Cyclic sets as a classifying topos

I. Moerdijk, Utrecht

This note is an exercise in the theory of classifying topoi. We define the notion of an "abstract circle" and prove the result that Connes' category of cyclic sets [C] is a classifying topos for abstract circles. This result is analogous to the wellknown fact that the category of simplicial sets is a classifying topos for abstract intervals (i.e., linear orders with two distinct end points), see e.g. [J] or [MM, §VIII.8]. These two results are in fact closely related, since the category of abstract circles with base point will turn out to be equivalent to that of abstract intervals.

§1. Cyclic sets. We recall the definition of the category $\mathbf{\Lambda}$ introduced by A. Connes [C]: The objects of $\mathbf{\Lambda}$ are the natural numbers $\mathbf{0}, \mathbf{1}, \ldots$. An arrow $\mathbf{n} \to \mathbf{m}$ is a pair (σ, u) where $\sigma \in C_{n+1}$ is a cyclic permutation of $\{0, \ldots, n\}$ and $u : \mathbf{n} \to \mathbf{m}$ is a non-decreasing function $\{0, \ldots, n\} \to \{0, \ldots, m\}$. Composition in $\mathbf{\Lambda}$ is defined by using the fact that for σ as above and any non-decreasing function $v : \mathbf{k} \to \mathbf{n}$, the composition $\sigma \circ v$ can be factored in a unique way as $v' \circ \sigma'$ where $\sigma' = v^*(\sigma) \in C_{k+1}$ is monotone on fibers of v and $v' = \sigma_*(v) : \mathbf{k} \to \mathbf{n}$ is non-decreasing. Then the composition of $(\sigma, u) : \mathbf{n} \to \mathbf{m}$ and $(\tau, v) : \mathbf{k} \to \mathbf{n}$ is $(\sigma' \circ \tau, u \circ v')$. Notice that $\mathbf{\Lambda}$ contains the simplicial category $\mathbf{\Delta}$ as the subcategory of those arrows (σ, u) where $\sigma = 1$ (we denote the cyclic group C_{n+1} multiplicatively).

An *(abstract) interval* is a linearly ordered set $(L, \leq)$ with distinct smallest and largest elements b and t. The intervals form a category with as arrows those functions which preserve $\leq$, b, and t. Recall [MM, p. 453] that the opposite category $\mathbf{\Delta}^{op}$ is equivalent to the category of *finite* intervals $\mathbf{n}^+ = (0 \leq 1 \leq \ldots \leq n+1)$.

The category $\mathbf{\Lambda}$ is equivalent to its dual $\mathbf{\Lambda}^{op}$, which can conveniently be viewed as follows: The object $\mathbf{n}^*$ of $\mathbf{\Lambda}^{op}$ (corresponding to $\mathbf{n}$ in $\mathbf{\Lambda}$) is the circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ with the cyclic group (roots of unity) $C_{n+1} \subseteq S^1$ as a set of marked points. An arrow $\mathbf{m}^* \to \mathbf{n}^*$ is a homotopy class of degree 1 maps $f : S^1 \to S^1$ which sends marked points to marked points. One can think of the marked points $\{0, \ldots, n\} = C_{n+1} \subseteq S$ as the linear order $\{0, \ldots, n+1\}$ with 0 and $n+1$ identified. Then any map of intervals $u : \mathbf{m}^+ \to \mathbf{n}^+$ determines a unique arrow $\bar{u} : \mathbf{m}^* \to \mathbf{n}^*$, with the additional property that it preserves the marked point 0. For an arbitrary arrow $f : \mathbf{m}^* \to \mathbf{n}^*$ in $\mathbf{\Lambda}^{op}$, let $\sigma \in C_{n+1}$ be the rotation of S^1 so that $\sigma \circ f$ preserves the marked point 0. Thus $f = \sigma^{-1} \circ \bar{u}$ for a map of intervals u. In this way, and dual to the description of $\mathbf{\Lambda}$ above, any arrow $f : \mathbf{m}^* \to \mathbf{n}^*$ is decomposed as $\sigma^{-1} \circ \bar{u}$ where $\sigma \in C_{n+1}$ and u is an arrow in $\mathbf{\Lambda}^{op}$.

§2. Abstract circles. An abstract circle C is a structure

$$C = (P, S, \partial_0, \partial_1, 0, 1, *, \cup).$$

Here P and S are sets, the elements of which will be called *points* and *segments* respectively. Furthermore, $\partial_0, \partial_1 : S \to P$ and $0, 1 : P \to S$ are functions, $* : S \to S$ is an involution, and $\cup : S \times S \to S$ is a partial function. (One can also think of $\cup$ as a ternary relation R, by

$R(a,b,c)$ iff $a \cup b = c$.) Elements of P are denoted $x, y, z, \ldots$ and elements of S are denoted $a, b, c, \ldots$. The axioms are the following.

1. (non-triviality) P contains at least one point, and for any two $x, y \in P$ there is at least one $a \in S$ with $\partial_0 a = x$ and $\partial_1 a = y$. For any $x \in P$, the segments $0(x) = 0_x$ and 1_x are distinct.

2. (equational axioms) $a^{**} = a$, $\partial_0(a^*) = \partial_1(a)$, $\partial_0(0_x) = x = \partial_1(0_x)$; $0_x^* = 1_x$; if $\partial_0 a = \partial_1 a = x$ then $a = 0_x$ or $a = 1_x$.

3. (axioms for concatenation)

   (i) $a \cup b$ exists only if $\partial_1 a = \partial_0 b$, and in that case $\partial_1(a \cup b) = \partial_1 b$, $\partial_0(a \cup b) = \partial_0 a$.

   (ii) $a \cup b = c$ iff $c^* \cup a = b^*$.

   (iii) if $a \cup b$ and $(a \cup b) \cup c$ exist then so do $b \cup c$ and $a \cup (b \cup c)$, and $(a \cup b) \cup c = a \cup (b \cup c)$.

   (iv) $a \cup b = 0_x \Rightarrow a = 0_x$.

   (v) if $\partial_0 a = x$ then $0_x \cup a = a$.

   (vi) if $\partial_1 a = \partial_0 b$ then at least one of $a \cup b$ and $b^* \cup a^*$ exists.

A *homomorphism* $f : C \to C'$ of abstract circles consists of two function $f : P \to P'$ and $f : S \to S'$ which commute with all the operations $\partial_0, \partial_1, 0, 1, *$ and $\cup$. (For $\cup$, this means that if $a \cup b \in S$ is defined then so is $f(a) \cup f(b)$ and the latter equals $f(a \cup b)$.)

It is easy to prove various elementary properties of abstract circles, such as the following ($a, b, c, \ldots$ denote segments, as before):

(i) If $\partial_0 a = \partial_0 b$ and $\partial_1 a = \partial_1 b$ then $a = b$ or $a = b^*$.

(ii) Define $a \subseteq b$ if $\exists u, v \in S : u \cup a \cup v = b$. If $\partial_0 a = \partial_0 b = x$, it then follows that $u = 0_x$. Furthermore, the subset $L_x = \{a \in S : \partial_0 a = x\}$ is linearly ordered by $\subseteq$, with smallest and largest elements 0_x and 1_x. This construction $(C, x) \mapsto L_x$ in fact defines an *equivalence of categories* between abstract circles with basepoint and abstract intervals.

(iii) ("Refinement") If $a \subseteq b \cup c$ then $a \subseteq b$ or $a \subseteq c$, or $\exists u, v \in S$ ($u \subseteq b$ and $v \subseteq c$ and $a = u \cup v$).

Example. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the "concrete" circle, and let $P \subseteq S^1$ be any set of points. Let S be the set of all positively oriented closed segments on S^1 with endpoints in P. Then (S, P) is an abstract circle, with structure maps defined as follows: ∂_0 and ∂_1 are the endpoints, $a^* = S^1 - Int(a)$, $0_x = \{x\}$ and $1_x = S^1$, while $a \cup b$ is the union of the segments a and b, defined only if $\partial_1 a = \partial_0 b$ and a, b have no interior point in common. We denote this abstract circle by $C(P)$.

This example is "universal" in the following sense. Consider the axioms for abstract circles as a first order theory. By the Löwenheim–Skolem theorem, every model has a countable elementary submodel. Furthermore, it is not difficult to see that any countable model is isomorphic to a model of the form $C(P)$ as constructed in the example, for a countable set $P \subseteq S^1$. (One way to prove this is to use the equivalence of categories in (ii) and the fact that every countable abstract interval can be embedded into $[0, 1] \subseteq \mathbf{R}$.) This observation yields:

Proposition. *Let φ be a first order formula in the language of abstract circles. Then φ is a consequence of the axioms, iff φ holds in all abstract circles of the form $C(P)$ where P is a (countable) subset of S^1.*

§3. Cyclic sets classifies abstract circles. We now turn our attention to (models of the axioms for) abstract circles in topoi – or in other words, sheaves of abstract circles on a site. For example, for a circle bundle $S^1 \subseteq E \to X$ on a paracompact space X, any subsheaf $P \subseteq \Gamma(E)$ defines a sheaf $C(P)$ of abstract circles on the site $\mathcal{C}(X)$ of closed subsets of X and locally finite covers. (For X a point, this is the example in §2.)

The topos (*cysets*) of *cyclic sets* is the topos of presheaves on $\mathbf{\Lambda}$ (i.e. functors $\mathbf{\Lambda}^{op} \to$ *sets*). The result announced in the title is the following.

Theorem. *For any topos $\mathcal{T}$, there is an equivalence of categories*

$$\mathrm{Hom}(\mathcal{T}, (cysets)) \cong (\text{Abstract circles in } \mathcal{T}), \tag{1}$$

natural in $\mathcal{T}$.

Proof. We first check that the equivalence (1) holds for the case where $\mathcal{T} = (sets)$. In this case, the proposition asserts that points of the topos (*cysets*) correspond to abstract circles (in sets).

It is known ("Diaconescu's Theorem", [MM, Ch.VII]) that points $(sets) \to (cysets)$ correspond to filtering functors $F : \mathbf{\Lambda} \to (sets)$. These are functors having the following three properties:

(i) $F(\mathbf{0}) \neq \phi$.

(ii) If $a \in F(\mathbf{n})$ and $f, g : \mathbf{n} \to \mathbf{m}$ are such that $F(f)(a) = F(g)(a)$ then there are an $h : \mathbf{k} \to \mathbf{n}$ and $b \in F(\mathbf{k})$ with $fh = gh$ and $F(h)(b) = a$.

(iii) If $a \in F(\mathbf{n})$ and $b \in F(\mathbf{m})$ then there are $f : \mathbf{k} \to \mathbf{n}$, $g : \mathbf{k} \to \mathbf{m}$ and $c \in F(\mathbf{k})$ so that $F(f)(c) = a$ and $F(g)(c) = b$.

Given an abstract circle C, one constructs such a functor $F = F_C$ as follows. Observe first that the dual category $\mathbf{\Lambda}^{op}$ (see §1) is exactly the category of finite abstract circles $\mathbf{n}^*$. Define F_C by $F_C(\mathbf{n}) = \mathrm{Hom}(\mathbf{n}^*, C)$, (where Hom is taken in the category of abstract circles). Thus $F_C(\mathbf{0}) = P$ is the set of points of C, while for $n > 0$ an element of $F_C(\mathbf{n})$ is the same as a sequence $(a_0, a_1, \ldots, a_n)$ of segments so that

$$a_0 \cup a_1 \cup \ldots \cup a_n = 1 \qquad (\text{where } 1 = 1_{\partial_0(a_0)}).$$

The conditions (i)-(iii) are easily verified: (i) is clear. For (ii), suppose

$$\mathbf{m}^* \underset{g}{\overset{f}{\rightrightarrows}} \mathbf{n}^* \xrightarrow{a} C$$

is a commutative diagram of abstract circles. Factor a as a surjection $h : \mathbf{n}^* \to D$ and an injection $b : D \to C$. Then D is a finite abstract circle, hence $D = \mathbf{k}^*$ for some $k \geq 0$. Thus $hf = hg$ in $\mathbf{\Lambda}^{op}$ and $b = ah$. Finally, (iii) follows by constructing a common refinement of the "covers" $a_0 \cup \ldots \cup a_n$ and $b_0 \cup \ldots \cup b_m$ using repeated application of the refinement property in §2.

For the converse construction, of an abstract circle C_F out of a filtering functor F, express F in the canonical way as a colimit of representables,

$$F = \lim_{\longrightarrow i \in I} \Lambda(U(i), -) ,$$

where I is the dual of the Grothendieck construction $\int_\Lambda F$ and $U : I \longrightarrow \Lambda^{op}$ is the dual of the projection $\pi : \int_\Lambda F \longrightarrow \Lambda$. Using again that Λ^{op} is the category of finite abstract circles, each $U(i)$ can be viewed as a finite abstract circle, and we define $C_F = \lim_{\rightarrow i \in I} U(i)$; this definition makes sense since the category of abstract circles has filtered (or directed) colimits.

These two constructions are inverse to eachother, and show the equivalence (1) of the proposition in case $\mathcal{T} = (sets)$.

For general $\mathcal{T}$, the proposition now follows using results of Makkai and Reyes. Indeed, the theory of abstract circles is a *coherent theory*, and has a coherent classifying topos, say $\mathcal{A}$. This means by definition that there is an equivalence like (1) with $\mathcal{A}$ for (*cysets*). We need to show that $\mathcal{A}$ is equivalent to the topos (*cysets*). Consider for each $\mathbf{n}$ the finite abstract circle $\mathbf{n}^*$. This defines a functor $C : \Lambda^{op} \longrightarrow$ (abstract circles), which is in fact an abstract circle C in the topos (*cysets*). Let $\chi_C : (cysets) \longrightarrow \mathcal{A}$ be its classifying morphism. To prove that χ_C is an equivalence, it suffices by [MR, Thm. 9.2.9] to prove that composition with χ_C induces an equivalence between the categories of points of these topoi. But Points($\mathcal{A}$) is the category of abstract circles by definition of $\mathcal{A}$, and the required equivalence Points(*cysets*) $\longrightarrow$ Points($\mathcal{A}$) is exactly the equivalence (1) of the proposition for $\mathcal{T} = (sets)$, already shown.

Remark. Recall that Λ contains the simplicial category Δ as a subcategory. Write $j : \Delta \hookrightarrow \Lambda$ for the inclusion functor, and also for the induced morphism of topoi

$$j : (ssets) \longrightarrow (cysets)$$

from *simplicial sets* to *cyclic sets*. Then j is a local homeomorphism. In fact, there is an equivalence of topoi $(cysets)/\Lambda(0) \cong (ssets)$. Given the proposition, this observation is equivalence to the fact (§2) that the category of pointed abstract circles is equivalent to the category of abstract intervals.

References

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