T-duality : a basic introduction

Type IIA $\Longleftrightarrow$ Type IIB

10th Geometry and Physics conference, *Quantum Geometry*

Anogia, 6-10 August 2012

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Collaborators and reference

joint work with:

- Peter Bouwknegt, ANU;

- Jarah Evslin, Beijing;

[BEM1]

P. Bouwknegt, J. Evslin and V. M.,
T-duality: Topology Change from H-flux,
Communications in Mathematical Physics,
249 no. 2 (2004) 383-415. [arXiv:hep-th/0306062]

[BEM2]

P. Bouwknegt, J. Evslin and V. M.,
On the Topology and Flux of T-Dual Manifolds,
Physical Review Letters, **92**, 181601 (2004)

Other related followup works

- D. Baraglia

- U. Bunke, T. Schick

- P. Bouwknegt, K. Hannabuss, V.M.

- V.M., J. Rosenberg

- V.M., Siye Wu

etc.

The idea of T-duality

The simplest example is a sigma model on a torus $\mathbb{T}^n = \mathbb{R}^n/\Gamma$ of radius equal to r, where Γ is a lattice in $\mathbb{R}^n$.
The (topological) partition function is a **theta function**,

$$Z_\Gamma(r) = \sum_{z \in \widehat{\Gamma}} e^{-2\pi^2 r |z|^2}$$

where $\widehat{\Gamma}$ is the dual lattice in the dual vector space $\widehat{\mathbb{R}}^n$.
By the **Poisson summation formula**, this is equivalent to the partition function $Z_{\widehat{\Gamma}}$ on the dual torus $\hat{\mathbb{T}}^n = \widehat{\mathbb{R}}^n/\widehat{\Gamma}$, and

$$r \Longleftrightarrow 1/r.$$

The situation however gets much more complicated when a background flux H is turned on.

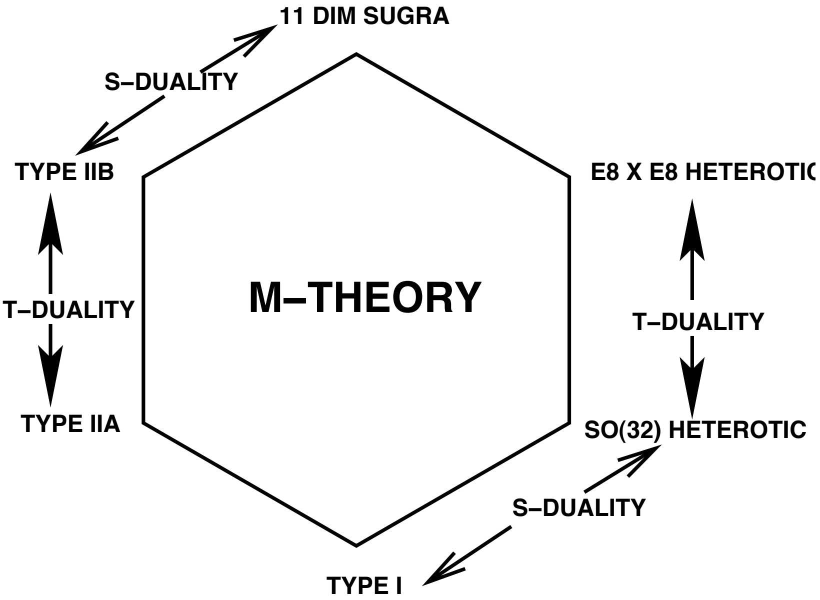
String theory in a background flux

String theory does not currently have a complete definition.
What we have instead are a set of partial definitions.
A question naturally arises given this state of affairs.

- Is each partial definition consistent with the others, via string dualities?

We will be concerned with 2 of the 6 known manifestations of string theory.

string theory and dualities



Type II A string theory

Data for a partial definition for Type IIA string theory is:

Let E be spacetime:

1. A **background H-flux** $H \in \Omega^3(E)$, $dH = 0$ with integral periods;
2. A **Riemannian metric** g on E satisfying the **Einstein-Maxwell field equations**,
$$\mathrm{Ric}_{ij} = \frac{1}{4} \sum_{p,q} H_{ipq} H_j^{pq};$$
3. A **Ramond-Ramond (RR) field** $G \in \Omega^{even}(E)$, satisfying the equations of motion, $(d - H\wedge)G = 0$;
4. A **complex-valued dilaton + axion**.

Type IIB string theory

Data for a partial definition for Type IIB string theory is:

Let E be spacetime:

1. A **background H-flux** $H \in \Omega^3(E)$, $dH = 0$ with integral periods;
2. A **Riemannian metric** g on E satisfying the **Einstein-Maxwell field equations**,

$$\mathrm{Ric}_{ij} = \frac{1}{4}\sum_{p,q} H_{ipq} H_j^{pq};$$

3. A **Ramond-Ramond (RR) field** $G \in \Omega^{odd}(E)$, satisfying the equations of motion, $(d - H\wedge)G = 0$;
4. A **complex-valued dilaton + axion**.

T-duality in the literature

Spacetime is $M \times \mathbb{T}$, with trivial background flux - then the T-dual is topologically the same space $M \times \hat{\mathbb{T}}$, and T-duality is realized by using the correspondence

$$
\begin{array}{ccccc}
 & & M \times \mathbb{T} \times \hat{\mathbb{T}} & & \\
 & \swarrow^{p} & & \searrow^{\hat{p}} & \\
M \times \mathbb{T} & & & & M \times \hat{\mathbb{T}}
\end{array}
$$

Poincaré line bundle $\mathcal{P}$:

There is a canonical line bundle defined on the 2D torus

$$\mathcal{P} \longrightarrow \mathbb{T} \times \widehat{\mathbb{T}},$$

called the Poincaré line bundle, defined as follows:
Consider the free action of $\mathbb{Z}$ on $\mathbb{R} \times \widehat{\mathbb{T}} \times \mathbb{C}$ given by,

$$\begin{aligned} \mathbb{Z} \times (\mathbb{R} \times \widehat{\mathbb{T}} \times \mathbb{C}) &\rightarrow \mathbb{R} \times \widehat{\mathbb{T}} \times \mathbb{C} \\ (n, (r, \rho, z)) &\rightarrow (r + n, \rho, \rho(n) z) \end{aligned}$$

The **Poincaré line bundle** is defined as the quotient

$$\mathcal{P} = (\mathbb{R} \times \widehat{\mathbb{T}} \times \mathbb{C})/\mathbb{Z}.$$

It has a **connection** $\Theta = \theta d\hat{\theta}$ whose **curvature** is $\mathcal{F} = d\theta \wedge d\hat{\theta}$.

T-duality in the literature

T-dualizing on $\mathbb{T}$, the **Buscher rules** for the RR fields can be conveniently encoded in the **Hori formula** on $M \times \mathbb{T} \times \hat{\mathbb{T}}$,

$$T_* G = \int_{\mathbb{T}} e^{\mathcal{F}} \, G \,, \qquad (1)$$

Here $\mathcal{F} = d\theta \wedge d\hat{\theta}$ is the curvature of the Poincaré line bundle $\mathcal{P}$ on $\mathbb{T} \times \hat{\mathbb{T}}$, so that $e^{\mathcal{F}} = ch(\mathcal{P})$ is the Chern character of $\mathcal{P}$.
$G \in \Omega^\bullet(M \times \mathbb{T})$ is the total RR fieldstrength,

$G \in \Omega^{even}(M \times \mathbb{T})$ for **Type IIA**;
$G \in \Omega^{odd}(M \times \mathbb{T})$ for **Type IIB**.

T-duality in the literature

Note that G is a closed form if and only if its T-dual $T_* G$ is a closed form. The Buscher rules (**??**) are interpreted as

$$T_* : H^\bullet(M \times \mathbb{T}) \xrightarrow{\cong} H^{\bullet+1}(M \times \hat{\mathbb{T}}). \quad (2)$$

That is, T-duality (no background field) gives an equivalence

Type IIA theory $\Longleftrightarrow$ **Type IIB theory**

N.B. **No** change in topology!

Remarks:This equivalence can be refined to K-theory

T-duality - The case of circle bundles

In [BEM], we isolated the geometry in the case when E is a principal $\mathbb{T}$-bundle over M

$$\begin{array}{ccc} \mathbb{T} & \longrightarrow & E \\ & & \downarrow \pi \\ & & M \end{array} \tag{3}$$

classified by its first Chern class $c_1(E) \in H^2(M, \mathbb{Z})$, with H-flux $H \in H^3(E, \mathbb{Z})$.

The **T-dual** is another principal $\mathbb{T}$-bundle over M, denoted by $\hat{E}$,

$$\begin{array}{ccc} \hat{\mathbb{T}} & \longrightarrow & \hat{E} \\ & & \downarrow \hat{\pi} \\ & & M \end{array} \tag{4}$$

which has first Chern class $c_1(\hat{E}) = \pi_* H$.

T-duality in a background flux

The Gysin sequence for E enables us to define a T-dual H-flux $\hat{H} \in H^3(\hat{E}, \mathbb{Z})$, satisfying

$$c_1(E) = \hat{\pi}_* \hat{H}\,, \tag{5}$$

where π_* and similarly $\hat{\pi}_*$, denote the pushforward maps.

N.B. $\hat{H}$ is not fixed by this data, since any integer degree 3 cohomology class on M that is pulled back to $\hat{E}$ also satisfies (**??**). However, $\hat{H}$ is determined uniquely (up to cohomology) upon imposing the condition $[H] = [\hat{H}]$ on the correspondence space $E \times_M \hat{E}$. Explicit formulae will be given shortly.

T-duality for **circle bundles** is the exchange,

background H-flux $\Longleftrightarrow$ Chern class

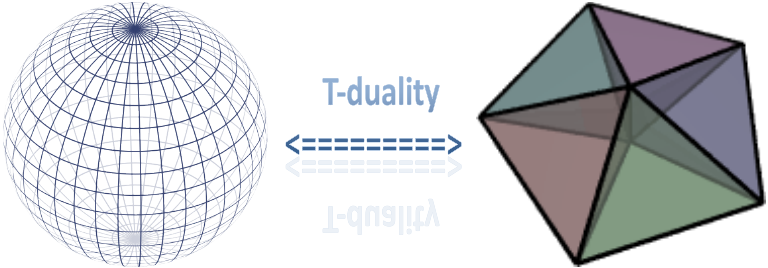
T-duality in a background flux

The surprising **new** phenomenon is that there is a **change in topology** when the H-flux is non-trivial.

An example is S^5 with trivial H-flux, is T-dual to $\mathbb{C}P^2 \times \mathbb{T}$ with H-flux $H = a \cup b$ where $a = \mathrm{vol} \in H^2(\mathbb{C}P^2, \mathbb{Z})$, b the generator of $H^1(\mathbb{T}, \mathbb{Z})$.

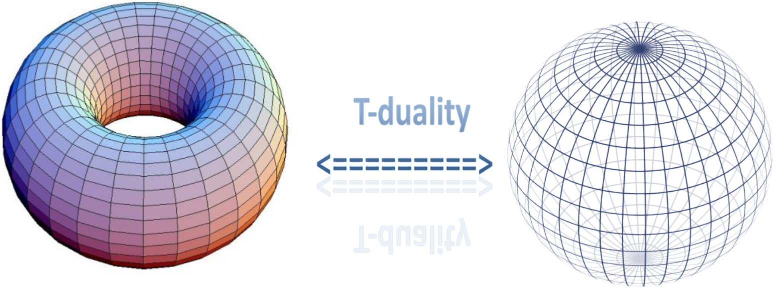
So $(AdS^5 \times S^5, H = 0)$ and $(AdS^5 \times \mathbb{C}P^2 \times \mathbb{T}, H = a \cup b)$ are T-dual spaces.

T-duality in a background flux



3-Sphere ; $H = 5$ $\Longleftrightarrow$ Lens Space $L(5, 1)$; $H = 1$

T-duality in a background flux



$$S^2 \times S^1 \ ; H = 1 \qquad \Longleftrightarrow \qquad \text{3-Sphere} \ ; H = 0$$

T-duality in a background flux

Lens space $L(p,1) = S^3/\mathbb{Z}_p$, where

$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ & $\mathbb{Z}_p$ action on S^3 is

$$\exp(2\pi i/p).(z_1, z_2) = (z_1, \exp(2\pi i/p) z_2).$$

$L(p,1)$ is the total space of the circle bundle over S^2 with Chern class equal to p times the generator of $H^2(S^2, \mathbb{Z}) \cong \mathbb{Z}$.

Then $L(p,1)$ is never homeomorphic to $L(q,1)$ whenever $p \neq q$.

Nevertheless

$$(L(j,1), H = k.\mathrm{vol}) \qquad \text{and} \qquad (L(k,1), H = j.\mathrm{vol})\,.$$

are T-dual pairs! Thus T-duality is the interchange

$$\boxed{j \Longleftrightarrow k}$$

Since $L(0,1) = S^2 \times \mathbb{T}$, we see the T-dual pairs:

$$(S^2 \times \mathbb{T}, H = k) \qquad \text{and} \qquad (L(k,1), H = 0)$$

T-duality in a background flux

Let $H_{\mathbb{Z}}(k)$ be the integer Heisenberg group,

$$H_{\mathbb{Z}}(k) = \left\{ \begin{pmatrix} 1 & x & \frac{1}{k}z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\},$$

which is a $\mathbb{Z}$-central extension of $\mathbb{Z}^2$,

$$0 \to \mathbb{Z} \to H_{\mathbb{Z}}(k) \to \mathbb{Z}^2 \to 0.$$

Also let $H_{\mathbb{R}}$ denote the Heisenberg group,

$$H_{\mathbb{R}} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\},$$

Clearly $H_{\mathbb{Z}}(k)$ is a discrete subgroup of $H_{\mathbb{R}}$.

T-duality in a background flux

The quotient space $H_{\mathbb{R}}/H_{\mathbb{Z}}(k)$, is a **Heisenberg nilmanifold**. It is a principal circle bundle over the torus $\mathbb{T}^2$ with Chern class equal to k-times the volume form of the torus.

$$(H_{\mathbb{R}}/H_{\mathbb{Z}}(k), H = j.\mathrm{vol}) \qquad \text{and} \qquad (H_{\mathbb{R}}/H_{\mathbb{Z}}(j), H = k.\mathrm{vol})$$

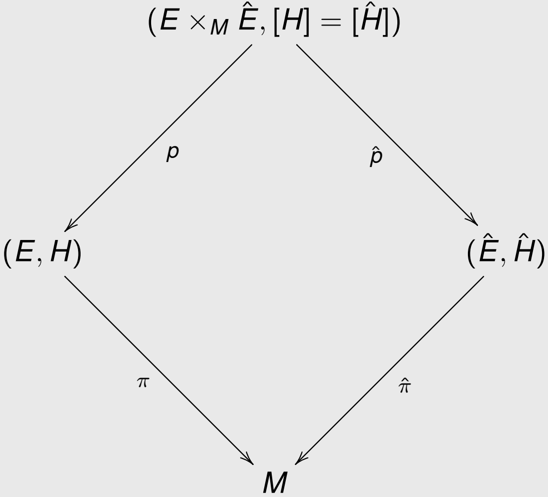
are T-dual pairs.

Thus T-duality is again the interchange

$$\boxed{j \Longleftrightarrow k}$$

A similar analysis can be done for circle bundles over all **Riemann surfaces**. The total spaces of such circle bundles are known as **Seifert fibered spaces**.

T-duality & correspondence spaces



T-duality in a background flux - cohomology

The correspondence space is defined as

$$E \times_M \hat{E} = \{(x, \hat{x}) \in E \times \hat{E} : \pi(x) = \hat{\pi}(\hat{x})\}.$$

By requiring

$$[H] = [\hat{H}] \in H^3(E \times_M \hat{E}, \mathbb{Z}),$$

determines $[\hat{H}] \in H^3(\hat{E}, \mathbb{Z})$ uniquely, via an application of the Gysin sequence.

A direct construction of $\hat{H}$ will be given shortly.

T-duality in a background flux - cohomology

Choosing connection 1-forms A and $\hat{A}$, on the $\mathbb{T}$-bundles E and $\hat{E}$, respectively, the rules for transforming the RR fields can be encoded in the **[BEM]** generalization of Hori's formula

$$T_* G = \int_{\mathbb{T}} e^{A \wedge \hat{A}} \, G \,, \tag{6}$$

where $G \in \Omega^\bullet(E)^{\mathbb{T}}$ is the total RR fieldstrength,

$$G \in \Omega^{even}(E)^{\mathbb{T}} \quad \text{for } \underline{\textbf{Type IIA}};$$
$$G \in \Omega^{odd}(E)^{\mathbb{T}} \quad \text{for } \underline{\textbf{Type IIB}},$$

and where the right hand side of (6) is an invariant differential form on $E \times_M \hat{E}$, and the integration is along the $\mathbb{T}$-fiber of E.

T-duality in a background flux

Let $F = dA$ and $\hat{F} = d\hat{A}$ be the curvatures of the connections, and we can assume wlog that H is $\mathbb{T}$-invariant. Then on E

$$H = A \wedge \hat{F} - \Omega\,, \tag{7}$$

for some $\Omega \in \Omega^3(M)$, while the T-dual $\hat{H}$ on $\hat{E}$ is given by

$$\hat{H} = F \wedge \hat{A} - \Omega\,. \tag{8}$$

We note that

$$d(A \wedge \hat{A}) = -H + \hat{H}\,, \tag{9}$$

T_* indeed maps d_H-closed forms G to $d_{\hat{H}}$-closed forms $T_* G$. Recall that the twisted cohomology is defined as

$$H^\bullet(E, H) = H^\bullet(\Omega^\bullet(E), d_H = d - H\wedge).$$

So T-duality T_* induces a map on twisted cohomologies,

$$T_* : H^\bullet(E, H) \to H^{\bullet-1}(\hat{E}, \hat{H}).$$

T-duality.

We define the Riemannian metrics on E and $\hat{E}$ by

$$g_E = \pi^* g_M + A \odot A, \qquad g_{\hat{E}} = \hat{\pi}^* g_M + \hat{A} \odot \hat{A}.$$

Theorem

Under the above choices of Riemannian metrics and flux forms,

$$T \colon \Omega^{\bar{k}}(E)^{\mathbb{T}} \to \Omega^{\overline{k+1}}(\hat{E})^{\hat{\mathbb{T}}},$$

for $k = 0, 1$, are isometries, inducing isometries on the spaces of twisted harmonic forms and hence on the twisted cohomology groups.

Proof of T-duality.

Proof.

For any $\omega = \pi^*\omega_1 + A \wedge \pi^*\omega_2 \in \Omega^\bullet(E)^{\mathbb{T}}$, where $\omega_1, \omega_2 \in \Omega^\bullet(M)$, we have $T(\omega) = \hat{\pi}^*\omega_2 + \hat{A} \wedge \hat{\pi}^*\omega_1$.

The isometry of T follows from

$$\int_E \omega \wedge *_E\, \omega = \int_M \omega_1 \wedge *_M\, \omega_1 + \int_M \omega_2 \wedge *_M\, \omega_2$$
$$= \int_{\hat{E}} T(\omega) \wedge *_{\hat{E}}\, T(\omega)$$

Since $d(p^*A \wedge \hat{p}^*\hat{A}) = -p^*H + \hat{p}^*\hat{H}$, we have $T \circ d_H = d_{\hat{H}} \circ T$. So T acts on the spaces of twisted harmonic forms and on the twisted cohomology groups. $\square$