12}}^{T} = -\frac{(15)(23)}{(52)(31)} \tag{17.8}$$

so

$$\frac{r_{e_{12}}^{T'}}{r_{e_{12}}^{T}} = \frac{(24)(31)}{(41)(23)} \tag{17.9}$$

On the other hand,

$$1 + r_e^T = \frac{(23)(41) + (34)(21)}{(41)(23)} \tag{17.10}$$

Now we recall a very useful relation: Any three vectors v_1, v_2, v_3 in a two-dimensional vector space must satisfy the relation:

$$(v_1 \wedge v_2)v_3 + (v_3 \wedge v_1)v_2 + (v_2 \wedge v_3)v_1 = 0 \tag{17.11}$$

and thereby conclude that

$$\begin{aligned} r_{e'}^{T'} &= \frac{1}{r_e^T} \\ r_{e_{12}}^{T'} &= r_{e_{12}}^{T}(1 + r_e^T) \end{aligned} \tag{17.12}$$

Indeed, Fock and Goncharov associate to each ideal triangulation $\mathcal{T}$ a coordinate chart $\mathcal{U}(\mathcal{T})$ on the moduli space of flat $SL(2, \mathbb{C})$ connections on C with flag structure at the punctures. To each edge e of an ideal triangulation they associate a coordinate r_e as above.

17.3 Relation of Darboux Coordinates to Fock-Goncharov coordinates for $K = 2$

In order to relate the Darboux coordinates to the Fock-Goncharov coordinates we consider a WKB spectral network for $K = 2$ and use the dual triangulation.

Fock and Goncharov choose monodromy invariant flags as part of their input data.

By contrast, a WKB triangulation picks out a distinguished flag at the punctures, given by "small flat sections." To explain this, suppose for simplicity that we have a regular singular point. Then we can solve for the flat sections in the neighborhood of the singular point in an expansion around $z = 0$. We find a two-dimensional vector space spanned by

$$\psi^{(1)} = z^{-R\zeta^{-1}\mathfrak{m}+\alpha}\bar{z}^{-R\zeta\bar{\mathfrak{m}}-\alpha}\begin{pmatrix}1\\0\end{pmatrix} + \cdots \tag{17.13}$$

$$\psi^{(2)} = z^{R\zeta^{-1}\mathfrak{m}-\alpha}\bar{z}^{R\zeta\bar{\mathfrak{m}}\alpha}\begin{pmatrix}1\\0\end{pmatrix} + \cdots \tag{17.14}$$

If we consider the growth/decay of these sections along a WKB path spiraling into the singular point then one section grows and one decays. The one which decays is the "small flat section." It is only determined up to scale, so it determines a distinguished flag at the puncture.

Let us now consider a piece of the ideal triangulation dual to a K=2 Spectral network shown in Figure 50.

Let ψ_a, $a = 1, 2, 3, 4$ be small flat sections defined near the singular points $\mathfrak{s}_a$. Again, they are only determined up to scale.

Note that all these sections can be continued, in a single-valued way, into the rectangle minus the cuts.

Now consider a closed path γ around the two branch points. We can write it as

$$\gamma = cl(\gamma_{12}(z) + \tilde{\gamma}_{21}(z)) \tag{17.15}$$

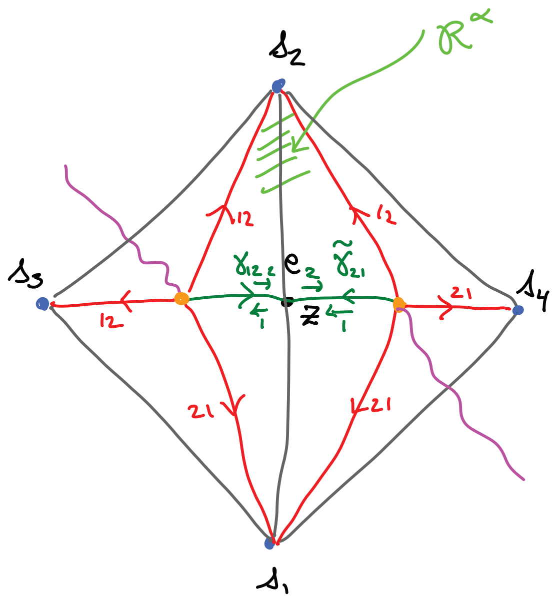

Figure 50: Two triangles in the triangulation dual to a spectral network. Fock and Goncharov associate a coordinate to the edge e. The spectral network technique associates the same coordinate to the amalgamation of the open parallel transports $\mathcal{Y}_{\gamma_{12}(z)}$ and $\mathcal{Y}_{\tilde{\gamma}_{21}(z)}$.

Here $\gamma_{12}(z)$ begins on $z^{(1)}$ and runs back to the ramification point above $\mathfrak{b}_L$ and returns on sheet 2 to $z^{(2)}$ while $\tilde{\gamma}_{21}(z)$ begins on $z^{(2)}$ and runs back to the ramification point above $\mathfrak{b}_R$ and returns on sheet 1 to $z^{(1)}$.

Now the main point is that the parallel transport operators $\mathcal{Y}_{\gamma_{12}(z)}$ and $\mathcal{Y}_{\tilde{\gamma}_{21}(z)}$ can be written explicitly in terms of flat sections as follows:

$$\mathcal{Y}_{\gamma_{12}(z)} := \frac{(\psi_1, \psi_3)}{(\psi_2, \psi_1)(\psi_2, \psi_3)} \psi_2(z) \otimes \psi_2(z) \tag{17.16}$$

where we define $\psi \otimes \chi \in End(E_z)$ by the operator which acts on $v \in E_z$ by

$$v \mapsto (v, \psi)\chi(z) \tag{17.17}$$

To prove (17.18) note that the operator must take $\psi_2 \to 0$ and ψ_1 to a multiple of ψ_2. It also must be independent of scale of ψ_1, ψ_2. This leaves an undetermined constant, which may be fixed by WKB methods using $\zeta \to 0$ asymptotics.

♣Is the invocation of WKB here a cheat? ♣

Similarly, we have

$$\mathcal{Y}_{\tilde{\gamma}_{21}(z)} := \frac{(\psi_2, \psi_4)}{(\psi_1, \psi_2)(\psi_1, \psi_4)} \psi_1(z) \otimes \psi_1(z) \tag{17.18}$$

Now $\mathcal{Y}_{\gamma_{12}(z)} \mathcal{Y}_{\tilde{\gamma}_{21}(z)}$ is an element of $\mathrm{Hom}(L_1|_z, L_1|_z)$, and a linear map of a complex line to itself is canonically a complex number. Indeed we have

$$\mathcal{Y}_{\gamma_{12}(z)} \mathcal{Y}_{\tilde{\gamma}_{21}(z)} = \mathcal{Y}_\gamma \tag{17.19}$$

On the other hand, a direct computation gives

$$\mathcal{Y}_{\gamma_{12}(z)} \mathcal{Y}_{\tilde{\gamma}_{21}(z)} = \frac{(\psi_1, \psi_3)(\psi_2, \psi_4)}{(\psi_4, \psi_1)(\psi_3, \psi_2)} \frac{\psi_2(z) \otimes \psi_1(z)}{(\psi_2, \psi_1)} \tag{17.20}$$

Now $\frac{\psi_2(z) \otimes \psi_1(z)}{(\psi_2, \psi_1)}$ is just the identity in $\mathrm{Hom}(L_1|_z, L_1|_z)$ and hence we identify:

$$\mathcal{Y}_\gamma = \frac{(\psi_1, \psi_3)(\psi_2, \psi_4)}{(\psi_4, \psi_1)(\psi_3, \psi_2)} \tag{17.21}$$

But this is essentially the Fock Goncharov coordinate

What we have shown is that in the dual spectral network there is a correspondence

$$e \to \gamma_e \in \Gamma \tag{17.22}$$

such that

$$r_e = \mathcal{Y}_{\gamma_e} \tag{17.23}$$

(Changing slightly the definition of r_e). Moreover, one can show that the cycles γ_e form a basis for Γ. So, in this case we know what the coordinate charts $\mathcal{U}_{\mathcal{W}}$ are.

Remark: We see an interesting idea here. Given two triangles with a branch point in each one we can glue the open parallel transports to get a coordinate $\mathcal{Y}_\gamma$ for a closed cycle. On the other hand, a single triangle with a single branch point was the triangulation dual to the "trivial" AD_1 theory with spectral curve $\lambda^2 = z(dz)^2$. This suggests there is a procedure of *amalgamation of theories*. A very similar kind of gluing was used by [58] to glue together three-dimensional theories associated with tetrahedra to get theories associated with three-manifolds. It would be nice to understand the physics of this better.

17.3.1 Computing parallel transport in Fock-Goncharov Coordinates

The fact that these *are* coordinates is nicely captured by the traffic rules for computing

$$\mathrm{Tr}_2 P \exp\left(\oint_{\wp} \nabla\right) \tag{17.24}$$

The rule - which can be easily derived by thinking about bases of flat sections and applying the identity (17.11) is the following: [41]

♣There must be a derivation in in FG somewhere. Put in the ref. ♣

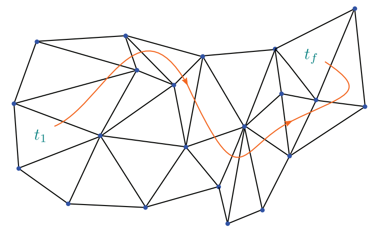

Figure 51: Computing parallel transport in Fock-Goncharov coordinates

Consider the path $\wp$ on the triangulated surface as in Figure 51

Then we multiply 2×2 matrices for each left and right turn within a triangle and for each edge we cross using the rules that left and right turns, as shown in Figure 52 contribute

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{17.25}$$

while transport across an edge E from face F to F' is given by the matrix

$$M_E = \begin{pmatrix} \sqrt{r_E} & 0 \\ 0 & 1/\sqrt{r_E} \end{pmatrix} \tag{17.26}$$

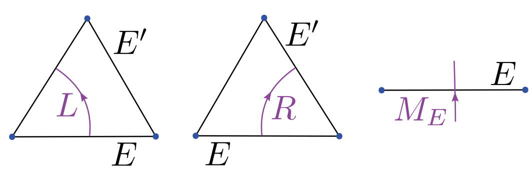

Figure 52: Traffic rules for computing parallel transport in Fock-Goncharov coordinates

Example: As an example, let us compute what will be interpreted as the Wilson line in the fundamental representation in pure $SU(2)$ gauge theory.

Now, the triangulation for generic ϑ looks like Figure 53.

There are two edges and we identify $r_{E_1} = \mathcal{Y}_{\gamma_1}$ and $r_{E_2} = \mathcal{Y}_{\gamma_2}$ with $\langle \gamma_1, \gamma_2 \rangle = +2$. The $SU(2)$ Wilson line is the line around the cylinder separating the two singular points. The traffic rules tell us to compute

$$\mathrm{Tr} M_{E_1} L M_{E_2} R = \sqrt{r_{E_1} r_{E_2}} + \frac{1}{\sqrt{r_{E_1} r_{E_2}}} + \sqrt{\frac{r_{E_1}}{r_{E_2}}} \tag{17.27}$$

When combined with $r_E = \mathcal{Y}_{\gamma_E}$ and the construction of $\mathcal{Y}_\gamma$ from the TBA below this gives the full nonperturbative expansion of the Wilson loop expectation value of for pure $SU(2)$ $\mathcal{N} = 2$ gauge theory.

[41] For a careful derivation see appendix A of [94] and appendix ??? or [96].

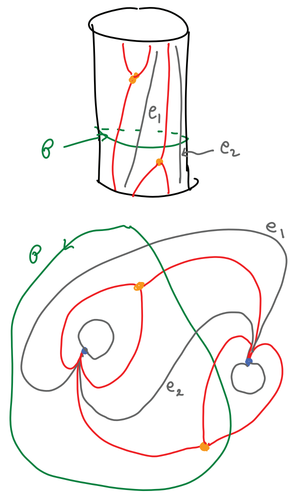

Figure 53: Triangulations given by the SU(2) spectral network.

17.3.2 Example: The AD_3 Moduli Space

Let us return to the AD_3 theory where we discussed the pentagon identity. There is a branch point at infinity and 5 Stokes matrices. The coordinates of the Stokes matrices are holomorphic functions on the moduli space $\mathcal{M}_F(\mathbb{C}, SL(2,\mathbb{C}), \mathcal{O}_\infty)$. The monodromy equation is

$$1 = \begin{pmatrix} 1 & \mathcal{Y}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mathcal{Y}_4 & 1 \end{pmatrix} \begin{pmatrix} 1 & \mathcal{Y}_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mathcal{Y}_5 & 1 \end{pmatrix} \begin{pmatrix} 1 & \mathcal{Y}_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{17.28}$$

The first 5 matrices are Stokes matrices and the last is the formal monodromy. The coordinates were chosen in the way shown so that the equations they satisfy are

$$\mathcal{Y}_{i-1}\mathcal{Y}_{i+1} = 1 + \mathcal{Y}_i \tag{17.29}$$

where we understand the index to be modulo 5.

There are 5 coordinate patches defined by the complement of $\mathcal{Y}_i = 0$. The coordinates on this patch are $\mathcal{Y}_i, \mathcal{Y}_{i+1}$ Note that if we assign coordinates to the vertices of a pentagon then neighboring vertices cannot simultaneously vanish. ♣??? ♣ ♣Nice. but so what? ♣

The 5 coordinate patches correspond to the 5 triangulations of the pentagon.

FIGURES OF DIFFERENT TRIANGULATIONS WITH BRANCH POINTS DETERMINING DIFFERENT BASES OF CYCLES

In one triangulation we can identify

$$\mathcal{Y}_1 = \mathcal{Y}_{\gamma_1} \qquad \mathcal{Y}_2 = \mathcal{Y}_{\gamma_2} \tag{17.30}$$

Then in the next patch we have coordinates $\mathcal{Y}_2, \mathcal{Y}_3$ corresponding to a flipped triangulation. The new coordinates are related to the old by a cluster transformation

$$\mathcal{Y}_3 = (1 + \mathcal{Y}_2)\mathcal{Y}_1^{-1} \tag{17.31}$$

The sequence of cluster transformations defined by (17.29) is 5-fold periodic.

♣Write out the whole 5 term orbit. ♣

♣There are some nice things to say about this being the moduli space of 5 points on a sphere which are not colinear. ♣

17.3.3 The spectrum generator for A_1 theories of class S

♣ Explain that $\vartheta \to \vartheta + \pi$ leads to a pop transformation at each vertex. So one can write directly the spectrum generator in terms of the combinatorics of the triangulation. ♣

17.4 A nontrivial class of $K > 2$ examples

The technique of spectral networks really comes into its own with higher rank theories. In this subsection we will discuss a class of higher rank examples, i.e. theories of class S with $\mathfrak{g} = su(K)$, for $K > 2$ by using spectral networks in a very nice region of the Coulomb branch.

17.4.1 Lifted theories and lifted locus

We will discuss *lifted theories*

The idea is very simple: Given a homomorphism $\rho : G \to G'$ we can mapp G Higgs bundles to G' Higgs bundles, or solutions of G Hitchin equations to G' Hitchin equations.

Let us apply this to the homomorphism $\rho : SU(2) \to SU(K)$ given by the K-dimensional irreducible representation of $SU(2)$.

If we start with an $SU(2)$ spectral curve:

$$\det(\lambda 1_{2\times 2} - \varphi) = 0 \tag{17.32}$$

which can be written as

$$\lambda^2 + \phi_2 = 0 \tag{17.33}$$

Then, the equation for the new "lifted" spectral curve will be

$$\det(\lambda 1_{K\times K} - \rho(\varphi)) = 0 \tag{17.34}$$

Now, at generic points, we can diagonalize

$$\varphi \sim \begin{pmatrix} \sqrt{-\phi_2} & 0 \\ 0 & -\sqrt{-\phi_2} \end{pmatrix}. \tag{17.35}$$

So, in the K-dimensional representation the eigenvalues of $\rho(\varphi)$ are

$$(K-1)\sqrt{-\phi_2},\ (K-3)\sqrt{-\phi_2},\ \dots,\ (3-K)\sqrt{-\phi_2}, (1-K)\sqrt{-\phi_2}, \tag{17.36}$$

So the lifted curve is the equation:

$$(\lambda^2 + (K-1)^2\phi_2)(\lambda^2 + (K-3)^2\phi_2)\cdots = 0 \tag{17.37}$$

To be explicit the first few polynomials are:

$$\begin{aligned}
K=2: &\qquad \lambda^2+\phi_2, \\
K=3: &\qquad \lambda^3+4\phi_2\lambda, \\
K=4: &\qquad \lambda^4+10\phi_2\lambda^2+9\phi_2^2, \\
K=5: &\qquad \lambda^5+20\phi_2\lambda^3+64\phi_2^2\lambda, \\
K=6: &\qquad \lambda^6+35\phi_2\lambda^4+259\phi_2^2\lambda^2+225\phi_2^3, \\
K=7: &\qquad \lambda^7+56\phi_2\lambda^5+784\phi_2^2\lambda^3+2304\phi_2^3\lambda, \\
K=8: &\qquad \lambda^8+84\phi_2\lambda^6+1974\phi_2^2\lambda^4+12916\phi_2^3\lambda^2+11025\phi_2^4.
\end{aligned} \tag{17.38}$$

This spectral curve is of course singular, but small perturbations of it will make it nonsingular.

We identify this with the spectral curve of an $SU(K)$ theory of class S, in a special region of its Coulomb branch.

Now, if the A1 theory gave us a spectral network like

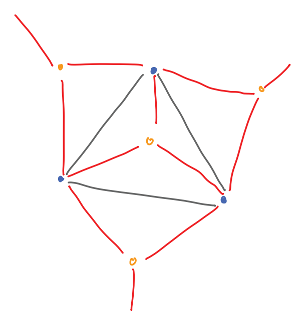

Figure 54: A piece of a triangulation associated with an A_1 theory of class S.

then, near the lift locus each branch point splits into $\frac{1}{2}K(K-1)$ branch points and the lines of the spectral network split into tightly bundled ”cables” as in Figure 55:

Now, we have argued that, in some sense, one could construct arbitrary A_1 theories of class S from a “gluing” or “amalgamation” procedure.

This suggests that we should focus on level K lifts of the AD_N theories which have

$$\phi_2 = -z^N (dz)^2 \tag{17.39}$$

Recall that these had a single irregular singular point at $z=\infty$ with $(N+2)$ marked points on the circle at infinity.

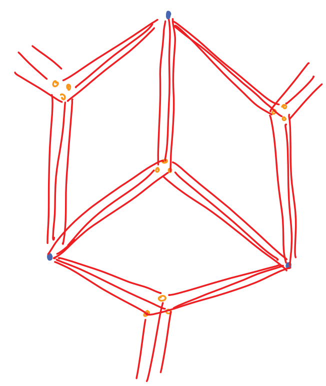

Figure 55: The spectral network of the level K lift of an A_1 theory (near the lift locus) is supported very near the spectral network of the parent A_1 theory. The single branch point (within each triangle) splits into groups of $\frac{1}{2}K(K-1)$ branch points and the S-walls are grouped into "cables" with support close to the original S-walls. ♣ Revise figure to write "cables" and "clusters" ♣

The moduli space of the lifted theory $L_K(AD_N)$ is the moduli space of $K \times K$ Stokes matrices:

$$\prod_{n=1}^{N+2} S_n = \mathrm{Diag}\{\mu_1, \dots, \mu_K\} \qquad N \qquad \textit{even} \tag{17.40}$$

$$\prod_{n=1}^{N+2} S_n = \mathrm{AntiDiag}\{\mu_1, \dots, \mu_K\} \qquad N \qquad \textit{odd.} \tag{17.41}$$

By analyzing these equations it is not difficult to show that for the charge lattice Γ we have

$$\mathrm{rank}(\Gamma) = N\frac{(K-1)(K-2)}{2} + (N-1)(K-1). \tag{17.42}$$

In the triangulation of the AD_N theory there are N triangles and $(N-1)$ internal edges. This suggests that

1. For each triangle we get $\frac{1}{2}(K-1)(K-2)$ coordinates.
2. For each internal edge we get $(K-1)$ coordinates.

We can learn about these new "triangle coordinates" by focussing on $N = 1$, the first AD theory.

So we are examining the moduli space of dimension:

$$\dim \mathcal{M}(L_K(AD_1)) = \frac{1}{2}(K-1)(K-2) \tag{17.43}$$

17.4.2 Minimal spectral networks for the level K lift of AD_1

In a region of the lifted locus of the Coulomb branch of $L_K(AD_1)$ we will have a triangular array of $\frac{1}{2}K(K-1)$ branch points as shown in Figure 56.

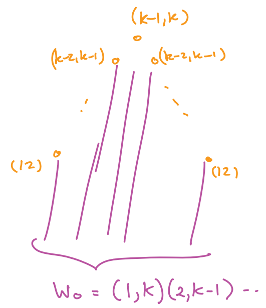

Figure 56: Triangulations given by the SU(2) spectral network.

We will assume that the branch points can be put in 1-1 correspondence with points (x, y, z) in an m-triangle for $m = K-2$. We will label them as $\mathfrak{b}^{x,y,z}$ where $x+y+z = K-2$. We follow Fock and Goncharov and make the definition: [42]

Definition (Fock-Goncharov): An m-triangle $T(m)$ is the set of points $(x, y, z) \in \mathbb{Z}_+^3$ with $x + y + z = m$.

An example for $m = 3$ is shown in Figure 57:

Now we introduce the idea of a *minimal spectral network*. That is a spectral network with the above triangular array of branch points and with no secondary lines. Actually, they come in two essentially different kinds which we call *Yin* and *Yang* spectral networks. Examples for $K = 3$ are shown in Figure 58:

Now one can show that there is a very natural homology basis $\gamma^{\mathfrak{v}}$ labeled by points $\mathfrak{v} = (x, y, z) \in T(K-3)$, i.e. $x + y + z = K - 3$. These are shown in Figure 59:

We will associate coordinates $\mathcal{Y}_{\gamma^{\mathfrak{v}}}$ to this basis of cycles.

17.4.3 Flags

Now suppose we have a solution to Hitchin's equations and

$$\mathcal{A} = \frac{R}{\zeta}\varphi + A + R\zeta\bar{\varphi} \tag{17.44}$$

[42] We have not produced explicit regions of the Coulomb branch for $K > 9$ and it remains a nice open problem to show explicitly how to perturb the lifted spectral curve, for all K so that the branch points can be labeled as we have done here and such that there is an (essentially) minimal spectral network.

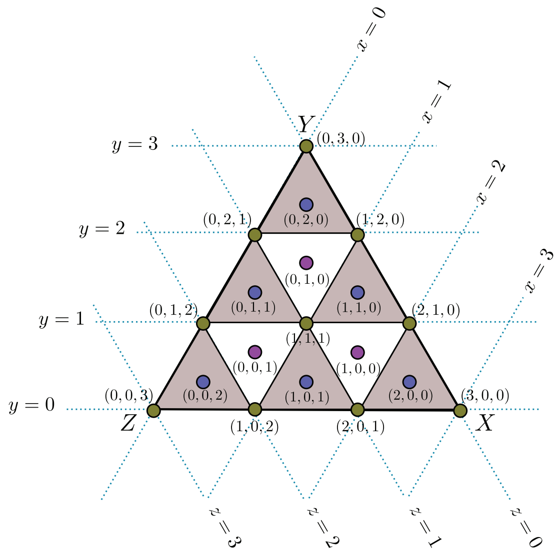

Figure 57: A 3-triangle.

Now let $\mathcal{E}$ be the vector space of global flat sections on the complement of the cuts.

The AD_1 theory has 3 marked points on the circle at infinity.

The level K lift $L_K(AD_1)$ still has the same three points on the circle at infinity.

We can use the asymptotics of the flat sections as $z \to \infty$ along the three cables shown in Figure 60:

For example, for $z \to \infty$ along the A-cable we have

$$\mathrm{Re} \int_{z_0}^{z} \frac{1}{\zeta} \lambda_1 \gg \mathrm{Re} \int_{z_0}^{z} \frac{1}{\zeta} \lambda_2 \gg \cdots \gg \mathrm{Re} \int_{z_0}^{z} \frac{1}{\zeta} \lambda_K, \tag{17.45}$$

and hence we can define a flag $A_\bullet$ in $\mathcal{E}$ via

$$\lim_{z \to \infty} \exp \left[2\, \mathrm{Re} \int_{z_0}^{z} \frac{R}{\zeta} \lambda_n \right] \psi(z) < \infty \Leftrightarrow s \in A_n \tag{17.46}$$

for $z \to \infty$ along the a-cable. Here A_n is n-dimensional and we define our flag so that

$$0 = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_{K-1} \subset A_K = \mathcal{E} \tag{17.47}$$

Similarly, the B and C cables define flags $B_\bullet$ and $C_\bullet$.

The relative positions of three flags $A_\bullet, B_\bullet$ and $C_\bullet$ in $\mathcal{E}$ completely determines the Stokes data. Indeed, one can write down very explicit Stokes matrices. This was done in [68]. (See also Appendix A of [102].)

In this way we identify the Seiberg-Witten moduli space $\mathcal{M}(L_K(AD_1))$ with the moduli space of three flags in a K-dimensional vector space.

17.4.4 Fock and Goncharov coordinates on moduli space of 3 flags

The moduli space of three flags $A_\bullet, B_\bullet, C_\bullet$ in a K-dimensional vector space V is

$$GL(L) \backslash (GL(K)/B)^3 \tag{17.48}$$

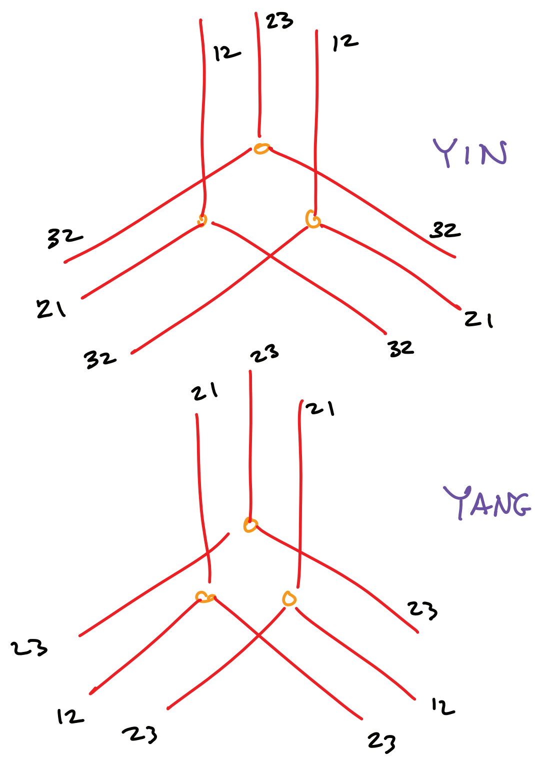

Figure 58: Yin and Yang minimal spectral networks for $K = 3$. Note that changing $\vartheta \to \vartheta + \pi$ exchanges Yin and Yang networks.

where B is the Borel subgroup of upper triangulat matrices. A nice check is that it has dimension $\frac{1}{2}(K-1)(K-2)$.

Fock and Goncharov defined some nice coordinates on this space.

First, define $A^x := A_{K-x}$ so that the superscript gives the codimension.

Note that we can define a collection of $\frac{1}{2}(K+1)K$ lines:

$$\mathfrak{L}^{x,y,z} := A^x \cap B^y \cap C^z, \qquad x+y+z = K-1, \tag{17.49}$$

where (x,y,z) is a lattice point in a $(K-1)$-triangle. Similarly, we define $\frac{1}{2}K(K-1)$ planes

$$\mathfrak{P}^{x,y,z} := A^x \cap B^y \cap C^z, \qquad x+y+z = K-2, \tag{17.50}$$

associated to points in a $(K-2)$-triangle, and $\frac{1}{2}(K-1)(K-2)$ spaces

$$\mathfrak{V}^{x,y,z} := A^x \cap B^y \cap C^z, \qquad x+y+z = K-3, \tag{17.51}$$

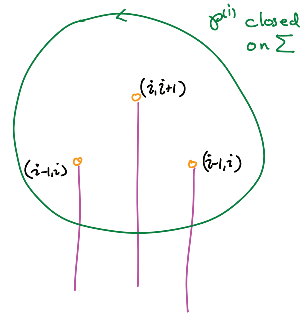

Figure 59: Yin and Yang minimal spectral networks for $K = 3$.

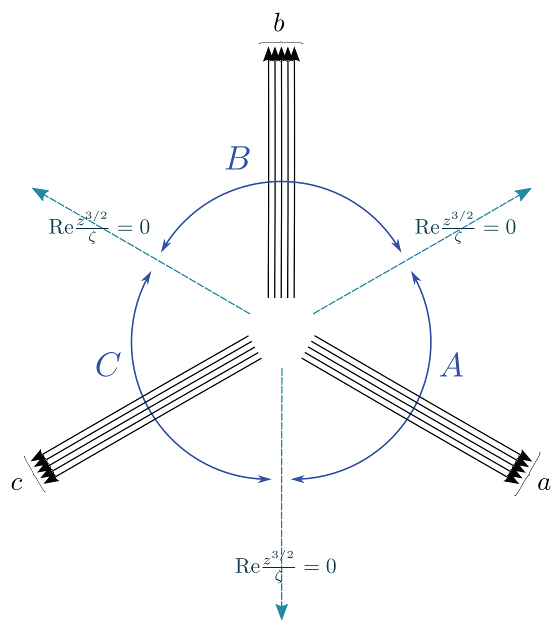

Figure 60: Yin and Yang minimal spectral networks for $K = 3$.

associated to points in a $(K-3)$-triangle.

These have incidence relations beautifully summarized by the $(K-1)$-triangle:

1. The upwards pointing shaded triangles correspond to points in $T(K-2)$. These are planes and the three lines at the vertices of the shaded triangle lie in the plane.

2. The downwards pointing unshaded triangles correspond to points in $T(K-3)$ and

correspond to the spaces. The three shaded triangles abutting the down pointing triangle are three planes in the common space.

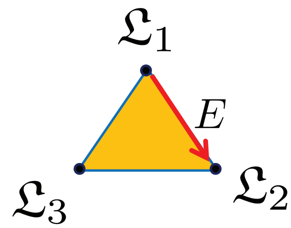

Figure 61: An oriented edge abuts a unique shaded triangle. Using the three lines associated to the vertices of the triangle we can define a canonical element $\mathrm{Hom}(\mathfrak{L}_1, \mathfrak{L}_2)$.

Next we define a *canonical homomorphism* of lines separated by an edge E of a shaded triangle as shown in Figure 61.

Definition: Given $v_1 \in \mathfrak{L}_1$ we define $x_E(v_1)$ to be the unique vector $v_2 \in \mathfrak{L}_2$ such that

$$\begin{cases} v_1 + v_2 \in \mathfrak{L}_3 & \text{clockwise} \\ v_1 - v_2 \in \mathfrak{L}_3 & \text{counterclockwise} \end{cases} \tag{17.52}$$

where the cases refer to whether E is oriented clockwise or counterclockwise around the shaded triangle.

Now, we associate a coordinate $r^{\mathfrak{v}}$ for $\mathfrak{v} = (x, y, z) \in T(K-3)$. We associate such a coordinate to each down pointing triangle. We do this by taking the three edges of the *unshaded* downwards pointing triangle and composing them:

$$x_{E_1} x_{E_2} x_{E_3} \in \mathrm{Hom}(\mathfrak{L}, \mathfrak{L}) \tag{17.53}$$

As before we can canonically identify $\mathrm{Hom}(\mathfrak{L}, \mathfrak{L}) \cong \mathbb{C}$ and so we define

$$r^{x,y,z} := x_{E_1} x_{E_2} x_{E_3} \tag{17.54}$$

Remark: These are not quite the same coordinates as those actually defined by Fock and Goncharov and in fact they are related somewhat nontrivially to the Fock and Goncharov coordinates. The relation involves considering the dual flags. However, this definition is better suited to our purposes.

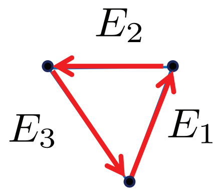

Figure 62: Composition of three canonical hom's counterclockwise around an unshaded, downwards-pointing triangle defines a Fock-Goncharov coordinate.

17.4.5 Darboux = Fock-Goncharov

Now, we have seen that for $L_K(AD_1)$ there is a set of three flags $A_\bullet, B_\bullet, C_\bullet$ in the vector space of flat sections $\mathcal{E}$.

We also noted that in each connected component $\mathcal{R}^\alpha$ of the complement of a spectral network there is a canonical decomposition

$$\mathcal{E} = \oplus_i \mathfrak{L}_i^{\mathcal{R}^\alpha} \tag{17.55}$$

It turns out that, for each region $\mathcal{R}^\alpha$ the lines $\mathfrak{L}_i^{\mathcal{R}^\alpha}$ indeed coincide with the lines $\mathfrak{L}^{x,y,z}$ in the $(K-1)$- triangle of lines defined by the three flags. This is nontrivial to show, but the argument is given in [102]

The lines in the noncompact regions form a configuration which Fock and Goncharov called *snakes*. The example of $K = 3$ is shown in

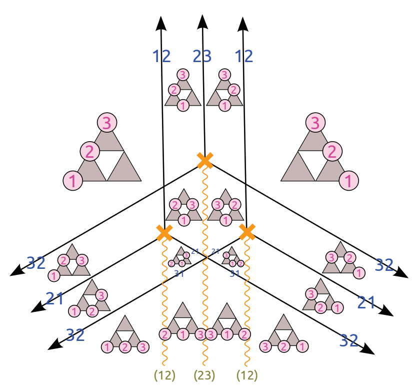

Figure 63: Showing the lines given in the spectral network decomposition of the space of flat sections for $K = 3$.

Now, the next step is to look at an S-wall. Consider an S-wall of type ij separating a region $\mathcal{R}^\beta$ on the left from $\mathcal{R}^\alpha$ on the right. Choose a point z on the S-wall and the soliton cycle $\gamma_{ij}(z)$ used in constructing the nonabelianization map. We found there a linear transformation

$$\mathcal{Y}_{\gamma_{ij}(z)} : \mathfrak{L}_i^{\mathcal{R}^\alpha} \to \mathfrak{L}_j^{\mathcal{R}^\beta} \tag{17.56}$$

and the key insight is that $\mathfrak{L}_i^{\mathcal{R}^\alpha}$ and $\mathfrak{L}_j^{\mathcal{R}^\beta}$ also sit on the triangle of lines separated by a single edge E and, moreover,

$$\mathcal{Y}_{\gamma_{ij}(z)} = x_E. \tag{17.57}$$

Now, on the one hand, composing the x_E's around a downwards pointing triangle gives the Fock-Goncharov coordinate $r^{x,y,z}$ and on the other it turns out that the composition of the three $\mathcal{Y}_{\gamma_{ij}(z)}$'s gives a closed cycle and in fact

$$\mathcal{Y}_{\gamma^{\mathfrak{v}}} = r^{\mathfrak{v}} \tag{17.58}$$

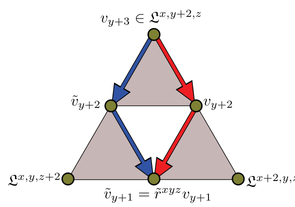

Figure 64: The basic move taking one snake to another.

17.4.6 Spectrum Generator

Definition: A snake is a path of length K from a vertex to the opposite side.

Using canonical homs it determines a projective basis of $\mathcal{E}$.

The transformation between basis from the basic move between snakes is of the form

$$M(x, y, z) = \begin{pmatrix} r^{x,y,z} 1_{(y+1)\times(y+1)} & 0 \\ 0 & 1 \end{pmatrix} (1 + e_{y+2,y+3}) \tag{17.59}$$

The product along the path of snakes gives a matrix $M^{BA\to BC}(r^{\mathfrak{v}})$.

Using the remark of Section 16.7 above to compute the spectrum generator we need to compute the relation between $r^{\mathfrak{v}}$ and $\check{r}^{\mathfrak{v}}$ defined by the corresponding matrix $\check{M}^{BA\to BC}(\check{r}^{\mathfrak{v}})$ for the dual flags. But one can show that

$$\check{M}^{BA\to BC}(\check{r}^{\mathfrak{v}}) = \beta \left(M^{BA\to BC}(r^{\mathfrak{v}})\right)^{tr,-1} \alpha \tag{17.60}$$

for antidiagonal matrices α, β. This equation determines both the matrices α and β as well as the relation of $\check{r}^{\mathfrak{v}}$ to $r^{\mathfrak{v}}$, and hence the spectrum generator.

The factorization of the resulting spectrum generator is not known, and it would be very interesting to know it.

17.5 Amalgamation

Now we can try to glue our triangles together, just the way we did for the A_1 case.

There are two cases depending on whether we try to glue Yin triangles to Yang triangles or not.

In either case we can produce a set of $(K-1)$ edge coordinates from expressions of the form

$$\mathcal{Y}_{\gamma_{ij}^L}\mathcal{Y}_{\gamma_{ji}^R} \sim \mathcal{Y}_{\gamma_e} \tag{17.61}$$

along the lines of what we did for $K = 2$ above.

This corresponds nicely to a procedure in cluster algebra theory which Fock and Goncharov called *amalgamation*.

17.6 A General Approach to the BPS Spectrum for class S

Now suppose that we are near the lift locus of a general level K lift of an A_1 theory of class S.

Consider Figure 65 which illustrates three general phenomenon that we encounter when discussing the BPS spectrum of the lifted theory in the neighborhood of the lifted locus:

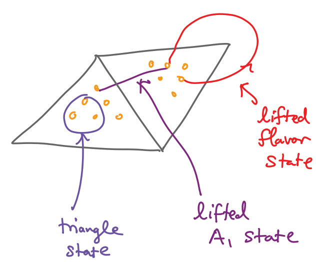

Figure 65: Three kinds of BPS states in a level K lift of an A_1 theory.

1. Each triangle of the A_1 spectral network produces $\frac{1}{2}(K-1)(K-2)$ BPS states analogous to those of the lift of the AD_1 theory.

2. There are also lifted flavor states which join an (ij) branch point to another (ij) branch point. We can count these

$$\sum_{s=1}^{K-1} s(s-1) = \frac{1}{3}K(K-1)(K-2) \tag{17.62}$$

3. There are also lifts of the "edge states" of the A_1 theory. These come from the two-way ij streets where an (ij) branchpoint in one triangle is connected to an ij branch point in the neighboring triangle. We can count these too:

$$\sum_{s=1}^{K} s(K-s) \qquad (Yin, Yin) \tag{17.63}$$

$$\sum_{s=1}^{K} s^2 \qquad (Yin, Yang) \tag{17.64}$$

In general we do not know whether or not there are other BPS states in the lifted theory, even in the lifted locus of the Coulomb branch. It would be great to fill in this gap.

18. Lecture 8a: Thursday, Oct. 11: Integral Equations and WKB Asymptotics

In this section we are going to use the spectral networks and their BPS degeneracies to write down solutions to the Hitchin equations. This procedure will clarify some interesting aspects of WKB asymptotics [43]

18.1 Review of physical motivation

Before launching into the mathematics it is perhaps worthwhile reviewing what the physical picture suggests to us.

The (2,0) theory tells us that, with a partial topological twist, to the data of a simple ADE Lie algebra $\mathfrak{g}$, a Riemann surface C with marked points $\mathfrak{s}_n$, and some data D at the marked points we have a 4d QFT with $\mathcal{N}=2$ superPoincaré symmetry, $S(\mathfrak{g}, C, D)$.

We now consider the Euclidean theory and its verkleinung on a circle S^1 of radius R with periodic boundary conditions for fermions. At energies $E \ll 1/R$ it is described by an effective d=3 sigma model with superPoincaré symmetry $\mathfrak{SP}(\mathbb{R}^{3|8})$ based on maps

$$\Phi : M_3 \to \mathcal{M} \tag{18.1}$$

where the target space, $\mathcal{M}$ is guaranteed by supersymmetry to have a hyperkähler structure.

For theories of class S there is a further identification of $\mathcal{M}$ with the moduli space of solutions to Hitchin's equations on C:

$$F + R^2[\varphi, \bar{\varphi}] = 0, \tag{18.2}$$

$$\bar{\partial}_A \varphi := d\bar{z}\,(\partial_{\bar{z}}\varphi + [A_{\bar{z}}, \varphi]) = 0, \tag{18.3}$$

$$\partial_A \bar{\varphi} := dz\,(\partial_z \bar{\varphi} + [A_z, \bar{\varphi}]) = 0. \tag{18.4}$$

[43] I am reviewing work done with D. Gaiotto and A. Neitzke. The viewpoint taken here however is slightly different from what is in the papers. The improvements are a result of continued discussions with Gaiotto and Neitzke.

Here A is a unitary connection on a complex topologically trivial vector bundle $V \to C$, $\varphi \in \Gamma(\mathcal{K}_C \otimes \mathrm{End}(V))$, and $\bar{\varphi}$ is its hermitian conjugate. The data of the "codimension two defects" D translates into certain singularity conditions on (φ, A) near the marked points.

Moreover, we have discussed surface defects $\mathbb{S}_z$, with $z \in C$ and these induce *line* defects in the three-dimensional sigma model since we can "reduce" a surface defect to a line defect by considering the surface defect on $\ell \times S^1_R$ where $\ell \subset M_3$ is a line. Again supersymmetry gives some strong constraints. The general supersymmetric line defect along ℓ must take the form

$$P \exp\left(\int_\ell \Phi^*(\Theta) + Fermions\right) \tag{18.5}$$

where Θ is a *hyperholomorphic connection on a vector bundle* $\mathcal{V} \to \mathcal{M}$.

The definition of a hyperholomorphic connection is that it is a connection such that the curvature $F = d\Theta + \Theta^2$ is of type $(1,1)$ in all the complex structures of $\mathcal{M}$. This nicely generalizes the instanton equations on four-dimensional hyperkähler manifolds.

In theories of class S, for $\mathbb{S}_z$ one gets the universal bundle over $C \times \mathcal{M}$ and hence, fixing a point $m \in \mathcal{M}$, we expect to be able to construct explicit solutions to Hitchin's equations.

♣This is not evident from anything said here. Is there a quick way to make it plausible? ♣

We are going to build solutions given the data of

1. A Higgs bundle $(V, \bar{\partial}, \varphi)$.
2. A family of WKB spectral networks $\mathcal{W}_\vartheta$, and the corresponding BPS degeneracies μ and ω.

In fact we will define a one-parameter family of solutions depending on a parameter $R > 0$, where R is the radius of the circle used in the verkleinung from four dimensions to three dimensions. [44]

The key new idea is to consider a RH problem in the complex plane where we have been drawing BPS rays. We call this the ζ-plane and we will eventually identify it with the stereographically projected twistor sphere of the Hitchin moduli space.

18.2 Reviewing Higgs bundles

First, let us recall some standard facts about Higgs bundles. (These are nicely summarized in Section 4 of [82].) The Higgs bundle gives us a spectral curve Σ which is a $K:1$ branched cover $\pi : \Sigma \to C$, together with a complex line bundle $\mathcal{L} \to \Sigma$, given by the eigenline of the Higgs field φ:

$$\mathcal{L} = \ker(\pi^*\varphi - \lambda) \tag{18.6}$$

One can compute that $\mathcal{L}$ has nonzero degree:

♣Frenkel-Witten [82] say cokernel. ♣

$$\deg \mathcal{L} = (K^2 - K)(g_C - 1) \tag{18.7}$$

Now let us assume that V is topologically trivial and consider $\pi^*(V)$. There is a pro-

♣Check. What about with punctures? ♣

[44] The procedure is only guaranteed to work for R sufficiently large and with certain restrictions on the behavior of the BPS degeneracies. See 18.7 below.

jection operator to $\mathcal{L}$, given by projecting to a given eigenbundle. Therefore there is a natural connection on $\mathcal{L}$ given by the projected connection. This connection has curvature concentrated at the ramification points.

Example: Suppose

$$\varphi = \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} dz \tag{18.8}$$

so that the spectral curve is just the double-cover $w^2 = z$. The $\pm w$ eigenlines on the w plane are generated by the sections

$$\psi_\pm = \begin{pmatrix} 1 \\ \pm w \end{pmatrix} \tag{18.9}$$

The projected connection $\nabla^{\rm proj} = \Pi_+ d$, where d is the trivial connection on the trivial bundle $\mathbb{C}_w \times \mathbb{C}^2$, is easily computed to be:

$$\nabla^{\rm proj}\psi_+ = \psi_+ \otimes \frac{dw}{2w} \tag{18.10}$$

Therefore, in these local coordinates and in this gauge we have

$$A = \frac{dw}{2w} \tag{18.11}$$

and therefore the holonomy around the ramification point $w = 0$ is $\oint_0 A = i\pi$. So there is a "half-unit of curvature" at the ramification point.

Now, we can make a degree zero line bundle $\mathcal{N} \to \Sigma$ by *choosing* spin structures on C and Σ and tensoring $\mathcal{L}$ with the difference of the spin structures:

$$\mathcal{N} = \mathcal{L} \otimes \pi^*(\mathcal{K}_C^{1/2}) \otimes \mathcal{K}_\Sigma^{-1/2} \tag{18.12}$$

One can eliminate the dependence on the choice of two spin structures by working with twisted local systems. This is what is done in the GMN papers.

The holomorphic line bundle is topologically trivial. It admits a unique (up to overall scale, i.e. one real number) hermitian metric so that the canonically associated Chern connection is *flat*. Let us call this ∇^h. For another approach to describing ∇^h see Section 13.1 of [94].

♣Need to discuss bc's at punctures. ♣

♣Check that these two connections really are the same. Second only applies to $K = 2$. ♣

18.3 A naive attempt to solve the Hitchin equations

The most naive attempt to solve the Hitchin equations would proceed by attempting to choose a gauge in which we diagonalize φ:

$$\varphi \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_K \end{pmatrix} \tag{18.13}$$

Then the commutator term $[\varphi, \bar{\varphi}]$ vanishes so our gauge field is flat. Moreover, since λ is meromorphic $\bar{\partial}\lambda = 0$, and generically the λ_i are all distinct so A should be diagonal in this gauge. Thus, locally we can simply choose a flat diagonal gauge field $A^{\rm diag}$, and diagonalize φ and we have solved the Hitchin equations.

It we try to make this solution more global the monodromy of the sheets of the cover $\pi : \Sigma \to C$ force $A^{\rm diag}$ to be the pushforward of a flat $U(1)$ gauge field α on Σ and hence the flat gauge field (16.8),

$$\mathcal{A} = \frac{R}{\zeta}\varphi + A^{\rm diag} + R\zeta\bar{\varphi} \tag{18.14}$$

is the pushforward of a flat $GL(1, \mathbb{C})$ abelian connection on Σ:

$$\frac{R}{\zeta}\lambda + \alpha + R\zeta\bar{\lambda} \tag{18.15}$$

where α is a flat unitary connection on Σ. This has parallel transport along a chain $a \in \Gamma(z_1, z_2)$ given by

$$\exp\left(\frac{R}{\zeta}\int_a \lambda + \int_a \alpha + R\zeta \int_a \bar{\lambda}\right) \tag{18.16}$$

We would like to pin down the ambiguity in the choice of α. This is done in principle by choosing an actual Higgs bundle. That chooses a point in the torus fiber of the Hitchin fibration. To be more precise about the flat $U(1)$ gauge field we take the unitary connection ∇^h on $\mathcal{N}$ described in equation (18.12) above.

This defines the family (parametrized by ζ, R) of *semiflat connections on* $\mathcal{N}$:

$$\nabla^{\rm sf} = \frac{R}{\zeta}\lambda + \nabla^h + R\zeta\bar{\lambda} \tag{18.17}$$

The parallel transport of this connection along a chain $a \in \Gamma(z_1, z_2)$ is:

$$\mathcal{Y}_a^{\rm sf}(\zeta) := \exp\left(\frac{R}{\zeta}\int_a \lambda + \int_a \nabla^h + R\zeta \int_a \bar{\lambda}\right) \tag{18.18}$$

We will somewhat informally write (18.18) in a unitary frame as

$$\mathcal{Y}_a^{\rm sf}(\zeta) := \exp\left(\frac{R}{\zeta}\int_a \lambda + i\theta_a + R\zeta \int_a \bar{\lambda}\right) \tag{18.19}$$

where θ_a is a collection of real numbers which satisfy

$$\theta_{a+b} - \theta_a - \theta_b = \xi_{a,b} \mod 2\pi \tag{18.20}$$

when the chains $a + b$ are composable. Here $\xi_{a,b}$ is an integer multiple of π and satisfies (mod 2π) the cocycle relation.

♣It can be trivialized by a choice of spin structures... ♣

The basic idea is that the semiflat parallel transports are not a bad approximation to the parallel transports of the true flat connection corresponding, in complex structure ζ, to the solution (φ, A) of the Hitchin system. For example, as $R \to \infty$ we expect that it becomes better and better throughout most of C on regions contracting to the branch points of $\Sigma \to C$. (This was shown, for $K = 2$, with some care in Section 13 of [94].

What we will do is show how the BPS degeneracies allow us systematically to correct this approximate solution to get a true solution of the Hitchin equations.

18.4 Constructing an abelian connection $\nabla^{\rm ab}$ on Σ

Of course φ is not globally diagonalizable and in the neigborhood of branch points it might not even be locally diagonalizable, as the example (18.8) clearly demonstrates.

In the next step we construct a family of flat $GL(1,\mathbb{C})$ connections $\nabla^{\rm ab}(\zeta)$ on the spectral line bundle $\mathcal{L}$, or rather on $\mathcal{N}$. We will try to motivate this somewhat strange construction below.

Recall that the 2d4d BPS rays are defined to be

$$\ell_\gamma := Z_\gamma \cdot \mathbb{R}_- \qquad\qquad \omega(\gamma;\cdot)\neq 0 \tag{18.21}$$

Think of them as rays in the ζ-plane.

We now write a system of integral equations for $a\in\Gamma(z_1,z_2)$:

$$\mathcal{Y}_a(\zeta) = \mathcal{Y}_a^{\rm sf}(\zeta)\exp\left[\sum_{\gamma_1\in\Gamma}\omega(\gamma';a)\frac{1}{4\pi {\rm i}}\int_{\ell_{\gamma_1}}\frac{d\zeta_1}{\zeta_1}\frac{\zeta_1+\zeta}{\zeta_1-\zeta}\log\left(1-\mathcal{Y}_{\gamma_1}(\zeta_1)\right)\right] \tag{18.22}$$

In order for this equation to make sense we must have ζ on the complement of the set of 2d4d BPS rays. Thus, the equation is only written in angular sectors of the complex ζ-plane as in Figure

FIGURE OF SECTORS IN THE COMPLEX ZETA PLANE

Moreover, the phase of ζ determines a spectral network:

$$\mathcal{W}_\zeta := \mathcal{W}_{\vartheta=\arg\zeta} \tag{18.23}$$

and we only write this equation for $z_1,z_2\in C-\mathcal{W}_\zeta$.

Let us try to understand this frightening equation. How can we compute the correction to $\mathcal{Y}_\gamma^{\rm sf}(\zeta)$? We need to know $\mathcal{Y}_\gamma(\zeta)$! To see how to determine $\mathcal{Y}_\gamma(\zeta)$ first recall the affine-linearity property (13.28) of the 2d-4d degeneracies:

$$\omega(\gamma;b+\gamma') = \omega(\gamma;b)+\langle\gamma,\gamma'\rangle\Omega(\gamma) \tag{18.24}$$

Therefore, consistency requires:

$$\log\mathcal{Y}_\gamma = \log\mathcal{Y}_\gamma^{\rm sf} + \frac{1}{4\pi{\rm i}}\sum_{\gamma_1}\Omega(\gamma_1)\langle\gamma_1,\gamma\rangle\int_{\ell_{\gamma_1}}\frac{d\zeta_1}{\zeta_1}\frac{\zeta_1+\zeta}{\zeta_1-\zeta}\log(1-\mathcal{Y}_{\gamma_1}(\zeta_1)) \tag{18.25}$$

where

$$\mathcal{Y}_\gamma^{\rm sf} := \exp\left(\pi R\frac{Z_\gamma}{\zeta}+i\theta_\gamma+\pi R\zeta\bar{Z}_\gamma\right). \tag{18.26}$$

and (18.25) is valid for ζ on the complement of the 4d BPS rays. Note that in this equation all the homology classes on Σ are closed, and the angles θ_γ (defined mod$2\pi\mathbb{Z}$) have a composition law correponding to the twisted multiplication law:

$$\mathcal{Y}_\gamma^{\rm sf}(\zeta)\mathcal{Y}_{\gamma'}^{\rm sf}(\zeta) = (-1)^{\langle\gamma,\gamma'\rangle}\mathcal{Y}_{\gamma+\gamma'}^{\rm sf}(\zeta) \tag{18.27}$$

This is a closed system of equations for the $\mathcal{Y}_\gamma(\zeta)$. Note that the BPS rays have the property that $\mathcal{Y}^{\rm sf}_{\gamma_1}(\zeta_1)$ is exponentially small when $\zeta_1 \in \ell_{\gamma_1}$: Along the BPS rays $Z_{\gamma_1}/\zeta_1 < 0$ and hence $\mathcal{Y}^{\rm sf}_{\gamma_1}(\zeta_1)$ decays exponentially for $\zeta_1 \to 0, \infty$ or for $R \to \infty$. [45] One may therefore hope to produce explicit solutions by iterating the equation, at least for sufficiently large R. We discuss this in §18.7 below.

Now, with $\mathcal{Y}_\gamma(\zeta)$ computed from (18.25) we can compute the parallel transports $\mathcal{Y}_a(\zeta)$ of $\nabla^{\rm ab}(\zeta)$ from (18.22).

What is the motivation for writing (18.22)? Note that it is the unique solution to a RH problem based on

1. *Asymptotics*: For $\zeta \to 0, \infty$ we have

$$\mathcal{Y}_a(\zeta) \sim \mathcal{Y}^{\rm sf}_a(\zeta) \tag{18.28}$$

2. *Discontinuities*: Moreover, if we vary ζ across a BPS ray ℓ_{γ_1} then we produce the discontinuity

$$\mathcal{Y}_a(\zeta^+) = (1 - \mathcal{Y}_{\gamma_1})^{\omega(\gamma_1;a)} \mathcal{Y}_a(\zeta^-) \tag{18.29}$$

   This is the kind of discontinuity we have seen when writing the coordinates for ∇ in terms of $\Psi_{\mathcal{W}_\vartheta}(\nabla^{\rm ab}(\vartheta))$. As ϑ varies through a critical angle ϑ_0 at which there are degenerate spectral networks containing string webs (i.e. 4d bps states of charge γ_1 and central charge phase $e^{{\rm i}\vartheta_0}$) the coordinates jump by an equation of the form (18.29).

These two properties define a Riemann-Hilbert problem in the ζ-plane and (18.22) is the solution. We are being a bit loose here. A very precise statement, for the closed cycles, is given in §18.4.1 below.

Remark: Note well: The 2d4d degeneracies $\omega(\gamma'; a)$ are discontinuous, as functions of z_1 and z_2, in a way described by the 2d4d wall-crossing-formula, across the walls of the degenerate spectral networks with two-way streets. This introduces extra, discontinuous z_1, z_2 dependence in the parallel transports (18.22).

18.4.1 RH problem for the closed cycles

Let us be a bit more precise about the Riemann-Hilbert problem at least for the $\mathcal{Y}_\gamma$ associated with the closed cycles:

5 basic properties of the Darboux functions

1. $\mathcal{Y}_\gamma(\zeta)$ are piecewise holomorphic on $\mathcal{M}^\zeta \times \mathbb{C}^*$. ♣Only discontinuous across the BPS walls. Right? ♣

2. They represent the twisted group algebra of Γ:

$$\mathcal{Y}_{\gamma_1}\mathcal{Y}_{\gamma_2} = (-1)^{\langle\gamma_1,\gamma_2\rangle}\mathcal{Y}_{\gamma_1+\gamma_2} \tag{18.30}$$

[45] This is the reason for the convention that BPS rays have $Z_\gamma/\zeta \in \mathbb{R}_-$ rather than with a plus sign.

3. They satisfy the reality condition

$$\overline{\mathcal{Y}_\gamma(\zeta)} = \mathcal{Y}_{-\gamma}(-1/\bar{\zeta}) \tag{18.31}$$

4. They have $R \to \infty$ and $\zeta \to 0$ asymptotics:

$$\mathcal{Y}_\gamma \sim \mathcal{Y}_\gamma^{\rm sf} \tag{18.32}$$

5. Across a BPS wall $W(\gamma_0)$ the $\mathcal{Y}_\gamma$ transform as

$$\mathcal{Y}_\gamma(\zeta^+) = (1-\mathcal{Y}_{\gamma_0}(\zeta))^{\langle\gamma,\gamma_0\rangle\Omega(\gamma_0)}\mathcal{Y}_\gamma(\zeta^-) \tag{18.33}$$

♣Need to say which direction you cross the wall here. ♣

These properties completely determine the $\mathcal{Y}_\gamma(\zeta)$. Moreover, they are easily checked to be properties of any solution to the integral equation (18.25). Therefore, that equation determines the $\mathcal{Y}_\gamma(\zeta)$.

♣Give the argument ♣

18.5 Constructing the nonabelian connection

Now let us apply the nonabelianization map for the spectral network $\mathcal{W}_\zeta$ with

$$\vartheta = \arg\zeta \tag{18.34}$$

Then

$$\Psi_{\mathcal{W}_\vartheta}(\nabla^{\rm ab}(\zeta)) \tag{18.35}$$

is a flat connection on some rank K complex vector bundle $E_{\mathcal{W}_\vartheta} \to C$.

Now, we want to compute a solution to the Hitchin equations for a connection on the hermitian vector bundle $V = \pi_*(\mathcal{N})$ where the hermitian structure is obtained from that on $\mathcal{N}$ as above. To this end we use the 2d soliton degeneracies $\mu(a)$ to construct a ζ-dependent family of isomorphisms

$$g(\zeta) : E_{\mathcal{W}_\vartheta}|_{C-\mathcal{W}} \to V|_{C-\mathcal{W}} \tag{18.36}$$

where again $\vartheta := \arg\zeta$ here and the isomorphism is only supposed to be valid off of the spectral network $\mathcal{W}_\vartheta$.

We choose any isomorphism $g^{\rm sf}$ which is independent of ζ. It could be the identity, since in our description via transitions functions, on $C-\mathcal{W}$ the bundle $E_{\mathcal{W}_\vartheta}$ is defined to be V.

♣Something strange here. The domain on which $g^{\rm sf}$ is defined is changing with ϑ so it can't be completely independent of ζ. Clarify this. Maybe avoid by contour deformation tricks. ♣

Having chosen $g^{\rm sf}$ at every $z \in C - \mathcal{W}_\vartheta$ we now use the 2d BPS rays, which we recall are

$$\ell_a := Z_a\mathbb{R}_- \qquad a \in \cup_{i\neq j}\Gamma_{ij}(z,z) \qquad \& \qquad \mu(a) \neq 0 \tag{18.37}$$

and for ζ on the complement of the 2d BPS rays we can write the integral equation:

$$g(\zeta) = g^{\rm sf} + \sum_{a\in\cup_{i\neq j}\Gamma_{ij}(z,z)} \frac{\mu(a)}{4\pi {\rm i}}\int_{\ell_a}\frac{d\zeta_1}{\zeta_1}\frac{\zeta_1+\zeta}{\zeta_1-\zeta}\mathcal{Y}_a(\zeta_1)g(\zeta_1) \tag{18.38}$$

which can be better written [46]

$$g(\zeta)=g_{+}+\zeta\sum_{a\in\cup_{i\neq j}\Gamma_{ij}(z,z)}\frac{\mu(a)}{2\pi\mathrm{i}}\int_{\ell_a}\frac{d\zeta_1}{\zeta_1}\frac{1}{\zeta_1-\zeta}\mathcal{Y}_a(\zeta_1)g(\zeta_1) \tag{18.39}$$

♣This is not a priori well-defined. Need to comment on the system with g_k. ♣

Note that for $a\in\Gamma(z,z)$ we consider $\mathcal{Y}_a(\zeta_1)\in\mathrm{End}(E_{\mathcal{W}_\vartheta})$

♣Note that for $\mathcal{Y}_a(\zeta_1)$ to be well-defined we are assuming that the 2d BPS rays do not overlap with the 2d4d BPS rays. ♣

Now we form the complex gauge field by taking $A^{\mathrm{ab}}(\zeta)$ to be the local one-form connection for $\pi_*(\nabla^{\mathrm{ab}}(\zeta))$ on $\pi_*(\mathcal{N})$ and defining:

$$\mathcal{A}(\zeta)=g(\zeta)^{-1}A^{\mathrm{ab}}(\zeta)g(\zeta)+g(\zeta)^{-1}dg(\zeta) \tag{18.40}$$

♣Careful. We are now putting succesive operations to the right. This is opposite to the convention in 2d4d sec. 5.6 ♣

Reading our linear transformations from left to right, this is a locally defined 1-form valued in End(V) and is a flat connection. To be concrete

$$A^{\mathrm{ab}}(\zeta)=\sum_i e_{ii}d\log\mathcal{Y}_{a_i} \tag{18.41}$$

where $a_i\in\Gamma_{i_0,i}(z_0,z)$ is any collection of chains from a basepoint $z_0^{(i_0)}$ to $z^{(i)}$.

Now, the BPS degeneracies satisfy some sign properties such as $\mu(-a)=-\mu(a)$ etc. (See [96], eqs. (5.34)-(5.39)) which allows us to show some important reality properties:

$$\overline{\mathcal{Y}_a(\zeta)}=\mathcal{Y}_a^{-1}(-1/\bar{\zeta}) \tag{18.42}$$

$$(g(\zeta))^{\dagger}=g(-1/\bar{\zeta})^{-1} \tag{18.43}$$

from which we deduce that

$$(\mathcal{A}(\zeta))^{\dagger}=-\mathcal{A}(-1/\bar{\zeta}) \tag{18.44}$$

Now we note the following:

1. From the integral equations and the formula for $\mathcal{Y}^{\mathrm{sf}}$ it follows that $\mathcal{A}$ has at most a simple pole at $\zeta\to 0$ and at most a simple pole at $\zeta\to\infty$.

2. Moreover, $\mathcal{A}$ is continuous across the BPS rays in the ζ plane. This follows because the discontinuities in $\mathcal{Y}_a(\zeta)$ as ζ crosses these BPS rays is given – by the residue theorem – by the 2d4d automorphisms we used in the 2d4d wall-crossing formula.

3. For example, if ζ crosses a 2d BPS ray ℓ_a where $a\in\Gamma(z,z)$ we have $g(\zeta^+)=(1+\mu(a)\mathcal{Y}_a(\zeta))g(\zeta^-)$ and

$$\mathcal{A}(\zeta)=g(\zeta)^{-1}(d+\mathcal{A}^{\mathrm{ab}}(\zeta))g(\zeta) \tag{18.45}$$

   is smooth because $\mathcal{Y}_a(\zeta)$ is a flat connection for $\mathcal{A}^{\mathrm{ab}}(\zeta)$ and because $\mu(a)$ is z-independent, where $a\in\Gamma(z,z)$. [47]

4. When ζ crosses a 2d4d BPS ray ℓ_γ there is a discontinuity $\mathcal{Y}_a\to(1-\mathcal{Y}_\gamma(\zeta))^{\omega(\gamma;a)}\mathcal{Y}_a$. If z is generic then the expression $\mathcal{A}^{\mathrm{ab}}(\zeta;z)=\sum_i d\log\mathcal{Y}_{a_i}$ remains invariant because $\omega(\gamma;a_i)$ is constant in z.

♣But what about the jumps in z? ♣

[46] This improvement is due to D. Gaiotto in an unpublished note. Note that g_+ is a function of z, if only because $\mu(a)$ is a function of z.

[47] We are assuming that 2d and 2d4d walls do not coincide.

It now follows - by Liouville's theorem - that, quite remarkably, the *a priori* very complicated ζ dependence of $\mathcal{A}(\zeta)$ is in fact a 3-term expansion of the form:

$$\mathcal{A}(\zeta) = \frac{R}{\zeta}\varphi + A + R\zeta\bar{\varphi} \tag{18.46}$$

where A is a unitary connection on V and $\bar{\varphi}$ is the hermitian conjugate of φ in the hermitian metric on V.

On the other hand, by construction, $\mathcal{A}(\zeta)$ is *flat*. Therefore it follows that (A,φ) are a solution to Hitchin's equations!

Moreover, using the 2d-4d WCF one can show that the flat connection $d+\mathcal{A}$ is continuous across the walls of marginal stability in $C \times \mathcal{B}^*$ and in particular extend from $C - \mathcal{W}_\zeta$ to all of C. In particular, it extends over the branch points.

♣Explain this more! ♣

Remarks

1. Expanding around $\zeta = 0$ we see that the matrix g_+ in (18.39) diagonalizes the Higgs field as in (18.13)

2. We separated the RH problems for the 4d and 2d BPS lines. The 4d for constructing $\nabla^{\rm ab}$ and the 2d for the diagonalization of $\mathcal{A}$.

♣So what. ♣

3. We began the construction starting with a Higgs bundle, and produced a solution of the Hitchin equations. On the other hand, there is supposed to be a 1-1 correspondence between Higgs bundles and solutions to the Hitchin equations....

♣Davide says they are the same. Explain. ♣

4. The discontinuities from $\omega(\gamma; a)$ should cancel. EXPLAIN.

18.5.1 A curious connection to integrable systems

The equation (18.25) is formally identical to Zamolodchikov's thermodynamic Bethe ansatz equation for computing the free energy and particle spectrum of $1+1$ dimensional systems from an exact scattering matrix satisfying the Yang-Baxter equation.

Suppose an integrable system has particles of mass m_a, where a is running over some index. The energy of such free particles with rapidity θ is

$$E_a(\theta) = m_a \cosh\theta \tag{18.47}$$

Now we introduce an integral equation for computing quasi-energies at inverse temperature β:

$$\epsilon_a(\theta) = m_a\beta\cosh\theta - \sum_b \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi}\phi_{ab}(\theta-\theta')\log\left(1+e^{\beta\mu_b-\epsilon_b(\theta')}\right) \tag{18.48}$$

where

$$\phi_{ab}(\theta) = -i\frac{\partial}{\partial\theta}\log S_{ab}(\theta) \tag{18.49}$$

and $Z_\gamma = e^{\mathrm{i}\alpha_\gamma}|Z_\gamma|$.

On a BPS ray we change variables using $\zeta_1 = -e^{\mathrm{i}\alpha_\gamma+\theta}$ and the ray is parametrized by $-\infty < \theta < \infty$ and

$$\log\mathcal{Y}^{\rm sf}_\gamma = -2\pi R|Z_\gamma|\cosh\theta + i\theta_\gamma \tag{18.50}$$

(Here there is a notation clash: θ_γ is an angle associated with the $U(1)$ connection and not to be confused with the rapidity.)

The analog of the kernel associated with the S-matrix is

$$K_{\gamma,\gamma'}(\theta-\theta') = i\langle\gamma,\gamma'\rangle \frac{\partial}{\partial\theta}\log\left[\sinh\left(\frac{1}{2}(\theta-\theta'+i\alpha_\gamma-i\alpha_{\gamma'})\right)\right] \tag{18.51}$$

and corresponds to a non-unitary scattering matrix.

In a similar way we can produce explicit solutions of the linear system $(d+\mathcal{A})\psi = 0$ via $g(\zeta)$, since it is the gauge transformation which effectively diagonalizes $\mathcal{A}$. Thus the integral equation (18.39) is an integral equation for producing solutions to a linear system of differential equations for flat sections. It is very similar to the use of the Gelfand-Levitan-Marchenko equation in inverse scattering theory and in some cases is directly related to that equation [134].

♣Explain this some more. ♣

18.6 WKB Asymptotics

A corollary of the above construction is interesting $\zeta\to 0,\infty$ asymptotics for the traces of holonomies of flat connections and for the flat sections.

When we wrap the line defects around S^1_R and take a trace we produce local operators in the $d=3$ sigma model which are supersymmetric. In particular, they have vevs which depend on the vacuum $m\in\mathcal{M}$ of the sigma model in a way which is holomorphic in complex structure ζ.

This leads to what was called in [96] the *Darboux expansion.* For the case of $\mathfrak{g} = su(K)$ this would be [48]

$$\langle L_{\wp,\vartheta}\rangle_m = \mathrm{Tr}_K P\exp\oint_\wp \mathcal{A}(\zeta) = \sum_{\gamma\in\Gamma}\underline{\overline{\Omega}}(L_{\wp,\vartheta},\gamma;u)\mathcal{Y}_\gamma(\zeta) \tag{18.52}$$

The expectation value $\langle L_{\wp,\vartheta}\rangle_m$ is not discontinuous in ϑ. Rather, the discontinuities in the framed BPS degeneracies cancel those of $\mathcal{Y}_\gamma(\zeta)$.

Now we can analytically continue in the ζ-plane away from the unit circle $\zeta = e^{i\vartheta}$.

The $\zeta\to 0,\infty$ asymptotics, holding $m = [(\varphi, A)]\in\mathcal{M}$ fixed is therefore

$$\mathrm{Tr}_K P\exp\oint_\wp \mathcal{A}(\zeta) \sim \sum_{\gamma\in\Gamma}\underline{\overline{\Omega}}(L_{\wp,\vartheta},\gamma;u)\mathcal{Y}_\gamma(\zeta)^{\mathrm{sf}} \tag{18.53}$$

Remarks:

1. There is a nice consistency check with the physical interpretation in terms of gauge theory with a line wrapped around the circle. Such an expectation value can also be written - from the d=4 viewpoint - as a trace in a Hilbert space (of the four-dimensional theory):

$$\langle L_{\wp,\vartheta}\rangle = \mathrm{Tr}_{\mathcal{H}_{u,L_\zeta}}(-1)^F e^{-2\pi RH} e^{i\theta\cdot\mathcal{Q}}\sigma(\mathcal{Q}). \tag{18.54}$$

From this viewpoint we very naturally recover the $R \to \infty$ asymptotics

$$\langle L_{\wp,\vartheta} \rangle \sim \sum_{\gamma} \underline{\overline{\Omega}}(L_\zeta, \gamma) \exp(2\pi R \,\mathrm{Re}(Z_\gamma/\zeta) + i\tilde{\theta}_\gamma), \tag{18.55}$$

♣There are two subtleties here.... explained in Framed BPS paper. ♣

where $e^{i\tilde{\theta}_\gamma} := e^{i\theta_\gamma}\sigma(\gamma)$ because the BPS bound for framed BPS states is

$$E = -\mathrm{Re}(Z_\gamma/\zeta) \tag{18.56}$$

So we conclude:

$$\mathcal{Y}_\gamma \sim \mathcal{Y}_\gamma^{\mathrm{sf}} := \exp\left(\pi R \frac{Z_\gamma}{\zeta} + i\theta_\gamma + \pi R \zeta \bar{Z}_\gamma\right). \tag{18.57}$$

for $R \to \infty$.

18.6.1 Tropical Labling of Line Defects

♣ Comment here on Framed BPS states Section 12.

18.6.2 Asymptotics of flat sections

♣ NEED TO FILL IN:

What you are taught in der Kindergarten.

Why $K > 2$ is hard.

♣

In the connected components of the complementary region $C - \mathcal{W}_\vartheta = \amalg_\alpha \mathcal{R}^\alpha$ the decomposition of the vector space of flat sections

$$\mathcal{E} \cong \oplus_{i=1}^{K} \mathfrak{L}_i^{\mathcal{R}^\alpha} \tag{18.58}$$

is a decomposition into lines of flat sections which have well-defined asymptotics for $\zeta \to 0, \infty$ or for $R \to \infty$.

That is, in $\mathcal{R}^\alpha$ we claim that if $\zeta \to 0$ in the half-plane $\mathbb{H}_\vartheta$ around $e^{\mathrm{i}\vartheta}$ then the asymptotics of $\psi \in \mathfrak{L}_i^{\mathcal{R}^\alpha}$ is

$$\psi \sim \psi_0 \exp\left(\int_{z_0}^{z} \frac{R}{\zeta}\lambda_i\right) \tag{18.59}$$

We conclude that the segments of the spectral network are Stokes lines.

18.7 Solving the TBA by iteration: The graph expansion

Now, to derive the tree expansion we expand the logarithm (Recall that $\mathcal{Y}_{\gamma_1}(\zeta_1)$ is small on the BPS ray) and rearrange the sum:

$$\begin{aligned}\log \mathcal{Y}_\gamma &= \log \mathcal{Y}_\gamma^{sf} + \sum_{\gamma_1} \sum_{d=1}^{\infty} \frac{\Omega(\gamma_1)\langle \gamma, \gamma_1 \rangle}{4\pi i d} \int_{\ell_{\gamma_1}} \frac{d\zeta_1}{\zeta_1} \frac{\zeta_1 + \zeta}{\zeta_1 - \zeta} \mathcal{Y}_{d\gamma_1}(\zeta_1) \\ &= \log \mathcal{Y}_\gamma^{sf} + \sum_{\gamma_1} c(\gamma_1)\langle \gamma, \gamma_1 \rangle \int_{\ell_{\gamma_1}} \frac{d\zeta_1}{\zeta_1} \frac{\zeta_1 + \zeta}{\zeta_1 - \zeta} \mathcal{Y}_{\gamma_1}(\zeta_1)\end{aligned} \tag{18.60}$$

[48] Here we have a sum over Γ but in fact the sum involves a sum over a Γ torsor Γ_L which depends on the line defect L.

where

$$c(\gamma_1) := \frac{1}{4\pi i}\sum_{d=1}^{\infty}\frac{\Omega(\gamma_1/d)}{d^2} = \frac{\bar{\Omega}(\gamma_1)}{4\pi i} \tag{18.61}$$

($\bar{\Omega}$ are the "rational invariants" which appear e.g. in the Joyce-Song approach to wall-crossing and in the physical discussion of [136].)

To derive the tree expansion we go one step further and write this as:

$$\begin{aligned}\log \mathcal{Y}_\gamma(\zeta) &= \log \mathcal{Y}^{sf}_\gamma(\zeta) + \sum_{\gamma_1} c(\gamma_1)\int_{\ell_{\gamma_1}} K(\zeta,\gamma:\zeta_1,\gamma_1)\mathcal{Y}^{sf}_{\gamma_1}(\zeta_1)\\ &+ \sum_{\gamma_1} c(\gamma_1)\int_{\ell_{\gamma_1}} K(\zeta,\gamma:\zeta_1,\gamma_1)\mathcal{Y}^{sf}_{\gamma_1}(\zeta_1)\left[\exp\left\{\sum_{\gamma_2} c(\gamma_2)\int_{\ell_{\gamma_2}} K(\zeta_1,\gamma_1:\zeta_2,\gamma_2)\mathcal{Y}_{\gamma_2}(\zeta_2)\right\} - 1\right]\end{aligned} \tag{18.62}$$

where we have defined

$$K(\zeta,\gamma:\zeta_1,\gamma_1) := \frac{\zeta_1+\zeta}{\zeta_1-\zeta}\langle\gamma,\gamma_1\rangle \tag{18.63}$$

and $\int_{\ell_\gamma}$ includes the measure $\frac{d\zeta}{\zeta}$.

Now, if we expand the exponential and then insert the integral equation in the terms generating $\mathcal{Y}_\gamma(\zeta)$ we see that we generate a sum over trees. The rules are:

1. The trees have a single basepoint denoted by $\odot$. It is decorated by ζ,γ. γ is not summed and ζ is not integrated. The tree has one edge to a basepoint with ζ_1,γ_1, where γ_1 will be summed and ζ_1 integrated. There is a natural outgoing orientation from this circled basepoint. Trees like this are sometimes called "planted trees." Since we only have one edge from $\odot$ to vertex 1 it could be eliminated and still leave a rooted tree. Note that if we do include this edge then the number of vertices excluding $\odot$ is the number of edges.

2. We are summing over *unordered* trees. There is a natural orientation on the tree with the vertex from the node outgoing. Suppose we draw the trees growing downward. Then there is an order on the downward pointing edges at each vertex. Even though two trees might be topologically the same they contribute different terms to the sum. It is important that we do *not* try to identify trees by their topological type and divide by the automorphism group. The effect of automorphisms is taken into account more easily by the $1/k!$ factors coming from expanding the exponential.

   FIGURE OF THE TREE

3. The trees are decorated by an element of Γ at all vertices. When we sum over decorated trees we sum over decorations at all vertices except $\odot$.

The weight associated to a decorated ordered planted tree is:

$$\prod_v \int_{\ell_{\gamma_v}} \frac{d\zeta_v}{\zeta_v}\prod_v\left(\frac{1}{k_v!}c(\gamma_v)\mathcal{Y}^{sf}_{\gamma_v}(\zeta_v)\right)\prod_e K(\zeta_s,\gamma_s:\zeta_t,\gamma_t) \tag{18.64}$$

where:

1. v runs over the vertices of the tree, except for the special one $\odot$ labeled by ζ, γ.

2. e runs over the edges of the tree and for an edge we use the notation s for the source vertex and t for the target vertex. Note that there is a special edge, beginning at the special vertex $\odot$.

Now, it is useful to separate out the dependence on the angle and base coordinates in the Seiberg-Witten fibration. We write the expansion as a sum

$$\log \mathcal{Y}_\gamma(\zeta) = \log \mathcal{Y}_\gamma^{sf} + \sum_{T \in \mathcal{DT}} e^{i\theta_T} \mathcal{F}_{T,\gamma}(u,\zeta) \tag{18.65}$$

where $\mathcal{DT}$ is the set of decorated ordered trees and for such a tree we define:

$$e^{i\theta_T} := \prod_v e^{i\theta_{\gamma_v}} \tag{18.66}$$

while

$$\mathcal{F}_{T,\gamma}(u,\zeta) = \prod_v \left(\frac{c(\gamma_v)}{k_v!} \right) \prod_e \langle \gamma_s, \gamma_t \rangle \ \ \mathcal{G}_T(u,\zeta) \tag{18.67}$$

and $\mathcal{G}_T(u,\zeta)$ is obtained from a special function - the *tree functions* $H_T(x,y)$ - associated to an (undecorated) tree:

$$H_T(x,y) := \int_{-\infty}^{+\infty} \prod_v d\rho_v \prod_v e^{-x_v \cosh \rho_v} \prod_e \frac{y_t e^{\rho_t} + y_s e^{\rho_s}}{y_t e^{\rho_t} - y_s e^{\rho_s}} \tag{18.68}$$

To get $\mathcal{G}_T(u,\zeta)$ we identify

$$\begin{aligned} x_v &= 2\pi R |Z_{\gamma_v}| \\ y_v &= e^{i\alpha_{\gamma_v}} \end{aligned} \tag{18.69}$$

and we also must give $y_\odot$ for the special vertex $\odot$ which is a source of the special edge. This is given by $y_\odot e^{\rho_\odot} = -\zeta$. To arrive at this expression one makes the change of variables $\zeta_v = -e^{i\alpha_{\gamma_v} + \rho_v}$ with $-\infty < \rho_v < \infty$ in the integrals over the BPS rays.

Now we want to establish a criterion for absolute convergence of the sum over decorated ordered trees. We will need to make some assumptions on the spectrum.

Assumption 1: Let $\mathcal{B}$ be the set of of pairs (γ_1, γ_2) with $\langle \gamma_1, \gamma_2 \rangle c(\gamma_1) c(\gamma_2) \neq 0$. Then:

$$\mathrm{Max}_{\mathcal{B}} \left\{ 1 + \frac{2|\cos \alpha_{12}|}{1 - \cos \alpha_{12}} \right\} = B_1 < \infty \tag{18.70}$$

N.B. This assumption is violated on the walls of marginal stability.

Next we introduce a constant B_2 so that the K-Bessel functions has a bound

$$\int_{-\infty}^{\infty} d\rho e^{-x \cosh \rho} \leq B_2 \sqrt{\frac{2\pi}{x}} e^{-x} \tag{18.71}$$

Assumption 2: There exists a Euclidean norm on Γ so that:

$$Max_{\mathcal{B}} \left\{ \frac{|\langle \gamma_1, \gamma_2 \rangle|}{\parallel \gamma_1 \parallel \cdot \parallel \gamma_2 \parallel} \right\} \leq B_3 < \infty \tag{18.72}$$

Now we make a third assumption, which can almost certainly be eliminated, but which we make just to simplify the argument:

Assumption 3: There is a set $\mathcal{S}$ so that if $\Omega(\gamma) \neq 0$ then $\gamma \in \mathcal{S}$, and then $\Omega(d\gamma) = 0$ for all integers d with $|d| > 1$.

At this point we must make our fourth and final assumption on the spectrum, and this is a critical assumption:

Assumption 4: We assume that there exist positive constants κ_0, R_0 so that, for $R \geq R_0$ and for all $k = 1, 2, 3, \dots$

$$\sum_{\gamma \in \mathcal{S}} \parallel \gamma \parallel^{k+1} \Omega(\gamma) e^{-2\pi R |Z_\gamma|} \leq e^{-\kappa_0 R} \tag{18.73}$$

Now we can establish absolute convergence from that of the sum over *undecorated trees*:

$$\sum_{t \in \mathcal{T}} (B_1 B_2 B_3 B_4 e^{-\kappa_0 R})^{V(t)} \tag{18.74}$$

Now we use the result from combinatorics to conclude that our series is absolutely convergent for

$$4 B_1 B_2 B_3 B_4 < e^{\kappa_0 R} \tag{18.75}$$

In sum: We can show that the sum from iterating the integral equation is absolutely convergent, for sufficiently large R, if the BPS spectrum satisfies the four assumptions stated above.

19. Lecture 8b: Thursday, October 11: Hyperkähler Geometry

19.1 Review of Hyperkahler geometry

Let us summarize a few basic definitions and facts about hyperkähler manifolds. A nice review of this beautiful subject can be found in the review of Hitchin [109]. See also [108].

Definition: A *hyperkähler* manifold is a Riemannian manifold M with three orthogonal transformations on the tangent bundle $J_a \in \mathrm{End}(TM)$, $a = 1, 2, 3$, such that

1. J_a satisfy the algebra of the quaternions:

$$J_a J_b = -\delta_{ab} + \epsilon_{abc} J_c \tag{19.1}$$

2. $\nabla J_a = 0$

Since the tangent space has a quaternionic structure the real dimension of M must be a multiple of four. Let us say that $\dim M = 4r$. Then, near any point $p \in M$ we can choose an orthonormal basis for the quaternionic vector space $T_p^* M$ so that, in complex structure, say, J_3 a basis of the cotangent space can be written as dz^I, dw_I, $I = 1, \dots, r$ and in this basis the complex structures act as:

$$\begin{aligned} J_3 &: (dz^I, dw_I) \to (i dz^I, i dw_I) \\ J_2 &: (dz^I, dw_I) \to (-\overline{dw_I}, \overline{dz^I}) \\ J_1 &: (dz^I, dw_I) \to (i\overline{dw_I}, -i\overline{dz^I}) \end{aligned} \tag{19.2}$$

For a hyperkähler manifold M one can show that it is Kähler with respect to each complex structure J_a and hence there are three Kähler forms ω_a, $a = 1, 2, 3$. In the local coordinates given above we can write:

$$\omega_3 = \frac{i}{2} dz^I \overline{dz^I} + \frac{i}{2} dw_I \overline{dw_I} \tag{19.3}$$

while

$$\omega_1 + i\omega_2 := \omega_+ = dz^I dw_I \tag{19.4}$$

is of type $(2, 0)$.

In fact, M has a whole sphere's worth of complex structures! If n^a is a real unit three-vector, $n^a n^a = 1$ then note that

$$(n^a J_a)^2 = -1 \tag{19.5}$$

This sphere of complex structures is known as the *twistor sphere*. The space $Z = M \times S^2$ is known as *twistor space*. It can be given the structure of a complex manifold by taking $S^2 = \mathbb{C}P^1$ and then, letting $\zeta \in \mathbb{C}$ be the inhomogeneous coordinate on $\mathbb{C}P^1$, the fiber above $\zeta \in \mathbb{C}P^1$ is M considered as a complex manifold in complex structure ζ, denoted M^ζ.

Let us choose the north pole to correspond to the complex structure J_3 and consider stereographic projection of $S^2 = \mathbb{C}P^1 \to \mathbb{C}$. the holomorphic symplectic form is

$$\omega_\zeta = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\omega_- \tag{19.6}$$

19.1.1 The twistor theorem

Hitchin's twistor theorem says - roughly speaking - that the holomorphic family of of holomorphic symplectic manifolds $\mathcal{M}^\zeta$ equipped with ω_ζ uniquely characterizes the hyperkähler metric. Indeed, some technical points in the statement of the theorem imply that ω_ζ has a three-term Laurent expansion in ζ together with an antiholomorphic symmetry under

$$\zeta \to -1/\bar{\zeta} \tag{19.7}$$

and from this one has the expansion (19.9), from which, in turn, one can extract the hyperkähler metric from the Kähler form ω_3.

To be a little more precise, Hitchin's theorem is an equivalence of holomorphic data for the twistor space $Z := \mathcal{M} \times S^2$ with the hyperkähler metric on $\mathcal{M}$:

Theorem[Hitchin]: If $(\mathcal{M}, g)$ is hyperkähler of real dimension $4r$ then

1. There is a holomorphic fibration

$$p : Z \to \mathbb{C}P^1 \tag{19.8}$$

   so that $\mathcal{M}^\zeta = p^{-1}(\zeta)$ is $\mathcal{M}$ in complex structure ζ.

2. There is a holomorphic section ϖ of $\Omega^2_{Z/\mathbb{C}P^1} \otimes \mathcal{O}(2)$ so that $\varpi_\zeta = \varpi|_{\mathcal{M}^\zeta}$ is the holomorphic symplectic form of $\mathcal{M}^\zeta$.

3. There is an anti-holomorphic map $\sigma : Z \to Z$ covering $\zeta \to -1/\bar{\zeta}$.

4. For all $x \in \mathcal{M}$, there is a holomorphic section $s_x : \mathbb{C}P^1 \to Z$ with normal bundle $\cong \mathcal{O}(1)^{\oplus 2r}$.

Conversely, given the above four pieces of holomorphic data we can reconstruct the hyperkähler metric. Concretely we find a family of holomorphic symplectic forms with a 3-term Laurent expansion

$$\omega_\zeta = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\omega_- \tag{19.9}$$

and then ω_+ is a holomorphic $(2,0)$ form in complex structure $\zeta = 0$ while ω_3 is the Kähler metric at $\zeta = 0$.

19.1.2 Hyperholomorphic connections

$F^{2,0} = 0$ in all complex structures. What more to say?

19.2 The construction of hyperkähler metrics

Finally, let us indicate how the hyperkähler metric on $\mathcal{M}$ can be constructed.

We saw that the holomorphic functions $\mathcal{Y}_\gamma$ on the moduli space of flat connections are Darboux functions with simple Poisson brackets. This means that in fact the holomorphic symplectic form ω_ζ is proportional to

$$\omega_\zeta = C^{st}\frac{d\mathcal{Y}_{\bar{\gamma}_s}}{\mathcal{Y}_{\bar{\gamma}_s}} \wedge \frac{d\mathcal{Y}_{\bar{\gamma}_t}}{\mathcal{Y}_{\bar{\gamma}_t}} \tag{19.10}$$

This is properly holomorphic symplectic, and moreover, it is *continuous* across walls of marginal stability, because the KS transformations are symplectic.

By the same kind of argument we used for solving for the Hitchin system, it follows that ϖ_ζ has a 3-term Laurent expansion as a function of ζ: (equation (19.9))

$$\omega_\zeta = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\omega_- \tag{19.11}$$

and therefore we can extract the hyperkähler metric on $\mathcal{M}$ from the Kähler form ω_3.

One should worry that the metric will be terribly discontinuous across the walls of marginal stability, that is across the image under the Hitchin fibration $\pi : \mathcal{M} \to \mathcal{B}$ of $MS(\gamma_1, \gamma_2)$. However, this is not the case, precisely because of the 4d KSWCF. Indeed, the original physical argument for the KSWCF was that - from physical considerations - the metric has to be continuous and that continuity implies the 4d KSWCF [91].

Let us now tie this together with the interpretation of $\mathcal{Y}_\gamma$ as cluster-like coordinates.

As we have remarked, the $\mathcal{Y}_\gamma$ satisfy the twisted group algebra

$$\mathcal{Y}_\gamma \mathcal{Y}_{\gamma'} = (-1)^{\langle \gamma, \gamma' \rangle} \mathcal{Y}_{\gamma+\gamma'} \tag{19.12}$$

They therefore generate an algebra isomorphic to the (twisted) holomorphic functions on the algebraic torus $T_c = \Gamma^* \otimes \mathbb{C}^*$.

Note that for T_c, contraction with γ gives a canonically defined function ("holomorphic Fourier mode") $Y_\gamma : T_c \to \mathbb{C}^*$. T_c is a holomorphic symplectic manifold with holomorphic symplectic form

$$\omega_{T_c} = C^{st} \frac{dY_{\bar\gamma_s}}{Y_{\bar\gamma_s}} \wedge \frac{dY_{\bar\gamma_t}}{Y_{\bar\gamma_t}} \tag{19.13}$$

where C^{st} is inverse to the intersection matrix computed in a basis $\{\bar\gamma_s\}$ for the symplectic quotient: $C_{st} := \langle \bar\gamma_s, \bar\gamma_t \rangle$.

We therefore view $\mathcal{Y}_{\bar\gamma_s}$ as local coordinates on $\mathcal{M}$ mapping to the algebraic torus T_c, as in the Fock-Goncharov story. The holomorphic symplectic form on $\mathcal{M}^\zeta$ is then just the pullback of the natural one on T_c.

19.3 Physical Interpretation

19.3.1 Semiflat geometry

The bosonic part of the low energy d=4 effective action of the $\mathcal{N} = 2$ theory $S(\mathfrak{g}, C, D)$ is:

$$-\frac{1}{4\pi} \int \mathrm{Im}\tau_{IJ} da^I * d\bar a^J + \mathrm{Im}\tau_{IJ} F^I * F^J + \mathrm{Re}\tau_{IJ} F^I F^J \tag{19.14}$$

where $I, J = 1, \dots r$, F^I is the 2-form fieldstrength and τ_{IJ} is the period matrix of Σ.

The compactification of the term with scalar fields is straightforward. The reduction of the gauge fields can be easily shown to be

$$\int -\frac{1}{2R} (\mathrm{Im}\tau)^{-1,IJ} dz_I * d\bar z_J \tag{19.15}$$

where $dz_I = d\theta_{m,I} - \tau_{IJ} d\theta_e^J$ where θ_e^I and $\theta_{m,J}$ are real scalar fields with period 1. They can be thought of as the electric and magnetic Wilson lines. We can define

$$\theta_e^I = \oint_{S^1} A^I \tag{19.16}$$

and then dualize the gauge field in three dimensions to define corresponding periodic scalars

$$d\theta_{m,I} := \mathrm{Re}\tau_{IJ} d\theta_e^J + * \left(\mathrm{Im}\tau_{IJ} F^J \right) \tag{19.17}$$

This formula comes from dualization. Actually, a better point of view is that we can define

♣Explain more details. They are in the PiTP notes. ♣

$$\theta = \oint_{S^1_R} \mathbb{A} \tag{19.18}$$

in terms of the self-dual formalism of Section *** above.

In any case, one finds a sigma model with, for each $I = 1, \dots, r$ a complex scalar field a^I and two real periodic fields θ_e^I and $\theta_{m,I}$. The strict compactification gives a model with the *semiflat metric*:

$$g^{\mathrm{sf}} = R(\mathrm{Im}\tau)|da|^2 + R^{-1}(\mathrm{Im}\tau)^{-1}|dz|^2 \tag{19.19}$$

The twistor sections for the semiflat metric are precisely $\mathcal{Y}^{\mathrm{sf}}_\gamma(\zeta)$ [145]. This can be verified quite straightforwardly by computing ϖ_ζ with the semiflat coordinates and extracting ω_3.

This simple reduction procedure ignores important quantum effects. There are instantons in the 3d sigma model with target $\mathcal{M}$ which correspond to the effects of worldlines of BPS particles wrapped around the compactification circle. These are the quantum effects which are taken into account from the corrections to the integral equation.

♣Say more. Explain about the weight $e^{-2\pi R|Z_\gamma|}$ expected for a worldline instanton effect. ♣

Why does the TBA give the exact physical metric?

In [91] it is argued that (after a rescaling ϖ by factor $(8\pi^2 R)^{-1}$) this is not only a hyperkähler metric but is indeed the proper physical metric which corrects the singular semi-flat metric at finite values of R to a smooth(er) hyperkähler metric on $\mathcal{M}$. As we have noted, the $\mathcal{Y}_\gamma$ when expanded around $\mathcal{Y}_\gamma^{\rm sf}$ have an expansion in quantities of the form $e^{-2\pi R|Z_\gamma|}$. As promised, we can now interpret that expansion as an exact instanton expansion for the quantum-corrected metric on $\mathcal{M}$.

♣There is also a natural mathematical interpretation of the semiflat metric. See Neitzke email Oct. 10 ♣

♣This is too brief. Add more details. ♣

19.4 Hyperholomorphic connection

The construction of the solutions of the Hitchin system really is a construction of a hyperholomorphic connection on the universal bundle over $C \times \mathcal{M}$.

♣which doesn't exist as an honest bundle.... comment. ♣

Again the physical interpretation of the sum over μ is that these are quantum corrections from one-dimensional "instantons" of the "reduced" theory coming from the worldlines of 2d solitons wrapping around the circle S^1_R.

19.5 Example: Periodic Taub-NUT

Section 6 of 2d4d paper.

20. Categorification

20.1 Motivation: Knot Homology

Summary of the Witten approach to knot homology.

Multiple M2 branes define surface defects $\mathbb{S}_{z_1,z_2,\dots,z_k}$.

Now C is replaced by $C^k - \Delta$, where Δ is the big diagonal $z_i = z_j$ for some i, j.

Gaiotto and Witten reproduced the Jones polynomial, but we would like to get the full knot homology.

20.2 Simplest case: Landau-Ginzburg models

Joint with Gaiotto and Witten:

More motivation: How do we describe A-branes in LG, i.e. boundary conditions? Hori-Iqbal-Vafa described some, but not all.

How do we describe susy groundstates $H_{\mathcal{A}\mathcal{B}}$ on an interval with boundary conditions $\mathcal{A}, \mathcal{B}$?

Reminder on LG solitons.

Morse theory problem.

Morse-Smale-Witten complex and grading.

Instantons: LG instanton equation.

The boosted soliton.

Fans of solitons at infinity.

Webs. Defining an algebraic structure.

Local boundary conditions on a half-line. New complex.

Boundary vertices and webs again.

Small A_∞ category whose objects are vacua.

Algebraic construction: Maurer-Cartan locus

Large A_∞ category of branes.

Janus/Domain Wall: Tensor product of A_∞ categories.

For susy interfaces $\wp$ we define a "parallel transport of brane categories" i.e. a functor $F(\wp)$ from the brane category at z_1 to the brane category at z_2.

Parallel transport across S-walls and mutation.

Theorem: Homotopy invariance.

A. Dimensions of complex orbits and dimensions of moduli spaces of local systems

B. A short primer on Stokes' phenomenon

Use pedagogical discussion from GGI lecture notes

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References

[1] O. Aharony, M. Berkooz and N. Seiberg, "Light cone description of (2,0) superconformal theories in six-dimensions," Adv. Theor. Math. Phys. **2** (1998) 119 [arXiv:hep-th/9712117].

[2] O. Aharony, "A Brief review of 'little string theories'," Class. Quant. Grav. **17**, 929 (2000) [hep-th/9911147].

[3] S. Alexandrov, D. Persson and B. Pioline, "Wall-crossing, Rogers dilogarithm, and the QK/HK correspondence," JHEP **1112**, 027 (2011) [arXiv:1110.0466 [hep-th]].

[4] L. F. Alday, D. Gaiotto and Y. Tachikawa, "Liouville Correlation Functions from Four-dimensional Gauge Theories," Lett. Math. Phys. **91**, 167 (2010) [arXiv:0906.3219 [hep-th]].

[5] L. F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa and H. Verlinde, "Loop and surface operators in N=2 gauge theory and Liouville modular geometry," JHEP **1001**, 113 (2010) [arXiv:0909.0945 [hep-th]].

[6] L. F. Alday, F. Benini and Y. Tachikawa, "Liouville/Toda central charges from M5-branes," Phys. Rev. Lett. **105**, 141601 (2010) [arXiv:0909.4776 [hep-th]].

[7] L. F. Alday and Y. Tachikawa, "Affine SL(2) conformal blocks from 4d gauge theories," Lett. Math. Phys. **94**, 87 (2010) [arXiv:1005.4469 [hep-th]].

[8] M. Alim, S. Cecotti, C. Cordova, S. Espahbodi, A. Rastogi and C. Vafa, "N=2 Quantum Field Theories and Their BPS Quivers," arXiv:1112.3984 [hep-th].

[9] M. Alim, S. Cecotti, C. Cordova, S. Espahbodi, A. Rastogi and C. Vafa, arXiv:1109.4941 [hep-th].

[10] L. Alvarez-Gaume, G. W. Moore, P. C. Nelson, C. Vafa and J. b. Bost, "Bosonization In Arbitrary Genus," Phys. Lett. B **178** (1986) 41.

[11] L. Alvarez-Gaume, J. B. Bost, G. W. Moore, P. C. Nelson and C. Vafa, "Bosonization on Higher Genus Riemann Surfaces," Commun. Math. Phys. **112**, 503 (1987).

[12] M. T. Anderson, C. Beem, N. Bobev and L. Rastelli, "Holographic Uniformization," arXiv:1109.3724 [hep-th].

[13] E. Andriyash, F. Denef, D. L. Jafferis, G. W. Moore, "Bound state transformation walls," [arXiv:1008.3555 [hep-th]].

[14] E. Andriyash, F. Denef, D. L. Jafferis, G. W. Moore, "Wall-crossing from supersymmetric galaxies," [arXiv:1008.0030 [hep-th]].

[15] P. C. Argyres and M. R. Douglas, "New phenomena in SU(3) supersymmetric gauge theory," Nucl. Phys. B **448**, 93 (1995) [hep-th/9505062].

[16] P. C. Argyres, M. R. Plesser, N. Seiberg and E. Witten, "New N=2 superconformal field theories in four-dimensions," Nucl. Phys. B **461**, 71 (1996) [hep-th/9511154].

[17] P. C. Argyres, M. R. Plesser and N. Seiberg, "The Moduli space of vacua of N=2 SUSY QCD and duality in N=1 SUSY QCD," Nucl. Phys. B **471**, 159 (1996) [hep-th/9603042].

[18] P. C. Argyres, A. Kapustin and N. Seiberg, "On S-duality for non-simply-laced gauge groups," JHEP **0606**, 043 (2006) [hep-th/0603048].

[19] P. C. Argyres and N. Seiberg, "S-duality in N=2 supersymmetric gauge theories," JHEP **0712**, 088 (2007) [arXiv:0711.0054 [hep-th]].

[20] N. Arkani-Hamed, A. G. Cohen, D. B. Kaplan, A. Karch and L. Motl, "Deconstructing (2,0) and little string theories," JHEP **0301**, 083 (2003) [hep-th/0110146].

[21] J. A. de Azcarraga, J. P. Gauntlett, J. M. Izquierdo and P. K. Townsend, "Topological Extensions of the Supersymmetry Algebra for Extended Objects," Phys. Rev. Lett. **63**, 2443 (1989).

[22] J. Wess and J. Bagger, "Supersymmetry and supergravity," Princeton, USA: Univ. Pr. (1992) 259 p

[23] J. Bagger, N. Lambert, "Gauge symmetry and supersymmetry of multiple M2-branes," Phys. Rev. **D77**, 065008 (2008). [arXiv:0711.0955 [hep-th]].

[24] T. Banks, *Modern Quantum Field Theory: A concise introduction*, Cambridge

[25] T. Banks and N. Seiberg, "Symmetries and Strings in Field Theory and Gravity," Phys. Rev. D **83**, 084019 (2011) [arXiv:1011.5120 [hep-th]].

[26] F. Bastianelli, S. Frolov and A. A. Tseytlin, "Conformal anomaly of (2,0) tensor multiplet in six-dimensions and AdS / CFT correspondence," JHEP **0002**, 013 (2000) [hep-th/0001041].

[27] D. Belov and G. W. Moore, "Classification of abelian spin Chern-Simons theories," arXiv:hep-th/0505235.

[28] D. Belov and G. W. Moore, "Holographic Action for the Self-Dual Field," hep-th/0605038.

[29] F. Benini, Y. Tachikawa, D. Xie, "Mirrors of 3d Sicilian theories," JHEP **1009**, 063 (2010). [arXiv:1007.0992 [hep-th]].

[30] E. Bergshoeff, E. Sezgin and A. Van Proeyen, "(2,0) tensor multiplets and conformal supergravity in D = 6," Class. Quant. Grav. **16**, 3193 (1999) [hep-th/9904085].

[31] M. Berkooz, M. Rozali and N. Seiberg, "Matrix description of M theory on T**4 and T**5," Phys. Lett. B **408**, 105 (1997) [hep-th/9704089].

[32] J. Bhattacharya, S. Bhattacharyya, S. Minwalla and S. Raju, "Indices for Superconformal Field Theories in 3,5 and 6 Dimensions," JHEP **0802**, 064 (2008) [arXiv:0801.1435 [hep-th]].

[33] A. Bilal, F. Ferrari, "The BPS spectra and superconformal points in massive N=2 supersymmetric QCD," Nucl. Phys. **B516**, 175-228 (1998). [hep-th/9706145].

[34] A. Bilal, F. Ferrari, "The Strong coupling spectrum of the Seiberg-Witten theory," Nucl. Phys. **B469**, 387-402 (1996). [hep-th/9602082].

[35] A. Bilal, F. Ferrari, "Curves of marginal stability, and weak and strong coupling BPS spectra in N=2 supersymmetric QCD," Nucl. Phys. **B480**, 589-622 (1996). [hep-th/9605101].

[36] C. G. Callan, Jr., J. A. Harvey and A. Strominger, "Worldbrane actions for string solitons," Nucl. Phys. B **367**, 60 (1991).

[37] S. Cecotti, C. Vafa, "Classification of complete N=2 supersymmetric theories in 4 dimensions," [arXiv:1103.5832 [hep-th]].

[38] S. Cecotti, A. Neitzke, C. Vafa, "R-Twisting and 4d/2d Correspondences," [arXiv:1006.3435 [hep-th]].

[39] S. Cecotti, C. Vafa, "2d Wall-Crossing, R-twisting, and a Supersymmetric Index," [arXiv:1002.3638 [hep-th]].

[40] S. Cecotti, C. Vafa, "BPS Wall Crossing and Topological Strings," [arXiv:0910.2615 [hep-th]].

[41] S. Cecotti, C. Vafa, "On classification of N=2 supersymmetric theories," Commun. Math. Phys. **158**, 569-644 (1993). [hep-th/9211097].

[42] S. Cecotti, P. Fendley, K. A. Intriligator, C. Vafa, "A New supersymmetric index," Nucl. Phys. **B386**, 405-452 (1992). [hep-th/9204102].

[43] O. Chacaltana, J. Distler, "Tinkertoys for Gaiotto Duality," JHEP **1011**, 099 (2010). [arXiv:1008.5203 [hep-th]].

[44] O. Chacaltana and J. Distler, "Tinkertoys for the D_N series," arXiv:1106.5410 [hep-th].

[45] O. Chacaltana, J. Distler and Y. Tachikawa, "Nilpotent orbits and codimension-two defects of 6d N=(2,0) theories," arXiv:1203.2930 [hep-th].

[46] P. Claus, R. Kallosh and A. Van Proeyen, "M five-brane and superconformal (0,2) tensor multiplet in six-dimensions," Nucl. Phys. B **518**, 117 (1998) [hep-th/9711161].

[47] P. Deligne, (Ed.), P. Etingof, (Ed.), D. S. Freed, (Ed.), L. C. Jeffrey, (Ed.), D. Kazhdan, (Ed.), J. W. Morgan, (Ed.), D. R. Morrison, (Ed.) and E. Witten, (Ed.), "Quantum fields and strings: A course for mathematicians. Vol. 1, 2," Providence, USA: AMS (1999) 1-1501

[48] F. Denef, "Supergravity flows and D-brane stability," JHEP **0008**, 050 (2000). [hep-th/0005049].

[49] F. Denef, "On the correspondence between D-branes and stationary supergravity solutions of type II Calabi-Yau compactifications," [hep-th/0010222].

[50] F. Denef, "Quantum quivers and Hall / hole halos," JHEP **0210**, 023 (2002). [hep-th/0206072].

[51] F. Denef, G. W. Moore, "Split states, entropy enigmas, holes and halos," [hep-th/0702146 [HEP-TH]].

[52] D. -E. Diaconescu, G. W. Moore and E. Witten, "E(8) gauge theory, and a derivation of K theory from M theory," Adv. Theor. Math. Phys. **6**, 1031 (2003) [hep-th/0005090].

[53] D. -E. Diaconescu, G. W. Moore and E. Witten, "A Derivation of K theory from M theory," hep-th/0005091.

[54] E. Diaconescu, G. W. Moore and D. S. Freed, "The M theory three form and E(8) gauge theory," hep-th/0312069.

[55] E. Diaconescu and G. W. Moore, "Crossing the wall: Branes versus bundles," Adv. Theor. Math. Phys. **14**, 1621 (2010) [arXiv:0706.3193 [hep-th]].

[56] P. Aspinwall, W.-Y. Chuang, D.-E. Diaconescu, J. Manschot, Gregory W. Moore, and Y. Soibelman, "Geometric Engineering of (Framed) BPS States," to appear.

[57] T. Dimofte, S. Gukov and Y. Soibelman, "Quantum Wall Crossing in N=2 Gauge Theories," Lett. Math. Phys. **95**, 1 (2011) [arXiv:0912.1346 [hep-th]].

[58] T. Dimofte, D. Gaiotto and S. Gukov, "Gauge Theories Labelled by Three-Manifolds," arXiv:1108.4389 [hep-th].

[59] T. Dimofte, D. Gaiotto and S. Gukov, "3-Manifolds and 3d Indices," arXiv:1112.5179 [hep-th].

[60] M. R. Douglas, "On D=5 super Yang-Mills theory and (2,0) theory," JHEP **1102**, 011 (2011). [arXiv:1012.2880 [hep-th]].

[61] N. Drukker, D. R. Morrison, T. Okuda, "Loop operators and S-duality from curves on Riemann surfaces," JHEP **0909**, 031 (2009). [arXiv:0907.2593 [hep-th]].

[62] N. Drukker, D. Gaiotto, J. Gomis, "The Virtue of Defects in 4D Gauge Theories and 2D CFTs," [arXiv:1003.1112 [hep-th]].

[63] N. Drukker and F. Passerini, "(de)Tails of Toda CFT," JHEP **1104**, 106 (2011) [arXiv:1012.1352 [hep-th]].

[64] H. Elvang, D. Z. Freedman, L. -Y. Hung, M. Kiermaier, R. C. Myers and S. Theisen, "On renormalization group flows and the a-theorem in 6d," arXiv:1205.3994 [hep-th].

[65] F. Ferrari, "The Dyon spectra of finite gauge theories," Nucl. Phys. **B501**, 53-96 (1997). [hep-th/9702166].

[66] G. Festuccia and N. Seiberg, "Rigid Supersymmetric Theories in Curved Superspace," JHEP **1106**, 114 (2011) [arXiv:1105.0689 [hep-th]].

[67] D. Fiorenza, H. Sati and U. Schreiber, "Multiple M5-branes, String 2-connections, and 7d nonabelian Chern-Simons theory," arXiv:1201.5277 [hep-th].

[68] V.V. Fock and A.B. Goncharov, "Moduli spaces of local systems and higher Teichmuller theory," eprint arXiv:math/0311149

[69] V.V. Fock and A.B. Goncharov, "Moduli spaces of convex projective structures on surfaces," eprint arXiv:math/0405348

[70] V.V. Fock and A.B. Goncharov, "Cluster X-varieties, amalgamation and Poisson-Lie groups," eprint arXiv:math/0508408

[71] V.V. Fock and A.B. Goncharov, "Dual Teichmuller and laminations spaces," in *Handbook of Teichmüller theory*, Vol.I, arXiv:math/0510312

[72] V.V. Fock and A.B. Goncharov, "Cluster ensembles, quantization and the dilogarithm ," in *Handbook of Teichmüller theory*, Vol.I, arXiv:math/0311245

[73] V.V. Fock and A.B. Goncharov, "The quantum dilogarithm and representations of quantum cluster varieties," arXiv:math/0702397 [math.QA]

[74] V.V. Fock and A.B. Goncharov, "Cluster ensembles, quantization and the dilogarithm II: The intertwiner," eprint arXiv:math/0702398

[75] V.V. Fock and A.B. Goncharov, "Cluster X-varieties at infinity," eprint arXiv:1104.0407

[76] D. S. Freed, "Special Kahler manifolds," Commun. Math. Phys. **203**, 31 (1999) [hep-th/9712042].

[77] D. S. Freed, "Dirac charge quantization and generalized differential cohomology," hep-th/0011220.

[78] D. S. Freed, "K theory in quantum field theory," math-ph/0206031.

[79] D. S. Freed, G. W. Moore, G. Segal, "The Uncertainty of Fluxes," Commun. Math. Phys. **271**, 247-274 (2007). [hep-th/0605198].

[80] D. S. Freed, G. W. Moore and G. Segal, "Heisenberg groups and noncommutative fluxes," Annals Phys. **322**, 236 (2007) [arXiv:hep-th/0605200].

[81] D. S. Freed, M. J. Hopkins, J. Lurie, C. Teleman, "Topological Quantum Field Theories from Compact Lie Groups," [arXiv:0905.0731 [math.AT]].

[82] E. Frenkel and E. Witten, "Geometric endoscopy and mirror symmetry," Commun. Num. Theor. Phys. **2** (2008) 113 [arXiv:0710.5939 [math.AG]].

[83] A. Gadde, E. Pomoni, L. Rastelli and S. S. Razamat, "S-duality and 2d Topological QFT," JHEP **1003**, 032 (2010) [arXiv:0910.2225 [hep-th]].

[84] A. Gadde, L. Rastelli, S. S. Razamat and W. Yan, "The Superconformal Index of the E6 SCFT," JHEP **1008**, 107 (2010) [arXiv:1003.4244 [hep-th]].

[85] A. Gadde, L. Rastelli, S. S. Razamat and W. Yan, "The 4d Superconformal Index from q-deformed 2d Yang-Mills," Phys. Rev. Lett. **106**, 241602 (2011) [arXiv:1104.3850 [hep-th]].

[86] A. Gadde, L. Rastelli, S. S. Razamat and W. Yan, "Gauge Theories and Macdonald Polynomials," arXiv:1110.3740 [hep-th].

[87] D. Gaiotto, L. Rastelli and S. S. Razamat, "Bootstrapping the superconformal index with surface defects," arXiv:1207.3577 [hep-th].

[88] D. Gaiotto, E. Witten, "Supersymmetric Boundary Conditions in N=4 Super Yang-Mills Theory," [arXiv:0804.2902 [hep-th]].

[89] D. Gaiotto, E. Witten, "Janus Configurations, Chern-Simons Couplings, And The theta-Angle in N=4 Super Yang-Mills Theory," JHEP **1006**, 097 (2010). [arXiv:0804.2907 [hep-th]].

[90] D. Gaiotto, E. Witten, "S-Duality of Boundary Conditions In N=4 Super Yang-Mills Theory," [arXiv:0807.3720 [hep-th]].

[91] D. Gaiotto, G. W. Moore and A. Neitzke, "Four-dimensional wall-crossing via three-dimensional field theory," arXiv:0807.4723 [hep-th].

[92] D. Gaiotto, "N=2 dualities," arXiv:0904.2715 [hep-th].

[93] D. Gaiotto, J. Maldacena, "The Gravity duals of N=2 superconformal field theories," [arXiv:0904.4466 [hep-th]].

[94] D. Gaiotto, G. W. Moore and A. Neitzke, "Wall-crossing, Hitchin Systems, and the WKB Approximation," arXiv:0907.3987 [hep-th].

[95] D. Gaiotto, "Surface Operators in N = 2 4d Gauge Theories," [arXiv:0911.1316 [hep-th]].

[96] D. Gaiotto, G. W. Moore, A. Neitzke, "Framed BPS States," [arXiv:1006.0146 [hep-th]].

[97] D. Gaiotto, G. W. Moore, A. Neitzke, "Wall-Crossing in Coupled 2d-4d Systems," [arXiv:1103.2598 [hep-th]].

[98] D. Gaiotto, G. W. Moore and Y. Tachikawa, "On 6d N=(2,0) theory compactified on a Riemann surface with finite area," arXiv:1110.2657 [hep-th].

[99] D. Gaiotto and S. S. Razamat, "Exceptional Indices," JHEP **1205**, 145 (2012) [arXiv:1203.5517 [hep-th]].

[100] D. Gaiotto, G. W. Moore and A. Neitzke, "Spectral networks," arXiv:1204.4824 [hep-th].

[101] D. Gaiotto, L. Rastelli and S. S. Razamat, "Bootstrapping the superconformal index with surface defects," arXiv:1207.3577 [hep-th].

[102] D. Gaiotto, G. W. Moore and A. Neitzke, "Spectral Networks and Snakes," arXiv:1209.0866 [hep-th].

[103] J. P. Gauntlett, N. Kim and D. Waldram, "M Five-branes wrapped on supersymmetric cycles," Phys. Rev. D **63**, 126001 (2001) [hep-th/0012195].

[104] A. Gustavsson, "Algebraic structures on parallel M2-branes," Nucl. Phys. **B811**, 66-76 (2009). [arXiv:0709.1260 [hep-th]].

[105] M. Henningson and K. Skenderis, "The Holographic Weyl anomaly," JHEP **9807**, 023 (1998) [hep-th/9806087].

[106] M. Henningson, "Self-dual strings in six dimensions: Anomalies, the ADE-classification, and the world-sheet WZW-model," Commun. Math. Phys. **257**, 291 (2005) [hep-th/0405056].

[107] Mans Henningson, "The partition bundle of type A_{N-1} (2, 0) theory," JHEP **1104**, 090 (2011). [arXiv:1012.4299 [hep-th]].

[108] N. J. Hitchin, A. Karlhede, U. Lindstrom and M. Rocek, "Hyperkahler Metrics and Supersymmetry," Commun. Math. Phys. **108**, 535 (1987).

[109] Nigel Hitchin, "Hyperkahler manifolds," Seminaire N. Bourbaki, 1991-1992, exp. no. 748, p.137-166. available online at http://www.numdam.org

[110] P. -M. Ho, K. -W. Huang, Y. Matsuo, "A Non-Abelian Self-Dual Gauge Theory in 5+1 Dimensions," [arXiv:1104.4040 [hep-th]].

[111] P. -M. Ho and Y. Matsuo, "Note on non-Abelian two-form gauge fields," arXiv:1206.5643 [hep-th].

[112] M. J. Hopkins, I. M. Singer, "Quadratic functions in geometry, topology, and M theory," J. Diff. Geom. **70**, 329-452 (2005). [math/0211216 [math-at]].

[113] K. Hosomichi, S. Lee and J. Park, "AGT on the S-duality Wall," JHEP **1012**, 079 (2010) [arXiv:1009.0340 [hep-th]].

[114] P. S. Howe, N. D. Lambert and P. C. West, "The three-brane soliton of the M-five-brane," Phys. Lett. B **419**, 79 (1998) [arXiv:hep-th/9710033].

[115] P. S. Howe, E. Sezgin and P. C. West, "The Six-dimensional selfdual tensor," Phys. Lett. B **400**, 255 (1997) [hep-th/9702111].

[116] D. Joyce, Y. Song, "A Theory of generalized Donaldson-Thomas invariants," [arXiv:0810.5645 [math.AG]].

[117] V. G. Kac, "Lie Superalgebras," Adv. Math. **26**, 8 (1977).

[118] V. G. Kac, "A Sketch of Lie Superalgebra Theory," Commun. Math. Phys. **53**, 31 (1977).

[119] V. G. Kac and J. W. van de Leur, "On classification of superconformal algebras," In *College Park 1988, Proceedings, Strings '88* 77-106. (see Conference Index)

[120] A. Kapustin, N. Saulina, "The Algebra of Wilson-'t Hooft operators," Nucl. Phys. **B814**, 327-365 (2009). [arXiv:0710.2097 [hep-th]].

[121] A. Kapustin, L. Rozansky, N. Saulina, "Three-dimensional topological field theory and symplectic algebraic geometry I," Nucl. Phys. **B816**, 295-355 (2009). [arXiv:0810.5415 [hep-th]].

[122] A. Kapustin, L. Rozansky, "Three-dimensional topological field theory and symplectic algebraic geometry II," [arXiv:0909.3643 [math.AG]].

[123] A. Kapustin, K. Setter, K. Vyas, "Surface operators in four-dimensional topological gauge theory and Langlands duality," [arXiv:1002.0385 [hep-th]].

[124] A. Kapustin, "Topological Field Theory, Higher Categories, and Their Applications," [arXiv:1004.2307 [math.QA]].

[125] A. Kapustin, N. Saulina, "Topological boundary conditions in abelian Chern-Simons theory," Nucl. Phys. **B845**, 393-435 (2011). [arXiv:1008.0654 [hep-th]].

[126] A. Kapustin, N. Saulina, "Surface operators in 3d Topological Field Theory and 2d Rational Conformal Field Theory," [arXiv:1012.0911 [hep-th]].

[127] A. Klemm, W. Lerche, P. Mayr, C. Vafa, N. P. Warner, "Selfdual strings and N=2 supersymmetric field theory," Nucl. Phys. **B477**, 746-766 (1996). [hep-th/9604034].

[128] M. Kontsevich and Y. Soibelman, "Stability structures, motivic Donaldson-Thomas invariants and cluster transformations," arXiv:0811.2435

[129] M. Kontsevich, Y. Soibelman, "Motivic Donaldson-Thomas invariants: Summary of results," [arXiv:0910.4315 [math.AG]].

[130] M. Kontsevich and Y. Soibelman, "Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants," Commun. Num. Theor. Phys. **5**, 231 (2011) [arXiv:1006.2706 [math.AG]].

[131] N. Lambert, C. Papageorgakis, "Nonabelian (2,0) Tensor Multiplets and 3-algebras," JHEP **1008**, 083 (2010). [arXiv:1007.2982 [hep-th]].

[132] N. Lambert, C. Papageorgakis, M. Schmidt-Sommerfeld, "M5-Branes, D4-Branes and Quantum 5D super-Yang-Mills," JHEP **1101**, 083 (2011). [arXiv:1012.2882 [hep-th]].

[133] H. Lin, O. Lunin, J. M. Maldacena, "Bubbling AdS space and 1/2 BPS geometries," JHEP **0410**, 025 (2004). [hep-th/0409174].

[134] S. L. Lukyanov and A. B. Zamolodchikov, "Quantum Sine(h)-Gordon Model and Classical Integrable Equations," JHEP **1007**, 008 (2010) [arXiv:1003.5333 [math-ph]].

[135] J. Lurie, "On the Classification of Topological Field Theories," arXiv:0905.0465

[136] J. Manschot, B. Pioline and A. Sen, "Wall Crossing from Boltzmann Black Hole Halos," JHEP **1107**, 059 (2011) [arXiv:1011.1258 [hep-th]].

[137] A. Mikhailov, "BPS states and minimal surfaces," Nucl. Phys. **B533**, 243-274 (1998). [hep-th/9708068].

[138] A. Mikhailov, N. Nekrasov and S. Sethi, "Geometric realizations of BPS states in N=2 theories," Nucl. Phys. B **531**, 345 (1998) [hep-th/9803142].

[139] G. W. Moore and N. Seiberg, "Polynomial Equations for Rational Conformal Field Theories," Phys. Lett. B **212**, 451 (1988).

[140] G. W. Moore and N. Seiberg, "Classical and Quantum Conformal Field Theory," Commun. Math. Phys. **123**, 177 (1989).

[141] G. W. Moore and N. Seiberg, "Lectures On Rcft," In *Trieste 1989, Proceedings, Superstrings '89* 1-129. (see Conference Index)

[142] G. W. Moore, "Anomalies, Gauss laws, and Page charges in M-theory," Comptes Rendus Physique **6**, 251-259 (2005). [hep-th/0409158].

[143] G. W. Moore, G. Segal, "D-branes and K-theory in 2D topological field theory," [hep-th/0609042].

[144] G. W. Moore and Y. Tachikawa, "On 2d TQFTs whose values are holomorphic symplectic varieties," arXiv:1106.5698 [hep-th].

[145] A. Neitzke and B. Pioline, unpublished.

[146] W. Nahm, "Supersymmetries and their Representations," Nucl. Phys. **B135**, 149 (1978).

[147] E. O Colgain, J. B. Wu and H. Yavartanoo, "On the generality of the LLM geometries in M-theory," JHEP **1104** (2011) 002 [arXiv:1010.5982 [hep-th]].

[148] B. Pioline, "Four ways across the wall," J. Phys. Conf. Ser. **346**, 012017 (2012) [arXiv:1103.0261 [hep-th]].

[149] J. Polchinski, "Renormalization and Effective Lagrangians," Nucl. Phys. B **231** (1984) 269.

[150] M. Rozali, "Matrix theory and U duality in seven-dimensions," Phys. Lett. B **400**, 260 (1997) [hep-th/9702136].

[151] Simons Center Workshop on Differential Cohomology, Jan.10-14, 2011. http://scgp.stonybrook.edu/archives/851

[152] M. Scheunert, W. Nahm and V. Rittenberg, "Classification of All Simple Graded Lie Algebras Whose Lie Algebra Is Reductive. 1.," J. Math. Phys. **17**, 1626 (1976).

[153] M. Scheunert, W. Nahm and V. Rittenberg, "Classification of All Simple Graded Lie Algebras Whose Lie Algebra Is Reductive. 2. Construction of the Exceptional Algebras," J. Math. Phys. **17**, 1640 (1976).

[154] N. Seiberg, "Naturalness versus supersymmetric nonrenormalization theorems," Phys. Lett. B **318**, 469 (1993) [hep-ph/9309335].

[155] N. Seiberg, "The Power of holomorphy: Exact results in 4-D SUSY field theories," arXiv:hep-th/9408013.

[156] N. Seiberg and E. Witten, "Monopole Condensation, And Confinement In N=2 Supersymmetric Yang-Mills Theory," Nucl. Phys. B **426**, 19 (1994) [Erratum-ibid. B **430**, 485 (1994)] [arXiv:hep-th/9407087].

[157] N. Seiberg and E. Witten, "Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD," Nucl. Phys. B **431**, 484 (1994) [hep-th/9408099].

[158] N. Seiberg and E. Witten, "Gauge dynamics and compactification to three dimensions," arXiv:hep-th/9607163.

[159] N. Seiberg, "Nontrivial fixed points of the renormalization group in six-dimensions," Phys. Lett. B **390**, 169 (1997) [arXiv:hep-th/9609161].

[160] N. Seiberg, "Notes on theories with 16 supercharges," Nucl. Phys. Proc. Suppl. **67**, 158 (1998) [arXiv:hep-th/9705117].

[161] N. Seiberg and W. Taylor, "Charge Lattices and Consistency of 6D Supergravity," JHEP **1106**, 001 (2011) [arXiv:1103.0019 [hep-th]].

[162] A. D. Shapere and C. Vafa, "BPS structure of Argyres-Douglas superconformal theories," hep-th/9910182.

[163] V. P. Spiridonov, Essays on the theory of elliptic hypergeometric functions, Russian Math. Surveys 63 (2008), no.3, 405–472.

[164] A. Strominger, "Open p-branes," Phys. Lett. **B383**, 44-47 (1996). [hep-th/9512059].

[165] Y. Tachikawa, "N=2 S-duality via Outer-automorphism Twists," J. Phys. A **A44**, 182001 (2011). [arXiv:1009.0339 [hep-th]].

[166] Y. Tachikawa, “On W-algebras and the symmetries of defects of 6d N=(2,0) theory,” JHEP **1103**, 043 (2011). [arXiv:1102.0076 [hep-th]].

[167] Y. Terashima, M. Yamazaki, “SL(2,R) Chern-Simons, Liouville, and Gauge Theory on Duality Walls,” [arXiv:1103.5748 [hep-th]].

[168] J. Teschner, “An Analog of a modular functor from quantized teichmuller theory,” math/0510174 [math-qa].

[169] A. A. Tseytlin, “Harmonic superpositions of M-branes,” Nucl. Phys. B **475**, 149 (1996) [hep-th/9604035].

[170] C. Vafa, “Geometric origin of Montonen-Olive duality,” Adv. Theor. Math. Phys. **1**, 158-166 (1998). [hep-th/9707131].

[171] S. Weinberg, “The quantum theory of fields. Vol. 3: Supersymmetry,” Cambridge, UK: Univ. Pr. (2000) 419 p

[172] E. Witten, “Topological Quantum Field Theory,” Commun. Math. Phys. **117**, 353 (1988).

[173] E. Witten, “Some comments on string dynamics,” arXiv:hep-th/9507121.

[174] E. Witten, “Five-brane effective action in M theory,” J. Geom. Phys. **22**, 103-133 (1997). [hep-th/9610234].

[175] E. Witten, “Solutions of four-dimensional field theories via M-theory,” Nucl. Phys. B **500**, 3 (1997) [arXiv:hep-th/9703166].

[176] E. Witten, “Duality relations among topological effects in string theory,” JHEP **0005**, 031 (2000). [hep-th/9912086].

[177] E. Witten, “Geometric Langlands From Six Dimensions,” arXiv:0905.2720 [hep-th].

[178] E. Witten, “Fivebranes and Knots,” arXiv:1101.3216 [hep-th].