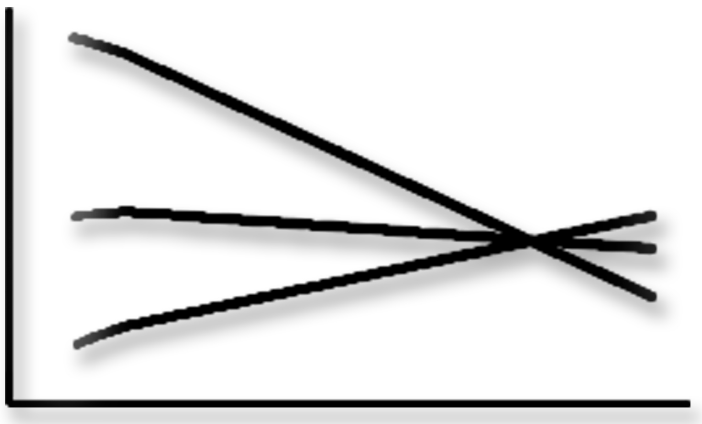
Supersymmetric Grand Unification



Sander Mooij
Master's Thesis in Theoretical Physics
Institute for Theoretical Physics
University of Amsterdam

Supervisor: Prof. Dr. Jan Smit

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Abstract

At an energy scale around 10^{16} GeV the Standard Model (SM) is assumed to give way to a theory in which all scalar, spinor and boson fields are in one universal gauge group, a so-called Grand Unified Theory (GUT). In this thesis we review the construction of supersymmetric GUTs and check in Mathematica how well its predictions, gauge coupling unification and Yukawa coupling unification, are met.

Thesis Outline

To provide a solid background we first review the Standard Model and its problematic features, especially the hierarchy problem. In the second chapter some necessary group theory, mainly weight vector analysis, is studied. Then, in the third chapter, we construct the two most popular GUTs, $SU(5)$ and $SO(10)$. We show that in these theories the SM hierarchy problem is still unresolved and see that gauge couplings do not meet well enough to fit low-energy measurements. Therefore, we are led to study supersymmetric GUTs. In the fourth chapter we introduce supersymmetry (SUSY). We review the minimal supersymmetric extension of the Standard Model (MSSM) and see how it solves the aforementioned problems. The theoretical part of the thesis is concluded by the construction of the most simple (elegant) SUSY GUT.
In the phenomenological part we convert electroweak scale observables, particle masses and quark and lepton mixing matrices to Yukawa matrices. Two-loop Yukawa Renormalisation Group Equations are solved in Mathematica. This enables us to check whether the SUSY GUT promise of Yukawa unification is met. In the last chapter a recent model by Dermisek and Raby that connects to the results of this Yukawa analysis is studied.

Chapter 1

The Standard Model

Dramatic progress in (special) relativity, quantum mechanics, quantum field theory and quantumchromodynamics has led us to the Standard Model of Elementary Particle Physics, or SM. It emerged in the early seventies and has been proven right in numerous experiments afterwards.
The SM lists all elementary particles and describes their interactions via the strong, weak interaction and $U(1)$ "hyper" interaction. After spontaneous symmetry breaking, caused by the so-called Higgs boson taking a nonzero vacuum expectation value, the weak and $U(1)$ hyperforce combine into the electromagnetic interaction. In a minute we will see how exactly this comes to be.

1.1 Introduction to the Standard Model

In this first section we will construct the full Standard Model Lagrangian.

1.1.1 Fundaments

The SM Lagrangian rests on the fundamental assumptions that it should be *gauge invariant*, *Lorentz invariant* and *renormalizable*.

$SU(3) \times SU(2) \times U(1)$ gauge invariance

To every field f appearing in the SM we associate a gauge transformation:

$$f \to e^{i\alpha^a T_s^a} e^{i\beta^b T_w^b} e^{i\gamma Y} f. \tag{1.1}$$

The first part $e^{i\alpha^a T_s^a}$ is the $SU(3)$ part. By definition, $SU(3)$ has 8 generators. These are the 8 matrices T_s^a. We can choose (infinitely) many representations for these generators, as long as they obey the commutation relations $[T_s^a, T_s^b] = i f^{abc} T_s^c$. (All over this work, a sum over repeated indices is assumed.) The structure constants are typical for $SU(3)$, they define the group. They can be looked up in appendix A.2.
The 8 α^a are infinitesimal displacement vectors specifying the gauge transformation.
Interesting possibilities are the so-called *fundamental representation*, where we take (1/2 times) the 8 traceless 3×3 Gell-Mann matrices λ^a to represent the T_s^a, the

trivial representation where we set all T_s^a to 0 so that this whole part disappears from the gauge transformation, and the *adjoint representation* where we have 8 8×8 matrices $(T_i)_{jk} = -if_{ijk}$ that represent the T_s^a.
The connection with physics now is that the strong force is mediated by the 8 gauge bosons (gluons) G_μ^a that transform in the adjoint representation of $SU(3)$. Every fermion that is sensitive to the strong force (that is, every quark) is represented by a field that transforms in the fundamental representation of $SU(3)$, while to the antiquarks we associate a field that transforms in the antifundamental representation: $T_s^a = -(\frac{\lambda^a}{2})^\star \equiv \frac{\overline{\lambda^a}}{2}$[1]. Fields representing particles that do not feel the strong interaction (particles that cannot couple to a gluon, leptons, that is) transform in the trivial representation of $SU(3)$.
The second part $e^{i\beta^b T_w^b}$ is the $SU(2)$ part. Everything works out in essentially the same way as in the $SU(3)$ case. The main difference is that $SU(2)$ has only 3 generators T_w^b, so there will be only 3 gauge bosons A_μ^b. These three bosons mediate the weak force. The fundamental representation is given by $T_w^b = \frac{\sigma^b}{2}$, where the σ^b are the well-known Pauli matrices. As we have $\sigma^2\sigma^b\sigma^2 = \sigma^b$, the antifundamental representation of $SU(2)$ is equivalent to its fundamental representation. Such a representation is called *real*.
The third part, $e^{i\gamma Y}$, that contains the $U(1)$ part is different in the sense that there is just one (one-dimensional) generator Y involved. Y is just a number and is called *hypercharge*. Such a gauge theory with just one generator (and, therefore, just one gauge boson B_μ) is called *Abelian* while a gauge theory with several non-commuting generators is *non-Abelian*.

Reps of non-Abelian groups are labeled by their dimension: the fundamental rep of $SU(3)$ for example, where generators are 3×3 matrices) for example is denoted as $\mathbf{3}$, the antifundamental rep as $\overline{\mathbf{3}}$ and the trivial rep as $\mathbf{1}$. The reps of the Abelian group $U(1)$ are simply denoted by the value of the one dimensional generator Y, the hypercharge.

Now for gauge invariance. Of each particle we know what forces it is subject to. So, for each of the associated fields we know the representation, the form of the matrices T_s^a, T_w^b, Y, in which it transforms. We can then try to write down those products of fields that as a whole are gauge invariant. Technically we want the decomposition of the tensor product to contain a $\mathbf{1}$ singlet. We will come back to this later, for the moment we just state that the tensor product of a rep and its conjugate always contain such a singlet:

$$\mathbf{r} \otimes \overline{\mathbf{r}} = \mathbf{1} \oplus \dots \tag{1.2}$$

Of course the next question is how to project out such a singlet. We will settle that when actually writing down the SM Lagrangian.
In the Abelian case gauge invariance is equivalent to a vanishing sum of $U(1)$ charges.

One last comment is that to write gauge invariant quantities including derivatives

[1] The concept of antimatter, or charge conjugation is reviewed in appendix A.1.

we had better replace the ordinary derivative operator ∂_μ by the gauge-covariant operator

$$D_\mu = \partial_\mu - ig_s G^a_\mu T^a_s - ig_w A^a_\mu T^a_w - ig' B_\mu Y. \tag{1.3}$$

It is easy to check that now $D_\mu f$ gets the same gauge transformation as f itself. The coupling constants g_s, g_w and g' indicate the strength of the strong force, weak force and $U(1)$ force respectively.

Lorentz invariance

The fields needed to describe all elementary particles come in different representations of the Lorentzgroup. We have fields of spin 0, 1/2, and 1, or *scalar*, *spinor* and *vector* fields. We look for combinations that are as a whole invariant under a Lorentz transformation.

Scalar fields $\phi(x)$ are in the trivial representation of the Lorentz group. A Lorentz transformation has a most simple effect: the transformed field, evaluated at the boosted point, has the same value as the original field at the point before boosting:

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x). \tag{1.4}$$

So, a scalar field is a Lorentz invariant quantity in itself.

Spinor fields $\psi(x)$ are four-dimensional anticommuting objects. The effect of a Lorentz transformation on a four-spinor is

$$\psi(x) \to \psi'(x) = e^{\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}\psi(\Lambda^{-1}x). \tag{1.5}$$

Here $\omega_{\mu\nu}$ specify the Lorentz tranformation. The rotation- and boost generators are contained in the antisymmetric $S^{\mu\nu}$: the boost generators K^i are on positions S^{0i}, the rotation generators are in the S^{ij} part. We now choose to represent $S^{\mu\nu}$ in the following way:

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{1.6}$$

(In this thesis we will exclusively use the Weyl representation, which can be looked up in Appendix A.1.)
Now we can figure out how to write down Lorentz invariant quantities. It is easy to verify that in this representation we have rotation terms that commute with γ^0 while the boost terms anticommute with γ^0. Furthermore we have that the rotation terms are hermitian and the boosts are antihermitian, as should be true in every representation. With all this in mind, we define

$$\overline{\psi} = \psi^\dagger\gamma^0. \tag{1.7}$$

Under a Lorentz transformation this quantity transforms as

$$\overline{\psi} \to \psi^\dagger(e^{\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger})\gamma^0 \tag{1.8}$$

$$= \psi^\dagger\gamma^0(e^{\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}), \tag{1.9}$$

so we conclude that the quantity $\overline{\psi}\psi$ transforms as a Lorentz scalar. In exactly the same way one can prove that $\overline{\psi}\gamma^\mu\psi$ transforms a a Lorentz vector. To have a Lorentz invariant quantity it should be contracted with another Lorentz vector.

Very often spinor fields $\psi(x)$ are decomposed in two two-dimensional parts, which are referred to as the lefthanded part $\psi_L(x)$ and the righthanded part $\psi_R(x)$:

$$\psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}, \qquad \psi_L = P_L\psi, \quad \psi_R = P_R\psi. \tag{1.10}$$

Here P_L and P_R are projection operators, defined in appendix A.1.
This decomposition does not change the above statements about Lorentz invariance. But let us see how this decomposition actually comes about.
The Lorentz group $SO(3,1)$ is similar to $SU(2) \times SU(2)$. This becomes clear when we re-order the 3 rotation generators J^i and the 3 boost generators K^i in the new generators $C^i = \frac{1}{2}(J^i + iK^i)$ and $D^i = \frac{1}{2}(J^i - iK^i)$. We then have

$$[C^i, C^j] = i\epsilon_{ijk}C^k, \quad [D^i, D^j] = i\epsilon_{ijk}D^k, \quad [C^i, D^j] = 0, \tag{1.11}$$

two separate copies of $SU(2)$.
We can now take $J^i = iK^i$ by taking $J^i = \frac{\sigma^i}{2}, K^i = -i\frac{\sigma^i}{2}$. We then have $D^i = 0$. Such a field, in which the C-generators are in the fundamental $SU(2)$ representation and the D-generators are in the trivial representation is called lefthanded.
The other possibility is of course to take $J^i = \frac{\sigma^i}{2}, K^i = i\frac{\sigma^i}{2}$ to have $J^i = -iK^i$ so that now $C^i = 0$. These are righthanded fields.
In the two-spinor formalism a Lorentz transformation takes the form

$$\psi_L(x) \to (1 - i\theta^i\frac{\sigma^i}{2} - \beta^i\frac{\sigma^i}{2})\psi_L(\Lambda^{-1}x), \tag{1.12}$$

$$\psi_R(x) \to (1 - i\theta^i\frac{\sigma^i}{2} + \beta^i\frac{\sigma^i}{2})\psi_R(\Lambda^{-1}x). \tag{1.13}$$

We have written this so explicitly because now, on using the identity $\sigma^2\left(\sigma^i\right)^\star = -\sigma^i\sigma^2$, it is easy to check that $\sigma^2\psi_R^\star$ transforms as a lefthanded spinor. Therefore we can safely represent righthanded fields by lefthanded antifields. This will prove to be very useful later on. Details are worked out in Appendix B.
Next we turn to vector fields $V^\mu(x)$, fields that are in the four-dimensional vector representation of the Lorentzgroup. These have transformation

$$V^\mu(x) \to V'^\mu(x) = \Lambda^\mu_\nu V^\nu(\Lambda^{-1}x). \tag{1.14}$$

Here Λ^μ_ν is an element of the Lorentz group. $V^\mu(x)$ transforms as a four-vector up to gauge transformations $(\partial^\mu\omega(x))$. For gauge fields we define

$$F_{\mu\nu} = \partial_\mu V^\nu(x) - \partial_\nu V^\mu(x) \tag{1.15}$$

$$F^a_{\mu\nu} = \partial_\mu V^a_\nu(x) - \partial_\nu V^a_\mu(x) + gf^{abc}V^b_\mu(x)V^c_\nu(x), \tag{1.16}$$

for the Abelian and non-Abelian case respectively.
To build a Lorentz scalar out of vector fields the crucial demand is that all indices

should be contracted. A Lorentz invariant quantity from which equations of motion can be deduced is $F_{\mu\nu}F^{\mu\nu}$ or $F^a_{\mu\nu}F^{a\mu\nu}$.

Renormalizability

We will treat this third fundament under the SM in a much quicker way. We want the Standard Model action not to blow up when we move to higher and higher energies. As the action is just the spacetime integral over the Lagrangian,

$$\mathcal{S}_{SM} = \int d^4x \mathcal{L}_{SM} \tag{1.17}$$

no term in the Lagrangian should have mass dimension larger than 4. Scalar and vector fields have mass dimension 1, spinor fields have mass dimension $\frac{3}{2}$. See for instance [14, chap. 4].

This ends the discussion on the fundamental Standard Model symmetries. Additional symmetries are described in C.

1.1.2 Contents of the Standard Model

As stated before, the Standard Model describes how the strong force, the weak force and the hyperforce affect all elementary particles. (The electromagnetic force is not one of the three fundamental forces,in the next section we will see how it results from the breaking of the weak force and the hyperforce.) The $8 + 3 + 1$ gauge bosons that mediate these forces all have spin 1. Gravity (and the graviton) are not described.

Fermions that are sensitive to the strong force are called quarks. The strong force is described by a $SU(3)$ gauge theory (QCD). That is why every quark q should be thought of as a "color triplet", it has a so-called "red", "blue" and "green" component $q = (q^r, q^b, q^g)$. The $SU(3)$ gauge transformation acts on this "color space".

Quarks come with two different electric charges, the "up quark" u has charge $+\frac{2}{3}$, the "down quark" d has charge $-\frac{1}{3}$. The same is true for the leptons (particles that are not subject to the strong force): the massless neutrino ν_e has charge 0, the electron e has charge -1.

All particles described so far are spin $\frac{1}{2}$ fermions. They come in lefthanded and righthanded versions. The only exception is that, at least at the time the SM was made up, a righthanded neutrino had never been detected. (Now there is strong evidence that righthanded neutrinos do exist, but we will settle that issue later.) A crucial observation now is that lefthanded particles have weak interactions but righthanded particles do not. This has led to the idea of putting the lefthanded quarks and leptons in weak $SU(2)$ doublets while keeping the righthanded particles in $SU(2)$ singlets.

The u, d, ν_e and e (up quark, down quark, neutrino and electron) together form the so-called "first generation" of SM particles. There two more generations, c, s, ν_μ, μ (charm quark, strange quark, μ-neutrino and muon) and t, b, ν_τ, τ (top-quark,

bottom quark, τ-neutrino and tau). The second and third generation are exactly like the first one except for the increasing masses.
The last, most mysterious, SM particle is the Higgs boson, which is needed to write down gauge invariant mass terms. A mass term for a field ψ should be bi-linear in ψ and $\overline{\psi}$. From the gauge transformations of the SM fields we can see that any potential mass term like $m_\psi \overline{\psi}_L \psi_R$ or $m_\psi \overline{\psi_L} \psi_L$ can never be gauge invariant. To overcome these troubles a new spin 0 particle has been postulated: the Higgs boson. The associated Higgs field $\phi(x)$ transforms as a $SU(2)$ doublet, thus rendering the combination $m_\psi \overline{\psi}_L \phi \psi_R$ gauge invariant. When ϕ takes a non-zero vacuum expectation value we are left with an effective mass term. The Higgs boson has never been detected (it has been excluded up to 114 GeV) but this situation may change within a few months from now when the LHC starts working. Without the Higgs particle the Standard Model is a sick theory.

So let us now write down all fields we have in the Standard Model and their $SU(3) \times SU(2) \times U(1)$ representations. At this stage the hypercharge assignments may seem strange, but in the next section we will see how after symmetry breaking these values lead to the correct electric charges.
In the first generation we have a lefthanded $\left(\mathbf{3}, \mathbf{2}, \frac{1}{6}\right)$ $SU(2)$ doublet of quark $SU(3)$ triplets:

$$Q_L(x) = \begin{pmatrix} u_L^r(x) & u_L^g(x) & u_L^b(x) \\ d_L^r(x) & d_L^g(x) & u_L^b(x) \end{pmatrix},$$

a righthanded $\left(\mathbf{3}, \mathbf{1}, \frac{2}{3}\right)$up quark triplet ($SU(2)$ singlet):

$$u_R(x) = \begin{pmatrix} u_R^r(x) & u_R^g(x) & u_R^b(x) \end{pmatrix},$$

a righthanded $\left(\mathbf{3}, \mathbf{1}, -\frac{1}{3}\right)$ down quark triplet:

$$d_R(x) = \begin{pmatrix} d_R^r(x) & d_R^g(x) & d_R^b(x) \end{pmatrix},$$

a lefthanded $\left(\mathbf{1}, \mathbf{2}, -\frac{1}{2}\right)$ lepton doublet ($SU(3)$ singlet):

$$L_L(x) = \begin{pmatrix} \nu_L(x) \\ e_L(x) \end{pmatrix}$$

and a righthanded $(\mathbf{1}, \mathbf{1}, -1)$ electron singlet

$$e_R(x).$$

The second and third generation fields have exactly the same structure.
Finally we have the $\left(\mathbf{1}, \mathbf{2}, \frac{1}{2}\right)$ Higgs doublet

$$\phi(x) = \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix}. \tag{1.18}$$

1.1.3 Writing down the SM Lagrangian

Let us start with the Higgs kinetic part:

$$\mathcal{L}_{HK} =(D^\mu\phi)^\dagger D_\mu\phi \tag{1.19}$$

$$= \left[(\partial^\mu + ig_w\tau^a A^{a\mu}(x) + ig'\frac{1}{2}B^\mu(x))\phi^\star(x)\right]^T$$
$$(\partial_\mu - ig_w\tau^a A^a_\mu(x) - ig'\frac{1}{2}B^\mu(x))\phi(x). \tag{1.20}$$

We directly see that all fundamental demands are met. ϕ is in $SU(2)$ rep **2** and has hypercharge $\frac{1}{2}$, while $\phi^\dagger$ transforms in the $\overline{\mathbf{2}}$ (which is equivalent to **2**) and carries hypercharge $-\frac{1}{2}$; henceforth the existence of a gauge invariance in the product is guaranteed [2]. From a Lorentz point of view we have two vectors that are contracted. The mass dimension of the whole expression equals $1+1+1+1=4$.
Let us elaborate a bit on gauge properties. The corresponding transformation is

$$\phi(x) \to e^{i\beta^b\tau^b}e^{i\gamma\frac{1}{2}}\phi(x). \tag{1.21}$$

We thus have four massless vector bosons $(A^1_\mu, A^2_\mu, A^3_\mu, B_\mu)$. One now assumes spontaneous symmetry breaking: ϕ takes a nonzero vacuum expectation value (vev):

$$< \phi(x) >= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{1.22}$$

It can be checked that now the particular gauge transformation $\beta^1=\beta^2=0, \beta^3=\gamma$ leaves the vacuum invariant while all others do not. We thus have only one symmetry left, with generator T^3+Y, in this case $\tau^3+\frac{1}{2}$. Goldstone's theorem now predicts that we are going to find three massive gauge boson fields that correspond with the "broken" generators and one that is still massless. Let us check that this is indeed the case.
Concentrating on the terms quadratic in vector boson fields we have

$$\Delta\mathcal{L}_{HK} = \frac{1}{2}\begin{pmatrix} 0 & v \end{pmatrix}\Big(g_w A^{a\mu}(x)\tau^a + \frac{g'}{2}B^\mu(x)\Big)\Big(g_w A^a_\mu(x)\tau^a + \frac{g'}{2}B_\mu(x)\Big)\begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{1.23}$$

which yields, after inserting all Pauli matrices,

$$\Delta\mathcal{L}_{HK} =\frac{1}{2}\frac{v^2}{4}\Big[g_w^2\big(A^{1\mu}(x)A^1_\mu(x) + A^{2\mu}(x)A^2_\mu(x) + A^{3\mu}(x)A^3_\mu(x)\big)$$
$$- 2g_w g' A^{3\mu}(x)B_\mu(x) + 2(g')^2 B^\mu(x)B_\mu(x)\Big]. \tag{1.24}$$

Now we change variables:

[2] We can project out this singlet term by using a Kronecker delta: $\delta^{ij}(\phi^\dagger)^i\phi^j$.

$$W_\mu^+ = \frac{1}{\sqrt{2}}(A_\mu^1 - iA_\mu^2) \tag{1.25}$$

$$W_\mu^- = \frac{1}{\sqrt{2}}(A_\mu^1 + iA_\mu^2) \tag{1.26}$$

$$Z_\mu^0 = \frac{1}{\sqrt{g_w^2 + g'^2}}(g_w A_\mu^3 - g' B_\mu) \equiv \cos\theta_w A_\mu^3 - \sin\theta_w B_\mu \tag{1.27}$$

$$A_\mu = \frac{1}{\sqrt{g_w^2 + g'^2}}(g' A_\mu^3 + g_w B_\mu) \equiv \sin\theta_w A_\mu^3 + \cos\theta_w B_\mu, \tag{1.28}$$

where θ_w is the so-called Weinberg angle. It sort of parametrizes the amount of mixing between the $SU(2)$ and the $U(1)$ part of the theory.
Inserting (the inverses) of these expressions we finally arrive at

$$\Delta\mathcal{L}_{HK} = \frac{1}{2}\frac{v^2}{4}\left[2g_w^2 W^{+\mu}(x)W_\mu^-(x) + (g_w^2 + g'^2)Z^{0\mu}(x)Z_\mu^0(x)\right]. \tag{1.29}$$

Note the absence of a $A^\mu(x)A_\mu(x)$ term. Bearing in mind that a mass term for a field f looks like $\frac{1}{2}m^2 ff$ we thus conclude that the breaking of the $SU(2)\times U(1)$ symmetry has led us to a decription of the W_μ^+ and W_μ^- bosons (with mass $\frac{g_w v}{2}$), the Z_μ^0 boson (with mass $\sqrt{g_w^2 + g'^2}\frac{v}{2}$) and the massless A_μ boson.
(Right from the start it has been clear that we are not going to have any term quadratic in gluon fields G_μ^a as the Higgs is in the trivial rep of $SU(3)$. This is how the masslessness of gluons is described.)
It is very instructive to check what the covariant derivative 1.3 looks like after this change of variables:

$$\begin{aligned} D_\mu =& \partial_\mu - i\frac{g_w}{\sqrt{2}}\big(W_\mu^+ T^+ + W_\mu^- T^-\big) - \frac{i}{\sqrt{g_w^2 + g'^2}} Z_\mu^0\big(g_w^2 T^3 - g'^2 Y\big) \\ & - i\frac{g_w g'}{\sqrt{g_w^2 + g'^2}} A_\mu\big(T^3 + Y\big). \end{aligned} \tag{1.30}$$

Here $T^\pm = T^1 \pm iT^2$ and we have for a moment switched back to write general generators T^3 and Y, although we know that for the Higgs boson these are represented by τ^3 and $\frac{1}{2}$ respectively because the Higgs is in the $\big(\mathbf{1}, \mathbf{2}, \frac{1}{2}\big)$ rep.
Because of its masslessness we should identify A_μ with the photon. Its coupling strength, e, equals $\frac{g_w g'}{\sqrt{g_w^2 + g'^2}}$. Its generator, Q, is seen to equal $T^3 + Y$. We now have a full description of the so-called electroweak breaking, caused by the Higgs mechanism, of $SU(2)$ weak symmetry and $U(1)$ symmetry to the electromagnetic $U(1)$ symmetry we observe in nature[3].

[3]Note that now we can check that we took the right hypercharge assignments in the preceding section, because for every field we have that $T^3 + Y$ equals their electric charge. For u_L we find, for example, $T^3 + Y = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, for u_R we have $T^3 + Y = 0 + \frac{2}{3} = \frac{2}{3}$, for d_L it is $T^3 + Y = -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3}$ etc.

Next fermion kinetic terms are considered:

$$\begin{aligned}\mathcal{L}_{FK} =&\overline{Q_L}(i\gamma^\mu D_\mu)Q_L + \overline{u_R}(i\gamma^\mu D_\mu)u_R + \overline{d_R}(i\gamma^\mu D_\mu)d_R \\ &+ \overline{L_L}(i\gamma^\mu D_\mu)L_L + \overline{e_R}(i\gamma^\mu D_\mu)e_R.\end{aligned} \tag{1.31}$$

Gauge invariance is, again, easily observable. If a field ψ is in a rep r , then $\overline{\psi}$ transforms in the conjugate $\overline{r}$ rep (this is shown in appendix A.1). Hypercharges trivially add to their conjugates to zero. Lorentz invariance is also ensured, fermion fields ψ are paired with $\overline{\psi}$ and no indices remain uncontracted. The mass dimension is $\frac{3}{2} + 1 + \frac{3}{2} = 4$.
Knowing the reps these fermions fields are in, we write out (for the last time!) the covariant derivatives:

$$\begin{aligned}\mathcal{L}_{FK} =&\overline{Q_L} i\gamma^\mu \big(\partial_\mu - ig_s G^a_\mu (\frac{\lambda^a}{2}) - ig_w A^a_\mu \tau^a - ig' \frac{1}{6} B_\mu\big) Q_L \\ &+ \overline{u_R} i\gamma^\mu \big(\partial_\mu - ig_s G^a_\mu (\frac{\lambda^a}{2}) - ig' \frac{2}{3} B_\mu\big) u_R \\ &+ \overline{d_R} i\gamma^\mu \big(\partial_\mu - ig_s G^a_\mu (\frac{\lambda^a}{2}) - ig' (-\frac{1}{3}) B_\mu\big) d_R \\ &+ \overline{L_L} i\gamma^\mu \big(\partial_\mu - ig_w A^a_\mu \tau^a - ig' (-\frac{1}{2}) B_\mu\big) L_L \\ &+ \overline{e_R} i\gamma^\mu \big(\partial_\mu - ig_w A^a_\mu \tau^a - ig' (-1) B_\mu\big) e_R.\end{aligned} \tag{1.32}$$

The partial derivative in each term is just the normal kinetic term (connected to the Dirac equation) we expect to appear for every fermion field.
The two $SU(3)$ terms (active in colour space) could be worked out by inserting the numerical values of the Gell-Mann matrices and thus render all kind of interactions between gluons and coloured quarks. At the moment this is not in our interest.
We choose to focus on the $SU(2) \times U(1)$ terms because they reveal some physics that will be of phenomenological importance in this work. On inserting the Pauli matrices and changing variables from (A^a_μ, B_μ) to $(W^+_\mu, W^-_\mu, Z^0_\mu, A_\mu)$ once again, we find

$$
\begin{aligned}
\Delta\mathcal{L}_{FK} =& g_w W_\mu^+ \left[\frac{1}{\sqrt{2}}\big(\overline{u_L}\gamma^\mu d_L + \overline{\nu_L}\gamma^\mu e_L\big)\right] \\
&+ g_w W_\mu^- \left[\frac{1}{\sqrt{2}}\big(\overline{d_L}\gamma^\mu u_L + \overline{e_L}\gamma^\mu \nu_L\big)\right] \\
&+ g_w Z_\mu^0 \Bigg[\frac{1}{\cos\theta_w}\big((\frac{1}{2} - \frac{2}{3}\sin^2\theta_w)\overline{u_L}\gamma^\mu u_L + (-\frac{1}{2} + \frac{1}{3}\sin^2\theta_w)\overline{d_L}\gamma^\mu d_L \\
&\quad + (-\frac{1}{2} + \sin^2\theta_w)\overline{e_L}\gamma^\mu e_L + \frac{1}{2}\overline{\nu_L}\gamma^\mu \nu_L \\
&\quad - \frac{2}{3}\sin^2\theta_w \overline{u_R}\gamma^\mu u_R + \frac{1}{3}\sin^2\theta_w \overline{d_r}\gamma^\mu d_R + \sin^2\theta_w \overline{e_r}\gamma^\mu e_R\big)\Bigg] \\
&+ eA_\mu \Bigg[\frac{2}{3}(\overline{u_L}\gamma^\mu u_L + \overline{u_R}\gamma^\mu u_R) - \frac{1}{3}(\overline{d_L}\gamma^\mu d_L + \overline{d_R}\gamma^\mu d_R) \\
&\quad - (\overline{e_L}\gamma^\mu e_L + \overline{e_R}\gamma^\mu e_R)\Bigg] \qquad (1.33) \\
\equiv& g_w W_\mu^+ J^{\mu+} + g_w W_\mu^- J^{\mu-} + g_w Z_\mu J^{\mu Z} + eA_\mu J^{\mu EM}. \qquad (1.34)
\end{aligned}
$$

Here we can identify the two charged weak currents, the neutral weak current and the electromagnetic current. We could use this Lagrangian to describe weak decay. Note that weak currents discriminate between left- and righthanded fields, but the electromagnetic current does not. The charged weak currents connect the two states of the $SU(2)$ doublets. This will be important when we come to generation mixing.

We now turn to the fermion mass terms. As noted before, we need the Higgs field $\phi(x)$ here to maintain gauge invariance. If there would be just one generation, we would have

$$\mathcal{L}_{FM} = -y_u \epsilon^{ab}\overline{Q_{La}}\phi_b u_R - y_d \overline{Q_L}\cdot\phi d_R - y_e \overline{L_L}\cdot\phi e_R + \text{h.c.}. \qquad (1.35)$$

The y are called Yukawa couplings. We have

$$\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (1.36)$$

(Technically speaking, there are two ways to project out the gauge invariant part of the product $\psi_L\phi$: by using a Kronecker delta or a Levi-Civita epsilon.)
Furthermore we see that hypercharges again add to zero, that Lorentz invariance is obvious as we only have $\overline{\psi}\psi$ combinations, and that the mass dimension is seen to equal $\frac{3}{2} + 1 + \frac{3}{2} = 4$.
Inserting the Higgs vev 1.22 yields

$$\mathcal{L}_{FM} = -\frac{v}{\sqrt{2}}\big(y_u\overline{u_L}u_R + y_d\overline{d_L}d_R + y_e\overline{e_L}e_R + h.c.\big) \qquad (1.37)$$

which leads to $m_u = \frac{y_u v}{\sqrt{2}}$, $m_d = \frac{y_d v}{\sqrt{2}}$, $m_e = \frac{y_e v}{\sqrt{2}}$.

Let us now deal with the three generations we observe in nature. This implies that we should think of the Yukawa couplings as 3×3 matrices. So we rather have

$$\mathcal{L}_{FM} = -\frac{v}{\sqrt{2}}\Big((y_u)_{ij}\overline{u_{Li}}u_{Rj} + (y_d)_{ij}\overline{d_{Li}}d_{Rj} + (y_e)_{ij}\overline{e_{Li}}e_{Rj} + h.c.\Big) \tag{1.38}$$

where i, j are generation indices.
So now the masses (apart from this factor $\frac{v}{\sqrt{2}}$) should follow from the 3×3 Yukawa matrices. We do not know anything about their form, but to extract masses we can of course diagonalize them. This wil affect other parts of the Lagrangian.
To diagonalize, say, y_u we start by constructing the Hermitian quantities $y_u y_u^\dagger$ and $y_u^\dagger y_u$. As any Hermitian matrix H can be decomposed in a unitary matrix U, containing H's (normalized) eigenvectors in its columns, and a diagonal matrix D wearing H's (real) eigenvalues as $H = UDU^{-1}$ we can write

$$y_u y_u^\dagger = U_u D_u^2 U_u^\dagger \qquad y_u^\dagger y_u = W_u D_u^2 W_u^\dagger. \tag{1.39}$$

We then see that we can decompose y_u as

$$y_u = U_u D_u W_u^\dagger, \tag{1.40}$$

where D_u is given by the positive square roots of D_u^2, that is, by the positive square roots of the eigenvalues of $y_u y_u^\dagger$.
The next step is to redefine the up-quark field in the following way:

$$u_{Li} \to (U_u)_{ij} u_{Lj} \qquad u_{Ri} \to (W_u)_{ij} u_{Rj} \tag{1.41}$$

The "up-part" of the mass terms in the Lagrangian now reads

$$\mathcal{L}_{FM} = -\sum_i \underbrace{\frac{v}{\sqrt{2}}(D_u)_{ii}}_{(m_u)_i} \overline{u_{Li}} u_{Ri}, \tag{1.42}$$

so we can read off the up, charm and top masses $(m_u)_1$,$(m_u)_2$ and $(m_u)_3$.
y_d and y_e can be treated in exactly the same way: the eigenvalues of $y_d y_d^\dagger$ and $y_e y_e^\dagger$ yield (up to this same factor of $\frac{v}{\sqrt{2}}$) the remaining six fermion masses.
Now let us investigate the price we pay for switching to mass eigenstates. We go back to the currents we encountered in writing out $\mathcal{L}_{FK}$ (see 1.33). The fermion fields used there have been redefined. In the quark sector we now have

$$\overline{u_{Li}}\gamma^\mu u_{Li} \to \overline{u_{Li}}(U_u)^\dagger_{ij}\gamma^\mu (U_u)_{jk} u_{Lk} = \overline{u_{Li}}\gamma^\mu u_{Li} \tag{1.43}$$

$$\overline{u_{Li}}\gamma^\mu d_{Li} \to \overline{u_{Li}}(U_u)^\dagger_{ij}\gamma^\mu (U_d)_{jk} d_{Lk} = \overline{u_{Li}}\gamma^\mu \left(U_u^\dagger U_d\right)_{ij} d_{Lj}. \tag{1.44}$$

In the lepton sector there will not be any new effect because there is just one type of U matrix (as long as we do not include righthanded neutrinos).
Comparing this with 1.33 we conclude that the quark sector of the charged weak interactions is affected by a factor

$$\left(U_u^\dagger U_d\right)_{ij} \equiv V_{ij}. \tag{1.45}$$

This V_{ij} thus describes so-called generation transitions, weak interactions between members of different generations. It is known as the Cabibbo-Kobayashi-Maskawa (CKM) matrix. By doing $U(1)$ phase rotations we can eliminate 5 of the 9 degrees of freedom in V. The remainders can be thought of as 3 rotation angles (between the 3 generations, the one between the first and second generation is called Cabibbo angle) and one phase factor.
Let us, for a moment, consider this Yukawa-analysis from a phenomenological point of view, as will be done intensively in chapter 6. Yukawa matrices describe the way in which fields couple to each other and thus induce masses, but they cannot be exactly determined. Physical observables are particle masses and primarily the absolute values of the CKM matrix elements[4]. That is, we can only deduce the eigenvalues of $yy^\dagger$ and the absolute value of the matrix product $U_u^\dagger U_d$, where $U_{u(d)}$ contain the eigenvalues of $y_{u(d)}y_{u(d)}^\dagger$. So, if we find expressions for Yukawa matrices that reproduce correct masses and CKM matrix elements, we can as well rotate these Yukawa-matrices by a unitary matrix without spoiling physical implications. This ambiguity in Yukawa matrices should disappear from every physical observable.
Now let us write down the kinetic terms for the gauge boson fields. Looking back at 1.15 and 1.16 we easily write down

$$\mathcal{L}_{GK} = -\frac{1}{4}\left(G^a_{\mu\nu}G^{a\mu\nu} + F^a_{\mu\nu}F^{a\mu\nu} + B_{\mu\nu}B^{\mu\nu}\right). \tag{1.46}$$

In the $SU(3)$ $(a = 1...8)$ $G^a_{\mu\nu}$ is constructed out of gluon fields G^a_μ, in the $SU(2)$ $(a = 1,2,3)$ $F^a_{\mu\nu}$ is constructed out of gluon fields A^a_μ and in the Abelian $U(1)$ part $B^{\mu\nu}$ contains the $U(1)$ gauge boson field B_μ. Gauge invariance is trivial now and Lorentz invariance is ensured as there are no uncontracted indices and the mass dimension of the whole expression equals 4 as it should do.

Finally we add a piece to our Lagrangian that should explain why the Higgs field tends to the vev 1.22. This is of course a rather speculative business, as we only have indirect clues for the nature of the Higss particle. But it looks promising to write

$$\mathcal{L}_{HP} = \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2. \tag{1.47}$$

This potential is seen to have a minimum that can be defined to be[5] at $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}$ with $v = \sqrt{\frac{\mu^2}{\lambda}}$. According to the discussion around 1.29 we could define the Higgs mass as $\sqrt{2}\mu = \sqrt{2\lambda}v$.

[4]Recently the relative phases of CKM elements have been determined in terms of the so-called Jarlskog parameter J.

[5]in the "unitary gauge"

1.2 Troubles and shortcomings of the Standard Model

1.2.1 Reasons to go beyond the SM

The Standard Model works. It is a perfect example of how a new scientific idea should be: it unifies several ideas, builds much more structure in the theoretical framework of particle physics and above all it is falsifiable. The Standard Model has been tested in numerous experiments and has survived all of them. But it cannot be the end of the story.
There is one issue in which the SM is just wrong: experiments have shown that neutrinos are not massless and there should be righthanded neutrinos as well.
The SM also has some features that are theoretically unsatisfying but nevertheless technically allowed. The first issue that comes to mind is the plethora of 19 arbitrarily free parameters the theory contains. (15 of the free parameters come from interactions involving the Higgs field. It is argued that the Higgs sector is nothing more than a "parametrization of our ignorance".) Anyhow, some more explanation or connections between the various coupling strengths, particle masses and electric charges would be very welcome. The alternative, invoking the anthropic principle (the parameters had to be tuned the way they are because if not, we had not been able to observe them because life would not have existed) seems a very weak statement, to me at least, and I really hope physics can come up with something better[6].
Then we have the so-called *hierarchy* or *fine tuning* problem. Below we will show that the Higgs mass is given by the difference of two $\mathcal{O}(10^{36})$GeV terms. To arrive at the $\mathcal{O}(10^2)$ GeV prediction we indeed need an incredible lot of fine tuning.

We will now elaborate on two reasons to look for physics beyond the SM that will be reconsidered in next chapters: the righthanded neutrino and the hierarchy problem.

1.2.2 Righthanded neutrinos

Only in the nineties of the previous century neutrino oscillation experiments (see below) have indicated that neutrinos are not massless. We thus need to add new mass terms to $\mathcal{L}_{SM}$. As we can not build a mass term out of lefthanded neutrino fields, the most logical (less exotic) approach then is to postulate the existence of (three generations of) righthanded neutrinos. We assume the righthanded neutrino field to be in the $SU(3) \times SU(2) \times U(1)$ representation $(\mathbf{1}, \mathbf{1}, 0)$. This actually gives a very promising explanation for the very small observed neutrino masses. So far we have only encountered fermion mass terms of the Dirac type, coupling two fields of different handedness. But there actually is another way of writing mass terms: Majorana[7] mass terms. Here a field is coupled to its own transpose rather than to

[6] Or shall we believe Lee Smolin who argues that new universes are born in black holes and there is some Darwinesque evolution in the parameters that make up the SM that has eventually culminated in the successful values that allow our existence today?

[7] After Ettore Majorana, the most promising physicist of the "ragazzi della via Panisperna" (the guys from Panisperna Street, where the Rome physics department was in the 1920s) until he disappeared without a trace in 1938.

a field of opposite handedness:

$$\mathcal{L} = \frac{1}{2} m \psi_L^T C \psi_L + h.c.. \tag{1.48}$$

(Appendix B explains why this expression is Lorentz invariant.)
We immediately understand that only ν_R, transforming trivially under all three SM gauge symmetries, can form such a mass term. We thus add to the SM a Dirac and a Majorana mass term:

$$\Delta\mathcal{L}_{FM} = -(\lambda_\nu)_{ij}\epsilon^{ab}(\overline{\nu_{La}})_i(\phi_b)_j\nu_R - \frac{1}{2}(\lambda)_{ij}(\nu_R^T)_i C(\nu_R)_j. \tag{1.49}$$

If we now give the Higgs field its usual vev and perform the usual switch to mass eigenstates we find

$$\Delta\mathcal{L}_{FM} = -\sum_i (m_\nu^{DIR})_i \overline{\nu_{Li}}\nu_{Ri} - (m_\nu^{MAJ})_i \nu_{Ri}^T C \nu_{Rj} + h.c.. \tag{1.50}$$

Writing everything in terms of lefthanded fields (see again appendix B we then end up with

$$\Delta\mathcal{L}_{FM} = -\frac{1}{2}\begin{pmatrix} \nu^T & (\nu^c)^T \end{pmatrix}_L C \begin{pmatrix} 0 & M^{DIR} \\ M^{DIR} & M^{MAJ} \end{pmatrix} \begin{pmatrix} \nu \\ \nu^c \end{pmatrix}_L + h.c. \tag{1.51}$$

To find two true mass eigenstates we look for the eigenvalues of this mass matrix [8]. These are $\frac{1}{2}(m^{MAJ} \pm \sqrt{(m^{MAJ})^2 + 4(M^{DIR})^2}$. On the assumption $M^{MAJ} \gg M^{DIR}$ we find a very light eigenstate and a very massive eigenstate:

$$m_- \approx = -\frac{(M^{DIR})^2}{M^{MAJ}}, \qquad m_+ \approx M^{MAJ}. \tag{1.52}$$

This is a beautiful "seesaw" mechanism: the larger the one eigenstate, the smaller the other one. In this way we give a powerful explanation for neutrino measurements: light mass eigenstates have hardly been detected because they are so light while the heavy ones are far too heavy to be observed in any experiment.
One cannot overemphasize the importance of the detection of righthanded neutrinos. It is the first clue we encounter in this thesis for new physics well above the SM scale.
Now that we assume the existence of righthanded neutrinos we expect generation mixing in the lepton sector of the charged weak currents as well. This indeed is the case. We here have the PMNS (Pontecorvo, Maki, Nakagawa, Sakata) matrix that does the same job as the CKM matrix in the quark sector. Actually, leptonic mixing lies at the basis of neutrino oscillations: neutrinos that are emitted in the one eigenstate are after a several hundred kilometres' trip detected in the other eigenstate. The probability of a neutrino switching from the one eigenstate to the other involves the PMNS matrix element describing this transition times (the exponent of) the difference in mass (squared) between the two states.

[8] In the Dirac cases we have encountered so far these mass matrices only carried two equal off-diagonal terms, which led to two eigenstates of equal mass.

1.2.3 Renormalization and the hierarchy problem in the SM

The hierarchy problem has to do with the renormalization of SM physics.
The basic idea of quantum field theory is quite easy: one is interested in transition amplitudes which are obtained by calculating the contributions from all Feynman diagrams that describe the transition.
If we take for example, two incoming particles with four-momenta p_1 and p_2 and, after interaction, two outgoing particles with four momenta k_1 and k_2 we have

$$\langle k_1 k_2 | iT | p_1 p_2 \rangle = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2) i\mathcal{M}(p_1, p_2 \to k_1, k_2) \tag{1.53}$$

where the righthandside is given by the sum of all Feynman diagrams with incoming p_1, p_2 and outgoing k_1 and k_2.
Feynman diagrams are written down easily once all Feynman rules have been obtained and these follow from the Lagrangian of the theory governing the process. So far so good.
Troubles arise when we go beyond tree level, that is, when we start studying processes that involve intermediate states (virtual particles). These processes are suppressed because they involve higher powers of coupling constants, but that is not going to save us when the momenta of these virtual particles can become arbitrarily large. And why would they not, being virtual particles that we cannot control at all. It is common use then to define a maximum allowed momentum Λ, the so called cut-off. The cut-off can be thought of as the maximal energy (or, invoking Heisenberg's uncertainty principle, the minimum distance) up to which the theory makes sense.
In some cases Λ cancels out from all observable quantities. Such a theory is called super-renormalizable. This is not the case in the SM, but it is at least a renormalizable theory: Λ shows up in just a small number of parameters, like the fermion mass for example. But as we can measure a fermion mass, we can in the end eliminate Λ from physical predictions.
Let us, for example, have a look at the calculation of the electron mass in QED (which is essentially the same in the SM). At tree level the electron propagator is given by

$$\frac{i(\not{p} + m_0)}{p^2 - m_0^2}. \tag{1.54}$$

The electron mass is given by the pole of the propagator so we conclude $m_e = m_0$. However, if we add higher order diagrams a long calculation reveals that the pole gets shifted by a divergent amount

$$\delta m_e = \frac{3\alpha}{4\pi} m_0 \log(\frac{\Lambda^2}{m_0^2}). \tag{1.55}$$

(As always $\alpha = \frac{g^2}{4\pi}$.)
We thus find $m_e = m_0 + \delta m \equiv m$.
The crucial insight now is that any measurement is to return the "physical" mass m, and not m_0. Nature does not know about perturbation theory, it is just our approach to describe her. Therefore, m_0 should be thought of as a "bare mass".

We should express measurable quantities in renormalized or "dressed" quantities like m. In a way we are just removing infinities from observables by putting them into abstract undetectable "bare" quantities. It may seem a very strange procedure but it certainly makes sense: after all an infinite electron mass has never been measured. The energy range in which we can perform measurements is bounded from above.
This is what renormalization is about, switching from bare to physical quantities. As long as we work at tree level there is no difference, but beyond the relation between bare and physical becomes non-trivial.

The hierarchy problem shows up once we calculate the one loop correction to the scalar field two point function, which is of course to give us the Higgs mass. We then find

$$\delta m_\phi = c\sqrt{\alpha}\Lambda^2 \tag{1.56}$$

where c is a dimensionless constant.
Now we are in trouble. Not only is the first order scalar mass contribution far more divergent than in the case of the electron (or any other fermion), but it is also independent from m_ϕ. Elaborating on this we mention that if we would set m_e to 0 the resulting theory would be more symmetric (we would gain an $U(1)$ symmetry between lefthanded and righthanded states). Therefore, the electron mass is a "natural" small parameter: putting it to zero increases the symmetry. The Higgs mass however is an "unnatural" parameter: putting it to zero does not yield any more symmetry (one could think of a scaling symmetry $\phi(x) \to l\phi(lx)$) because the first order correction does not vanish.
In short, we do not have any control over the Higgs mass. If we set $\Lambda = m_{Planck} = \mathcal{O}(10^{18})$ GeV, the highest scale known in physics, we have an $\mathcal{O}(10^{36})$ contribution that should be subtracted from the naked Higgs mass to return a physical mass μ of $\mathcal{O}(10^2)$ GeV. Now we clearly see how much fine tuning is needed to have a reasonable Higgs mass. Of course nothing really forbids as much fine tuning as necessary but... who does all this tuning? It certainly is a very unsatisfying way of stabilizing the Higgs scale (and, in fact, the whole electroweak scale because we have seen that μ enters in the vev v of the Higgs field).

1.3 Running of Coupling Constants: a first clue for Grand Unification

By now we have a clear picture of the Standard Model. Three gauge groups, three forces, three coupling constants g_s, g_w and g'. After symmetry breaking we are left with g_s and $g_{EM} \equiv e = \frac{g_w g'}{\sqrt{g_w^2+g'^2}}$. Nothing signals that we might look for more unification.
But let us go back to the discussion on renormalization. We have seen that physical, observable quantities can depend on the cut-off scale Λ. Let us instead of the electron mass think of its charge. Stating that its physical electric charge depends on the maximum allowed energy scale implies that this charge depends on the minimum probing distance. And why would it not. If we think of an electron as a naked point particle with a cloud of electron-positron pairs around it, the

positron pointing to the electron of course, we understand that, as we get closer to the naked kernel, the shielding effect of the positrons decreases which renders the actual charge we observe larger and larger. We thus conclude that we can expect observables like coupling constants to depend on the actual energy scale (or distance scale) at which they are measured: coupling constants can run!
The remainder of this section is devoted to calculate this running. The goal of this thesis being Grand Unification, we hope to see the three coupling constants meet somewhere.
One last remark before we set off: g_w and g' are of course already connected by the Weinberg angle. We have $g' = \tan\theta_w g_w$, as can be checked from (1.27) or (1.28). So we in fact have two options: either we check for what value of θ_w the three coupling constants meet (two lines and a third adjustable line will of course always meet as long as they are not parallel), or we borrow from the next chapter the general GUT result $\tan^2\theta_w = \frac{3}{5}$ and check how well coupling constants meet now[9]. We choose to follow the latter approach. We are thus to investigate the behaviour of g_s, g_w and $\sqrt{\frac{5}{3}}g'$ that from now on will be labeled g_3, g_2 and g_1 respectively.

1.3.1 Renormalized Perturbation Theory

The running of coupling constants could be summarized in two or three equations but I would like to provide some more framework. First we have to elaborate a bit on our description of renormalization. “Renormalized perturbation theory” is based on the same logic I described before, but deals with divergences in a better organised way. The central idea is to (after having established the relations between bare and physical quantities) write the Lagrangian in terms of these physical quantities. Leftover bare quantities are collected in so-called “counterterms”.
For example, a fermion - fermion - gauge boson vertex has Feynman rule $ig_0 T^a\gamma^\mu\overline{\psi}\psi A^a_\mu$ (see for example [14]). After switching to physical quantities we have $ig(1+\delta_g)T^a\gamma^\mu\overline{\psi_r}\psi_r A^a_{\mu r}$. The term with δ_g is the counterterm, the other is the bare term.
In a general non-Abelian gauge theory we need three counterterms: δ_1 describes the aformentioned fermion-gauge boson vertex correction, δ_2 fixes the fermion self-energy (its two point function) and δ_3 takes care of the gauge boson self-energy.
So far we have only been doing bookkeeping: we have swept all the unknown, all divergences in our counterterms.
Now we state renormalization conditions: we define the theory at a certain scale M (high enough that we canneglect particle masses). At this scale we want to remove all divergences. (This is a very reasonable demand because we have already switched from bare to physical quantities.)
Let us start with the fermion-gauge boson vertex. If we work up to one loop order we have three contributing diagrams. A fourth one involves δ_1 and is to cancel all divergences.

[9] This value of θ_w of course only applies at the GUT energy scale!

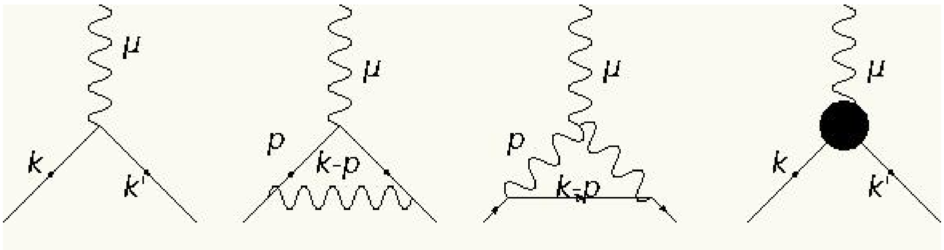


Figure 1.1: Vertex diagrams

The first diagram describes the tree level process and does not contain any divergence. The second and third ones do contain an internal loop momentum p that will blow up. But now we can require that at $p = M$ the divergent parts of these diagrams cancel against the fourth diagram that corresponds to the counterterm $igT^a\gamma^\mu\delta_1$. (If the gauge boson is a gluon, T^a will be a $SU(3)$ generator, in case of a W-boson T^a is a $SU(2)$ generator and so forth.)

Now for the fermion self-energy. Up to first order we have a tree-level diagram, a potentially divergent diagram (internal loop momentum p again) and a third one with Feynman rule $i\not{k}\delta_2$ that is to save the situation at $p = M$ (see 1.3 2).
The renormalization of the gauge boson self energy involves more diagrams. Apart

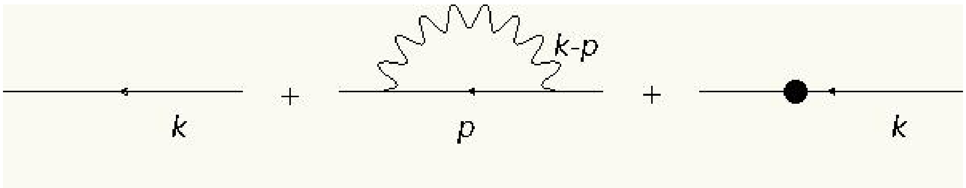


Figure 1.2: Fermion self-energy

from the tree level diagram there is a diagram with an internal fermionic loop, one with an internal (scalar) bosonic loop, two diagrams with a (vector) bosonic loop, one with a Faddeev-Popov [10] ghost loop and then finally a diagram involving δ_3. See 1.3.

The last diagram has Feynman rule $-i(k^2g^{\mu\nu} - k^\mu k^\nu)\delta^{ab}\delta_3$. (The tensorial structure is dictated by the Ward identity.)
In an Abelian gauge theory the three one loop pure gauge diagrams are all zero.

1.3.2 Calculation of counterterms

The strategy is very simple. To calculate δ_1 for example we compute the second and third diagram of figure 1, isolate the divergent part, set $p = M$, peel off a factor

[10]Faddeev Popov ghosts are treated in, for example, Peskin and Schroeder ([14]). In this work we take their existence for granted, let them participate in 1.3 and that is it.

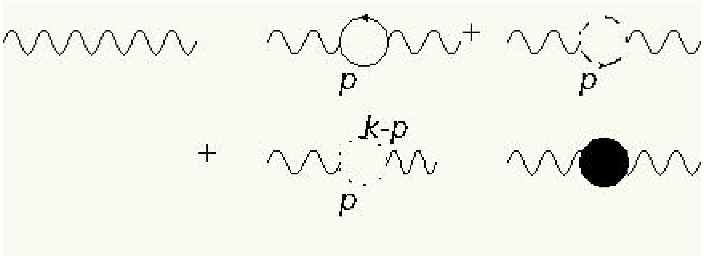


Figure 1.3: Gauge boson self-energy. The second diagram contains a fermionic loop. In the third one there is the Higgs boson in the loop. The whole pure gauge sector, that is, vector bosonic loops and Faddeev Popov loops is informally summarized in the fourth diagram.

$igT^a\gamma^\mu$, and add a minus sign so that the fourth term cancels this divergence instead of doubling it. This yields

$$\delta_1 = -\frac{g^2}{16\pi^2}\frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-\frac{d}{2}}}\left[C_2(r) + C_2(G)\right] \tag{1.57}$$

Here d denotes the number of spacetime dimensions we are working in, it will be put to 4 later on. Γ is the factorial function: $\Gamma(n) = (n-1)!$ so $n\Gamma(n) = \Gamma(n+1)$. Depending on which gauge theory we are working on, g equals g_s, g_w or g'.
The Casimir operator C_2 can be calculated for every rep r of every Lie group: if in this rep generators are given by T^a then $T^aT^a = C_2(r)\mathbf{1}$. If we are in the adjoint representation G we can write an equivalent definition invoking the structure constants: $f^{acd}f^{bcd} = C_2(G)\delta^{ab}$. For $SU(N)$ we have $C_2 = \frac{N^2-1}{2N}$ for the fundamental rep and $C_2 = N$ for the adjoint rep.
Without doing the actual calculation we can thus understand where these factors of C_2 stem from. Both one loop diagrams contain three gauge boson-fermion-fermion vertices. From the SM-Lagrangian (part $\mathcal{L}_{FK}$) we read off that such a vertex yields a factor T^a. a is a vectorboson index. Vectorbosons are all in the adjoint rep. We thus have a product of three generators of the adjoint representation that can be manipulated into the form we have in 1.57.
In the same way we calculate δ_2. From figure 2 we see that now we have two factors of T^a, both with the same index a because they come from vertices at the two ends of the same vectorboson. We thus find a factor of $C_2(r)$. The exact answer is

$$\delta_2 = -\frac{g^2}{16\pi^2}\frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-\frac{d}{2}}} \times C_2(r). \tag{1.58}$$

Now for δ_3. The second and third diagram of figure 3 are the most interesting ones, for reasons that will become clear later. In the second diagram every fermion that is in the fundamental rep of the gauge group we are interested in can take part in the loop diagram. In $SU(3)$ for example we thus have four (up, down, left, right) quark triplets that enter the loop. We thus expect a sum over fermion multiplets (triplets or doublets). In the same way the third diagram should generate a sum over scalar boson multiplets. (But as there is no $SU(3)$ scalar boson in the SM we only expect such a bosonic sum in the $SU(2)$ part.) In both cases we have a vectorboson with index a from the left and one with index b from the right so there will be a factor $T^a T^b$ in every diagram. After many cancellations (we really need all five divergent diagrams) we find

$$\delta_3 = \frac{g^2}{16\pi^2} \frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-\frac{d}{2}}} \left[\frac{5}{3} C_2(G) - \frac{4}{3} \sum_f C(r_f) - \frac{1}{3} \sum_b C(r_b) \right]. \tag{1.59}$$

Here $C(r)\delta^{ab} = \mathrm{Tr}[T^a T^b]$. In the fundamental rep we have $C = \frac{1}{2}$, in the adjoint rep $C = N$ (for $SU(N)$).
We should also calculate δ_3 for the $U(1)$ part of the SM. This is very similar to QED, we only have to replace the QED coupling constant e by $g'Y$. However, QED does not know any scalar boson loop diagram. But we can read off its Feynman rules from the SM Lagrangian (part $\mathcal{L}_1$). We clearly expect two factors of the hypercharge in every diagram. The result is

$$\delta_3 = \frac{g^2}{16\pi^2} \frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-\frac{d}{2}}} \left[-\frac{4}{3} \sum_f Y_f^2 - \frac{1}{3} \sum_b Y_b^2 \right] \tag{1.60}$$

We now have found all necessary counterterms. In the next section we will see how to use them in order to calculate the running of the three SM coupling constants.

1.3.3 Callan-Symanzik equation and renormalization equation

The renormalization scale M is of course totally arbitrary. We thus do not want physical quantities to depend on it. The *Callan-Symanzik equation* states that if we shift M, in ϕ^4-theory for example, we should also shift ("re-renormalize") the coupling constant g and the scalar field ϕ in such a way that the bare n-point function $G^{(n)}(x_1, \ldots x_n)$ remains fixed:

$$\left[M\frac{\partial}{\partial M} + \beta \frac{\partial}{\partial g} + n\gamma \right] G^{(n)}(x_1, \ldots x_n, M, g) = 0. \tag{1.61}$$

Here we have defined dimensionless parameters β and γ:

$$\beta \equiv \tfrac{M}{\delta M} \delta g, \quad \gamma \equiv -\tfrac{M}{\delta M} \delta\eta. \tag{1.62}$$

(The parameter η has to do with the rescaling of ϕ, see [14, chap.12].)
From this definition it is a small step towards the *renormalization equation*:

$$M \frac{\partial g_i}{\partial M} = \beta_i(g). \tag{1.63}$$

We thus see that once we have the three betafunctions β_i in hand, we can solve the renormalization equation to find the three running coupling constants $g_i(M)$.
To find these betafunctions we solve CS equations. That is, we write down the n-point function, including counterterms, and apply derivatives with respect to M

and λ on it.
For non-Abelian gauge theories the result can be written in the form

$$\beta(g) = gM\frac{\partial}{\partial M}\left(-\delta_1 + \delta_2 + \frac{1}{2}\delta_3\right). \tag{1.64}$$

In the Abelian case we have $\delta_1 = \delta_2$ (in QED this in fact is a direct consequence the Ward-Takahashi identity) so then

$$\beta(g) = gM\frac{\partial}{\partial M}\left(\frac{1}{2}\delta_3\right). \tag{1.65}$$

1.3.4 Running Coupling Constants

We now insert our expressions for counterterms in 1.65, perform the derivative and take the limit $d \to 4$. If we set

$$\beta_i = -\frac{g_i^3}{16\pi^2}b_i \tag{1.66}$$

we can summarize our results as

$$b_1 = -\frac{2}{3}\sum_f \frac{3}{5}Y_f^2 - \frac{1}{3}\sum_b \frac{3}{5}Y_b^2 \tag{1.67}$$

$$b_{2,3} = \frac{11}{3}C_2(G) - \frac{2}{3}\sum_f C(r_f) - \frac{1}{3}\sum_b C(r_b). \tag{1.68}$$

Note that M has dropped out of these equations.
The factors of $\frac{3}{5}$ in 1.67 come from the rescaling of the $U(1)$ force described earlier this section. In 1.68 the ratio between the fermionic and the bosonic contribution has decreased from 4 (as in 1.59) to 2 because we are discussing two-spinors, not four-spinors.
Collecting hypercharges of each Standard Model field we conclude that for N_g generations and N_h Higgs doublets we have

$$\begin{aligned} b_1 = &-\frac{2}{5}\left[3\times 2\times(\frac{1}{6})^2 + 3\times(\frac{2}{3})^2 + 3\times(-\frac{1}{3})^2 + 2\times(-\frac{1}{2})^2 + (-1)^2\right]\times N_g \\ &-\frac{1}{5}\times 2\times(\frac{1}{2})^2\times N_h \\ = &-\frac{4}{3}N_g - \frac{1}{10}N_h. \end{aligned} \tag{1.69}$$

Now for b_2. In the adjoint representation of $SU(2)$ we have $C_2 = 2$. We have four lefthanded fermion doublets (three in the quark sector and one in the lepton sector) and one scalar boson doublet (the Higgs field). All fields are in the fundamental representation, that is, $C(r_f) = C(r_b) = \frac{1}{2}$. We find

$$\begin{aligned} b_2 = &\frac{11}{3}\times 2 - \frac{2}{3}\times 4\times\frac{1}{2}\times N_g - \frac{1}{3}\times\frac{1}{2}\times N_h \\ = &\frac{22}{3} - \frac{4}{3}N_g - \frac{1}{6}N_h. \end{aligned} \tag{1.70}$$

Note that now the sum runs over doublets rather than over fields as in the $U(1)$ case.

In the $SU(3)$ sector we have $C_2 = 3$ for the adjoint representation. In each generation there are four triplets (up and down, LH and RH) that contribute to the betafunction and no scalar boson fields, just because the Higgs field is a $SU(3)$ singlet. So we simply have

$$\begin{aligned} b_3 =& \frac{11}{3} \times 3 - \frac{2}{3} \times 4 \times \frac{1}{2} \times N_g \\ =& 11 - \frac{4}{3} N_g. \end{aligned} \tag{1.71}$$

If we now plug in the general expression 1.66 into the renormalization equation 1.63 and solve for $\alpha_i (\equiv \frac{g_i^2}{4\pi})$ we finally have what we were actually looking for:

$$\frac{1}{\alpha_i(M)} = \frac{1}{\alpha(M_{GUT})} - \frac{b_i}{2\pi} \log\left(\frac{M_{GUT}}{M}\right). \tag{1.72}$$

Here $\frac{1}{\alpha(M_{GUT})}$ has entered the story as a universal integration constant.
In these equations we read the beautiful idea of Grand Unification. They suggest that at a certain energy scale M_{GUT} all three coupling constants are equal. It might very well be, then, that at this scale there is just one gauge group. Below, this symmetry gets broken to the $SU(3) \times SU(2) \times U(1)$ symmetry we observe in the SM. The difference that we observe at SM energy scales between the three coupling constants would be due to their different behaviour when dialing down to SM energies, that is, to their different betafunctions.

This idea of merging coupling constants, of Grand Unification, looks tempting and fascinating. But we had better look for numerical evidence first instead of already trumpeting nature's beauty. After all, it has been our choice to take three equal integration constants and we need to justify this assumption.
Here is our strategy. Assuming that the three coupling constants indeed meet at some energy scale M_{GUT} and have universal value α there, we use our betafunctions (with $N_g = 3$ and $N_h = 1$) to predict their values at energy scale $M_Z = 91.19$ GeV. There we can compare with the experimental values (obtained from [13]) for $\alpha_i^{EXP}(M_Z)$:

$$\frac{1}{\alpha_1^{EXP}(M_Z)} = 59.00 \pm 0.02, \quad \frac{1}{\alpha_2^{EXP}(M_Z)} = 29.57 \pm 0.02, \quad \frac{1}{\alpha_3^{EXP}(M_Z)} = 8.50 \pm 0.14. \tag{1.73}$$

Note that from these values we easily infer the value of the Weinberg angle at $M = M_Z$:

$$\sin^2\theta_w(M_Z) = \frac{\alpha'(M_Z)}{\alpha'(M_Z) + \alpha_w(M_Z)} = \frac{\frac{3}{5}\alpha_1(M_Z)}{\frac{3}{5}\alpha_1(M_Z) + \alpha_2(M_Z)} = 0.23119 \tag{1.74}$$

The relative errors will be denoted γ_i from now. So

$$\gamma_1 = \frac{0.02}{59}, \quad \gamma_2 = \frac{0.02}{29.57}, \quad \gamma_3 = \frac{0.14}{8.50}. \tag{1.75}$$

We now define a χ^2-function that measures the error in this approach. That is, at $M = M_Z$ we take a weighted sum over the three relative differences between experimental values of coupling constants α_i^{EXP} and the GUT-predicted values α_i. This still is a function of α and M_{GUT}. So we have

$$f(\alpha, M_{GUT}) = \sum_i \frac{1}{(\gamma_i)^2} \frac{1}{(\alpha_i^{EXP}(M_Z))^2} \left(\frac{1}{\alpha_i(M_Z)} - \frac{1}{\alpha_i^{EXP}(M_Z)} \right)^2 \tag{1.76}$$

$$= \sum_i \frac{1}{(\gamma_i)^2} \frac{1}{(\alpha_i^{EXP}(M_Z))^2} \left(\frac{1}{\alpha} - \frac{b_i}{2\pi} \ln \frac{M_{GUT}}{M_Z} - \frac{1}{\alpha_i^{EXP}(M_Z)} \right)^2. \tag{1.77}$$

This function can be minimized with respect to α and M_{GUT}. This yields

$$\tfrac{1}{\alpha} = 42.39, \quad M_{GUT} = 1.246 \times 10^{13}\text{GeV} \leftrightarrow \log(M_{GUT}) = 30.15. \tag{1.78}$$

(All calculations for this thesis were done in Mathematica.)
But now we can finally put the GUT assumption to the test: inserting this α and M_{GUT} we obtain the "best fit predictions" for the α_i at scale $M = M_Z$:

$$\tfrac{1}{\alpha_1(M_Z)} = 59.06, \quad \tfrac{1}{\alpha_2(M_Z)} = 29.41, \quad \tfrac{1}{\alpha_3(M_Z)} = 13.75, \tag{1.79}$$

which implies $\sin^2\theta_w = 0.2300$.
Thus, (cf 1.73), the coupling constants do not meet well enough to have them all within their error bars at $M = M_Z$. The strong coupling constant is most off, but that of course results from the large relative error in its measured value at M_Z.
Another way to express this failure in the meeting of the coupling constants, is to choose, when solving the renormalization group equation, three different integration constants in such a way that the running couplings α_i exactly meet the experimental values at $M = M_Z$.

$$\frac{1}{\alpha_i(M)} = \frac{1}{\alpha^{EXP}(M_Z)} - \frac{b_i}{2\pi} \log\left(\frac{M_Z}{M}\right). \tag{1.80}$$

Now we can just plot the three running coupling constants and see how well they meet in the region around our proposed M_{GUT}: see figure 1.4.
We conclude, anyhow, that the meeting of the running coupling constants is far from perfect. A cynical approach now would be to reject the Grand Unifying hypothesis. But let us view the glass as half full: coupling constants surely converge, even if they do not exactly meet. This is the second clue for new physics at a very high energy scale, after the discovery of righthanded neutrinos in the previous section. So let us start exploring Grand Unifying Theories. After that we will investigate in what way we can improve on the result obtained in this section.

Threshold effects

One final remark before we leave this section: in order to not interrupt last section's discussion I have omitted threshold effects, although they were taken into account in the Mathematica calculation. Including threshold effects means realizing that below M_{top} the top quark does not contribute to the various sums in 1.69, 1.70 and

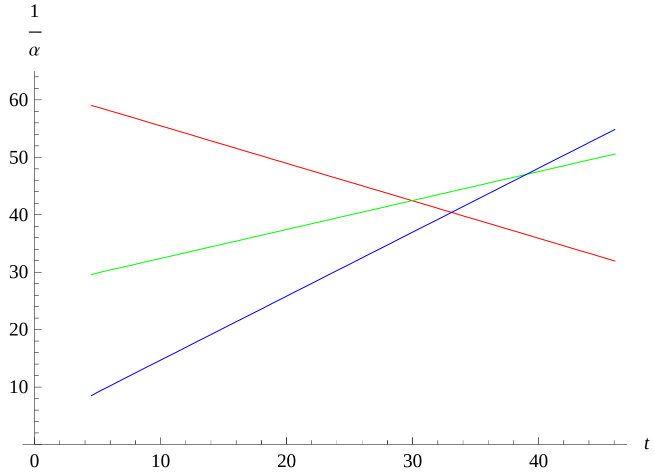


Figure 1.4: One-loop evaluation of the coupling constants α_i from M_Z to 10^{20} GeV. The horizontal scale is logarithmic: $t = \log M$. At M_Z we have $\frac{1}{\alpha_3} < \frac{1}{\alpha_2} < \frac{1}{\alpha_1}$.

1.71. The same goes for the Higgs boson (which was put at 120 GeV).
The expression 1.72 should be modified to

$$\begin{aligned}
\frac{1}{\alpha_i(M)} =& \frac{1}{\alpha(M_{GUT})} - \frac{b_i a}{2\pi} \log\left(\frac{M_{GUT}}{M}\right) \qquad (M > M_{top}) \\
=& \frac{1}{\alpha(M_{GUT})} - \frac{b_i a}{2\pi} \log\left(\frac{M_{GUT}}{M_{top}}\right) - \frac{b_i b}{2\pi} \log\left(\frac{M_{top}}{M}\right) \\
& (M_{Higgs} < M < M_{top}) \\
=& \frac{1}{\alpha(M_{GUT})} - \frac{b_i a}{2\pi} \log\left(\frac{M_{GUT}}{M_{top}}\right) - \frac{b_i b}{2\pi} \log\left(\frac{M_{top}}{M_{Higgs}}\right) - \frac{b_i c}{2\pi} \log\left(\frac{M_{Higgs}}{M}\right) \\
& (M_Z < M < M_{Higgs}), \qquad (1.81)
\end{aligned}$$

where we have full betafunctions on the a-trajectory down to M_{top} ($N_G = 3$, $N_H = 1$),

$$\begin{aligned}
b_1 a =& -\frac{4}{3} \times 3 - \frac{1}{10} \times 1 \\
b_2 a =& \frac{22}{3} - \frac{4}{3} \times 3 - \frac{1}{6} \times 1 \\
b_3 a =& 11 - \frac{4}{3} \times 3, \qquad (1.82)
\end{aligned}$$

betafunctions without top contribution on the b-trajectory between M_{top} and M_{Higgs},

$$\begin{aligned}
b_1 b =& -\frac{4}{3} \times 2 - \frac{23}{30} \times 1 - \frac{1}{10} \times 1 \\
b_2 b =& \frac{22}{3} - \frac{4}{3} \times 2 - \frac{5}{6} \times 1 - \frac{1}{6} \times 1 \\
b_3 b =& 11 - \frac{4}{3} \times 2 - \frac{2}{3} \times 1, \qquad (1.83)
\end{aligned}$$

and betafunctions without top and Higgs contribution between M_{Higgs} and M_Z

$$\begin{aligned} b_1 c =& -\frac{4}{3} \times 2 - \frac{23}{30} \times 1 \\ b_2 c =& \frac{22}{3} - \frac{4}{3} \times 2 - \frac{5}{6} \times 1 \\ b_3 c =& 11 - \frac{4}{3} \times 2 - \frac{2}{3} \times 1. \end{aligned} \tag{1.84}$$

In this SM analysis the threshold effects are very small. But we will encounter more severe threshold effects in a next chapter.

Chapter 2

Group Theoretical Backgrounds

Before we dive into Grand Unification we had better organise our equipment. Group theory is the essential tool for building the SM structure into a bigger underlying framework. Therefore, this chapter offers an overview of useful group theoretical results. We do not bother too much with exactly proving all statements, our aim is to gain an understanding of how we might use them.
The reader already familiar with group theory might want to skip this chapter. However, as we wish to study material beyond master courses in physics, which is of fundamental importance in building unified theories, putting all this in a Appendix would too much cut the story we want to bring up.

2.1 Roots

Every Lie algebra has a fixed number of generators. A certain number of them can be diagonalized simultaneously. This number, the *rank*, is also characteristic for a Lie algebra. For $SU(N)$ for example we have $N^2 - 1$ generators, $N - 1$ of them can be diagonalized simultaneously. If we call these generators H_i, we have

$$[H_i, H_j] = 0 \qquad i, j = 1 \ldots l, \tag{2.1}$$

where l equals the rank of the group. We then write linear combinations of the remaining generators in such a way that these E_α satisfy

$$[H_i, E_\alpha] = \alpha_i E_\alpha. \tag{2.2}$$

This basis of a Lie algebra, with generators H_i and E_α is called Cartan-Weyl basis. We now see that to every generator E_α we can associate an l-dimensional vector α_i. These vectors are the *root vectors* of the algebra, they live in l-dimensional *root space*.
Let us briefly apply this Cartan-Weyl approach to $SU(3)$. In the usual basis, consisting of ($\frac{1}{2}$ times) Gell-Mann matrices λ^a, we have two diagonal generators λ^3 and λ^8. As the rank of $SU(3)$ is 2, we have already found the diagonal part of the algebra. So, writing $T^i = \frac{\lambda^i}{2}$, we pick

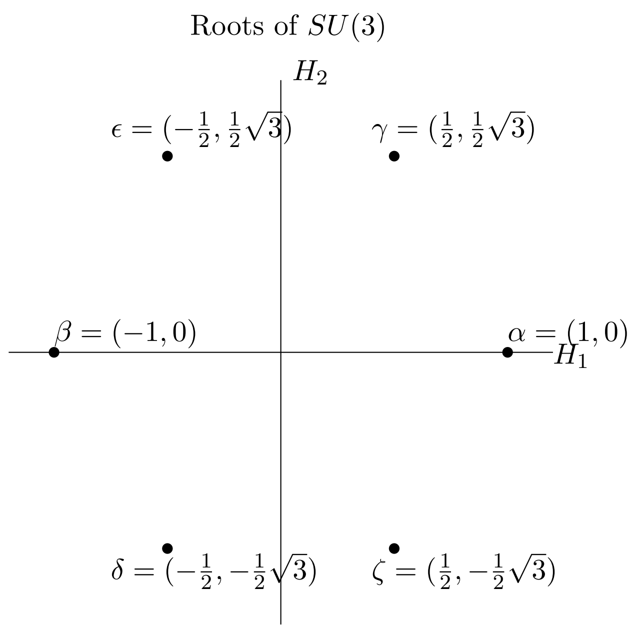
$$H_1 = T^3 \qquad H_2 = T^8, \tag{2.3}$$

$$E_{\alpha(\beta)} = \frac{T^1 + (-)iT^2}{\sqrt{2}} \qquad E_{\gamma(\delta)} = \frac{T^4 + (-)iT^5}{\sqrt{2}} \qquad E_{\epsilon(\zeta)} = \frac{T^6 + (-)iT^7}{\sqrt{2}}. \tag{2.4}$$

Let us calculate the root vector associated to E_α. From commutation relations or direct computation we get that

$$[H_1, E_\alpha] = E_\alpha \qquad [H_2, E_\alpha] = 0, \tag{2.5}$$

so this root α equals $(1, 0)$. In the same way we calculate the other five roots and depict them in a root diagram.



2.1.1 Simple roots

We now choose l of these roots to span this two dimensional root space. These are called *simple roots*. In $SU(3)$ it is common use to pick α and ϵ[1].
Different coordinate systems lead to different simple roots, but the angles between them and their relative lengths are always the same. They determine the complete root system and thus the whole algebra. Therefore some clever ways have been invented to write them down. One method is to write down the Cartan matrix A_{ij}, with

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}. \tag{2.6}$$

Here α_i and α_j denote the ith and jth simple root (i and j are no component indices). The $(,)$ inner product is just Euclidian. The Cartan matrix is a group character, different choices for generators H_i and E_α yield the same result. For

[1]One takes “positive” roots that cannot be written as linear combinations with positive coefficients of the other positive roots. In this case, positive roots are defined as root vectors with a positive second coordinate (or zero second coordinate and positive first coordinate), but the choice of the “positivity defining coordinate’ is arbitrary.

$SU(3)$ we easily obtain

$$A_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{2.7}$$

2.1.2 Dynkin labels

Just having established our root coordinates a minute ago, we already switch to new ones, called *Dynkin labels*. At first hand this may seem just a waste of time, all these new coordinate systems, but the use of Dynkin labels is going to simplify many calculations a lot because of its great property that all root coordinates are integers now. For any root vector a we obtain its Dynkin labels a_1 and a_2 by

$$a_1 = \frac{2(a,\alpha_1)}{(\alpha_1,\alpha_1)} \qquad a_2 = \frac{2(a,\alpha_2)}{(\alpha_2,\alpha_2)}, \tag{2.8}$$

where α_1 and α_2 are still the simple roots of $SU(3)$. It is easy to check that the Dynkin labels of the simple roots themselves are given, by definition, by the rows (or columns) of the Cartan matrix A. From now on we will distinguish between "H-roots", roots written in the original H-basis and "D-roots", roots expressed in Dynkin labels.
As we can easily observe that simple roots are in general not orthogonal, the inner product between D-roots cannot be Euclidian as before. Instead we have that the inner product between two D-roots α and β is given by

$$(\alpha,\beta) = \frac{1}{2}\alpha_i G_{ij}\beta_j, \tag{2.9}$$

where G is the inverse of the Cartan matrix A.
Instead of giving a rigorous proof of this result, which is not that hard but involves more new coordinate systems, let us check explicitly that it works by using the coordinate independence of the inner product. As an example we again take $SU(3)$. We take two arbitrary H-roots a_H and b_H:

$$a_H = \begin{bmatrix} a_1 & a_2 \end{bmatrix}, \qquad b_H = \begin{bmatrix} b_1 & b_2 \end{bmatrix}. \tag{2.10}$$

Their inner product obviously equals $a_1b_1 + a_2b_2$.
Now we express these same roots in Dynkin labels, we write them as D-roots. This yields

$$a_D = \left[\frac{2(\begin{bmatrix} a_1 & a_2 \end{bmatrix},\begin{bmatrix} 1 & 0 \end{bmatrix})}{(\begin{bmatrix} 1 & 0 \end{bmatrix},\begin{bmatrix} 1 & 0 \end{bmatrix})} \qquad \frac{2(\begin{bmatrix} a_1 & a_2 \end{bmatrix},\begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix})}{(\begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix},\begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix})} \right], \tag{2.11}$$

and the same for b_D. All inner products are still Euclidean. We thus conclude

$$a_D = \begin{bmatrix} 2a_1 & -a_1 + a_2\sqrt{3} \end{bmatrix} \qquad b_D = \begin{bmatrix} 2b_1 & -b_1 + b_2\sqrt{3} \end{bmatrix}. \tag{2.12}$$

The inverse of A is given by

$$G_{ij} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{2.13}$$

So now we can check our prescription for the inner product between D-roots:

$$
\begin{aligned}
(a_D, b_D) =& \frac{1}{2} a_i G_{ij} b_j \\
=& \frac{1}{2} \left[\ 2a_1 \quad -a_1 + a_2\sqrt{3}\ \right] \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} 2b_1 \\ -b_1 + b_2\sqrt{3} \end{bmatrix} \\
=& a_1 b_1 + a_2 b_2.
\end{aligned} \tag{2.14}
$$

.

2.2 Weights and representations

We now apply our framework of roots and root space to representation theory. In the first chapter it was mentioned already that there are infinitely many representations for every Lie-group. Every choice of generators that obeys the group-characteristic commutation relations is allowed. Every representation has the same number of generators, but there is no restriction to their dimensionality. Taking $SU(3)$ as our familiar example, we have that in the fundamental representation the generators are given by 3×3 matrices. These generators (or better, the Lie group elements derived from them) act on three dimensional states. The fundamental represention thus acts on a triplet of states, while the adjoint representation, whose generators are 8×8 matrices, acts on an octet of states. In this section we find out how to identify these states, these representation vectors.
The key idea is that every state, no matter to which representation it belongs, can be characterized by the l eigenvalues of the diagonal operators H_i:

$$
H_i|\lambda\rangle = \lambda_i|\lambda\rangle. \tag{2.15}
$$

The l-dimensional vector λ_i is called the *weight* of the representation vector $|\lambda\rangle$. Thus, we use the H-eigenvalues to label representation vectors.
Weight vectors live in the same space as root vectors. The most important notion of this whole chapter is that if a generator E_α acts on a state with weight λ, we get a state with weight $\lambda + \alpha$. (α is the root associated to the generator E_α.)

$$
\begin{aligned}
H_i(E_\alpha|\lambda\rangle) =& E_\alpha H_i|\lambda\rangle + [H_i, E_\alpha]|\lambda\rangle \\
=& E_\alpha \lambda_i|\lambda\rangle + \alpha_i E_\alpha|\lambda\rangle \\
=& (\lambda_i + \alpha_i) E_\alpha|\lambda\rangle
\end{aligned} \tag{2.16}
$$

We now state that by applying operators E_α we move through root space (or weight space) from one representation vector to another. As the simple roots suffice to span the whole root space, we can restrict ourselves to simple roots.
So here is the way, pointed out by Dynkin, to explore the various states in an irrep. Every irrep is uniquely identified by its state of highest weight Λ. These have been listed for many irreps of many groups. When working in the Dynkin basis (which is highly recommended) all components of Λ are non-negative integers. To get to the other states we subtract the simple roots. The number of times a simple root can be subtracted from a given state is given by its weight component. That is, if the ith component of the weight of a state equals n, we can subtract the ith simple

root n times. This process continues until we reach a state without any positive weight component, this is the "lowest state" of the irrep.
For example, the highest weight of the fundamental rep of $SU(3)$ is $(1 \quad 0)$. We thus subtract the first simple root $\alpha_1 = [2 \quad -1]$ once. This gives $(-1 \quad 1)$. Now we can subtract the second simple root $\alpha_2 = [-1 \quad 2]$. No simple root can be subtracted from the resulting state $(0 \quad -1)$, so we are at the end already. We found three states, which is exactly what we expected for this three dimensional representation. Later on we will have more complicated irreps, like the 16 dimensional irrep of $SO(10)$, but the approach to explore all states is always as it is in this example.
Some tools are useful to check the correctness of the pattern of states obtained in this procedure. The *level* of a state in an irrep is the number of simple roots that should be subtracted from the state of highest weight to get that state. In the preceding example the level of the state $(-1 \quad 1)$ equals 1. The level of the lowest state of an irrep is called the *height* $T(\Lambda)$ of that irrep. (It depends on the state of highest weight Λ of that irrep, as the whole pattern of the irrep depends on Λ.)
We have

$$T(\Lambda) = \sum_i \overline{R}_i a_i, \tag{2.17}$$

where $\overline{R}_i = 2\sum_j G_{ij}$, the so called *level vector* and a is the state of highest weight written in Dynkin coordinates.
In cases more complicated than $SU(3)$ we will see that there can be several states on the same level of an irrep. This naturally comes about when a certain state has several positive Dynkin labels, so that we can subtract several simple roots. Another check of the obtained weight diagram is then that it should be "spindle shaped": on every level k there should be as many states as on level $T(\Lambda) - k$.
It is even possible that certain states are degenerate. Degenerate states are always on the same level. Degeneracies can be checked with the Freudenthal recursion formula. The degeneracy of a state λ is defined in terms of the degeneracies of all states λ' that are above this one, up to the state of highest weight Λ:

$$n_\lambda = 2\frac{\sum_{\lambda'} n_{\lambda'}(\lambda', \alpha)}{(\Lambda+\delta, \Lambda+\delta) - (\lambda+\delta, \lambda+\delta)}. \tag{2.18}$$

Here $\delta = (1, 1...1)$ (as long as we work in a Dynkin basis) and α is that root that we should add (several times sometimes) to λ to get to λ'.

2.3 Applications

2.3.1 Casimir operators

In the previous chapter we already introduced the quadratic Casimir operator $C_2(r)$ and the related invariant $C(r)$. These can now be given for every irrep in terms of the state of highest weight of that irrep Λ:

$$C_2(\Lambda) = (\Lambda, \Lambda + 2\delta) \tag{2.19}$$

$$C(\Lambda) = \frac{N(\Lambda)}{N(adj)} C_2(\Lambda), \tag{2.20}$$

where N denotes the number of states in the considered irrep.
So let us check these formulae for the fundamental rep of $SU(3)$, with $\Lambda = (1 \quad 0)$.

$$C_2(1 \quad 0) = \frac{1}{2}\begin{pmatrix} 1 & 0 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \frac{4}{3} \tag{2.21}$$

$$C(1 \quad 0) = \frac{3}{8} \times \frac{4}{3} = \frac{1}{2} \tag{2.22}$$

in accordance with the results in chapter 1.

2.3.2 Eigenstates

The diagonal generators H_i form the axes of root space. In our $SU(3)$ example we have, working in H-coordinates for a moment, that the H_1 eigenvalue of a weight vector $(a \quad b)$ is simply given by a and the H_2 eigenvalue by b. We are just taking inner products between the H_1 axis $[1 \quad 0]$, or the H_2 axis $[0 \quad 1]$, and the weight vector $(a \quad b)$.
We could as well consider linear combinations of H_1 and H_2. If we define $Q = \alpha H_1 + \beta H_2$ the Q eigenvalue of weight vector $(a \quad b)$ equals $a\alpha + b\beta$. In the SM we have that the electric charge of a state is given by $T^3 + Y$, which are both diagonal generators. So we already understand how weight vectors can give eigenvalues of physical interest.
Things get more cumbersome when we use Dynkin labels. We then have to express the H axes in Dynkin coordinates and use the Dynkin inner product 2.9. Conceptually that is not such a big deal. The problem is that in general the Dynkin labels of simple roots are listed in many tables (in the form of Cartan matrices) but the H coordinates are not. So instead of finding H coordinates of simple roots of complicated groups like $SU(5)$ or $SO(10)$ I propose to calculate eigenvalues of any diagonal generator Q of a weight vector λ by applying

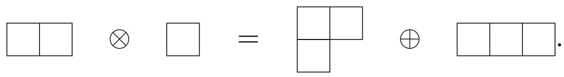
$$Q(\lambda) = q_i \lambda_i, \tag{2.23}$$

where q is an as yet undefined axis and λ_i are the Dynkin labels of λ. If we then know the Q eigenvalues of some weight vectors we can find an expression for the q axis. From there we calculate the Q eigenvalues of all other weight vectors.

2.3.3 Decomposition of tensor products

To build gauge invariant quantities in GUT Lagrangians we have to be able to identify the various parts of a tensor product between two irreps. One quick way to do this is with the use of Young tableaux. This works especially well for $SU(N)$. To every irrep of $SU(N)$ we associate a diagram of boxes. For such a diagram we can calculate two useful numbers, the Ferrers factor F and the Hooks factor H. Every box has a F value and an H value, the values of the whole diagram are just products of the values of all boxes. F values are assigned in the following way: the top left box gets N and from there the values increase by 1 when moving to the right and decrease by 1 when moving downwards. The H factor of a box is just 1 plus the number of boxes to the right and below that one. The dimensionality of the irrep is then given by $\frac{F}{H}$.

To multiply two diagrams we follow an easy algorithm. The boxes in the first row of the right diagram are filled with αs, the boxes in the second row get βs, and so forth. First we paste the α boxes to the left diagram. We can only add boxes to the right or to the bottom of a diagram, and the number of boxes in rows and columns should never increase when moving from the top left to the bottom right corner of the resulting diagram. Next we add the βs, the γs, all in the same way. In the end we write down the sequence of αs, βs when reading from right to left and from top to bottom. We only keep diagrams with sequences that never contain more βs than αs left of any symbol, or more γs than βs.
So let us construct. We have for example



(2.24)

In $SU(3)$ F factors are 12, 3, 24 and 60 respectively while the Hooks factors yield 2, 1, 3 and 6. We thus conclude that the tensor product of the sixdimensional and the three dimensional irrep of $SU(3)$ can be decomposed in an eight dimensional irrep and a ten dimensional irrep:

$$\mathbf{6} \otimes \mathbf{3} = \mathbf{8} \oplus \mathbf{10}. \tag{2.25}$$

The construction 2.24 is legitimate in every special unitary group, but the dimensions of the irreps represented by the diagrams can change. In $SU(5)$ 2.24 denotes

$$\mathbf{15} \otimes \mathbf{5} = \mathbf{40} \oplus \mathbf{35}. \tag{2.26}$$

Now let us investigate what the structure of roots and weights has to say about these matters.
Working in $SU(3)$ we already found the weight system of the fundamental representation. The state of highest weight of the six dimensional represention is $(2 \quad 0)$ (as we can check from the literature). Applying the simple roots $a_1 = [2 \quad -1]$ and $a_2 = [-1 \quad 2]$ we easily find the other five states. We have

$$\begin{aligned} &\mathbf{3}: (1 \quad 0) \quad (-1 \quad 1) \quad (0 \quad -1) \\ &\mathbf{6}: (2 \quad 0) \quad (0 \quad 1) \quad (-2 \quad 2) \quad (1 \quad -1) \quad (-1 \quad 0) \quad (0 \quad -2) \ . \end{aligned} \tag{2.27}$$

Now we perform the tensor multiplication by adding Dynkin labels of all combinations of one state from the **3** and one state from the **6**. We thus find 18 new states:

$$\begin{array}{cccccc} (3 \quad 0) & (1 \quad 1) & (-1 \quad 2) & (2 \quad -1) & (0 \quad 0) & (1 \quad -2) \\ (1 \quad 1) & (-1 \quad 2) & (-3 \quad 3) & (0 \quad 0) & (-2 \quad 1) & (-1 \quad 1) \\ (2 \quad -1) & (0 \quad 0) & (-2 \quad 1) & (1 \quad -2) & (-1 \quad -1) & (0 \quad -3). \end{array}$$

To decompose this product we first look for the state of highest weight. The $SU(3)$ level vector $\overline{R}$ is [22]. The maximum value of $R_i a_i$ is 6, for the state $a = (3 \quad 0)$. We thus conclude that this a is (the weight of) the state of highest weight. Subtracting simple roots from this state should yield an irrep of height 6 (which means that we should have to subtract 6 simple roots from the state of highest weight to get to the lowest state). So let us try to construct this irrep.

$$
\begin{array}{ccccc}
 & & (3 \quad 0) & & \\
 & & \downarrow a_1 & & \\
 & & (1 \quad 1) & & \\
 & \swarrow a_1 & & \searrow a_2 & \\
(-1 \quad 2) & & & & (2 \quad -1) \\
\downarrow a_1 & & \searrow a_2 & & \downarrow a_1 \\
(-3 \quad 3) & & & & (0 \quad 0) \\
\downarrow a_2 & & \swarrow a_1 & & \downarrow a_2 \\
(-2 \quad 1) & & & & (1 \quad -2) \\
 & \searrow a_2 & & \swarrow a_1 & \\
 & & (-1 \quad -1) & & \\
 & & \downarrow a_2 & & \\
 & & (0 \quad -3) & &
\end{array}
$$

Well, that certainly looks like a perfectly spindle shaped weight diagram of height 6!
In the same way we can organize the 8 remaining states. That is going to give the adjoint representation with highest weight $(1 \quad 1)$.
N.B. We have to subtract both roots from $(-1 \quad 2)$, as can be seen from the weight diagram: we should subtract root a_2 because the second weight component of $(-1 \quad 2)$ is positive and root a_1 because that one should be subtracted three times from the top state $(3 \quad 0)$.

2.3.4 Branching rules and projection matrices

The first step in Grand Unification is to find a group in which the SM $SU(3) \times SU(2) \times U(1)$ gauge group can be embedded. Therefore we want to study a method of classifying (maximal) subalgebras of simple algebras in this section.
Subalgebras can be found by looking for subsets of generators of the overlying algebra. For example, in $SU(3)$ we have that the generators T^1, T^2, T^3 and T^8 form a subalgebra. (This means that all Lie products (commutators) between T^1, T^2, T^3 and T^8 can be written as linear combinations of T^1, T^2, T^3 and T^8.) These four generators can form an algebra of $SU(2) \times U(1)$. Another subalgebra, consisting of T^3, T^4 and T^5 is an algebra of $SO(3)$.
Once we know a possible subgroup of a Lie group, we can study the breaking of the original group to this subgroup. We want to find out to which state of which irrep of that subgroup an arbitrary state of the original group "branches to". To this end, we construct a projection matrix that transforms weight vectors.
As an example we take the $SU(3) \times SU(2)$ subgroup of $SU(5)$[2]. We need some reasonable assumptions to define the projection matrix. The fundamental rep of $SU(5)$ contains five states. The biggest $SU(3) \times SU(2)$ rep it can thus contain is a $SU(3)$ singlet, $SU(2)$ doublet. We then demand that the highest weight of the **5** of $SU(5)$, which is $(1 \quad 0 \quad 0 \quad 0)$, branches to the highest weight of the fundamental

[2]The existence of this subgroup follows from an analysis of $SU(5)$ generators that will be done in the next chapter.

3 rep of $SU(3)$, $(1 \quad 0)$, and to the (0) weight of the trivial one dimensional rep of $SU(2)$. Thus, denoting $SU(3) \times SU(2)$ weight vectors by $SU(3)$ weights between first parentheses and $SU(2)$ weights between second parentheses, we demand

$$(1 \quad 0 \quad 0 \quad 0) \rightarrow (1 \quad 0)(0). \tag{2.28}$$

Now it seems very reasonable that the highest state in the $\overline{\mathbf{5}}$ branches to the highest state in the $SU(3)$ antitriplet $\overline{\mathbf{3}}$ and $SU(2)$ singlet:

$$(0 \quad 0 \quad 0 \quad 1) \rightarrow (0 \quad 1)(0). \tag{2.29}$$

We are proceeding well but we do not have enough constraints yet. After having exploited the five dimensional rep of $SU(5)$ we turn to the next simplest irrep of $SU(5)$, the **10**. In these 10 states there is room to embed a $SU(3)$ triplet, $SU(2)$ doublet. The highest weight of **10** is $(0 \quad 1 \quad 0 \quad 0)$. The $\overline{\mathbf{10}}$ has highest weight $(0 \quad 0 \quad 1 \quad 0)$. The $SU(2)$ doublet has weights (1) and (-1). We thus demand

$$(0 \quad 1 \quad 0 \quad 0) \rightarrow (1 \quad 0)(1) \tag{2.30}$$

$$(0 \quad 0 \quad 1 \quad 0) \rightarrow (0 \quad 1)(1). \tag{2.31}$$

Now we have fixed all parameters of the projection matrix P. We conclude

$$P(SU(5)) \subset SU(3) \times SU(2) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \tag{2.32}$$

With the projection matrix we can now derive every $SU(5) \rightarrow SU(3) \times SU(2)$ branching rule of every irrep we are interested in.

Chapter 3

Grand Unification

After investigating in the first chapter what exactly we are trying to "grand unify" and getting used to our tools to do this in the second, we are ready to launch our first try.

If we are to embed the whole SM in one gauge group, then the most obvious demand is of course that there should be enough room in this group to accomodate all the SM contents. The ranks of $SU(3)$, $SU(2)$ and $U(1)$ are two, one and one respectively, so the unifying group should at least have rank four. The smallest Lie group with rank four is $SU(5)$. It thus seems a well suited starting point.

3.1 Embedding the SM fields in $SU(5)$

Let us first try to place all matter fields. We thus assume that at the GUT scale of 10^{13} GeV all fields are in $SU(5)$ reps. Due to spontaneous symmetry breaking that we still have to explain but that is described already by branching rules we only observe an $SU(3) \times SU(2) \times U(1)$ symmetry at the SM scale (or weak scale) of around 100 GeV.

We choose to work with only lefthanded fields, so we express all righthanded fields f as lefthanded antifields f^c (see Appendix B). For the moment we will take care of just one generation. To embrace the whole SM we should just copy thrice.

The simplest non-trivial reps of $SU(5)$ are the fivedimensional fundamental and antifundamental reps $\mathbf{5}$ and $\overline{\mathbf{5}}$. As we stated in the preceding chapter, we thus find room to accomodate one $SU(3)$ triplet, $SU(2)$ singlet and one $SU(3)$ singlet, $SU(2)$ doublet. We are thus uniquely led to postulate that this fivedimensional rep should contain an antiquark triplet (in the language of chapter 1: a righthanded quark triplet) and a lepton doublet. But as we have already built our $SU(5) \to SU(3) \times SU(2)$ projection matrix in such a way that the highest weight of the $\overline{\mathbf{5}}$ rep branches to the highest weight of the $\overline{\mathbf{3}}$ rep, which is a very reasonable demand, it is seems clear that we should sweep these five states in a $\overline{\mathbf{5}}$ rep rather than in a $\mathbf{5}$ rep.

We now have ten fields left. The next simplest $SU(5)$ irrep is actually the tendimensional one. That is a wonderful match. [1] It has lost much of its glance since the

[1]It even led Georgi and Glashow to state that "we are led inescapably to the conclusion that $SU(5)$ is the gauge group of the world."

discovery of righthanded neutrinos though. OK, we could always couple an $SU(5)$ singlet to the other reps, but we had better forget about righthanded neutrinos at all for the moment and explore $SU(5)$ further as a useful toy model. Note, by the way, that we cannot put these ten remaining states in two more fivedimensional reps because the $SU(3)$ triplet, $SU(2)$ doublet of lefthanded quarks does not fit in such a rep. Moreover, the assumptions on which we built the projection matrix force us to use the **10** rep and not its complex conjugate.
So let us explore these two reps and check whether we can see them branching to the known SM fields. To this end we need to know the weight diagrams of the relevant reps of $SU(2)$, $SU(3)$ and $SU(5)$.
In $SU(2)$ there is just one root, $a = 2$. The fundamental twodimensional rep consists of states with onedimensional weight vectors (1) and (-1). The threedimensional rep consists of (2), (0) and (-2). Such a rep, where the weights on level $T(\Lambda) - k$ are the negatives of the weights on level k is called *selfconjugate*. If, moreover, $T(\Lambda)$ is even, the rep is *real*: it is equivalent to its complex conjugate rep. So $\mathbf{2} \equiv \overline{\mathbf{2}}$, just as we saw in the first chapter when we focused on the generators. If the height of a selfconjugate rep is odd, it is *pseudoreal*.
We have already seen the fundamental rep of $SU(3)$: it consists of (10), (-1 1) and (0 -1)). Similarly, the antifundamental rep contains states with weight vectors (0 1), (1 -1) and (-1 0). These are both *complex* reps, the weights on level $T(\Lambda) - k$ are not the negatives of the ones on the kth level.
Now we have to construct the weight diagrams of the $\overline{\mathbf{5}}$ and **10** rep of $SU(5)$. First we need the four simple roots of $SU(5)$. These follow from its Cartan matrix that we look up in the literature:

$$A_{SU(5)} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}. \tag{3.1}$$

Now we could calculate the level vector and use some more tricks to calculate the state of highest weight of the $\overline{\mathbf{5}}$ rep, but states of highest weight have also been extensively listed so let us not go too deep into the mathematics and just state that the highest weight is (0 0 0 1).
So here we go:

$$\begin{array}{c} (0 \quad 0 \quad 0 \quad 1) \\ \downarrow a_4 \\ (0 \quad 0 \quad 1 \quad -1) \\ \downarrow a_3 \\ (0 \quad 1 \quad -1 \quad 0) \\ \downarrow a_2 \\ (1 \quad -1 \quad 0 \quad 1) \\ \downarrow a_1 \\ (-1 \quad 0 \quad 0 \quad 0). \end{array} \tag{3.2}$$

The highest weight in the **10** is given by (0 1 0 0). Its weight diagram is a bit more complicated:

$$
\begin{array}{ccccc}
 & & (0\quad 1\quad 0\quad 0) & & \\
 & & \downarrow a_2 & & \\
 & & (1\quad -1\quad 1\quad 0) & & \\
 & \swarrow a_1 & & \searrow a_3 & \\
(-1\quad 0\quad 1\quad 0) & & & & (1\quad 0\quad -1\quad 1) \\
\downarrow a_3 & & \swarrow a_1 & & \downarrow a_4 \\
(-1\quad 1\quad -1\quad 1) & & & & (1\quad 0\quad 0\quad -1) \\
\downarrow a_2 & & \searrow a_4 & & \downarrow a_1 \\
(0\quad -1\quad 0\quad 1) & & & & (-1\quad 1\quad 0\quad -1) \\
 & \searrow a_4 & & \swarrow a_2 & \\
 & & (0\quad -1\quad 1\quad -1) & & \\
 & & \downarrow a_3 & & \\
 & & (0\quad 0\quad -1\quad 0). & &
\end{array}
$$

Next we want to check whether these fifteen states really branch to the full particle content of the SM. Applying our projection matrix 2.32 we find

$SU(5)$ weight	$SU(3)\times SU(2)$ weight	SM multiplet
(0001)	(010)	$(\overline{\mathbf{3}},\ \mathbf{1},\frac{1}{3})$
(001-1)	(001)	$(\mathbf{1},\ \mathbf{2},-\frac{1}{2})$
(01-10)	(1-10)	$(\overline{\mathbf{3}},\ \mathbf{1},\frac{1}{3})$
(1-100)	(00-1)	$(\mathbf{1},\ \mathbf{2},-\frac{1}{2})$
(-1000)	(-100)	$(\overline{\mathbf{3}},\ \mathbf{1},\frac{1}{3})$
(0100)	(101)	$(\mathbf{3},\ \mathbf{2},\frac{1}{6})$
(1-110)	(010)	$(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3}$
(-1010)	(-111)	$(\mathbf{3},\ \mathbf{2},\frac{1}{6})$
(10-11)	(10-1)	$(\mathbf{3},\ \mathbf{2},\frac{1}{6})$
(-11-11)	(000)	$(\mathbf{1},\ \mathbf{1},1)$
(100-1)	(1-10)	$(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3}$
(0-101)	(-11-1)	$(\mathbf{3},\ \mathbf{2},\frac{1}{6})$
(-110-1)	(0-11)	$(\mathbf{3},\ \mathbf{2},\frac{1}{6})$
(0-11-1)	(-100)	$(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3}$
(00-10)	(0-1-1)	$(\mathbf{3},\ \mathbf{2},\frac{1}{6})$.

(3.3)

The hypercharge assignments follow from stating that the hypercharge axis should be $\frac{1}{6}[-21-12]$. (We could for example demand that the lefthanded quark fields have hypercharge $\frac{1}{6}$. This uniquely defines the hypercharge axis.) We now understand that the triplet of antifields in the $\overline{\mathbf{5}}$ describes anti-down quarks. The anti-up quarks are in the **10**.

In passing by we note that it is equally easy to define an electric charge axis: $Q=\frac{1}{3}[-1211]$.

We now turn to gauge boson fields. They should, as always, follow from the adjoint representation. In $SU(5)$ the adjoint contains 24 states. Subtracting many simple roots (the height of this irrep is 8) from the state of highest weight (1001) we find these 24 weight vectors. On applying the projection matrix we find back a $\mathbf{8},\mathbf{1}$ gluon octet, a **1**, **3** weak boson triplet, a **1**, **1** $U(1)$ boson singlet and twelve more states: a **3**, **2** doublet of colour triplets, one with electric charge $-\frac{1}{3}$, one with electric charge $-\frac{4}{3}$, and a $\overline{\mathbf{3}}$, **2** doublet of colour antitriplets with electric charges $\frac{4}{3}$ and $\frac{1}{3}$. We choose to call the colour antitriplets X and Y bosons, the two fundamental triplets will be X and Y antibosons. As X and Y bosons have never been detected, we should look for a way to extend the Higgs mechanism to put their masses on the GUT-scale.

We now propose to organise the fivedimensional state vector (transforming in the $\overline{\mathbf{5}}$ rep) as

$$\psi = \begin{pmatrix} (d^1)^c \\ (d^2)^c \\ (d^3)^c \\ e \\ \nu \end{pmatrix}. \tag{3.4}$$

Something very funny is going on here: now that we have three down quarks, an electron and a neutrino in the same multiplet, we are naturally led to $Q(d) = -\frac{1}{3}Q(e)$. If not, a traceless electric charge generator Q could never be defined. This $SU(5)$ GUT therefore naturally explains the fractional charges of the quarks! This particular choice 3.4 allows for a convenient representation of generators. Let us check immediately how that comes to be. We will organise the tendimensional state vector χ (transforming in the **10** rep) afterwards.

The $SU(5)$ covariant derivative reads

$$D_\mu = \partial_\mu - i\frac{g}{2}V_\mu^a T_\mu^a, \qquad a = 1\ldots 24. \tag{3.5}$$

Here g denotes the one and only universal $SU(5)$ GUT coupling constant. We thus have to write down representations for the 24 Hermitian 5×5 generators T^a that define the **5** rep of $SU(5)$. We could of course look up some standard generators from the literature, as we did for $SU(3)$ and $SU(2)$ for example, or try to construct them ourselves from the $SU(5)$ commutation relations, but there is a more convenient way. We will combine our knowledge of the $SU(3)$ and $SU(2)$ fundamental reps with the fact, obtained from weight vector analysis, that the **5** of $SU(5)$ can accomodate a **3** of $SU(3)$ and a **2** of $SU(2)$.

First we take

$$T^{1\ldots 8} = \begin{pmatrix} & & & 0 & 0 \\ & \lambda^{1\ldots 8} & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.6}$$

The gauge fields $V_\mu^{1\ldots 8}$ are now identified with the 8 gluons.

Note that here we once and for all have defined the normalization of our $SU(5)$ generators to be (see Appendix A.2)

$$\mathrm{Tr}(T^a T^b) = 2\delta^{ab}. \tag{3.7}$$

As the Pauli matrices have the same normalization, we can place the $SU(2)$ generators in the bottom right corner:

$$T^{9,10,11} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \multicolumn{2}{c}{\sigma^{1,2,3}} \\ 0 & 0 & 0 & & \end{pmatrix}. \tag{3.8}$$

Thus, $V_\mu^{9,10,11}$ will be the weak boson fields after symmetry breaking on the GUT scale. (After symmetry breaking on the electroweak scale V^9 and V^{10} will combine to charged W boson fields while V^{11} and V^{12} form the Z boson and photon.) The twelfth generator should therefore act as a $U(1)$ generator. As hypercharges are the same for all triplet and doublet members it must necessarily be of the form $\text{Diag}(a, a, a, b, b)$. Normalization then requires

$$T^{12} = \frac{1}{\sqrt{15}}\text{Diag}(-2, -2, -2, 3, 3). \tag{3.9}$$

Now for the remaining generators, that are to correspond to the X and Y bosons. From their weight vectors we saw that these form "six-plets", or "tridoublets" : $SU(3)$ triplets, $SU(2)$ doublets, just like the quark fields. They should thus have nonzero entries outside the pure $SU(3)$ 3×3 box in the top left corner and the pure $SU(2)$ 2×2 box in the bottom left corner. So let us try the easiest solution that respects the demanded normalization condition:

$$T^{13} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, T^{14} = \begin{pmatrix} 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.10}$$

$$T^{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.11}$$

and so forth.
We now want to find out the connection between the 12 gauge fields $V_\mu^{13...24}$ and the X and Y bosons. From our group-theoretical analysis we have already found that these are charged bosons. In the first chapter we found that we have to take linear combinations of the $SU(2)$ gauge fields to describe charged W-bosons. Here we are in the same situation. The most common definition is to denote

$$\begin{aligned} X_\mu^1 =& \frac{1}{\sqrt{2}}(V_\mu^{13} + iV_\mu^{14}) \\ \overline{X_\mu^1} =& \frac{1}{\sqrt{2}}(V_\mu^{13} - iV_\mu^{14}) \end{aligned} \tag{3.12}$$

and so forth. $V^{15...18}$ provide X^2 and X^3 boson fields and antifields while $V^{19...24}$ yield all Y bosons.
In the last part of this embedding operation we assign the ten remaining SM fields to χ, a tendimensional irrep of $SU(5)$. We could compose a tendimensional row vector χ but then we are forced to write out representations of the 24 $SU(5)$ generators. In principle that can be done straightforwardly, using the decomposition of the **10** rep, but the result, 24 10×10 matrices, might not prove that illuminating.
Instead we choose to accomodate these 10 states in a 5×5 antisymmetric matrix (having 10 degrees of freedom). This is a reasonable choice because it arises in the decomposition of the simplest $SU(5)$ tensor product:

$$\Box \otimes \Box = \Box\Box \oplus \begin{array}{c}\Box\\[-4pt]\Box\end{array}, \tag{3.13}$$

or

$$\mathbf{5} \otimes \mathbf{5} = \mathbf{15} \oplus \mathbf{10} \tag{3.14}$$

or even

$$\left((\mathbf{3}, \mathbf{1} \oplus \mathbf{1}, \mathbf{2}) \otimes (\mathbf{3}, \mathbf{1} \oplus \mathbf{1}, \mathbf{2})\right)_{AS} = \overline{\mathbf{3}}, \mathbf{1} \oplus \mathbf{3}, \mathbf{2} \oplus \mathbf{1}, \mathbf{1} \tag{3.15}$$

where the subscript AS indicates that we focus on the antisymmetric parts of the tensor product. On the righthandside of 3.15 we recognise a doublet of quark triplets (in the language of the first chapter: the lefthanded quark fields), another quark triplet (these must be anti-up fields as the anti-downs are already contained in the $\overline{\mathbf{5}}$ rep) and a singlet (the neutrino field).
So it seems quite reasonable to denote the tendimensional rep as

$$\chi = \begin{pmatrix} 0 & (u_b)^c & -(u_g)^c & -u_r & -d_r \\ -(u_b)^c & 0 & (u_r)^c & -u_g & -d_g \\ (u_g)^c & -(u_r)^c & 0 & -u_b & -d_b \\ u_r & u_g & u_b & 0 & -e^c \\ d_r & d_g & d_b & e^c & 0 \end{pmatrix}. \tag{3.16}$$

3.2 Writing down the $SU(5)$ Lagrangian

Now that all the embedding is done we can write down the $SU(5)$ generalization of the Standard Model Lagrangian.

3.2.1 Fermion kinetic terms

The $\overline{\mathbf{5}}$ part has almost the same structure as 1.31 in the SM:

$$\mathcal{L}^5_{FK} = \overline{\psi}(i\gamma^\mu D^5_\mu)\psi, \qquad D^5_\mu \psi = [\partial_\mu + \frac{ig}{2} V^a_\mu (T^a)^\star]\psi. \tag{3.17}$$

Here T^a denote the 5×5 $SU(5)$ generators we have just composed. We take their conjugates $\overline{T^a} = -(T^a)^\star$ because we are working in the $\overline{\mathbf{5}}$ rep rather than the **5** rep.
In the gauge boson part of D_μ there is, as compared to the SM structure, an extra factor of $\frac{1}{2}$. It compensates the fact that whereas we had $\frac{\lambda^a}{2}$ and $\frac{\sigma^a}{2}$ matrices in

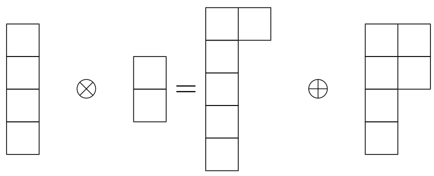
our SM generators, we have chosen the $SU(5)$ generators to contain λ^a and σ^a matrices.
The **10** part of the $SU(5)$ kinetic term looks different because we have chosen to represent χ_{10} as an antisymmetric 5×5 tensor instead of a 10-dimensional row vector. We have

$$\mathcal{L}_{FK}^{10} = -\frac{1}{2}\mathrm{Tr}\left[\overline{\chi} i \gamma^\mu \left(\partial_\mu \mathbf{1} - i g V_\mu^a T^a\right)\chi\right], \tag{3.18}$$

where again T^a denote the $SU(5)$ generators in the fundamental rep **5**, matrix multiplication is implied and extra factors of -1 and $\frac{1}{2}$ were included to compensate for the antisymmetry of χ and the double counting of kinetic terms that this matrix multiplication implies.

3.2.2 Fermion mass terms

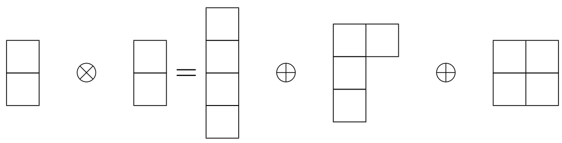
To recover all SM mass terms we need a term like $\psi^T C \chi H + h.c.$ (for down and electron masses) and a term like $\mathrm{Tr}\left(\chi^T C \chi H + h.c.\right)$ (for up masses). Building mass terms from LH fields only is reviewed in Appendix B.
To check a Lagrangian on renormalizablility and Lorentz invariance is as straightforward as in the SM case, but building a gauge invariant term is a bit less trivial now. First we use Young tableaux to check how we can decompose these products between $\psi(\overline{\mathbf{5}})$ and $\chi(\mathbf{10})$:



(3.19)

$$\overline{\mathbf{5}} \otimes \mathbf{10} = \mathbf{5} \oplus \overline{\mathbf{45}} \tag{3.20}$$

and



(3.21)

$$\mathbf{10} \otimes \mathbf{10} = \overline{\mathbf{5}} \oplus \mathbf{45} \oplus \mathbf{50}. \tag{3.22}$$

In minimal $SU(5)$ one keeps only the fivedimensional Higgs multiplet but we can as well maintain the **45**. The **50** is useless because it does not contain a colour singlet after electroweak (SM) symmetry breaking. (Which means that it will lead to $SU(3)$ breaking mass terms, discriminating between quark colours)
So let us write down gauge invariant mass terms:

$$\mathcal{L}_{FM}^{\psi\chi} = (\psi_i^\alpha)^T(\overline{\mathbf{5}}) C \chi^{kl\beta}(\mathbf{10})\left[(y_{de}^5)^{\alpha\beta}\left(\delta_k^i H_l(\overline{\mathbf{5}}) - \delta_l^i H_k(\overline{\mathbf{5}})\right) \oplus (y_{de}^{45})^{\alpha\beta} H_{kl}^i(\mathbf{45})\right] + \text{h.c.} \tag{3.23}$$

and

$$\mathcal{L}_{FM}^{\chi\chi} = (\chi^{kl\alpha})^T(\mathbf{10}) C \chi^{pq\beta}(\mathbf{10})\left[(y_u^5)^{\alpha\beta}\epsilon_{klpqr} H^r(\mathbf{5}) \oplus (y_u^{45})^{\alpha\beta}\epsilon_{klpst} H_q^{st}(\overline{\mathbf{45}})\right] + \text{h.c.}. \tag{3.24}$$

Here we have used α and β as generation indices. As before, the gauge singlet terms are projected out using Kronecker deltas and Levi-Civita epsilons. Roman indices denote ψ and χ multiplet members.
We see that we have lost one Yukawa matrix. In the SM we had y_u, y_d and y_e, now there are only two: y_{de} couples ψ to χ and will thus yield down and electron masses, y_u couples χ to χ and will therefore give rise to up mass terms. The fact that down quarks and charged leptons are in the same $SU(5)$ multiplet means that there can be no differences between their Yukawa matrices anymore. However, mass differences are still possible, because the various Higgs fields can still take vevs discriminating between multiplet members. (Remember that masses are eventually given by $\frac{yv}{\sqrt{2}}$.)
A natural extension of the Higgs vev 1.22 is

$$< H >= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v_5 \end{pmatrix}. \tag{3.25}$$

We then easily find mass terms for down quarks and electrons from 3.23 and a mass term for up quarks from 3.24. The values of these masses can be manipulated by changing the diagonal terms of the y matrices and by adjusting v_5 of course but, as this Higgs vev treats downquarks and electrons in an equal way, there is no freedom left to distinguish between their masses: $m_e = m_d$ in all generations[2].
This is why we consider Higgs fields in the **45** rep as well. To give the field H^i_{kl} its 45 degrees of freedom we demand that it should be antisymmetric in k and l and that the sum H^i_{il} vanishes. If the 45 dimensional Higgs field develops a vev

$$< H^a_{b5} >= v_{45}(\delta^a_b - 4\delta^a_4\delta^4_b) \tag{3.26}$$

we conclude (omitting the $H(\mathbf{5})$ contributions for a moment) $m_e = 3m_d$ and the same for the other generations. If we allow both reps of the H field we can in principle achieve any ratio between these masses.
In the "up" mass term 3.24 we basically take the same vevs. There is one subtlety: y^{45}_u should be antisymmetric because it multiplies the antisymmetric ϵ-tensor. (We do not have this problem for y^5_u: in that term antisymmetry is guaranteed by the presence of *all* χ-indices in the ϵ-tensor.) As eigenvalues of antisymmetric matrices always come in opposite pairs the best fitting possibility would be $m_u = 0, m_c = m_t$.
Finally we want to report very briefly on generation mixing in minimal $SU(5)$. In the SM we redefined all quark and lepton fields in order to have mass terms diagonal in generation space. This switching to mass eigenstates led to generation mixing in the charged currents (see 1.44). The difference in $SU(5)$ is that there are just two fermion fields left to redefine: ψ and χ. This is still enough to reproduce the CKM matrix. A detailed analysis (from [17]) yields that these rotations of ψ and χ lead to two more CP violating phases for the anti-up fields (righthanded fields in the language of chapter 1). These will be observable in nucleon decay processes only.

[2] These relations apply at the GUT scale of course. One needs renormalisation group equations to extrapolate to lower energies. We will come to that later.

3.2.3 Gauge kinetic terms

Writing out the gauge kinetic term is quite straightforward.

$$\mathcal{L}_{GK} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}. \tag{3.27}$$

Now $F^a_{\mu\nu}$ is composed from the 24 vector fields V^a_μ.
We have not fully completed the $SU(5)$ Lagrangian. The $SU(5)$ generalizations of the Higgs kinetic term and the Higgs mass term will be treated in the next section however, when we study GUT-symmetry breaking.

3.3 The Higgs mechanism in $SU(5)$

In $SU(5)$ the Higgs mechanism is more delicate than in the SM. It should not only repeat the Standard Model trick of providing masses to weak bosons, the breaking of the $SU(5)$ GUT-symmetry should also be described by it. The Higgs mechanism should act as a prism, breaking almost[3] perfect white light at the GUT-scale (the unified multiplets) to the variety of colours (the different SM multiplets) that blind our eyes at the electroweak scale.

$$SU(5) \rightarrow^{10^{13}\mathrm{GeV}} SU(3) \times SU(2) \times U(1) \rightarrow^{10^{2}\mathrm{GeV}} SU(3) \times U(1). \tag{3.28}$$

That is, the SM gauge symmetry is recovered at 10^{13} GeV already. The difference with the SM is that the various fields are in different reps of a different gauge group now. But effectively no difference can be seen: no non-SM like interaction can occur below M_{GUT} because the gauge bosons responsible for these interactions are too heavy. The first stage of symmetry breaking, at the GUT scale, should therefore provide GUT masses for the twelve new gauge bosons while the twelve SM-like gauge fields should remain massless. To this end we introduce a field Σ that is in the adjoint rep of $SU(5)$. Thus, it is a 24-plet. On combining the usual form of the covariant derivative $D_\mu = \partial_\mu - igV^a_\mu T^a$, where T^a are arbitrary reps of the generators of the gauge group, and the definition of the adjoint representation, $(T^a)_{kj} = -if_{ajk}$ we infer

$$D_\mu \Sigma_p = \partial_\mu \Sigma_p - gV^a_\mu f_{aqp} \Sigma_q. \tag{3.29}$$

Now we introduce

$$\Sigma = \frac{1}{\sqrt{2}} \Sigma^a T^a. \tag{3.30}$$

We then have

$$D_\mu \Sigma = \partial_\mu \Sigma - gV^a_\mu [T^a, \Sigma]. \tag{3.31}$$

From this last expression we can understand how to write a SU(5) generalisation of the SM term 1.19. We take

$$\mathcal{L}_{HK} = (D_\mu \Sigma_p)^\star D^\mu \Sigma_p \tag{3.32}$$

[3] We still have two distinct multiplets.

Now we assume that at the GUT scale this Σ takes a nonzero vev $< \Sigma >$. Following the same strategy as in the SM we then find that the mass squared of the gauge boson V_μ^a will be given by

$$2g^2\mathrm{Tr}[T^a, < \Sigma >]^2. \tag{3.33}$$

We pick

$$< \Sigma >= v_{24}\mathrm{Diag}(1, 1, 1, -\frac{3}{2}, -\frac{3}{2}), \tag{3.34}$$

which yields $m_X^2 = m_Y^2 = 25g^2v_{24}^2$ while the first twelve bosons remain massless.
Note that this mechanism of symmetry breaking can also be described by a more group theoretical analysis. We can just write out all the weights in the adjoint **24** rep of $SU(5)$. Then we can find a "Σ Higgs axis", that does the same job as for example the hypercharge axes described before. That is, the (Euclidean) inner product squared between a weight vector and this axis yields the mass of the corresponding gauge boson (up to a factor $g^2v_{24}^2$ after symmetry breaking. I calculated this axis to be $[-2, 1, -1, 2]$.
The second stage of symmetry breaking takes place at the electroweak scale. We already wrote out mass terms. We understand that in minimal $SU(5)$ we need a **5** Higgs field H. Then the term

$$\mathcal{L}_{HK} = (D_\mu H)^\star D^\mu H \tag{3.35}$$

will generate masses for the W and Z bosons, just as 1.19 did in the SM.
Finally, we have to write down generalizations of the $\mathcal{L}_{HM}$ term in the SM. We need a scalar potential that can generate the vevs 3.34 as well as 3.25. We write

$$\mathcal{L}_{HP} = -V(\Sigma) - V(H) - V(\Sigma, H), \tag{3.36}$$

where

$$V(\Sigma) =\nu^2\mathrm{Tr}(\Sigma)^2 + a[\mathrm{Tr}(\Sigma^2)]^2 + b\mathrm{Tr}(\Sigma^4) \tag{3.37}$$

$$V(H) =\mu^2H^\dagger H - \lambda(H^\dagger H)^2. \tag{3.38}$$

I have not written down the exact calculation because I want to stress another, more conceptual problem. After GUT symmetry breaking the Higgs field will decouple in a colour triplet and a weak doublet. The weak doublet is just the SM Higgs boson field. But in the electroweak regime there certainly is no place for this new Higgs triplet. Moreover, on examining the mass terms in the next section we can see that this triplet will mediate proton decay. To have a consistent theory its mass should be around the GUT scale. The potential 3.38 can of course only generate matrices on the weak scale. So we are forced to introduce a third part of the potential, one that contains terms connecting Σ and H.

$$V(\Sigma, H) = \alpha H^\dagger H\mathrm{Tr}\Sigma^2 + \beta H^\dagger\Sigma^2H. \tag{3.39}$$

This potential lifts the triplet mass as it should but it contributes to the doublet mass as well. One can calculate this new contribution to be $(\frac{15}{2}\alpha + \frac{9}{2}\beta^2)v_{24}^2$. We are thus back to the hierarchy problem, in a more concrete way this time: we need an extreme amount of fine tuning between α and β to prevent that the Higgs doublet gets GUT-like heavy.

3.4 Consequences of $SU(5)$ unification

A true unification in theoretical physics should not only show that entities that were thought to be completely different are in fact manifestations of the same phenomenon, it should also do new predictions. It is fascinating to regard quarks and leptons as parts of the same multiplet and to connect the third integral quark charges with the fact that there are three quark colours. But what new physics does this embedding predict?
First, unifying g_w and g' on the GUT scale must of course yield a prediction for the Weinberg angle, because this is just a number describing the proportionality between these coupling constants. From the field assignment 3.4 we deduce that the electromagnetic charge generator should be given by

$$Q = \mathrm{Diag}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1, 0) = -\frac{1}{2}(T^{11} + \sqrt{\frac{5}{3}}T^{12}). \tag{3.40}$$

On the other hand we know from the first chapter that the field that couples to Q is

$$A_\mu = \sin\theta_w A_\mu^3 + \cos\theta_w B_\mu. \tag{3.41}$$

Thus, in $SU(5)$ generator language we have

$$Q = \sin\theta_w T^{11} + \cos\theta_w T^{12}. \tag{3.42}$$

Now we understand that $\tan\theta_w$ equals the ratio of the T^{11} coefficient and T^{12} coefficient. Therefore we conclude, on equating 3.40 and 3.42

$$\tan\theta_w = \sqrt{\frac{3}{5}} \tag{3.43}$$

or

$$\sin^2\theta_w = \frac{3}{8}. \tag{3.44}$$

Does this result reveal new physics? I am tempted to say that it only provides a better formulation for the $SU(5)$ GUT hypothesis. We assume that an energy scale M_{GUT} exists where there is just one force with coupling constant g that is related to the SM coupling constants as:

$$g = g_s = g_w = \sqrt{\frac{5}{3}}g'. \tag{3.45}$$

Now for a real prediction. The generators $T^{13...24}$ have nonzero entries outside the "pure" $SU(3)$ and $SU(2)$ parts. So, in the Lagrangian terms $\overline{\psi}i\gamma^\mu D_\mu\psi$ and $\overline{\chi}i\gamma^\mu D_\mu\chi$ we will have that the X and Y bosons connect these parts, thus describing processes like $uu \to \overline{d}e^+$. That is, GUTs predict baryon and lepton number (see Appendix C) violating processes, proton decay for example. Omitting some more technical details on renormalization and complications in the calculation of the proton decay width [17] we simply state that the $SU(5)$ GUT predicts an upper limit on the proton lifetime of 10^{33} years. This prediction has been checked extensively. Instead of waiting 10^{33} years to see a proton decay, the Kamiokande experiment in Japan has been monitoring an ensemble of 10^{33} protons but in several years no decay was observed. This mismatch between theory and experiment has ruled out the $SU(5)$ GUT as candidate "GUT of the world". However, we go on, by considering $SO(10)$.

3.5 SO(10) unification: embedding of SM fields

The analysis of $SU(5)$ has provided much more connection between fields that are distinct in the SM. The third integral quark charges have emerged in a natural way and in the first chapter we already saw that the predicted value $\sin^2\theta_w = \frac{3}{8}$ yields not a perfect, but a very encouraging convergence of the three SM coupling constants. The fields are still in two different representations though. We also have not included a righthanded neutrino. The best next step would be considering a gauge group in which all 16 fermion fields fit in one representation. $SO(10)$ is the most elegant option. It contains a desired 16 dimensional rep and it has a $SU(5)$ subgroup.

3.5.1 Exploring the 16 rep

Most textbooks (for example, [17]) use a spinor representation of $SO(10)$. I prefer trying a more straightforward group-theoretical approach. Let us first construct the weight diagram of this **16** rep. $SO(10)$ has rank 5. Its simple roots are conventionally given by

$$\begin{array}{rccccc} \alpha_1 = & (2 & -1 & 0 & 0 & 0) \\ \alpha_2 = & (-1 & 2 & -1 & 0 & 0 \\ \alpha_3 = & (0 & -1 & 2 & -1 & -1) \\ \alpha_4 = & (0 & 0 & -1 & 2 & 0) \\ \alpha_5 = & (0 & 0 & -1 & 0 & 2), \end{array} \tag{3.46}$$

while the state of highest weight is $(0 \quad 0 \quad 0 \quad 0 \quad 1)$.
So let us construct:

$$
\begin{array}{ccc}
 & (0 \quad 0 \quad 0 \quad 0 \quad 1) & \\
 & \downarrow \alpha_5 & \\
 & (0 \quad 0 \quad 1 \quad 0 \quad -1) & \\
 & \downarrow \alpha_3 & \\
 & (0 \quad 1 \quad -1 \quad 1 \quad 0) & \\
\swarrow \alpha_2 & & \searrow \alpha_4 \\
(1 \quad -1 \quad 0 \quad 1 \quad 0) & & (0 \quad 1 \quad 0 \quad -1 \quad 0) \\
\downarrow \alpha_1 & \searrow \alpha_4 & \downarrow \alpha_2 \\
(-1 \quad 0 \quad 0 \quad 1 \quad 0) & & (1 \quad -1 \quad 1 \quad -1 \quad 0) \\
\downarrow \alpha_4 & \swarrow \alpha_1 & \downarrow \alpha_3 \\
(-1 \quad 0 \quad 1 \quad -1 \quad 0) & & (1 \quad 0 \quad -1 \quad 0 \quad 1) \\
\downarrow \alpha_3 & \swarrow \alpha_1 & \downarrow \alpha_5 \\
(-1 \quad 1 \quad -1 \quad 0 \quad 1) & & (1 \quad 0 \quad 0 \quad 0 \quad -1) \\
\downarrow \alpha_2 & \swarrow \alpha_5 & \downarrow \alpha_1 \\
(0 \quad -1 \quad 0 \quad 0 \quad 1) & & (-1 \quad 1 \quad 0 \quad 0 \quad -1) \\
\searrow \alpha_5 & & \swarrow \alpha_2 \\
 & (0 \quad -1 \quad 1 \quad 0 \quad -1) & \\
 & \downarrow \alpha_3 & \\
 & (0 \quad 0 \quad -1 \quad 1 \quad 0) & \\
 & \downarrow \alpha_4 & \\
 & (0 \quad 0 \quad 0 \quad -1 \quad 0) & .
\end{array}
$$

3.5.2 Branching to the SM

We now have various routes from $SO(10)$ to the SM. I want to briefly discuss two of these.
A first, very natural breaking scheme is

$$SO(10) \to^{M^1_{GUT}} SU(5) \times U(1) \to^{M^2_{GUT}} SU(3) \times SU(2) \times U(1) \to^{M_Z} SU(3) \times U(1). \tag{3.47}$$

In this scheme there are two GUT-symmetry breaking scales. Let us just try to find out how the states in the **16** branch to $SU(5)$ states. From there we could simply repeat our analysis (see 3.3) from the preceding sections. Following the approaches explained in the second chapter we infer the $SO(10) \to SU(5)$ projection matrix from branching rules[4]:

$$P(S0(10) \subset SU(5)) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}. \tag{3.48}$$

[4] For simplicity we omit $U(1)$ charges. These can easily be achieved by writing out suitable hypercharge axes.

So there we go:

$SO(10)$ weight	$SU(5)$ weight	$SU(5)$ multiplet
(00001)	(0100)	**10**
(0010-1)	(0001)	$\overline{\mathbf{5}}$
(01-110)	(1-110)	**10**
(1-1010)	(001-1)	$\overline{\mathbf{5}}$
(010-10)	(10-11)	**10**
(-10010)	(-1010)	**10**
(1-11-10)	(01-10)	$\overline{\mathbf{5}}$
(-101-10)	(-11-11)	**10**
(10-101)	(100-1)	**10**
(-11-101)	(0000)	**1**
(1000-1)	(1-100)	$\overline{\mathbf{5}}$
(0-1001)	(-110-1)	**10**
(-1100-1)	(0-101)	**10**
(0-110-1)	(-1000)	$\overline{\mathbf{5}}$
(00-110)	(0-11-1)	**10**
(000-10)	(0001)	**10**.

(3.49)

We thus see the full $SU(5)$ spectrum emerge from the **16** rep of $SO(10)$. We also recognise a new state: the (0000) weight represents the non-interacting righthanded neutrino.
Now we turn to the second breaking scheme:

$$\begin{aligned} SO(10) &\rightarrow^{M_{GUT}^1} SU(4)\times SU(2)\times SU(2)\\ &\rightarrow^{M_{GUT}^2} SU(3)\times SU(2)\times SU(2)\times U(1)\\ &\rightarrow^{M_{GUT}^3} SU(3)\times SU(2)\times U(1)\times U(1)\\ &\rightarrow^{M_{GUT}^4} SU(3)\times SU(2)\times U(1)\\ &\rightarrow^{M_Z} SU(3)\times U(1). \end{aligned} \tag{3.50}$$

This unification model has been considered by Pati and Salam [12] even before $SU(5)$ unification came into fashion. Working out its breaking scheme takes some time but nevertheless we want to show the details. The only new weight systems we need are those of the **4** and the $\overline{\mathbf{4}}$ of $SU(4)$:

$$\mathbf{4}:\begin{pmatrix}(100)\\(-110)\\(0-11)\\(00-1)\end{pmatrix},\quad \overline{\mathbf{4}}:\begin{pmatrix}(001)\\(01-1)\\(1-10)\\(-100)\end{pmatrix}. \tag{3.51}$$

From these reps and the easiest branching rules we find projection matrices:

$$P(S0(10)\subset SU(4)\times SU(2)\times SU(2)) = \begin{pmatrix} 0&0&1&0&1\\ 1&1&0&0&0\\ 0&0&1&1&0\\ 0&1&1&1&1\\ 0&1&1&0&0 \end{pmatrix} \tag{3.52}$$

and

$$P(SU(4) \subset SU(3)) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.53}$$

So we are ready to branch:

$SO(10)$	$SU(4) \times SU(2) \times SU(2)$	$SU(3) \times SU(2) \times SU(2)$	$SU(3) \times SU(2) \times U(1)$
(00001)	(10010)	(1010)	(101)$\frac{1}{6}$
(0010-1)	(00101)	(0101)	(010)$\frac{1}{3}$
(01-110)	(-11010)	(0010)	(001)$-\frac{1}{2}$
(1-1010)	(0010-1)	(010-1)	(010)$-\frac{2}{3}$
(010-10)	(01-101)	(1-101)	(1-10)$\frac{1}{3}$
(-10010)	(0-1110)	(-1110)	(-111)$\frac{1}{6}$
(1-11-10)	(100-10)	(10-10)	(10-1)$\frac{1}{6}$
(-101-10)	(1-1001)	(0001)	(000)1
(10-101)	(01-10-1)	(1-10-1)	(1-10)$-\frac{2}{3}$
(-11-101)	(00-110)	(0-110)	(0-11)$\frac{1}{6}$
(1000-1)	(-110-10)	(00-10)	(00-1)$-\frac{1}{2}$
(0-1001)	(1-100-1)	(000-1)	(000)0
(-1100-1)	(-10001)	(-1001)	(-100)$\frac{1}{3}$
(0-110-1)	(0-11-10)	(-11-10)	(-11-1)$\frac{1}{6}$
(00-110)	(-1000-1)	(-100-1)	(-100)$-\frac{2}{3}$
(000-10)	(00-1-10)	(0-1-10)	(0-1-1)$\frac{1}{6}$

(3.54)

In the first rows we have omitted the $U(1)$ charges. The first one emerges in the breaking from $SU(4)$ to $SU(3) \oplus U(1)$, the second one is the result from $SU(2) \to U(1)$, as can be checked from 3.50. In the final step, $U(1) \otimes U(1) \to U(1)$ the charges simply add. We have listed these resulting charges because we need them to discriminate between states with identical $SU(3) \times SU(2)$ weight vectors.

So what can we learn from this exercise? One thing is that it depends on the breaking scheme to which SM state a $SO(10)$ state branches to. In both schemes we find back the complete SM including a righthanded neutrino (in the first approach it comes from the state $(-11-101)$, in the second one from $(0-1001)$.

Another interesting feature is that in the second approach left-right symmetry only breaks at the third stage of symmetry breaking.

In the second column we can identify two $SU(4) \times SU(2) \times SU(2)$ octets: $(\mathbf{4}, \mathbf{2}, \mathbf{1})$ (that is to branch to all lefthanded states) and $(\overline{\mathbf{4}}, \mathbf{1}, \mathbf{2})$ (righthanded states). So, we have colour "quartets" now, the leptons are on the same level as the quarks. In the next step these respectively branch to $(\mathbf{3}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1})$ and $(\overline{\mathbf{3}}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2})$. Only in the third step lefthanded fields remain $SU(2)$ doublets while righthanded fields (or lefthanded antifields) become singlets under $SU(2)$.

Let us now have a look at gauge bosons. As $SO(10)$ has 45 generators, we have 45 gauge bosons now. They are in the adjoint rep of course. How do they branch to the SM scale? A direct calculation would be straightforward but rather lengthy so we will just give the result at the most illuminating breaking moment (M^2_{GUT}):

$$\begin{aligned} SO(10) &\to SU(3)\times SU(2)\times SU(2) \\ \mathbf{45} &\to (\mathbf{3},\mathbf{2},\mathbf{2})\oplus(\overline{\mathbf{3}},\mathbf{2},\mathbf{2})\oplus(\mathbf{8},\mathbf{1},\mathbf{1})\oplus(\overline{\mathbf{3}},\mathbf{1},\mathbf{1})\oplus(\mathbf{3},\mathbf{1},\mathbf{1})) \\ &\quad \oplus(\mathbf{1},\mathbf{3},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{3})\oplus(\mathbf{1},\mathbf{1},\mathbf{1}). \end{aligned} \tag{3.55}$$

The first two terms together contain eight colour triplets. They can be thought of as $X^\alpha, X'^\alpha, Y^\alpha$ and Y'^α gauge boson fields plus antifields. These are exactly two copies of the $SU(5)$ heavy gauge bosons. The next term will eventually branch to the gluon fields. Then we have two more new triplets called $\overline{X_s}$ and X_s boson fields. In the next two terms we recognise $SU(2)$ bosons: we have twice more than in the SM. They both can be rewritten in two charged and one chargeless boson, just as we did in the SM. The last term can be identified with the chargeless $U(1)$ boson of the SM.
In both schemes we need Higgs fields that give GUT-scale masses to all new, non SM like gauge fields but leave the SM gauge bosons massless. That is, we need fields like the Σ field in the $SU(5)$ theory. This is most easily done in terms of weight vectors. We can just write down all 45 weight vectors corresponding to the gauge fields. For every state of symmetry breaking we can then find its defining axis, that is, the axis whose inner product with a weight vector gives the mass of the field corresponding to that weight vector after symmetry breaking (M_X or 0).

3.5.3 Ordering of generators and fields

A clear disadvantage of $SO(10)$ is that it is very hard to write down exactly where all fields end up in its unified fermion multiplet and, consequently, how to organize the 45 generators. All textbooks and articles we have studied ([17], [18], [9]) simply omit this job, stating that it is hard and not very illuminating work. In the preceding subsection it was already shown that we can describe the breaking of $SO(10)$ symmetry to SM symmetry without matrix representations for the $SO(10)$ generators. As long as we are not after exact descriptions of proton decay processes this will suffice. But trying to write out explicitly the field contents of the fundamental **16** rep (from which we can at least find expressions for twelve of the generators) could be useful.
Inspired by the $SU(4)$-colour breaking scheme we propose to organise the **16** rep quartet by quartet. We let the "up" part of the quartet fields occupy the first four entries, the "down" part take the next four entries and in the second half we do the same with the quartet antifields. In SM language the 16 dimensional state vector then reads

$$\psi_{16} = \left(\left(u^r u^g u^b \nu\right)\left(d^r d^g d^b e\right)\left((u^r)^c (u^g)^c (u^b)^c (\nu)^c\right)\left((d^r)^c (d^g)^c (d^b)^c e^c\right)\right)^T, \tag{3.56}$$

or

$$\psi_{16} = \left(\left(u^r_L u^g_L u^b_L \nu_L\right)\left(d^r_L d^g_L d^b_L e_L\right)\left(\overline{u}^r_R \overline{u}^g_R \overline{u}^b_R \overline{\nu}_R\right)\left(\overline{d}^r_R \overline{d}^g_R \overline{d}^b_R \overline{e}_R\right)\right)^T \tag{3.57}$$

Now we can at least write down the first twelve, SM-like, generators in the same way as we did in section 3.1.

The first 8 generators should be generalizations of the $SU(3)$ Gell-Mann matrices. They have to act on the quark triplets. So for $a = 1 \dots 8$ we propose

$$T^a = \begin{pmatrix} \begin{matrix} & & & 0 \\ & \lambda^a & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & & & \\ & \begin{matrix} & & & 0 \\ & \lambda^a & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & & \\ & & \begin{matrix} & & & 0 \\ & -\lambda^{a\star} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \\ & & & \begin{matrix} & & & 0 \\ & -\lambda^{a\star} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \end{pmatrix} . \quad (3.58)$$

Next we come to $T^{9,10,11}$, the weak $SU(2)$ generators. They act on lefthanded doublets and are therefore only expected in the upper left corner of the generator matrices. All three are on the same four entries. The thing is, there are four doublets they act on. We thus need four copies of σ^1 in T^9, four copies of σ^2 in T^{10} and four σ^3 copies in T^{11}. We will always have that the first copy takes entries $(11), (15), (51)$ and (55), the second one $(22), (26), (62)$ and (66), the third one $(33), (37), (73)$ and (77) and the last one $(44), (48), (84)$ and (88).
Now we are on the right way to extract a value for $\sin^2\theta_w$. In the discussion in section 3.4 we stated that this depends on the ratio of T^{11} and T^{12}. Well, this time we have

$$T^{11} = \frac{1}{2}\text{Diag}\begin{pmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.59)$$

and

$$T^{12} = \sqrt{\frac{3}{5}}\text{Diag}\begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{2} & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix} . \quad (3.60)$$

(This last expression follows from simply inserting hypercharges and maintaining $SU(5)$ normalisation.)
By inserting charges we obtain the electromagnetic charge generator:

$$Q = \text{Diag}\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -1 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix} . \quad (3.61)$$

We now conclude

$$Q = T^{11} + \sqrt{\frac{5}{3}}T^{12} \quad (3.62)$$

which explicitly shows that the prediction obtained in $SU(5)$ $\sin^2\theta_w = \frac{3}{8}$ also holds in $SO(10)$.

Finally we mention that now, even if we have not specified the remaining 33 generators, we can be sure that the term

$$\mathcal{L}_{FK} = i\psi_{16}^T(D_\mu\gamma^\mu)\psi_{16} \tag{3.63}$$

will return all SM kinetic terms plus many new baryon and lepton number violating processes like proton decay. But let us not plunge any deeper into these details.

3.6 The $SO(10)$ Higgs mechanism

Again we have two stages of breaking (one of which, at the high scale, may consist of several sub-stages).
This high-scale breaking is done by a new field Σ that is in the adjoint rep **45**, the idea is just the same as in the SM. After getting a vev the term

$$\mathcal{L}_{HK} = (D_\mu\Sigma_p)^\star D^\mu\Sigma_p \tag{3.64}$$

yields mass terms for 33 gauge boson fields.
Now for the electroweak symmetry breaking. Let us first consider just one generation. All mass terms should follow from

$$\mathcal{L}_{FM} \propto \psi_{16}^T CH\psi_{16}. \tag{3.65}$$

We thus have to find suitable reps (and vevs) for the Higgs field H. From Young tableaux (or very non straightforward weight vector analysis) we find

$$\mathbf{16}\otimes\mathbf{16} = \mathbf{10}\oplus\mathbf{120}\oplus\mathbf{126}. \tag{3.66}$$

Thus, to build gauge invariant mass terms we have to use Higgs fields that are in the $\overline{\mathbf{10}}$, $\overline{\mathbf{120}}$ or $\overline{\mathbf{126}}$. Assigning vevs to these large irreps and tracing back mass terms seems quite complicated. Fortunately, there is a way out. Whether $SO(10)$ branches to the SM via $SU(5)$ or via any other route, the fact remains that $SU(5)$ is a subgroup of $SO(10)$ and we know how to write down mass terms in $SU(5)$. So let us break these Higgs fields to $SU(5)$ as well:

$$\begin{aligned}
\overline{\mathbf{10}} &\rightarrow \mathbf{5}+\overline{\mathbf{5}} \\
\overline{\mathbf{120}} &\rightarrow \mathbf{5}+\overline{\mathbf{5}}+\mathbf{10}+\overline{\mathbf{10}}+\mathbf{45}+\overline{\mathbf{45}} \\
\overline{\mathbf{126}} &\rightarrow \mathbf{1}+\mathbf{5}+\overline{\mathbf{10}}+\mathbf{15}+\overline{\mathbf{45}}+\mathbf{50}.
\end{aligned} \tag{3.67}$$

As before, we only keep reps that can break to a $SU(3)$ singlet and a $SU(2)$ non-singlet, that is, the five and forty-five dimensional ones. (Just because the fields that should be coupled in mass terms are in the same $SU(3)$ rep but in a different $SU(2)$ rep.) Majorana mass terms for neutrinos form an exception to this rule.
In $SU(5)$ matter fields are in the reps **1** (RH neutrino), $\overline{\mathbf{5}}$ and **10**. So we investigate what mass terms we can build:

$$\mathbf{1}\otimes\mathbf{1} = \mathbf{1} \rightarrow \psi_1^T CH_1^{\overline{126}}\psi_1 \tag{3.68}$$

$$\overline{\mathbf{5}}\otimes\overline{\mathbf{5}} = \overline{\mathbf{10}}+\overline{\mathbf{15}} \rightarrow \psi_{\overline{5}}^T CH_{15}^{\overline{126}}\psi_{\overline{5}} \tag{3.69}$$

$$\mathbf{10}\otimes\mathbf{10} = \overline{\mathbf{5}}+\mathbf{45}+\mathbf{50} \rightarrow \psi_{10}^T C\big(H_5^{\overline{10}} + H_5^{\overline{120}} + H_5^{\overline{126}} + H_{\overline{45}}^{\overline{120}} + H_{\overline{45}}^{\overline{120}}\big)\psi_{10} \tag{3.70}$$

$$\mathbf{1}\otimes\overline{\mathbf{5}} = \overline{\mathbf{5}} \rightarrow \psi_1^T C\big(H_5^{\overline{10}} + H_5^{\overline{120}} + H_5^{\overline{126}}\big)\psi_{\overline{5}} \tag{3.71}$$

$$\overline{\mathbf{5}}\otimes\mathbf{10} = \mathbf{5}+\overline{\mathbf{45}} \rightarrow \psi_{\overline{5}}^T C\big(H_{\overline{5}}^{\overline{10}} + H_{\overline{5}}^{\overline{120}} + H_{\overline{45}}^{\overline{120}}\big)\psi_{10}. \tag{3.72}$$

Here we just state that vevs could be chosen such that gauge singlets are projected out. We do not give precise prescriptions.
The first term 3.68 can be used for a Majorana mass term for the righthanded neutrino field. 3.69 is not needed. The next term 3.70 provides the same Dirac masses for the up quark as we had in the original $SU(5)$ case, but with much more freedom as they arise now from linear combinations of five Higgs terms instead of two. A Dirac mass for neutrinos can arise from 3.71. Finally, 3.72 provides $SU(5)$-like mass terms for down quarks and electrons, with three Higgs fields involved instead of the usual two.
From these mass terms we see that we can also define some "minimal" variant of $SO(10)$: with only a $\overline{\mathbf{10}}$ Higgs field we can obtain all minimal $SU(5)$ relations plus a Dirac neutrino mass term. The $\overline{\mathbf{10}}$ can be chosen such that all 16-plet members acquire the same vev (unlike the **45** case in $SU(5)$ for example that made down quarks and leptons differ by a factor of three). If we then add a singlet Higgs field we have a Majorana mass term for righthanded neutrinos as well, which is enough for the seesaw mechanism.
Now we include all three families. In the minimal scenario we have just one Yukawa coupling, rendering Dirac masses as well as a righthanded neutrino Majorana mass:

$$\mathcal{L}_{FM} = y^{\alpha\beta}(\psi_{16}^{\alpha})^T C H_{10+1}\psi_{16}^{\beta}. + h.c. \tag{3.73}$$

Within a generation, differences in particle masses can only rise from asymmetric Higgs vevs. A direct consequence is that the CKM matrix equals the unit matrix now: flavour changing processes are predicted to disappear at the GUT-scale.
In $SU(5)$ we found that the H_5 coupling should be family symmetric and the H_{45} coupling antisymmetric. This followed from the tensor structure of the mass terms. Now we have that the H_{10} and H_{126} couplings are symmetric and the H_{120} coupling is antisymmetric. (We could find this result by explicitly writing out all mass terms 3.68 - 3.72 but actually it follows already from 3.66: a 16×16 matrix can be split in a symmetric part with 136 degrees of freedom and a antisymmetric part with 120 free parameters.) The complete Lagrangian should be symmetric under exchange of ψ^T and ψ, so the terms with antisymmetric Higgs term should be multiplied by an antisymmetric Yukawa matrix and vice versa.

Finally we mention that the gauge kinetic term looks exactly as in $SU(5)$ and in the SM, but now the index a runs up to 45. When constructing Higgs potential terms we meet the same hierarchy problem as in $SU(5)$: the GUT-like masses for the heavy Higgs bosons Σ should be kept away from the field H "by hand" as to provide all fermions and light bosons from taking GUT-like masses.

3.7 SO(10): a short summary

So what picture do we have of the minimal $SO(10)$ GUT?
At the GUT scale we find three perfect 16-plets, one for each generation. All plet-members are on exactly the same footing. There is a tendimensional Higgs plet. These fields interact in kinetic terms 3.63 and mass terms 3.73. To mediate the unified $SU(5)$ force between the 16-plet members there are 45 gauge boson fields

V^a_μ. These matter kinetic terms are contained in 3.63. Gauge kinetic terms are simply $V^a_\mu V^{a\mu}$

Below M_{GUT}, perfect symmetry breaks. Whatever the intermediate steps are, we are left with an effective $SU(3)\times SU(2)\times U(1)$ symmetry. Due to this spontaneous symmetry breakdown, 33 of the 45 gauge bosons acquire GUT-like masses. This process is most easily described in terms of weight vectors and symmetry breaking axes. Fields are still in a 16-plet, but only the 12 SM gauge bosons are still interacting. This breaks the symmetry between the 16-plet members, effectively we are left with the same division in doublets and triplets as in the SM.

Now we start running down from M_{GUT}. Yukawa unification breaks, the matrix entries evaluate in different ways. As we pass its (Majorana) rest mass, the righthanded neutrino is integrated out from the theory (the Higgs singlet takes its vev.).

Around 250 GeV, we reach the second stage of spontaneous symmetry breakdown. The Higgsfield **10** (or, equivalently, **5** and $\overline{\mathbf{5}}$) takes its vev. Only now we can really speak of “masses” for the various fermion fields.

Chapter 4

Supersymmetry and the Minimal Supersymmetric Standard Model

So far our investigation of the possibility of Grand Unification has been encouraging but far from complete. Although Grand Unification could provide a powerful explanation of all SM physics while reducing the number of free parameters, the main proof, exactly meeting coupling constants, has not been achieved yet. Moreover, the hierarchy problem seems even more urgent than in the SM. We are clearly missing something.

4.1 Introduction to supersymmetry

Supersymmetry, or SUSY, promises to solve both of these problems by yet a new unification, far more revolutionary than all others we have seen so far. It suggests a certain equality between bosons and fermions or, following chapter 1, an equality of force and matter. It certainly meets all criteria on "elegance in unification" we might hope for. However, this thesis is about physics, not arts, so caution is needed. The "Minimal Supersymmetric Standard Model" (MSSM) predicts twice as many particles and many more degrees of freedom than we have in the Standard Model. In this way it thus seems to contradict the idea of Grand Unification. A more realistic objection is that up to today none of these predicted new particles have been detected.

Our hopes not affected by these two problems, we will explore SUSY and the MSSM in this chapter. In the next chapter we will find some clues to solve the first (aesthetic) problem, the plethora of free parameters. The second (very serious) problem could only be solved at the CERN laboratories in Geneva in the next year.

4.1.1 A new symmetry

In 1967 Coleman and Mandula stated that the four translation operators P_μ and the operators $M_{\mu\nu}$, that contains boosts and rotations, together generating the Poincare group, cannot be combined with internal symmetries other than in a trivial way. The Poincare Lie algebra reads [8, chap. 3]

$$\begin{aligned}[P_\mu, P_\nu] =&0 \\ [M_{\mu\nu}, P_\rho] =&i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu) \\ [M_{\mu\nu}, M_{\rho\sigma}] =&-i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}),\end{aligned} \tag{4.1}$$

and that is about it: for a generator T_a of an internal symmetry we will always have

$$[T_a, P_\mu] = [T_a, M_{\mu\nu}] = 0. \tag{4.2}$$

Eight years later Haag, Łopuszanski and Sohnius found an ingenious loophole in this no-go theorem: what if we also consider anticommuting (odd) symmetry generators instead of only commuting (even) ones? And well, at first it may seem a very odd assumption, but given the fact that we already study commuting and anticommuting *fields*, why would we not consider antisymmetric coordinates and generators as well? The concept of a Lie algebra is then generalized to that of a Lie *super*algebra, or *graded* Lie algebra. To 4.1 we add

$$\begin{aligned}[Q_a, P_\mu] =&0 \\ [Q_a, M_{\mu\nu}] =&(\Sigma_{\mu\nu})_{ab}Q_b \\ \{Q_a, Q_b\} =&-2(\gamma^\mu C)_{ab}P_\mu.\end{aligned} \tag{4.3}$$

($\Sigma_{\mu\nu}$ is defined in appendix A.)
Here Q_a is the fourdimensional spinorial generator of a SUSY transformation. It does exactly what seemed forbidden by Coleman and Mandula: it mixes the particle content of the Poincare group, that is, fermions and bosons. From the last anticommutator we see that Q can be thought of as a "square root of a translation", just as i that is the square root of -1.
To connect a fermionic, anticommuting spinor field $\xi(x)$ and a bosonic, commuting field $\phi(x)$ (scalar) or $V(x)$ (vector) we need to extend the bosonic coordinate x with a fermionic coordinate θ. Only then we can build symmetric quantities out of all these antisymmetric variables, like $\xi(x)\theta$, that are on the same footing as $\phi(x)$ for example. Note that we are in no way postulating new spacetime dimensions: the four component spinorial coordinate θ_a only serves as a "parametrization of the anti-commutative part of space" and will be integrated out from every physical prediction, just in the same way in which complex numbers are used in physics.
As four dimensional spinor reps of the Lorentz group come in two twodimensional types, lefthanded and righthanded, or $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ (see section 1.1.1), we also need two kinds of twodimensional anticommuting coordinates, θ_A and $\overline{\theta}^{\dot{A}}$, and two kinds of twodimensional anticommuting variables, $\xi_A(x)$ and $\overline{\xi}^{\dot{A}}(x)$. As usual, lefthanded and righthanded spinors can be combined in a Dirac four-spinor.
We can formally write down the extension of spacetime:

$$x^\mu \to (x^\mu, \theta_A, \overline{\theta}^{\dot{A}}). \tag{4.4}$$

We could also have more anticommuting coordinates but in this thesis $N = 1$ supersymmetry will suffice.

Now we define a SUSY transformation. We postulate that its effect should be

$$
\begin{aligned}
x^\mu \to & x^\mu - i\theta\sigma^\mu\overline{\epsilon} + i\epsilon\sigma^\mu\overline{\theta} \\
\theta \to & \theta + \epsilon \\
\overline{\theta} \to & \overline{\theta} + \overline{\epsilon}
\end{aligned}
\tag{4.5}
$$

where ϵ and $\overline{\epsilon}$ are infinitesimal spinorial parameters. We have suppressed spinor indices.
Now we are in the position to find expressions for the SUSY transformation generators Q_A and $\overline{Q}^{\dot{A}}$. (We prefer working in the two component spinor formalism.) A SUSY transformation on a function f of supercoordinates $(x, \theta, \overline{\theta})$ should bring it to $f + \delta f$ with

$$
\delta f = \delta x^\mu \partial_\mu f + \delta\theta^A \partial_A f + \delta\overline{\theta}_{\dot{A}} \overline{\partial}^{\dot{A}} f \equiv i(\epsilon Q + \overline{\epsilon}\overline{Q})f. \tag{4.6}
$$

So on equating 4.5 and 4.6 we find what we were looking for

$$
\begin{aligned}
Q_A =& - i(\partial_A + i\sigma^\mu_{A\dot{B}}\overline{\theta}^{\dot{B}}\partial_\mu) \\
Q^{\dot{A}} =& - i(\overline{\partial}^{\dot{A}} + i\theta^B\sigma^\mu_{B\dot{B}}\epsilon^{\dot{B}\dot{A}}\partial_\mu).
\end{aligned}
\tag{4.7}
$$

To conclude this section we write down expressions for supersymmetric covariant derivatives. Just as gauge covariant derivatives are constructed in such a way that maintains covariance under a gauge transformation, SUSY covariant derivatives are covariant under a SUSY transformation. Working out this requirement yields

$$
\begin{aligned}
\mathcal{D}_A =& \partial_A - i\sigma^\mu_{A\dot{B}}\overline{\theta}^{\dot{B}}\partial_\mu \\
\overline{\mathcal{D}}^{\dot{A}} =& \overline{\partial}^{\dot{A}} - i\overline{\sigma}^{\mu\dot{A}B}\theta_B\partial_\mu.
\end{aligned}
\tag{4.8}
$$

4.1.2 Superfields

In the SUSY formalism ordinary fields that are functions of x are generalized to superfields that are functions of x, θ and $\overline{\theta}$. However, the fact that θ and $\overline{\theta}$ are anticommuting variables strongly constrains the form of a superfield. For example, if we would consider just one onedimensional spinorial coordinate θ we would have $\theta^2 = 0$ and we could decompose a general superfield as follows

$$
f(x, \theta) = f_1(x) + f_2(x)\theta. \tag{4.9}
$$

In our case we have two twodimensional anticommuting coordinates so a general superfield will be a bit more complicated, but it still has a finite number (9) of terms.
To connect to SM physics we will demand one more restriction. The generalization of a lefthanded SM fermion field will be a superfield Φ that satisfies $\overline{\mathcal{D}}_{\dot{A}} = 0$. Working in the lefthanded representation, the LH antifields that denote the abandoned RH fields are generalized to superfields $\Phi^\dagger$ that are subject to $\mathcal{D}_A\Phi^\dagger = 0$. Φ and $\Phi^\dagger$ will be referred to as left and right chiral superfields respectively.

Chiral superfields can now be parametrized in terms of just three x-dependent fields ϕ, ξ and F that have spin 0, $\frac{1}{2}$ and 1 respectively:

$$\begin{aligned}\Phi(x,\theta,\overline{\theta}) =& \phi(x) - i\theta\sigma^\mu\overline{\theta}\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\overline{\theta\theta}\partial^\mu\partial_\mu\phi(x) + \sqrt{2}\theta\xi(x) \\ &+ \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\xi(x)\sigma^\mu\overline{\theta} + \theta\theta F(x) \end{aligned} \tag{4.10}$$

$$\begin{aligned}\Phi^\dagger(x,\theta,\overline{\theta}) =& \phi^\star(x) + i\theta\sigma^\mu\overline{\theta}\partial_\mu\phi^\star(x) - \frac{1}{4}\theta\theta\overline{\theta\theta}\partial^\mu\partial_\mu\phi^\star(x) + \sqrt{2}\overline{\theta\xi}(x) \\ &- \frac{i}{\sqrt{2}}\overline{\theta\theta}\theta\sigma^\mu\partial_\mu\overline{\xi}(x) + \overline{\theta\theta}F^\star(x). \end{aligned} \tag{4.11}$$

Here we can understand the bottom line of the SUSY formalism. In the SM a fermion field is described by a spinor field $\xi(x)$ and a scalar field by a field $\phi(x)$. In SUSY these fields are just components of a more fundamental quantity, a chiral superfield. So, to every SM scalar and spinor field we associate a chiral superfield now. One of its physical components is the original SM field, the other component is totally new. It is the "superpartner" of the original SM field. SUSY thus postulates that there are twice as many elementary fields than in the SM: every SM fermion field has a scalar partner, (a "sfermion" field) and every SM scalar field has a fermionic partner (a "bosino" field). The field $F(x)$ is auxiliary, we will integrate it out later.
There is one more type of fields in the SM: vector fields. To describe them we need a new kind of superfields that are, not that surprisingly, called vector superfields $V(x,\theta,\overline{\theta})$. Their main property is that they should satisfy

$$V = V^\dagger. \tag{4.12}$$

This already restricts the number of free parameters in the expansion of a vector superfield, but we can also exploit its "supergauge" freedom. That is, if V is a vector superfield, then $V + i\Lambda - i\Lambda^\dagger$, with Λ an arbitrary left chiral superfield, is too. In the so-called Wess-Zumino gauge that we will use throughout this thesis a vector superfield is maximally restricted to

$$V_{WZ}(x,\theta,\overline{\theta}) = \theta\sigma^\mu\overline{\theta}A_\mu(x) + \theta\theta\overline{\theta\lambda}(x) + \overline{\theta\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\overline{\theta\theta}D(x). \tag{4.13}$$

The component $A_\mu(x)$ is a vector field. In this expansion of a vector superfield it is accompanied by a superpartner $\lambda(x)$, a "gaugino" field of spin $\frac{1}{2}$. The field $D(x)$ is auxiliary.

4.2 Construction of the MSSM

In this section we build the SUSY Lagrangian density that leads to the MSSM.

4.2.1 The rules of the game

A SUSY Lagrangian density built out of superfields should be Lorentz invariant, $SU(3)\times SU(2)\times U(1)$ supergauge invariant, renormalizable and also supersymmetric invariant. We will focus on supergauge and supersymmetric invariance, as the concept of Lorentz invariance in SUSY is unchanged from the SM and can easily be checked once we have written down the SUSY action and renormalizability is still guaranteed as long as the mass dimensions of all terms in the SUSY Lagrangian density do not exceed four[1].

To reproduce the non-Abelian gauge symmetric structure we have in the SM (with coupling strength g and generators T^a) we introduce a two-component chiral superfield Φ_i with gauge transformation

$$\begin{aligned} \Phi_i &\to \left[e^{-2igT^a\Lambda^a}\right]_{ij}\Phi_j \\ \Phi_i^\dagger &\to \Phi_j^\dagger\left[e^{2igT^a\Lambda^{a\dagger}}\right]_{ji} \end{aligned} \tag{4.14}$$

Here Λ^a is a left chiral superfield specifying the gauge transformation.

Next we turn to vector superfields. Their aforementioned gauge freedom yields a simple gauge transformation:

$$e^V \to e^{-i\Lambda^\dagger}e^V e^{i\Lambda}, \tag{4.15}$$

which in the non-Abelian case generalizes to[2]

$$e^{2gV^aT^a} \to e^{-2igT^a\Lambda^{a\dagger}}e^{2gV^aT^a}e^{2igT^a\Lambda^a}. \tag{4.16}$$

Exponentiating vector superfields is not as cumbersome as it may seem: in WZ gauge we have that $V^n = 0$ for every $n \geq 2$ so we easily find

$$e^V = 1+\theta\sigma^\mu\overline{\theta}A_\mu(x)+\theta\theta\overline{\theta}\overline{\lambda}(x)+\overline{\theta}\overline{\theta}\theta\lambda(x)+\frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}\left(D(x)+\frac{1}{2}A^\mu(x)A_\mu(x)\right). \tag{4.17}$$

Now that we understand the behaviour under gauge transformations of our superfields, we investigate how to build supersymmetric invariant actions.

Let us first consider a chiral superfield (4.10). Performing a supersymmetric transformation (4.7) yields new expressions for components ϕ, ξ and F. For example, if θ transforms to $\theta + \epsilon$ the term $\sqrt{2}\theta\xi$ will yield a contribution $\sqrt{2}\epsilon\xi$ to the θ-independent component ϕ.

So, we can write out transformation rules for all three components of the chiral superfield. We then find that the new contribution to the F-term can be written as a total spacetime derivative. If we can discard surface terms, which we assume to be the case, we conclude that the F-term of a chiral superfield is a supersymmetric invariant.

If we move from chiral superfields Φ to arbitrary superfields $\mathcal{F}$ we find, reasoning along the same lines, that the only the term multiplying $\theta\theta\overline{\theta\theta}$ is a SUSY invariant. This term is usually called $D(x)$. So we conclude that the terms $[\Phi]_F$ and $[\mathcal{F}]_D$ are suitable for SUSY invariant actions.

How do we find these D- and F terms? A practical answer would be: they are just the expressions that multiply $\theta\theta\overline{\theta\theta}$ and $\theta\theta$ respectively. A more formal approach is

[1] That is, once that all factors of θ and $\overline{\theta}$ have been integrated out

[2] This yields quite a complicated transformation rule for the superfield V^a itself but we will not worry about that here.

to project them out:

$$F(x) = \frac{1}{4}\mathcal{D}\mathcal{D}\Phi(x,\theta,\overline{\theta})|_{\theta=\overline{\theta}=0} \tag{4.18}$$

$$D(x) = \frac{1}{16}\mathcal{D}\mathcal{D}\overline{\mathcal{D}\mathcal{D}}\mathcal{F}(x,\theta,\overline{\theta})|_{\theta=\overline{\theta}=0}. \tag{4.19}$$

We can also write these last expressions as

$$F(x) = \frac{1}{4}\int d^2\theta\Phi(x,\theta,\overline{\theta}) = \frac{1}{16}\int d^4\theta\Phi(x,\theta,\overline{\theta})\delta(2)(\overline{\theta}) \tag{4.20}$$

$$D(x) = \frac{1}{16}\int d^4\theta\mathcal{F}(x,\theta,\overline{\theta}). \tag{4.21}$$

Now that we understand what a SUSY Lagrangian density should look like we are ready to build the MSSM.

4.2.2 Matter kinetic terms

We look for a supersymmetric Lagrangian that, after integrating out all factors of θ and $\overline{\theta}$ reproduces, among new other terms, the kinetic part of the SM Lagrangian 1.31.

We introduce five left chiral superfields and specify their $SU(3) \times SU(2) \times U(1)$ gauge transformations:

$$\begin{array}{ccccccc} \Phi_{Lq} & \rightarrow & e^{-2ig_sT^a\Lambda_s^a} & e^{-2ig_w\tau^a\Lambda_z^a} & e^{-2ig'Y_{Lq}\Lambda_y} & \Phi_{Lq} \\ \Phi_{Ru} & \rightarrow & e^{-2ig_s\overline{T}^a\Lambda_s^a} & & e^{-2ig'Y_{Ru}\Lambda_y} & \Phi_{Ru} \\ \Phi_{Rd} & \rightarrow & e^{-2ig_s\overline{T}^a\Lambda_s^a} & & e^{-2ig'Y_{Rd}\Lambda_y} & \Phi_{Rd} \\ \Phi_{Ll} & \rightarrow & & e^{-2ig_w\tau^a\Lambda_s^a} & e^{-2ig'Y_{Ll}\Lambda_y} & \Phi_{Ll} \\ \Phi_{Re} & \rightarrow & & & e^{-2ig'Y_{Re}\Lambda_y} & \Phi_{Re} \end{array} \tag{4.22}$$

Here T^a ($\overline{T}^a$) are the generators of the fundamental (antifundamental) rep of $SU(3)$, τ^a are generators of $SU(2)$ (fundamental and antifundamental rep) and every chiral superfield i carries its own hypercharge Y_i which generates its $U(1)$ transformation. (We are working in the lefthanded representation again.)

For notational clarity we have suppressed multiplet indices. The quark superfields are color triplets while the lefthanded superfields are weak doublets. So in the first equation Φ_{Lq} on the LHS carries a color index i and a weak index k, T^a has indices ij, τ^a has indices kl and Φ_{Lq} on the RHS has indices jl. The hypercharge Y_{Lq} is multiplied by $\delta_{ij}\delta_{kl}$.

Note furthermore that we are labeling chiral superfields by their fermionic contents. We do so just to maintain a clear connection with the SM. But every superfield Φ_i has a bosonic component ϕ_i as well as a fermionic component ξ_i.

Now we define our 8 + 3 + 1 vector superfields V_s^a, V_w^b and V_y. V_s^a, for example, has components $(A_s^a)_\mu$ (gluon vectorboson field), λ_s^a and $\overline{\lambda_s^a}$ (gluino, or gaugino in general, fermion field) and D_s^a (auxiliary field).

We combine them in five new vector superfields that connect with the five chiral superfields we have defined before:

$$V_{Lq} = 2g_s V_s^a T^a + 2g_w V_w^b \tau^b + 2g' V_y Y_{Lq} \tag{4.23}$$

$$V_{Ru} = 2g_s V_s^a \overline{T}^a + 2g' V_y Y_{Ru} \tag{4.24}$$

$$V_{Rd} = 2g_s V_s^a \overline{T}^a + 2g' V_y Y_{Rd} \tag{4.25}$$

$$V_{Ll} = 2g_w V_w^b \tau^b + 2g' V_y Y_{Ll} \tag{4.26}$$

$$V_{Re} = 2g' V_y Y_{Re}. \tag{4.27}$$

All terms in 4.23 - 4.27 transform as in 4.16.

From now on we will contract our notation even more: equations 4.22 and 4.23 - 4.27 are summarized in

$$\Phi_i \to e^{-2ig_s (T_i^3)^a \Lambda_s^a} e^{-2ig_w (T_i^2)^a \Lambda_z^a} e^{-2ig' Y_i \Lambda_y} \Phi_i \tag{4.28}$$

and

$$V_i = 2g_s V_s^a (T_i^3)^a + 2g_z V_w^b (T_i^2)^b + 2g' V_y Y_i. \tag{4.29}$$

The label i takes values Lq, Ru, Rd, Ll and Re and we have

i	Lq	Ru	Rd	Ll	Re
$(T_i^3)^a$	T^a	$\overline{T}^a$	$\overline{T}^a$	0	0
$(T_i^2)^a$	τ^a	0	0	τ^a	0
Y_i	Y_{Lq}	Y_{Ru}	Y_{Rd}	Y_{Ll}	Y_{Re}

. (4.30)

We can now write out the MSSM kinetic term:

$$\mathcal{L}_K = \sum_i \left[\Phi_i^\dagger e^{V_i} \Phi_i \right]_D \tag{4.31}$$

Working out this term is quite non-trivial. When the smoke finally clears we find

$$\mathcal{L}_{\mathcal{K}} = 2 \sum_i \Delta_i^{\mu\dagger} \phi_i^\star(x) \Delta_{\mu i} \phi_i(x) \tag{4.32}$$

$$+ i\xi_i(x) \sigma^\mu \left[\partial_\mu\right] \overline{\xi}_i(x) \tag{4.33}$$

$$- \sqrt{2}\phi_i^\star(x) \Big(g_s (T_i^3)^a \lambda_s^a(x) + g_w (T_i^2)^b \lambda_w^b(x) + g' Y_i \lambda_y(x) \Big) \xi(x) \tag{4.34}$$

$$- \sqrt{2}\overline{\xi}_i(x) \Big(g_s (T_i^3)^a \overline{\lambda}_s^a(x) + g_w (T_i^2)^b \overline{\lambda}_w^b(x) + g' Y_i \overline{\lambda}_y(x) \Big) \phi_i(x) \tag{4.35}$$

$$- \xi_i(x) \sigma^\mu \overline{\xi}_i(x) \Big(g_s (T_i^3)^a (A_s^a)_\mu(x) + g_w (T_i^2)^b (A_w^b)_\mu(x) + g' Y_i (A_y)_\mu(x) \Big) \tag{4.36}$$

$$+ \phi_i^\star(x) \Big(g_s (T_i^3)^a D_s^a(x) + g_w (T_i^2)^b D_w^b(x) + g' Y_i D_y(x) \Big) \phi_i(x) \tag{4.37}$$

$$+ F_i^\star(x) F_i(x). \tag{4.38}$$

Here we have used $\Delta_{\mu i} = \partial_\mu + ig_s (T_i^3)^a (A_s^a)_\mu(x) + ig_w (T_i^2)^b (A_w^b)_\mu(x) + ig' Y_i (A_y)_\mu(x)$.
In 4.33 $A[\partial_\mu]B$ denotes $\frac{1}{2}(A\partial_\mu B - B\partial_\mu A)$.
On combining 4.33 and 4.36 we recover the SM kinetic term 1.31.

4.2.3 Higgs kinetic terms

One MSSM novelty is that we introduce two Higgs chiral superfields Φ_{H_1} and Φ_{H_2} instead of one. (Later in this chapter we will see that with just one Higgs field we cannot obtain all SM mass terms.) Both are color singlets and $SU(2)$ doublets. They carry hypercharge $Y_{H_1} = \frac{1}{2}$ and $Y_{H_2} = -\frac{1}{2}$.
SUSY connects fermion fields and boson fields, so it is very easy to generalize all discussions from the previous subsection to these Higgs chiral superfields. We just have to add two more labels, $i = H_1$ and $i = H_2$ to 4.30.

i	H_1	H_2
$(T_i^3)^a$	0	0
$(T_i^2)^a$	τ^a	τ^a
Y_i	Y_{H_1}	Y_{H_2}

. (4.39)

Working out 4.31 for $i = H_1$ and $i = H_2$ yields the same result of course. In 4.32 we now recognise the SM Higgs kinetic term 1.20. So, for the first time we can make out some of the unifying beauty in this supersymmetric approach: we have combined the SM Higgs and fermion kinetic terms.

4.2.4 Gauge kinetic terms

In this section we try to supersymmetrically reproduce the gauge kinetic terms 1.46. To this end we construct left and right spinorial field strengths out of our vector superfields.
In the Abelian part of the MSSM we construct

$$W_{yA} = -\frac{1}{4}\overline{\mathcal{D}\mathcal{D}}\mathcal{D}_A V_y \tag{4.40}$$

$$\overline{W}_y^{\dot{A}} = \frac{1}{4}\mathcal{D}\mathcal{D}\overline{\mathcal{D}}^{\dot{A}} V_y. \tag{4.41}$$

Their left (right) chirality is manifest: acting with a third $\overline{\mathcal{D}}(\mathcal{D})$ on them automatically yields 0. Moreover, both terms are invariant under a supergauge transformation $V \to V + i\Lambda - i\Lambda^\dagger$.
In non-Abelian super gauge theory things are a bit more complicated. Generalizing 4.40 and 4.41 gives

$$W_A = -\frac{1}{4}\overline{\mathcal{D}\mathcal{D}}e^{-V}\mathcal{D}_A e^V \tag{4.42}$$

$$\overline{W}^{\dot{A}} = \frac{1}{4}\mathcal{D}\mathcal{D}e^V\overline{\mathcal{D}}^{\dot{A}} e^{-V}. \tag{4.43}$$

Using the chiral properties of Λ and $\Lambda^\dagger$ we find that under a supergauge transformation (4.15) we have

$$W_A \to e^{-i\Lambda} W_A e^{i\Lambda} \tag{4.44}$$

$$\overline{W}^{\dot{A}} \to e^{i\Lambda^\dagger}\overline{W}^{\dot{A}} e^{-}i\Lambda^\dagger \tag{4.45}$$

so we conclude that $\mathrm{Tr}\left[W^A W_A + \overline{W}_{\dot{A}}\overline{W}^{\dot{A}}\right]$ is supergauge invariant.
In this notation the connection with the non-Abelian MSSM vector superfields V_s^a

and V_W^b is not very clear. We had better decompose the general vector superfield in $2g_s V_s^a T^a$ and $2g_w V_w^b \tau^b$. That means that we are in fact decomposing W_A in $2g_s W_s^a T^a$ and $2g_w W_w^b \tau^b$. If we then expand $e^{-V}\mathcal{D}e^V$ we arrive at the $SU(3)$ chiral field strengths

$$W_{sA}^a = -\frac{1}{4}\overline{\mathcal{D}\mathcal{D}}\left[\mathcal{D}_A V_s^a + i g_s f_s^{abc}\left(D_A V_s^b\right) V_s^c\right] \quad (4.46)$$

$$\overline{W}_s^{a\dot{A}} = -\frac{1}{4}\mathcal{D}\mathcal{D}\left[-\overline{\mathcal{D}}^{\dot{A}} V_s^a + i g_s f_s^{abc}\left(\overline{\mathcal{D}}^{\dot{A}} V_s^b\right) V_s^c\right] \quad (4.47)$$

and the $SU(2)$ chiral field strengths

$$W_{wA}^a = -\frac{1}{4}\overline{\mathcal{D}\mathcal{D}}\left[\mathcal{D}_A V_w^a + i g_w f_w^{abc}\left(D_A V_w^b\right) V_w^c\right] \quad (4.48)$$

$$\overline{W}_w^{a\dot{A}} = -\frac{1}{4}\mathcal{D}\mathcal{D}\left[-\overline{\mathcal{D}}^{\dot{A}} V_w^a + i g_w f_w^{abc}\left(\overline{\mathcal{D}}^{\dot{A}} V_w^b\right) V_w^c\right]. \quad (4.49)$$

Here f_s^{abc} and f_w^{abc} denote $SU(3)$ and $SU(2)$ structure constants respectively. We see that every single W^a depends on all 8 (3) V_s^a (V_w^a).
Finally we are in a position to write down a supersymmetric super gauge invariant gauge kinetic term:

$$\frac{1}{4}\left[W_s^{aA} W_{sA}^a + \overline{W}_{s\dot{A}}^a \overline{W}_s^{a\dot{A}} + W_w^{aA} W_{wA}^a + \overline{W}_{w\dot{A}}^a \overline{W}_w^{a\dot{A}} + W_Y^A W_{YA} + \overline{W}_{Y\dot{A}} \overline{W}_Y^{\dot{A}}\right]_F. \quad (4.50)$$

This eventually leads to

$$\mathcal{L}_{GK} = \frac{1}{2} D_s^a(x) D_s^a(x) - \frac{1}{4} F_{s\mu\nu}^a(x) F_s^{\mu\nu a}(x) + i\lambda_s^a(x)\sigma^\mu[\Delta_\mu]\overline{\lambda}_s^a(x) \quad (4.51)$$

$$+ \frac{1}{2} D_z^b(x) D_z^b(x) - \frac{1}{4} F_{w\mu\nu}^a(x) F_w^{\mu\nu a}(x) + i\lambda_w^a(x)\sigma^\mu[\Delta_\mu]\overline{\lambda}_w^a(x) \quad (4.52)$$

$$+ \frac{1}{2} D_y(x) D_y(x) - \frac{1}{4} F_{y\mu\nu}^a(x) F_y^{\mu\nu a}(x) + i\lambda_y^a(x)\sigma^\mu[\partial_\mu]\overline{\lambda}_y^a(x). \quad (4.53)$$

Here the $SU(3)$, $SU(2)$ and $U(1)$ parts have been ordered line by line, as can be seen from the labels s, w and y of the vector superfield components. This implies that Δ_μ denotes a $SU(3)$ covariant derivative in 4.51 and a $SU(2)$ covariant derivative in 4.52. Identifying $A_{s\mu}^a$ with the gluon fields G_μ^a, $A_{z\mu}^a$ with the weak boson fields W_μ^a and $A_{y\mu}$ with the $U(1)$ boson field B_μ yields the SM gauge kinetic terms 1.46.

4.2.5 Superpotential terms

We still need to recover the fermion mass terms of the SM 1.35. To this end we construct a superpotential. To ensure supersymmetric invariance superpotential terms cannot contain both lefthanded and righthanded chiral superfields. In the SM the Higgs field takes a vev 1.22 which easily generates mass terms for down quarks and electrons but to have an up quark mass term we needed a tensor ϵ^{ab}. This is just a technical way of taking the conjugate of the Higgs field. That will not be allowed in the MSSM Lagrangian. We are therefore led to introduce two

Higgs chiral superfields, Φ_{H_1} and Φ_{H_2}[3]. They take vevs

$$< \Phi_{H_1} >= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \qquad < \Phi_{H_2} >= \frac{1}{\sqrt{2}} \begin{pmatrix} v_2 \\ 0 \end{pmatrix}. \tag{4.54}$$

Now that we have two Higgs fields, the ratio between their vevs becomes a new free parameter:

$$\frac{v_2}{v_1} \equiv \tan\beta. \tag{4.55}$$

The contribution to the MSSM Lagrangian can now be written

$$\Big[\mathcal{W}_{MSSM} + \mathcal{W}^{\dagger}_{MSSM}\Big]_F, \tag{4.56}$$

with

$$\mathcal{W}_{MSSM} = \mu\Phi_{H_1}\cdot\Phi_{H_2} + y^e_{ij}(\Phi_{Ll})_i\Phi_{H_1}(\Phi_{Re})_j + y^d_{ij}(\Phi_{Lq})_i\Phi_{H_1}(\Phi_{Rd})_j + y^u_{ij}(\Phi_{Lq})_i\Phi_{H_2}(\Phi_{Ru})_j \tag{4.57}$$

In the first RHS term a $SU(2)$ inproduct is assumed, the other terms contain just ordinary Euclidean inproducts.
In this case projecting out the SUSY invariant F-component is rather easy. We find

$$\begin{aligned}
\mathcal{L}_{FM} =& \mu\Big[\phi_{H_1}(x)\cdot F_{H_2}(x) + F_{H_1}(x)\cdot\phi_{H_2}(x) - \xi_{H_1}(x)\cdot\xi_{H_2}(x)\Big] && (4.58)\\
&+ y^e_{ij}\Big[(\phi_{Ll})_i(x)\phi_{H_1}(x)(F_{Re})_j(x) - (\phi_{Ll})_i(x)\xi_{H_1}(x)(\xi_{Re})_j(x)\\
&\quad + (\phi_{Ll})_i(x)F_{H_1}(x)(\phi_{Re})_j(x) - (\xi_{Ll})_i(x)\phi_{H_1}(x)(\xi_{Re})_j(x)\\
&\quad - (\xi_{Ll})_i(x)\xi_{H_1}(x)(\phi_{Re})_j(x) + (F_{Ll})_i(x)\phi_{H_1}(x)(\phi_{Re})_j(x)\Big] && (4.59)\\
&+ y^d_{ij}\Big[(\phi_{Lq})_i(x)\phi_{H_1}(x)(F_{Rd})_j(x) - (\phi_{Lq})_i(x)\xi_{H_1}(x)(\xi_{Rd})_j(x)\\
&\quad + (\phi_{Lq})_i(x)F_{H_1}(x)(\phi_{Rd})_j(x) - (\xi_{Lq})_i(x)\phi_{H_1}(x)(\xi_{Rd})_j(x)\\
&\quad - (\xi_{Lq})_i(x)\xi_{H_1}(x)(\phi_{Rd})_j(x) + (F_{Lq})_i(x)\phi_{H_1}(x)(\phi_{Rd})_j(x)\Big] && (4.60)\\
&+ y^u_{ij}\Big[(\phi_{Lq})_i(x)\phi_{H_2}(x)(F_{Ru})_j(x) - (\phi_{Lq})_i(x)\xi_{H_2}(x)(\xi_{Ru})_j(x)\\
&\quad + (\phi_{Lq})_i(x)F_{H_2}(x)(\phi_{Ru})_j(x) - (\xi_{Lq})_i(x)\phi_{H_2}(x)(\xi_{Ru})_j(x)\\
&\quad - (\xi_{Lq})_i(x)\xi_{H_2}(x)(\phi_{Ru})_j(x) + (F_{Lq})_i(x)\phi_{H_2}(x)(\phi_{Ru})_j(x)\Big] + \text{h.c.} && (4.61)
\end{aligned}$$

Every fourth term between brackets corresponds to one of the three SM mass terms 1.35. Now that we have two Higgs fields, we should repeat the SM calculation that yields the mass of the W and Z bosons. It turns out that we only have to replace

[3] Another important aspect of this is that only with two chiral Higgs superfields the MSSM is anomaly free.

v, the SM Higgs vev, by $\sqrt{v_1^2 + v_2^2}$.
At this point we are almost done with the supersymmetric part of the MSSM. We have seen that the supersymmetric Lagrangian that consists of 4.31, 4.50 and 4.56 can be written in terms of superfields components which was done in 4.32 - 4.38, 4.51 - 4.53 and in 4.58 - 4.61.
Finally we want to get rid of the auxiliary F- and D fields. This is rather easy. As they never appear in any derivative, the Euler-Lagrange equations dictate that the derivative of the MSSM component Lagrangian to any of these fields vanishes. For the $SU(3)$ D-fields $D_s^a(x)$ for example this yields

$$\frac{\delta}{\delta D_s^a(x)}\left[\phi_i^\star(x) g_s (T_i^3)^a D_s^a(x)\phi_i(x) + \frac{1}{2}D_s^a(x)D_s^a(x)\right] = 0, \tag{4.62}$$

which enables to eliminate $D_s^a(x)$:

$$D_s^a(x) = g_s \phi_i^\star(x)(T_i^3)^a \phi_i(x). \tag{4.63}$$

All other auxiliary fields can be eliminated in this same way.

4.3 How sparticles solve the hierarchy problem

In the first chapter we found that in the framework of the SM the Higgs mass is “unprotected”. The first order correction to the Higgs propagator contains a fermionic loop (two Higgs-fermion-antifermion couplings) that can raise the Higgs mass squared to $\mathcal{O}(10^{36}\ \mathrm{GeV}^2)$. We then need an incredible amount of fine tuning between the bare Higgs mass and this first order correction to have a physical Higgs mass of $\mathcal{O}(10^2\ \mathrm{GeV})$.
Supersymmetry was originally invented to get rid of these quadratic divergences. Now that we have sparticles around there are new contributions to the Higgs propagator. From the superpotential terms 4.58 - 4.61 we read that now we have Higgs - sfermion - sfermion and Higgs - Higgs - sfermion - sfermion couplings. The two sfermions can only be both lefthanded or both righthanded.
In exact, unbroken supersymmetry the Higgs propagator receives only one new sfermion loop contribution, the one with a quartic Higgs - sfermion coupling. (The one with two cubic Higgs - sfermion couplings gives a zero contribution as long as sparticles have the same mass as their superpartners, which is the case in exact supersymmetry.) The non-vanishing contribution is quadratic and cancels the quadratic divergence from the fermionic loop contribution. This cancellation (non-renormalization) is the essence of the use of supersymmetry. If supersymmetry is exact, scalar masses do not receive any correction at all.
Next, we consider what happens when supersymmetry does break. If, in one way or another, a particle and its corresponding sparticle acquire different masses and if we allow Higgs - LH sfermion - RH sfermion couplings as well, quadratic divergences still cancel. We now have two sfermion loop diagrams with two cubic couplings that do induce a divergence but only a logarithmic one. That is why this kind of supersymmetry breaking is called *soft* supersymmetry breaking and we will need it right away.

4.4 Breaking of Supersymmetry

So far we have been telling a beautiful story in this chapter. Unfortunately, nature is not supersymmetric. That is, we do not observe unbroken supersymmetry, no superpartners have been found. To describe nature we should look for a way to break supersymmetry in such a way that connects to experiment but maintains the advantage of solving the hierarchy problem.
The most elegant way to explain the mass asymmetry between particles and sparticles would be postulating that supersymmetry gets broken spontaneously. However, detailed analysis [8, chap. 7] then leads to

$$STrM_e^2 + STrM_\nu^2 = STrM_u^2 + STrM_d^2 = 0. \tag{4.64}$$

Here we have introduced the supertrace, the spin weighted graded trace of the squared mass matrix of a chiral multiplet:

$$STrM_i^2 = \sum_{J=0}^{J=\frac{1}{2}} (-1)^{2J}(2J+1)m_J^2. \tag{4.65}$$

If for every chiral supermultiplet the bosonic component is far heavier than the fermionic component, which is, to use an understatement, clearly observed, each of these four supertraces is positive. Spontaneously broken supersymmetry can be in accordance with experiment because of 4.64.
We are thus forced to raise sparticle masses out of experimental reach "by hand", by introducing (soft) supersymmetry breaking terms. There is no overlying chiral superfield, we insert bare component fields. To comfort themselves, people like to view these terms as an effective theory, resulting from complicated interactions somewhere between M_{GUT} and M_{PLANCK}. Well, that can be, of course, but let us postpone this "hidden sector physics" until we have a working SUSY GUT. As long as they do not ruin the solution of the hierarchy problem, that is, as long as they do not cause new quadratic (or quartic) divergencies, they are more than welcome to save the theory.
The most general soft SUSY breaking terms that meet these criteria are gaugino mass terms, scalar mass terms and scalar quadratic and cubic interaction terms. Thus, the scalar component $\phi(x)_i$ of every chiral superfield $\Phi_i(x)$ and the gaugino components $\lambda_i(x)$, $\overline{\lambda}(x)$ of every vector superfield $V_i(x)$ receive an additional mass term, and every quadratic and cubic interaction between scalar components will receive a new contribution that adds to the one present in the supersymmetric part of the MSSM.
We have

$$\begin{aligned}-\mathcal{L}_{SOFT} = &\sum_i m_i^2\phi_i^2 + \Big(\sum_{i=1,2,3} M_i\lambda_i\lambda_i - B\mu\phi_{H1}\phi_{H2} \\ &+ \sum_{i,j}\big[A_{ik}^e y_{kj}^e(\phi_{Ll})_i\phi_{H_1}(\phi_{Re})_j + A_{ik}^d y_{kj}^d(\phi_{Lq})_i\phi_{H_1}(\phi_{Rd})_j \\ &+ A_{ik}^u y_{kj}^u(\phi_{Lq})_i\phi_{H_2}(\phi_{Ru})_j\big] + h.c.\Big).\end{aligned} \tag{4.66}$$

Here the first sum is over all chiral superfields. The second sum is over all gauginos, introducing a $SU(3)$ gaugino mass, a $SU(2)$ gaugino mass and a $U(1)$ gaugino mass. The last sums are just over generation indices.
The introduction of these soft breaking terms induces drastic changes in the SUSY spectrum. Most are purely in the as yet unexplored SUSY energy region above 200 GeV and therefore impossible to detect directly. (Although they might induce new contributions to flavour changing neutral currents.) Without soft breaking the sfermion mass matrices, written in the L-R basis, had only two equal off-diagonal terms, just like the fermion mass matrices. Now there are many new terms, thus leading to complicated mass eigenstates. The charged[4] Higgsinos form new, "chargino" eigenstates with the charged gauginos (the λ-superparters of the $SU(2)$ gauge bosons W^+_μ en W^-_μ. Neutral Higgsinos and gauginos mix into "neutralino" eigenstates.
The only directly observable implications of the soft breaking terms are the contributions to the Higgs boson masses. With the inclusion of soft breaking the situation is as follows. Before the electroweak $SU(2) \times U(1)$ breaking we have two Higgs doublets, that is, eight real fields. Just as in the SM, three of them get "eaten" by the weak W and Z bosons in the breaking process. We are then left with four heavy mass eigenstates and one light eigenstate, which is assumed to be around 127 GeV [8, chap. 10]. As soon the LHC starts working, this last statement actually makes SUSY falsifiable.
A clear disadvantage of the soft SUSY breaking terms is the huge number of freely adjustable parameters they bring in. Therefore [4] some constraints are assumed at the GUT scale:

$$\begin{aligned}-\mathcal{L}_{SOFT}|_{MGUT} =& m_0^2 \sum_i \phi_i^2 + \Big(M_{\frac{1}{2}} \sum_{i=1,2,3} \lambda_i \lambda_i - B\mu \phi_{H1}\phi_{H2} \\ &+ A_0 \sum_{i,j} \big[y^e_{ij} (\phi_{Ll})_i \phi_{H_1} (\phi_{Re})_j + y^d_{ij} (\phi_{Lq})_i \phi_{H_1} (\phi_{Rd})_j \\ &+ y^u_{ij} (\phi_{Lq})_i \phi_{H_2} (\phi_{Ru})_j \big] + h.c. \Big). \end{aligned} \quad (4.67)$$

The gaugino unification (the introduction of one universal gaugino mass $M_{\frac{1}{2}}$) is motivated by the fact that the ratios of gauge couplings to gaugino masses are scale invariant [8, chap. 11]. Therefore, if gauge couplings meet, gaugino masses should do so as well. The other assumptions have no such solid bases of origin. Most motivations come from "hidden sector arguments"
Much more could be said and calculated about these soft SUSY breaking analysis, but that would take us to far out of our Grand Unification theme. So let us just take home the bottom line: SUSY implies equal masses for members of the same SUSY multiplet. Observations clearly rule out this scenario. Therefore we assume soft SUSY breaking: we add new terms to the SUSY Lagrangian that lift sfermion and gaugino masses while only producing logarithmic divergencies. In the remainder of this thesis we will mainly concentrate on that SUSY-fields that were already present in the SM (that is, no superpartners). We will assume a universal gaugino

[4]The charged Higgs field is that part of the doublet that gets vev v. That is, in our notation, the lower entry of Φ_{H1} and the upper entry of Φ_{H2}.

and Higgsino mass of 200 GeV, a Higgs boson mass of 700 GeV (except for the light Higgs) and a universal sfermion mass of 4500 GeV [5].

4.5 Running of the Coupling Constants in the MSSM: a true clue for Grand Unification

In the first chapter we have investigated the possibility of meeting coupling constants. The result was rather poor: assuming that there is an energy scale, somewhere around 10^{13} GeV, where they indeed meet, we can never fit all three within their error bars at $M = M_Z$.
Now that we have brought supersymmetry into play this situation improves a lot. Sparticles contribute significantly to the betafunctions that dictate the running of the coupling constants. Equation 1.69 now reads

$$\begin{aligned} b_1 =& -\frac{2}{5}\big(\frac{120}{36}N_g + 2\times\frac{1}{4}N_h\big) - \frac{1}{5}\big(2\times\frac{1}{4}\times N_h + \frac{120}{36}N_g\big) \\ =& -2N_g - \frac{3}{10}N_h. \end{aligned} \tag{4.68}$$

Here the first term $\frac{120}{36}N_g$ is the sum of squared hypercharges of all SM fermionic fields, nothing new. The second term stems from the fact that for every Higgs doublet we now have two Higgsinos with hypercharge $\pm\frac{1}{2}$. Their bosonic partners, present in the SM as well, are in the third term. The last term contains all sfermions, all having the same hypercharge as their superpartners.
Meanwhile the supersymmetric version of the $SU(2)$ betafunction 1.70 is

$$\begin{aligned} b_2 =& \frac{11}{3}\times 2 - \frac{2}{3}\big(4\times\frac{1}{2}\times N_g + N_h\times\frac{1}{2} + 2\big) - \frac{1}{3}\big(4\times\frac{1}{2}\times N_g + N_h\times\frac{1}{2}\big) \\ =& 6 - 2N_g - \frac{1}{2}N_h. \end{aligned} \tag{4.69}$$

In the fermionic part we have new contributions from Higgsinos (no factor 2, the sum is over $SU(2)$ multiplets) and from the $SU(2)$ gauginos ("Winos and Zino") that are in the adjoint rep. The bosonic part receives a contribution coming from sfermions.
For the $SU(3)$ betafunction we now have, instead of 1.71

$$\begin{aligned} b_3 =& \frac{11}{3}\times 3 - \frac{2}{3}\big(4\times\frac{1}{2}\times N_g + 3\big) - \frac{1}{3}\big(4\times\frac{1}{2}\times N_g\big) \\ =& 9 - 2N_g. \end{aligned} \tag{4.70}$$

Here we recognise a new gluino contribution and a new sfermion contribution.
We now repeat the analysis of the first chapter, taking a supersymmetric approach this time. That is, above a universal sfermion mass (4500 GeV), we work with the full superysmmetric betafunctions 4.68, 4.69 and 4.70. Below this sfermion mass scale we use betafunctions without sfermionic contributions. Below 700 GeV we freeze out the contribution of the second Higgs boson doublet. At 200 GeV we also

[5] These choices were taken from the article [5] reviewed in chapter 7.

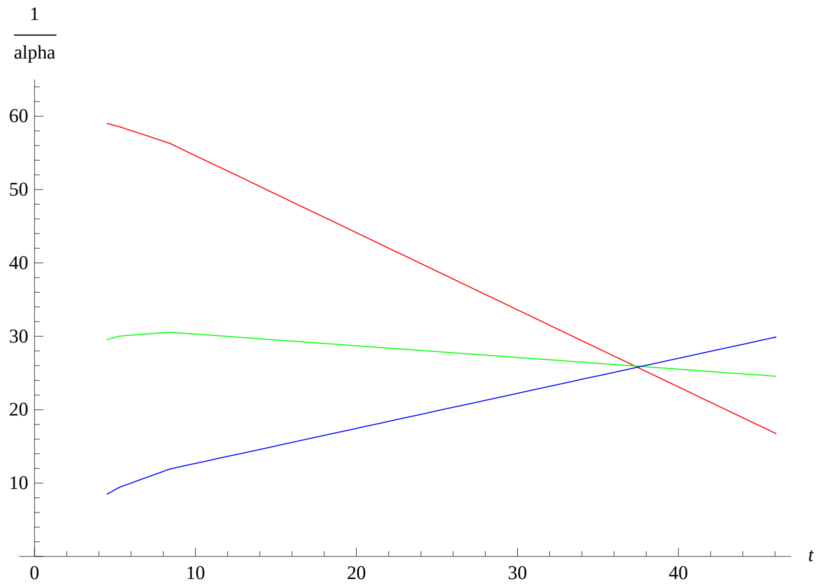


Figure 4.1: One-loop evaluation of the SUSY coupling constants α_i from M_Z to 10^{20} GeV. The horizontal scale is logarithmic: $t = \log M$. At M_Z we have $\frac{1}{\alpha_3} < \frac{1}{\alpha_2} < \frac{1}{\alpha_1}$.

do away with gaugino and Higgsino contributions. That means that we are left with the SM betafunctions.
Again we assume the existence of a universal coupling constant α at energy scale M_{GUT}. We check for which values (α, M_{GUT}) we obtain the best fit with the measured $\alpha_i^{EXP}(M_Z)$ once we run down from M_{GUT} to M_Z with our supersymmetrically modified betafunctions.
In this supersymmetric case the result is quite spectacular. We obtain best fits

$$\frac{1}{\alpha} = 24.3, \quad M_{GUT} = 1.5849 \times 10^{16}\text{GeV}, \leftrightarrow \log(M_{GUT}) = 37.30. \tag{4.71}$$

So, in SUSY Grand Unification the GUT scale is 3 orders of magnitude higher. This may explain the failure of the quest for proton decay at Kamiokande.
What we see now is that at $M = M_Z$ the discrepancy between predicted and measured values of the α_i has almost vanished. We find (cf 1.73 and 1.79)

$$\frac{1}{\alpha_1(M_Z)} = 59.0007, \quad \frac{1}{\alpha_2(M_Z)} = 29.5684, \quad \frac{1}{\alpha_3(M_Z)} = 8.54538, \tag{4.72}$$

which implies $\sin^2\theta = 0.23113$.
As we did in the SM analysis, another way to picture this greatly improved meeting of coupling constants is to solve the RGE in such a way that the three couplings exactly fit the experimental values at the weak scale. This yields a very encouraging plot: see figure 4.1. We actually need to zoom in a lot more to see that the coupling constants do not precisely meet: see figure 4.2.
Note that from the plots we see that these results do not depend crucially on the choice of mass scales for superpartners that we discussed before. The "winning ingredients", so to speak, are the slopes in the very long traject where all superpartners are present.

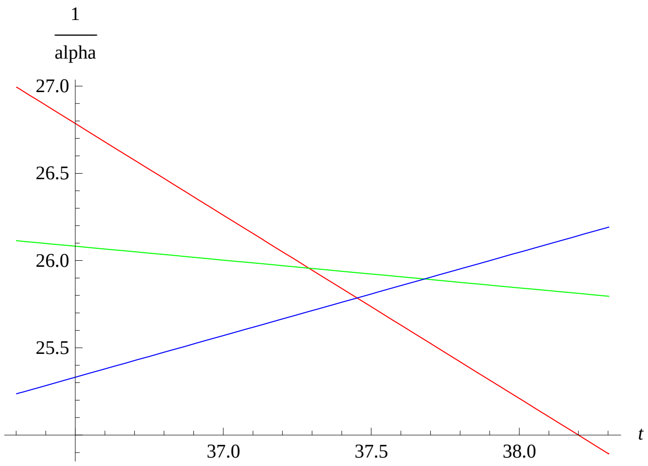


Figure 4.2: A closer look at the SUSY coupling constants around M_{GUT}.

We thus conclude that once we assume supersymmetry the hypothesis of Grand Unification around 10^{16} GeV is compatible with experimental values of coupling constants at the weak scale.

Chapter 5

SUSY SO(10)

Let us now “supersymmetrize” the $SO(10)$ results from chapter 3 as to have, at an energy level of about 1.6×10^{16} GeV, a Grand Unifying Theory that does not suffer from the hierarchy problem. We naively try to keep things as simple (as elegant) as possible, in the next chapter we will see to what extent our assumptions are phenonemenologically viable.

5.1 The situation at the GUT-scale

At the GUT scale we assume to have three 16-plets of chiral superfields Φ_{16}, a 45-plet of vector superfields V_{45} and a 10-plet if Higgs chiral superfields Φ_{H10}.
The fermion kinetic term is given by

$$\mathcal{L}_{FK} = \left(\Phi_{16} e^{V_M} \Phi_{16}\right)_D ; \qquad V_M = 2gV^a T^a \tag{5.1}$$

where T^a are the 45 $SO(10)$ generators written as 16×16 matrices. This expression should be integrated over $d^4\theta$ as well as over d^4x. Writing out this action in component form yields an expression equivalent to 4.32-4.38. In this GUT case there is no sum over i anymore as there is just one type of chiral supermultiplet and there are not three different T^a generators anymore as there is just one gauge group. The terms 4.33 and 4.36 become precisely equal to the $SO(10)$ kinetic term 3.63.
Then there is the Higgs kinetic term:

$$\mathcal{L}_{HK} = \left(\Phi_{10} e^{V_M} \Phi_{10}\right)_D ; \qquad V_H = 2gV^a T^a. \tag{5.2}$$

Here the generators T^a are written as 45 10×10 matrices. Now it is the bosonic part of the component action (the GUT version of 4.32 and 4.37) that looks like the kinetic Higgs term from $SO(10)$.
To build gauge kinetic terms, we first construct SUSY GUT field strengths

$$W^a_A = -\frac{1}{4}\overline{\mathcal{D}\mathcal{D}}\left[\mathcal{D}_A V^a + igf^{abc}\left(D_A V^b\right)V^c\right] \tag{5.3}$$

$$\overline{W}^{a\dot{A}}_s = -\frac{1}{4}\mathcal{D}\mathcal{D}\left[-\overline{\mathcal{D}}^{\dot{A}} V^a + igf^{abc}\left(\overline{\mathcal{D}}^{\dot{A}} V^b\right)V^c\right]. \tag{5.4}$$

Here f^{abc} are $SO(10$ structure constants. The gauge kinetic term is now given by

$$\mathcal{L}_{GK} = \frac{1}{4}\Big[W^{aA}W^a_A + \overline{W}^a_{\dot{A}}\overline{W}^{a\dot{A}} + \Big]_F . \tag{5.5}$$

Now we write down a mass term. The simplest option is to supersymmetrize 3.73:

$$-\mathcal{L}_{FM} = \Big[y^{\alpha\beta}(\Phi^\alpha_{16})^T C H_{10+1}\Phi^\beta_{16} + \text{h.c.}\Big]_F . \tag{5.6}$$

Again, the GUT hypothesis dictates that there is just one Yukawa coupling y. There is no constraint on the actual values of the entries. The positive square roots of the eigenvalues of $yy^\dagger$ (that lead to particle masses) could be anything, but for every generation there is just one universal value. Note, however, that we do not predict equal (particle and sparticle) masses within one generation, as the bosonic components of Φ_{16} receive additional mass terms from soft SUSY breaking terms and because, even with a Higgs superfield in the **10**, the 16 components do not receive equal vevs as long as $\tan\beta \neq 1$. But we will come to this.
So much for the supersymmetric part of our SUSY GUT. As we have argued before, if we want to connect with the very basic observation "There are no sparticles at the electroweak scale" we have to invoke a SUSY-breaking mechanism that provides additional mass constraints for sfermions, sleptons and gauginos. These soft breaking are postulated to result from some "higher scale physics", the GUT scale is not the highest scale in the theory. At this higher scale it might be possible to describe this soft breaking in terms of interactions between unified superfields, but at the GUT scale we can only write component terms. For our convenience, to have a complete overview, we will just repeat them.

$$\begin{aligned}-\mathcal{L}_{SOFT} =& m_0^2\sum_i \phi_i^2 + \Big(M_{\frac{1}{2}}\sum_{i=1,2,3}\lambda_i\lambda_i - B\mu\phi_{H1}\phi_{H2} \\ &+ A_0\sum_{i,j}\big[y^e_{ij}(\phi_{Ll})_i\phi_{H_1}(\phi_{Re})_j + y^d_{ij}(\phi_{Lq})_i\phi_{H_1}(\phi_{Rd})_j \\ &+ y^u_{ij}(\phi_{Lq})_i\phi_{H_2}(\phi_{Ru})_j\big] + \text{h.c.}\Big).\end{aligned} \tag{5.7}$$

We furthermore have a Majorana mass term for neutrinos and a Higgs mass term.

5.2 Down from the GUT scale to the electroweak scale

Below M_{GUT} $SO(10)$ symmetry breaks to $SU(3)\times SU(2)\times U(1)$. The vector and gaugino components of 33 of the 45 vector superfields acquire GUT-like masses. Therefore, their interactions are frozen out. The chiral superfields are still in one multiplet but this is statement does not have any physical implications: on checking the generator matrices corresponding with the still massless vector superfields (just like $T^1 - T^{12}$ in the $SU(5)$ case) one understands that effectively the chiral superfields are in SM multiplets already. The **10** Higgs chiral superfield first breaks in a **5** and a $\overline{\mathbf{5}}$. The triplet parts of both 5-plets become GUT-like heavy, the doublet parts are still massless.
Around 10^{10} GeV the Higgs singlet gets a vev, providing a Majorana mass term

for righthanded neutrinos. (This Majorana mass term could also be formed without the inclusion of a Higgs field, but that would not be in accordance with the GUT-picture we have of mass terms.)
Due to the soft SUSY breaking all sfermions, sleptons, gauginos, Higgsinos and four out of the five Higgs bosons acquire masses in the region between the TeV scale and, say, 200 GeV. Below that , we have only SM fields and interactions left. In the SM regime fermions, the remaining Higgs boson and the weak bosons become massive. A subtle inheritance from SUSY physics applies here: up quarks and neutrinos get a Higgs vev v_1, down quarks and electrons get a Higgs vev v_2. So, again, Yukawa eigenvalue unification does not necessarily imply particle mass unification. As the two Higgs SUSY vevs v_1 and v_2 are related to the Higgs SM vev v as

$$\sqrt{v_1^2 + v_2^2} = v, \tag{5.8}$$

we will set

$$v_1 = \sin\beta, \qquad v_2 = \cos\beta. \tag{5.9}$$

β is a free parameter. In this notation, there is a factor $\tan\beta$ difference between the (Dirac) mass value for up quarks and neutrinos and the mass value for downs and charged leptons resulting from the same Yukawa eigenvalue.
One concluding remark: in fact it only makes physical sense to refer to particle masses at an energy scale low enough for the Higgs fields to take their vacuum expectation value configurations. Above this scale, by “mass” we technically mean “product of actual Yukawa eigenvalue and Higgs vev” (times $\frac{1}{\sqrt{2}}$, see the discussion around 1.37).

Chapter 6

Yukawa unification in SUSY SO(10)

After having constructed our minimal $SO(10)$ SUSY GUT, in the phenomenological part of this thesis we want check how well these GUT-scale assumptions match low-energy observables. Just like the coupling constants, the running of all relevant parameters is described by appropiate renormalization group equations (RGE). We will focus on the RGE evaluation of Yukawa matrices y_u, y_d, y_e and y_n. We want to investigate to what extent the postulated Yukawa unification is phenomenologically viable.
We do so because we consider this Yukawa unification, after the gauge coupling unification perhaps, the most appealing prediction made by supersymmetric Grand Unification Theories. Once we have expressions for GUT-scale Yukawa matrices, we can immediately calculate the quark and lepton (Dirac) mass spectrum. (To actually predict squark and slepton masses, we would also need the RGE evaluated mass parameters of all soft SUSY breaking parameters. That seems a much harder job, beyond this thesis: as these soft parameters start running outside phenomenological reach, we do not have any boundary conditions to put into their RGE.)

6.1 Experimental input at the electroweak scale

First we need our feet put firmly on the ground. At $M_Z = 91.19$ GeV the Particle Data Group [3] can provide us with all SM fermion masses (except for neutrinos of course) and parametrizations of the CKM quark mixing matrix and PMNS lepton mixing matrix.
Throughout this chapter we will always present masses in matrix notation. In the rows we have the up quark mass, down quark mass, electron mass and Dirac neutrino mass. There are three columns, one for each generation.

$$\begin{pmatrix} m_u & m_c & m_t \\ m_d & m_s & m_b \\ m_e & m_\mu & m_\tau \\ m^D_{\nu_e} & m^D_{\nu_\mu} & m^D_{\nu_\tau} \end{pmatrix} \tag{6.1}$$

All masses will be denoted in MeV.
The Particle Data Group gives

$$m(M_Z)_{PDG} = \begin{pmatrix} 1.27 & 619 & 171700 \\ 2.9 & 55 & 2890 \\ 0.48657 & 102.718 & 1746.24 \end{pmatrix}. \tag{6.2}$$

In the neutrino sector assigning initial values is quite problematic. We believe in the seesaw mechanism, an electron-sized Dirac mass state and a very heavy Majorana mass state together result in very light and very heavy eigenstates. However, as there have only been (rather unprecise) observations of these light eigenstates, nothing could be said. One option now is to leave out the neutrinos out altogether. The other option is to invent "reasonable" (that is, electron-scale and increasing through the three generations) Dirac masses:

$$m_\nu^D(M_Z) = \begin{pmatrix} 10 & 100 & 1000 \end{pmatrix}. \tag{6.3}$$

We will investigate both. The worth of the second approach will not be in its predictions for Dirac neutrino masses, but it might show to what extent the high-scale evaluation of the other nine fermions is affected by the neutrino presence.
Now for the mixing matrices. The CKM matrix V (and the PMNS matrix P too) can be parametrized in terms of three angles and a phase factor:

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \tag{6.4}$$

Here $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$. In defining the CKM matrix, it is common use to parametrize even further:

$$\sin\theta_{12} = \lambda, \quad \sin\theta_{13} = A\lambda^2, \quad \theta_{13} = \frac{A\lambda^3\eta}{\sin\delta}, \quad \tan\delta = \frac{\eta}{\rho}. \tag{6.5}$$

The most recent PDG fit that we will work with has

$$\lambda = 0.2272, \quad A = 0.818, \quad \rho = 0.221, \quad \eta = 0.34. \tag{6.6}$$

The PMNS matrix is not as well known as its quarklike equivalent. The most recent Particle Data Group fit has (in rad)

$$\theta_{12} = 0.589921, \quad \theta_{23} = \frac{\pi}{4}, \quad \theta_{13} = \delta = 0. \tag{6.7}$$

The only observables are the absolute values of the matrix entries. Using these parametrisations, they read

$$|V| = \begin{pmatrix} 0.973841 & 0.227198 & 0.003890 \\ 0.227085 & 0.972959 & 0.042225 \\ 0.008172 & 0.041609 & 0.999101 \end{pmatrix} \tag{6.8}$$

and

$$|P| = \begin{pmatrix} 0.830984 & 0.556296 & 0 \\ 0.393360 & 0.587595 & 0.707107 \\ 0.393360 & 0.587595 & 0.707107 \end{pmatrix}. \tag{6.9}$$

We realize that given the uncertainty in fit parameters we do not have any control over the fifth and sixth digits in these expressions. We only write these down as to be able to observe tiny changes after renormalization group evolution.
In the first chapter we already saw how to get from Yukawa mass eigenvalues y (the entries of the diagonalized Yukawa matrices) to masses m:

$$m = y \times \frac{v}{\sqrt{2}}. \tag{6.10}$$

Again, v is the vacuum expectation value of the Higgs field. Anticipating a bit on the next section, we mention that, in order to get rid of some factors of 4π, we have solved the RGE for a new variable y' that is related to y as $y = 4\pi y'$. There is no harm at all in this renaming, we just mention it to state clearly that (dropping the primes already), in all calculations we have used

$$y = m \times s, \qquad s = \frac{1}{4\pi} \times \frac{\sqrt{2}}{v}. \tag{6.11}$$

So let us convert the PDG observables to initial values for our Yukawa matrices. To investigate the effect of quark and lepton mixing we will write down a set of Yukawa boundary conditions ignoring mixing effects and another one that does take CKM and PMNS effects into account.

6.1.1 No mixing

In this case Yukawa matrices start out diagonal and remain so all the way up. Therefore, boundary conditions are quite innocent:

$$y_u(M_Z) = \text{Diag}(1.27 \times s, 619 \times s, 171700 \times s), \tag{6.12}$$

and the same for the other Yukawas.

6.1.2 Mixing

At the electroweak scale we have twelve Dirac masses. These are the square roots of the eigenvalues of the matrices $y_u y_u^\dagger$, $y_d y_d^\dagger$, $y_e y_e^\dagger$ and $y_\nu y_\nu^\dagger$. In the first chapter, in 1.39, we saw how to decompose these matrixproducts:

$$yy^\dagger = UD^2U^\dagger. \tag{6.13}$$

The U matrices are connected to the quark and lepton mixing matrices:

$$V = U_u^\dagger U_d \qquad P = U_n^\dagger U_e. \tag{6.14}$$

Now we are ready to write down initial weak scale values for the Yukawa matrices. We are, however, faced with some ambiguity. Rewriting 6.13 we could end up with

$$U_u^\dagger y_u y_u^\dagger U_u = D_u^2, \qquad U_u^\dagger y_d y_d^\dagger U_u = VD_d^2V^\dagger, \tag{6.15}$$

or

$$U_d^\dagger y_u y_u^\dagger U_d, = V^\dagger D_u^2 V \qquad U_d^\dagger y_d y_d^\dagger U_d = D_d^2. \tag{6.16}$$

Expressed either way, the RHS contains only observables. At the LHS we still have the freedom to design U_u and U_d the way we like, only $U_u^\dagger U_d$ is fixed. The first expression, 6.15 suggests taking $U_u = \mathbf{1}$ (which implies $U_d = V$), that is, to work in a basis where the up Yukawa matrix is diagonal. The second expression, 6.16 invites us to take U_d as the unit matrix and thus work in a down Yukawa diagonal basis. In the lepton sector we have exactly the same ambiguity. This is just a reflection of the fact that whereas the Yukawa eigenvalues and (the rephasing invariants of) the quark and lepton mixing matrices are physical observables, the Yukawa matrices are not. We have checked extensively that all four approaches (up or down diagonal, electron or neutrino diagonal) lead to different solutions for Yukawa matrices at the GUT scale but that the aforementioned observables do end up at the same values, precisely as we expected.
Having chosen one of these four routes, we see that there are even more choices to be made. We still only know the products $yy^\dagger$. Every solution for a Yukawa matrix can be right multiplied by a unitary matrix, thus rendering a new solution. Again, just for checking the robustness of the equation framework, several routes were explored. Using the "FindRoot" commando we had Mathematica suggest three different sets of initial Yukawa values leading to correct eigenvalues and (absolute) mixing elements at the weak scale. A fourth one, the easiest approach in fact, yielded taking $y_u = D_u$, $y_d = VD_dV^\dagger$ (or, in the basis where downs start out diagonal, $y_u = V^\dagger D_u V$, $y_d = D_d$). A fifth approach consisted of rewriting the RGEs in such a way that only combinations $yy^\dagger$ remain while all single y's disappear. This is done by right multypling 6.18 by $y_d^\dagger$ and left multiplying the hermitian conjugate of 6.18 by y_d. The LHS of this equation equals the time (log M) derivative of $y_d y_d^\dagger$. We are rather relieved to report that, after solving the RGE, Mathematica every time coughed up the same twelve physically meaningful eigenvalues, just as it should. (From now, all "mixed" calculations will be done following the aforementioned fourth approach.)
Once the RGE have been solved, the GUT-scale Yukawa matrices are reconverted into mass values and mixing matrices.

6.2 Standard Model Yukawa running

To warm ourselves up, we first study Yukawa evolution in the framework of the Standard Model. In the first chapter we computed that, if it would exist at all, the evaluation of coupling constants does not suggest so, such a non supersymmetric GUT would live at a scale around 1.2×10^{13} GeV (see 1.78), so that is where we will evaluate the RGE solutions. In the SM we have $v = 246000$ MeV. As SUSY GUT addicts, we again (after the coupling constants analysis) hope to see some slight tendencies to unification that will be hugely improved on in the supersymmetric case.

6.2.1 SM Renormalization Group Equations

We get our Yukawa RGEs from the literature, from [10]. After converting to our own notation (in [10] $y^\dagger$ multiplies $\overline{\psi}_L \psi_R$ whereas we have $y\overline{\psi}_L \psi_R$ in our Lagrangian

1.35), these RGEs read as follows:

$$\dot{y_u} = \left(C_u^d y_d y_d^\dagger + C_u^u y_u y_u^\dagger + \alpha_u \right) y_u \tag{6.17}$$

$$\dot{y_d} = \left(C_d^d y_d y_d^\dagger + C_d^u y_u y_u^\dagger + \alpha_d \right) y_d \tag{6.18}$$

$$\dot{y_e} = \left(C_e^e y_e y_e^\dagger + C_e^\nu y_\nu y_\nu^\dagger + \alpha_e \right) y_e \tag{6.19}$$

$$\dot{y_\nu} = \left(C_\nu^e y_e y_e^\dagger + C_\nu^\nu y_\nu y_\nu^\dagger + \alpha_\nu \right) y_\nu . \tag{6.20}$$

The derivative is with respect to $t = \log M$.
We have

$$\begin{array}{cccc} C_u^d = -\frac{3}{2} & C_u^u = \frac{3}{2} & C_d^d = \frac{3}{2} & C_d^u = -\frac{3}{2} \\ C_e^e = \frac{3}{2} & C_e^\nu = -\frac{3}{2} & C_\nu^e = -\frac{3}{2} & C_\nu^\nu = \frac{3}{2} \end{array} \tag{6.21}$$

and

$$\begin{aligned} \alpha_u &= -\frac{17}{20}\left(\frac{g_1(M)}{4\pi}\right)^2 - \frac{9}{4}\left(\frac{g_2(M)}{4\pi}\right)^2 - 8\left(\frac{g_3(M)}{4\pi}\right)^2 + \mathrm{Tr}\left[3 y_d y_d^\dagger + 3 y_u y_u^\dagger + y_e y_e^\dagger + y_\nu y_\nu^\dagger\right] \\ \alpha_d &= -\frac{1}{4}\left(\frac{g_1(M)}{4\pi}\right)^2 - \frac{9}{4}\left(\frac{g_2(M)}{4\pi}\right)^2 - 8\left(\frac{g_3(M)}{4\pi}\right)^2 + \mathrm{Tr}\left[3 y_d y_d^\dagger + 3 y_u y_u^\dagger + y_e y_e^\dagger + y_\nu y_\nu^\dagger\right] \\ \alpha_e &= -\frac{9}{4}\left(\frac{g_1(M)}{4\pi}\right)^2 - \frac{9}{4}\left(\frac{g_2(M)}{4\pi}\right)^2 + \mathrm{Tr}\left[3 y_d y_d^\dagger + 3 y_u y_u^\dagger + y_e y_e^\dagger + y_\nu y_\nu^\dagger\right] \\ \alpha_\nu &= -\frac{9}{20}\left(\frac{g_1(M)}{4\pi}\right)^2 - \frac{9}{4}\left(\frac{g_2(M)}{4\pi}\right)^2 + \mathrm{Tr}\left[3 y_d y_d^\dagger + 3 y_u y_u^\dagger + y_e y_e^\dagger + y_\nu y_\nu^\dagger\right], \end{aligned} \tag{6.22}$$

Note that in these equations the ambiguity in Yukawa matrices we found before is reflected: we can freely right multiply a Yukawa matrix without changing the physical contents of these RGE.
We have used the prescriptions for the coupling constants that exactly fit experimental values at the electroweak scale (recall the discussion around 1.80). Threshold effects at 120 GeV (Higgs boson) and 171.7 GeV (top quark) were included.

6.2.2 Results

No mixing, no neutrino contribution

At the GUT scale, the nine fermion masses have run to

$$m(M_{GUT}) = \begin{pmatrix} 0.588194 & 286.687 & 89279.5 \\ 1.37626 & 26.1014 & 1221.61 \\ 0.492252 & 103.918 & 1766.68 \end{pmatrix}. \tag{6.23}$$

No mixing, neutrino contribution

Now we have twelve running fermions masses, that at M_{GUT} are found at

$$m(M_{GUT}) = \begin{pmatrix} 0.588198 & 286.689 & 89280.3 \\ 1.37627 & 26.1016 & 1221.62 \\ 0.492256 & 103.918 & 1766.68 \\ 10.8843 & 108.843 & 1088.41 \end{pmatrix}. \tag{6.24}$$

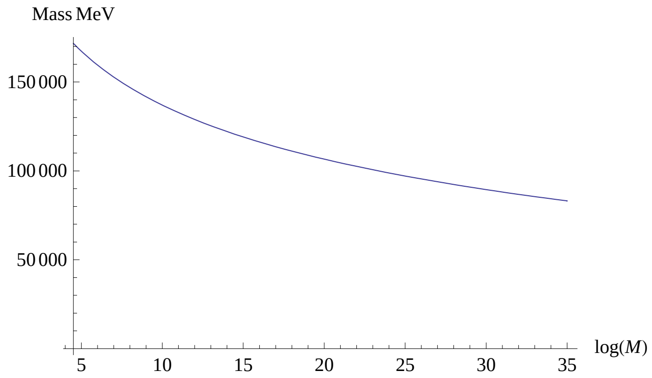


Figure 6.1: One-loop running of the top quark mass.

Mixing, no neutrino contribution

We find

$$m(M_{GUT}) = \begin{pmatrix} 0.588194 & 286.687 & 89279.5 \\ 1.37624 & 26.0955 & 1221.9 \\ 0.492252 & 103.918 & 1766.68 \end{pmatrix}. \quad (6.25)$$

Mixing, neutrino contributions

In this ultimate case we find

$$m(M_{GUT}) = \begin{pmatrix} 0.588198 & 286.689 & 89280.3 \\ 1.37625 & 26.0957 & 1221.91 \\ 0.492255 & 103.918 & 1766.68 \\ 10.8843 & 108.841 & 1088.42 \end{pmatrix}. \quad (6.26)$$

The absolute values of the CKM matrix have run, (or better, have crept) to

$$|V|(M_{GUT}) = \begin{pmatrix} 0.973832 & 0.227226 & 0.004367 \\ 0.227083 & 0.972721 & 0.047402 \\ 0.009175 & 0.046710 & 0.998866 \end{pmatrix}, \quad (6.27)$$

while the PMNS matrix now reads

$$|P|(M_{GUT}) = \begin{pmatrix} 0.830983 & 0.556297 & 1.5 \times 10^{-8} \\ 0.393355 & 0.587583 & 0.707119 \\ 0.393369 & 0.587604 & 0.707094 \end{pmatrix}. \quad (6.28)$$

We also plot the running of the top mass (6.1) and combine the bottom and tau evolution in one graph (6.2). (Recall that in the SM case $\log M_{GUT} = 30.15$.)

6.2.3 Conclusions

Three conclusions can be made.
First, the inclusion of Dirac masses for neutrinos (as compared to not taking neutrinos into account at all) affects the running of other observables only in a very small way: the relative effect is always smaller than 10^{-6}.

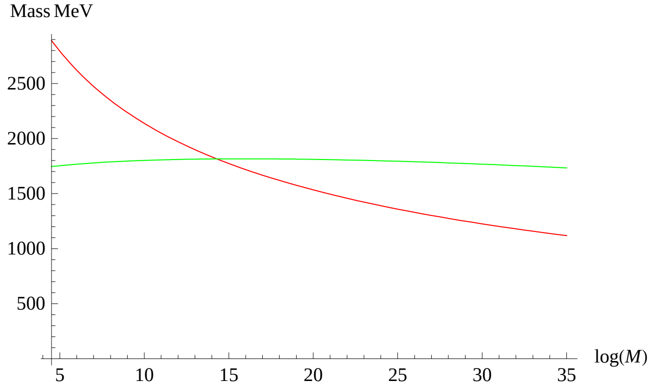


Figure 6.2: One-loop running of the bottom quark and the tau lepton mass. (The bottom quark starts out highest.)

Second, mixing effects are very tiny as well. This is reflected in the fact that the CKM matrix has hardly run (compare 6.8 and 6.27). The same goes for the PMNS matrix. The most significant effect of mixing inclusion is in the down quark sector. As the top quark mass is by far the largest number around, it has the biggest effect on the running of other fermion masses. On examing the RGE we see that the top quark is most closely related to the bottom quark. So, once we allow for quark mixing, the down sector is affected by the large value of m_t, while the up sector is much more insensitive to bottom quark contributions.
Third, hardly a hint for Grand Unification can be found in the running of particle masses. A very optimistic look at the running of the bottom quark and the tau lepton might yield small hope... could these be connected? Well, at least we can be sure that there is much room for improvement in a supersymmetric Yukawa analysis.

6.3 MSSM Yukawa running

Here we are. In a SUSY treatment of Yukawa evolution the RGE look different. However, the most striking new effect is the presence of *two* Higgs fields now. As we have seen in subsection 4.2.5, the masses of down quarks and electrons result from interactions with the Higgs field H_1 that takes a vev v_1 while up quark and neutrino masses invoke the Higgs field H_2 with its vev v_2. Their ratio, commonly denoted as

$$\tan\beta = \frac{v_2}{v_1}, \tag{6.29}$$

is a new free parameter. So, the mass difference between, say, the top and bottom quark mass can now be explained by the ratio of their vevs instead of by "Yukawa non-unification". We will solve RGE for $\tan\beta = 1$ (which may seem most natural), $\tan\beta = 10$ and $\tan\beta = 50$ (which is quite popular in the literature, it will come back in the next chapter).
As we have $\sqrt{v_1^2 + v_2^2} = v$, with v the SM Higgs vev of 246 GeV, we will work with

$$v_1 = v\cos\beta, \qquad v_2 = v\sin\beta. \tag{6.30}$$

6.3.1 MSSM Renormalization Group Equations

At the one loop level the SUSY RGE have the same structure as the SM ones (6.17-6.20). The coefficients now read:

$$\begin{array}{llll} C_u^u = 3 & C_d^u = 1 & C_u^d = 1 & C_d^d = 3 \\ C_e^e = 3 & C_e^\nu = 1 & C_\nu^e = 1 & C_\nu^\nu = 3. \end{array} \tag{6.31}$$

Moreover, we now have

$$\begin{aligned} \alpha_u &= -\frac{13}{15}\left(\frac{g_1}{4\pi}\right)^2 - 3\left(\frac{g_2}{4\pi}\right)^2 - \frac{16}{3}\left(\frac{g_3}{4\pi}\right)^2 + \mathrm{Tr}\left[3y_u y_u^\dagger + y_\nu y_\nu^\dagger\right] \\ \alpha_d &= -\frac{7}{15}\left(\frac{g_1}{4\pi}\right)^2 - 3\left(\frac{g_2}{4\pi}\right)^2 - \frac{16}{3}\left(\frac{g_3}{4\pi}\right)^2 + \mathrm{Tr}\left[3y_d y_d^\dagger + y_e y_e^\dagger\right] \\ \alpha_e &= -\frac{9}{5}\left(\frac{g_1}{4\pi}\right)^2 - 3\left(\frac{g_2}{4\pi}\right)^2 + \mathrm{Tr}\left[3y_d y_d^\dagger + y_e y_e^\dagger\right] \\ \alpha_\nu &= -\frac{3}{5}\left(\frac{g_1}{4\pi}\right)^2 - 3\left(\frac{g_2}{4\pi}\right)^2 + \mathrm{Tr}\left[3y_u y_u^\dagger + y_\nu y_\nu^\dagger\right]. \end{aligned} \tag{6.32}$$

We also wrote a Mathematica program that solves two-loop Renormalization Group Equations. Its essentials can be found in the Appendix. The equations were taken from a very recent article by Martin and Vaughn [11]. These are quite lengthy expressions [1]. The parts in parentheses in 6.17-6.20 now contain additional products of four Yukawa matrices. The same goes for the traces.

6.3.2 Results

RGE were solved for three different values of $\tan\beta$, mixed and non-mixed intial values, with or without neutrino contributions and at one-loop or two-loop level, thus making up for 24 different solutions. As their complete ensemble might not too much increase the readibility of this thesis, we will present the most interesting sample. Let us just start off with the one we, from a GUT-point of view, look most forward to (two-loop, mixing, neutrinos). After that we can investigate the effects of these assumptions.

Two-loop, mixing, neutrino contributions

First we take $\tan\beta = 1$. This turns out to be a unfortunate choice. Around 10^{10} GeV a fixed point (Landau pole) is reached, where y_u blows up and nothing can be said anymore. The problem is in the up quark sector. As the Higgs vevs v_1 and v_2 are equal now, the ratio between y_u and y_d eigenvalues is as big as the ratio of their masses.

We quickly turn to the case $\tan\beta = 10$. Now the masses behave well all the way up to M_{GUT}. At the GUT scale we find the following masses:

$$m(M_{GUT}) = \begin{pmatrix} 0.624838 & 304.549 & 132527 \\ 0.930383 & 17.649 & 1082.4 \\ 0.333333 & 70.37 & 1201.82 \\ 11.5123 & 115.211 & 1152.14 \end{pmatrix}. \tag{6.33}$$

[1] Martin and Vaughn did not include neutrino RGE. They (neutrino RGE) actually seem to be quite impopular, hardly any author mentions them, and only up to one-loop. Therefore we only included one-loop neutrino RGE.

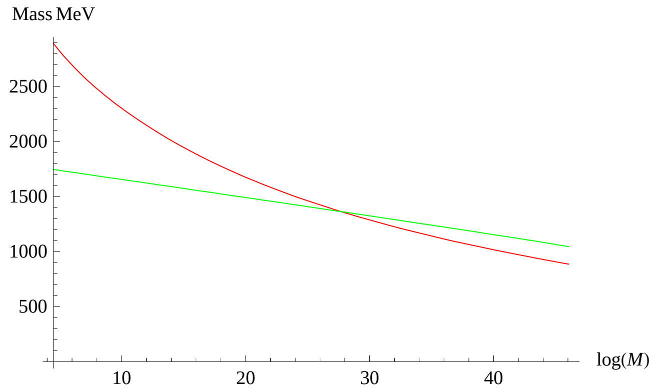


Figure 6.3: Two-loop bottom-tau running for $\tan\beta = 10$. The bottom starts out highest.

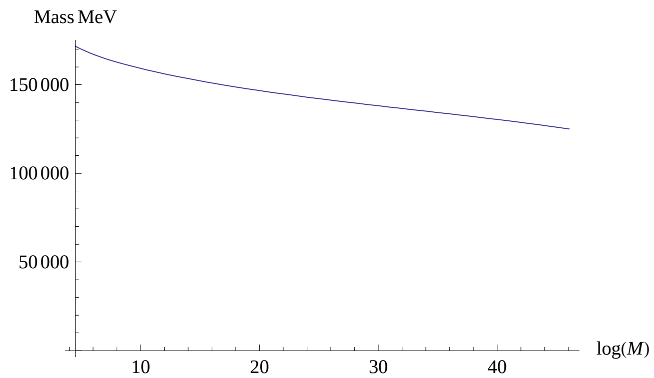


Figure 6.4: Two-loop top quark running for $\tan\beta = 50$.

(Recall that in the supersymmetric approach $M_{GUT} = 1.58 \times 10^{16}$ GeV, and $\log M_{GUT} = 37.30$.)
The bottom mass seems to have come pretty close to the tau mass (it has passed it). At the other hand, for complete third generation Yukawa unification we would require $\frac{m_t}{m_b} = 10$, while we find a ratio of 122. In other generations we again do not observe any unifying tendencies at all. But let us plot the bottom-tau evolution in figure 6.3. This yields quite a thrill: that looks a lot more like unification than the result in the SM case!
We also plot the running of the top quark mass (figure 6.4), just to check its behaviour. We see it neatly decreasing all the way up to M_{GUT}.
Now for $\tan\beta = 50$. We curiously calculate the GUT-scale masses:

$$m(M_{GUT}) = \begin{pmatrix} 0.683217 & 333.055 & 173973 \\ 1.33216 & 25.2726 & 2128.2 \\ 0.47728 & 100.813 & 2079.14 \\ 12.6186 & 130.222 & 1305.06 \end{pmatrix}. \tag{6.34}$$

Now the bottom quark and tau lepton end up even closer! Moreover, they have not crossed (yet). We instantly plot their running and acquire, in figure 6.5, this chapter's most illuminating, GUT-allowing plot. There seems to be a convincing

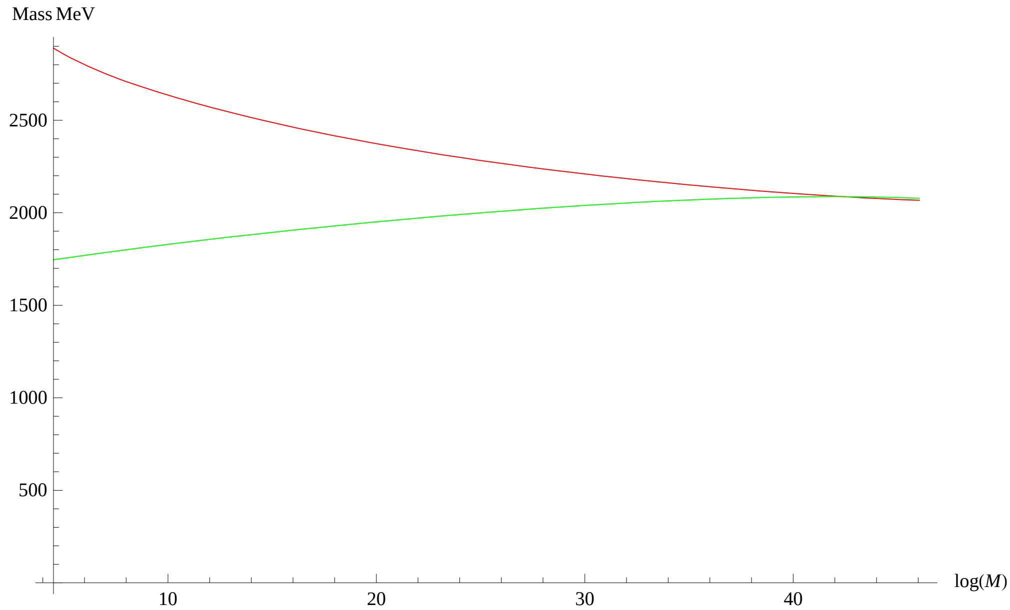


Figure 6.5: Two-loop bottom-tau running for $\tan\beta = 50$.

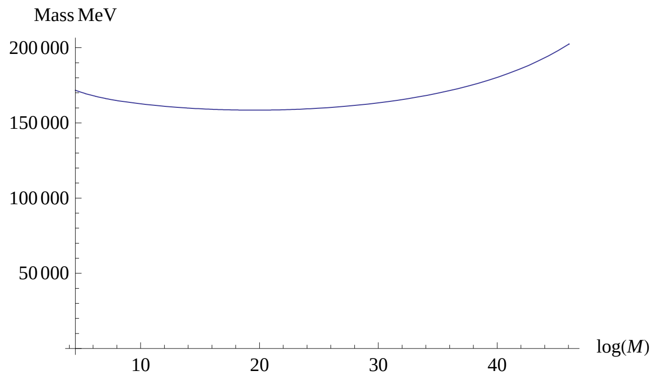


Figure 6.6: Two-loop top quark running for $\tan\beta = 50$.

unification, at a scale a bit higher perhaps than the GUT-scale we work with. The situation concerning full third generation Yukawa unification has also improved: now we find $\frac{m_t}{m_b} = 82$ instead of 50.
We also plot the running of the top quark in this framework, in figure 6.6. It curiously endsup at almost the same scale as where it started, after having made a dip however.

One-loop versus two-loop

To investigate the effect of *two*-loop RGE (for quarks and charged leptons), we solve the one-loop version (still taking into account mixing effects and neutrino running) and compare results.
For $\tan\beta = 1$ the troubles are the same. Now the fixed point is reached at 10^6 GeV already. The up quark blows up and takes the other fields with it. To show

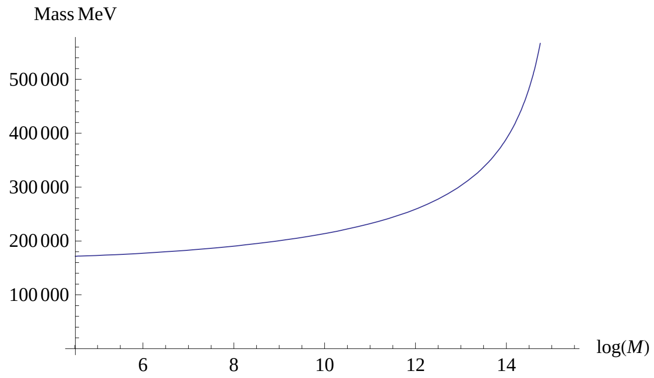


Figure 6.7: One-loop top quark running for $\tan\beta = 1$.

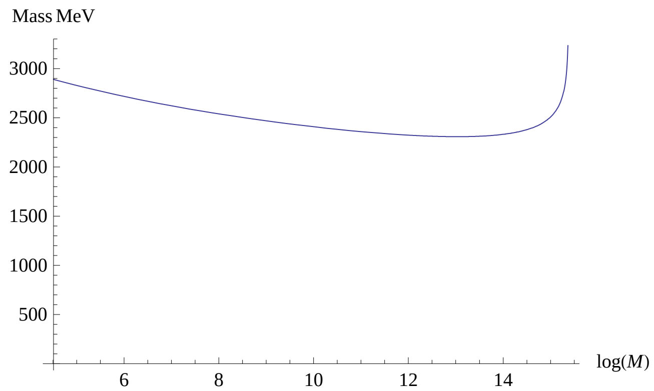


Figure 6.8: One-loop bottom quark running for $\tan\beta = 1$.

this effect for once, we plot the runnning of the top and bottom quark in figures 6.7 and 6.8.
Now for $\tan\beta = 10$. At the GUT-scale we find the following masses (that should be compared with 6.33):

$$m(M_{GUT}) = \begin{pmatrix} 0.614689 & 299.603 & 130646 \\ 0.925745 & 17.5611 & 1078.94 \\ 0.331925 & 70.0727 & 1196.69 \\ 11.4596 & 114.684 & 1146.87 \end{pmatrix}. \tag{6.35}$$

That is quite a small effect[2]. Let us see what we get for $\tan\beta = 50$:

$$m(M_{GUT}) = \begin{pmatrix} 0.677924 & 330.475 & 175124 \\ 1.32054 & 25.0523 & 2129.12 \\ 0.473478 & 100.01 & 2062.4 \\ 12.6392 & 130.401 & 1306.81 \end{pmatrix}. \tag{6.36}$$

On comparing with 6.34 we see that the corrections are a bit larger than in the case $\tan\beta = 10$, about 1% at most. The most notorious change is achieved for the tau lepton, and points right into the direction of bottom-tau unification. We

[2]especially when considering all the efforts made in the two-loop battle with Mathematica!

understand why two-loop corrections are more significant now: due to the larger ratio of v_2 and v_1 the Yukawa eigenvalues differ less and therefore the tau lepton mass is more heavily influenced by the top quark mass.

Mixing effects

In the SM we saw there was hardly any mixing at all. Let us investigate what is the case now.
For $\tan\beta = 10$ (by now we have understood that $\tan\beta = 1$ simply does not yield any physically relevant result at all) we compare GUT-scale masses acquired under diagonal Yukawa assumptions with 6.33. We find

$$m(M_{GUT}) = \begin{pmatrix} 0.62484 & 304.55 & 132529 \\ 0.930376 & 17.6451 & 1082.66 \\ 0.333333 & 70.3699 & 1201.82 \\ 11.5123 & 115.124 & 1153.02 \end{pmatrix}. \tag{6.37}$$

Again, hardly any change at all. No mass has changed more than $\frac{1}{1000}$ We thus expect to find GUT-scale CKM and PMNS mixing matrices almost unaffected. We find

$$|V|(M_{GUT}) = \begin{pmatrix} 0.973852, & 0.227165 & 0.003023 \\ 0.227086 & 0.973197 & 0.0363322 \\ 0.00703068 & 0.0358022 & 0.999334 \end{pmatrix} \tag{6.38}$$

and

$$|P|(M_{GUT}) = \begin{pmatrix} 0.830987 & 0.556292 & 3.76 \times 10^{-8} \\ 0.393365 & 0.588055 & 0.706554 \\ 0.393050 & 0.587137 & 0.707659 \end{pmatrix}. \tag{6.39}$$

which are to be compared with 6.8 and 6.9 respectively. The most influential diagonal terms have hardly changed, but small off-diagonal terms are recieving significant effects, about 15%. In an article by Allanach et.al. [1], very similar results were found. (Recall from 6.6 that we did include the CP violating phase in our CKM parametrization.)

We repeat this analysis for $\tan\beta = 50$. This yields (compare with 6.34)

$$m(M_{GUT}) = \begin{pmatrix} 0.683377 & 333.088 & 174085 \\ 1.33232 & 25.2699 & 2129.4 \\ 0.477339 & 100.826 & 2079.45 \\ 12.6209 & 126.233 & 1347.05 \end{pmatrix}, \tag{6.40}$$

which is, again, an indication for the smallness of quark and lepton mixing effects. For completeness we plot the GUT-scale versions of the CKM and PMNS matrix for this $\tan\beta = 50$ case:

$$|V|(M_{GUT}) = \begin{pmatrix} 0.973852 & 0.227164 & 0.00296113 \\ 0.227098 & 0.973341 & 0.0321457 \\ 0.00622026 & 0.0316769 & 0.999479 \end{pmatrix} \tag{6.41}$$

and

$$|P|(M_{GUT}) = \begin{pmatrix} 0.831034 & 0.556221 & 8.21 \times 10^{-7} \\ 0.406118 & 0.60677 & 0.683299 \\ 0.380066 & 0.567845 & 0.730138 \end{pmatrix}. \tag{6.42}$$

We conclude that mixing effects are small, also in this case, but a bit more significant than for $\tan\beta = 10$.

Dirac neutrino influences

Here we compare (two-loop, mixing included) GUT scale masses calculated without any neutrino contribution with 6.33 (for $\tan\beta = 10$) and 6.34 $\tan\beta = 50$).
Taking $\tan\beta = 10$ now yields

$$m(M_{GUT}) = \begin{pmatrix} 0.624828 & 304.545 & 132524 \\ 0.930383 & 17.649 & 1082.4 \\ 0.333333 & 70.3698 & 1201.81 \end{pmatrix}, \tag{6.43}$$

while $\tan\beta = 50$ results in

$$m(M_{GUT}) = \begin{pmatrix} 0.683204 & 333.048 & 173967 \\ 1.33216 & 25.2726 & 2128.19 \\ 0.47728 & 100.813 & 2079.12 \end{pmatrix}. \tag{6.44}$$

So, the neutrino corrections on other masses are tiny: all relative effects are smaller than 10^{-4}.

6.3.3 Conclusions

We have solved Renormalization Group equations under a variety of assumptions. We have seen beautiful plots suggesting bottom-tau unification, for $\tan\beta = 10$ as well as for $\tan\beta = 50$. The latter case yields the most illuminating plot, but in the former case the running of the GUT scale appears less close to the Landau pole.
There are no indications for other unifications.
The case $\tan\beta = 1$ should not be taken under consideration at all, it inescapably leads to fixed points which induce exploding masses for all fermion fields.
The difference between one-loop and two-loop results is small, but not negligible. It actually improves a lot on bottom-tau unification for the case $\tan\beta = 50$.
Quark and lepton mixing matrices are only slightly affected by the running. Therefore, solutions obtained under initial diagonal Yukawa assumptions are as suitable to work with as solutions resulting from the more complicated mixed case.
Many authors do not write down RGE for neutrinos. As long as we do not know its light and heavy mass eigenstates and, therefore, their Dirac masses, this indeed makes sense. We showed that lepton-sized neutrino Dirac masses do run considerably, just like all other masses, but their effects on other Yukawa couplings can safely be discarded.

Chapter 7

Dermisek and Raby's family symmetric approach

If we understand one thing from our Yukawa analysis, it is that we can never arrive at complete Yukawa unification at the GUT scale. There is some hope in the third generation: the third (largest) eigenvalues of $y_u y_u^\dagger$, $y_d y_d^\dagger$ and $y_e y_e^\dagger$ seem to end up at pretty much the same level. (This would induce $\frac{m_t}{\tan[\beta]} = m_b = m_\tau$.) In the first and second generation such a unification seems impossible.
From 2002 Radovan Dermisek and Stuart Raby (DR) have been constructing realistic SUSY GUT models ([5], [6], [7]). In this last chapter we want to discuss and slightly check their ideas. We mainly follow their 2006 article [5].

7.1 Construction of DR GUT-scale Yukawa matrices

DR assume that at the GUT level there is an additional D_3 symmetry. The third generation is in a singlet rep of this discrete group, the first and second generation are in a doublet rep. By assuming interactions with a **45** adjoint rep of $SO(10)$ and several "flavon" fields ($SO(10)$ singlets, but non-trivial in D_3), they are able to construct Yukawa matrices y_u, y_d, y_e and y_ν that can be parametrized with just 11 real parameters (Yukawa textures). These Yukawa matrices discriminate between $SO(10)$ **16**-plet members and therefore Dermisek and Raby state that after RGE evolution back to the electroweak scale (this is a "top-down" approach) their Yukawa matrices can connect with experimental values. So let us investigate how this comes about.

7.1.1 D_3

The group D_3 is formed by all possible rotations in three dimensions that leave an equilateral triangle *abc* invariant. It has six elements, divided in three classes: the first class contains the identity operation (denoted as E), the second class has two rotations ($C_{\frac{2\pi}{3}}$ and $C_{\frac{4\pi}{3}}$) and three mirroring operations (C_a, C_b, C_c) are in the third class. As there are three classes, there should be three non-equivalent irreps. From (finite) group theory we have a relation between the number of elements g of a group and the dimensionality d_ν of its irreps ν:

$$\sum_\nu d_\nu^2 = g. \tag{7.1}$$

So there is just one possibility: we have two one dimensional irreps, $\mathbf{1_A}$ and $\mathbf{1_B}$, and one two-dimensional irrep $\mathbf{2}$. As we are interested in D_3 invariant objects, we want to find out how we can decompose the tensor products of these reps.
To this end we first look, in each of the three reps $\mathbf{1_A}$, $\mathbf{1_B}$ and $\mathbf{2}$, for representations of the elements E, $C_{\frac{2\pi}{3}}$ and C_a[1] by examining the products of all six group elements. This yields

D_3	E	C_3	C_a
$\mathbf{1_A}$	1	1	1
$\mathbf{1_B}$	1	1	-1
$\mathbf{2}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(7.2)

where $\epsilon = e^{\frac{2\pi i}{3}}$. From these reps we find characters (the trace of the reps, these are class invariant objects) and from there we find the decompositions we were looking for. We get

$$\mathbf{1_A} \otimes \mathbf{1_A} = \mathbf{1_A}, \quad \mathbf{1_A} \otimes \mathbf{1_B} = \mathbf{1_B}, \quad \mathbf{1_B} \otimes \mathbf{1_B} = \mathbf{1_A} \tag{7.3}$$

$$\mathbf{1_A} \otimes \mathbf{2_A} = \mathbf{2_A}, \quad \mathbf{1_B} \otimes \mathbf{2_A} = \mathbf{2_A} \tag{7.4}$$

$$\mathbf{2_A} \otimes \mathbf{2_A} = \mathbf{1_A} \oplus \mathbf{1_B} \oplus \mathbf{2_A}. \tag{7.5}$$

Let us try to find an actual prescription for finding the two singlets and the doublet in the decomposition 7.5. If we take $\psi_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\psi_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ we have[2]

$$(\psi_1 \otimes \psi_2)_{\mathbf{1_A}} = x_1 y_2 + y_1 x_2 \tag{7.6}$$

$$(\psi_1 \otimes \psi_2)_{\mathbf{1_B}} = x_1 y_2 - y_1 x_2 \tag{7.7}$$

$$(\psi_1 \otimes \psi_2)_{\mathbf{2}} = \begin{pmatrix} y_1 y_2 \\ x_1 x_2 \end{pmatrix}. \tag{7.8}$$

These are the decompositions we were looking for. We now know how to extract an D_3 invariant ($\mathbf{1_A}$) quantity from the product of two D_3 doublets. The product of three D_3 doublets $\psi_1 \otimes \psi_2 \otimes \psi_3$ is also seen to possess a $\mathbf{1_A}$ part: using 7.6 we take the $\mathbf{1_A}$ part of the product of the $\mathbf{2_A}$ (using 7.8) part of $\psi_1 \otimes \psi_2$ and ψ_3:

$$(\psi_1 \otimes \psi_2 \otimes \psi_3)_{\mathbf{1_A}} = x_1 x_2 x_3 + y_1 y_2 y_3. \tag{7.9}$$

[1] Once we have reps for these three elements we can construct the other element's reps by $C_{\frac{4\pi}{3}} = C_{\frac{2\pi}{3}}^2$, $C_b = C_a C_{\frac{2\pi}{3}}$ and $C_c = C_a C_{\frac{2\pi}{3}}^2$

[2] To check these statements one can multiply ψ_1 and ψ_2 by E, $C_{\frac{2\pi}{3}}$ and C_a. The quantity $x_1 y_2 + y_1 x_2$ is then seen to transform under these operations in the same way as does the $\mathbf{1_A}$ rep (it does not change at all), the quantity $x_1 y_2 - y_1 x_2$ transforms as the $\mathbf{1_B}$ etcetera.

7.1.2 The DR superpotential

Although we are more interested in the final expressions for the four Dirac Yukawa matrices proposed by DR, we will provide some basic motivation in this section. DR get their Dirac Yukawa matrices from the following superpotential:

$$\begin{aligned} W_{\mathrm{DIRAC}} =& (\Phi_{16})_3\Phi_{10}(\Phi_{16})_3 + (\Phi_{16})_a\Phi_{10}\chi_a \\ &+ \overline{\chi}_a\left(M_\chi\chi_a + \Phi_{45}\frac{\phi_a}{\hat{M}}(\Phi_{16})_3 + \Phi_{45}\frac{\tilde{\phi}_a}{\hat{M}}(\Phi_{16})_a + \mathbf{A}(\Phi_{16})_a\right). \end{aligned} \quad (7.10)$$

The third generation SUSY 16-plet is in a D_3 $\mathbf{1_A}$ singlet. Therefore it can combine with a Higgs 10-plet ($\mathbf{1_A}$ as well) in the simplest form of a mass term. This is the first term of 7.10, leading to Yukawa unification in the third generation (all four Yukawa matrices will have equal (33)-elements).
The 16-plets carrying chiral superfields of the first and second generation are together in a D_3 $\mathbf{2}$ multiplet $(\Phi_{16})_a$. To form $SO(10)$ and D_3 invariant mass terms interactions with non-trivial D_3 fields are needed.
The fields χ_a ($\mathbf{16}$, $\mathbf{2}$) and $\overline{\chi}_a$ ($\overline{\mathbf{16}}$,$\mathbf{2}$) (Froggatt-Nielsen states) are supposed to have an M_χ mass eigenvalue even above the GUT scale. Below, we can integrate these fields out. Using the identity

$$\int \mathcal{D}\chi\mathcal{D}\overline{\chi}e^{-\int \omega\chi+\overline{\omega}\overline{\chi}+\overline{\chi}M\chi} = \mathrm{det}Me^{\omega M^{-1}\overline{\omega}} \quad (7.11)$$

(which follows from completing the exponent and using translation invariance) we arrive at an effective superpotential

$$W_{\mathrm{eff}} = \frac{1}{M_\chi}\left((\Phi_{16})_a\Phi_{10}\left[\Phi_{45}\frac{\phi_a}{\hat{M}}(\Phi_{16})_3 + \Phi_{45}\frac{\tilde{\phi}_a}{\hat{M}}(\Phi_{16})_a + \mathbf{A}(\Phi_{16})_a\right]\right). \quad (7.12)$$

Now all fields except for the Φ_{16} multiplets take vevs. The vev of M_χ is supposed to discriminate between 16-plet members:

$$< M_\chi >= M_0\left(1 + \alpha X + \beta Y\right). \quad (7.13)$$

Here X denotes the $U(1)$ charge of a superfield in the symmetry breaking process $SO(10) \to SU(5) \times U(1)$, Y is the standard hypercharge. The parameters α and β can be adjusted to fit. This is the crucial point in the analysis, right here DR explain for the differences between up, down, electron and neutrino Yukawa matrices that from our phenomenological approach seem impossible to evade.
The flavon superfields ϕ_a and $\tilde{\phi}_a$ take vevs $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ and $\begin{pmatrix}0\\ \tilde{\phi}_2\end{pmatrix}$ respectively. $\mathbf{A}$ is a $\mathbf{1_B}$ singlet taking a vev A.
Now we should look for D_3 invariants. For example, the leading term of the last part of 7.12 reads $\frac{1}{M_0}(\Phi_{16})_a\Phi_{10}\mathbf{A}(\Phi_{16})_a$, which has D_3 transformation

$$\mathbf{2} \otimes \mathbf{1_A} \otimes \mathbf{1_B} \otimes \mathbf{2} = \mathbf{2} \otimes \mathbf{2}. \quad (7.14)$$

So, on examining 7.6 we expect that this term will contribute to the (12) and (21) entries of the four Yukawa matrices. DR actually have a parameter $\epsilon' = \frac{A}{M_0}$ on

that entry.
This has only been a very hand-waving picture of the analysis that brings DR to their postulated Yukawa matrices. The precise calculation is very complicated. I propose to take out the bottom line: by assuming GUT symmetry breaking vevs like 7.13 and a D_3 family symmetry that treats generations in different ways Dermisek and Raby argue that in their SUSY GUT model there can be differences between the Yukawa matrices y_u, y_d, y_e and y_ν.

7.1.3 DR Dirac Yukawa matrices

When the smoke around their superpotential clears, Dermisek and Raby come up with their Yukawa matrices depending on just 7 parameters, four of which are complex, making up for a total of 11 adjustable real Yukawa textures:

$$y_u = \begin{pmatrix} 0 & \epsilon'\rho & -\epsilon\xi \\ -\epsilon'\rho & \tilde{\epsilon}\rho & -\epsilon \\ \epsilon\xi & \epsilon & 1 \end{pmatrix} \lambda \tag{7.15}$$

$$y_d = \begin{pmatrix} 0 & \epsilon' & -\epsilon\xi\sigma \\ -\epsilon' & \tilde{\epsilon} & -\epsilon\sigma \\ \epsilon\xi & \epsilon & 1 \end{pmatrix} \lambda \tag{7.16}$$

$$y_e = \begin{pmatrix} 0 & -\epsilon' & 3\epsilon\xi \\ \epsilon' & 3\tilde{\epsilon} & 3\epsilon \\ -3\epsilon\xi\sigma & -3\epsilon\sigma & 1 \end{pmatrix} \lambda \tag{7.17}$$

$$y_\nu = \begin{pmatrix} 0 & -\epsilon'\omega & \frac{3}{2}\epsilon\xi\omega \\ \epsilon'\omega & 3\tilde{\epsilon}\omega & \frac{3}{2}\epsilon\omega \\ -3\epsilon\xi\sigma & -3\epsilon\sigma & 1 \end{pmatrix} \lambda. \tag{7.18}$$

Just for the record, we mention the connection between these Yukawa textures and the vevs of the fields in 7.12:

$$\begin{array}{lll} \xi = \frac{\phi_2}{\phi_1} & \epsilon \propto \frac{\phi_1}{\hat{M}} & \sigma = \frac{1+\alpha}{1-3\alpha} \\ \tilde{\epsilon} \propto \frac{\phi_2}{\hat{M}} & \epsilon' \sim \frac{A}{M_0} & \rho \sim \beta. \end{array} \tag{7.19}$$

7.1.4 χ^2 analysis

Dermisek and Raby have succeeded to write a complete SUSY GUT in terms of just 23 parameters. First we have three parameters that set the scale of Grand Unification: M_{GUT}, α_{GUT} and a third parameter that describes a small correction of α_3 as to make coupling constants really unify[3]. Then we have the 11 Yukawa textures. In the sparticle region we have universal sfermion and gaugino masses m_{16}

[3] In other words: DR define the GUT scale at the exact meeting of α_1 and α_2 and manipulate α_3 a bit. This yields $M_{GUT} = 1.5646 \times 10^{16}$ GeV, instead of our all-three-compromis value of 1.5849×10^{16} GeV

and $M_{\frac{1}{2}}$, the soft SUSY breaking parameter A_0 and the Higgs mass parameter μ. In the Higgs sector we furthermore have the two Higgsino masses and $\tan\beta$. Finally we have the three Majorana neutrino masses that provide the seesaw mechanism. Using a (not published) two-loop SUSY RGE framework that runs from M_{GUT} down to the electroweak scale (or even lower, including three loop QCD effects, for light fermion masses) Dermisek and Raby are able to connect their 23 GUT-scale parameters to 23 low-scale observables such as quark and lepton masses, their ratios, low-energy values of coupling constants and CKM and PMNS matrix elements. They then perform a χ^2 analysis as to determine the best fitting values of the GUT-scale parameters and manage to find a surprisingly low χ^2 value. Having these best fitting GUT-scale parameter values in hand, they use them to predict SUSY and Higgs spectra, neutrino masses (light eigenstates), remaining CKM and PMNS elements, a leptogenesis governing parameter ϵ_1, leptonic dipole moments and branching ratios for lepton flavor violating decays.
At the moment more research to the connction between DR's fitted GUT-scale variables and new low energy variables like branching ratios from B-physics is being done [2]. It is not clear whether the DR model will survive all these new tests, but the composition of a predictive SUSY GUT is a major achievement in itself.
In the last bit of this thesis we will try to connect our own results from Yukawa RGE evolution with the DR Yukawa matrices 7.15-7.19.

7.2 Analysis of DR Yukawa matrices

DR fit $\tan\beta = 50.34$. So let us take our working two-loop solution, including quark and lepton mixing, and neutrinos, and check what we get.
DR use the following fit:

$$\begin{aligned}(\lambda, \lambda\epsilon, \sigma, \lambda\tilde{\epsilon}, \rho, \lambda\epsilon', \lambda\epsilon\xi) =&(0.62, 0.03, 0.87, 0.0063, -0.0059, -0.0021, 0.04)\\ (\Phi_\sigma, \Phi_{\tilde{\epsilon}}, \Phi_\rho \Phi_\xi) =&(0.637, 0.453, 0.709, 3.609),\end{aligned} \tag{7.20}$$

where the complex phases are in rad.
On constructing the DR Yukawa matrices and calculating its associated mass eigenstates (taking care of that factor $\frac{1}{4\pi}$ we once relieved our Yukawa matrices from), we arrive at the following mass spectrum:

$$m(M_{GUT}) = \begin{pmatrix} 1.24449 & 240.58 & 108083 \\ 1.85628 & 27.9108 & 2146.4 \\ 0.499149 & 103.085 & 2180.51 \\ 43.794 & 6651.63 & 109324 \end{pmatrix}. \tag{7.21}$$

That is quite a bit off from our fit 6.34. All quark masses have run in the same direction, but in some cases (the up quark, most notably) the distance covered is calculated much smaller by DR than by us. A positive point is that bottom-tau unification, resulting from the GUT-scale DR Yukawa matrices in this case, takes place at almost the same mass and with the same accuracy as we calculated, although in the DR case the bottom mass has already crossed the tau mass.

The most severe troubles are in the up sector. Therefore we have thought of considerably raising the Dirac tau neutrino mass which is, after all, still an unobserved, and therefore adjustable, parameter. We hoped that a huge neutrino contribution could tame the top running. Perhaps they could even meet at a reasonable mass, thus fulfilling DR's top - tau neutrino unification.
However, this turned out to be an unrealizable hypothesis. As we raise the tau neutrino mass, the top mass increases as well. We can actually make these masses meet, but at an absurd mass level. (Remember that DR's top-bottom-tau-tauneutrino unification predicts $m_t = m_{\nu_\tau} = 50m_b = 50m_\tau$). Taking electroweak Dirac neutrino masses of 10, 1000 and 100000 MeV for example yields

$$m(M_{GUT}) = \begin{pmatrix} 1.0809 & 526.92 & 361345 \\ 1.34274 & 25.4754 & 2380.81 \\ 0.494205 & 108.4 & 2414.65 \\ 20.1524 & 2080.18 & 396049 \end{pmatrix}. \qquad (7.22)$$

We can safely conclude that heavy Dirac neutrinos are not going to close the gap between our analysis and DR's. Differences are much more likely to result from different input values or, most probably, from the DR calculation taking into account more effects in their Yukawa running. It would be great to have a look at their code.

Chapter 8

Conclusions

We have come at the end of our trip from the electroweak to the exotic... and while understanding the one issue, the next one would already pop up, leaving us with more insight in Supersymmetric Grand Unification and many more questions. Let us try to make up our mind in this concluding chapter.
We started off by studying the Standard Model. Floating on the fundamental assumptions of gauge invariance, Lorentz invariance and renormalizibility it very accurately describes interactions of all known particles and forces at and around the electroweak scale of 91 GeV. However, apart from the unpleasant arbitrariness in SM parameters, we saw a potential danger: scalar boson masses are unprotected, and should be kept "by hand" from acquiring immense masses. When also the understanding came that parameters such as masses and coupling "constants" actually are functions of the energy probing scale M, we had found a way and a reason to look for physics beyond the Standard Model.
Even if in the SM framework the coupling constants could not be made to fit, (the best fit implied $M_{GUT} = 10^{13}$ GeV), we started exploring a $SU(5)$ GUT, where all fermion fields are in two different multiplets. In the more compact $SO(10)$ framework all fermion fields are embedded in just one irrep. Armed with a great deal of representation theory, we investigated how to compose these unified multiplets, gauge group generators and GUT-Lagrangians. When writing gauge invariant mass terms, we saw how minimal $SU(5)$ predicts partial Yukawa unification, $y_d = y_e$. Full Yukawa unification, $y_u = y_d = y_e = y_\nu$ is promised by $SO(10)$ (the righthanded neutrino can still form a Majorana mass term). We predicted the value of $\sin^2\theta_w$ and connected the third integral quark charges to the fact that quarks come in three colors. However, we found that the hierarchy problem is still present in a GUT.
Then we included supersymmetry. The very elegant SUSY framework, that starts off from extending the Poincare group by anticommuting supergenerators predicts the existence of superpartners (of equal mass) for each known SM field. As these have never been observed, we have to add some SUSY breaking effects. However, these can be chosen such that they lift superpartner masses into regions still to explore while not too much damaging SUSY's greatest prediction: the solution of the hierarchy problem by protecting scalar boson masses by Higgs-sparticle interactions. From the GUT-point of view, these new sparticles were very welcome, their inclusion makes coupling constants really meet, at an energy scale of 10^{16} GeV.

From then, we studied SUSY GUTs. Composing a minimal SUSY GUT was not that hard after having studied non-SUSY GUTs and the SUSY extension of the SM, the Minimal Supersymmetric Standard Model, as long as we focused on the fields already present in the SM rather then the results of all new SUSY interactions. It was proved quite a technical job, however, to investigate whether we could as well make particle masses meet at the GUT-scale. After reviewing how exactly masses are related to Yukawa and mixing matrices, we had Mathematica solve two-loop Renormalization Group Equations. We concluded that, for the case $\tan\beta = 10$ as well as $\tan\beta = 50$, there is strong evidence for partial Yukawa unification in the third generation: at the GUT-scale the bottom quark tends to unify with the tau lepton mass. Full third generation Yukawa unification was not observed: the best result was obtained for $\tan\beta = 50$ where we found $\frac{m_t}{m_b} = 82$ instead of 50. In other generations tendencies towards unification were not seen. We furthermore observed that corrections due to two loop corrections are very small, at most 1%. Corrections resulting from the running of quark and lepton mixing matrices were seen to be even smaller, about $\frac{1}{1000}$.
Led by the observation of some Yukawa unification in the third generation and the absence of any in the first two generations we finally studied the family symmetric model by Dermisek and Raby. Assuming complex interactions, and discriminating between generations, they write down expressions for Yukawa matrices that bear third generation unification. They claim that starting from these Yukawa matrices they can solve RGEs as to very accurately fit and predict low-energy variables. However, there are serious discrepancies between their results and ours. From our point of view, it seems impossible to unify the top mass with the tau neutrino mass at a value only 50 times higher than the bottom-tau mass.

We will conclude these conclusions by again stating that the running of MSSM coupling constants and bottom and tau masses strongly suggest Superysmmetric Grand Unification.

We also look ahead. At the one hand we would like to study the running behaviour of the supersymmetric MSSM parameters as to build a tighter, more sophisticated RGE framework that eventually should be able to do the same as Dermisek and Raby's: connecting a Supersymmetric Grand Unified Theory to an accurate description of low-energy observables. At the other hand we should, once we have properly arrived at the GUT-scale, continue to look up. We still need an explanation for the nature of soft SUSY breaking terms as well as the GUT-scale form of the Yukawas that even in the DR case contain eleven real parameters. Mechanisms at an even higher scale (strings? non-commutative spacetime?) should be studied as to set the next step in the ongoing quest for unification.

Bibliography

[1] B. C. Allanach, A. Dedes, and H. K. Dreiner. 2-loop supersymmetric renormalization group equations including R-parity violation and aspects of unification. *Physical Review D*, 60(5), September 1999.

[2] W. Altmannshofer. Flavor Changing Neutral Current Processes in a SO(10) SUSY GUT with Family Symmetry. *ArXiv e-prints*, 710, October 2007.

[3] C. Amsler. Particle Data Group Live. *Phys. Lett. B*, 1, 2008.

[4] C. Csáki. The Minimal Supersymmetric Standard Model. *Modern Physics Letters A*, 11, 1996.

[5] R. Dermíšek, M. Harada, and S. Raby. SO(10) SUSY GUT for fermion masses: Lepton flavor and CP violation. *Physical Review D*, 74(3), August 2006.

[6] R. Dermíšek and S. Raby. Fermion masses and neutrino oscillations in SO(10) supersymmetric grand unified theory with D3 x U(1) family symmetry. *Physical Review D*, 62(1), July 2000.

[7] R. Dermíšek and S. Raby. Bi-large neutrino mixing and CP violation in an SO(10) SUSY GUT for fermion masses [rapid communication]. *Physics Letters B*, 622, September 2005.

[8] Manuel Drees, Rohini M. Godbole, and Probir Roy. *Theory and Phenomenology of Sparticles.* World Scientific, Singapore, 2004.

[9] P. Langacker. Grand unified theories and proton decay. *Physics Reports*, 72, 1981.

[10] M. Lindner, M. Ratz, and M. A. Schmidt. Renormalization group evolution of Dirac neutrino masses. *Journal of High Energy Physics*, 9, September 2005.

[11] S. P. Martin and M. T. Vaughn. Two-loop renormalization group equations for soft supersymmetry-breaking couplings. *Physical Review D*, 50:2282–2292, August 1994.

[12] J. C. Pati. Grand Unification as a Bridge Between String Theory and Phenomenology. *International Journal of Modern Physics D*, 15, 2006.

[13] M. E. Peskin. Supersymmetry in Elementary Particle Physics. *ArXiv e-prints*, 801, January 2008.

[14] Michael E. Peskin and Daniel V. Schroeder. *An introduction to Quantum Field Theory*. Westview Press, Boulder (Colorado), 1995.

[15] S. Raby. SUSY GUT Model Building. *ArXiv e-prints*, 807, July 2008.

[16] R. Rattazzi, U. Sarid, and L. J. Hall. Yukawa Unification: The Good, The Bad and The Ugly. *ArXiv High Energy Physics - Phenomenology e-prints*, May 1994.

[17] Graham G Ross. *Grand Unified Theories*. The Benjamin/Cummings Publishing Company, Boulder (Colorado), 1984.

[18] A.N. Schellekens. Beyond the Standard Model, lecture notes. February 2006.

[19] R. Slansky. Group theory for unified model building. *Physics Reports*, 79, December 1981.

Appendix A

Conventions

A.1 Essentials

Throughout this thesis we use $+---$ metric:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.1}$$

We thus have $\gamma^0 = \gamma_0$, $\gamma^i = -\gamma_i$.
Next we actually define our gamma-matrices. We choose to employ the Weyl representation:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{A.2}$$

(NB: We are using a shorthand notation: these are four-by-four matrices! Every **1** denotes a two-dimensional unit-matrix.)
We see that in Weyl representation we have

$$\gamma^0 = \left(\gamma^0\right)^\star = \left(\gamma^0\right)^\dagger = \left(\gamma^0\right)^T = \left(\gamma^0\right)^{-1}. \tag{A.3}$$

Next we define the γ^5 operator:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{A.4}$$

We have $\{\gamma^5, \gamma^\mu\} = 0$ and $(\gamma^5)^2 = 1$. From the matrix representation of γ^5 we understand its use in the construction of projection operators. We define

$$P_L = \frac{1-\gamma^5}{2} \qquad P_R = \frac{1+\gamma^5}{2}, \tag{A.5}$$

which, when working on $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, project out the left- and righthanded components:

$$P_L\psi = \psi_L \qquad P_R\psi = \psi_R. \tag{A.6}$$

Next we define the charge conjugation matrix C:

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}. \tag{A.7}$$

We have $C = C^\star = -C^T = -C^\dagger = -C^{-1}$.
C's meaning in life is to define ψ^c, the charge conjugated spinor of ψ:

$$\psi^c \equiv C\overline{\psi}^T \tag{A.8}$$
$$= C\gamma^0\psi^\star. \tag{A.9}$$

Now we want to check the behaviour of ψ^c under gauge transformations. In the first chapter we have seen that the gauge transformation of a quantity f can generally be written as $f \to e^{iT^a\alpha^a}f$, where T^a are the generators of the gauge transformation and α^a specify the gauge transformation.
So, if this is the way in which ψ transforms, ψ^c behaves like:

$$\psi^c \to e^{-i(T^a)^\star\alpha^a}\psi^c = e^{i(-T^a)^\star\alpha^a}\psi^c \equiv e^{i\overline{T^a}\alpha^a}\psi^c. \tag{A.10}$$

We thus conclude that charge conjugated spinors transform in a way "opposite" (or "conjugated") to the transformation of the spinor fields they stem from. For example, up quark fields are in the **3** rep of $SU(3)$, so their charge conjugates, "anti-up" fields, are in the $\overline{\mathbf{3}}$ rep.
(Note that A.10 also goes for fields $\overline{\psi}$.)

Additional useful objects (still working in shorthand notation) are

$$\sigma^\mu = (1, \sigma^i) \tag{A.11}$$
$$\overline{\sigma^\mu} = (1, -\sigma^i). \tag{A.12}$$

From there we define

$$\sigma^{\mu\nu} = \frac{i}{4}\left(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu\right) \tag{A.13}$$
$$\overline{\sigma}^{\mu\nu} = \frac{i}{4}\left(\sigma^\mu\overline{\sigma}^\nu - \overline{\sigma}^\nu\sigma^\mu\right) \tag{A.14}$$

and

$$\Sigma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \overline{\sigma}^{\mu\nu} \end{pmatrix} \tag{A.15}$$

A.2 $SU(2)$ and $SU(3)$

A Lie group is fully characterized by the commutation relations between its generators T^i which are summarized in structure constants f^{ijk}:

$$[T^a, T^b] = f^{abc}T^c. \tag{A.16}$$

$SU(2)$ has structure constants $f^{ijk} = i\epsilon^{ijk}$, where ϵ^{ijk} is the antisymmetric Levi-Civita tensor with $\epsilon^{123} = 1$.
The fundamental rep **2** is chosen $T^i = \frac{\sigma^i}{2}$. σ^i are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.17}$$

The normalization of the Pauli matrices is $\mathrm{Tr}\left(\sigma^a\sigma^b\right) = 2\delta^{ab}$.

The structure constants of $SU(3)$ are given by

$$f^{123} = 1, \qquad f^{147} = f^{516} = f^{246} = f^{257} = f^{345} = f^{637} = \frac{1}{2}, \qquad f^{458} = f^{678} = \frac{1}{2}\sqrt{3} \tag{A.18}$$

plus all even and odd (minus sign) permutations. The fundamental rep **3** is commonly chosen $T^i = \frac{\lambda^i}{2}$. λ^i are the Gell-Mann matrices:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{A.19}$$

The Gell-Mann matrices have the same normalization as the Pauli matrices: $\mathrm{Tr}\left(\lambda^a\lambda^b\right) = 2\delta^{ab}$.

Appendix B

Associating LH fields to RH fields

In 1.1.1 it was shown that $\sigma^2\psi_R^\star$ transforms as a lefthanded spinor and vice versa. This allows us to switch from a representation with LH and RH fields to one with only LH fields, which is very useful when we want to combine several SM multiplets in one GUT-multiplet.
I hope it is clear that we are in no way stating that $\sigma^2\psi_R^\star$ *is* ψ_L, we just remark that $\sigma^2\psi_R^\star$ (or $5\sigma^2\psi_R^\star$, or $-i\sigma^2\psi_R^\star$), has the same Lorentz transformation as a lefthanded spinor χ_L.
We are thus free to associate to each RH field ψ_R a LH field χ_L in the following way:

$$\chi_L = i\sigma^2\psi_R^\star \qquad \Leftrightarrow \qquad \psi_R = -i\sigma^2\chi_L^\star, \tag{B.1}$$

which reads in four-spinor notation

$$\chi_L = C\gamma^0\psi_R^\star \qquad \Leftrightarrow \qquad \psi_R = -\gamma^0 C\chi_L^\star. \tag{B.2}$$

From this last expression we see that this relation between χ and ψ is just charge conjugation.
Now we can ask ourselves how the structure of a kinetic term that involves righthanded fields in the Lagrangian changes under this transformation. We have, in four-spinor notation,

$$\begin{aligned}\overline{\psi_R} i\not{D}\psi_R \to & \overline{\left(-\gamma^0 C\chi_L^\star\right)} i\not{D}\left(-\gamma^0 C\chi_L^\star\right) \\ = & -\chi_L^T C^\dagger \gamma^{0T}\gamma^0 i\not{D}\left(-\gamma^0\right) C\chi_L^\star \\ = & -\chi_L^T C i\left(\partial_\mu - igT^a A_\mu^a\right)\gamma^\mu\gamma^0 C\chi_L^\star;\end{aligned}$$

where in the last step we have explicitly written out $\not{D}$.
As the object under consideration is a scalar after all , we can freely transpose it. One minus sign however does slip in because we are interchanging two fermionic quantities. The T^a are Hermitian.

$$\begin{aligned}= & \chi_L^\dagger C^T\gamma^{0T}\left(i[\overleftarrow{\partial_\mu} - igT^a A_\mu^a]\right)^T C^T\chi_L \\ = & \chi_L^\dagger C\gamma^0 i\gamma^{\mu T}[\overleftarrow{\partial_\mu} - ig(T^a)^\star A_\mu^a] C\chi_L\end{aligned}$$

Now we partially integrate this whole expression as to have the derivative operator work to the right again:

$$
\begin{aligned}
&= -\,\chi_L^\dagger C\gamma^0 i\gamma^{\mu T}\left(\partial_\mu + ig(T^a)^\star A_\mu^a\right)C\chi_L \\
&= \chi_L^\dagger C\gamma^0 C^{-1} C i\gamma^{\mu T}\left(\partial_\mu - ig\overline{T^a}A_\mu^a\right)C^{-1}\chi_L.
\end{aligned}
$$

Here we use the relation $C\gamma^{\mu T}C^{-1} = -\gamma^\mu$ twice:

$$
\begin{aligned}
&= \chi_L^\dagger \gamma^0 i\overline{\not{D}}\chi_L \\
&= \overline{\chi_L} i\overline{\not{D}}\chi_L. \qquad \text{(B.3)}
\end{aligned}
$$

We thus conclude that the form of the kinetic term does not change. The covariant derivative does change, but we expected that: if we replace, for example, d_R fields that are in the **3** rep of $SU(3)$ to d_L^c fields, we expect these fields to transform in the $\overline{\mathbf{3}}$ rep and that is exactly what the new form of the covariant derivative implies.

Finally we investigate what a mass term looks like after a $\psi_R \to \chi_L$ transformation. Writing out, for once, the hermitian conjugate term as well, we have

$$
\begin{aligned}
\overline{\psi_L}M\psi_R - \overline{\psi_R}M^\dagger\psi_L \to &-\,\psi_L^\dagger\gamma^0 M\gamma^0 C\chi_L^\star + \overline{\gamma^0 C\chi_L^\star}M^\dagger\psi_L \\
&= -\,\psi_L^\dagger M C\chi_L^\star + \chi_L^T C^\dagger\gamma^{0\dagger}\gamma^0 M^\dagger\psi_L \\
&= -\,\psi_L^\dagger M C\chi_L^\star - \chi_L^T C M^\dagger\psi_L.
\end{aligned}
$$

Again, we are free to transpose because we have two scalars here. We choose to transpose the second term.

$$
= -\,\psi_L^\dagger M C\chi_L^\star - \psi_L^T M^\star C\chi_L \qquad \text{(B.4)}
$$

We thus conclude that we can rewrite a theory containing left- and righthanded spinors ψ_L and ψ_R to a new one that contains only lefthanded fields ψ_L and χ_L and their conjugates, with the connection between ψ_R and χ_L given by B.2, without spoiling Lorentz invariance. As the new fields χ_L are the charge conjugates of the replaced fields ψ_R, they transform in the opposite rep. The old theory's kinetic and mass terms get replaced by $\overline{\psi_L}i\not{D}\psi_L$ plus B.3 plus B.4.

Appendix C

Additional SM symmetries

We now shortly mention some additional symmetries that have not been postulated but are nevertheless observed in the SM. At first sight it may seem strange that the SM does not explicity demand baryon number B and lepton number L conservation, concepts that seem very natural. ($B(q) = \frac{1}{3}$ for all quarks q, $B(\overline{q}) = -\frac{1}{3}$ for all antiquarks $\overline{q}$ and $B = 0$ for all others. $L = 1$ for all leptons, $L = -1$ for all antileptons.) As long as we do not have righthanded neutrinos we even observe conservation of the "family specific" lepton numbers and L_e, L_μ and L_τ. These symmetries are *not* crucial building blocks of the SM, they just follow from the postulated fundamental symmetries Lorentz invariance, gauge invariance and the demand for renormalizability. Thus, an eventual breakdown would not damage the SM.

Apart from charge conjugation there are two other basic operations we can apply on a Lorentz rep which are denoted P and T.

P describes a parity inverting operation:

$$\begin{aligned} \phi(x) &\rightarrow \phi^P(x) = \eta_P \phi(\tilde{x}) \\ \psi(x) &\rightarrow \psi^P(x) = \eta_P \gamma^0 \psi(\tilde{x}) \\ T^a V^a_\mu(x) &\rightarrow \text{Diag}(1, -1, -1, -1)[T^a V^a_\mu(\tilde{x})], \end{aligned} \tag{C.1}$$

where $\tilde{x}$ denotes $(t, -\overrightarrow{x})$ and η_C are phases.

We also have the operation T of time inversion:

$$\begin{aligned} \phi(x) &\rightarrow \phi^T(x) = \eta_T \phi(-\tilde{x}) \\ \psi(x) &\rightarrow \psi^T(x) = \eta_T \gamma_5 C \psi(-\tilde{x}) \\ T^a V^a_\mu(x) &\rightarrow \text{Diag}(-1, 1, 1, 1)[T^a V^a_\mu(-\tilde{x})]. \end{aligned} \tag{C.2}$$

These are important (but still accidental) symmetries, as can be seen for example from the fact that if $\psi(x)$ obeys the Dirac equation, so do its three C,P and T conjugates.

The Lagrangian of the SM is not invariant under each of these symmetries separately. For example, P converts a left handed fermion field in a right handed one, that is, a field that couples to $SU(2)$ gauge bosons to one that does not.

Troubles arise from the question whether the SM is invariant under the combined operation of C and P. In nature CP violating processes do occur (as was observed in K^0 meson decays) but its origin is still an open question. The only CP violating term in the SM is the one phase factor in the CKM-matrix. (If the CKM matrix would be 2×2 instead of 3×3 there would not be such a factor. This is why the existence of a third generation was actually proposed to explain CP violation.) But it is not clear whether just this one phase factor can account for all CP violation that is observed. We might even have to turn to a more complicated form of the Higgs sector to solve the situation.

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The manuscript, finally, was cheerfully ("neutronis... exciting!") corrected for English grammar and spelling by Laetitia Kolstee.