THE NEWLANDER-NIRENBERG THEOREM

BEN MCMILLAN

Abstract. For any kind of geometry on smooth manifolds (Riemannian, Complex, foliation, ...) it is of fundamental importance to be able to determine when two objects are isomorphic. The method of equivalence using G-structures is a systematic way to classify the local invariants of a particular geometry. For example, these ideas can be used to show that the only local invariants of a Riemannian manifold are the curvature form and its derivatives.

In the case of almost complex structures we arrive at the Nijenhuis tensor. The Newlander-Nirenberg theorem states that this tensor vanishes exactly when an almost complex structure arises from a complex structure.

In this talk I introduce G-structures and use them to develop a sketch of the proof of the Newlander-Nirenberg theorem.

1. G-structures

One of the fundamental problems in geometry is that of equivalence, the problem of determining when two objects in a geometric category are isomorphic. While there is no general definition of what a 'geometric' structure on manifolds is, there are common features which all of the classical geometric structures have. One of the fundamental ways to capture what is meant by a 'geometric' structure is a G-structure, where G is a Lie group. The choice of G determines a kind of geometry, in the sense that G is the group of 'local symmetries' of your geometry, or the group which preserves the framings compatible with a geometry. To define what a G-structure is we need first

Definition. Given a smooth manifold M^n, a coframe at $x \in M$ is an isomorphism $u\colon T_xM \to \mathbb{R}^n$. We denote the set of coframes based at x by $\mathcal{F}_x^{\mathrm{GL}}$. The frame bundle $\mathcal{F}^{\mathrm{GL}}$ is given by $\cup_{x\in M}\mathcal{F}_x^{\mathrm{GL}} \subset T^*M \otimes \mathbb{R}^n$. This embedding gives us the subset topology and projection map π which sends $u \in \mathcal{F}_x^{\mathrm{GL}}$ to x.

It is not hard to see that this is a principal $\mathrm{GL}(\mathbb{R}^n)$ bundle. Indeed, for $u, u' \in \mathcal{F}_x^{\mathrm{GL}}$ the map $A = u'u^{-1} \in \mathrm{GL}(\mathbb{R}^n)$ is the unique element of $\mathrm{GL}(\mathbb{R}^n)$ so that $Au = u'$.

Definition. Let $G \subseteq \mathrm{GL}(\mathbb{R}^n)$ be a Lie subgroup. A G-structure on M^n is a principal G subbundle of $\mathcal{F}^{\mathrm{GL}}$.

Example. Suppose M^{2n} has an almost complex structure J, i.e. an endomorpism of TM so that $J^2 = -I$ at each point. Recall that $\mathbb{R}^{2n} \cong \mathbb{C}^n$ has a canonical complex structure given by multiplication with i. We consider the subset $\mathcal{F}$ of coframes $u \in \mathcal{F}^{\mathrm{GL}}$ so that $u(Jv) = iu(v)$ for all $v \in T_{\pi(u)}M$. We may write any two coframes in $\mathcal{F}$ as u and Au for some $A \in GL(\mathbb{R}^{2n})$, and then

$$iAu(v) = Au(Jv) = Aiu(v).$$

Since this holds for all v, and u is onto, we have $A \in \mathrm{GL}(n, \mathbb{C})$. Thus we see that $\mathcal{F}$ is a principal $\mathrm{GL}(n, \mathbb{C})$ subbundle of $\mathcal{F}^{\mathrm{GL}}$. Conversely, given a principal $\mathrm{GL}(n, \mathbb{C})$ subbundle $\mathcal{F}$ we can determine an almost complex structure J at each point by requiring $u(Jv) = iu(v)$ for some $u \in \mathcal{F}_x$ and all $v \in T_xM$. Because $\mathcal{F}$ is a $\mathrm{GL}(n, \mathbb{C})$ bundle, J will be independent of the choice of coframing u.

At the moment this seems like a case of taking a simple structure and turning it into a complicated bundle, but there are several benefits to taking this perspective.

2. The fundamental structure equation

Given a G-structure B we can define a very natural 1-form, the tautological form $\omega\colon TB \to \mathbb{R}^n$, by

$$\omega(v) = u(\pi'_u(v)) = (\pi^* u)(v)$$

for $v \in T_uB$. Since ω maps to $\mathbb{R}^n$, this is a vector of n 1-forms, $\omega = (\omega^1, \ldots, \omega^n)^T$.

The tautological form gives us a partial coframing of B, and we would like to extend it to a full coframing. By definition this form is zero exactly on the vertical vectors[1]. But B is a principal G-bundle, so the obvious way to parameterize the vertical directions is with a $\mathfrak{g}$ valued connection. And here we have the theorem

Theorem (Cartan's first structure equation)**.** *For a G-structure B with tautological form ω and any psuedo-connection $\theta \in \Omega^1(B, \mathfrak{g})$ there is a map $T\colon B \to \mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$, called torsion, so that*

$$d\omega = -\theta \wedge \omega + T(\omega \wedge \omega)$$

at each point of B.

A few remarks:

1) I remind the reader that $\omega \wedge \omega$ is not zero, since ω takes values in $\mathbb{R}^n$. In general, for two $V \cong \mathbb{R}^n$ valued 1-forms ω and η, the wedge product $\omega \wedge \eta$ takes values $V \otimes V$. However, since we wedge ω with itself, the image will lie in the subspace $\Lambda^2 V$.

2) There is no guarantee of uniqueness. A different choice θ will result in different torsion T. In general we try to find a choice of θ which absorbs T, or failing that, one which leaves it as reduced as possible and hopefully in some canonical form.

3) In a sense which will soon be illustrated by the almost complex case, torsion is a first order measure of the failure for B to be 'integrable'. The Newlander-Nirenberg theorem says that an almost complex structure satisfies the integrability condition to be a complex structure exactly if there is a choice of θ for which the torsion is 0.

Similarly, for an almost-symplectic structure (I.e. a manifold with non-degenerate 2-form, not necessarily closed. Here G is the symplectic group) the space of possible torsions is naturally isomorphic to $\Lambda^3\mathbb{R}^n$. Under this isomorphism the torsion is the exterior derivative of the almost symplectic 2 form. Presupposing all of this, it is clear that an almost symplectic structure is symplectic exactly when its torsion vanishes.

Example (Reduction of torsion in an almost complex manifold)**.** We continue to follow the example of our almost complex manifold M. The fiber of $\mathcal{F}$ is $G = \mathrm{GL}(n, \mathbb{C})$, which has Lie algebra $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. In this case the form $\omega = (\omega^1, \ldots, \omega^n)^T$ where each ω^i is complex valued. Cartan's structure equation takes the form

$$d\omega^i = -\theta^i_j \wedge \omega^j + T^i_{jk}\omega^j \wedge \omega^k + T^i_{j\bar{k}}\omega^j \wedge \bar{\omega}^k + T^i_{\bar{j}\bar{k}}\bar{\omega}^j \wedge \bar{\omega}^k.$$

Now, by judicious choice of $a^i_{jk}, a^i_{j\bar{k}} \in \mathbb{C}$ we can find a new pseudo-connection $\tilde{\theta}^i_j = \theta^i_j + a^i_{jk}\omega^k + a^i_{j\bar{k}}\bar{\omega}^k$ to cancel out all of the T^i_{jk} and $T^i_{j\bar{k}}$ terms. In other words, there is a psuedo-connection $\tilde{\theta}$ so that

$$d\omega^i = -\tilde{\theta}^i_j \wedge \omega^j + T^i_{\bar{j}\bar{k}}\bar{\omega}^j \wedge \bar{\omega}^k.$$

[1]In a general fiber bundle $\pi\colon E \to M$ the vertical vectors are the elements annihilated by π'. In particular, these are the directions which correspond to staying in the fiber.

No further reduction of the torsion is possible since no choice of θ can produce $\bar{\omega}^j \wedge \bar{\omega}^k$ terms.

3. The Newlander-Nirenberg theorem

In general it is not possible to choose a psuedo-connection for which the torsion vanishes completely. However, if our almost complex structure J comes from a complex chart then the torsion does vanish. Indeed, at any point $p \in M$ let U be a neighborhood and $\phi\colon U \to \mathbb{C}^n$ a J-holomorphic map. By definition the map $d\phi\colon TU \to \mathbb{C}^n$ satisfies $d\phi(Jv) = i \cdot d\phi(v)$ for all $v \in T_pM$, so $\eta(x) = d\phi_x$ defines a section of $\mathcal{F}|_U$. Then the tautological form can be written[2] as $\omega_u = g(u)^{-1}\eta_x$, where $\pi(u) = x$ and $g(u)$ is the unique element of $\mathrm{GL}(n, \mathbb{C})$ so that $g(u)^{-1}\eta_x = u$. The fundamental structure equation is then of the form

$$d\omega = -g^{-1}dgg^{-1} \wedge \eta = -\theta \wedge \omega.$$

We choose $\theta = g^{-1}dg$ so that $T \equiv 0$.

Conversely, we have

Theorem (Newlander-Nirenberg). *For an analytic almost complex structure J, if $T \equiv 0$, then there is a complex atlas on M holomorphic with respect to J, i.e. one so that $d\phi(Jv) = i \cdot d\phi(v)$.*

sketch of proof. Fix η^i, a J-holomorphic framing of M. Our plan is to show that we can 'integrate' this to a new coframing $dz^1, \ldots, dz^n$ which is also J-holomorphic.

Consider the manifold $M \times \mathbb{C}^n$ with coordinates $p_1, \ldots, p_n$ in the second factor and the ideal $\mathcal{I} = \{d\zeta\}$ where $\zeta = p_1\eta^1 + \ldots + p_n\eta^n$. The form ζ is holomorphic as a linear combination of holomorphic forms. For notation I will take e_i be the i^{th} coordinate vector of $\mathbb{C}^n$.

Claim: If $T \equiv 0$ then through each point of $M \times \mathbb{C}^n$ there is an integral $2n$-manifold[3] of $\mathcal{I}$ on which $\eta^1 \wedge \bar{\eta}^1 \wedge \ldots \wedge \eta^n \wedge \bar{\eta}^n \neq 0$.

Suppose first that the claim is true. Fix a point $q \in M$ and consider the integral manifold through (q, e_1). Because of the independence condition $\eta^1 \wedge \bar{\eta}^1 \wedge \ldots \wedge \eta^n \wedge \bar{\eta}^n \neq 0$ this manifold is the graph of a function $(\tilde{p}_1, \ldots, \tilde{p}_n)\colon M \to \mathbb{C}^n$. Then the form $\zeta^1 = \tilde{p}_1\eta^1 + \ldots + \tilde{p}_n\eta^n$ is closed, so in some neighborhood U of q there is a function $z^1 : U \to \mathbb{C}$ so that $dz^1 = \zeta^1$. Note that $(\tilde{p}_1, \ldots, \tilde{p}_n)(q) = e_1 = (1, 0, \ldots, 0)$, so $dz^1(q) = \eta^1(q)$. Similarly, from the integral manifold through (q, e_i) we get holomorphic z^i so that $dz^i(q) = \eta^i(q)$. In particular we have $dz^1 \wedge \ldots \wedge dz^n(q) = \eta^1 \wedge \ldots \wedge \eta^n(q) \neq 0$, so $dz^1 \wedge \ldots \wedge dz^n \neq 0$ in a neighborhood of q. So we have found a holomorphic chart near q.

[2]Any section of a principal bundle gives a trivialization. It is an exercise in unwinding the definitions to show that $\omega_u = g(u)^{-1}\eta_x$. One point of caution: η_x lives on M, but we can pull it back to $\mathcal{F} = M \times \mathrm{GL}(n, \mathbb{C})$ by the projection onto the M term. Traditionally we omit pullbacks when the context allows it.

[3]An integral manifold manifold is one on which the form $d\zeta$ restricts to be zero on each tangent plane.

To prove the claim we use Cartan's test[4]. From Cartan's structure equation we compute that

$$\begin{aligned} d\zeta &= dp_i \wedge \eta^i + p_j d\eta^j \\ &= dp_i \wedge \eta^i + p_j(-\theta_i^j \wedge \eta^i + \cancel{T^i_{\bar{j}\bar{k}}\bar{\eta}^j \wedge \eta^k}) \\ &= \pi_i \wedge \eta^i \end{aligned}$$

where $\pi_i = dp_i - p_j\theta_i^j$ are independent of the η^i and $\bar{\eta}^i$. Without going into any more detail about exterior differential systems, this is a very nice form for the equations to take. It means that by a basic application of Cartan's test this system is involutive, which is a fancy way of saying that there is an integral manifold through each point. $\square$

In the case where torsion is not zero we would have instead

$$d\zeta = \pi_i \wedge \eta^i + p_i T^i_{\bar{j}\bar{k}}\bar{\eta}^j \wedge \bar{\eta}^k.$$

The extra term means that the claim in the proof does not hold. For example, if $T^1_{\bar{1}\bar{2}} = 1$ at a point q then an integral manifold through (q, e_1) would simultaneously have $\bar{\eta}^1 \wedge \bar{\eta}^2 = 0$ and $\eta^1 \wedge \bar{\eta}^1 \wedge \ldots \wedge \eta^n \wedge \bar{\eta}^n \neq 0$ at q, a contradiction. It is in this sense that torsion is the obstruction to integrating an almost complex structure.

4. The classical Newlander-Nirenberg theorem

There is a more classical statement of the Newlander-Nirenberg theorem which says that an almost complex structure is integrable exactly if the Nijenhuis tensor vanishes. To get this we need to go through a but of complex geometry.

Since $J^2 = -I$, the eigenvalues of J must be $\pm i$. We would like to split TM into its eigenspaces, so we complexify. Consider $T^{\mathbb{C}}M = \mathbb{C} \otimes TM$ and extend the bracket on TM to sections of $T^{\mathbb{C}}M$ by $\mathbb{C}$-linearity. We define

$$T^{1,0}M = \{X - iJX \colon X \in \mathfrak{X}(M)\}$$

and

$$T^{0,1}M = \{X + iJX \colon X \in \mathfrak{X}(M)\},$$

the eigenspaces of J. For example,

$$J(X + iJX) = JX - iX = -i(X + iJX).$$

We thus have the decomposition $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$. There is a dual decomposition $\mathbb{C} \otimes \Omega^1(M) = \Omega^{0,1}(M) \oplus \Omega^{1,0}(M)$

We continue to take η^i a J-holomorphic framing of M. Extend η^i to $T^{\mathbb{C}}M$ by $\mathbb{C}$-linearity and consider it as an element of $\mathbb{C} \otimes \Omega^1(M)$. For $W \in T^{0,1}M$ we have

$$i\eta^i(W) = \eta^i(JW) = \eta(-iW) = -i\eta^i(W),$$

so each η^i is zero on $T^{0,1}M$. By dimension count we see that $T^{0,1}M = \{\eta^1, \ldots, \eta^n\}^{\perp}$. In other words, a 1-form is holomorphic if and only if it is zero for all $W \in T^{0,1}M$ and $W \in T^{0,1}M$ if and only if $\eta(W) = 0$ for all holomorphic η.

[4]This is different from Cartan's structure equation. If you do not know Cartan's test, or anything about the theory of exterior differential systems you can take the claim on faith. Or check out my crash course introduction at `http://math.berkeley.edu/~bmcmilla/Talks/EDS%20-%20Riemannian%20surfaces%20embed.pdf` which also has the punchline of proving that any Riemannian manifold has an isometric embedding in Euclidean space. The definitive source is Robert Bryant's '9 Lectures in Exterior Differential Systems,' googleable.

Proposition. *For an almost complex structure, $T \equiv 0$ if and only if $[T^{0,1}M, T^{0,1}M] \subseteq T^{0,1}M$.*

Proof. We fix $W_1, W_2 \in T^{0,1}M$ and evaluate $d\eta^i(W_1, W_2)$ in 2 different ways:

$$\begin{aligned} d\eta^i(W_1, W_2) &= -\theta^i_j \wedge \eta^j(W_1, W_2) + T^i_{\bar{j}\bar{k}} \bar{\eta}^j \wedge \bar{\eta}^k(W_1, W_2) \\ &= W_1\eta^i(W_2) - W_2\eta^i(W_1) - \eta^i([W_1, W_2]) \end{aligned}$$

Because η^i are holomorphic this simplifies to

$$T^i_{\bar{j}\bar{k}} \bar{\eta}^j \wedge \bar{\eta}^k(W_1, W_2) = \eta^i([W_1, W_2]).$$

If $T \equiv 0$ then $\eta^i([W_1, W_2]) = 0$ for all i, which implies $[W_1, W_2] \in T^{0,1}M$ for all $W_1, W_2 \in T^{0,1}M$. Conversely, if $[W_1, W_2] \in T^{0,1}M$ for all $W_1, W_2 \in T^{0,1}M$ then $T^i_{\bar{j}\bar{k}} \bar{\eta}^j \wedge \bar{\eta}^k(W_1, W_2) = 0$ for all W_1, W_2. □

To derive the Nijenhuis tensor, suppose we have $X, Y \in \mathfrak{X}(M)$. Then

$$\begin{aligned} W &= [X + iJX, Y + iJY] \\ &= [X, Y] - [JX, JY] + i([X, JY] + [JX, Y]). \end{aligned}$$

The vector W is in $T^{0,1}M$ exactly if $JW = -iW$, or

$$J[X, Y] - J[JX, JY] + i(J[X, JY] + J[JX, Y]) = [X, JY] + [JX, Y] - i([X, Y] - [JX, JY])$$

So, if we define

$$N(X, Y) = J[X, Y] - J[JX, JY] - [X, JY] - [JX, Y])$$

we have $[T^{0,1}M, T^{0,1}M] \subseteq T^{0,1}M$ exactly when $N(X, Y) = 0$ for all $X, Y \in \mathfrak{X}(M)$.