Jacob Lurie's: **Chromatic Homotopy Theory**

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Lecture notes, 2010

Introduction (Lecture 1)

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A major goal of algebraic topology is to study topological spaces by means of algebraic invariants (such as homology or cohomology). There is a balance to be struck here: we would like our invariants to be simple enough to be tractable and computable, but rich enough to convey interesting information about topology. In this course, we are going to study one example where both of these demands can be satisfied. This is the so-called "chromatic" picture of stable homotopy theory, and it begins with Quillen's work on the relationship between cohomology theories and formal groups.

Let E be a multiplicative cohomology theory. For any topological space X, one can attempt to compute the E-cohomology groups $E^*(X)$ by means of the Atiyah-Hirzebruch spectral sequence

$$\mathrm{H}^p(X; E^q(*)) \Rightarrow E^{p+q}(X).$$

If X is the infinite dimensional projective space $\mathbf{C}\mathrm{P}^\infty$, then its ordinary cohomology groups are given by $\mathrm{H}^*(X; \mathbf{Z}) \simeq \mathbf{Z}[t]$, where $t \in \mathrm{H}^2(X; \mathbf{Z})$ is a generator. We say that E is *complex-orientable* if the Atiyah-Hirzebruch spectral sequence degenerates at the second page. In this case, we get an isomorphism $E^*(\mathbf{C}\mathrm{P}^\infty) \simeq E^*(*)[[t]]$ for some generator $t \in E^2(*)$.

In ordinary cohomology, we can define $t \in \mathrm{H}^2(\mathbf{C}\mathrm{P}^\infty; \mathbf{Z})$ to be the first Chern class $c_1(\mathcal{O}(1))$, where $\mathcal{O}(1)$ denotes the universal line bundle on $\mathbf{C}\mathrm{P}^\infty$. Conversely, if we are given t then we can define the first Chern class in general, using the fact that $\mathbf{C}\mathrm{P}^\infty$ is a classifying space for complex line bundles. Namely, if $\mathcal{L}$ is *any* complex line bundle on a (nice) space X, then there exists a continuous map $f : X \to \mathbf{C}\mathrm{P}^\infty$ (well-defined up to homotopy) and an isomorphism $\mathcal{L} \simeq f^* \mathcal{O}(1)$. We can then define $c_1(\mathcal{L}) = f^* t \in \mathrm{H}^2(X; \mathbf{Z})$.

If E is a complex-orientable cohomology theory, then the isomorphism $E^*(\mathbf{C}\mathrm{P}^\infty) \simeq E^*(*)[[t]]$ permits us to define a Chern class which takes values in E-cohomology. Namely, if $\mathcal{L}$ is a line bundle on a space X and $f : X \to \mathbf{C}\mathrm{P}^\infty$ is defined as above, then we can define $c_1^E(\mathcal{L}) = f^* t \in E^2(X)$.

Warning 1. The definition of the Chern class $c_1^E(\mathcal{L})$ depends not only on the cohomology theory E, but also on the choice of isomorphism $E^*(\mathbf{C}\mathrm{P}^\infty) \simeq E^*(*)[[t]]$ (that is, on the choice of t). A complex-orientable cohomology theory E together with a choice of generator $t \in E^2(\mathbf{C}\mathrm{P}^\infty)$ is called a *complex-oriented cohomology theory.*

We now ask: how well-behaved is this theory of E-valued Chern classes? For example, ordinary Chern classes satisfy a multiplicativity formula

$$c_1(\mathcal{L} \otimes \mathcal{L}') \simeq c_1(\mathcal{L}) + c_1(\mathcal{L}').$$

Does the analogous formula hold in E-cohomology? Generally, the answer is no. However, we can say that there is always *some* formula which allows us to express $c_1^E(\mathcal{L} \otimes \mathcal{L}')$ in terms of $c_1^E(\mathcal{L})$ and $c_1^E(\mathcal{L}')$. To see this, it suffices to consider the universal example of a space with two complex line bundles. This is the space $\mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty$. Using the Atiyah-Hirzebruch spectral sequence, we get an isomorphism $E^*(\mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty) \simeq E^*(*)[[u, v]]$. Here u and v denote the pullbacks of $t \in E^2(\mathbf{C}\mathrm{P}^\infty)$ along the two projection maps $\pi_1, \pi_2 : \mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty \to \mathbf{C}\mathrm{P}^\infty$; in other words, we can identify u and v with the Chern classes of the universal line bundles $\pi_1^* \mathcal{O}(1)$ and $\pi_2^* \mathcal{O}(1)$ on $\mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty$. We also have a third line bundle $\mathcal{O} = \pi_1^* \mathcal{O}(1) \otimes \pi_2^* \mathcal{O}(1)$. Then

$c_1^E(\mathcal{O}) = f(u,v) \in E^*(\mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty) \simeq E^*[[u,v]]$. By the method of the universal example, we deduce that $c_1^E(\mathcal{L} \otimes \mathcal{L}') \simeq f(c_1^E \mathcal{L}, c_1^E \mathcal{L}')$ for any pair of line bundles $\mathcal{L}$ and $\mathcal{L}'$ on any space X (here we must take some care with the meaning of this statement, since f is a power series and not a polynomial in general).

Example 2. For the usual theory of Chern classes, f is given by the formula $f(u,v) = u + v$.

What can we say about the power series f? In general it is a power series rather than a polynomial, and can be quite complicated. However, it is not arbitrary: it satisfies certain identities, which reflect the idea that the tensor product of complex line bundles is commutative and associative up to isomorphism. More precisely, we have

$$f(u,0) = u = f(0,u)$$

$$f(u,v) = f(v,u)$$

$$f(u, f(v,w)) = f(f(u,v), w).$$

In general, if R is a commutative ring, then a power series $f(u,v) \in R[[u,v]]$ satisfying the identities above is called a *formal group law* over R.

We can summarize our discussion as follows: every complex-oriented cohomology theory E determines a formal group law over the commutative ring $E^{even}(*)$. This assignment fits into the general paradigm of algebraic topology. A cohomology theory E should be regarded as a topological object: it can be represented by a spectrum, which is a variation on the notion of a space. To this cohomology theory we assign an algebraic object: a formal group law over a commutative ring. This assignment satisfies both of the requirements posited at the beginning of this lecture:

(a) Though somewhat more complicated than an abelian group or a vector space, a formal group law is a reasonably tractable mathematical object. In particular, formal group laws have been thoroughly studied by algebraic geometers and number theorists.

(b) The formal group law associated to a complex-oriented cohomology theory E remembers a great deal about E. In fact, one can often reconstruct E from its formal group law.

To elaborate on these points, we first note that there is a *universal* example of a formal group law. That is, there is a commutative ring L and a formal group law $f(u,v) \in L[[u,v]]$ which is "maximally complicated", in the sense that any other formal group law over a commutative ring R is obtained from $f(u,v)$ by means of a ring homomorphism $L \to R$. The ring L is called the *Lazard ring* in honor of Lazard, who proved that L is a polynomial ring (in infinitely many generators).

According to a theorem of Quillen, the Lazard ring L has another incarnation: it is the coefficient ring of the cohomology theory MU of *complex bordism* (which is *universal* among complex-oriented cohomology theories). One can attempt to use this observation to construct an "inverse" to the above constructions. Namely, suppose we are given a commutative ring R and a formal group law $f(u,v) \in R[[u,v]]$, classified by a map $L \to R$. We can then attempt to define a new cohomology theory E (having coefficient ring R) by the formula $E^*(X) \simeq MU^*(X) \otimes_L R$ for finite complexes X (for this to be sensible, R should be equipped with a suitable grading; we will suppress mention in the discussion which follows). This construction does not always work: that is, E^* does not always have the excision and Mayer-Vietoris exact sequences that are required of cohomology theories. However, a fundamental result of Landweber gives a purely algebraic criterion on ϕ which, if satisfied, guarantees that E^* is a cohomology theory. One can use this criterion to produce many interesting examples of cohomology theories.

Example 3. One can take R to be the ring of integers and $f(u,v)$ to be the *multiplicative* formal group given by $f(u,v) = u + v + uv$. In this case, Landweber's theorem applies and produces a cohomology theory, namely, complex K-theory.

Motivated by Example 3, it is natural to ask what other cohomology theories can be produced by means of Landweber's theorem: that is, starting with a map of affine schemes $\phi : \operatorname{Spec} R \to \operatorname{Spec} L = \mathbf{A}^\infty$. First, we should note that the map ϕ is not really fundamental. The formal group law associated to a cohomology theory E depends not only on E, but also on a choice of complex orientation $t \in E^2(\mathbf{CP}^\infty)$. The collection of all such choices is acted on by the group G of coordinate changes

$$t \mapsto t + a_1 t^2 + a_2 t^3 + \ldots$$

Consequently, our real interest is not in the moduli space $\operatorname{Spec} L$ of formal group laws, but in the quotient $\operatorname{Spec} L/G$. This is a kind of algebraic stack, called the *moduli stack of formal groups* (more precisely, it is the moduli stack of formal groups with trivialized Lie algebras). The main thrust of this course can be stated as follows:

- The structure of the stable homotopy category is controlled by the geometry of the stack $\operatorname{Spec} L/G$.

For example, every complex-orientable cohomology theory E determines a commutative ring $R = E^{even}(*)$ and a formal group over R, which we can think of as a map $\operatorname{Spec} R \to \operatorname{Spec} L/G$. This construction provides the beginning of a rough dictionary:

Multiplicative cohomology theories	Affine schemes over $\operatorname{Spec} L/G$
Cohomology theories	Quasi-coherent sheaves on $\operatorname{Spec} L/G$
Complex bordism	Moduli Space $\operatorname{Spec} L \simeq \mathbf{A}^\infty$ of formal group laws
$\cdots$	$\cdots$

As we will see over the course of the semester, these ideas give an extremely useful picture of the stable homotopy category.

Lazard's Theorem (Lecture 2)

April 27, 2010

Let R be a commutative ring. We recall that a *formal group law* over R is a power series $f(x,y) \in R[[x,y]]$ satisfying the identities

$$f(x,0) = f(0,x) = x$$
$$f(x,y) = f(y,x)$$
$$f(x, f(y,z)) = f(f(x,y), z).$$

We let $\mathrm{FGL}(R)$ denote the subset of $R[[x,y]]$ consisting of all formal group laws over R. Note that a map of commutative rings $R \to R'$ induces, by substitution, a map $\mathrm{FGL}(R) \to \mathrm{FGL}(R')$. In other words, FGL is a functor from the category of commutative rings to the category of sets.

Any power series $f(x,y) \in R[[x,y]]$ can be written as a formal sum $f(x,y) = \sum c_{i,j} x^i y^j$, for some coefficients $c_{i,j} \in R$. However, for f to be a formal group law, the coefficients $c_{i,j}$ must satisfy some constraints. For example, the first condition gives

$$c_{i,0} = c_{i,0} = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

and the second condition gives $c_{i,j} = c_{j,i}$. The third condition imposes more complicated constraints on the coefficients $c_{i,j}$, which we will not write out in detail. However, we note that these constraints are simply given by polynomial equations that the coefficients $c_{i,j}$ are forced to satisfy. We can summarize the discussion as follows:

- Giving a formal group law over a ring R is equivalent to giving a collection of elements $c_{i,j} \in R$ satisfying certain polynomial equations.

Let L denote the commutative ring $\mathbf{Z}[c_{i,j}]/Q$, where Q is the ideal in $\mathbf{Z}[c_{i,j}]$ generated by the polynomial constraints mentioned above. By construction, the power series $f(x,y) = \sum c_{i,j} x^i y^j$ defines a formal group law over L. We can restate the previous assertion as follows:

- There is a formal group law $f \in \mathrm{FGL}(L)$ with the following universal property: for every commutative ring R, evaluation on f determines a bijection $\mathrm{Hom}(L, R) \to \mathrm{FGL}(R)$.

The commutative ring L is called the *Lazard ring*. Our goal in this lecture is to describe the structure of L.

Remark 1. The existence of L is equivalent to the assertion that the functor FGL is *corepresentable*. By general nonsense, the representability of FGL is equivalent to the following pair of properties, which are easy to verify directly:

(1) The functor FGL carries limits of commutative rings to limits of sets.

(2) The functor FGL carries κ-filtered colimits of commutative rings to κ-filtered colimits of sets, provided that κ is sufficiently large (in fact, we can take κ to be any uncountable cardinal: this reflects the fact that a formal group law is determined by a countable number of parameters).

We first note that the commutative ring $\mathbf{Z}[c_{i,j}]$ has a natural grading, where we define the degree of $c_{i,j}$ to be $2(i+j-1)$. This grading is dictated by the requirement that if we let x and y have degree -2, then the expression

$$f(x,y) = \sum_{i,j} c_{i,j} x^i y^j$$

again has degree -2. Then the power series $f(f(x,y),z)$ and $f(x,f(y,z))$ also have degree -2. It follows that the coefficients of $x^i y^j z^k$ in the $f(f(x,y),z)$ and $f(x,f(y,z))$ both have degree $2(i+j+k)-2$ in the ring $\mathbf{Z}[c_{i,j}]$. Consequently, the grading on $\mathbf{Z}[c_{i,j}]$ descends to a grading on the quotient ring $L = \mathbf{Z}[c_{i,j}]/Q$: that is, L has the structure of a graded ring. Since $c_{0,0} = 0$ and $c_{1,0} = c_{0,1} = 1$ in L, it is actually a *nonnegatively* graded ring, with $L_0 \simeq \mathbf{Z}$.

Remark 2. Our convention that the grading of L is *even* is irrelevant for this lecture. We introduce this convention in order to be compatible with the gradings which appear in topology.

Remark 3. The existence of the above grading on L can be explained more abstractly as follows. The collection of formal group laws admits an action of the multiplicative group $\mathbb{G}_m$. That is, for every commutative ring R, there is a canonical action of $R^\times$ on $\mathrm{FGL}(R)$, given by

$$f^\lambda(x,y) = \lambda^{-1} f(\lambda x, \lambda y).$$

This determines an action of $\mathbb{G}_m$ on the affine scheme $\operatorname{Spec} L$ representing the functor FGL, which is the same as the data of a grading of L. The nonnegativity of the grading reflects the observation that the action of $R^\times$ on $\mathrm{FGL}(R)$ extends to an action of the multiplicative monoid $(R, \times)$ on FGL (that is, $f(\lambda x, \lambda y)$ is formally divisible by λ). The isomorphism $L_0 \simeq \mathbf{Z}$ reflects the observation that for any formal group f, we have $f^\lambda(x,y) = x+y$ when $\lambda = 0$).

Our goal in this lecture is to begin the proof of the following result:

Theorem 4 (Lazard)**.** *The Lazard ring L is isomorphic to a polynomial ring $\mathbf{Z}[t_1, t_2, \ldots]$, where each t_i has degree $2i$.*

Theorem 4 implies that it is easy to write down formal group laws over a commutative ring R: one just needs to select a countable sequence of elements in R. In particular, formal group laws exist in abundance. Where do these formal group laws come from? We can get a good supply by combining the following pair of observations:

(a) The power series $f(x,y) = x+y$ is a formal group law (over any ring R).

(b) If $f(x,y)$ is a formal group law over the ring R and we are given some substitution $g(x) = x + b_1 x^2 + b_2 x^3 + \cdots$, then the power series $gf(g^{-1}(x), g^{-1}(y))$ is also a formal group law over R.

In particular:

(c) If g is defined as above, then $g(g^{-1}(x) + g^{-1}(y))$ is a formal group law over the polynomial ring $\mathbf{Z}[b_1, b_2, \ldots]$.

This formal group law is classified by a map $\phi : L \to \mathbf{Z}[b_1, b_2, \ldots]$. We will soon learn that this map is an isomorphism over the rational numbers (Lemma 10). That is, in characteristic zero, every formal group law is obtained from the additive formal group law $f(x,y) = x+y$ by a change of variables. This is not true in positive characteristic (otherwise, this course would be very short).

Remark 5. The map $\phi : L \to \mathbf{Z}[b_1, b_2, \ldots]$ is compatible with the gradings, if we let each b_i have degree $2i$. To see this, it suffices to note that if each b_i has degree $2i$, then $g(g^{-1}(x) + g^{-1}(y))$ has degree -2 when x and y are both given degree -2.

Let I denote the ideal in L consisting of elements of positive degree, and let J denote the ideal in $\mathbf{Z}[b_1, b_2, \ldots]$ generated by elements of positive degree (that is, the ideal generated by $b_1, b_2, \ldots$). Then J/J^2 can be identified with the free abelian group on generators $\{b_i\}_{i>0}$. Note that the quotient I/I^2 inherits a grading from the grading of L. The main step in the proof of Theorem 4 is the following calculation:

Lemma 6. *For every integer $n > 0$, the ring homomorphism map $\phi : L \to \mathbf{Z}[b_1, b_2, \ldots]$ induces an injection $(I/I^2)_{2n} \to (J/J^2)_{2n} \simeq \mathbf{Z}$. The image of this map is $p\mathbf{Z}$ if $n+1$ is a prime power p^f, and $\mathbf{Z}$ otherwise.*

We will prove Lemma 6 in the next lecture. For now, let us collect some of the consequences.

Corollary 7. *For every integer $n > 0$, the abelian group $(I/I^2)_{2n}$ is (canonically) isomorphic to $\mathbf{Z}$.*

In particular, we can choose homogeneous elements $t_n \in I_{2n} = L_{2n}$ lifting generators for $(I/I^2)_{2n} \simeq \mathbf{Z}$. This choice of generators determines a map of graded rings $\theta : \mathbf{Z}[t_1, t_2, \ldots] \to L$.

Lemma 8. *The map θ is surjective.*

Proof. We prove by induction on n that θ induces a surjection in degree $2n$. The inductive hypothesis shows that the image of θ contains $(I^2)_{2n}$. Since the image of θ contains a generator for $(I/I^2)_{2n} \simeq \mathbf{Z}$, it contains $I_{2n} = L_{2n}$. □

We now complete the proof of Theorem 4 as follows:

Lemma 9. *The composite map $\psi : \mathbf{Z}[t_1, t_2, \ldots,] \xrightarrow{\theta} L \xrightarrow{\phi} \mathbf{Z}[b_1, b_2, \ldots]$ is injective. In particular, the map θ is injective.*

Since the polynomial rings $\mathbf{Z}[t_1, t_2, \ldots]$ and $\mathbf{Z}[b_1, b_2, \ldots]$ are torsion-free, they inject into their rationalizations $\mathbf{Q}[t_1, t_2, \ldots]$ and $\mathbf{Q}[b_1, b_2, \ldots]$. Lemma 9 is therefore an immediate consequence of the following:

Lemma 10. *The map $\psi_{\mathbf{Q}} : \mathbf{Q}[t_1, t_2, \ldots] \to \mathbf{Q}[b_1, b_2, \ldots]$ is an isomorphism of commutative rings.*

Proof. Let J' denote the ideal in $\mathbf{Q}[t_1, t_2, \ldots]$ generated by the elements t_i. Then $J'/(J')^2$ is isomorphic to the free $\mathbf{Q}$-vector space generated by $t_1, t_2, \ldots$. Using Lemma 6, we see that $\phi_{\mathbf{Q}}$ induces a surjection $J'/(J')^2 \to (J/J^2)_{\mathbf{Q}}$. Repeating the proof of Lemma 8, we see that $\psi_{\mathbf{Q}}$ is surjective. Since the vector spaces $\mathbf{Q}[t_1, t_2, \ldots]$ and $\mathbf{Q}[b_1, b_2, \ldots]$ have the same dimension in every graded degree, we deduce that $\psi_{\mathbf{Q}}$ is also injective. □

Lazard's Theorem (Continued) (Lecture 3)

January 28, 2010

Our goal in this lecture is to complete the proof of Lazard's theorem. In the last lecture, we were reduced to proving the following result:

Lemma 1. *Let $\phi : L \to \mathbf{Z}[b_1, b_2, \ldots]$ be the ring homomorphism classifying the formal group law $g(g^{-1}(x) + g^{-1}(y))$, where g is the power series $g(x) = x + b_1 x^2 + b_2 x^3 + \cdots$. Let $I \subseteq L$ be the ideal consisting of elements of positive degree, and let $J \subseteq \mathbf{Z}[b_1, b_2, \ldots]$ be defined likewise. Then, for every integer $n > 0$, ϕ induces an injection $(I/I^2)_{2n} \to (J/J^2)_{2n} \simeq \mathbf{Z}$. The image of this map is $p\mathbf{Z}$ if $n+1$ is a prime power p^f, and $\mathbf{Z}$ otherwise.*

We regard n as a positive integer which is fixed throughout this lecture. Recall that for any commutative ring R, there is a canonical bijection $\epsilon : \mathrm{Hom}(L, R) \to \mathrm{FGL}(R)$, where FGL denotes the collection of formal group laws $f(x, y) \in R[[x, y]]$ over R. Suppose now that R is a graded ring, and let $\mathrm{Hom}^{gr}(L, R) \subseteq \mathrm{Hom}(L, R)$ denote the collection of all graded ring homomorphisms from L to R. Then ϵ restricts to a bijection $\mathrm{Hom}^{gr}(L, R) \simeq \mathrm{FGL}^{gr}(R)$, where $\mathrm{FGL}^{gr}(R)$ denotes the collection of formal group laws $f(x, y) = \sum a_{i,j} x^i y^j \in R[[x, y]]$ where the coefficients $a_{i,j}$ have degree $2(i + j - 1)$ (in other words, the collection of all formal group laws where $f(x, y)$ is homogeneous of degree -2, when we regard the variables x and y as having degree -2).

The main point of Lemma 1 is to show that the abelian group $(I/I^2)_{2n}$ is isomorphic to $\mathbf{Z}$: in other words, that it is free on one generator. Equivalently, we wish to show that for any abelian group M, the collection of group homomorphisms $\mathrm{Hom}((I/I^2)_{2n}, M)$ can be identified with M. Let us denote this collection of group homomorphisms by $F(M)$: that is, we let F be the functor corepresented by $(I/I^2)_{2n}$ (from the category of abelian groups to the category of sets). To proceed further, we would like to relate F to the functor corepresented by L. To this end, let us regard $\mathbf{Z} \oplus M$ as a graded commutative ring, with the "square zero" multiplication law $(a, m)(b, m') = (ab, am' + bm)$ and the grading

$$(\mathbf{Z} \oplus M)_k = \begin{cases} \mathbf{Z} & \text{if } k = 0 \\ M & \text{if } k = 2n \\ 0 & \text{otherwise.} \end{cases}$$

Unwinding the definitions, we see that evaluation in degree $2n$ induces a bijection $\mathrm{Hom}^{gr}(L, \mathbf{Z} \oplus M) \to \mathrm{Hom}((I/I^2)_{2n}, M) = F(M)$. In other words, $F(M)$ can be identified with the set $\mathrm{FGL}^{gr}(\mathbf{Z} \oplus M)$ of (homogeneous) formal group laws over $\mathbf{Z} \oplus M$. Any such formal group law can be written in the form

$$f(x, y) = x + y + \sum_{i+j=n+1} m_{i,j} x^i y^j.$$

In order for such a polynomial to define a formal group law, the coefficients $m_{i,j}$ need to satisfy some conditions. Since the multiplication on $\mathbf{Z} \oplus M$ is square-zero, it is possible to make these conditions very explicit. For example, the requirement that $f(x, 0) = f(0, x) = x$ translates into equations $m_{0,n+1} = m_{n+1,0} = 0$, while the commutativity of f is the requirement $m_{i,j} = m_{j,i}$. Associativity is only slightly more complicated: we require that for every triple of integers i, j, and k, the coefficient of $x^i y^j z^k$ appearing in the expressions $f(f(x,y),z)$ and $f(x,f(y,z))$ are the same. This follows immediately from the earlier conditions if i, j, or k is equal to zero. If $i,j,k>0$, then a simple computation (using the fact that $M^2=0$) shows that the coefficient in $f(f(x,y),z)$ is given by $\binom{i+j}{j}m_{i+j,k}$ if $i+j+k=n+1$ (and is zero otherwise). Similarly, the relevant coefficient in $f(x,f(y,z))$ is given by $\binom{j+k}{j}m_{i,j+k}$. We can summarize our discussion as follows:

Lemma 2. *The functor F carries an abelian group M to the collection of all sequences $\{m_{i,j}\in M\}_{i+j=n+1}$ satisfying the conditions*

$$m_{0,n+1}=m_{n+1,0}=0 \qquad m_{i,j}=m_{j,i}$$

$$\binom{i+j}{j}m_{i+j,k}=\binom{j+k}{j}m_{i,j+k} \text{ if } i,j,k>0.$$

We want to understand how to find all solutions to the equations appearing in Lemma 2. We can start by considering the solutions that we get using the homomorphism $\phi: L\to\mathbf{Z}[b_1,b_2,\ldots]$ appearing in Lemma 1. This homomorphism induces a map $(I/I^2)_{2n}\to(J/J^2)_{2n}\simeq\mathbf{Z}$, and therefore gives rise to a map

$$\lambda: M=\mathrm{Hom}(\mathbf{Z},M)\to\mathrm{Hom}((J/J^2)_{2n},M)\to\mathrm{Hom}((I/I^2)_{2n},M)=F(M).$$

To understand this map more explicitly, we note that $M\simeq\mathrm{Hom}((J/J^2)_{2n},M)$ can be identified with $\mathrm{Hom}^{gr}(\mathbf{Z}[b_1,b_2,\ldots],\mathbf{Z}\oplus M)$ by assigning to each $m\in M$ the ring homomorphism $\psi_m:\mathbf{Z}[b_1,\ldots]\to\mathbf{Z}\oplus M$ which carries b_n to m and all other b_i to zero. In this case, the change-of-variable transformation $g(x)=x+b_1x^2+\cdots$ can be written as $g(x)=x+mx^{n+1}$. Since $m^2=0$ in $\mathbf{Z}\oplus M$, the inverse transformation is simply given by $g^{-1}(x)=x-mx^{n+1}$. Then g defines the formal group law

$$f(x,y)=g(g^{-1}(x)+g^{-1}(y))=g(x-mx^{n+1}+y-my^{n+1})=x+y+m((x+y)^{n+1}-x^{n+1}-y^{n+1}).$$

We conclude that the map $\lambda: M\to F(M)$ carries an element $m\in M$ to the sequence $\{m_{i,j}\}_{i+j=n+1}$ given by

$$m_{i,j}=\begin{cases}0 & \text{if } i=0 \text{ or } j=0\\ \binom{n+1}{i}m & \text{otherwise.}\end{cases}$$

These are the "obvious" solutions to the equations of Lemma 2.

But sometimes there are more solutions. For example, if the binomial coefficients $\{\binom{n+1}{i}\}_{0<i<n+1}$ have greatest common divisor d, then we can write down another solution given by

$$m_{i,j}=\begin{cases}0 & \text{if } i=0 \text{ or } j=0\\ \frac{\binom{n+1}{i}}{d}m & \text{otherwise.}\end{cases}$$

It is therefore of interest to determine d. For this, we will need the following combinatorial fact:

Lemma 3. *Let p be a prime number, and suppose that a and b are nonnegative integers with base p expansions*

$$a=\sum a_ip^i \qquad b=\sum b_ip^i$$

Then $\binom{a}{b}$ is congruent to the product $\prod\binom{a_i}{b_i}$ modulo p.

Proof. Let S be a set of size a. We can partition S into subsets S_α whose sizes are powers of p, with exactly a_i subsets of size p^i. Regard each S_α as acted on by the cyclic group $G_\alpha=\mathbf{Z}/p^i\mathbf{Z}$. These actions together determine an action of $G=\prod_\alpha G_\alpha$ on S. Let T be the collection of all b-element subsets of S, so that $\binom{a}{b}=|T|$. The set T is acted on by G. Since G is a p-group, every nontrivial orbit of G has size divisible by p. Thus $|T|$ is congruent modulo p to the cardinality of T^G, the set of G-fixed points of T. Note that a G-fixed point of T is a subset $S_0\subseteq S$ of cardinality b which is a union of some of the subsets S_α. There are precisely $\prod\binom{a_i}{b_i}$ ways that these subsets can be chosen. $\square$

Corollary 4. *Let i and j be nonnegative integers, and let p be a prime number. Then the binomial coefficient $\binom{i+j}{i}$ is not divisible by p if and only if each digit in the base p expansion of $i+j$ is at least as large as the corresponding digit of i in base p: in other words, if and only if the sum $i+j$ can be computed in base p "without carrying".*

Corollary 5. *Let d be the greatest common divisor of the binomial coefficients $\{\binom{n+1}{i}\}_{0<i<n+1}$. Then* $d = \begin{cases} p & \text{if } n+1 = p^f \\ 1 & \text{otherwise.} \end{cases}$

Proof. If $n+1$ is not a power of p, then we can nontrivially decompose $n+1$ as a sum $i+j$, where the sum of i and j is computed in base p without carrying; it follows that $\binom{n+1}{i}$ is not divisible by p. If $n+1 = p^f$, then there is no such decomposition, so that p is a common divisor of $\{\binom{n+1}{i}\}_{0<i<n+1}$. To see that it is the greatest common divisor, we note that p^2 does not divide the binomial coefficient $\binom{p^f}{p^{f-1}}$. □

We let $\lambda' : M \to F(M)$ be the map which carries $m \in M$ to the sequence

$$m_{i,j} = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \frac{\binom{n+1}{i}}{d} m & \text{otherwise.} \end{cases}$$

We will prove the following:

Proposition 6. *The map λ' is an isomorphism.*

It follows from Proposition 6 that the functor $F(M)$ is corepesentable by the abelian group $\mathbf{Z}$: that is, we get an isomorphism $(I/I^2)_{2n} \simeq \mathbf{Z}$. Moreover, the map λ factors as a composition

$$M \xrightarrow{d} M \xrightarrow{\lambda'} F(M),$$

so that the map

$$\mathbf{Z} \simeq (I/I^2)_{2n} \to (J/J^2)_{2n} \simeq \mathbf{Z}$$

is given by multiplication by d. This completes the proof of Lemma 1.

To prove Proposition 6, it suffices to show that λ' induces an isomorphism $M_{(p)} \to F(M)_{(p)} \simeq F(M_{(p)})$ after localizing at every prime p. In other words, we may assume that M is a $\mathbf{Z}_{(p)}$-module.

Lemma 7. *Let $\{m_i = m_{i,j}\}_{i+j=n+1}$ be an element of $F(M)$. Then:*

(a) If $m_i = 0$, then $m_{n+1-i} = 0$.

(b) If $m_i = 0$ and the sum $i+j$ is computed in base p without carrying, then $m_{i+j} = 0$ vanishes.

Proof. Assertion (a) follows by symmetry. To prove (b), we use the associativity formula

$$\binom{n+1-i}{j} m_i = \binom{i+j}{j} m_{i+j}.$$

If m_i vanishes, then the left hand side vanishes, so (since $\binom{i+j}{j}$ is not divisible by p, by Corollary 4) we conclude that m_{i+j} vanishes. □

Proof of Proposition 6 when $n+1 = p^f$. Let $\chi : F(M) \to M$ be given by extracting the coefficient $m_{p^{f-1}}$. Then the composition $\chi \circ \lambda' : M \to M$ is given by multiplication by $\frac{\binom{p^f}{p^{f-1}}}{p}$, which is not divisible by p. Consequently, $\chi \circ \lambda'$ is an isomorphism, which proves that λ' is injective. To show that λ' is surjective, it suffices to show that χ is injective. Let $\{m_i\} \in F(M)$ belong to the kernel of χ, so that $m_{p^{f-1}}$ vanishes. Part (b) of Lemma 7 shows that m_k vanishes for $p^{f-1} \leq k < p^f$. Using symmetry, we deduce that m_k vanishes for all $0 < k < p^f$. □

Proof of Proposition 6 when $n+1 \neq p^f$. Let p^e be the largest power of p which divides $n+1$. We let $\chi : F(M) \to M$ be given by extracting the coefficient of m_{p^e}. Then $\chi \circ \lambda' : M \to M$ is given by multiplication by $\frac{\binom{n+1}{p^e}}{d}$; here d is either 1 or some prime distinct from p, and the binomial coefficient $\binom{n+1}{p^e}$ is not divisible by p by Corollary 4. As before, we deduce that $\chi \circ \lambda'$ is an isomorphism, λ' is injective, and we are reduced to proving that χ is injective. Suppose that $\{m_i\} \in F(M)$ belongs to the kernel of χ. Then $m_{p^e} = 0$.

Assume $e > 0$ (if not, ignore this step). By symmetry, we get $m_{n+1-p^e} = 0$. Since $n+1-p^{e-1}$ can be obtained as a sum of $n+1-p^e$ and $(p-1)p^{e-1}$ in base p without carrying, we deduce that $m_{n+1-p^{e-1}} = 0$. By symmetry, we get $m_{p^{e-1}} = 0$.

Now choose any nontrivial decomposition $n+1 = i+j$. We wish to prove that $m_i = m_j = 0$. Since $n+1$ has a nontrivial coefficient on p^e in its base p expansion, we conclude that either i or j must contain a nonzero coefficient on p^e or p^{e-1} in its base p expansion. Without loss of generality, we may suppose that i has a nonzero p^a coefficient in its base p-expansion, with $a \in \{e-1, e\}$. Then we can write $i = p^a + (i - p^a)$ in base p without carrying. Since m_{p^a} vanishes by the above argument, we conclude from Lemma 7 that $m_i = 0$. □

Complex-Oriented Cohomology Theories (Lecture 4)

February 1, 2010

In this lecture, we will introduce the notion of a *complex-oriented* cohomology theory E. We will generally not distinguish between a cohomology theory E and the spectrum that represents it. The E-cohomology groups of a space X are given by

$$E^n(X) = \pi_{-n} E^X = [X, \Omega^{\infty - n} E] = \mathrm{Hom}(\Sigma^\infty X, \Sigma^n E),$$

while the E-homology groups of X are given by $E_n(X) = \pi_n(E \otimes \Sigma^\infty X)$.

Warning 1. In this class, we will not employ the usual notations in dealing with spectra. Instead we will denote the smash product with the symbol $\otimes$, and the coproduct by $\oplus$.

We will say that a cohomology theory is *multiplicative* if its representing spectrum E is equipped with a multiplication

$$E \otimes E \to E$$

which is associative and unital up to homotopy. We will generally also assume that E is homotopy commutative, though it is sometimes convenient to relax this assumption.

Definition 2. A multiplicative cohomology theory E is *complex-orientable* if the map $E^2(\mathbf{C}\mathrm{P}^\infty) \to E^2(S^2)$ is surjective. Here we identify the 2-sphere S^2 with $\mathbf{C}\mathrm{P}^1 \subseteq \mathbf{C}\mathrm{P}^\infty$.

We will henceforth regard S^2 and $\mathbf{C}\mathrm{P}^\infty$ as pointed spaces. A choice of base point gives canonical decompositions

$$E^2(\mathbf{C}\mathrm{P}^\infty) \simeq \widetilde{E}^2(\mathbf{C}\mathrm{P}^\infty \oplus E^2(*) \qquad E^2(S^2) \simeq \widetilde{E}^2(S^2) \oplus E^2(*);$$

here the $\widetilde{E}$ denotes reduced cohomology with coefficients in E. Note that $\widetilde{E}^2(S^2) \simeq E^0(*) \simeq \pi_0 E$ is equipped with a canonical unit element $\bar{t}$. Since the image of the map $\theta : \widetilde{E}(\mathbf{C}\mathrm{P}^\infty) \to \widetilde{E}^2(S^2)$ is a $(\pi_0 E)$-module, θ is surjective if and only if its image contains $\bar{t}$. In other words:

- A multiplicative cohomology theory E is complex-orientable if and only if there exists an element $t \in \widetilde{E}^2(\mathbf{C}\mathrm{P}^\infty)$ such that $\theta(t) = \bar{t}$ is the canonical generator of $\widetilde{E}^2(S^2)$.

We will refer to a choice of $t \in \widetilde{E}^2(\mathbf{C}\mathrm{P}^\infty) \subseteq E^2(\mathbf{C}\mathrm{P}^\infty)$ as a *complex orientation* of E.

Remark 3. Let E be a multiplicative cohomology theory and let E' be its connective cover. Then the canonical map $\widetilde{E'}^2(X) \to \widetilde{E}^2(X)$ is an isomorphism whenever X is simply connected. It follows that E is complex orientable if and only if E' is complex-orientable: better yet, there is a bijection between complex orientations of E and complex orientations of E'.

Remark 4. We can think of $\bar{t}$ as encoding a pointed map $S^2 \to \Omega^\infty E$. A complex orientation of E is an extension of this map to $\mathbf{C}\mathrm{P}^\infty$. The existence of such a map can often be established by obstruction theory. For example, if we are already given an extension of $\bar{t}$ to $\mathbf{C}\mathrm{P}^n$, then there is an obstruction to further extending to $\mathbf{C}\mathrm{P}^{n+1}$ which lies in the homotopy group $\pi_{2n+1}\Omega^\infty E = \pi_{2n+1} E = E^{-2n-1}(*)$. In particular, if we have $\pi_3 E = \pi_5 E = \ldots$, then E is complex-orientable.

Example 5. Ordinary cohomology (with coefficients in any commutative ring R) is complex-orientable. In fact, the restriction map $\mathrm{H}^2(\mathbf{C}\mathrm{P}^\infty; R) \to \mathrm{H}^2(S^2; R)$ is an isomorphism.

Example 6. Complex K-theory is complex-orientable. This follows from Remark 4, since $\pi_i K = 0$ whenever i is odd. In this case, the complex orientation is not unique. However, there is a canonical complex orientation, given by the class $t \in K^2(\mathbf{C}\mathrm{P}^\infty) \simeq K^0(\mathbf{C}\mathrm{P}^\infty) = [\mathcal{O}(1)] - 1$, where the first map is Bott periodicity and $\mathcal{O}(1)$ denotes the universal complex line bundle on $\mathbf{C}\mathrm{P}^\infty$.

We next show that the existence of a complex orientation on E often forces the Atiyah-Hirzebruch spectral sequence for E to degenerate. We begin with a degeneration criterion (not the most general, but sufficient for our purposes).

Proposition 7. *Let X be a space and assume that each of the homology groups $\mathrm{H}_n(X;\mathbf{Z})$ is a free abelian group on generators $\{h_{\alpha,n}\}_{\alpha \in B_n}$. Let $c_{\alpha,n} \in \mathrm{H}^n(X;\mathbf{Z}) \simeq \mathrm{Hom}(\mathrm{H}_n(X;\mathbf{Z}), Z)$ be defined by the formula*

$$c_{\alpha,n}(h_{\beta,n}) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{otherwise.} \end{cases}$$

Let E be a multiplicative cohomology theory and let $\tau_{\leq 0} E$ denote its truncation, so that $\pi_i \tau_{\leq 0} E = \begin{cases} \pi_i E & \text{if } i \leq 0 \\ 0 & \text{otherwise.} \end{cases}$. The unit $S \to E$ determines a map of spectra $H\mathbf{Z} \simeq \tau_{\leq 0} S \to \tau_{\leq 0} E$. Under this map, the homology classes $h_{n,\alpha}$ have images $h'_{n,\alpha} \in (\tau_{\leq 0}E)_n(X)$ and the cohomology classes $c_{n,\alpha}$ have images $c'_{n,\alpha} \in (\tau_{\leq 0}E)^n(X)$. Assume that one of the following conditions is satisfied:

- $(*)$ *Each of the homology classes $h'_{n,\alpha}$ can be lifted to a class $h''_{n,\alpha} \in E_n(X)$.*
- $(*')$ *Each of the groups $\mathrm{H}_n(X;\mathbf{Z})$ is finitely generated, and each of the cohomology classes $c'_{n,\alpha}$ can be lifted to a class $c''_{n,\alpha} \in E^n(X)$.*

Then:

1. *The smash product $E \otimes \Sigma^\infty X_+$ is equivalent, as an E-module, to a coproduct $\bigoplus_{n,\alpha \in B_n} \Sigma^n E$.*
2. *The function spectrum E^X is equivalent to a product $\prod_{n,\alpha \in B_n} \Sigma^{-n} E$.*
3. *We have (noncanonical) isomorphisms $E_*(X) \simeq \pi_* E \otimes \mathrm{H}_*(X)$ and $E^*(X) \simeq \mathrm{Hom}(\mathrm{H}_*(X), \pi_*(E))$.*

Proof. We will prove (1); assertions (2) and (3) are obvious consequences. Let Y denote the suspension spectrum $\Sigma^\infty X_+$. In what follows, we will not use that Y is a suspension spectrum: only that Y is connective with freely generated homology. We construct a sequence of spectra

$$Y_0 \to Y_1 \to \dots$$

having colimit Y, with the following additional properties:

- (a) The map $Y_n \to Y$ induces an isomorphism in homology in degrees $\leq n$. In particular, Y is homotopy equivalent to the colimit of the sequence $\{Y_n\}$.
- (b) The spectrum Y_n is build from finitely many spheres of dimension $\leq n$; in particular, the cohomology groups $\mathrm{H}^k(Y_n;\mathbf{Z})$ vanish for $k > n$.

Assume that Y_{n-1} has been constructed, and let Z_n denote the cofiber of the map $Y_{n-1} \to Y$. Then Z_n is $(n-1)$-connected, and the map $\mathrm{H}_n(Y;\mathbf{Z}) \to \mathrm{H}_n(Z_n;\mathbf{Z})$ is an isomorphism. By the Hurewicz theorem, the image of each of the homology classes $h_{n,\alpha}$ is represented by a map $S^n \to Z_n$. Let $Z'_n = \bigoplus_{\alpha \in B_n} S^n$ and let $\phi_n : Z'_n \to Z_n$ be the induced map, so that we have a cofiber sequence

$$Z'_n \to Z_n \to Z''_n.$$

We now define Y_n to be the homotopy fiber product $Y \times_{Z_n} Z'_n$; in other words, Y_n is the homotopy fiber of the composite map $Y \to Z_n \to Z''_n$. It is easy to see that (a) and (b) hold.

Now suppose that $(*)$ is satisfied. Each $h''_{n,\alpha}$ is represented by a map of E-modules $\Sigma^n E \to E \otimes Y$. We will prove:

(c) The map $\theta : \bigoplus_{n,\alpha \in B_n} \Sigma^n E \to E \otimes Y$ is a homotopy equivalence.

To prove (c), it suffices to show that θ is k-connected for every value of k. This is obvious for $k = 0$. Assume that $k > 0$. Note that that ϕ_0 induces an E-module map $\bigoplus_{\alpha \in B_0} E \simeq E \otimes Z'_0 \to E \otimes Y$, which we can identify with a sequence of homology classes $b_{\alpha,0} \in E_0(Y)$. By construction, the classes $b_{\alpha,0}$ lift the classes $h'_{\alpha,0}$. Since Y is connective, we have $(\tau_{\geq 1} E)_0(Y) \simeq 0$ so that the map $E_0(Y) \to (\tau_{\leq 0} E)_0(Y)$ is injective; it follows that $b_{\alpha,0} = h''_{\alpha,0}$. We therefore have a map of cofiber sequences

$$\begin{tikzcd}
\bigoplus_{\alpha \in B_0} E \arrow[r] \arrow[d, "\theta'"] & \bigoplus_{n,\alpha \in B_n} \Sigma^n E \arrow[r] \arrow[d, "\theta"] & \bigoplus_{n>0,\alpha \in B_n} \Sigma^n E \arrow[d, "\theta''"] \\
Z'_0 \arrow[r] & Y \arrow[r] & Z''_0.
\end{tikzcd}$$

Since θ is a homotopy equivalence, to prove that θ' is k-connective it suffices to show that θ'' is k-connective. This follows from the inductive hypothesis, applied to the connective spectrum $\Sigma^{-1} Z''_0$.

Now suppose that condition $(*')$ is satisfied. We will prove, using induction on n, that each of the maps $E \otimes Y \to E \otimes Z_n$ admits a splitting $s_n : E \otimes Z_n \to E \otimes Y$, so that the cohomology classes $c''_{\alpha,n}$ give maps

$$\phi_\alpha : Z_n \to E \otimes Z_n \to E \otimes Y \xrightarrow{c''_{\alpha,n}} \Sigma^n E.$$

Using (b), we deduce that the map $(\tau_{\leq 0} E)^n Z_n \to (\tau_{\leq 0} E)^n Y$ is injective, so each the image of $\psi_\alpha \in E^n(Z_n) \to (\tau_{\leq 0} E)^n(Z_n)$ coincides with the image of $c_{\alpha,n} \in \mathrm{H}^n(Y; \mathbf{Z}) \simeq \mathrm{H}^n(Z_n; Z) \to (\tau_{\leq 0} E)^n(Z_n)$.

Assume that s_{n-1} has been constructed. The maps $\{\psi_\alpha\}_{\alpha \in B_{n-1}}$ together yield a map $Z_n \to \bigoplus_\alpha \Sigma^n E \simeq E \otimes Z'_n$, which we can identify with an E-module map $s_n : E \otimes Z_n \to E \otimes Z'_n$. Moreover, the compatibility of the classes ϕ_α with c_{α_n} shows that the composition

$$E \otimes Z'_n \xrightarrow{\psi} E \otimes Z_n \xrightarrow{\phi} E \otimes Z'_n$$

is the identity; that is, s_n is a splitting of the projection $E \otimes Y \to E \otimes Z_n$.

It now follows that $E \otimes Y \simeq \varinjlim (E \otimes Y_n) \simeq \varinjlim_n \bigoplus_{m \leq n} E \otimes Z'_m$. □

Example 8. Let $X = \mathbf{C}\mathrm{P}^n$, and let $t \in E^2(X)$ be a complex orientation on a multiplicative cohomology theory E. Then the cohomology classes $\{1, t, t^2, \ldots, t^n\}$ satisfy the hypotheses of Proposition 7. It follows that the classes $1, t, t^2, \ldots, t^n$ form a basis for $E^*(\mathbf{C}\mathrm{P}^n)$ over $\pi_* E$. We claim that $t^{n+1} = 0$. To prove this, we may replace E by its connective cover and thereby assume that E is connective: then $t^{n+1} \in E^{2n+2}(\mathbf{C}\mathrm{P}^n)$ vanishes since $\mathbf{C}\mathrm{P}^n$ has dimension $< 2n + 2$. It follows that we have a ring isomorphism $E^*(\mathbf{C}\mathrm{P}^n) \simeq (\pi_* E)[t]/(t^{n+1})$. Writing $\mathbf{C}\mathrm{P}^\infty = \varinjlim \mathbf{C}\mathrm{P}^n$, we get

$$E^*(\mathbf{C}\mathrm{P}^\infty) = \varprojlim E^*(\mathbf{C}\mathrm{P}^n) \simeq \varprojlim (\pi_* E)[t]/(t^{n+1}) \simeq (\pi_* E)[[t]].$$

Here the potential $\lim^1$-terms vanish because the maps $(\pi_* E)[t]/(t^{n+1}) \to (\pi_* E)[t]/(t^{m+1})$ are surjective.

Example 9. If $X = \mathbf{C}\mathrm{P}^m \times \mathbf{C}\mathrm{P}^n$, the same reasoning gives an isomorphism $E^*(X) \simeq (\pi_* E)[x, y]/(x^{m+1}, y^{n+1})$. Passing to the limit as before, we get an isomorphism $E^*(\mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty) = (\pi_* E)[[x, y]]$.

The space $\mathbf{C}\mathrm{P}^\infty$ is an Eilenberg-MacLane space $K(\mathbf{Z}, 2)$, and can therefore be realized as a topological abelian group. In fact, it is easy to realize $\mathbf{C}\mathrm{P}^\infty$ as a topological monoid: we can define $\mathbf{C}\mathrm{P}^\infty$ to be the projectivization $(V - \{0\})/\mathbf{C}^*$ for any complex vector space V of infinite dimension. Taking V to be the underlying vector space of the ring $\mathbf{C}[x]$, we get a commutative and associative multiplication on $\mathbf{C}\mathrm{P}^\infty$. The multiplication map

$$\mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty \to \mathbf{C}\mathrm{P}^\infty$$

classifies the operation of forming tensor products of complex line bundles. If E is a complex-oriented cohomology theory, then we get a pullback map on E-cohomology

$$(\pi_* E)[[t]] \simeq E^*(\mathbf{C}\mathrm{P}^\infty) \to E^*(\mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty) \simeq (\pi_* E)[[x, y]].$$

We let $f(x, y) \in (\pi_* E)[[x, y]]$ denote the image of t under this map. (The map is entirely determined by $f(x, y)$, since it is continuous with respect to the "inverse limit" topologies on the power series rings in question.)

The associativity and commutativity of the multiplication $\mathbf{C}\mathrm{P}^\infty$ imply the following:

Proposition 10. *Let E be a complex-oriented multiplicative cohomology theory. Then the above construction determines a formal group law $f(x, y) \in R[[x, y]]$, where R is the commutative ring $\oplus_n \pi_{2n} E$. This formal group law is compatible with the natural grading of R: that is, the expression $f(x, y)$ has degree -2, if we let x and y have degree -2.*

We close by describing another application of Proposition 7. Fix an integer $n \geq 0$, and let $X = BU(n)$ be the classifying space of the unitary group $U(n)$. There is a canonical map

$$\theta : (\mathbf{C}\mathrm{P}^\infty)^n \simeq BU(1) \times \cdots \times BU(1) \to BU(n).$$

This map classifies the construction $(\mathcal{L}_1, \ldots, \mathcal{L}_n) \mapsto \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$, which takes the direct sum of a collection of complex line bundles. Since the formation of direct sums is commutative up to isomorphism, the map θ is Σ_n-equivariant, up to homotopy. It therefore induces a map $\mathrm{H}^*(BU(n); \mathbf{Z}) \to \mathrm{H}^*((\mathbf{C}\mathrm{P}^\infty)^n; \mathbf{Z}) \simeq \mathbf{Z}[t_1, \ldots, t_n]$, whose image is contained in the subgroup $\mathbf{Z}[t_1, \ldots, t_n]^{\Sigma_n}$ of symmetric polynomials in n-variables. This ring of invariants is given by $\mathbf{Z}[c_1, c_2, \ldots, c_n]$, where c_i is the ith elementary symmetric function on $(t_1, \ldots, t_n)$. In fact, this construction yields an isomorphism $\mathrm{H}^*(BU(n), \mathbf{Z}) \to \mathbf{Z}[c_1, \ldots, c_n]$; under this isomorphism, the cohomology class c_i corresponds to the ith Chern class of the universal bundle.

Dually, can write $\mathrm{H}_*(\mathbf{C}\mathrm{P}^\infty; \mathbf{Z}) = \mathbf{Z}\{\beta_0, \beta_1, \ldots\}$, where $\{\beta_i\}$ is the dual basis to $\{t_i\}$. Then $\mathrm{H}_*(\mathrm{BU}(n); \mathbf{Z})$ is given by $\mathrm{H}_*(\mathbf{C}\mathrm{P}^\infty; \mathbf{Z})^{\otimes n}_{\Sigma_n} = \mathrm{Sym}^n\, \mathrm{H}_*(\mathbf{C}\mathrm{P}^\infty; \mathbf{Z})$. In particular, it is a free $\mathbf{Z}$-module whose generators can be lifted to $\mathrm{H}_*((\mathbf{C}\mathrm{P}^\infty)^n; \mathbf{Z})$.

Let E be a complex-oriented multiplicative cohomology theory. Then we have a canonical isomorphism $E^*(\mathbf{C}\mathrm{P}^\infty) \simeq (\pi_* E)[[t]]$. The (topological) basis $\{t^i\}$ for this cohomology has a dual basis $\{\beta_i\}$ for $E_*(\mathbf{C}\mathrm{P}^\infty)$ over $\pi_* E$. Using the map θ, we get homology classes $\{\beta_{i_1} \beta_{i_2} \ldots \beta_{i_n}\}_{i_1 \leq \cdots \leq i_n}$ in $E_*(BU(n))$ which lift the corresponding basis for the $\mathbf{Z}$-homology of $BU(n)$. It follows from Proposition 7 that $E_*(BU(n))$ is freely generated by the classes $\{\beta_{i_1} \beta_{i_2} \ldots \beta_{i_n}\}_{0 \leq i_1 \leq \cdots \leq i_n}$ over $\pi_* E$.

The same argument shows that $E_*(BU(n) \times BU(n))$ is given by $E_*(BU(n)) \otimes_{\pi_* E} E_*(BU(n))$. The diagonal map $BU(n) \to BU(n) \times BU(n)$ determines a comultiplication on $E_*(BU(n))$. When $n = 0$, this comultiplication is dictated by the structure of the multiplication on $E^*(BU(1)) = (\pi_* E)[[t]]$: namely, it is given by $\delta_{\beta_n} = \sum_{i+j=n} \beta_i \otimes \beta_j$. Since θ induces a map of coalgebras $E_*(BU(1)^n) \to E_*(BU(n))$, this completely determines the comultiplication on $E_*(BU(n))$. More informally, we can say that the comultiplication on $E_*(BU(n))$ is given by the same formulas as in the case of integral homology. It follows that multiplication on $E^*(BU(n))$ can be described in the same way as the multiplication on the ordinary cohomology of $E^*(BU(n))$. More precisely, we have a canonical isomorphism

$$E^*(BU(n)) \simeq (\pi_* E)[[c_1, \ldots, c_n]]$$

where c_i is dual to β_1^i (with respect to the basis consisting of monomials in the β_i). We can think of the c_i as analogues of the Chern classes in E-cohomology.

Complex Bordism (Lecture 5)

February 4, 2010

In this lecture, we will introduce another important example of a complex-oriented cohomology theory: the cohomology theory MU of *complex bordism*. In fact, we will show that MU is *universal* among complex-oriented cohomology theories.

We begin with a general discussion of orientations. Let X be a topological space and let ζ be a vector bundle of rank n on X. We may assume without loss of generality that ζ is equipped with a metric, so that the unit ball bundle $B(\zeta) \to X$ and the unit sphere bundle $S(\zeta) \to X$ are well-defined. If E is an arbitrary cohomology theory, we can define the twisted E-cohomology $E^{*-\zeta}(X)$ to be the relative cohomology $E^*(B(\zeta), S(\zeta))$.

Example 1. If ζ is the trivial bundle of rank n, then $B(\zeta) \simeq B^n \times X$ and $S(\zeta) \simeq S^{n-1} \times X$. In this case, we have a canonical isomorphism $E^{*-\zeta}(X) = E^*(X \times B^n, X \times S^{n-1}) \simeq E^{*-n}(X)$.

If E is a multiplicative cohomology theory, then $E^{*-\zeta}(X)$ is a module over $E^*(X)$.

Definition 2. Let ζ be a vector bundle of rank n on a space X, and let E be a multiplicative cohomology theory. An *E-orientation of* ζ is a cohomology class $u \in E^{n-\zeta}(X) \simeq E^n(B(\zeta), S(\zeta))$ such that:

$(*)$ For every point $x \in X$, the restriction $x^*(u) \in E^{n-\zeta_x}(\{x\}) \simeq E^0(\{x\})$ is a generator of $E^*(\{x\}) \simeq \pi_* E$ (as a $\pi_* E$-module).

In this case, we say that u is a *Thom class* for ζ in E-cohomology.

Remark 3. The identification $E^{n-\zeta_x}(\{x\}) \simeq E^0(\{x\})$ is noncanonical: it depends on a trivialization of the fiber ζ_x. This dependence is not very strong, since the orthogonal group $O(n)$ has only two components: the resulting elements of $E^0(\{x\})$ are off by a sign if we choose trivializations with different orientations.

Remark 4. A consequence of Definition 2 is that the Leray-Serre spectral sequence for the fibration $S(\zeta) \to X$ degenerates and gives an identification $E^*(X) \simeq E^{*+n-\zeta}(X)$, given by multiplication by u.

Remark 5. In the setting of Definition 2, it suffices to check the condition at one point x in each connected component of X.

Our next goal is to show that if E is a complex-oriented cohomology theory, then all complex vector bundles have a canonical E-orientation. To prove this, it suffices to consider the universal case: that is, the case of a universal bundle ζ of (complex) rank n over the classifying space $BU(n)$. Recall that $BU(n)$ can be realized as the quotient of a contractible space $EU(n)$ by a free action of the unitary group $U(n)$. In this case, the subgroup $U(n-1)$ also acts freely on $EU(n)$, so the quotient $EU(n)/U(n-1)$ is a model for the classifying space $BU(n-1)$. This realization gives us a fibration $BU(n-1) \to BU(n)$, whose fiber is the quotient group $U(n)/U(n-1) \simeq S^{2n-1}$. In fact, this sphere bundle can be identified with the unit sphere bundle $S(\zeta)$. Since $B(\zeta) \simeq BU(n)$, we get canonical isomorphisms $E^{*-\zeta}(BU(n)) = E^*(BU(n), BU(n-1))$.

We have computed these groups: the group $E^*(BU(n))$ is a power series ring $(\pi_* E)[[c_1, \ldots, c_n]]$ and the group $E^*(BU(n-1))$ is a power series ring $(\pi_* E)[[c_1, \ldots, c_{n-1}]]$. The restriction map $\theta : E^*(BU(n)) \to E^*(BU(n-1))$ is a surjective ring homomorphism. It follows that the relative cohomology group $E^*(BU(n), BU(n-1))$ can be identified with the kernel of θ: that is, with the ideal $c_n(\pi_* E)[[c_1, \ldots, c_n]]$. This is in fact a free module over $E^*(BU(n))$, which suggests the following:

Proposition 6. *The cohomology class $c_n \in E^{2n}(BU(n), BU(n-1)) \simeq E^{2n-\zeta}(BU(n))$ is a Thom class (so that the universal bundle ζ on $BU(n)$ has a canonical E-orientation).*

Proof. We must check that condition $(*)$ holds at every point of $BU(n)$. Since $BU(n)$ is connected, it will suffice to check $(*)$ at *any* points of $BU(n)$. We may therefore replace ζ by its pullback along the map $f : BU(1)^n \to \mathrm{BU}(n)$ and c_n by its image in

$$E^{*-f^*\zeta}(BU(1)^n) \simeq (t_1 \dots t_n)(\pi_* E)[[t_1, \dots, t_n]] \subseteq (\pi_* E)[[t_1, \dots, t_n]] \simeq E^*(BU(1)^n),$$

which can be identified with the product $t_1 \dots t_n$. Since $f^*\zeta$ is a direct sum $\oplus_{1 \le i \le n} p_i^* \mathcal{O}(1)$ of pullbacks of the universal line bundle $\mathcal{O}(1)$ on $BU(1) \simeq \mathbf{C}\mathrm{P}^\infty$ along the projection maps $p_i : BU(1)^n \to BU(1)$, we can reduce to proving the assertion in the case $n = 1$. In this case, $E^{*-\zeta}(BU(1))$ can be identified with the reduced cohomology $\widetilde{E}^*(\mathbf{C}\mathrm{P}^\infty)$, and the condition that $u \in \widetilde{E}^2(\mathbf{C}\mathrm{P}^\infty)$ be an orientation of $\mathcal{O}(1)$ is that it maps to a unit when restricted to $\widetilde{E}^2(S^2) \simeq \pi_0 E$. Our complex orientation is even better: it maps to $1 \in \pi_0 E$. □

If ζ' is any complex vector bundle of rank n on any (nice) space X, then we can write $\zeta' = f^*\zeta$ for some classifying map $f : X \to BU(n)$. We can then define an orientation $u_{\zeta'} \in E^{2n-\zeta'}(X)$ to be the pullback of $c_n \in E^{2n-\zeta}(BU(n))$.

By construction, our Chern classes in E-cohomology have the same behavior with respect to direct sums of vector bundles as the usual Chern classes: namely, we have

$$c_n(\zeta \oplus \zeta') = \sum_{i+j=n} c_i(\zeta) c_j(\zeta').$$

In particular, if ζ and ζ' have ranks a and b, then we have $c_{a+b}(\zeta + \zeta') = c_a(\zeta) c_b(\zeta')$. We conclude from this:

(1) If ζ and ζ' are complex vector bundles of rank a and b on a space X, then the Thom classes $u_\zeta \in E^{2a-\zeta}(X)$ and $u_{\zeta'} \in E^{2b-\zeta'}(X)$ have product $u_{\zeta+\zeta'} \in E^{2a+2b-(\zeta \oplus \zeta')}(X)$.

We also have the following

(2) Let ζ be the trivial bundle of rank 1 on a space X. Then the Thom class $u_\zeta \in E^{2-\zeta}(X) \simeq E^0(X)$ coincides with the unit. This is a translation of our assumption that $t \in \widetilde{E}^2(\mathbf{C}\mathrm{P}^\infty)$ restricts to the unit in $\widetilde{E}^2(S^2) \simeq \pi_0 E$.

Definition 7. For each integer n, we let $\mathrm{MU}(n)$ denote the Thom spectrum $\Sigma^{-2n} BU(n)^{\zeta_n} = \Sigma_+^{\infty-2n} BU(n)/BU(n-1)$, where ζ_n denotes the universal bundle of rank n. The restriction of ζ_n to $BU(n-1)$ is the sum of a trivial bundle $\mathbf{1}$ of rank 1 with a bundle ζ_{n-1}. We therefore have a canonical map

$$\mathrm{MU}(n-1) \simeq \Sigma^{2-2n} BU(n-1)^{\zeta_{n-1}} = \Sigma^{-2n} BU(n-1)^{\zeta_{n-1} \oplus \mathbf{1}} \to \Sigma^{-2n} BU(n)^{\zeta_n} = \mathrm{MU}(n).$$

The universal Thom class $c_n \in E^n(BU(n)/BU(n-1))$ can be interpreted as a map of spectra $\phi_n : \mathrm{MU}(n) \to E$. It follows from (1) and (2) that the restriction of this map to $\mathrm{MU}(n-1)$ is homotopic to ϕ_{n-1}. In the next lecture, we will see that the maps $\{\phi_n\}_{n \ge 0}$ therefore determine a map from the colimit

$$S \simeq \mathrm{MU}(0) \to \mathrm{MU}(1) \to \mathrm{MU}(2) \to \dots$$

into E.

Definition 8. The colimit $\varinjlim \mathrm{MU}(n)$ is denoted by MU; it is called the *complex bordism spectum.*

Remark 9. The complex bordism spectrum MU can be described as a Thom spectrum for the space $BU = \varinjlim BU(n)$. However, it is not a Thom spectrum for a vector bundle of any particular rank: rather, it is the Thom spectrum for a virtual bundle of rank 0, whose restriction to each $BU(n)$ is a formal difference $\zeta_n - \mathbf{1}^n$.

Remark 10. The complex bordism spectrum has a natural geometric interpretation. Namely, each homotopy group $\pi_n E$ can be identified with the group of bordism classes of n-dimensional manifolds M equipped with a stable almost complex structure (that is, a complex structure on the direct sum of the tangent bundle M with a trivial vector bundle of sufficiently large rank). More generally, if X is any space, we can identify the homology groups $E_n X$ with bordism groups of stably almost complex n-manifolds equipped with a map to X.

MU and Complex Orientations (Lecture 6)

February 4, 2010

In the last lecture, we defined spectra $\mathrm{MU}(n) = \Sigma^{\infty-2n} BU(n)/BU(n-1)$ which form a direct system

$$\mathrm{MU}(0) \to \mathrm{MU}(1) \to \mathrm{MU}(2) \to \cdots$$

The (homotopy) colimit of this sequence is called the *complex bordism spectrum* and is denoted by MU.

Example 1. The spectrum $\mathrm{MU}(0)$ is equivalent to the sphere spectrum.

Example 2. The spectrum $\mathrm{MU}(1)$ is the desuspension $\Sigma^{\infty-2}\,\mathbf{C}\mathrm{P}^\infty$ of $\mathbf{C}\mathrm{P}^\infty = \mathrm{BU}(1)$.

Remark 3. In terms of the above identifications, the inclusion $\mathrm{MU}(0) \to \mathrm{MU}(1)$ is given by

$$\mathrm{MU}(0) \simeq S \simeq \Sigma^{\infty-2} S^2 \to \Sigma^{\infty-2}\,\mathbf{C}\mathrm{P}^\infty = \mathrm{MU}(1).$$

Remark 4. The direct sum of complex vector bundles is classified by a multiplication $m_{a,b} : BU(a) \times BU(b) \to BU(a+b)$. Passing to Thom spectra, we get a multiplication $\mathrm{MU}(a) \otimes \mathrm{MU}(b) \to \mathrm{MU}(a+b)$. We note that the inclusion $\mathrm{MU}(n) \to \mathrm{MU}(n+1)$ can be identified with the map

$$\mathrm{MU}(n) \simeq S \otimes \mathrm{MU}(n) = \mathrm{MU}(0) \otimes \mathrm{MU}(n) \to \mathrm{MU}(1) \otimes \mathrm{MU}(n) \to \mathrm{MU}(n+1).$$

Remark 5. Taking the limit in a and b, we get a multiplication $\mathrm{MU} \otimes \mathrm{MU} \to \mathrm{MU}$. That is, MU has the structure of a ring spectrum. In fact, this multiplication is commutative and associative up to homotopy. It has a unit, given by the map $S \simeq \mathrm{MU}(0) \to \mathrm{MU}$.

In fact, the situation is much better: the multiplication on MU is commutative and associative up to all higher homotopies. That is, MU has the structure of an E_∞-ring spectrum.

Let E be a complex-oriented cohomology theory. In the last lecture, we saw that every complex vector bundle is E-orientable. In fact, for each integer n we can write down a canonical orientation of the universal bundle ζ_n of rank n on the classifying space $BU(n)$: it is classified by a map $\phi_n : \mathrm{MU}(n) \to E$. These maps ϕ_n are uniquely characterized by the following requirements:

(1) The map $\phi_1 : \mathrm{MU}(1) \to E$ is given by the complex orientation of E (note that we can identify the set of homotopy classes of maps $[\mathrm{MU}(1), E]$ with $\widetilde{E}^2(\mathbf{C}\mathrm{P}^\infty)$).

(2) The maps ϕ_n are multiplicative in the following sense: for every pair of integers m and n, the diagram

$$\begin{tikzcd} \mathrm{MU}(m) \otimes \mathrm{MU}(n) \arrow[r] \arrow[d, "\phi_m \otimes \phi_n"] & \mathrm{MU}(m+n) \arrow[d, "\phi_{m+n}"] \\ E \otimes E \arrow[r] & E \end{tikzcd}$$

commutes up to homotopy.

To prove assertion (2), we recall that $E^*(\mathrm{MU}(m+n))$ can be identified (up to a shift in grading) with the ideal in $E^*(\mathrm{BU}(m+n)) \simeq (\pi_* E)[[c_1, \ldots, c_{m+n}]]$ generated by the Chern class c_{m+n}. Similarly, $E^*(\mathrm{MU}(m) \otimes \mathrm{MU}(n))$ can be identified with the ideal in $E^*(\mathrm{BU}(m) \times \mathrm{BU}(n)) \simeq (\pi_* E)[[c_1, \ldots, c_m, c'_1, \ldots, c'_n]]$ generated by the product $c_m c'_n$. The commutativity of the diagram now follows from the equation $c_{m+n}(\zeta \oplus \zeta') = c_m(\zeta) c_n(\zeta')$, where ζ and ζ' are vector bundles of rank m and n.

We claim that the composite map

$$\mathrm{MU}(n) \to \mathrm{MU}(n+1) \stackrel{\phi_{n+1}}{\to} E$$

coincides with ϕ_n. Using (2) and Remark 4, we deduce that this composite map is given by

$$\mathrm{MU}(n) \simeq \mathrm{MU}(n) \otimes \mathrm{MU}(0) \to \mathrm{MU}(n) \otimes \mathrm{MU}(1) \stackrel{\phi_n \phi_1}{\to} E.$$

We are therefore reduced to proving that $\phi_1 | \mathrm{MU}(0)$ coincides with ϕ_0 (which is the unit map $S \to E$). According to Remark 3, this map is given by the class in $\pi_0 E$ given by restricting our complex orientation $t \in \widetilde{E}^2(\mathbf{C}\mathrm{P}^\infty)$ to $\widetilde{E}^2(S^2) \simeq \pi_0 E$, which we have assumed to be the unit in E.

The mapping spectrum E^{MU} can be obtained as a homotopy limit of mapping spectra $E^{\mathrm{MU}(n)}$. We therefore have a Milnor long exact sequence

$$\varprojlim{}^1 \{E^{-1}(\mathrm{MU}(n))\} \to E^0(\mathrm{MU}) \to \varprojlim E^0(\mathrm{MU}(n)) \to \varprojlim{}^1 \{E^0(\mathrm{MU}(n))\}.$$

The outer groups vanish, since each of the restriction maps $E^*(\mathrm{MU}(n+1)) \to E^*(\mathrm{MU}(n))$ is surjective (it corresponds, under our choice of Thom isomorphisms, to the restriction map $E^*(BU(n+1)) \to E^*(BU(n))$, which is obtained by killing c_{n+1} in the power series ring $E^*(BU(n+1)) \simeq (\pi_* E)[[c_1, \ldots, c_{n+1}]]$). It follows that the maps $\phi_n : \mathrm{MU}(n) \to E$ can be uniquely amalgamated to give a map $\phi : \mathrm{MU} \to E$.

Proposition 6. *The map ϕ is a map of ring spectra.*

Proof. We must show that the diagram

$$\begin{tikzcd}
\mathrm{MU} \otimes \mathrm{MU} \arrow[r, "\phi \otimes \phi"] \arrow[d] & \mathrm{MU} \arrow[d, "\phi"] \\
E \otimes E \arrow[r] & E
\end{tikzcd}$$

commutes. Repeating the above argument, we conclude that $E^0(\mathrm{MU} \otimes \mathrm{MU})$ can be obtained as the inverse limit of the cohomology groups $E^0(\mathrm{MU}(a) \otimes \mathrm{MU}(b))$. The desired result now follows from the commutativity of each of the squares

$$\begin{tikzcd}
\mathrm{MU}(a) \otimes \mathrm{MU}(b) \arrow[r] \arrow[d] & \mathrm{MU}(a+b) \arrow[d] \\
E \otimes E \arrow[r] & E
\end{tikzcd}$$

(see Remark 4). $\square$

The inclusion $\Sigma^{\infty-2} \mathbf{C}\mathrm{P}^\infty \simeq \mathrm{MU}(1) \to \mathrm{MU}$ determines a class $t \in \widetilde{\mathrm{MU}}^2(\mathbf{C}\mathrm{P}^\infty)$. By construction, the ring spectrum map $\phi : \mathrm{MU} \to E$ carries t to our chosen complex orientation of E.

Remark 7. The class $t \in \widetilde{\mathrm{MU}}^2(\mathbf{C}\mathrm{P}^\infty)$ is a complex orientation of MU. To see this, we note that the restriction of t to $\widetilde{\mathrm{MU}}^2(S^2) \simeq \pi_0 \mathrm{MU}$ is given by the map $S \simeq \mathrm{MU}(0) \to \mathrm{MU}(1) \to \mathrm{MU}$, which is the unit for the ring spectrum MU.

We can summarize our discussion as follows:

Theorem 8. *Let E be a commutative ring spectrum, and let $t \in \widetilde{\mathrm{MU}}^2(\mathbf{CP}^\infty)$ be the complex orientation described above. The construction $(\phi : \mathrm{MU} \to E) \mapsto \phi(t)$ determines a bijection between complex orientations of E and ring spectrum maps $\mathrm{MU} \to E$.*

In other words, the complex bordism spectrum MU is the *universal* complex-oriented cohomology theory.

Proof. The above analysis shows that given any complex orientation of E, we can construct a ring spectrum map $\phi : \mathrm{MU} \to E$ which carries our complex orientation to the specified complex orientation of E. It remains to prove injectivity. Let $\phi, \phi' : \mathrm{MU} \to E$ be two ring spectrum maps which determine the same complex orientation of E; we wish to prove that ϕ and ϕ' are homotopic. The condition that ϕ and ϕ' determine the same complex orientation tells us that $\phi| \mathrm{MU}(1) \simeq \phi'| \mathrm{MU}(1)$. Since E is complex-orientable, the preceding argument shows that $E^0(\mathrm{MU}) \simeq \varprojlim E^0(\mathrm{MU}(n))$. It will therefore suffice to show that ϕ and ϕ' have the same restriction to $\mathrm{MU}(n)$ for every integer n. Since ϕ is a ring map, the composition

$$\mathrm{MU}(1)^{\otimes n} \to \mathrm{MU}(n) \xrightarrow{\phi} E$$

is a product of n copies of $\phi| \mathrm{MU}(1)$ in $E^0(\mathrm{MU}(1))$, and therefore coincides with the composition

$$\mathrm{MU}(1)^{\otimes n} \to \mathrm{MU}(n) \xrightarrow{\phi'} E.$$

It will therefore suffice to show that the restriction maps $E^0(\mathrm{MU}(n)) \to E^0(\mathrm{MU}(1)^{\otimes n})$. Using our Thom isomorphisms (provided by any complex orientation of E), this is equivalent to the injectivity of the map $E^0(\mathrm{BU}(n)) \to E^0(\mathrm{BU}(1)^n)$, which is the "spliting principle" we discussed earlier. $\square$

Theorem 8 suggests that if we are interested in complex-oriented cohomology theories and the associated formal group laws, then we should focus our attention on the complex bordism spectrum MU. The universal complex orientation determines a (graded) formal group law $f(x, y) \in (\pi_* \mathrm{MU})[[x, y]]$. As we have seen, this formal group law is given by a map of graded rings $L \to \pi_* \mathrm{MU}$.

Our goal next week will be to prove the following theorem:

Theorem 9 (Quillen)**.** *The map $L \to \pi_* \mathrm{MU}$ is an isomorphism. (In particular, the spectrum* MU *has homotopy groups only in even degrees.)*

The Homology of MU (Lecture 7)

February 9, 2010

Last week, we defined the complex bordism spectrum MU and showed that it was a universal complex oriented cohomology theory. In particular, there is a formal group law $f(x,y)$ over the ring $\pi_* \mathrm{MU}$. This formal group law is classified by a map $L \to \pi_* \mathrm{MU}$, where L is the Lazard ring. Our goal this week is to prove the following fundamental result:

Theorem 1 (Quillen)**.** *The map $L \to \pi_* \mathrm{MU}$ is an isomorphism. (In particular, the spectrum* MU *has homotopy groups only in even degrees.)*

The obstacle to overcome in the proof of Theorem 1 is that homotopy groups are typically difficult to compute. In this lecture, we will consider the much easier problem of computing the *homology* groups $\mathrm{H}_*(\mathrm{MU}; \mathbf{Z})$. In fact, we will do something a little more general: namely, we compute the homology $E_*(\mathrm{MU})$, where E is an arbitrary complex oriented cohomology theory.

Since MU is the (homotopy) colimit of the sequence $\mathrm{MU}(n)$, we have $E_*(\mathrm{MU}) \simeq \varinjlim E_*(\mathrm{MU}(n))$. Since every complex vector bundle has a canonical E-orientation, we obtain a canonical isomorphism of $E_*(\mathrm{MU}(n))$ with $E_*(\mathrm{BU}(n))$. Recall that $E_*(\mathrm{BU}(n))$ can be identified with the symmetric power $\mathrm{Sym}^n E_*(\mathrm{BU}(1)) = \mathrm{Sym}^n(\pi_* E\{\beta_0, \beta_1, \ldots\})$, where $\{\beta_i\}$ is the dual basis to topological basis $\{t^i\}$ for $E^*(BU(1)) \simeq (\pi_* E)[[t]]$. Correspondingly, we we identify $E_*(\mathrm{MU}(n))$ with the symmetric power $\mathrm{Sym}^n E_*(\mathrm{MU}(1)) \simeq \mathrm{Sym}^n(\pi_* E\{b_0, b_1, b_2, \ldots\})$, where the $\{b_i\}$ are a dual basis to the basis $\{t^{i+1}\}$ for the cohomology

$$E^*(\mathrm{MU}(1)) \simeq \widetilde{E}^*(\mathbf{C}\mathrm{P}^\infty) \simeq t(\pi_* E)[[t]] \subseteq (\pi_* E)[[t]] \simeq E^*(\mathbf{C}\mathrm{P}^\infty).$$

To pass to the bordism spectrum MU, we need to understand the transition maps $E_*(\mathrm{MU}(n)) \to E_*(\mathrm{MU}(n+1))$. These maps are induced by the composition

$$\mathrm{MU}(n) \simeq S \otimes \mathrm{MU}(n) \simeq \mathrm{MU}(0) \otimes \mathrm{MU}(n) \to \mathrm{MU}(1) \otimes \mathrm{MU}(n) \to \mathrm{MU}(n+1).$$

In the case $n = 0$, the inclusion $\mathrm{MU}(0) \to \mathrm{MU}(1)$ induces a map $\pi_* E = E_*(\mathrm{MU}(0)) \to E_*(\mathrm{MU}(1))$, which simply corresponds to the element b_0 in our chosen basis for $E_*(\mathrm{MU}(1))$. We conclude:

- For each $n \geq 0$, the map on homology $\mathrm{Sym}^n(\pi_* E)\{b_0, b_1, \ldots\} \simeq E_*(\mathrm{MU}(n)) \to E_*(\mathrm{MU}(n+1)) \simeq \mathrm{Sym}^{n+1}(\pi_* E)\{b_0, b_1, \ldots\}$ is given by multiplication by the class b_0.

There is a map of polynomial algebras $(\pi_* E)[b_0, b_1, b_2, \ldots] \to (\pi_* E)[b_1, b_2, \ldots]$ which carries b_0 to 1. This map induces an isomorphism from $\mathrm{Sym}^n(\pi_* E\{b_0, b_1, \ldots\})$ to $\mathrm{Sym}^{\leq n}(\pi_* E\{b_1, b_2, \ldots\})$. Under these isomorphisms, the map $E_*(\mathrm{MU}(n)) \to E_*(\mathrm{MU}(n+1))$ simply corresponds to the inclusion $\mathrm{Sym}^{\leq n}(\pi_* E\{b_1, b_2, \ldots\}) \hookrightarrow \mathrm{Sym}^{\leq n+1}(\pi_* E\{b_1, b_2, \ldots\})$. Passing to the limit as n grows, we obtain the following:

Proposition 2. *Let E be a complex oriented cohomology theory, and let $\{b_i\} \subseteq E_*(\mathrm{MU}(1))$ be dual to the topological basis $\{t^{i+1}\}$ for $E^*(\mathrm{MU}(1)) \simeq t(\pi_* E)[[t]]$. Then the images of the b_i in $E_*(\mathrm{MU})$ determine a ring isomorphism $(\pi_* E)[b_1, b_2, \ldots] \simeq E_*(\mathrm{MU})$ (note that the image of b_0 is the identity of $E_*(\mathrm{MU})$).*

Corollary 3. *There is a canonical isomorphism $\mathrm{H}_*(\mathrm{MU}; \mathbf{Z}) \simeq \mathbf{Z}[b_1, b_2, \ldots]$.*

To use this observation in the proof of Theorem 1, we need to understand the composition $L \to \pi_* \mathrm{MU} \to \mathrm{H}_*(\mathrm{MU}; \mathbf{Z}) \simeq \mathbf{Z}[b_1, b_2, \ldots]$ (here the second map is the Hurewicz homomorphism). This map classifies a formal group law over the commutative ring $\mathbf{Z}[b_1, b_2, \ldots]$. We will see in a moment that this is the same formal group law that we studied in Lecture 2.

It will be convenient to again consider a slightly more general problem. Let E be any complex oriented cohomology theory. The smash product $\mathrm{MU} \otimes E$ is another multiplicative cohomology theory, with $\pi_*(\mathrm{MU} \otimes E) = E_*(\mathrm{MU}) \simeq (\pi_* E)[b_1, b_2, \ldots]$. This multiplicative cohomology theory has *two* complex orientations: one coming from our given complex orientation on E, and one from the universal complex orientation on MU. In other words, we can find two classes $t_E, t_{\mathrm{MU}} \in \widetilde{\mathrm{MU} \otimes E}^2(\mathbf{C}\mathrm{P}^\infty)$, which determine isomorphisms

$$(\pi_* E)[b_1, b_2, \ldots][[t_E]] \simeq (\mathrm{MU} \otimes E)^*(\mathbf{C}\mathrm{P}^\infty) \simeq (\pi_* E)[b_1, b_2, \ldots][[t_{\mathrm{MU}}]].$$

In particular, we can write t_{MU} as a power series

$$\sum_{i \geq 1} a_i t_E^{i+1}$$

for some coefficients $a_i \in (\pi_* E)[b_1, b_2, \ldots]$.

Claim 4. *We have $a_i = b_i$: that is, we can write $t_{\mathrm{MU}} = t_E + b_1 t_E^2 + b_2 t_E^3 + \ldots$.*

To prove the claim, note that we can think of a class in $\widetilde{\mathrm{MU} \otimes E}^2(\mathbf{C}\mathrm{P}^\infty)$ as a map of spectra $\mathrm{MU}(1) \to \mathrm{MU} \otimes E$. By general nonsense, this is the same thing as a map of E-module spectra from $\mathrm{MU}(1) \otimes E$ to $\mathrm{MU} \otimes E$. Consequently, t_E and t_{MU} correspond to a pair of maps $\phi_{\mathrm{MU}}, \phi_E : \mathrm{MU}(1) \otimes E \to \mathrm{MU} \otimes E$.

For every integer i, the class $b_i \in E_{2i}(\mathrm{MU}(1))$ determines a map of E-modules $\Sigma^{2i} E \to \mathrm{MU}(1) \otimes E$. Taking the coproduct, we obtain an equivalence of E-module spectra

$$\oplus_{i \geq 0} \Sigma^{2i} E \simeq \mathrm{MU}(1) \otimes E.$$

Consequently, to describe a map of spectra from $\mathrm{MU}(1) \otimes E$ to $\mathrm{MU} \otimes E$, we just need to specify its restriction to $\Sigma^{2i} E$ for every integer i.

The map ϕ_E is given by the composition

$$E \otimes \mathrm{MU}(1) \xrightarrow{\lambda} E \xrightarrow{u} E \otimes \mathrm{MU},$$

where λ classifies the complex orientation of E and u is the unit map $E \to E \otimes \mathrm{MU}$. Since the $\{b_i\}$ are chosen to be the dual basis to $\{t^{i+1}\}$, we see that λ vanishes on $\Sigma^{2i} E$ for $i > 0$, and restricts to the identity map $\Sigma^{2i} E \simeq E$ when $i = 0$.

The map ϕ_{MU} is given by smashing with E the canonical map $\mathrm{MU}(1) \to \mathrm{MU}$. In particular, ϕ_{MU} can be identified with the coproduct of the family of maps $\phi^i_{\mathrm{MU}} : \Sigma^{2i} E \to \mathrm{MU} \otimes E$ classified by $b_i \in E_{2i}(\mathrm{MU})$.

Note that the tensor product $\mathrm{MU}(1) \otimes E$ is acted on by the function spectrum $E^{\mathbf{C}\mathrm{P}^\infty}$: at the level of homology, this is given by the action of the cohomology ring $E^*(\mathbf{C}\mathrm{P}^\infty)$ on the reduced homology $\widetilde{E}_*(\mathbf{C}\mathrm{P}^\infty)$ (via the cap product). In particular, our complex orientation t induces a map $\Sigma^{-2}(\mathrm{MU}(1) \otimes E) \to \mathrm{MU}(1) \otimes E$, which we will denote by T. In terms of our identification $\mathrm{MU}(1) \otimes E \simeq \oplus_{i \geq 0} \Sigma^{2i} E$, the map T carries $\Sigma^{-2} \Sigma^{2i} E$ to $\Sigma^{2(i-1)} E$ by the identity map for $i > 0$, and is zero otherwise.

It follows that ϕ_{MU} can be written as a formal sum $\sum_i \phi^i_{\mathrm{MU}}$, where ϕ^i_{MU} is given by the composition

$$\mathrm{MU}(1) \otimes E \xrightarrow{T^i} \mathrm{MU}(1) \otimes E \xrightarrow{\lambda} E \xrightarrow{b_i} \mathrm{MU} \otimes E.$$

In other words, we have the formula

$$\phi_{\mathrm{MU}} = \sum_i b_i \phi_E \circ T^i.$$

Identifying ϕ_{MU} and ϕ_E with classes in $(\mathrm{MU} \otimes E)^0(\mathrm{MU}(1)) \simeq t_E(\pi_*)[b_1, \ldots][[t_E]]$, we see that T^i is given by multiplication by t_E. It follows that we have

$$t_{\mathrm{MU}} = \sum_i t_E^i(b_i t_E) = \sum_i b_i t_E^{i+1}.$$

This completes the proof of Claim 4.

Let R be the graded-commutative ring $\pi_*(\mathrm{MU} \otimes E) \simeq E_*(\mathrm{MU}) \simeq (\pi_* E)[b_1, b_2, \ldots]$. The complex orientations t_E and t_{MU} give rise to a pair of formal group laws $f_E, f_{\mathrm{MU}} \in R[x, y]$. These formal group laws can be characterized as follows: if $\pi_1, \pi_2 : \mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty \to \mathbf{C}\mathrm{P}^\infty$ are the two projection maps and $m : \mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty \to \mathbf{C}\mathrm{P}^\infty$ denotes the multiplication, then we have

$$m^* t_E = f_E(\pi_1^* t_E, \pi_2^* t_E) \qquad m^* t_{\mathrm{MU}} = f_{\mathrm{MU}}(\pi_1^* t_{\mathrm{MU}}, \pi_2^* t_{\mathrm{MU}})$$

in the cohomology ring $(\mathrm{MU} \otimes E)^*(\mathbf{C}\mathrm{P}^\infty)$. This immediately gives the following result:

Proposition 5. *Let E be a complex oriented cohomology theory and let R, $f_E, f_{\mathrm{MU}} \in R[[x, y]]$ be defined as above. Let $g(x) \in R[[x]]$ denote the power series $g(x) = x + b_1 x^2 + b_2 x^3 + \cdots$, so we have the formal identity $t_{\mathrm{MU}} = g(t_E)$. Then f_{MU} is given by the formula*

$$f_{\mathrm{MU}}(x, y) = g \circ f_E(g^{-1}(x), g^{-1} y).$$

Specializing to the case where E is an Eilenberg-MacLane spectrum $H\mathbf{Z}$, we deduce:

Corollary 6. *Let $E = \mathrm{MU} \otimes H\mathbf{Z}$, equipped with the complex orientation coming from* MU. *Then $\pi_* E \simeq H_*(\mathrm{MU}; \mathbf{Z}) \simeq \mathbf{Z}[b_1, b_2, \ldots]$, and the formal group law over $\mathbf{Z}[b_1, b_2, \ldots]$ is given by the formula $f(x, y) = g(g^{-1}(x) + g^{-1}(y))$, where $g(x) = x + b_1 x^2 + b_2 x^3 + \ldots$.*

It follows that the composition $L \to \pi_* \mathrm{MU} \to \mathrm{H}_*(\mathrm{MU}; \mathbf{Z})$ is the homomorphism studied in Lecture 2. We conclude:

Corollary 7. *The composite map $L \to \pi_* \mathrm{MU} \to \mathrm{H}_*(\mathrm{MU}; \mathbf{Z})$ is an isomorphism after tensoring with $\mathbf{Q}$.*

Since the Hurewicz map $\pi_* \mathrm{MU} \to \mathrm{H}_*(\mathrm{MU}; \mathbf{Z})$ is *always* a rational isomorphism, we deduce the following baby version of Theorem 1:

Corollary 8. *The map $L \to \pi_* \mathrm{MU}$ induces an isomorphism after tensoring with $\mathbf{Q}$.*

Since MU is a connective spectrum whose homology groups $\mathrm{H}_n(\mathrm{MU}; \mathbf{Z})$ are finitely generated, we conclude that the homotopy groups $\pi_n \mathrm{MU}$ are finitely generated abelian groups. Consequently, to prove Theorem 1 holds integrally, it will suffice to show that the map $L \to \pi_* \mathrm{MU}$ becomes an isomorphism after p-adic completion, for every prime number p. We will prove this later in the week, using the Adams spectral sequence.

The Adams Spectral Sequence (Lecture 8)

April 27, 2010

Recall that our goal this week is to prove the following result:

Theorem 1 (Quillen)**.** *The universal complex orientation of the complex bordism spectrum* MU *determines a formal group law over* $\pi_* \mathrm{MU}$*. This formal group law is classified by an isomorphism of commutative rings* $L \to \pi_* \mathrm{MU}$*.*

To prove this theorem, we need a method for calculating the homotopy groups $\pi_* \mathrm{MU}$. In the last lecture, we computed the homology groups $\mathrm{H}_*(\mathrm{MU}; \mathbf{Z}) \simeq \mathbf{Z}[b_1, b_2, \ldots]$. The universal coefficient theorem then gives $\mathrm{H}_*(\mathrm{MU}; R) \simeq R[b_1, b_2, \ldots]$ for any commutative ring R. In this lecture, we review a general method for passing from information about the homology of a spectrum to information about its homotopy groups: the *Adams spectral sequence*.

Fix a prime number p, and let R denote the Eilenberg-MacLane spectrum $H\mathbf{F}_p$. Then E admits a coherently associative multiplication. In particular, we can define a functor $\overline{R}^\bullet$ from finite linearly ordered sets to spectra, given by $\{0, 1, \ldots, n\} \mapsto R \otimes \cdots \otimes R \simeq R^{\otimes n+1}$. In other words, we can view $\overline{R}^\bullet$ as an *augmented cosimplicial spectrum*. Restricting to *nonempty* finite linearly ordered sets, we get a cosimplicial spectrum which we will denote by $R^\bullet$. If X is any other spectrum, we can define an augmented cosimplicial spectrum $\overline{X}^\bullet = X \otimes \overline{R}^\bullet$; let $X^\bullet$ denote the underlying cosimplicial spectrum. The augmented cosimplicial spectrum $\overline{X}^\bullet$ determines a map

$$X \simeq X \otimes S \simeq X \otimes \overline{R}^{-1} = \overline{X}^{-1} \to \mathrm{Tot}\, X^\bullet.$$

Put more concretely, we have the canonical Adams resolution of X, which is a chain complex of spectra

$$X \to X \otimes R \overset{d^0 - d^1}{\to} X \otimes R \otimes R \overset{d^0 - d^1 + d^2}{\to} X \otimes R \otimes R \otimes R \to \cdots$$

Any cosimplicial spectrum $X^\bullet$ determines a spectral sequence $\{E_r^{a,b}\}$. Here $E_1^{a,b} = \pi_a X^b$, and the differential on the first page is the differential in the chain complex

$$\pi_a X^0 \overset{d^0 - d^1}{\to} \pi_a X^1 \overset{d^0 - d^1 + d^2}{\to} \pi_a X^2 \to \cdots$$

In good cases, this spectral sequence will converge to information about the totalization $\mathrm{Tot}\, X^\bullet$. In our case, we have the following result:

Theorem 2 (Adams)**.** *Let X be a connective spectrum whose homotopy groups $\pi_n X$ are finitely generated for every integer n. Fix a prime number p and let $R = H\mathbf{F}_p$, and let $X \to \mathrm{Tot}\, X^\bullet$ be the map constructed above. Then:*

(1) *For every integer n, we have a canonical decreasing filtration*

$$\cdots \subseteq F^2 \pi_n X \to F^1 \pi_n X \to F^0 \pi_n X = \pi_n X$$

where $F^i \pi_n X$ is the kernel of the map $\pi_n X \to \pi_n \mathrm{Tot}^{i-1} X^\bullet$.

(2) *The decreasing filtration $F^i\pi_n X$ is commensurate with the p-adic filtration. That is, for each $i \geq 0$, there exists $j \gg i$ such that $p^i\pi_n X \subseteq F^j\pi_n X \subseteq p^j\pi_n X$. In particular, we have a canonical isomorphism*

$$\varprojlim(\pi_n X/F^j\pi_n X) \simeq \varprojlim(\pi_n X/p^i\pi_n X) \simeq (\pi_n X)^\vee,$$

where $\vee$ denotes the functor of p-adic completion.

(3) *For fixed a and b, the abelian groups $\{E_r^{a,b}\}_{r\geq 0}$ stabilize to some fixed value $E_\infty^{a,b}$ for $r \gg 0$. Moreover, we have a canonical isomorphism*

$$F^b\pi_{a-b}(X)/F^{b+1}\pi_{a-b}X \simeq E_\infty^{a,b}.$$

If X is a ring spectrum, then $X^\bullet$ has the structure of a cosimplicial ring spectrum. In this case, we have the following additional conclusions:

(4) *For integers m and n, the multiplication map $\pi_m X \times \pi_n X \to \pi_{n+m} X$ carries $F^i\pi_m X \times F^j\pi_n X$ into $F^{i+j}\pi_{m+n}X$. In particular, we get a bilinear multiplication $E_\infty^{a,b} \times E_\infty^{a',b'} \to E_\infty^{a+a',b+b'}$.*

(5) *The spectral sequence $\{E_r^{a,b}\}$ is multiplicative. That is, for each r, we have bilinear maps $E_r^{a,b} \times E_r^{a',b'} \to E_r^{a+a',b+b'}$. These maps are associative in the obvious sense and compatible with the differential (i.e., the differential satisfies the Leibniz rule). Moreover, when $r \gg 0$ so that $E_r^{a,b} = E_\infty^{a,b}$ and $E_r^{a',b'} = E_\infty^{a',b'}$, these multiplications agree with the multiplications defined in* (4).

To apply Theorem 2 in practice, we would like to understand the initial pages of the spectral sequence $\{E_r^{a,b}\}$. When $r = 1$, we have $E_r^{a,b} \simeq \pi_a(X \otimes R^{\otimes b+1})$. In particular, $E_1^{*,0} \simeq \pi_*(X \otimes R) = \mathrm{H}_*(X; \mathbf{F}_p)$ is the mod p homology of the spectrum X. To understand the next term, we write $X^1 = X \otimes R \otimes R = (X \otimes R) \otimes_R (R \otimes R)$. This gives a canonical isomorphism $E_1^{*,1} \simeq \mathrm{H}_*(X; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \pi_*(R \otimes R)$.

Definition 3. Let R be the ring spectrum $H\mathbf{F}_p$. The graded-commutative ring $\pi_*(R \otimes R)$ is called the *dual Steenrod algebra*, and will be denoted by $\mathcal{A}^\vee$.

More generally, we can write

$$X^b = X \otimes R^{\otimes b+1} = (X \otimes R) \otimes_R (R \otimes R) \otimes_R \cdots \otimes_R (R \otimes R).$$

This identification gives a canonical isomorphism

$$E_1^{*,b} \simeq \mathrm{H}_*(X; \mathbf{F}_p) \otimes_{\mathbf{F}_p} (\mathcal{A}^\vee)^{\otimes b}$$

. We have a chain complex of graded abelian groups

$$\mathrm{H}_*(X; \mathbf{F}_p) \to \mathrm{H}_*(X; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathcal{A}^\vee \to \mathrm{H}_*(X; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathcal{A}^\vee \otimes_{\mathbf{F}_p} \mathcal{A}^\vee \to \cdots$$

associated to a cosimplicial graded abelian group $\mathrm{H}_*(X; \mathbf{F}_p) \otimes (\mathcal{A}^\vee)^{\otimes \bullet}$.

To describe the second page of the spectral sequence $\{E_r^{a,b}\}$, we would like understand the differentials in this chain complex. We begin by noting that $\mathcal{A}^\vee$ is actually a Hopf algebra. That is, there is a comultiplication $c : \mathcal{A}^\vee \to \mathcal{A}^\vee \otimes_{\mathbf{F}_p} \mathcal{A}^\vee$, which is induced by the map of ring spectra

$$R \otimes R \simeq R \otimes S \otimes R \to R \otimes R \otimes R \simeq (R \otimes R) \otimes_R (R \otimes R)$$

by passing to homotopy groups. Moreover, this coalgebra *acts* on $\mathrm{H}_*(X; \mathbf{F}_p)$ for any spectrum X: that is, we have a canonical map $a : \mathrm{H}_*(X; \mathbf{F}_p) \to \mathrm{H}_*(X; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathcal{A}^\vee$. This map is induced by the map of spectra

$$X \otimes R \simeq X \otimes S \otimes R \to X \otimes R \otimes R \simeq (X \otimes R) \otimes_R (R \otimes R).$$

Remark 4. Passing to (graded) vector space duals, we see that the dual of $\mathcal{A}^\vee$ is an algebra (called the *Steenrod algebra*, which acts on the cohomology $\mathrm{H}^*(X; \mathbf{F}_p)$ of any spectrum.

Unwinding the definitions, we see that each of the differentials

$$\mathrm{H}_*(X;\mathbf{F}_p)\otimes_{\mathbf{F}_p}(\mathcal{A}^\vee)^{\otimes n-1}\to \mathrm{H}_*(X;\mathbf{F}_p)\otimes_{\mathbf{F}_p}(\mathcal{A}^\vee)^{\otimes n}$$

is given by an alternating sum $\sum_{0\leq i\leq n} d^i$, where:

- The map d^0 is induced by the action map $a : \mathrm{H}_*(X;\mathbf{F}_p)\to \mathrm{H}_*(X;\mathbf{F}_p)\otimes_{\mathbf{F}_p}\mathcal{A}^\vee$.
- The maps $d^1,\ldots,d^{n-1}$ are induced by the comultiplication $c:\mathcal{A}^\vee\to\mathcal{A}^\vee\otimes_{\mathbf{F}_p}\mathcal{A}^\vee$, applied to each factor of $\mathcal{A}^\vee$.
- The map d^n is given by the inclusion of the unit $\mathbf{F}_p\to\mathcal{A}^\vee$.

It is convenient to describe the above analysis in the language of algebraic geometry. For simplicity, we will henceforth assume that $p=2$, so that graded-commutative rings are actually commutative.

Proposition 5. (1) *Let $\mathbb{G}$ denote the spectrum of the commutative ring $\mathcal{A}^\vee$. The comultiplication $\mathcal{A}^\vee\to\mathcal{A}^\vee\otimes_{\mathbf{F}_2}\mathcal{A}^\vee$ determines a multiplication $\mathbb{G}\times_{\operatorname{Spec}\mathbf{F}_2}\mathbb{G}\to\mathbb{G}$, which endows $\mathbb{G}$ with the structure of an affine group scheme over* $\operatorname{Spec}\mathbf{F}_2$.

(2) *For any spectrum X, the action map $\mathrm{H}_*(X;\mathbf{F}_2)\to\mathrm{H}_*(X;\mathbf{F}_2)\otimes_{\mathbf{F}_2}\mathcal{A}^\vee$ endows the vector space $V=\mathrm{H}_*(X;\mathbf{F}_2)$ with the structure of a representation of the group scheme $\mathbb{G}$.*

(3) *The E_1-term of the Adams spectral sequence can be identified with the canonical cochain complex*

$$V\to\Gamma(\mathbb{G};V\otimes\mathcal{O}_{\mathbb{G}})\to\Gamma(\mathbb{G}^2;V\otimes\mathcal{O}_{\mathbb{G}^2})\to\cdots$$

which encodes the action of $\mathbb{G}$ on V.

(4) *The E_2-term of the Adams spectral sequence can be identified with the cohomologies of this cochain complex. In other words, we have*

$$E_2^{*,b}\simeq\mathrm{H}^b(\mathbb{G};\mathrm{H}_*(X;\mathbf{F}_2)).$$

In the special case where X is a ring spectrum, we can say more. In this case, the multiplication on X endows the homology $\mathrm{H}_*(X;\mathbf{F}_2)$ with the structure of a commutative $\mathbf{F}_2$-algebra. Then the spectrum $\operatorname{Spec}\mathrm{H}_*(X;\mathbf{F}_2)$ is an affine scheme Y. The action $\mathrm{H}_*(X;\mathbf{F}_2)\to\mathrm{H}_*(X;\mathbf{F}_2)\otimes_{\mathbf{F}_2}\mathcal{A}^\vee$ is a map of commutative rings, which determines a map of affine schemes $\mathbb{G}\times_{\operatorname{Spec}\mathbf{F}_2}Y\to Y$. In other words, the affine scheme Y is *acted on* by the group scheme $\mathbb{G}$. Moreover, the cohomology groups $\mathrm{H}^b(\mathbb{G};\mathrm{H}_*(X;\mathbf{F}_2))$ are simply the cohomology groups of the quotient stack $Y/\mathbb{G}$. In particular, we get an isomorphism of commutative algebras $E_2^{*,*}\simeq\mathrm{H}^*(Y/\mathbb{G};\mathcal{O}_{Y/\mathbb{G}})$.

To apply this information in practice, we need to understand the algebraic group $\mathbb{G}$. For each integer n, let $X(n)$ denote the function spectrum $S^{\mathbf{R}P^n}$, where $\mathbf{R}P^n$ denotes real projective space of dimension n. Then $X(n)$ is a commutative ring spectrum (in fact, an E_∞-ring spectrum), and we have a canonical isomorphism

$$\mathrm{H}_*(X(n);\mathbf{F}_2)\simeq\mathrm{H}^*(\mathbf{R}P^n;\mathbf{F}_2)\simeq\mathbf{F}_2[x]/(x^{n+1}).$$

In particular, we get an action of $\mathbb{G}$ on the affine scheme $\operatorname{Spec}\mathbf{F}_2[x]/(x^{n+1})$. Passing to the limit as n grows, we get an action of $\mathbb{G}$ on the formal scheme

$$\varinjlim\operatorname{Spec}\mathbf{F}_2[x]/(x^{n+1})\simeq\operatorname{Spf}\mathbf{F}_2[[x]]=\operatorname{Spf}H^*(\mathbf{R}P^\infty,\mathbf{F}_2)$$

This action is not arbitrary. Note that $\mathbf{R}P^\infty$ has a commutative multiplication. For example, we can realize $\mathbf{R}P^\infty$ as the projectivization of the *real* vector space $\mathbb{R}[t]$, and $\mathbf{R}P^n$ as the projectivization of the subspace of $\mathbb{R}[t]$ spanned by polynomials of degree $\leq n$. The multiplication on $\mathbb{R}[t]$ induces a multiplication $\mathbf{R}P^\infty\times\mathbf{R}P^\infty\to\mathbf{R}P^\infty$, which is the direct limit of multiplication maps $\mathbf{R}P^m\times\mathbf{R}P^n\to\mathbf{R}P^{m+n}$. Each of these multiplication maps induces a map of spectra $X(m+n) \to X(m) \otimes X(n)$, which induces a $\mathbb{G}$-equivariant map

$$\mathrm{Spec}(\mathbf{F}_2[x]/(x^{m+1})) \times_{\mathrm{Spec}\,\mathbf{F}_2} \mathrm{Spec}(\mathbf{F}_2[x]/(x^{n+1})) \to \mathrm{Spec}(\mathbf{F}_2[x]/x^{m+n+1}).$$

In concrete terms, this is just given by the map of commutative rings $\mathbf{F}_2[x]/(x^{m+n+1}) \to \mathbf{F}_2[x, x']/(x^{m+1}, x'^{n+1})$ given by $x \mapsto x + x'$. Passing to the limit as m and n grow, we get a map of formal schemes

$$\mathrm{Spf}\,\mathbf{F}_2[[x]] \times_{\mathrm{Spec}\,\mathbf{F}_2} \mathrm{Spf}\,\mathbf{F}_2[[x]] \to \mathrm{Spf}\,\mathbf{F}_2[[x]].$$

This map encodes a formal group law over the ring $\mathbf{F}_2$, which is given by the power series $f(x, y) = x + y \in \mathbf{F}_2[[x, y]]$.

By construction, the action of $\mathbb{G}$ on $\mathrm{Spf}\,\mathbf{F}_2[[x]]$ preserves the group structure given by $f(x, y) = x + y$. That is, we can regard $\mathbb{G}$ as acting by *automorphisms* of the formal group law f. This gives a description of $\mathbb{G}$ which is very convenient for our purposes:

Theorem 6. *For every commutative $\mathbf{F}_2$-algebra A, the above construction yields a canonical bijection of* $\mathrm{Hom}(\mathcal{A}^\vee, A) \simeq \mathrm{Hom}(\mathrm{Spec}\,A, \mathbb{G})$ *with the group of all power series*

$$x \mapsto x + a_1 x^2 + a_2 x^4 + a_3 x^8 + \ldots,$$

where $a_i \in A$, regarded as automorphisms of the formal group $\mathrm{Spec}\,A \times_{\mathrm{Spec}\,F_2} \mathrm{Spf}\,\mathbf{F}_2[[x]] = \mathrm{Spf}\,A[[x]]$.

The Adams Spectral Sequence for MU (Lecture 9)

February 18, 2010

In this lecture, we will apply the Adams spectral sequence to obtain information about the homotopy ring $\pi_* \mathrm{MU}$. Let us begin by recalling the major conclusions of the last lecture:

(1) Let X be a commutative ring spectrum. Then $\mathrm{H}_*(X; \mathbf{F}_2)$ is a commutative $\mathbf{F}_2$-algebra; we may therefore regard $\operatorname{Spec} \mathrm{H}_*(X; \mathbf{F}_2)$ as a scheme Z over $\operatorname{Spec} \mathbf{F}_2$.

(2) The scheme $\operatorname{Spec} \mathrm{H}_*(X; \mathbf{F}_2)$ is acted on by the group scheme $\mathbb{G} = \operatorname{Spec} \mathcal{A}^\vee$ of automorphisms of the formal additive group which are equal to the identity to first order. In other words, $\mathbb{G}$ is the group scheme characterized by the formula

$$\mathrm{Hom}_{\mathbf{F}_2}(\operatorname{Spec} R, \mathbb{G}) = \{f(x) \in R[[x]] : f(x) = x + a_1 x^2 + a_2 x^4 + a_3 x^8 + \cdots\}.$$

(3) The cohomology groups $\mathrm{H}^b(Z/\mathbb{G}, \mathcal{O}_{Z/\mathbb{G}}) = \mathrm{H}^b(\mathbb{G}; \mathrm{H}_*(X; \mathbf{F}_2))$ can be identified with E_2^{*b}, the second page of the Adams spectral sequence for computing the homotopy groups $\pi_* X$.

We would like to apply this in the situation where X is the complex bordism spectrum MU. In this case, we already have a good understanding of the homology $\mathrm{H}_*(\mathrm{MU}; \mathbf{F}_2)$: it can simply be identified with a polynomial ring $\mathbf{F}_2[b_1, b_2, \ldots]$ on infinitely many generators. Consequently, the scheme $Z = \operatorname{Spec} \mathrm{H}_*(\mathrm{MU}; \mathbf{F}_2)$ can be described as an infinite dimensional affine space over $\mathbf{F}_2$. However, our study of formal group laws gives a more conceptual description of Z: namely, if R is any $\mathbf{F}_2$-algebra, then $\mathrm{Hom}(\operatorname{Spec} R, Z) = \mathrm{Hom}(\mathbf{F}_2[b_1, \ldots], R) = R^\infty$ can be identified with the set of formal expressions of the form $y + b_1 y^2 + b_2 y^3 + \ldots$ in the power series ring $R[[y]]$. In other words, we can identify $\mathrm{Hom}(\operatorname{Spec} R, Z)$ with the set of coordinates on the formal additive group over R, which agree with the standard coordinate y to first order.

Since $\mathbb{G}$ is the automorphism group of the formal additive group which preserves the standard coordinate to first order, there is an obvious action of $\mathbb{G}$ on the scheme Z. However, this is *not* the action described in (2) above. The natural action of $\mathbb{G}$ on the formal additive group comes from studying the action of the Steenrod algebra on the cohomology ring $\mathrm{H}^*(\mathbf{R}P^\infty; \mathbf{F}_2) \simeq \mathbf{F}_2[[x]]$. On the other hand, we can relate Z to the space of coordinates on the formal additive group by considering the cohomology ring $\mathrm{H}^*(\mathbf{C}\mathrm{P}^\infty; \mathbf{F}_2) \simeq \mathbf{F}_2[[y]]$. The power series rings $\mathbf{F}_2[[x]]$ and $\mathbf{F}_2[[y]]$ are abstractly isomorphic (even better, by an isomorphism which respects the formal group structures). However, this isomorphism isn't relevant to our picture: for example, it does not respect gradings (the coordinate x has cohomological degree 1, while the coordinate y has cohomological degree 2). Instead, they are related by the existence of a complexification $\mathbf{R}P^\infty \to \mathbf{C}\mathrm{P}^\infty$. This induces a map on cohomology rings $\mathrm{H}^*(\mathbf{C}\mathrm{P}^\infty; \mathbf{F}_2) \to \mathrm{H}^*(\mathbf{R}P^\infty, \mathbf{F}_2)$, which is given concretely by the formula $y \mapsto x^2$. Consequently, if $f(x) = x + a_1 x^2 + a_2 x^4 + \ldots$ is an R-point of $\mathbb{G}$, then f acts on $\mathrm{H}^*(\mathbf{C}\mathrm{P}^\infty; \mathbf{F}_2) \simeq \mathbf{F}_2[[y]]$ by the formula

$$f'(y) = f(x)^2 = x^2 + a_1^2 x^4 + a_2^2 x^8 + \ldots = y + a_1^2 y^2 + a_2^2 y^3 + \cdots$$

To clarify the situation, it is convenient to introduce a bit of notation. The Frobenius map $F : \mathbb{G} \to \mathbb{G}$ determines an exact sequence of group schemes

$$0 \to \ker(F) \to \mathbb{G} \xrightarrow{F} \mathbb{G}' \to 0.$$

Here $\mathbb{G}'$ denotes the group scheme $\mathbb{G}$, but regarded as the quotient $\mathbb{G}/\ker(F)$. In other words, we should think of $\mathbb{G}$ as acting on the power series ring $\mathrm{H}^*(\mathbf{R}P^\infty; \mathbf{F}_2) \simeq \mathbf{F}_2[[x]]$, and $\mathbb{G}'$ as acting on the subring $\mathrm{H}^*(\mathbf{C}\mathrm{P}^\infty; \mathbf{F}_2) \simeq \mathbf{F}_2[[y]] \simeq \mathbf{F}_2[[x^2]]$. The action of $\mathbb{G}$ on the scheme Z factors through the Frobenius map F: in other words, it is trivial on the normal subgroup $\ker(F) \subseteq \mathbb{G}$.

To understand the Adams spectral sequence, we need to study the quotient stack $Z/\mathbb{G}$. We first consider the quotient $Z/\mathbb{G}'$.

Proposition 1. *Let $Z_0 = \operatorname{Spec} \mathbf{F}_2[b_2, b_4, b_5, b_6, b_8, \ldots]$ be the closed subscheme of Z whose R-points consist of those formal coordinates $f(y) = y + b_1 y^2 + b_2 y^3 + \cdots \in R[[x]]$ for which the coefficients $b_{2^i - 1}$ of y^{2^i} vanish for $i > 0$. Then the action of $\mathbb{G}$ on Z determines an isomorphism of schemes*

$$a : \mathbb{G}' \times Z_0 \to Z.$$

In particular, $\mathbb{G}'$ acts freely on Z, and the composition

$$Z_0 \to Z \to Z/\mathbb{G}'$$

is an isomorphism of schemes.

Proof. We must show that for every $\mathbf{F}_2$-algebra R, the map a induces a bijection on R-points. In other words, we must show that if $h(y) = y + c_1 y^2 + c_2 y^3 + c_3 y^4 + \cdots$ is an arbitrary R-point of Z, then h can be written uniquely as a composition $h(y) = (f \circ g)(y)$, where g has the form $g(y) = y + a_1 y^2 + a_2 y^4 + \cdots$ and f has the form $f(y) = y + b_2 y^3 + b_4 y^5 + b_5 y^6 + b_6 y^7 + b_8 y^9 + \cdots$. In fact, we claim that the coefficients $\{a_i, b_i\}_{i<n}$ are uniquely determined by the requirement that the equation $h(y) = (f \circ g)(y)$ holds modulo y^{n+1}. Assuming this, we note that a_n or b_n (whichever is defined) is uniquely determined by examining the y^{n+1}-coefficients of $h(y)$ and $(f \circ g)(y)$. □

The quotient $Z/\mathbb{G}$ can be identified with the quotient $(Z/\ker(F))/\mathbb{G}'$. Since $\ker(F)$ acts trivially on Z, we can identify $Z/\ker(F)$ with the product $Z \times B\ker(F)$. The action of $\mathbb{G}'$ determines a composite map

$$\beta : \mathbb{G}' \times Z_0 \times B\ker(F) \hookrightarrow \mathbb{G}' \times (Z \times B\ker(F)) \simeq \mathbb{G}' \times (Z/\ker(F)) \to Z/\ker(F).$$

This is pullback of the map appearing in Proposition 1, and therefore also an isomorphism. We therefore obtain an isomorphism of stacks

$$Z/\mathbb{G} \simeq (Z/\ker(F))/\mathbb{G}' \simeq (\mathbb{G}' \times Z_0 \times B\ker(F))/\mathbb{G}' \simeq Z_0 \times B\ker(F).$$

In other words, we can identify the cohomology $\mathrm{H}^b(Z/\mathbb{G}; \mathcal{O}_{Z/\mathbb{G}})$ with the tensor product $\mathbf{F}_2[b_2, b_4, b_5, \ldots] \otimes_{\mathbf{F}_2} \mathrm{H}^b(\ker(F); \mathbf{F}_2)$. It therefore remains only to compute the cohomology of $\ker(F)$.

Unwinding the definitions, we can describe the group scheme $\ker(F)$ as follows: an R-point of $\ker(F)$ is a power series of the form

$$g(x) = x + a_1 x^2 + a_2 x^4 + \cdots$$

where $a_i^2 = 0$ for each i. It is very easy to compose such power series: if $g'(x)$ is given by $x + a_1' x^2 + a_2' x^4 + \cdots$, then the composition $(g' \circ g)(x)$ is given by $x + (a_1 + a_1')x^2 + (a_2 + a_2')x^4 + \ldots$ In other words, we can identify $\ker(F)$ with a *product* of infinitely many copies of the group scheme $\alpha_2 = \operatorname{Spec} \mathbf{F}_2[a]/(a^2)$, whose R-points are given by elements $a \in R$ such that $a^2 = 0$ (regarded as a group with respect to addition). We are therefore reduced to computing the cohomology of the group scheme α_2.

To understand this cohomology, we need to understand what it means for a vector space V to have an action of the group α_2. By definition, this is just a map $V \to V \otimes_{\mathbf{F}_2} \mathbf{F}_2[a]/(a^2)$ compatible with the comultiplication $a \mapsto a \otimes 1 + 1 \otimes a$ on $\mathbf{F}_2[a]/(a^2)$. Note that this category depends *only* on the comultiplication on $\mathbf{F}_2[a]/(a^2)$, not on its multiplication. There is an isomorphism of coalgebras

$$\theta : \mathbf{F}_2[a]/(a^2) \simeq \mathbf{F}_2^{\mathbf{Z}/2\mathbf{Z}},$$

carrying 1 to $(1,1)$ and a to $(0,1)$. It follows that the category of representations of α_2 is equivalent to the category of representations of the group $\mathbf{Z}/2\mathbf{Z}$ (note that θ is *not* an isomorphism of algebras: this means that our equivalence of categories does not respect tensor products). Under this equivalence of categories, the trivial representation V of α_2 goes to a 1-dimensional representation of $\mathbf{Z}/2\mathbf{Z}$, which must itself be trivial. It follows that we have canonical isomorphisms

$$\mathrm{H}^*(\alpha_2; \mathbf{F}_2) = \mathrm{Ext}^*(V, V) = \mathrm{H}^*(\mathbf{Z}/2\mathbf{Z}; \mathbf{F}_2) = \mathrm{H}^*(\mathbf{R}P^\infty; \mathbf{F}_2)$$

is a polynomial algebra $\mathbf{F}_2[\epsilon]$, where ϵ has cohomological degree 1.

It follows that $\mathrm{H}^*(\ker(F); \mathbf{F}_2)$ can be identified with a polynomial ring $\mathbf{F}_2[\epsilon_1, \epsilon_2, \ldots]$, where each ϵ_i has cohomological degree 1. However, there is another grading on this cohomology ring, coming from the grading on the ring of functions $\mathbf{F}_2[a_1, a_2, \ldots]/(a_1^2, a_2^2, \ldots)$. This grading is determined by the requirement that the expression $x + a_1 x^2 + a_2 x^4 + \ldots$ has total degree -1, where x has degree -1: in other words, each a_i has degree $2^i - 1$.

We can summarize our discussion as follows:

Proposition 2. *The second page of the mod 2 Adams spectral sequence for* MU *is given by*

$$E_2^{*,*} \simeq \mathbf{F}_2[b_2, b_4, b_5, b_6, b_8, \ldots, \epsilon_1, \epsilon_2, \ldots].$$

Here each b_i has bidegree $(2i, 0)$, while each ϵ_j has bidegree $(2^j - 1, 1)$.

Note that the total degree of each of the polynomial generators in Proposition 2 is even. It follows that a group $E_2^{a,b}$ can be nonzero only when the total degree $a - b$ is even. Consequently, there can be no nontrivial differentials in the Adams spectral sequence in the second page or beyond.

The Proof of Quillen's Theorem (Lecture 10)

February 19, 2010

At the end of the last lecture, we arrived at the following conclusion for the prime $p = 2$:

Proposition 1. *The second page of the mod p Adams spectral sequence for* MU *is given by*

$$E_2^{*,*} \simeq \mathbf{F}_p[b_i, \epsilon_j].$$

Here i ranges over nonnegative integers such that $i+1$ is not a power of p, and b_i has bidegree $(2i, 0)$, while ϵ_j is defined for all $j \geq 0$ and has bidegree $(p^j - 1, 1)$.

This calculation is valid for all p, not just the case $p = 2$. The proofs are essentially the same, but our use of algebraic geometry needs to be replaced by "super" algebraic geometry.

We define $c_i = \begin{cases} \epsilon_j & \text{if } i+1 = p^j \\ b_i & \text{otherwise.} \end{cases}$ so that we have an isomorphism $E_2^{*,*} \simeq \mathbf{F}_p[c_0, c_1, \ldots]$. Here each c_i has total degree $2i$. In particular, every nonzero element of $E_2^{*,*}$ has even total degree, so the Adams spectral sequence degenerates at the second page. We deduce the following:

Proposition 2. *The (mod p) Adams filtration on $\pi_* \mathrm{MU}$ has the following property:*

$(*)$ *The associated graded ring $\mathrm{gr}(\pi_* \mathrm{MU})$ is isomorphic to a polynomial algebra $\mathbf{F}_p[c_0, c_1, \ldots]$, with $c_i \in \mathrm{gr}_0(\pi_{2i} \mathrm{MU})$ for $i + 1 \neq p^j$, and $c_i \in \mathrm{gr}_1(\pi_{2i} \mathrm{MU})$ for $i + 1 = p^j$.*

Remark 3. In particular, the class c_0 can be lifted to an element of $F^1 \pi_0 \mathrm{MU}$, which is the kernel of the Hurewicz map

$$\pi_0 \mathrm{MU} \simeq \mathrm{H}_0(\mathrm{MU}; \mathbf{Z}) \simeq \mathbf{Z} \to \mathbf{F}_p \simeq \mathrm{H}_0(\mathrm{MU}; \mathbf{F}_p).$$

This map is nonzero modulo elements of Adams filtration 2, which includes the subgroup $p^2 \pi_0 \mathrm{MU} \simeq p^2 \mathbf{Z}$. It follows that (after modifying by a suitable scalar) we may assume that c_0 is represented by $p \in p\mathbf{Z} \simeq F^1 \pi_* \mathrm{MU}$.

Let R denote the polynomial ring $\mathbf{Z}[u_1, u_2, \ldots]$. We regard R as a graded ring where each class u_i has degree $2i$. We also regard R as filtered, where $F^i R$ is generated by monomials of the form $p^{m_1} u_1^{m_2} u_2^{m_3} \ldots$ for which $m_1 + m_p + m_{p^2} + \ldots \geq i$. Choose a map of commutative rings $\phi : R \to \pi_* \mathrm{MU}$ with the following properties:

(1) If $i+1$ is not a power of p, then $\phi(u_i)$ is an element of $\pi_{2i} \mathrm{MU}$ representing $c_i \in \mathrm{gr}_0(\pi_{2i} \mathrm{MU})$.

(2) If $i + 1 = p^v$, then $\phi(u_i)$ is an element of $F^1 \pi_{2i} \mathrm{MU}$ representing $c_i \in \mathrm{gr}_1 \pi_{2i} \mathrm{MU}$.

Then ϕ is compatible with both the grading and the filtrations on R and $\pi_* \mathrm{MU}$. It follows from Proposition 2 that ϕ induces an isomorphism of associated graded rings $\mathrm{gr}\, R \to \mathrm{gr}(\pi_* \mathrm{MU})$. It follows by induction on i that ϕ induces an isomorphism of quotients $R/F^i R \to (\pi_* \mathrm{MU})/F^i \pi_* \mathrm{MU}$. Passing to the inverse limit, we get an isomorphism of graded rings

$$\mathbf{Z}_p[u_1, u_2, \ldots] \simeq \varprojlim R/F^i R \simeq \varprojlim (\pi_* \mathrm{MU})/F^i \pi_* \mathrm{MU} \simeq (\pi_* \mathrm{MU})^\vee.$$

Here $\vee$ denotes the functor of p-adic completion (applied in each graded degree).

We are now ready to prove Quillen's theorem:

Theorem 4 (Quillen)**.** *Let $\theta : L \to \pi_* \mathrm{MU}$ be the ring homomorphism classifying the formal group law coming from the universal complex orientation on* MU*. Then θ is an isomorphism.*

Proof. We have already seen that the composite map $L \xrightarrow{\theta} \pi_* \mathrm{MU} \to \mathrm{H}_*(\mathrm{MU}; \mathbf{Z}) \simeq \mathbf{Z}[b_1, \ldots]$ is an injection. We show that θ is surjective. Since each homotopy group of MU is finitely generated, it will suffice to show that θ induces a surjection $L^\vee \to (\pi_* \mathrm{MU})^\vee$ after p-adically completing at every prime p.

Using Lazard's theorem we can identify $L^\vee \simeq \mathbf{Z}_p[t_1, t_2, \ldots]$, and the above analysis gives an isomorphism of graded rings $(\pi_* \mathrm{MU})^\vee \simeq \mathbf{Z}_p[u_1, u_2, \ldots]$. Let I denote the ideal of $L^\vee$ generated by homogeneous elements of positive degree and let $K \subseteq (\pi_* \mathrm{MU})^\vee$, $J \subseteq \mathbf{Z}_2[b_1, b_2, \ldots]$ be defined similarly. As in the proof of Lazard's theorem, it will suffice to show that the map $I/I^2 \to K/K^2$ is surjective in each degree. In each degree, we have identifications $(I/I^2)_{2n} \simeq \mathbf{Z}_p t_n$ and $(K/K^2)_{2n} \simeq \mathbf{Z}_p u_n$, so that $\theta(t_n) = \lambda u_n +$ decomposables . We wish to prove that λ is a p-adic unit. The Hurewicz map carries u_n to $\lambda' b_n +$ decomposables . In the proof of Lazard's theorem, we saw that

$$\lambda\lambda' = \begin{cases} p & \text{if } n+1 = p^v \\ 1 & \text{otherwise.} \end{cases}$$

If $n+1$ is not a power of p, it follows immediately that λ is a p-adic unit. If $n+1 = p^v$, then we need to work a little harder: namely, we need to show that λ' is divisible by p. It will suffice to show that the image of u_n vanishes in the ring $\mathrm{H}_*(\mathrm{MU}; \mathbf{F}_p) \simeq \mathbf{F}_p[b_1, b_2, \ldots]$. This is equivalent to the requirement that u_n have Adams filtration ≥ 1, which is true by construction. □

Let us now return to the construction of the Adams spectral sequence. Let X be an arbitrary spectrum. Since the complex bordism spectrum MU is coherently associative, we can define a cosimplicial spectrum $X^\bullet$ by the formula $X^n = X \otimes \mathrm{MU}^{\otimes n+1}$. If X is connective, then one can show that the map $X \to \mathrm{Tot}\, X^\bullet \simeq \varprojlim \mathrm{Tot}^n X^\bullet$ is an equivalence. We therefore obtain a spectral sequence $\{E_r^{a,b}, d_r\}$ which carries information about the homotopy groups of X. This spectral sequence is called the *Adams-Novikov spectral sequence*. It has slightly different behavior than the classical Adams spectral sequence: since $\pi_0 \mathrm{MU} \simeq \pi_0 S \simeq \mathbf{Z}$, the convergence is very fast. Namely, if we define $F^n \pi_* X$ to be the kernel of the map

$$\pi_* X \to \pi_* \mathrm{Tot}^{n+1} X^\bullet,$$

then we have for every integer $n \geq 0$ a *finite* filtration

$$0 = F^{n+1}\pi_n X \subseteq F^n \pi_n X \cdots \subseteq F^1 \pi_n X \subseteq F^0 \pi_n X = \pi_n X.$$

The E_1-term of the Adams-Novikov spectral sequence is given by the chain complex of graded abelian groups

$$\mathrm{MU}_*(X) \to (\mathrm{MU} \otimes \mathrm{MU})_*(X) \to (\mathrm{MU} \otimes \mathrm{MU} \otimes \mathrm{MU})_* X \to \cdots$$

We would like to understand this chain complex in algebro-geometric terms.

For any complex-oriented cohomology theory E, we have a canonical isomorphism $\pi_*(E \otimes \mathrm{MU}) \simeq (\pi_* E)[b_1, b_2, \ldots]$. Assuming that the homotopy groups of E are concentrated in even degrees (so that $\pi_* E$ is a commutative ring R), we conclude that $\mathrm{Spec}\, \pi_*(E \otimes \mathrm{MU})$ is an infinite dimensional affine space over $\mathrm{Spec}\, R$: more precisely, it is the affine space parametrizing all coordinates

$$g(t) = t + b_1 t^2 + b_2 t^3 + \cdots$$

on the formal power series ring $R[[t]]$ which agree with the standard coordinate to first order.

Let $G = \mathrm{Spec}\, \mathbf{Z}[b_1, b_2, \ldots]$ be the scheme whose R-points are power series $g(t) = t + b_1 t^2 + b_2 t^3 + \cdots \in R[[t]]$, regarded as a group under composition. We conclude that $\pi_*(E \otimes \mathrm{MU})$ is the ring of functions on the affine scheme $G \times \mathrm{Spec}\, \pi_* E$. In particular, Quillen's theorem gives $\mathrm{Spec}\, \pi_*(\mathrm{MU} \otimes \mathrm{MU}) \simeq G \times \mathrm{Spec}\, L$. Here the two natural inclusions $\mathrm{MU} \to \mathrm{MU} \otimes \mathrm{MU} \leftarrow \mathrm{MU}$ induce a pair of maps $G \times \mathrm{Spec}\, L \to \mathrm{Spec}\, L$. In concrete terms, this means that given a formal group $f(x,y) \in R[[x,y]]$ and a power series $g(t) = t + b_1 t^2 + \ldots$, we can naturally construct *two* formal groups over R: the first is given by f itself, and the second by the formula $gf(g^{-1}(x), g^{-1}(y))$. In other words, the group G of coordinate changes acts on the moduli space $\operatorname{Spec} L$ parametrizing formal groups.

More generally, the same reasoning shows that $\pi_* \mathrm{MU}^{\otimes n+1}$ can be identified with the ring of functions on the product scheme $G^n \times \operatorname{Spec} L$. In particular, the cosimplicial spectrum $\mathrm{MU}^{\otimes \bullet+1}$ gives rise to a simplicial scheme $\operatorname{Spec} \pi_*(\mathrm{MU}^{\otimes \bullet+1})$, which encodes the canonical action of G on $\operatorname{Spec} L$.

Definition 5. We let $\mathcal{M}_{\mathrm{FG}}^s$ denote the quotient stack $\operatorname{Spec} L/G$. We will refer to $\mathcal{M}_{\mathrm{FG}}^s$ as the moduli stack of formal groups and *strict* isomorphisms.

More precisely, $\mathcal{M}_{\mathrm{FG}}^s$ is a functor which assigns to each commutative ring R the category whose objects are formal groups $f \in R[[x,y]]$, where a morphism from f to f' is a power series $g(t) = t + b_1 t^2 + \ldots$ such that $f(g(x), g(y)) = gf'(x,y)$. Here the word "strict" refers to the requirement that $g(t)$ have leading coefficient t. (One can show that $\mathcal{M}_{\mathrm{FG}}^s$ is in fact a stack: that is, the groupoids defined above satisfy descent with respect to the flat topology.)

Now suppose that X is an arbitrary spectrum. It is clear that $\pi_* X^n = \pi_*(X \otimes \mathrm{MU}^{\otimes n+1})$ is a module over the commutative ring $\pi_* \mathrm{MU}^{\otimes n+1}$, and can therefore be identified with a quasi-coherent sheaf on the affine scheme $\operatorname{Spec} L \times G^n$. These quasi-coherent sheaves are compatible with one another under base change. In the language of algebraic stacks, this means:

- Let X be any spectrum. Then $\pi_* X$ can be regarded as a quasi-coherent sheaf $\mathcal{F}_X$ on the quotient stack $\mathcal{M}_{\mathrm{FG}}^s$.

Put more concretely, the abelian group $\mathrm{MU}_*(X)$ is a module over $\pi_* \mathrm{MU} \simeq L$, and can therefore be regarded as a quasi-coherent sheaf on $\operatorname{Spec} L$. This quasi-coherent sheaf carries an action of the group scheme G defined above, compatible with the action of G on $\operatorname{Spec} L$.

The cochain complex

$$\mathrm{MU}_*(X) \to (\mathrm{MU} \otimes \mathrm{MU})_*(X) \to (\mathrm{MU} \otimes \mathrm{MU} \otimes \mathrm{MU})_* X \to \cdots$$

now admits a natural interpretation: it is simply the standard complex for computing the cohomology of $\mathcal{M}_{\mathrm{FG}}^s$ with coefficients in $\mathcal{F}_X$. In other words, we have the following:

Proposition 6. *Let X be any spectrum. The second page of the Adams-Novikov spectral sequence is given by*

$$E_2^{*b} \simeq \mathrm{H}^b(\mathcal{M}_{\mathrm{FG}}^s; \mathcal{F}_X) \simeq \mathrm{H}^b(G; \mathrm{MU}_*(X)).$$

Formal Groups (Lecture 11)

April 27, 2010

We begin by recalling our discussion of the Adams-Novikov spectral sequence:

Claim 1. *Let X be any spectrum. Then $\mathrm{MU}_*(X)$ is a module over the commutative ring $L = \pi_* \mathrm{MU}$, and can therefore be understood as a quasi-coherent sheaf on the affine scheme $\mathrm{Spec}\, L$ which parametrizes formal group laws (here L denotes the Lazard ring). This quasi-coherent sheaf admits an action of the affine group scheme $G = \mathrm{Spec}\, \mathbf{Z}[b_1, b_2, \ldots]$ which assigns to each commutative ring R the group $\{g \in R[[t]] : g(t) = t + b_1 t^2 + b_2 t^3 + \cdots\}$, compatible with the action of G on $\mathrm{Spec}\, L$ by the construction*

$$(g \in G(R), f(x,y) \in \mathrm{FGL}(R) \subseteq R[[x,y]]) \mapsto gf(g^{-1}(x), g^{-1}(y)) \in \mathrm{FGL}(R) \subseteq R[[x,y]]$$

There is a spectral sequence $\{E_r^{p,q}, d_r\}$, called the Adams-Novikov spectral sequence, with the following properties. If X is connective, then $\{E_r^{p,q}, d_r\}$ converges to a finite filtration of $\pi_{p-q} X$. Moreover, the groups $E_2^{*,q}$ are given by the cohomology groups $\mathrm{H}^q(G; \mathrm{MU}_* X)$.

Equivalently, we can think of $E_2^{*,q}$ as the cohomology of the stack $\mathcal{M}^s_{\mathrm{FG}} = \mathrm{Spec}\, L/G$ with coefficients in the sheaf $\mathcal{F}_X$ determined by $\mathrm{MU}_*(X)$ with its G-action.

To be more precise, we should observe that the ring L, and the ring $\mathbf{Z}[b_1, \ldots,]$ are all equipped a canonical grading. In geometric terms, this grading corresponds to an action of the multiplicative group $\mathbb{G}_m$. This group acts on L by the formula

$$(\lambda \in R^\times, f(x,y) \in \mathrm{FGL}(R)) \mapsto \lambda f(\lambda^{-1} x, \lambda^{-1} y).$$

In fact, we can identify both $\mathbb{G}_m$ and G with subgroups of a larger group G^+, with $G^+(R) = \{g \in R[[x]] : g(t) = b_0 t + b_1 t^2 + \cdots, b_0 \in R^\times\}$. This group can be identified with a semidirect product of the subgroup $\mathbb{G}_m$ (consisting of those power series with $b_i = 0$ for $i > 0$) and G (consisting of those power series with $b_0 = 1$), and this semidirect product acts on $\mathrm{Spec}\, L$ by substitution.

For any spectrum X, $\mathrm{MU}_*(X)$ is a graded L-module, and the action of G on $\mathrm{MU}_*(X)$ is compatible with the grading. In the language of algebraic geometry, this means that $\mathrm{MU}_{\mathrm{even}}(X) = \oplus_n \mathrm{MU}_{2n}(X)$ can be regarded as a representation of the group G^+, compatible with the action of G^+ on $\mathrm{Spec}\, L$. In the language of stacks, this means that $\mathrm{MU}_{\mathrm{even}}(X)$ can be regarded as a quasi-coherent sheaf on the quotient stack $\mathrm{Spec}\, L/G^+$.

Definition 2. The quotient stack $\mathrm{Spec}\, L/G^+$ is called the *moduli stack of formal groups* and will be denoted by $\mathcal{M}_{\mathrm{FG}}$.

To understand $\mathcal{M}_{\mathrm{FG}}$, it will be useful to have a more conceptual way of thinking about formal group laws. Let R be a commutative ring and let $f(x,y) \in R[[x,y]]$ be a formal group law over R. We let Alg_R denote the category of commutative R-algebras. We can associate to f a functor $\mathcal{G}_f : \mathrm{Alg}_R \to \mathrm{Ab}$ from R to the category of abelian groups: namely, we let $\mathcal{G}_f(A) = \{a \in A : (\exists n) a^n = 0\} \subseteq A$, with the group structure given by $(a,b) \mapsto f(a,b)$. Note that this expression makes sense: though f has infinitely many terms, if a and b are nilpotent then only finitely many terms are nonzero. We will call $\mathcal{G}_f$ the *formal group* associated to f.

Remark 3. The condition that $f \in R[[x, y]]$ define a formal group law is *equivalent* to the requirement that the above formula defines a group structure on $\mathcal{G}_f(A)$ for every R-algebra A.

Suppose that we are given two formal group laws $f, f' \in R[[x, y]]$ and an isomorphism $\alpha : \mathcal{G}_f \simeq \mathcal{G}_{f'}$ of the corresponding formal groups. In particular, for every R-algebra A, α determines a bijection α_A from the set $\{a \in A : a \text{ is nilpotent }\}$ with itself. To understand this bijection, let us treat the universal case where A contains an element a such that $a^{n+1} = 0$. This is the truncated polynomial ring $A = R[t]/t^{n+1}$. In this case, α carries t to another nilpotent element, necessarily of the form $b_0 t + b_1 t^2 + \ldots + b_{n-1} t^n$. Since α is functorial, it follows that for *any* commutative R-algebra A containing an element a with $a^n = 0$, we have $\alpha_A(a) = b_0 a + b_1 a^2 + \cdots + b_{n-1} a^n$. In particular, if $A = R[t]/t^n$, we deduce that $\alpha_A(t) = b_0 t + b_1 t^2 + \ldots + b_{n-2} t^{n-1}$. In other words, the coefficients b_i which appear are independent of n. We conclude that there exists a power series $g(t) = b_0 t + b_1 t^2 + \cdots$ such that $\alpha_A(a) = g(a)$ for every commutative ring a. Since α_A is a bijection for any A, we conclude that g is an invertible power series. Since α_A is a group homomorphism, we deduce that g satisfies the formula $f'(g(x), g(y)) = gf(x, y)$: that is, the formal group laws f and f' differ by the change-of-variable g.

Definition 4. Let R be a commutative ring. An *coordinatizable formal group* over R is a functor $\mathcal{G} : \mathrm{Alg}_R \to \mathrm{Ab}$ which has the form $\mathcal{G}_f$, for some formal group law $f \in R[[x, y]]$.

We regard the coordinatizable formal group laws (and isomorphisms between them) as a subcategory of the category $\mathrm{Fun}(\mathrm{Alg}_R, \mathrm{Ab})$ of functors from Alg_R to abelian groups. We have just seen that this subcategory admits a less invariant description: it is equivalent to a category whose objects are formal group laws $f \in R[[x, y]]$, and whose morphisms are maps g such that $f'(g(x), g(y)) = gf(x, y)$.

The coordinatizable formal group laws over R do *not* satisfy descent in R. Consequently, it is convenient to make the following more general definition:

Definition 5. Let R be a commutative ring. A *formal group law* over R is a functor $\mathcal{G} : \mathrm{Alg}_R \to \mathrm{Ab}$ satisfying the following conditions:

(1) The functor $\mathcal{G}$ is a sheaf with respect to the Zariski topology. In other words, if A is a commutative R-algebra with a pair of elements x and y such that $x + y = 1$, then $\mathcal{G}(A)$ can be described as the subgroup of $\mathcal{G}(A[\frac{1}{x}]) \times \mathcal{G}(A[\frac{1}{y}])$ consisting of pairs which have the same image in $\mathcal{G}(A[\frac{1}{xy}])$.

(2) The functor $\mathcal{G}$ is a coordinatizable formal group law locally with respect to the Zariski topology. That is, we can choose elements $r_1, r_2, \ldots, r_n \in R$ such that $r_1 + \cdots + r_n = 1$, such that each of the composite functors
$$\mathrm{Alg}_{R[\frac{1}{r_i}]} \to \mathrm{Alg}_R \to \mathrm{Ab}$$
has the form $\mathcal{G}_f$ for some formal group law $f \in R[\frac{1}{r_i}][[x, y]]$.

By definition, the moduli stack of the formal groups $\mathcal{M}_{\mathrm{FG}}$ is the functor which assigns to each commutative ring R the category of formal group laws over R (the morphisms in this category are given by isomorphisms).

There is a canonical map of stacks $\mathcal{M}^s_{\mathrm{FG}} = \operatorname{Spec} L/G \to \operatorname{Spec} L/G^+ = \mathcal{M}_{\mathrm{FG}}$. To understand this map (and the failure of general formal groups to be coordinatizable) it is useful to introduce a definition.

Definition 6. Let $\mathcal{G}$ be a formal group over R. The *Lie algebra* of $\mathcal{G}$ is the abelian group $\mathfrak{g} = \ker(\mathcal{G}(R[t]/(t^2)) \to \mathcal{G}(R))$.

Note that if $\mathcal{G} = \mathcal{G}_f$ for some formal group law f, we get a group isomorphism $\mathfrak{g} \simeq tR[t]/(t^2) \simeq R$ (since $f(x, y) = x + y$ to order 2). In fact, $\mathfrak{g}$ is not just an abelian group: for each $\lambda \in R$, the equation $t \mapsto \lambda t$ determines a map from $R[t]/(t^2)$ to itself, which induces a group homomorphism $\mathfrak{g} \to \mathfrak{g}$. When $\mathcal{G}$ is coordinatizable, this is the usual action of R on itself by multiplication. It follows by descent that the above formula always determines an action of R on $\mathfrak{g}$. Since $\mathfrak{g} \simeq R$ locally for the Zariski topology, we deduce that $\mathfrak{g}$ is an *invertible* R-module: that is, it determines a line bundle on the affine scheme $\operatorname{Spec} R$.

Proposition 7. (1) *A formal group $\mathcal{G}$ over R is coordinatizable if and only if its Lie algebra $\mathfrak{g}$ is isomorphic to R.*

(2) *The quotient stack $\mathcal{M}^s_{\mathrm{FG}}$ parametrizes pairs $(\mathcal{G}, \alpha)$, where $\mathcal{G}$ is a formal group and $\alpha : \mathfrak{g} \simeq R$ is a trivialization of its Lie algebra.*

Proof. We have already established that $\mathfrak{g} \simeq R$ when $\mathcal{G}$ is coordinatizable. Conversely, fix an isomorphism $\mathfrak{g} \simeq R$. After localizing $\operatorname{Spec} R$, the group $\mathcal{G}$ becomes coordinatizable: that is, we can write $\mathcal{G} \simeq \mathcal{G}_f$ for some $f \in R[[x, y]]$. Modifying f by the action of $\mathbb{G}_m$, we may assume that this isomorphism is compatible with our trivialization of $\mathfrak{g}$. The trouble is that these isomorphisms might not glue. The obstruction to gluing them determines a cocycle representing a class in $H^1_{\mathrm{Zar}}(\operatorname{Spec} R, G)$. We claim that this group vanishes. This is because the group G is an iterated extension of copies of the additive group $(A \in \mathrm{Alg}_R) \mapsto (A, +)$, which has no cohomology on affine schemes.

Assertion (2) is just a translation of the following observation: if $f, f' \in R[[x, y]]$ are formal group laws, then an isomorphism of formal groups $\mathcal{G}_f \simeq \mathcal{G}_{f'}$ respects the trivializations of the Lie algebras of $\mathcal{G}_f$ and $\mathcal{G}_{f'}$ if and only if it is given by a power series of the form $g(t) = t + b_1 t^2 + \cdots$ (a power series of the form $g(t) = b_0 t + \cdots$ acts on the Lie algebras by multiplication by the scalar b_0). □

We can think of the assignment $(R, \mathcal{G}) \mapsto \mathfrak{g}^{-1}$ as defining a line bundle ω on the moduli stack $\mathcal{M}_{\mathrm{FG}}$. In fact, $\mathcal{M}^s_{\mathrm{FG}}$ is just the total space of ω with the zero section removed (equivalently, the moduli stack of trivializations of ω).

We can now be a little bit more precise about the E_2-term of the Adams-Novikov spectral sequence. Translating our gradings into algebraic geometry, we get the following result:

Claim 8. *For any spectrum X, the bordism groups $\mathrm{MU}_{\mathrm{even}}(X)$ form a module over the Lazard ring $L \simeq \pi_* \mathrm{MU}$ which carries a compatible action of the group scheme G^+, and therefore determines a sheaf $\mathcal{F}^{\mathrm{even}}$ on $\mathcal{M}_{\mathrm{FG}} = \operatorname{Spec} L/G^+$. The E_2-term of the Adams-Novikov spectral sequence satisfies*

$$E_2^{2a,b} = \mathrm{H}^b(\mathcal{M}_{\mathrm{FG}}; \mathcal{F}^{\mathrm{even}} \otimes \omega^a).$$

Similarly, the odd homotopy groups $\mathrm{MU}_{\mathrm{odd}}(X)$ determine a sheaf $\mathcal{F}^{\mathrm{odd}}$ on $\mathcal{M}_{\mathrm{FG}}$ satisfying

$$E_2^{2a+1,b} = \mathrm{H}^b(\mathcal{M}_{\mathrm{FG}}; \mathcal{F}^{\mathrm{odd}} \otimes \omega^a).$$

In order to exploit Claim 8, we will need to understand the structure of the moduli stack $\mathcal{M}_{\mathrm{FG}}$. This will be our goal in the next lecture.

Heights of Formal Groups (Lecture 12)

April 27, 2010

Our next goal in this course is to understand the structure of the moduli stack $\mathcal{M}_{\mathrm{FG}}$ of formal groups. Our starting point is the following result from Lecture 2:

Proposition 1. *Let R be a ring of characteristic zero. Then, for every formal group law $f \in R[[x,y]]$, there exist a unique power series $g(t) = t + b_1 t^2 + b_2 t^3 + \ldots$ such that $f(x,y) = g(g^{-1}(x) + g^{-1}(y))$.*

Corollary 2. *The quotient stack $\mathcal{M}^s_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Q} = (\operatorname{Spec} L/G) \times \operatorname{Spec} \mathbf{Q}$ is isomorphic to $\operatorname{Spec} \mathbf{Q}$.*

Corollary 3. *The quotient stack $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Q} = (\operatorname{Spec} L/G^+) \times \operatorname{Spec} \mathbf{Q}$ is isomorphic to the classifying stack $B\mathbb{G}_m$ (over $\operatorname{Spec} \mathbf{Q}$). In other words, if R is a ring of characteristic zero, then every formal group over R is determined (up to unique isomorphism) by its Lie algebra.*

Example 4. Let $f(x,y) = x + y + xy$ be the multiplicative formal group law. If R is a ring of characteristic zero, then f is isomorphic to the additive formal group law via the isomorphism $g(t) = e^t - 1 = t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots$.

The coefficients of the power series $e^t - 1$ are not integral. This suggests that over rings which are not of characteristic zero, the additive and multiplicative formal groups are not isomorphic. To prove this, we need an invariant which can be used to tell two formal groups apart. First, we need a brief digression concerning endomorphisms of a formal group law.

Definition 5. Let $f \in R[[x,y]]$ be a formal group law over R. An *endomorphism* of f is a power series $g(t) \in tR[[t]]$ such that $f(g(x), g(y)) = gf(x,y)$.

To prove Proposition 1, we need to introduce the notion of a *translation invariant* differential on $\operatorname{Spf} R[[t]]$. First, let $R[[t]]dt$ denote a free module of rank 1 over $R[[t]]$; we will call elements of $R[[t]]$ *differentials*. Given a differential $g(t)dt$, we write $f^*(g(t)dt) = g(f(x,y))(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy) \in R[[x,y]]\{dx, dy\}$. We will say that $g(t)dt$ is an *translation invariant* differential if we have $f^*(g(t)dt) = g(x)dx + g(y)dy$.

Example 6. Let $f(x,y) = x + y$ be the additive formal group law. Then $dt \in R[[t]]$ is a translation invariant differential.

Example 7. Let $f(x,y) = x + y + xy$ is a multiplicative formal group law. Then $\frac{dt}{1+t} = dt - tdt + t^2 dt + \cdots$ is a translation invariant differential.

There exists a unique translation invariant differential of the form $\omega = dt + c_1 t dt + \ldots$. Moreover, $R[[t]]dt$ can be identified with the free module $R[[t]]\omega$.

Now suppose that $h(t) = a_1 t + a_2 t^2 + \cdots \in tR[[t]]$. Composition with h determines a map h^* from $R[[t]]dt$ to itself, given by $h^*(g(t)dt) = (g \circ h)(t)dh$, where $dh = a_1 dt + 2a_2 t dt + \cdots$. Note that $h^* = 0$ if and only if each coefficient $ia_i = 0$: since $p = 0$ in R, this is equivalent to the requirement that a_i vanishes for i not divisible by p. Equivalently, $h^* = 0$ if and only if we can write $h(t) = h'(t^p)$ for some other power series h'.

Suppose that f and f' are formal groups over R, and that h is a morphism from f to f': that is, h satisfies $hf(x,y) = f'(h(x), h(y))$. Then h^* carries invariant differentials with respect to f' to invariant differentials with respect to f. In particular, if we let ω_f and $\omega_{f'}$ be defined as above, then we have $h^*\omega_{f'} = \lambda\omega_f$ for some constant $\lambda \in R$. Unwinding the definitions, we see that $h(t) \equiv \lambda t \mod (t^2)$. We conclude the following:

Claim 8. *Let $f(x,y), f'(x,y) \in R[[x,y]]$ be formal group laws over a ring R such that $p = 0$ in R, and let $h \in tR[[t]]$ satisfy $hf(x,y) = f'(h(x),h(y))$. Then one of the following conditions holds:*

(1) *There exists $\lambda \neq 0 \in R$ such that $h(t) = \lambda t + \cdots$.*

(2) *There exists another power series h' such that $h(t) = h'(t^p)$.*

Let $f'(x,y)$ be the power series defined by the equation $f'(x^p, y^p) = f(x,y)^p$ (that is, f' is obtained from f by raising all coefficients to the pth power). In the second case, we get

$$f'(h_0(x^p), h_0(y^p)) = f'(h(x), h(y)) = hf(x,y) = h_0(f(x,y)^p) = h_0 f^p(x^p, y^p),$$

so that $h_0 f^p(x,y) = f'(h_0(x), h_0(y))$. that is, h_0 can be regarded as a morphism from f^p into f'. Repeating the above argument, we arrive at the following:

Claim 9. *Let R be a commutative ring with $p = 0$, let $f(x,y), f'(x,y) \in R[[x,y]]$ be formal group laws, and let h be a power series satisfying $hf(x,y) = f'(h(x),h(y))$. If $h \neq 0$, then there exists $n \geq 0$ such that $h(t) = h'(t^{p^n})$ with $h'(t) = \lambda t + O(t^2)$, $\lambda \neq 0$.*

Definition 10. Let $f(x,y) \in R[[x,y]]$ be a formal group law over a commutative ring R. For every nonnegative integer n, we define the *n-series* $[n](t) \in R[[t]]$ as follows:

(1) If $n = 0$, we set $[n](t) = 0$.

(2) If $n > 0$, we set $[n](t) = f([n-1](t), t)$.

Remark 11. For every integer n, the n-series $[n]$ determines a homomorphism from the formal group f to itself. That is, we have $f([n](x), [n](y)) = [n]f(x,y)$.

Since $f(x,y) = x + y + \cdots$, we immediately deduce that $[n](t) = nt + O(t^2)$. Consequently, if p is a prime number such that $p = 0$ in R, then the linear term of $[p](t)$ vanishes: that is, we can write $[p](t) = ct^k + O(t^{k+1})$ for some $k > 1$.

Since $[p]$ is an endomorphism of f, we immediately obtain the following:

Proposition 12. *Let R be a commutative ring in which $p = 0$ and let f be a formal group law over R. Then either $[p](t) = 0$, or $[p](t) = \lambda t^{p^n} + O(t^{p^n+1})$ for some $n > 0$.*

Definition 13. Let f be a formal group law over a commutative ring R, and fix a prime number p. We let v_n denote the coefficient of t^{p^n} in the p-series $[p]$. We will say that f has *height* $\geq n$ if $v_i = 0$ for $i < n$. We will say that f has *height exactly* n if it has height $\geq n$ and $v_n \in R$ is invertible.

Remark 14. We have $v_0 = p$. Thus f has height ≥ 1 if and only if $p = 0$ in R, and height exactly zero if and only if p is invertible in R.

Remark 15. Let f and f' be formal group laws over a commutative ring R, having p-series $[p]_f$ and $[p]_{f'}$. If $g(t)$ is an isomorphism between f and f', then we have $[p]_{f'}(t) = (g \circ [p]_f \circ g^{-1})(t)$. It follows immediately that the heights of f and f' are the same.

Example 16. Let $f(x,y) = x + y + xy$ be the formal multiplicative group. Then $[n](t) = (1+t)^n - 1$. If $p = 0$ in R, then $[p](t) = (1+t)^p - 1 = t^p$; thus f has height exactly 1.

Example 17. Let $f(x,y) = x + y$ be the formal multiplicative group over a commutative ring R with $p = 0$. Then $[p](t) = 0$, so f has infinite height. In the next lecture, we will see that the converse holds: if f is a formal group law of infinite height, then f is isomorphic to the additive group.

It follows from Examples 16 and 17 that the additive and multiplicative formal group laws are not isomorphic over any commutative ring in which $p = 0$.

The Stratification of $\mathcal{M}_{\mathrm{FG}}$ (Lecture 13)

April 27, 2010

Let p be a prime number, fixed throughout this lecture. Our goal is to describe the structure of the moduli stack $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$ of formal groups over p-local rings.

We begin by recalling a few definitions from the previous lecture. If $f(x,y) \in R[[x,y]]$ is a formal group law over a $\mathbf{Z}_{(p)}$-algebra R, we let v_n denote the coefficient on t^{p^n} in the p-series $[p](t)$. We obtain a sequence of elements $v_0 = p, v_1, \ldots \in R$. We say that f has *height* $\geq n$ if the elements v_i vanish for $i < n$, and *height exactly* n if it has height $\geq n$ and v_n is invertible.

Restricting our attention to the universal case, we can regard $v_0, v_1, \ldots$ as elements of the Lazard ring L. We now describe the relationship between these elements and our presentation of L as a polynomial ring $\mathbf{Z}[t_1, t_2, \ldots]$. In our earlier discussion, the coordinates t_i were not canonically determined. What is canonically determined is the isomorphism $(I/I^2)_{2n} \simeq \mathbf{Z} t_n$, where I is the ideal generated by elements of positive degree. We can regard each v_i as an element of $L_{2(p^i-1)}$, so that v_i has a canonically defined image in $(I/I^2)_{2(p^i-1)} \simeq \mathbf{Z} t_{p^i-1}$.

Proposition 1. *The image of $v_n \in (I/I^2)_{2(p^n-1)} \simeq \mathbf{Z}$ is $p^{p^n-1} - 1$. That is, we can write $v_n = -t_{p^n-1} + p^{p^n-1} t_{p^n-1} +$ decomposables.*

Proof. Let $k = p^n - 1$. The homomorphism $L \to \mathbf{Z} \oplus (I/I^2)_{2k} \simeq \mathbf{Z} \oplus \mathbf{Z} t_k$ classifies the formal group law

$$f(x,y) = x + y + \sum_{0 < i < p^n} \frac{1}{p} \binom{p^n}{i} t_k x^i y^{p^n - i}.$$

We obtain formally $f(x,y) = x + y + \frac{t_k}{p}((x+y)^{p^n} - x^{p^n} - y^{p^n})$. It follows by induction on a that the series $[a]$ is given by $[a](t) = at + \frac{t_k}{p}((at)^{p^n} - at^{p^n})$. In particular, the coefficient of t^{p^n} in $[p](t)$ is $\frac{t_k}{p}(p^{p^n} - p) = (p^{p^n-1} - 1) t_k$. $\square$

It follows that after localizing at the prime p, we can choose another isomorphism $L_{(p)} \simeq \mathbf{Z}_{(p)}[t_1, t_2, \ldots]$, where each t_{p^n-1} is given by v_n. In other words, the elements v_n in L can be regarded as the "interesting" generators of L (under the isomorphism $L \simeq \pi_* \mathrm{MU}$ of Quillen's theorem, these are the generators of Adams filtration 1).

Corollary 2. *Let k be a field of characteristic p. Then, for every integer $1 \leq n \leq \infty$, there exists a formal group law of height n over k.*

Proof. If $n = \infty$, we can take $f(x,y) \in k[[x,y]]$ to be the additive formal group law $f(x,y) = x + y$. If $n < \infty$, we take f to be any formal group law classified by a map $L \simeq \mathbf{Z}[t_1, t_2, \ldots] \to k$ such that $t_i \mapsto 0$ for $i < p^n - 1$, but $t_{p^n-1} \mapsto 1$. $\square$

Recall that the condition that a formal group $f(x,y) \in R[[x,y]]$ have height $\geq n$ depends only on the isomorphism class of f. Moreover, it is a *local* condition: that is, if we are given a collection of elements $a_1, \ldots, a_k \in R$ with $a_1 + \cdots + a_k = 1$, then f has height $\geq n$ over R if and only if f has height $\geq n$ over $R[\frac{1}{a_i}]$ for all i. Consequently, if $\mathcal{F} : \mathrm{Alg}_R \to \mathrm{Ab}$ is a formal group over R which is not necessarily coordinatizable, it makes sense to demand that $\mathcal{F}$ has height $\geq n$: this is the requirement that $\mathcal{F} \,|\, \mathrm{Alg}_{R'}$ have height $\geq n$, whenever R' is an R-algebra such that $\mathcal{F} \,|\, \mathrm{Alg}_{R'}$ is coordinatizable.

Remark 3. Here is another interpretation of the height of a formal group. Let $\mathcal{F} : \mathrm{Alg}_R \to \mathrm{Ab}$ be a formal group of height exactly n. Then $\mathcal{F}[p] = \ker(\mathcal{F} \xrightarrow{p} \mathcal{F})$ is representable by a finite flat group scheme over R, of rank p^n. To see this, it suffices to work locally: we may therefore assume that $\mathcal{F}$ is defined by a formal group law $f(x,y) \in R[[x,y]]$ with p-series $[p](t) = \lambda t^{p^n} + \cdots$ where λ is invertible. Then $\mathcal{F}[p] = \operatorname{Spec} R[[t]]/(\lambda t^{p^n} + \cdots)$.

For example, if $\mathcal{F}$ is the formal multiplicative group, then $\mathcal{F}[p]$ is the group scheme μ_p, defined by $\mu_p(A) = \{a \in A : a^p = 1\}$. We have $\mu_p = \operatorname{Spec} R[a]/(a^p - 1)$, which has rank p.

We can define a closed substack $\mathcal{M}_{\mathrm{FG}}^{\geq n}$ of $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$ as follows: for every commutative $\mathbf{Z}_{(p)}$-algebra R, $\mathcal{M}_{\mathrm{FG}}^{\geq n}(R)$ is the category of formal groups of height $\geq n$ over R (with morphisms given by isomorphisms). We have $\mathcal{M}_{\mathrm{FG}}^{\geq n} = \operatorname{Spec}(L_{(p)}/(v_0, \ldots, v_{n-1}))/G^+$, where G^+ is the group scheme of coordinate transformations defined in the previous lecture. This makes sense because the ideal $(v_0, \ldots, v_{n-1})$ is G^+-invariant: this is a translation of the statement that the condition of having height $\geq n$ is an isomorphism invariant condition.

Remark 4. The elements $v_i \in L$ are not themselves G-invariant: that is, if f and f' are isomorphic formal group laws over a commutative ring R, then the p-series $[p]_f(t)$ and $[p]_{f'}(t)$ are generally different. However, if we assume that f and f' have height $\geq n$ and $g(t) = b_0 t + b_1 t^2 + \ldots$ is an invertible power series such that $gf(x,y) = f'(g(x), g(y))$, then $g \circ [p]_f \simeq [p]_{f'} \circ g'$. If $[p]_f(t) = v_n t^{p^n} + \cdots$ and $[p]_{f'} = v'_n t^{p^n} + \cdots$, then examining leading terms gives $b_0 v_n = b_0^{p^n} v'_n$. In other words, as an element in the quotient ring $L/(v_0, \ldots, v_{n-1})$, v_n is invariant under the subgroup $G \subseteq G^+$, and is acted on by the quotient $G^+/G \simeq \mathbb{G}_m$ by the character $\mathbb{G}_m \xrightarrow{p^n - 1} \mathbb{G}_m$. In more invariant terms, this means that we can descend v_n to a section of the line bundle $\omega^{p^n - 1}$ on the moduli stack $\mathcal{M}_{\mathrm{FG}}^{\geq n}$.

For $0 \leq n < \infty$, we let $\mathcal{M}_{\mathrm{FG}}^n$ denote the locally closed substack

$$\mathcal{M}_{\mathrm{FG}}^{\geq n} - \mathcal{M}_{\mathrm{FG}}^{\geq n+1} = (\operatorname{Spec} L_{(p)}[v_n^{-1}]/(v_0, \ldots, v_{n-1}))/G^+$$

of $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$. Also let $\mathcal{M}_{\mathrm{FG}}^{\infty} = \mathcal{M}_{\mathrm{FG}}^{\geq \infty} = (\operatorname{Spec} L/(v_0, v_1, \ldots))/G^+$ denote the moduli stack of formal groups having infinite height. Thus $\mathcal{M}_{\mathrm{FG}}^n$ are the open strata for a stratification of the moduli stack $\mathcal{M}_{\mathrm{FG}}$. We will see that each stratum as a relatively simple structure.

Example 5. The moduli stack $\mathcal{M}_{\mathrm{FG}}^0$ of formal groups of height 0 can be identified with $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Q} \simeq B\mathbb{G}_m$.

Note that $\mathcal{M}_{\mathrm{FG}}^{\geq 1} = \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{F}_p$. For the remainder of the discussion, we will work with commutative rings R which have characteristic p: that is, we will assume that $p = 0$ in R.

The following characterization of height is convenient:

Proposition 6. *Let R be a commutative ring such that $p = 0$ in R, and let $f(x,y) \in R[[x,y]]$ be a formal group law over R. For $1 \leq n \leq \infty$, the following conditions are equivalent:*

(1) *The formal group law f has height $\geq n$.*

(2) *There exists a formal group law f' which is isomorphic to f such that $f'(x,y) \equiv x + y \mod (x,y)^{p^n}$ is congruent to $x + y$ modulo $(x,y)^{p^n}$.*

Lemma 7. *Let R be a commutative ring and let $f, f' \in R[[x,y]]$ be formal group laws. Suppose that $f(x,y)$ is congruent to $f'(x,y)$ modulo the ideal $(x,y)^m$. Let*

$$d = \begin{cases} p & \text{if } m = p^n \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists a unique constant $\lambda \in R$ such that $f(x,y)$ is congruent to $f'(x,y) + \sum_{0<i<m} \frac{\lambda}{d} \binom{m}{i} x^i y^{m-i}$. modulo $(x,y)^{m+1}$.

Proof. There exist constants $\{\lambda_{i,j}\}_{i+j=m}$ such that $f(x,y)$ is congruent to $f'(x,y)+\sum_{0\leq i\leq m}\lambda_{i,m-i}x^i y^{m-i}$ modulo $(x,y)^{m+1}$. Since $f(x,0)=f'(x,0)=x$, we conclude that $\lambda_{m,0}=0$; similarly $\lambda_{0,m}=0$.

We have

$$f(f(x,y),z)-f'(f'(x,y),z)=f(x,f(y,z))-f'(x,f'(y,z)).$$

Extracting the coefficient of $x^i y^j z^k$ when $i+j+k=m$ and $i,j,k>0$, we conclude that

$$\binom{i+j}{j}\lambda_{i+j,k}=\binom{j+k}{j}\lambda_{i,j+k}.$$

In Lecture 3, we saw that all solutions to these equations are as stated in the Lemma. $\square$

Proof of Proposition 6. First assume that n is finite. We prove by induction on $m<p^n$ that, after a change of variable, we can assume that $f(x,y)$ is congruent to $x+y$ modulo $(x,y)^m$. By the inductive hypothesis, we may assume that the congruence holds modulo $(x,y)^{m-1}$. Let d be defined as in Lemma 7, so that $f(x,y)$ is congruent to $x+y+\sum_{0<i<m}\frac{\lambda}{d}\binom{m}{i}x^i y^{m-i}$ for some $\lambda\in R$. If m is not a power of p, then we define $f'(x,y)=g^{-1}f(g(x),g(y))$ where $g(t)=t+\frac{\lambda t^m}{d}$; a simple calculation shows that $f'(x,y)$ is congruent to $x+y$ modulo $(x,y)^m$. If $m=p^{n'}$ then we necessarily have $n'<n$. We claim that $f(x,y)$ is automatically congruent to $x+y$ modulo $(x,y)^m$. This follows from the calculation of the previous lecture: f is classified by a homomorphism $L\simeq \mathbf{Z}[t_1,\ldots,]\rightarrow R$, and we wish to show that the image of each $t_{m'}$ is equal to zero for $m'<m$. By the inductive hypothesis, this holds for $m'<m-1$. Then the image of t_{m-1} is given by $-v_{n'}$ (here $v_{n'}$ is the coefficient of $t^{p^{n'}}$ in the p-series $[p](t)$), and therefore vanishes since we have assumed that f has height $\geq n$.

Suppose now that n is infinite. Using the above construction, we define a sequence of formal group laws $f_m(x,y)$ which are isomorphic to f such that $f_m(x,y)$ is congruent to $x+y$ modulo $(x,y)^m$. We have $f_m(x,y)=g_m^{-1}f(g_m(x),g_m(y))$. By construction, the power series $g_m(t)$ converge in the t-adic topology to an invertible power series $g(t)$; then $g^{-1}f(g(x),g(y))=x+y$ is the additive formal group. $\square$

Corollary 8. *Let f be a formal group law of infinite height over a commutative ring R (necessarily with $p=0$ in R). Then f is isomorphic to the additive formal group law $f'(x,y)=x+y$.*

Remark 9. It follows that we can identify $\mathcal{M}_{\mathrm{FG}}^{\infty}$ with the classifying stack for the group of automorphisms of the additive formal group $f(x,y)=x+y\in\mathbf{F}_p[[x,y]]$. This is the group scheme whose R-points are given by power series of the form

$$g(t)=a_0t+a_1t^p+a_2t^{p^2}+\ldots\in R[[t]],$$

where a_0 is invertible. This group scheme is closely related to the structure of the (mod p) Steenrod algebra.

We now study formal groups of height n where $0<n<\infty$. The basic result is the following:

Theorem 10 (Lazard)**.** *Let k be an algebraically closed field of characteristic p. Then two formal group laws $f(x,y),f'(x,y)\in k[[x,y]]$ are isomorphic if and only if they have the same height.*

Here the condition that k be algebraically closed can be weakened, but not completely removed. To prove Theorem 10 we need to write down an isomorphism between f and f': that is, we need to find an invertible power series $g(t)=b_0t+b_1t^2+\ldots$ such that $gf(x,y)=f'(g(x),g(y))$. This identity amounts to a system of equations that the coefficients b_i must satisfy. Theorem 10 asserts that these equations can be solved with values in an algebraically closed field. In fact, we can be much more precise. Let $f(x,y),f'(x,y)\in R[[x,y]]$ be formal group laws of height exactly $n>0$ over a commutative ring R. Then we can define a ring $R'=R[b_0^{\pm1},b_1,\ldots]/I$ which parametrizes isomorphisms between f and f': take I to be the ideal generated by the coefficients on x^iy^j in the expression $gf(x,y)-f'(g(x),g(y))$. A more precise version of Theorem 10 can be formulated as follows:

Theorem 11. *Let $f(x, y), f'(x, y) \in R[[x, y]]$ be formal group laws of height exactly $n > 0$ and let R' be defined as above. Then R' isomorphic to the direct limit of a system of (injective) finite etale maps*

$$R = R(0) \hookrightarrow R(1) \hookrightarrow R(2) \hookrightarrow \cdots$$

When R is an algebraically closed field k, each $R(i)$ is a product of copies of k. It follows that we can choose a compatible system of ring homomorphisms $R(i) \to k$, which together define a map $R' \to k$ giving rise to the desired isomorphism of f with f'. In fact, we need not assume that k is algebraically closed: it is enough to suppose that k is separably closed or, more generally, that k is a strictly Henselian ring.

We will prove Theorem 11 in the next lecture.

Classification of Formal Groups (Lecture 14)

April 27, 2010

Our goal in this lecture is to prove Lazard's theorem, which asserts that a formal group law over an algebraically closed field is determined up to isomorphism by its height. We will prove this result in the following more precise form:

Theorem 1. *Let $f(x,y), f'(x,y) \in R[[x,y]]$ be formal group laws of height exactly $n > 0$ and let R' be the ring which classifies isomorphisms between f and f': that is, $R' = R[b_0^{\pm 1}, b_1, b_2, \ldots]/I$, where I is the ideal generated by all coefficients in the power series $f(g(x), g(y)) - g(f'(x,y))$, where $g(t) = b_0 t + b_1 t^2 + \cdots$. Then R' is isomorphic to the direct limit of a system of (injective) finite etale maps*

$$R = R(1) \hookrightarrow R(2) \hookrightarrow \cdots$$

We will regard f and f' as fixed for the duration of the proof. Since $f'(x,y)$ has height exactly n, we may assume without loss of generality that

$$f'(x,y) \equiv x + y + \sum_{0<i<p^n} \lambda \frac{\binom{p^n}{i}}{p} x^i y^{p^n - i} \mod (x,y)^{p^n+1},$$

where λ is invertible in R.

Our first step is to choose a more convenient set of polynomial generators for the ring $R[b_0^{\mp 1}, b_1, b_2, \ldots]$.

Construction 2. *Let A be a commutative R-algebra and suppose we are given a sequence of elements $c_0, c_1, \ldots \in A$ with c_0 invertible. We define a sequence of formal group laws $f_m(x,y)$ by induction as follows:*

1. *Set $f_1(x,y) = f(x,y)$.*
2. *If m is not a power of p, we let $f_m(x,y) = g_m^{-1} f_{m-1}(g_m(x), g_m(y))$, where $g_m(x) = x + c_{m-1}x^m$.*
3. *If $m = p^{n'}$ for $n' < n$, we let $f_m = f_{m-1} = g_m^{-1} f_{m-1}(g_m(x), g_m(y))$ where $g_m(t) = t$.*
4. *If $m = p^n$, we let $f_m = g_m^{-1} f_{m-1}(g_m(x), g_m(y))$ where $g_m(t) = c_0 t$.*
5. *If $m = p^{n+n'}$ for $n' > 0$, we let $f_m = g_m^{-1} f_{m-1}(g_m(x), g_m(y))$ where $g_m(t) = f_{m-1}(t, c_{p^{n'}-1} t^{p^{n'}})$.*

We note that $f_m(x,y)$ tends to a limit $f_\infty(x,y) = g^{-1} f(g(x), g(y))$ where $g(t)$ denotes the infinite (convergent) infinite composition $g_2 \circ g_3 \circ g_4 \circ \cdots$. Note that $g(t) = b_0 t + b_1 t^2 + b_2 t^3 + \cdots$ where $b_i = c_i +$decomposables. This gives an identification of polynomial rings

$$R[b_0^{\pm 1}, b_1, b_2, \ldots] \simeq R[c_0^{\pm 1}, c_1, \ldots].$$

We can therefore identify the ring R' of Theorem 1w ith $R[c_0^{\pm 1}, c_1, \ldots]/I$, where I is the ideal generated by all coefficients in the power series $f_\infty(x,y) - f'(x,y)$.

Lemma 3. *Let $c_0, c_1, \ldots \in A$ be as above. Assume that $f_{m-1}(x,y)$ is congruent to $f'(x,y)$ modulo $(x,y)^m$. Then $f_m(x,y)$ is congruent to $f'(x,y)$ modulo $(x,y)^m$.*

Proof. In cases (1) through (3), we have $g_m(t) \equiv t \mod t^m$ so it is clear that

$$f_m(x,y) \equiv f_{m-1}(x,y) \equiv f'(x,y) \mod (x,y)^m.$$

In case (4), we have $f_{m-1}(x,y) \equiv x+y \mod (x,y)^m$ so that

$$f_m(x,y) = c_0^{-1} f_{m-1}(c_0 x, c_0 y) \equiv x+y \mod (x,y)^m$$

The tricky part is case (5).

The tricky part is case (5). Let $m = p^{n+n'}$ for $n' > 0$, and let $c = c_{p^{n'}-1}$, so that $g_m(t) = f_{m-1}(t, ct^{p^{n'}})$. For any sequence of variables $x_1, x_2, \ldots, x_a$, we let $f_{m-1}(x_1, x_2, \ldots, x_a) = f_{m-1}(x_1, f_{m-1}(x_2, \ldots f_{m-1}(x_{a-1}, x_a)) \ldots)$ (this is unambiguous since f_{m-1} is a formal group law).

We have

$$g_m f_m(x,y) = f_{m-1}(g_m(x), g_m(y)) = f_{m-1}(x, y, cx^{p^{n'}}, cy^{p^{n'}}.$$

Let $z = z(x,y)$ be such that $cf_m(x,y)^{p^{n'}} = f_{m-1}(z, cx^{p^{n'}}, cy^{p^{n'}})$, so that $f_{m-1}(f_m(x,y), z) = f_{m-1}(x,y)$. We prove the following by simultaneous induction on $m' \leq m$:

(a) We have $z \equiv 0 \mod ((x,y)^{m'})$.

(b) We have $f_m(x,y) \equiv f_{m-1}(x,y) \equiv f'(x,y) \mod ((x,y)^{m'})$.

These claims are obvious when $m' = 1$, and the implication $(a) \Rightarrow (b)$ is clear. Assume that (a) and (b) hold for some integer $m' < m$. The inductive hypothesis gives $f_{m-1}(z, cx^{p^{n'}}, cy^{p^{n'}}) \equiv z + f_{m-1}(cx^{p^{n'}}, cy^{p^{n'}})$ mod $(x,y)^{m'+1}$. Thus $z \equiv cf_m(x,y)^{p^{n'}} - f_{m-1}(cx^{p^{n'}}, cy^{p^{n'}}) \mod (x,y)^{m'+1}$. The inductive hypothesis gives $f_m(x,y)^{p^{n'}} \equiv f_{m-1}(x,y)^{p^{n'}} \mod (x,y)^{p^{n'}m'}$, so we get

$$z \equiv cf_{m-1}(x,y)^{p^{n'}} - f_{m-1}(cx^{p^{n'}}, cy^{p^{n'}}) \mod (x,y)^{m'+1}$$

By assumption, we have $f_{m-1}(x,y) \equiv f'(x,y) \equiv x+y \mod (x,y)^{p^n}$. It follows that

$$cf_{m-1}(x,y)^{p^{n'}} - f_{m-1}(cx^{p^{n'}}, cy^{p^{n'}}) \equiv c(x+y)^{p^{n'}} - cx^{p^{n'}} - cy^{p^{n'}} \equiv 0 \mod (x,y)^{p^{n+n'}}.$$

Since $m'+1 \leq m = p^{n+n'}$, we conclude that $z \equiv 0 \mod (x,y)^{m'+1}$ as desired. $\square$

We now return to the proof of Theorem 1. By Lemma 3, we have $f_\infty(x,y) = f'(x,y)$ if and only if $f_m(x,y) \equiv f'(x,y) \mod (x,y)^{m+1}$ for all m. Note that $f_m(x,y)$ depends only on the parameters c_i where i belongs to the set $S_m = \{i < m : i \neq p^k - 1\} \cup \{p^k - 1 : p^{n+k} \leq m\}$. $R(m)$ denote the quotient ring $R[c_i]_{i \in S_m}/I(m)$ for $m < p^n$, and the quotient ring $R[c_i, c_0^{-1}]_{i \imath S_m}/I(m)$ for $m \geq p^n$, where $I(m)$ is the ideal generated by the coefficients of $x^i y^j$ in $f_m(x,y) - f'(x,y)$ where $i + j \leq m$. Then R' is the colimit of the sequence

$$R = R(1) \to R(2) \to R(3) \to \cdots$$

To prove Theorem 1, it will suffice to show that each of the inclusions $R(m-1) \to R(m)$ is a finite etale extension (which is injective). There are several cases to consider:

(a) Suppose that m is not a power of p. Then $R(m) = R(m-1)[c_{m-1}]/J$, where J is the ideal generated by coefficients of total degree m in the expression $f_m(x,y) - f'(x,y)$. Note that $f_{m-1}(x,y) \equiv f'(x,y)$ mod $(x,y)^m$, so (by the lemma of the previous lecture) we can write

$$f'(x,y) \equiv f_{m-1}(x,y) + \mu \sum_{0<i<m} \frac{\binom{m}{i}}{d} x^i y^{m-i} \mod (x,y)^{m+1}$$

where d is the greatest common divisor of the binomial coefficients $\binom{m}{i}$. Since m is not a power of p, the integer d is invertible in R. A simple calculation gives $f_m(x,y) \equiv f_{m-1}(x,y) + c_m(x^m + y^m - (x+y)^m)$ mod $(x,y)^{m+1}$. Thus $f_m(x,y) \equiv f'(x,y)$ if and only if $c_m = -\frac{\mu}{d}$. It follows that $R(m) \simeq R(m-1)$ (that is, the coefficient c_m is uniquely determined by the requirement that $f'(x,y) \equiv f_m(x,y)$ mod $(x,y)^{m+1}$.

(b) Suppose that $m = p^{n'}$, $n' < n$. Then $R(m) = R(m-1)/J$, where J is the ideal generated by coefficients of degree m in the difference $f_m(x,y) - f'(x,y)$. We have $f_m(x,y) = f_{m-1}(x,y) \equiv f'(x,y) \equiv x + y \mod (x,y)^{p^m}$. It follows from the lemma of the last lecture that $f_m(x,y) = x+y+\mu\sum_{0<i<m} \frac{\binom{p^{n'}}{i}}{p} x^i y^{m-i}$ for some uniquely determined constant μ. Since f_m is isomorphic to f, it has height exactly n, and therefore $\mu = 0$. It follows that $f_m(x,y) \equiv x+y \equiv f'(x,y) \mod (x,y)^{p^m+1}$, so that again $R(m) \simeq R(m-1)$.

(c) Suppose that $m = p^n$. Then $R(m) = R(m-1)[c_0^{\pm 1}]/J$ where J is the ideal generated by coefficients of degree m in $f_m(x,y) - f'(x,y)$. We have $f_{m-1}(x,y) \equiv f'(x,y) \equiv x+y \mod (x,y)^{p^m}$ so that

$$f_{m-1}(x,y) \equiv x + y + \lambda' \sum_{0<i<m} \frac{\binom{m}{i}}{p} x^i y^{m-j} \mod (x,y)^{m+1}$$

for some constant λ'. It follows that

$$f_m(x,y) \equiv x + y + c_0^{p^n-1}\lambda' \sum_{0<i<m} \frac{\binom{m}{i}}{p} x^i y^{m-j} \mod (x,y)^{m+1}.$$

Consequently, $f_m(x,y) \equiv f'(x,y) \mod (x,y)^{m+1}$ if and only if $c_0^{p^n-1}\lambda' = \lambda$. Since f and f' have height exactly n, the constants λ and λ' are invertible; thus $R(m) \simeq R(m-1)[c_0]/(c_0^{p_n-1} - \frac{\lambda}{\lambda'})$.

(d) Suppose that $m = p^{n+n'}$ for $n' > 0$. Let $c = c_{p^{n'}-1}$, so that $R(m) \simeq R(m-1)[c]/J$, where J is the ideal generated by coefficients on monomials of degree m in $f_m(x,y) - f'(x,y)$. This is the tricky part.

Since $f_{m-1}(x,y) \equiv f'(x,y) \mod (x,y)^m$, we can write

$$f_{m-1}(x,y) \equiv f'(x,y) + \mu \sum_{0<i<m} \frac{\binom{m}{i}}{p} x^i y^{m-i}$$

for some constant μ. Let $z = z(x,y)$ be as in the proof of Lemma 3, so that $z(x,y) \in (x,y)^m$. We have

$$f_{m-1}(x,y) = f_{m-1}(f_m(x,y), z) \equiv f_m(x,y) + z \mod (x,y)^{m+1}.$$

Consequently, we have $f_m(x,y) \equiv f'(x,y) \mod (x,y)^{m+1}$ if and only if $z \equiv \mu\sum_{0<i<m} \frac{\binom{m}{i}}{p} x^i y^{m-i} \mod (x,y)^{m+1}$.

The proof of Lemma 3 gives

$$z \equiv c f_{m-1}(x,y)^{p^{n'}} - f_{m-1}(cx^{p^{n'}}, cy^{p^{n'}}) \mod (x,y)^{m+1}.$$

We have

$$f_{m-1}(x,y) \equiv f'(x,y) \equiv x + y + \lambda \sum_{0<j<p^n} \frac{\binom{p^n}{j}}{p} x^j y^{p^n-j} \mod (x,y)^{p^n+1}.$$

It follows that

$$z \equiv (c\lambda^{p^{n'}} - \lambda c^{p^n}) \sum_{0<j<p^n} \frac{\binom{p^n}{j}}{p} x^{p^{n'}j} y^{m-p^{n'}j} \mod (x,y)^{m+1}.$$

Thus $f_m(x,y) \equiv f'(x,y) \mod (x,y)^{m+1}$ if and only if the following conditions are satisfied:

(i) The coefficients $\mu\frac{\binom{p^{n+n'}}{i}}{p}$ vanishes when i is not divisible by p^n.

(ii) For $0 < j < p^{n'}$, we have

$$\mu \frac{\binom{p^{n+n'}}{p^n j}}{p} = (\lambda^{p^{n'}} c - \lambda c^{p^n}) \frac{\binom{p^{n'}}{j}}{p}$$

We claim that these conditions are satisfied if and only if $c^{p^n} - \lambda^{p^{n'}-1} c + \frac{\mu}{\lambda} = 0$. It follows that $R(m) = R(m-1)[c]/(c^{p^n} - \lambda^{p^{n'}-1} c + \frac{\mu}{\lambda})$ is a finite étale extension of $R(m-1)$. To complete the proof, we verify the following combinatorial identity:

Lemma 4. *Let n be an integer. Then*

$$\binom{p^n}{i} \equiv \begin{cases} \binom{p}{j} & \text{if } i = p^{n-1} j \\ 0 & \text{otherwise} \end{cases} \mod p^2.$$

Proof. Let $G = \mathbf{Z}/p^n\mathbf{Z}$ be a cyclic group. Then G acts by translation on the set S of all i-element subsets of G. Let G' be the subgroup $p\mathbf{Z}/p^n\mathbf{Z}$. Any point of S is either fixed by G', or is fixed by a smaller subgroup and therefore has size divisible by p^2. It follows that the cardinality $|S|$ is congruent modulo p^2 to the cardinality of the fixed point set $|S^{G'}|$, which is the number of ways to choose a subset of the quotient G/G' having cardinality $j = \frac{i}{p^{n-1}}$. $\square$

Flat Modules over $\mathcal{M}_{\mathrm{FG}}$ (Lecture 15)

April 27, 2010

We have seen that if E is a complex oriented cohomology theory, then the coefficient ring $\pi_* E$ has the structure of an algebra over the Lazard ring $L \simeq \pi_* \mathrm{MU}$. Our next goal is to address the converse: suppose we are given a graded ring R equipped with a homomorphism $L \to R$ (corresponding to a graded formal group over R). When can we find a complex oriented cohomology theory E such that $R = \pi_* E$?

There is an obvious way to try to write down such a cohomology theory. Namely, for any space X, we can attempt to define the E-homology of X by the formula $E_*(X) = \mathrm{MU}_*(X) \otimes_{\pi_* \mathrm{MU}} R = \mathrm{MU}_*(X) \otimes_L R$. However, this prescription does not always work: in order to get a homology theory, we to know that certain sequences of abelian groups are exact. The functor $\bullet \mapsto \bullet \otimes_L R$ generally does not preserve exact sequences. Of course, if R is flat over L, there is no problem. However, the condition of flatness over the Lazard ring L is very restrictive, because $L \simeq \mathbf{Z}[t_1, t_2, \ldots]$ is very large. Fortunately, we can get by with much less: we do not need to assume that R is flat over L, only that R is flat over the moduli stack $\mathcal{M}_{\mathrm{FG}}$.

We begin by reviewing the notion of quasi-coherent sheaves on a stack.

Definition 1. A *quasi-coherent sheaf* on the moduli stack $\mathcal{M}_{\mathrm{FG}}$ is a rule which specifies, for every R-point $\eta \in \mathcal{M}_{\mathrm{FG}}(R)$ (corresponding to a formal group over R), an R-module $M(\eta)$. This rule is required to be functorial in the following sense: given a homomorphism $R \to R'$ carrying η to $\eta' \in \mathcal{M}_{\mathrm{FG}}(R')$, we have a canonical isomorphism $M(\eta') \simeq M(\eta) \otimes_R R'$.

Remark 2. There is an obvious analogue of Definition 1 if the moduli stack $\mathcal{M}_{\mathrm{FG}}$ is replaced by any other stack.

The collection of quasi-coherent sheaves on $\mathcal{M}_{\mathrm{FG}}$ forms an abelian category, which we will denote by $\mathrm{QCoh}(\mathcal{M}_{\mathrm{FG}})$.

Definition 3. Let M be a quasi-coherent sheaf on $\mathcal{M}_{\mathrm{FG}}$. We will say that M is *flat* if, for every R-point $\eta \in \mathcal{M}_{\mathrm{FG}}$, the R-module $M(\eta)$ is flat over R. Similarly, we will say that M is *faithfully flat* if each $M(\eta)$ is faithfully flat over R.

Remark 4. The condition that an R-module be flat (or faithfully flat) is local with respect to the Zariski topology on $\mathrm{Spec}\, R$. Consequently, if M is a quasi-coherent sheaf on $\mathcal{M}_{\mathrm{FG}}$, then to verify the flatness (or faithful flatness) of M it suffices to test the condition of Definition 3 in the case where $\eta \in \mathcal{M}_{\mathrm{FG}}(R)$ classifies a coordinatizable formal group over R. In this case, η is the image of the point $\eta_0 \in \mathcal{M}_{\mathrm{FG}}(L)$ classifying the universal formal group law. In other words, M is flat (or faithfully flat) if and only if $M(\eta_0)$ is flat (faithfully flat) as a module over the Lazard ring L.

Let R be any ring, and suppose we are given a map $q : \mathrm{Spec}\, R \to \mathcal{M}_{\mathrm{FG}}$ corresponding to a point $\eta \in \mathcal{M}_{\mathrm{FG}}(R)$. Then the forgetful functor $M \mapsto M(\eta)$ can be identified with the pullback functor $q^* : \mathrm{QCoh}(\mathcal{M}_{\mathrm{FG}}) \to \mathrm{QCoh}(\mathrm{Spec}\, R) \simeq \mathrm{Mod}_R$. This functor has a right adjoint $q_* : \mathrm{Mod}_R \to \mathrm{QCoh}(\mathcal{M}_{\mathrm{FG}})$. In concrete terms, if N is an R-module, then $q_*(N)$ is a quasi-coherent sheaf on $\mathcal{M}_{\mathrm{FG}}$ given by the formula $q_*(N)(\eta') = q'_*(p')^* M = M \otimes_R B$, where $\eta' \in \mathcal{M}_{\mathrm{FG}}(R')$ classifies a map $p : \mathrm{Spec}\, R' \to \mathcal{M}_{\mathrm{FG}}$ and B is the $(R \otimes R')$-algebra which classifies the universal isomorphism between the formal groups over R and R'

determined by η and η', so that we have a pullback square

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, "q'"] \arrow[d, "p'"] & \operatorname{Spec} R' \arrow[d, "p"] \\ \operatorname{Spec} R \arrow[r, "q"] & \mathcal{M}_{\mathrm{FG}} . \end{tikzcd}$$

Given $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$, we say that an R-module N is *flat* (or *faithfully flat*) over $\mathcal{M}_{\mathrm{FG}}$ if $q_* N$ is flat (faithfully flat) over $\mathcal{M}_{\mathrm{FG}}$. We will say that q is *flat* (*faithfully flat*) over $\mathcal{M}_{\mathrm{FG}}$ if the R-module R is flat (faithfully flat) over $\mathcal{M}_{\mathrm{FG}}$.

The usefulness of this notion to us is expressed by the following:

Proposition 5. *Let $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$ be a map (classifying a formal group $\eta \in \mathcal{M}_{\mathrm{FG}}(R)$) and let N be an R-module which is flat over $\mathcal{M}_{\mathrm{FG}}$. Then the functor $M \mapsto M(\eta) \otimes_R N = q^* M \otimes_R N$ is an exact functor from $\operatorname{QCoh}(\mathcal{M}_{\mathrm{FG}})$ to Mod_R.*

Proof. The question is local on $\operatorname{Spec} R$; we may therefore assume that η classifies a coordinatizable formal group. Let $p : \operatorname{Spec} L \to \mathcal{M}_{\mathrm{FG}}$ classify the universal formal group law, so we have a pullback diagram

$$\begin{tikzcd} \operatorname{Spec} R[b_0^{\pm 1}, b_1, b_2, \ldots] \arrow[r, "q'"] \arrow[d, "p'"] & \operatorname{Spec} L \arrow[d, "p"] \\ \operatorname{Spec} R \arrow[r, "q"] & \mathcal{M}_{\mathrm{FG}} \end{tikzcd}$$

Since p' is a faithfully flat map, it will suffice to show that the functor

$$M \mapsto {p'}^*(q^* M \otimes_R N) \simeq (qp')^* M \otimes_{R[b_0^{\pm 1}, b_1, \ldots]} {p'}^* N \simeq (p^* M) \otimes_L {p'}^* N$$

is exact. The flatness of N is precisely the condition that ${p'}^* N = N[b_0^{\pm 1}, b_1, \ldots]$ is a flat L-module. $\square$

Corollary 6. *Let M be a graded module over the Lazard ring L. If M is flat over $\mathcal{M}_{\mathrm{FG}}$, then the functor $X \mapsto \mathrm{MU}_*(X) \otimes_L M$ is a homology theory, representable by some spectrum E.*

Example 7. Fix a prime number p, and let $R \simeq \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ be the L-module obtained by taking the quotient of $L_{(p)} \simeq \mathbf{Z}_{(p)}[t_1, t_2, \ldots]$ by the ideal generated by $\{t_i\}_{i+1 \neq p^k}$. We claim that the map $\operatorname{Spec} R \to \operatorname{Spec} L \to \mathcal{M}_{\mathrm{FG}}$ is flat. To prove this, we must show that if we form the fiber product

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, "q"] \arrow[d] & \operatorname{Spec} L \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \mathcal{M}_{\mathrm{FG}}, \end{tikzcd}$$

the map q comes from a flat ring homomorphism $L \to B$. Note that we have two ring homomorphisms $\phi_0, \phi_1 : L \to L[b_0^{\pm 1}, b_1, \ldots]$; ϕ_0 is the obvious map, and ϕ_1 classifies the formal group over $L[b_0^{\pm 1}, b_1, \ldots]$ obtained from the universal formal group by the change of variables $g(t) = b_0 t + b_1 t^2 + \cdots$. Unwinding the definitions, we see that B can be identified with the quotient of $L_{(p)}[b_0^{\pm 1}, b_1, b_2, \ldots]$ by the ideal generated by $\{\phi_1(t_i)\}_{i+1 \neq p^k}$. The proof of Lazard's theorem shows that if $i+1$ is not a power of p, then the image of t_i under the composite map

$$L \xrightarrow{\phi_1} L_{(p)}[b_0^{\pm 1}, b_1, b_2,] \to \mathbf{Z}_{(p)}[b_1, b_2, \ldots]$$

is given by $d b_i$ + *decomposables*, where d is invertible in $\mathbf{Z}_{(p)}$. It follows that we can replace b_i by $\phi_1(t_i)$ in our set of polynomial generators for $L_{(p)}[b_0^{\pm 1}, b_1, b_2, \ldots]$, so that B is a polynomial ring over $L_{(p)}[b_0^{\pm 1}]$ and in particular flat over $L_{(p)}$.

It follows from Corollary 6 that the construction $X \mapsto \mathrm{MU}_*(X) \otimes_L R \simeq \mathrm{MU}_*(X)_{(p)}/(t_i)_{i+1 \neq p^k}$ is a homology theory. This homology theory is called *Brown-Peterson homology* and is denoted by $\mathrm{BP}_*(X)$.

Remark 8. The above proof gives more: not only is $\operatorname{Spec} \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ flat over $\mathcal{M}_{\mathrm{FG}}$, it is faithfully flat over the localized moduli stack $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$.

Corollary 6 highlights the importance of the condition of flatness over the moduli stack $\mathcal{M}_{\mathrm{FG}}$. In the next lecture, we will establish the following criterion for flatness:

Theorem 9 (Landweber)**.** *Let M be a module over the Lazard ring L. Then M is flat over $\mathcal{M}_{\mathrm{FG}}$ if and only if, for every prime number p, the sequence $v_0 = p, v_1, v_2, \ldots \in L$ is a regular sequence for M.*

An L-module M satisfying the hypothesis of Theorem 9 is said to be *Landweber-exact.* Every Landweber-exact graded L-module M determines a homology theory E, given by $E_*(X) = \mathrm{MU}_*(X) \otimes_L M$. In particular we have $\pi_* E = E_*(*) \simeq M$.

Remark 10. Recall that if R is a commutative ring and M is an R-module, then a sequence of elements $x_0, x_1, x_2, \ldots \in R$ is said to be *regular for M* if x_0 is not a zero divisor on M, x_1 is not a zero divisor on $M/x_0 M$, x_2 is not a zero divisor on $M/(x_0 M + x_1 M)$, and so forth. Note that if a module M is trivial, then every element of R is a non zero-divisor on M.

Example 11. Let M be an L-module which is a rational vector space. Then M is Landweber-exact: for every prime p, $v_0 = p$ acts invertibly on M, so $M/v_0 M \simeq 0$.

Example 12. Let $R = \mathbf{Z}[\beta, \beta^{-1}]$, where β has degree 2. We have a graded formal group law $f(x, y) = x + y + \beta xy$ over R (a graded version of the multiplicative formal group over $\mathbf{Z}$), which determines a map of graded rings $L \to R$. We claim that R is Landweber exact. Fix a prime p. Then $v_0 = p$ is a non zero-divisor on R. Modulo p, the p-series $[p](t)$ for f is given by the formula $[p](t) = \beta^{p-1} t^p$, so that $v_1 \equiv \beta^{p-1} \mod p$ and therefore v_1 acts invertibly on R/pR (and so every element of R is a nonzero divisor on $R/(v_0 R + v_1 R)$).

Using Landweber's theorem, we deduce the existence of a homology theory $E_*(X) = \mathrm{MU}_*(X) \otimes_L \mathbf{Z}[\beta, \beta^{-1}]$ with $\pi_* E \simeq R = \mathbf{Z}[\beta, \beta^{-1}]$. We will later see that E_* is given by complex K-theory.

Example 13. Let R be a commutative ring and let E be an elliptic curve defined over R. We can associate to E a formal group $\widehat{E}$, where $\widehat{E}(A) \subseteq \operatorname{Hom}(\operatorname{Spec} A, E)$ is the collection of all A-points of E for which the diagram

$$\begin{tikzcd} \operatorname{Spec} A/\mathfrak{m} \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec} R \arrow[r, "0"] & E \end{tikzcd}$$

commutes; here $\mathfrak{m}$ denotes the nilradical of A. This construction determines a map from the moduli stack $\mathcal{M}_{\mathrm{Ell}}$ of elliptic curves to the moduli stack $\mathcal{M}_{\mathrm{FG}}$ of formal groups.

If we fix a prime number p and a trivialization of the Lie algebra of E, then $v_1 \in R/pR$ can be identified with the classical *Hasse invariant*: it vanishes precisely on the closed subscheme of $\operatorname{Spec} R/pR$ over which E is supersingular. Moreover, v_2 is invertible in $R/(pR + v_1 R)$: that is, the formal group of an elliptic curve is everywhere of height ≤ 2.

To satisfy Landweber's criterion, the pair (E, R) must satisfy the following:

(1) Every prime number p is a non zero-divisor in R: that is, R is flat over $\mathbf{Z}$.

(2) For every prime number p, the Hasse invariant of E is a non zero-divisor in R/pR.

If these conditions are satisfied, then we can define a homology theory $\mathrm{Ell}_*(X)$, where $\mathrm{Ell}_*(X) = \mathrm{MU}_*(X) \otimes_L R[\beta, \beta^{-1}]$. The representing spectrum Ell is sometimes called *elliptic cohomology.*

Remark 14. Conditions (1) and (2) satisfied in "universal" cases; that is, for elliptic curves over $\operatorname{Spec} R$ which define an étale map from $\operatorname{Spec} R$ to the moduli stack $\mathcal{M}_{\mathrm{Ell}}$. In other words, the map of stacks $\mathcal{M}_{\mathrm{Ell}} \to \mathcal{M}_{\mathrm{FG}}$ is flat.

The Landweber Exact Functor Theorem (Lecture 16)

April 27, 2010

Our goal in this lecture is to prove the following result:

Theorem 1. *Let M be a module over the Lazard ring. Then M is flat over $\mathcal{M}_{\mathrm{FG}}$ if and only if, for every prime number p, the elements $v_0 = p, v_1, v_2, \ldots \in L$ form a regular sequence for M.*

We first note that M is flat over $\mathcal{M}_{\mathrm{FG}}$ if and only if, for every prime number p, the localization $M_{(p)} = M \otimes \mathbf{Z}_{(p)}$ is flat over $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$. We therefore fix a prime number p and work locally at p.

Lemma 2. *Let $q : \operatorname{Spec} \mathbf{Z}_{(p)}[v_1, v_2, \ldots] \to \mathcal{M}_{\mathrm{FG}}$ be the flat map considered in the previous lecture. Let M be a quasi-coherent sheaf on $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$. Then M is flat over $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$ if and only if $q^* M$ is a flat $\mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ module.*

Proof. The "only if" direction is immediate from the definitions. Conversely, suppose that $q^* M$ is flat. Fix any map $f : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$; we wish to prove that $f^* M$ is a flat R-module. Form a pullback diagram

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d, "f"] \\ \operatorname{Spec} \mathbf{Z}_{(p)}[v_1, v_2, \ldots] \arrow[r, "q"] & \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}. \end{tikzcd}$$

We saw in the last lecture that q is faithfully flat, so $R \to B$ is a faithfully flat map of commutative rings. Consequently, it will suffice to show that $f^* M \otimes_R B$ is a flat B-module. But $f^* M \otimes_R B = q^* M \otimes_{\mathbf{Z}_{(p)}[v_1, v_2, \ldots]} B$, which if flat over B since $q^* M$ is flat over $\mathbf{Z}_{(p)}[v_1, v_2, \ldots]$. $\square$

Let us now return to the proof of Theorem 1. Let M be a module over the localized Lazard ring $L_{(p)}$ such that $v_0 = p, v_1, v_2, \ldots$ is a regular sequence on M. We wish to prove that the pushforward of M along the map $\operatorname{Spec} L_{(p)} \to \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$ is flat. Form a pullback square

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] & \operatorname{Spec} L_{(p)} \arrow[d] \\ \operatorname{Spec} \mathbf{Z}_{(p)}[v_1, v_2, \ldots] \arrow[r] & \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}. \end{tikzcd}$$

By the Lemma, it will suffice to show that $M_B = M \otimes_{L_{(p)}} B$ is flat as a module over the ring $\mathbf{Z}_{(p)}[v_1, v_2, \ldots]$. In other words, we wish to prove that for every R-module N, the groups $\operatorname{Tor}_i^{\mathbf{Z}_{(p)}[v_1, \ldots]}(M_B, N)$ vanish for $i > 0$.

Since the functor $N \mapsto \operatorname{Tor}_i^{\mathbf{Z}_{(p)}[v_1, v_2, \ldots]}(M_B, N)$ commutes with filtered colimits, it will suffice to show that the groups $\operatorname{Tor}_i^R(M_B, N)$ vanish when $i > 0$ and N is a finitely presented $\mathbf{Z}_{(p)}[v_1, v_2, \ldots]$-module (every module is a filtered colimit of finitely presented modules). Note that a finite presentation of an $\mathbf{Z}_{(p)}[v_1, v_2, \ldots]$-module can reference only finitely many of the polynomial generators $v_1, v_2, \ldots$. In other words, we may assume that there exists an integer $n \geq 1$ such that $N \simeq N_0[v_{n+1}, v_{n+2}, v_{n+3}, \ldots]$, where N_0 is a module over the ring $\mathbf{Z}_{(p)}[v_1, \ldots, v_n]$. In this case, we have $\mathrm{Tor}_i^{\mathbf{Z}_{(p)}[v_1,v_2,\ldots]}(M_B, N) \simeq \mathrm{Tor}_i^{\mathbf{Z}_{(p)}[v_1,\ldots,v_n]}(M_B, N_0)$. In other words, we are reduced to proving that M_B is a flat module over $\mathbf{Z}_{(p)}[v_1, \ldots, v_n]$ for all n.

Let us now address a potentially confusing point. By construction, the ring B is equipped with homomorphisms $\phi' : \mathbf{Z}_{(p)}[v_1, v_2, \ldots] \to B$ and $\phi'' : L_{(p)} \to B$. Consequently, we obtain two *different* sequences of elements

$$v'_0, v'_1, v'_2, \ldots \qquad v''_0, v''_1, v''_2, \ldots$$

in B, given by $v'_i = \phi'(v_i)$ and $v''_i = \phi''(v_i)$. It follows that, for each $m \geq 0$, the finite sequences $(v'_0, v'_1, v'_2, \ldots, v'_{m-1})$ and $(v''_0, v''_1, \ldots, v''_{m-1})$ generate the same ideal $I_m \subseteq B$.

We will prove the following:

Claim 3. *For $m \leq n+1$, the quotient $M_B/I_m M_B$ is a flat module over the ring $\mathbf{Z}_{(p)}[v_1, v_2, \ldots, v_n]/(p, v_1, \ldots, v_{m-1})$.*

When $m = 0$, Claim 3 reduces to what we need to know. We will prove Claim 3 by descending induction on m. Note that if $m = n+1$, then $\mathbf{Z}_{(p)}[v_1, \ldots, v_n]/(p, v_1, \ldots, v_n) \simeq \mathbf{F}_p$ is a field and there is nothing to prove. To carry out the inductive step, we need the following algebraic lemma:

Lemma 4. *Let R be a commutative ring containing a non zero-divisor x, and let M be an R-module. Then M is flat over R if and only if the following conditions are satisfied:*

(1) *The element x is a non zero-divisor on M.*

(2) *The quotient M/xM is a flat $R/(x)$-module.*

(3) *The module $M[x^{-1}]$ is flat over $R[x^{-1}]$.*

Proof. The necessity of conditions (1) through (3) is easy (and not needed for our application). Let us assume that conditions (1), (2), and (3) are satisfied. We wish to prove that M is flat over R: that is, for any R-module N, the groups $\mathrm{Tor}_i^R(M, N)$ vanish for $i > 0$. We carry out the proof in several steps:

(a) Suppose that N is annihilated by x: that is, N is a module over $R/(x)$. Assumption (1) gives $\mathrm{Tor}_i^R(M, N) \simeq \mathrm{Tor}_i^{R/(x)}(M/xM, N)$, which vanishes for $i > 0$ by assumption (2).

(b) Suppose that N is annihilated by x^k for some k. We prove by induction on k that $\mathrm{Tor}_i^R(M, N) \simeq 0$ for $i > 0$. We have an exact sequence

$$0 \to K \to N \to xN \to 0,$$

where K is the kernel of the map $N \to xN$. Since $\mathrm{Tor}_i^R(M, K) \simeq 0$ by (a) and $\mathrm{Tor}_i^R(M, xN) \simeq 0$ by the inductive hypothesis, we deduce from the exact sequence

$$\mathrm{Tor}_i^R(M, K) \to \mathrm{Tor}_i^R(M, N) \to \mathrm{Tor}_i^R(M, xN)$$

that $\mathrm{Tor}_i^R(M, N) \simeq 0$.

(c) Suppose that N consists of x-power torsion: that is, every element $n \in N$ satisfies $x^k n = 0$ for $k \gg 0$. Then N is a filtered colimit of submodules annihilated by x^k, so that $\mathrm{Tor}_i^R(M, N) \simeq 0$ for $i > 0$ by part (b).

(d) Let N be arbitrary, and let K be the kernel of the map $N \to N[x^{-1}]$. Then K satisfies the hypothesis of (c), so that $\mathrm{Tor}_i^R(M, K) \simeq 0$ for $i > 0$. Consequently, to prove that $\mathrm{Tor}_i^R(M, N) \simeq 0$, it suffices to show that $\mathrm{Tor}_i^R(M, N/K) \simeq 0$; that is, we may replace N by N/K and thereby assume that the map $N \to N[x^{-1}]$ is injective.

(e) Let N be as in (d), and let K' be the cokernel of the injection $N \to N[x^{-1}]$. Then K' satisfies the condition of (c), so that $\mathrm{Tor}_i^R(M, K') \simeq 0$ for $i > 0$. Consequently, to prove that $\mathrm{Tor}_i^R(M, N) \simeq 0$, it will suffice to show that $\mathrm{Tor}_i^R(M, N[x^{-1}]) \simeq 0$.

(f) We are now reduced to the case where $N \simeq N[x^{-1}]$: that is, N is a module over $R[x^{-1}]$. We then have $\mathrm{Tor}_i^R(M,N) \simeq \mathrm{Tor}_i^{R[x^{-1}]}(M[x^{-1}],N)$, which vanishes for $i>0$ by assumption (3).

□

Let us now return to the proof of Claim 3. Let $m \leq n$; we wish to prove that $M_B/I_m M_B$ is flat over $R = \mathbf{Z}_{(p)}[v_1,\ldots,v_n]/(p,v_1,\ldots,v_{m-1})$. Note that $M_B/I_m M_B$ can be identified with the tensor product

$$B \otimes_{L_{(p)}} (M/(v_0,\ldots,v_{m-1})).$$

By assumption, v_m is a non zero-divisor on the quotient $M/(v_0,\ldots,v_{m_1})$. Since B is flat over $L_{(p)}$, we have an exact sequence

$$0 \to M_B/I_m M_B \xrightarrow{v''_m} M_B/I_m M_B \to M_B/I_{m+1}M_B \to 0.$$

Since v''_m is congruent to an invertible multiple of v'_m moduli I_m, we deduce that $v_m \in R$ is a non zero-divisor on $M_B/I_m M_B$. Moreover, the quotient $M_B/(I_m,v_m)M_B \simeq M_B/I_{m+1}M_B$ is flat over $R/(v_m)$ by the inductive hypothesis. By the Lemma, we are reduced to proving that $(M_B/I_m M_B)[v_m^{-1}]$ is flat over $R[v_m]^{-1}$. We will prove the following stronger statement:

Claim 5. *For every integer $m \geq 0$, the module $(M_B/I_m M_B)[v_m^{-1}]$ is flat over $(\mathbf{Z}_p[v_1,v_2,\ldots]/(p,v_1,\ldots,v_{m-1}))[v_m^{-1}]$.*

We have a pullback diagram of stacks

$$\begin{tikzcd}
\mathrm{Spec}\, B/(p,v_1,\ldots,v_{m-1})[v_m^{-1}] \arrow[r] \arrow[d] & \mathrm{Spec}\, L_{(p)}/(v_0,\ldots,v_{m-1})[v_m^{-1}] \arrow[d] \\
\mathrm{Spec}(\mathbf{Z}_{(p)}[v_1,v_2,\ldots]/(p,v_1,\ldots,v_{m-1}))[v_m^{-1}] \arrow[r] & \mathcal{M}_{\mathrm{FG}}^m .
\end{tikzcd}$$

Claim 5 is a special case of the assertion that the $L_{(p)}/(v_0,\ldots,v_{m-1})[v_m^{-1}]$-module $(M/(v_0,\ldots,v_{m-1})M)[v_m^{-1}]$ is flat over $\mathcal{M}_{\mathrm{FG}}^m$. This in turn follows from:

Claim 6. *Every quasi-coherent sheaf on the stack $\mathcal{M}_{\mathrm{FG}}^m$ is flat.*

We will prove this claim when $m>0$; the proof when $m=0$ is similar. Let X be a quasi-coherent sheaf on $\mathcal{M}_{\mathrm{FG}}^m$. We wish to prove that q^*X is a flat A-module, for any map $\mathrm{Spec}\, A \to \mathcal{M}_{\mathrm{FG}}^m$ classifying a formal group height exactly m on A. Working locally on $\mathrm{Spec}\, A$, we may assume that the formal group is coordinatizable. Choose a formal group law of height m over $\mathbf{F}_p$ classified by a map $f : \mathrm{Spec}\, \mathbf{F}_p \to \mathcal{M}_{\mathrm{FG}}^m$, and form a pullback diagram

$$\begin{tikzcd}
\mathrm{Spec}\, A' \arrow[r] \arrow[d] & \mathrm{Spec}\, \mathbf{F}_p \arrow[d, "f"] \\
\mathrm{Spec}\, A \arrow[r] & \mathcal{M}_{\mathrm{FG}}^m
\end{tikzcd}$$

In Lecture 14, we proved that A' is a direct limit of a sequence of injective finite etale ring extensions; in particular, A' is faithfully flat over A. Consequently, it will suffice to prove that $q^*X \otimes_A A'$ is flat over A'. But $q^*X \otimes_A A' \simeq f^*X \otimes_{\mathbf{F}_p} A'$. We are therefore reduced to proving that f^*X is flat over $\mathbf{F}_p$, which is obvious since $\mathbf{F}_p$ is a field.

Phantom Maps (Lecture 17)

April 27, 2010

We begin by recalling Adams' variant of the Brown representability theorem:

Theorem 1 (Adams). *Let E be a spectrum and let h_* be a homology theory. Suppose we are given a map of homology theories $\alpha : E_* \to h_*$ (that is, a collection of maps $E_*(X,Y) \to h_*(X,Y)$, depending functorially on a pair of spaces $(Y \subseteq X)$ and compatible with boundary maps). Then there is a map of spectra $\beta : E \to E'$ and an isomorphism of homology theories $E'_* \simeq h'_*$ such that α is given by the composition $E_* \to E'_* \simeq h_*$.*

Corollary 2 (Adams). *Let E and E' be spectra, and let $\alpha : E_* \to E'_*$ be a map between the corresponding homology theories. Then α is induced by a map of spectra $\overline{\alpha} : E \to E'$.*

Proof. Let $h_* = E'_*$. Applying Theorem 1 the evident map $\alpha : E_* \oplus E'_* \to h_*$, we get a spectrum F and a map $E \oplus E' \to F$ inducing α. This comes from a pair of spectrum maps $f : E \to F$ and $g : E' \to F$. The map g induces an isomorphism $\pi_* E' = h_*(*) = \pi_* F$ and is therefore a homotopy equivalence. Then $\overline{\alpha} : g^{-1} \circ f$ is the desired map of spectra from E to E'. □

Corollary 3 (Adams). *Every homology theory h_* is represented by a spectrum E, which is uniquely defined up to (nonunique) homotopy equivalence.*

Proof. The existence of E follows from Theorem 1. For the uniqueness, we note that if E and E' are two spectra with $E_* \simeq h_* \simeq E'_*$, then the isomorphism $E_* \simeq E'_*$ is induced by a map of spectra $E \to E'$ (Corollary 2), which is automatically a homotopy equivalence. □

In the situation of Corollary 2, the map $\overline{\alpha}$ is generally not determined by α, even up to homotopy. This is due to the existence of *phantom maps*:

Definition 4. Let $f : E \to E'$ be a map of spectra. We say that f is a *phantom* if the underlying map of homology theories $E_* \to E'_*$ is zero: that is, for every space X, the map $E_*(X) \to E'_*(X)$ is identically zero.

Lemma 5. *Let $f : E \to E'$ be a map of spectra. The following conditions are equivalent:*

(1) *The map f is a phantom.*

(2) *For every spectrum X, the map $E_*(X) \to E'_*(X)$ is zero.*

(3) *For every finite spectrum X, the map $E_*(X) \to E'_*(X)$ is zero.*

(4) *For every finite spectrum X, the map $E^*(X) \to E'^*(X)$ is zero.*

(5) *For every finite spectrum X and every map $g : X \to E$, the composition $f \circ g : X \to E'$ is nullhomotopic.*

Proof. The implication (2) ⇒ (1) is obvious, and the converse follows from the fact that every spectrum X can be written as a filtered colimit $\varinjlim \Sigma^{\infty - n} \Omega^{\infty - n} X$. The implication (2) ⇒ (3) is obvious, and the converse follows from the fact that every spectrum is a filtered colimit of finite spectra. The equivalence of (4) and (5) follows by Spanier-Whitehead duality, and the equivalence of (4) and (5) is a tautology. □

Let us now return to the setting of the previous lectures. Let $L \simeq \mathbf{Z}[t_1, \ldots]$ denote the Lazard ring, and let M be a graded L-module. Assume that the grading on M is *even*: that is, $M_k \simeq 0$ for every odd number k. In the last lecture, we saw that if M satisfies Landweber's criterion: that is, if the sequence $v_0 = p, v_1, v_2, \ldots \in L$ is M-regular for every prime number p, then the construction

$$X \mapsto \mathrm{MU}_*(X) \otimes_L M$$

is a homology theory. It follows from Corollary 3 that this homology theory is represented by a spectrum E, which is unique up to homotopy equivalence. We will say that a spectrum E is *Landweber-exact* if it arises from this construction. Our goal in this lecture is to show that, as an object of the homotopy category of spectra, E is functorially determined by M. This is a consequence of the following assertion:

Theorem 6. *Let E be a Landweber-exact spectrum, and let E' be a spectrum such that $\pi_k E' \simeq 0$ for k odd. Then every phantom map $f : E \to E'$ is nullhomotopic.*

Corollary 7. *Let E and E' be Landweber exact spectra. Then every phantom map $f : E \to E'$ is nullhomotopic. In particular, every nontrivial endomorphism of E acts nontrivially on the homology theory E_*.*

To prove Theorem 6, we introduce two new notions:

Definition 8. We will say that a finite spectrum X is *even* if the homology groups $\mathrm{H}_k(X; \mathbf{Z})$ are free abelian groups, which vanish when k is odd. Equivalently, a finite spectrum X is *even* if it admits a finite cell decomposition using only even-dimensional cells.

We say that a spectrum E is *evenly generated* if, for every map $X \to E$ where X is a finite spectrum, there exists a factorization $X \to X' \to E$ where X' is a finite even spectrum.

Theorem 6 is a consequence of the following two assertions:

Proposition 9. *Every Landweber exact spectrum E is evenly generated.*

Proposition 10. *Let E be an evenly generated spectrum and let E' be a spectrum whose homotopy groups are concentrated in even degrees. Then every phantom map $f : E \to E'$ is null.*

We begin by proving Proposition 9. Let E be a Landweber-exact spectrum, associated to a graded L-module M, and let $f : X \to E$ be a map where X is a finite spectrum. We can associate to f an element of $E^0(X) = E_0(DX) = \mathrm{MU}_0(DX) \otimes_L M = \mathrm{MU}^0(X) \otimes_L M$, which can be written as $\sum c_i m_i$ where $c_i \in \mathrm{MU}^{d_i}(X)$ and $m_i \in M_{d_i}$. Then f factors as a composition

$$X \stackrel{\{c_i\}}{\to} \bigoplus \Sigma^{d_i} \mathrm{MU} \stackrel{m_i}{\to} E.$$

We may therefore replace E by $\bigoplus \Sigma^{d_i} \mathrm{MU}$: that is, it suffices to prove that $\bigoplus \Sigma^{d_i} \mathrm{MU}$ is evenly generated. Since M is evenly graded, each of the integers d_i is even. We can therefore reduce to showing that MU itself is evenly generated.

Since $\mathrm{MU} \simeq \varinjlim \mathrm{MU}(n)$, it suffices to show that each $\mathrm{MU}(n)$ is evenly generated. Recall that $\mathrm{MU}(n)$ is the Thom complex of the virtual bundle $\zeta - \mathbf{C}^n$, where ζ is the tautological vector bundle on $BU(n)$. We can write $BU(n) \simeq \varinjlim_m \mathrm{Grass}(n, n+m)$, wehre $\mathrm{Grass}(n, n+m)$ denotes the Grassmannian of n-dimensional subspaces of $\mathbf{C}^{n+m}$. It follows that $\mathrm{MU}(n)$ is a direct limit of Thom spectra associated to the finite-dimensional Grassmannians $\mathrm{Grass}(n, n+m)$. It therefore suffices to show that each of these Thom complexes is an even finite spectrum. We now note that the space $\mathrm{Grass}(n, n+m)$ admits a finite cell decomposition with cells of even dimension: for example, we can take the Bruhat decomposition. This proves Proposition 9.

We now prove Proposition 10. Let E be an evenly generated spectrum. We begin by describing the structure of phantom maps from E to other spectra. Let A be a set of representatives for all homotopy equivalence classes of maps $X_\alpha \to E$, where X_α is an even finite spectrum, and form a fiber sequence

$$K \to \bigoplus_{\alpha} X_\alpha \xrightarrow{u} E.$$

This sequence is classified by a map $u' : E \to \Sigma(K)$. Since E is evenly generated, every map from a finite spectrum X into E factors through u, so the composite map $X \to E \to \Sigma(K)$ is null: in other words, u' is a phantom map. Conversely, if $f : E \to E'$ is *any* phantom map, then $f \circ u$ is nullhomotopic, so that f factors as a composition $E \to \Sigma(K) \to E'$. Consequently, to prove Proposition 10, it will suffice to prove that every map $\Sigma(K) \to E'$ is nullhomotopic: that is, that the group $E'^{-1}(K)$ is zero.

Since the homotopy groups of E' are concentrated in even degrees, the Atiyah-Hirzebruch spectral sequence shows that $E'^{-1}(X) \simeq 0$ whenever X is a finite even spectrum. It will therefore suffice to prove the following:

($*$) The spectrum K is a retract of a direct sum of even finite spectra.

To prove ($*$), we will compare the cofiber sequence

$$K \to \bigoplus_{\alpha \in A} X_\alpha \to E$$

with another cofiber sequence of spectra. Let B be the collection of triples (α, α', f), where $\alpha, \alpha' \in A$ and f ranges over all homotopy classes of maps fitting into a commutative diagram

$$\begin{tikzcd} X_\alpha \arrow[rr, "f"] \arrow[dr] & & X_{\alpha'} \arrow[dl] \\ & E. & \end{tikzcd}$$

For each $\beta = (\alpha, \alpha', f) \in B$, we let $Y_\beta = X_\alpha$. We have a canonical map $\phi : \bigoplus_{\beta \in B} Y_\beta \to \bigoplus_{\alpha \in A} X_\alpha$, whose restriction to Y_β for $\beta = (\alpha, \alpha', f)$ given by the difference of the maps $Y_\beta = X_\alpha \to \bigoplus_{\alpha \in A} X_\alpha$ and

$$Y_\beta = X_\alpha \xrightarrow{f} X_{\alpha'} \to \bigoplus_{\alpha \in A} X_\alpha.$$

Let F be the cofiber of the map ϕ. By construction, we have a map of fiber sequences

$$\begin{tikzcd} \bigoplus_{\beta \in B} Y_\beta \arrow[r] \arrow[d] & \bigoplus_{\alpha \in A} X_\alpha \arrow[r, "u"] \arrow[d] & F \arrow[d] \\ K \arrow[r] & \bigoplus_{\alpha \in A} X_\alpha \arrow[r] & E. \end{tikzcd}$$

We now construct a map of spectra $q : E \to F$. By Corollary 2, it will suffice to define a map of homology theories $E_* \to F_*$. We will give a map $E_*(X) \to F_*(X)$ defined for every spectrum X. Since homology theories commute with filtered colimits, it will suffice to consider the case where X is a finite spectrum. Replacing X by its Spanier-Whitehead dual, we are reduced to the problem of producing a map $q(f) : X \to F$ for every map of spectra $f : X \to E$ for X finite.

Here is our construction. Since E is evenly generated, every map $f : X \to E$ factors through some map $X \xrightarrow{f'} X_{\alpha'} \to E$ for $\alpha' \in A$. We define $q(f)$ to be the composite map $X \xrightarrow{f'} X_{\alpha'} \to \bigoplus_{\alpha \in A} X_\alpha \to F$. We must show that this construction is well-defined; that is, it does not depend on the choice of f'. To this end, suppose we are given another factorization of f $X \xrightarrow{f''} X_{\alpha''} \to E$, where $\alpha'' \in A$. Let Y denote the pushout $X_{\alpha'} \coprod_X X_{\alpha''}$. Then Y is a finite spectrum, and our data gives a canonical map $Y \to E$. Since E is evenly generated, this map factors as a composition

$$Y \xrightarrow{g} X_\alpha \to E$$

for some $\alpha \in A$. Let h' denote the composite map $X_{\alpha'} \to X' \xrightarrow{g} X_\alpha$ and let h' be defined similarly. Then (α', α, h) and (α'', α, h) can be identified with elements of B. It follows that the composite maps

$$X \to X_{\alpha'} \to \bigoplus_{\alpha \in A} X_\alpha \to F$$

$$X \to X_{\alpha''} \to \bigoplus_{\alpha \in A} X_\alpha \to F$$

both coincide with the map

$$X \to Y \xrightarrow{g} X_\alpha \to \bigoplus_{\alpha \in A} X_\alpha \to F,$$

which proves that q is well-defined.

We now have a larger commutative diagram of fiber sequences

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & \bigoplus_{\alpha \in A} X_\alpha \arrow[r] \arrow[d] & E \arrow[d] \\
\bigoplus_{\beta \in B} Y_\beta \arrow[r] \arrow[d] & \bigoplus_{\alpha \in A} X_\alpha \arrow[r] \arrow[d] & F \arrow[d] \\
K \arrow[r] & \bigoplus_{\alpha \in A} \arrow[r] & E.
\end{tikzcd}$$

The right vertical composition induces the identity map on the underlying homology theory E_*: that is, it differs from id_E by a phantom map. In particular, it is an equivalence, so that the left vertical composition is an equivalence of K with itself. It follows that K is a retract of $\bigoplus_{\beta \in B} Y_\beta$, which proves $(*)$.

Even Periodic Cohomology Theories (Lecture 18)

April 27, 2010

Definition 1. Let R be a commutative ring and let $\mathcal{L}$ be an invertible R-module. An *$\mathcal{L}$-twisted formal group law* is a formal series

$$f(x,y) = \sum a_{i,j} x^i y^j$$

where $a_{i,j} \in \mathcal{L}^{\otimes(i+j-1)}$ which satisfies the identities

$$f(x,y) = f(y,x) \qquad f(x,0) = x \qquad f(x, f(y,z)) = f(f(x,y), z).$$

When $\mathcal{L} = R$, an $\mathcal{L}$-twisted formal group law is the same thing as a formal group law over R. Every $\mathcal{L}$-twisted formal group law $f(x,y)$ determines a formal group $\mathcal{G}_f$. More precisely, f defines a group structure on the functor $\operatorname{Spf} R[[\mathcal{L}]] = \operatorname{Spf}(\prod_n \mathcal{L}^{\otimes n})$ given by $A \mapsto \operatorname{Hom}_R(\mathcal{L}, \sqrt{A})$, where $\sqrt{A}$ denotes the ideal consisting of nilpotent elements of A. Note that the fiber of the map

$$(\operatorname{Spf} R[[\mathcal{L}]])(R[\epsilon/\epsilon^2]) \to (\operatorname{Spf} R[[\mathcal{L}]])(R)$$

is the collection of R-linear maps $\mathcal{L} \to \epsilon R/\epsilon^2 R$: that is, it is the R-module $\mathcal{L}^{-1}$. In other words, if f is any $\mathcal{L}$-twisted formal group law, there is a canonical isomorphism $\eta_f : \mathfrak{g}_{\mathcal{G}_f} \simeq \mathcal{L}^{-1}$, where $\mathfrak{g}_{\mathcal{G}_f}$ denotes the Lie algebra over $\mathcal{G}_f$. Conversely, we have the following:

Lemma 2. *Let R be a commutative ring and let $\mathcal{G}$ be a formal group over R with Lie algebra $\mathfrak{g}$. Then there exists a $\mathfrak{g}^{-1}$-twisted formal group law f and an isomorphism $\mathcal{G}_f \simeq \mathcal{G}$ lifting the isomorphism $\eta_f : \mathfrak{g}_{\mathcal{G}_f} \simeq \mathfrak{g}$.*

Proof. We first suppose that $\mathcal{G}$ is coordinatizable. In particular, we can choose an isomorphism $\alpha : \mathfrak{g} \simeq R$. We also have an isomorphism $\beta : \mathcal{G} \simeq \mathcal{G}_f$ for some formal group law $f(x,y) \in R[[x,y]]$. Replacing f by $\lambda^{-1} f(\lambda x, \lambda y)$ for some invertible constant λ, we can ensure that the composite map

$$R \overset{\alpha}{\simeq} \mathfrak{g} \overset{\beta}{\simeq} \mathcal{G}_{\mathfrak{f}} \overset{\eta_f}{\simeq} R$$

is the identity.

Let G denote the affine R-scheme which carries every R-algebra A to the group of power series of the form

$$g(t) = t + b_1 t^2 + b_2 t^3 + \cdots$$

where $b_n \in \mathcal{L}^{\otimes n}$, and let P be the affine R-scheme which carries every R-algebra A to the collection of all pairs (f, β), where f is an $(\mathcal{L} \otimes_R A)$-twisted formal group law and β is an isomorphism of $\mathcal{G}_f \simeq \mathcal{G}$ over $\operatorname{Spec} A$ which lifts the isomorphism η_f. There is an obvious action of G on P, and the above argument shows that P is a locally trivial G-torsor with respect to the Zariski topology. To prove the Lemma, we wish to show that $P(R)$ is trivial.

For each $n \geq 1$, we let G_n denote the subgroup scheme of G consisting of those power series such that $b_i = 0$ for $i \leq n$. Then $P \simeq \varprojlim P/G_n$, and $P/G_0 \simeq *$. To prove that $P(R)$ is nonempty, it will suffice to show that each of the maps $P/G_n(R) \to P/G_{n-1}(R)$ is surjective. The obstruction to surjectivity lies in the group

$$\mathrm{H}^1(\operatorname{Spec} R; G_n/G_{n-1}) \simeq \mathrm{H}^1(\operatorname{Spec} R; \mathcal{L}^{\otimes n})$$

. This group is trivial, since $\mathcal{L}^{\otimes n}$ is a quasi-coherent sheaf on $\operatorname{Spec} R$. $\square$

Remark 3. Let R be a commutative ring and let $\mathcal{L}$ be an invertible R-module. The data of an $\mathcal{L}$-twisted formal group law over R is equivalent to the data of a graded formal group law over the ring $\bigoplus_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n}$, where $\mathcal{L}^{\otimes n}$ has degree $2n$. That is, it is equivalent to giving a map of graded rings $L \to \bigoplus_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n}$.

Remark 4. Let f be an $\mathcal{L}$-twisted formal group law over a commutative ring R. The following conditions are equivalent:

(1) The associated formal group $\mathcal{G}_f$ is classified by a flat map $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$.

(2) The graded L-module $\bigoplus \mathcal{L}^{\otimes n}$ is Landweber-exact.

By Landweber's theorem, condition (2) is equivalent to the sum $\bigoplus \mathcal{L}^{\otimes n}$ being flat over $\mathcal{M}_{\mathrm{FG}}$. In particular, this implies that $\mathcal{L}^{\otimes 0} \simeq R$ is flat over $\mathcal{M}_{\mathrm{FG}}$, so that (2) $\Rightarrow$ (1). The converse follows from the observation that $\bigoplus \mathcal{L}^{\otimes n}$ is flat over R.

In the situation of Remark 4, we can apply Landweber's theorem to obtain a spectrum E_R, whose underlying homology theory is given by $(E_R)_*(X) = \mathrm{MU}_*(X) \otimes_L (\bigoplus_n \mathcal{L}^{\otimes n})$.

Example 5. Let R be the Lazard ring L and let $\mathcal{L} = R$ be trivial, so that $\bigoplus_n \mathcal{L}^{\otimes n}$ can be identified with $L[\beta^{\pm 1}]$. Then the above construction applies to produce a spectrum E_L whose homology theory is given by

$$(E_L)_*(X) = \mathrm{MU}_*(X) \otimes_L L[\beta^{\pm 1}] \simeq \mathrm{MU}_*(X)[\beta^{\pm 1}].$$

This spectrum is called the *periodic complex bordism spectra*, and will be denoted by MP. Just as MU can be realized as the Thom spectrum of the universal virtual complex bundle of rank 0 over BU, MP can be realized as the Thom spectrum of the universal virtual complex bundle of arbitrary rank over the space $BU \times \mathbf{Z}$. We have $\mathrm{MP}_0(X) = \mathrm{MU}_{\mathrm{even}}(X)$.

Now suppose more generally, we are given a $\mathcal{L}$-twisted formal group law f over a commutative ring R satisfying the conditions of Remark 4. If we choose an isomorphism $\mathcal{L} \simeq R$, then we can identify f with a formal group law classified by a map $L \to R$, and $\bigoplus_n \mathcal{L}^{\otimes n}$ with the ring $R[\beta^{\pm 1}]$. Then the homology theory E_R is given by

$$(E_R)_*(X) = \mathrm{MU}_*(X) \otimes_L R[\beta^{\pm 1}] \simeq \mathrm{MP}_*(X) \otimes_L R.$$

In particular, we have $(E_R)_0(X) = \mathrm{MP}_0(X) \otimes_L R = \mathrm{MU}_{\mathrm{even}}(X) \otimes_L R$.

The above calculation can be expressed in a more invariant way. Recall that to any spectrum X we can associate a quasi-coherent sheaf $\mathcal{F}_X$ on $\mathcal{M}_{\mathrm{FG}}$, whose restriction to $\operatorname{Spec} L$ is given by $\mathrm{MU}_{\mathrm{even}} X$. Then $(\mathrm{MU}_{\mathrm{even}} X) \otimes_L R$ is the pullback of $\mathcal{F}_X$ along the map $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$. From this description, it is clear that the homology theory $(E_R)_*$ depends only on the formal group $\mathcal{G}_f$ (or equivalently, the map q), and not on the particular choice of formal group law f. This calculation globalizes as follows:

Proposition 6. *Let $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$ be a flat map. Then there exists a spectrum E_R which is determined up to canonical isomorphism (in the homotopy category of spectra) by its underlying homology theory, which is given by $(E_R)_0(X) = q^* \mathcal{F}_X$ (so that, more generally, $(E_R)_n(X) = (E_R)_0(\Sigma^{-n} X) = q^* \mathcal{F}_{\Sigma^{-n} X}$).*

Remark 7. Suppose we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R' \arrow[rr] \arrow[dr, "q'"'] & & \operatorname{Spec} R \arrow[dl, "q"] \\ & \mathcal{M}_{\mathrm{FG}} & \end{tikzcd}$$

where q and q' are flat: that is, we have a Landweber-exact formal group over R whose restriction along a map of commutative rings $R \to R'$ is also Landweber-exact. Then we get an evident map $E_R \to E_{R'}$ (which is unique up to homotopy, by the results of the previous lecture).

Proposition 8. *Suppose we are given flat maps* $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$ *and* $q' : \operatorname{Spec} R' \to \mathcal{M}_{\mathrm{FG}}$*. Then the smash product* $E_R \otimes E_{R'}$ *is homotopy equivalent to* E_B*, where* B *fits into a pullback diagram*

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ \operatorname{Spec} R' \arrow[r] & \mathcal{M}_{\mathrm{FG}} . \end{tikzcd}$$

Proof. It is clear that $\operatorname{Spec} B$ is flat over $\mathcal{M}_{\mathrm{FG}}$. For simplicity, we will suppose that q and q' classify formal groups which admit coordinates, given by maps $L \to R$ and $L \to R'$. Note that

$$\mathrm{MP}_0(\mathrm{MP}) \simeq \mathrm{MU}_{\mathrm{even}}(\mathrm{MP}) \simeq \mathrm{MU}_*(\mathrm{MU})[b_0^{\pm 1}] \simeq \mathrm{MU}_*[b_0^{\pm 1}, b_1, \ldots].$$

Using this calculation, one sees that the diagram

$$\begin{tikzcd} \operatorname{Spec} \mathrm{MP}_0 \mathrm{MU} \arrow[r] \arrow[d] & \operatorname{Spec} L \arrow[d] \\ \operatorname{Spec} L \arrow[r] & \mathcal{M}_{\mathrm{FG}} \end{tikzcd}$$

is a pullback square, so that $B \simeq R \otimes_L \mathrm{MP}_0 \mathrm{MP} \otimes_L R'$.

Now let X be any spectrum. We have

$$\begin{aligned} (E_R \otimes E_{R'})_0(X) &\simeq (E_R)_0(E_{R'} \otimes X) && (1)\\ &\simeq R \otimes_L \mathrm{MP}_0(E_{R'} \otimes X) && (2)\\ &\simeq R \otimes_L (E_{R'})_0(\mathrm{MP} \otimes X) && (3)\\ &\simeq R \otimes_L \mathrm{MP}_0(\mathrm{MP} \otimes X) \otimes_L R' && (4)\\ &\simeq R \otimes_L (\mathrm{MP} \otimes \mathrm{MP})_0 X \otimes_L R'. && (5) \end{aligned}$$

where $(\mathrm{MP} \otimes \mathrm{MP})_0 X$ is the pullback of $\mathcal{F}_X$ to $\operatorname{Spec} \mathrm{MP}_0 \mathrm{MP} \simeq \operatorname{Spec} L \times_{\mathcal{M}_{\mathrm{FG}}} \operatorname{Spec} L$. It follows that $(E_R \otimes E_{R'})_0 X$ is the pullback of $\mathcal{F}_X$ to $\operatorname{Spec} B$, thus giving a canonical homotopy equivalence $E_R \otimes E_{R'} \simeq E_B$. □

Corollary 9. *For any flat map* $\operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$*, there is a canonical multiplication* $E_R \otimes E_R \to E_R$*, making* E_R *into a commutative and associative algebra in the homotopy category of spectra.*

Proof. Form a pullback diagram

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \mathcal{M}_{\mathrm{FG}} . \end{tikzcd}$$

There is an evident diagonal map $\operatorname{Spec} R \to \operatorname{Spec} B$. By Remark 7, this induces a map

$$E_R \otimes E_R \simeq E_B \to E_R.$$

The commutativity and associativity properties of this construction are evident. □

Let $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$ be a flat map classifying a formal group with Lie algebra $\mathfrak{g}$, and let E_R the associated ring spectrum. By construction, we have

$$\pi_n E_R \simeq \begin{cases} \mathfrak{g}^k & \text{if } n = -2k \\ 0 & \text{if } n = -2k+1. \end{cases}$$

Let us now axiomatize this structural phenomenon:

Definition 10. Let E be a ring spectrum. We will say that E is *even periodic* if the following conditions are satisfied:

(1) The homotopy groups $\pi_i E$ vanish when i is odd.

(2) The map $\pi_2 E \otimes_{\pi_0 E} \pi_{-2} E \to \pi_0 E$ is an isomorphism (so that, in particular, $\pi_2 E$ is an invertible E-module $\mathcal{L}$, and we have $\pi_{2n} E \simeq \mathcal{L}^{\otimes n}$ for all n).

If E is an even periodic ring spectrum, then E is automatically complex-orientable, so we obtain a formal group $\mathcal{G}$ over $\pi_* E$. However, in the periodic case we can do better: since $E^*(\mathbf{CP}^\infty) \simeq E^0(\mathbf{CP}^\infty) \otimes_{\pi_0 E} \pi_* E$, we get a formal group $\operatorname{Spf} E^0(\mathbf{CP}^\infty)$ over the commutative ring $R = \pi_0 E$, whose restriction to $\pi_* E$ is the formal group we have been discussing earlier in this course. This formal group is classified by a map $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$.

We can summarize the situation as follows:

Proposition 11. *Let $\mathcal{C}$ be the category of pairs (R, η), where R is a commutative ring and $\eta : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$ is a flat map (that is, η corresponds to a Landweber-exact formal group over* $\operatorname{Spec} R$*). Then the construction $R \mapsto E_R$ determines a fully faithful embedding Φ of $\mathcal{C}$ into the category of commutative algebras in the homotopy category of spectra. A ring spectrum E belongs to the essential image of this embedding if and only if E is even periodic, and the induced map $\pi_0 E \to \mathcal{M}_{\mathrm{FG}}$ is flat.*

To prove Proposition 11, we note that the construction $E \mapsto (\pi_0 E, \operatorname{Spf} E^0(\mathbf{CP}^\infty))$ provides a left inverse to Φ. What is not entirely clear is that this construction is also right-inverse to Φ: that is, if E is an even periodic ring spectrum which determines a map $q : \operatorname{Spec} \pi_0 E = \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$, can we identify E with the ring spectrum E_R? Choose a complex orientation on E, given by a map of ring spectra $\mathrm{MU} \to E$ which induces a map of graded rings $\phi : L \to \pi_* E$. Then the homology theory E_R is given by

$$(E_R)_*(X) = \mathrm{MU}_*(X) \otimes_L (\pi_* E).$$

We get an evident map of homology theories $(E_R)_*(X) \to E_*(X)$. This map is an isomorphism by construction when X is a point. Since E is even and E_R is Landweber exact, the results of the previous lecture show that we get a map of spectra $E_R \to E$ which is well-defined up to homotopy equivalence. This map induces an isomorphism $\pi_* E_R \to \pi_* E$ by construction, and is therefore an equivalence of spectra; it is easy to see that this equivalence is compatible with the ring structures on E_R and E.

Morava Stabilizer Groups (Lecture 19)

March 24, 2010

Fix a prime number p and an integer $0 < n < \infty$. Our goal in this lecture is to understand the structure of the moduli stack $\mathcal{M}_{\mathrm{FG}}^n$, whose R-points are formal groups of height exactly n over R.

Let $\overline{\mathbb{F}}_{\mathrm{p}}$ denote the algebraic closure of the field $\mathbf{F}_p$. We have seen that there exists a formal group law $f(x, y) \in \overline{\mathbb{F}}_{\mathrm{p}}[[x, y]]$ of height n, which is unique up to isomorphism. The map $\operatorname{Spec} \overline{\mathbb{F}}_{\mathrm{p}} \to \mathcal{M}_{\mathrm{FG}}^n$ is faithfully flat: for any commutative ring R and any formal group law $f'(x, y)$ over R of height exactly n, we have a pullback diagram

$$\begin{tikzcd} \operatorname{Spec} R' \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ \operatorname{Spec} \overline{\mathbb{F}}_{\mathrm{p}} \arrow[r] & \mathcal{M}_{\mathrm{FG}}^n \end{tikzcd}$$

where R' is a direct limit of finite etale extensions of $R \otimes \overline{\mathbb{F}}_{\mathrm{p}}$ (and therefore faithfully flat over R). Consequently, we can regard $\overline{\mathbb{F}}_{\mathrm{p}}$ as an atlas for $\mathcal{M}_{\mathrm{FG}}^n$. To understand $\mathcal{M}_{\mathrm{FG}}^n$, we form a pullback diagram

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] & \operatorname{Spec} \overline{\mathbb{F}}_{\mathrm{p}} \arrow[d] \\ \operatorname{Spec} \overline{\mathbb{F}}_{\mathrm{p}} \arrow[r] & \mathcal{M}_{\mathrm{FG}}^n . \end{tikzcd}$$

The ring $\operatorname{Spec} B$ is a direct limit of finite etale extensions of $\overline{\mathbb{F}}_{\mathrm{p}}$. Since $\overline{\mathbb{F}}_{\mathrm{p}}$ is an algebraically closed field, each of these etale extensions is just a product of finitely many copies of $\overline{\mathbb{F}}_{\mathrm{p}}$. Consequently, we can identify $\operatorname{Spec} B$ (as a topological space) with an inverse limit of a tower of finite sets

$$\cdots \to X_2 \to X_1 \to X_0.$$

We will denote this inverse limit by $\mathbb{G}$. Unwinding the definitions, a point of $\mathbb{G}$ is given by an isomorphism class of maps $B \to k$, where k is an algebraic closure of $\mathbf{F}_p$ (noncanonically isomorphic to $\overline{\mathbb{F}}_{\mathrm{p}}$. To give such a map is equivalent to giving the following data:

(1) A pair of maps $\eta, \eta' : \overline{\mathbb{F}}_{\mathrm{p}} \to k$.

(2) An isomorphism between the formal groups $\eta(f)$ and $\eta'(f)$ over k.

Since we are interested in classifying such data up to isomorphism, we may as well assume that $k = \overline{\mathbb{F}}_{\mathrm{p}}$ and η' is the identity. Then η is an an automorphism of $\overline{\mathbb{F}}_{\mathrm{p}}$: that is, we can think of η as an element of the Galois group $\operatorname{Gal}(\overline{\mathbb{F}}_{\mathrm{p}} / \mathbf{F}_p) \simeq \widehat{\mathbf{Z}}$. The data of (2) is then an isomorphism of f with $\eta(f)$, where $\eta(f)$ denotes the formal group law obtained by applying η to each coefficient in f. In other words, we can identify $\mathbb{G}$ with the automorphism group $\operatorname{Aut}(\overline{\mathbb{F}}_{\mathrm{p}}, f)$ of the pair $\overline{\mathbb{F}}_{\mathrm{p}}$, $f \in \mathrm{FGL}(\overline{\mathbb{F}}_{\mathrm{p}})$. This group sits in an exact sequence

$$0 \to \operatorname{Aut}(f) \to \operatorname{Aut}(\overline{\mathbb{F}}_{\mathrm{p}}, f) \to \operatorname{Gal}(\overline{\mathbb{F}}_{\mathrm{p}}, \mathbf{F}_p) \to 0,$$

where $\operatorname{Aut}(f)$ is the automorphism group of the formal group law f (keeping the field $\overline{\mathbb{F}}_{\mathrm{p}}$ fixed). The group $\mathbb{G} = \operatorname{Aut}(\overline{\mathbb{F}}_{\mathrm{p}}, f)$ is called the *Morava stabilizer group*. We arrive at the following conclusion:

Proposition 1. *The moduli stack $\mathcal{M}_{\mathrm{FG}}^n$ can be identified with the quotient (with respect to the flat topology)* $(\operatorname{Spec} \overline{\mathbb{F}}_{\mathrm{p}}) / \operatorname{Aut}(\overline{\mathbb{F}}_{\mathrm{p}}, f)$*, where* $\operatorname{Aut}(\overline{\mathbb{F}}_{\mathrm{p}}, f)$ *acts via the map* $\operatorname{Aut}(\overline{\mathbb{F}}_{\mathrm{p}}, f) \to \operatorname{Gal}(\overline{\mathbb{F}}_{\mathrm{p}} / \mathbf{F}_p)$.

To understand the stack $\mathcal{M}_{\mathrm{FG}}^n$ better, we need to understand the group $\operatorname{Aut}(\overline{\mathbb{F}}_{\mathrm{p}}, f)$. We begin by analyzing the subgroup $\operatorname{Aut}(f)$. By definition, $\operatorname{Aut}(f)$ can be identified with the group of units in the ring $\operatorname{End}(f)$ of endomorphisms of f: that is, elements of $\operatorname{End}(f)$ are power series $g(t) \in \overline{\mathbb{F}}_{\mathrm{p}}[[t]]$ such that $gf(x, y) = f(g(x), g(y))$.

Let f^p denote the formal group law over $\overline{\mathbb{F}}_{\mathrm{p}}$ obtained by applying the Frobenius map $a \mapsto a^p$ to each coefficient of f. Then f^p is another formal group law of height n over $\overline{\mathbb{F}}_{\mathrm{p}}$, so there exists a *noncanonical* isomorphism ν of f with f^p: that is, a power series ν satisfying $\nu f^p(x, y) = f(\nu(x), \nu(y))$. Note that $f(x, y)^p \simeq f^p(x^p, y^p)$, so that

$$\nu f(x, y)^p = \nu f^p(x^p, y^p) = f(\nu(x^p), \nu(y^p)).$$

Consequently, we deduce that the power series $\pi(t) = \nu(t^p)$ is an endomorphism of f, and belongs to the ring $\operatorname{End}(f)$.

Let $g \in \operatorname{End}(f)$ be arbitrary, and write $g(t) = b_0 t + b_1 t^2 + \dots$ If $b_0 \neq 0$, then g is invertible and belongs to $\operatorname{Aut}(f)$. Otherwise, we have seen that $g(t) = g_0(t^p)$ for some uniquely defined power series g_0, and that g_0 is an endomorphism of the formal group law f^p. Then $g_0 \circ \nu^{-1}$ is an endomorphism of f, and we have $g = g_0 \circ \nu^{-1} \circ \nu \circ (t \mapsto t^p) = (g_0 \circ \nu^{-1})\pi$. In other words:

Proposition 2. *Every non-invertible element g of the ring* $\operatorname{End}(f)$ *can be written uniquely in the form $g'\pi$, where $\pi(t) = \nu(t^p)$ is the endomorphism defined above. In particular,* $\operatorname{End}(f)$ *is a (noncommutative) local ring: the collection of non-invertible elements of* $\operatorname{End}(f)$ *is a two-sided ideal, which is the left ideal generated by π.*

More generally, we saw in lecture 12 that every nonzero endomorphism g of f can be written uniquely in the form $u\pi^k$ for some $k \geq 0$; here k is the smallest integer for which the coefficient of t^{p^k} in $g(t)$ is nonzero. We will refer to k as the *valuation* of g and write $k = v(g)$. By convention we set $v(0) = \infty$. Note that $v(gg') = v(g) + v(g')$. In particular, $v(p) = n$ where n is the height of f (this is the definition of height).

Remark 3. There is an evident ring homomorphism $\lambda : \operatorname{End}(f) \mapsto \overline{\mathbb{F}}_{\mathrm{p}}$ given by differentiation: more precisely, λ carries $g(t) = b_0 t + b_1 t^2 + \dots$ to the element $b_0 \in \overline{\mathbb{F}}_{\mathrm{p}}$. The kernel of λ is the collection of noninvertible power series: that is, the ideal $\operatorname{End}(f)\pi$. Since the p-series for f is given by $[p](t) = \mu t^{p^n} + \cdots$ for some μ, any endomorphism g of f satisfies $g([p](t)) = [p](g(t))$, so that

$$b_0 \mu t^{p^n} + \cdots = b_0^{p^n} \mu t^{p^n} + \cdots .$$

It follows that the image of λ is contained in the subfield $\mathbf{F}_{p^n} \subseteq \overline{\mathbb{F}}_{\mathrm{p}}$. Conversely, in Lecture 14 we showed that any solution to the equation $b_0 = b_0^{p^n}$ can be extended to an automorphism of f: that is, the map $\lambda : \operatorname{End}(f) \to \mathbf{F}_{p^n}$ is surjective.

Remark 4. Since an endomorphism $g(t)$ of f is determined knowing all of its reductions modulo t^{p^k}, we deduce that $\operatorname{End}(f) \simeq \varprojlim (\operatorname{End}(f) / \operatorname{End}(f)\pi^k)$. Each of the quotients $\operatorname{End}(f) / \operatorname{End}(f)\pi^k$ has finite cardinality p^{nk}, so this inverse limit exhibits $\operatorname{End}(f)$ as a profinite set. The induced topology on the closed subset $\operatorname{Aut}(f)$ agrees with Zariski topology on $\operatorname{Spec} B = \operatorname{Aut}(\overline{\mathbb{F}}_{\mathrm{p}}, k)$.

We have a canonical map

$$\mathbf{Z}_p \simeq \varprojlim \mathbf{Z}/p^k\mathbf{Z} \to \varprojlim \operatorname{End}(f) / \operatorname{End}(f)\pi^k \simeq \operatorname{End}(f)$$

whose image is central in $\operatorname{End}(f)$.

In other words, we can think of $\operatorname{End}(f)$ as a noncommutative discrete valuation ring, having commutative residue field $\mathbf{F}_{p^n}$. Let $D = \operatorname{End}(f)[p^{-1}]$. Since $p = u\pi^n$ for some invertible constant u, π is invertible in D, so that D is a division algebra over $\mathbf{Z}_p[p^{-1}] \simeq \mathbf{Q}_p$. The valuation v extends to D formally by the formula $v(\frac{\lambda}{p^k}) = v(\lambda) - nk$.

Note that p is not a zero-divisor in $\operatorname{End}(f)$, so that $\operatorname{End}(f)$ can be identified with a subset of D.

Lemma 5. *We have* $\mathrm{End}(f) = \{x \in D : v(x) \geq 0\}$.

Proof. It is clear that $v(x) \geq 0$ if $x \in \mathrm{End}(f)$. Conversely, suppose that $x = \frac{\lambda}{p^k}$ for some $\lambda \in \mathrm{End}(f)$. If $v(x) \geq 0$, then $v(\lambda) \geq nk$ so that $\lambda = \lambda' \pi^{nk}$. It will therefore suffice to show that $\frac{\pi^{nk}}{p^k} \in \mathrm{End}(f)$. Since $\mathrm{End}(f)$ is closed under products, it suffices to show that $\frac{\pi^n}{p} \in \mathrm{End}(f)$. This is clear, since $v(p) = n$ implies that $p = u\pi^n$ for some invertible $u \in \mathrm{End}(f)$. $\square$

Lemma 6. *As a vector space over* $\mathbf{Q}_p$, D *has dimension* n^2.

Proof. Let $\{\overline{x}_i\}_{0 \leq i < n}$ be a basis for $\mathbf{F}_{p^n}$ over $\mathbf{F}_p$. Choose elements $x_i \in \mathrm{End}(f)$ with $\lambda(x_i) = \overline{x_i}$. Then the elements $\{\pi^j x_i\}_{0 \leq i,j < n}$ form a basis for D over $\mathbf{Q}_p$. $\square$

To identify D further, we note that conjugation by any $g \in D^\times$ is an automorphism of D which preserves $\mathrm{End}(f) \subseteq D$ and therefore acts on the quotient $\mathrm{End}(f)/\pi$.

Lemma 7. *Let* $g \in D$. *The conjugation action of* g *on* $\mathrm{End}(f)/\pi \simeq \mathbf{F}_{p^n}$ *is given by* $b \mapsto b^{p^{v(g)}}$.

Proof. Without loss of generality we may assume that $g \in \mathrm{End}(f)$, so that $g(t) = \lambda t^{p^{v(g)}}$ for some $\lambda \neq 0$. Fix $b \in \mathbf{F}_{p^n}$, and let $h \in \mathrm{End}(f)$ be a power series given by $h(t) = b_0 t + \cdots$. Let $h'(t) = (g \circ h \circ g^{-1})(t) = b't + \cdots \in \mathrm{End}(f)$. The equation $g \circ h = h' \circ g$ gives

$$\lambda b^{p^{v(g)}} t^{p^{v(g)}} + \cdots = b' \lambda t^{p^{v(g)}} + \cdots$$

so that $b' = b^{p^{v(g)}}$. $\square$

Lemma 8. *The center of* D *is* $\mathbf{Q}_p$.

Proof. Let g be in the center of D; we wish to prove that $g \in \mathbf{Q}_p$. Multiplying by a power of p if necessary, we may assume that $g \in \mathrm{End}(f)$; we wish to prove that $g \in \mathbf{Z}_p$. Since $\mathbf{Z}_p$ is closed in $\mathrm{End}(f)$, it will suffice to show that there exists an integer m such that $g \equiv m \mod p^k$ for all k. We work by induction on k. Since $\pi g \pi^{-1} = g$, Lemma 7 implies that the reduction of g moduli π belongs to $\mathbf{F}_p \subseteq \mathbf{F}_{p^n}$. Subtracting an integer from g, we may suppose that $v(g) > 0$. Lemma 7 implies that $v(g)$ is divisible by n, so that $v(g) \geq n$ and therefore $g = g'p$ for some g' belonging to the center of $\mathrm{End}(f)$. Then g' is congruent to an integer modulo p^{k-1} by the inductive hypothesis, so that g is congruent to an integer modulo p^k. $\square$

Remark 9. It follows from the above analysis that the division algebra D can be identified with an element of the Brauer group $\mathrm{Br}(\mathbf{Q}_p)$. There is a canonical isomorphism $\mu : \mathrm{Br}(\mathbf{Q}_p) \simeq \mathbf{Q}/\mathbf{Z}$, which is defined as follows. Every Brauer class over $\mathbf{Q}_p$ is represented by a central division algebra D' over $\mathbf{Q}_p$, which contains a ring of integers $\mathcal{O}$ and maximal ideal $\mathfrak{m}$. There is a valuation $v : D' - \{0\} \to \mathbf{Z}$ with $\mathcal{O} = v^{-1}\mathbf{Z}_{\geq 0}$ and $\mathfrak{m} = v^{-1}\mathbf{Z}_{\geq 1}$. Conjugation induces a surjective homomorphism $D' - \{0\} \to \mathrm{Gal}((\mathcal{O}/\mathfrak{m})/\mathbf{F}_p)$. In particular, the Frobenius map $x \mapsto x^p$ on the residue field $\mathcal{O}/\mathfrak{m}$ is given by conjugation by x, for some $x \in D'$. Then $\mu(D') = \frac{v(x)}{v(p)}$ (modulo $\mathbf{Z}$, this invariant does not depend on the choice of x).

In the case $D = D'$, we can take $x = \pi$, so that D is the unique central division algebra over $\mathbf{Q}_p$ with $\mu(D) = \frac{1}{n}$.

By construction, there is a canonical isomorphism $\mathrm{End}(f)^\times \simeq \mathrm{Aut}(f)$. In fact, we can extend this to a map $\chi : D^\times \to \mathrm{Aut}(\overline{\mathbb{F}}_p, f)$. Here χ is defined on $\mathrm{End}(f) - \{0\}$ by carrying a nonzero endomorphism $g(t)$ of f to the pair $(F^{v(g)}, g_0)$, where $F^{v(g)}$ is a power of the Frobenius automorphism $x \mapsto x^p$ of $\overline{\mathbb{F}}_p$, and g_0 is the isomorphism of f with $f^{p^{v(g)}}$ characterized by the formula $g(t) = g_0(t^{p^{v(g)}})$.

We have a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{End}(f)^\times \arrow[r] \arrow[d] & D^\times \arrow[r, "v"] \arrow[d] & \mathbf{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Aut}(f) \arrow[r] & \mathrm{Aut}(\overline{\mathbb{F}}_p, f) \arrow[r] & \mathrm{Gal}(\overline{\mathbb{F}}_p, \mathbf{F}_p) \arrow[r] & 0.
\end{tikzcd}$$

The left vertical map is an isomorphism, and the right vertical map is *almost* an isomorphism (the group $\mathrm{Gal}(\overline{\mathbb{F}}_{\mathrm{p}}, \mathbf{F}_p)$ is the profinite completion $\widehat{\mathbf{Z}}$ of the $\mathbf{Z}$). Consequently, the Morava stabilizer group is *almost* the group of units in the division algebra $D^\times$ (they differ by a completion procedure).

We can use the above picture to study the problem of *descending* the formal group law defined by f to a finite field $\mathbf{F}_{p^k} \subseteq \overline{\mathbb{F}}_{\mathrm{p}}$. By descent theory, this is equivalent to giving an action of $\mathrm{Gal}(\overline{\mathbb{F}}_{\mathrm{p}} / \mathbf{F}_{p^k}) \simeq k\widehat{\mathbf{Z}}$ on the formal group, compatible with the action of $k\widehat{\mathbf{Z}}$ on $\overline{\mathbb{F}}_{\mathrm{p}}$ itself. In other words, we need to give a *splitting* of the projection map $\mathrm{Aut}(\overline{\mathbb{F}}_{\mathrm{p}}, f) \to \mathrm{Gal}(\overline{\mathbb{F}}_{\mathrm{p}} / \mathbf{F}_p)$ over the subgroup $k\widehat{\mathbf{Z}} \subseteq \mathrm{Gal}(\overline{\mathbb{F}}_{\mathrm{p}} / \mathbf{F}_p)$. Since $k\widehat{\mathbf{Z}}$ is topologically cyclic, this is equivalent to giving a single element of $\mathrm{Aut}(\overline{\mathbb{F}}_{\mathrm{p}}, f)$ lying over the integer k: that is, giving an element of $x \in D^\times$ with $v(x) = k$.

Such an element exists for every integer $k \geq 1$. However, when $k = 1$ there is a canonical choice $x = p$, which belongs to the center of D. Unwinding the definitions, this proves the following:

Proposition 10. *The formal group of height n over $\overline{\mathbb{F}}_{\mathrm{p}}$ has a canonical form over the finite field $\mathbf{F}_{p^n}$. This formal group over $\mathbf{F}_{p^n}$ has the property that every endomorphism (and, in particular, every automorphism) is defined over $\mathbf{F}_{p^n}$.*

It follows that the moduli stack $\mathcal{M}_{\mathrm{FG}}^n$ can also be identified with the quotient $\mathrm{Spec}\, \mathbf{F}_{p^n} / \mathbb{G}'$, where $\mathbb{G}' \simeq D^\times / p^{\mathbf{Z}}$ fits into an exact sequence

$$0 \to \mathrm{End}(f)^\times \to \mathbb{G}' \to \mathbf{Z}/n\mathbf{Z} \to 0.$$

The group $\mathbb{G}'$ is also sometimes called the Morava stabilizer group.

Bousfield Localization (Lecture 20)

April 27, 2010

Let $\mathcal{C}$ be a full subcategory of the category Sp of spectra, which is closed under shifts and homotopy colimits and satisfies the following technical condition:

$(*)$ There exists a small subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ which generates $\mathcal{C}$ under homotopy colimits.

In this case, the inclusion $\mathcal{C} \subseteq \mathrm{Sp}$ preserves homotopy colimits; using a version of the adjoint functor theorem one deduces that this inclusion admits a right adjoint G (at the level of homotopy categories). We can think of G as a functor from Sp to itself, which takes values in $\mathcal{C}$.

Remark 1. Roughly speaking, if X is a spectrum then we want to define $G(X)$ to be the homotopy colimit of all objects $Y \in \mathcal{C}$ with a map to X. Condition $(*)$ is used to make this homotopy colimit sensible (that is, to replace it by a homotopy colimit indexed by a small category).

For every spectrum X, we have a counit map $v : G(X) \to X$. We let $L(X)$ denote the cofiber of v, so that we have a cofiber sequence

$$G(X) \to X \to L(X).$$

By construction, for every object $Y \in \mathcal{C}$, the map of function spectra $G(X)^Y \to X^Y$ is a homotopy equivalence; it follows that $L(X)^Y \simeq 0$.

Definition 2. A spectrum X is $\mathcal{C}$-*local* if every map $Y \to X$ is nullhomotopic when $Y \in \mathcal{C}$. We denote the category of $\mathcal{C}$-local spectra by $\mathcal{C}^{\perp}$.

Remark 3. The full subcategory $\mathcal{C}^{\perp} \subseteq \mathrm{Sp}$ is stable under shifts and homotopy limits.

The above analysis shows that for every X, the spectrum $L(X)$ is $\mathcal{C}$-local. Moreover, for every $\mathcal{C}$-local spectrum Z, we have $Z^{G(X)} \simeq 0$, so that the map $Z^{L(X)} \to Z^X$ is a homotopy equivalence. It follows that L can be viewed as a left adjoint to the inclusion $\mathcal{C}^{\perp} \subseteq \mathrm{Sp}$.

Example 4 (Bousfield)**.** Fix a spectrum E. We say that another spectrum X is E-*acyclic* if the smash product $X \otimes E$ is zero. The collection $\mathcal{C}_E$ of E-acyclic spectra is clearly stable under shifts and homotopy colimits, and one can show that it satisfies $(*)$. We say that a spectrum X is E-*local* if every map $Y \to X$ is nullhomotopic whenever Y is E-acyclic. The above analysis shows that every spectrum X sits in an essentially unique cofiber sequence

$$G_E(X) \to X \to L_E(X)$$

where $G(X)$ is E-acyclic and $L_E(X)$ is E-local. The functor L_E is called *Bousfield localization* with respect to E. The map $X \to L_E(X)$ is characterized up to equivalence by two properties:

(a) The spectrum $L_E(X)$ is E-local.

(b) The map $X \to L_E(X)$ is an E-equivalence: that is, it induces an isomorphism on E-homology groups $E_*(X) \simeq E_* L_E(X)$.

Example 5. Let E be a ring spectrum. If X is an E-module spectrum, then X is E-local. Indeed, suppose that Y is E-acyclic and we are given a map $f : Y \to X$. Then f can be written as the composition

$$Y \xrightarrow{f} X \to E \otimes X \to X.$$

The composition of the first pair of morphisms factors as a composition

$$Y \to E \otimes Y \xrightarrow{\mathrm{id} \otimes f} E \otimes X,$$

and is therefore nullhomotopic since $E \otimes Y \simeq 0$.

Remark 6. Let E be an A_∞-ring spectrum and X an arbitrary spectrum, and let $X^\bullet$ be the cosimplicial spectrum given by $X^n = E^{\otimes n+1} \otimes X$. Each X^n is E-local, so the totalization $\varprojlim X^\bullet$ of $X^\bullet$ is E-local. It follows that the canonical map $X \to \varprojlim X^\bullet$ factors through a map $\alpha : L_E X \to \varprojlim X^\bullet$. In many cases, one can show that α is a homotopy equivalence: that is, the cosimplicial object $X^\bullet$ is a means of computing the E-localization of X.

Example 7. Let E be the Eilenberg-MacLane spectrum $H\,\mathbf{Q}$. Then a spectrum X is E-acyclic if and only if the homotopy groups $\pi_* X$ consist entirely of torsion. A spectrum X is E-local if and only if the homotopy groups $\pi_* X$ are rational vector spaces.

Example 8. The theory of Bousfield localization works in a very general context. For example, rather than working with spectra, we can work with chain complexes of abelian groups. Fix a prime number p. We say that a projective chain complex $A_\bullet$ is $\mathbf{Z}/p\mathbf{Z}$-acyclic if $A_\bullet \otimes \mathbf{Z}/p\mathbf{Z}$ is nullhomotopic: equivalently, $A_\bullet$ is $\mathbf{Z}/p\mathbf{Z}$-acyclic if each homology group $\mathrm{H}_n(A_\bullet)$ is a $\mathbf{Z}[\frac{1}{p}]$-module. We say that $A_\bullet$ is $\mathbf{Z}/p\mathbf{Z}$-local if every map from a projective $\mathbf{Z}/p\mathbf{Z}$-acyclic chain complex into $A_\bullet$ is nullhomotopic.

For any projective chain complex $A_\bullet$, we define its completion $\widehat{A}_\bullet$ to be the homotopy limit

$$\varprojlim_n A_\bullet \otimes \mathbf{Z}/p^n\mathbf{Z}.$$

As a homotopy limit of $\mathbf{Z}/p\mathbf{Z}$-local chain complexes, we conclude that $\widehat{A}_\bullet$ is $\mathbf{Z}/p\mathbf{Z}$-local. On the other hand, a simple calculation shows that the map $A_\bullet \to \widehat{A}_\bullet$ induces a quasi-isomorphism modulo p, so that $\widehat{A}_\bullet$ can be identified with the $\mathbf{Z}/p\mathbf{Z}$-localization of $A_\bullet$.

In general, it is good to think of Bousfield localization as involving a mix of Examples 7 and 8. In algebro-geometric terms, it can behave sometimes like restriction to an open subscheme (as in Example 7) and sometimes like completion along a closed subscheme (Example 8). Our next goal is to describe Bousfield localizations of the first type more precisely.

Lemma 9. *Let $\mathcal{C}$, $\mathcal{C}^\perp$, G, and L be as above. The following conditions are equivalent:*

(1) *The subcategory $\mathcal{C}^\perp \subseteq \mathrm{Sp}$ is stable under homotopy colimits.*

(2) *The functor L preserves homotopy colimits.*

(3) *The functor G preserves homotopy colimits.*

(4) *The functor L has the form $L(X) = K \otimes X$ for some spectrum K.*

Proof. We first prove (1) $\Rightarrow$ (2). Assume $\mathcal{C}^\perp$ is stable under homotopy colimits. For any diagram of spectra $\{X_\alpha\}$, we have canonical maps

$$\varinjlim X_\alpha \xrightarrow{\gamma} \varinjlim L(X_\alpha) \xrightarrow{\beta} L \varinjlim X_\alpha.$$

The fiber of γ belongs to $\mathcal{C}$ (since $\mathcal{C}$ is stable under homotopy colimits), and $\varinjlim L(X_\alpha) \in \mathcal{C}^\perp$ by (1). It follows that β is an equivalence.

To prove that (2) $\Rightarrow$ (1), we note that if $\{X_\alpha\}$ is a diagram in $\mathcal{C}^\perp$, then $L(\varinjlim X_\alpha) \simeq \varinjlim L(X_\alpha) \simeq \varinjlim X_\alpha$ so that $\varinjlim X_\alpha \in \mathcal{C}^\perp$.

The equivalence of (2) and (3) follows from the cofiber sequence of functors

$$G \to \mathrm{id} \to L.$$

Finally, the equivalence of (2) and (4) follows from from the following observation: every functor $F : \mathrm{Sp} \to \mathrm{Sp}$ which preserves homotopy colimits has the form $F(X) \simeq K \otimes X$, for some spectrum K. $\square$

We say that a Bousfield localization L is *smashing* if it satisfies the equivalent conditions of Lemma 9.

Remark 10. In the situation of Lemma 9, the spectrum K can be recovered as the image $L(S)$ of the sphere spectrum S under the localization functor L.

Remark 11. Let $\mathcal{C} \subseteq \mathrm{Sp}$ be a subcategory satisfying the conditions of Lemma 9. Then a spectrum X belongs to $\mathcal{C}$ if and only if $L(X) = L(S) \otimes X \simeq 0$. In other words, $\mathcal{C}$ can be identified with the collection of $L(S)$-acyclic spectra, so that $L = L_E$ for $E = L(S)$.

Example 12. Let $\mathcal{C} \subseteq \mathrm{Sp}$ be a subcategory which is stable under shifts and homotopy colimits, which is generated under homotopy colimits by a subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ consisting of *finite* spectra. Then it is easy to see that $\mathcal{C}$ satisfies condition (1) of Lemma 9, so that $\mathcal{C}$ determines a smashing localization functor.

Lubin-Tate Theory (Lecture 21)

April 27, 2010

We have seen that the moduli stack $\mathcal{M}_{\mathrm{FG}}$ of formal groups admits a stratification. The open strata are locally closed substacks $\mathcal{M}^n_{\mathrm{FG}} \subseteq \mathcal{M}^n_{\mathrm{FG}}$ classifying formal groups of height exactly n (at some fixed prime p). These strata are relatively well understood: for $0 < n < \infty$, the stratum $\mathcal{M}^n_{\mathrm{FG}}$ can be identified with a quotient $\operatorname{Spec} \overline{\mathbb{F}}_{\mathrm{p}_p} / \mathbb{G}$, where $\mathbb{G}$ is a certain profinite group (the Morava stabilizer group). To understand the moduli stack $\mathcal{M}_{\mathrm{FG}}$ itself, we want to know how these strata fit together. In other words, we would like to understand what $\mathcal{M}_{\mathrm{FG}}$ looks like in a small neighborhood of some point of $\mathcal{M}^n_{\mathrm{FG}}$. This is the subject of *Lubin-Tate theory*.

Let us fix a perfect field k of characteristic p and a formal group law $f(x, y) \in k[[x, y]]$ of height n over k. We would like to understand formal group which are, in some sense, "close" to f.

Definition 1. An *infinitesimal thickening* of k is a commutative ring A with a surjective map $\phi : A \to k$ whose kernel $\mathfrak{m}_A = \ker(\phi)$ has the following properties:

(1) The ideal $\mathfrak{m}_A^a = 0$ for $a \gg 0$.

(2) Each quotient $\mathfrak{m}_A^a / \mathfrak{m}_A^{a+1}$ is a finite-dimensional vector space over k.

In other words, A is a local Artin ring having residue field k.

Definition 2. Let A be an infinitesimal thickening of k. A *deformation of f over A* is a formal group law f_A over A, whose image under the map $\mathrm{FGL}(A) \to \mathrm{FGL}(k)$ is f. We say that two deformations of f are *isomorphic* if they differ by an invertible power series $g(t) \in A[[t]]$ such that $g(t) \equiv t \mod \mathfrak{m}_A$. We will denote the collection of isomorphism classes of deformations of f over A by $\mathrm{Def}(A)$.

Remark 3. A priori, we expect that deformations of a formal group law f over A should form a groupoid. However, this groupoid is actually discrete. In other words, if f_A is a deformation of f over A, then any automorphism of f_A which is the identity modulo $\mathfrak{m}_A$ is automatically trivial. To prove this, we can replace f by the image of f_A in $\mathrm{FGL}(k)$ and thereby reduce to the case $\eta = \mathrm{id}$. Let $g(x) = b_0 x + b_1 x^2 + \cdots$ be an automorphism of the formal group law f_A. We will prove by induction on a that $g(x) \equiv x \mod \mathfrak{m}_A^a$. When $a = 1$, this is true by hypothesis; for a sufficiently large, we have $\mathfrak{m}_A^a = 0$ so that we will have proven $g(x) = x$. To complete the proof, we carry out the inductive step. Let A' be the quotient of $A[b_0^{\pm 1}, b_1, \ldots]$ which classifies automorphisms of f_A. The map g is classified by a ring homomorphism $\psi : A' \to A$, while the identity automorphism is classified by $\psi_0 : A' \to A$. Assume that the composite maps

$$\psi, \psi' : A' \to A \to A/\mathfrak{m}_A^a$$

agree. Then, modulo $\mathfrak{m}_A^{a+1}$, the difference $\psi - \psi'$ is a map $d : A' \to V$, where V is the k-vector space $\mathfrak{m}_A^a / \mathfrak{m}_A^{a+1}$. The map d is an A-linear derivation, and factors as a composition

$$A' \to A' \otimes_A k \xrightarrow{d'} V$$

where d' is a k-linear derivation. But $A' \otimes_A k$ is the ring classifying automorphisms of the formal group f of height n, and is therefore etale over k: it follows that $d' = 0$ so that $\psi \equiv \psi' \mod \mathfrak{m}_A^{a+1}$.

Remark 4. The set $\mathrm{Def}(A)$ can be identified with the set of isomorphism classes of formal groups $\mathcal{F}$ over A lifting the formal group $\mathcal{G}_f$ associated to f. To see this, we note that since A is local the formal group $\mathcal{F}$ automatically has the form $\mathcal{F}_{f'}$ for some $f' \in \mathrm{FGL}(A)$. By assumption, the image f'_k of f' in $\mathrm{FGL}(k)$ is isomorphic to f, via some invertible power series $g(t) \in k[[t]]$. Lifting the coefficients of g arbitrarily, we can assume that g is the image of a power series $\overline{g}(t) \in A[[t]]$ (automatically invertible). Conjugating f' by g, we obtain the desired deformation of f.

We would like to understand the deformation functor $A \mapsto \mathrm{Def}(A)$. We begin by writing down a specific deformation of f. Let $W(k)$ denote the ring of Witt vectors of k, and let $R = W(k)[[v_1, \ldots, v_{n-1}]]$. There is a canonical map $R \to k$, whose kernel is the maximal ideal $\mathfrak{m}_R = (p, v_1, \ldots, v_{n-1})$. The formal group f over k is classified by a map $\phi_0 : L_{(p)} \to k$, where $L_{(p)} \simeq \mathbf{Z}_{(p)}[t_1, t_2, \ldots]$. We may assume without loss of generality that $t_{p^i-1} = v_i$ for $1 \leq i \leq n-1$. Since f has height n, we conclude that $t_{p^i-1} \mapsto 0 \in k$ for $1 \leq i \leq n-1$. Let $\phi : L_{(p)} \to R$ be *any* homomorphism which lifts ϕ_0, and carries t_{p^i-1} to v_i for $0 < i < m$. This homomorphism determines a formal group law $\overline{f} \in \mathrm{FGL}(R)$ whose image in $\mathrm{FGL}(k)$ is f.

Theorem 5 (Lubin-Tate)**.** *The formal group law $\overline{f}$ over $R = W(k)[[v_1, \ldots, v_{n-1}]]$ is a universal deformation of f in the following sense: for every infinitesimal thickening A of k, $\overline{f}$ gives a bijection*

$$\mathrm{Hom}_{/k}(R, A) \to \mathrm{Def}(A).$$

The proof rests on the following pair of observations:

(1) The functor $A \mapsto \mathrm{Def}(A)$ is formally smooth: that is, if $A \to A'$ is a surjective map between infinitesimal thickenings of k, then the induced map $\mathrm{Def}(A) \to \mathrm{Def}(A')$ is surjective (this is because any formal group law over A' extends to a formal group law over A, since the Lazard ring L is polynomial).

(2) Given a pair of surjective maps $A \to B \leftarrow C$ between infinitesimal thickenings of k, the canonical map $\mathrm{Def}(A \times_B C) \to \mathrm{Def}(A) \times_{\mathrm{Def}(B)} \mathrm{Def}(C)$ is a bijection. To see this, it is best to think in terms of formal groups (Remark 4): $\mathrm{Spec}(A \times_B C)$ is obtained by gluing $\mathrm{Spec}\, A$ and $\mathrm{Spec}\, C$ along the common closed subscheme $\mathrm{Spec}\, B$, so giving a formal group over $\mathrm{Spec}(A \times_B C)$ is equivalent to giving a formal groups over $\mathrm{Spec}\, A$ and $\mathrm{Spec}\, C$, together with an isomorphism between their restrictions to $\mathrm{Spec}\, B$.

To prove Theorem 5 we work by induction on the length of the Artinian ring A. If A has length 1, then $A \simeq k$ and both $\mathrm{Hom}_{/k}(R, A)$ and $\mathrm{Def}(A)$ consist of a single element. If A has length > 1, then we can choose an element $x \in A$ which is annihilated by $\mathfrak{m}_A$. Let us study the relationship between $\mathrm{Def}(A)$ and $\mathrm{Def}(A/x)$. Using (2), we have a pullback diagram

$$\begin{tikzcd}
\mathrm{Def}(A \times_{A/x} A) \arrow[r] \arrow[d] & \mathrm{Def}(A) \arrow[d, "p"] \\
\mathrm{Def}(A) \arrow[r] & \mathrm{Def}(A/x).
\end{tikzcd}$$

Note that $A \times_{A/x} A \simeq k[x]/(x^2) \times_k A$. There is an addition map

$$k[x]/(x^2) \times_k k[x]/(x^2) \to k[x]/(x^2)$$

which, by (2), determines a group structure on $\mathrm{Def}(k[x]/(x^2))$. The multiplication

$$k[x]/(x^2) \times_k A \to A$$

determines an action of $\mathrm{Def}(k[x]/(x^2))$ on $\mathrm{Def}(A)$, and the pullback square above shows that p determines an embedding $\mathrm{Def}(A)/\mathrm{Def}(k[x]/(x^2)) \hookrightarrow \mathrm{Def}(A/x)$. It follows from (1) that this map is surjective: that is, $\mathrm{Def}(A)$ is a principal homogeneous space for $\mathrm{Def}(k[x]/(x^2))$ over $\mathrm{Def}(A/x)$. The same reasoning shows that $\mathrm{Hom}_{/k}(R, A)$ is a torsor for $\mathrm{Hom}_{/k}(R, k[x]/(x^2))$ over $\mathrm{Hom}_{/k}(R, A/x)$. Since $\mathrm{Hom}_{/k}(R, A/x) \simeq \mathrm{Def}(A/x)$ by the inductive hypothesis, we are reduced to proving the following special case of Theorem 5:

Lemma 6. *The canonical map $\theta : \mathrm{Hom}_{/k}(R, k[x]/(x^2)) \to \mathrm{Def}(k[x]/(x^2))$ is bijective.*

To prove this, we construct a map $\theta' : \mathrm{Def}(k[x]/(x^2)) \to k^{n-1}$ as follows. Every deformation f' is classified by a map ϕ from the Lazard ring L into $k[x]/(x^2)$ and we have $\phi(v_i) = c_i x$ for $0 < i < n$. Set $\theta'(\phi) = (c_1, c_2, \ldots, c_{n-1})$.

Claim 7. *The sequence $(c_1, c_2, \ldots, c_{n-1})$ depends only on the isomorphism class of the deformation ϕ.*

To see this, let us suppose that f' and f'' are deformations of the formal group law f over $k[x]/(x^2)$ which differ by an autommorphism $g(t) = (1 + b_0 x)t + b_1 x t^2 + b_2 x t^3 + \cdots$. These formal group laws have p-series which we will denote by $[p]'(t)$ and $[p]''(t)$, which are related by the formula

$$g([p]'(g^{-1}(t))) = [p]''(t).$$

Since f has height $\geq n$, the power series $[p]'(t)$ and $[p]''(t)$ are divisible by x modulo t^{p^n}. Since $x^2 = 0$ and $g(t) \equiv t \mod (x)$, we deduce that $[p]'(t) \equiv [p]''(t) \mod (t^{p^n})$, thereby proving the claim.

It is not hard to see that θ' is a group homomorphism. Moreover, the composition

$$\mathrm{Hom}_{/k}(R, k[x]/(x^2)) \xrightarrow{\theta} \mathrm{Def}(k[x]/(x^2)) \xrightarrow{\theta'} k^{n-1}$$

is an isomorphism by construction. This proves that θ is injective. To prove that θ is surjective, it will suffice to show that θ' is injective. A deformation f' of f belongs to the kernel of θ' if and only if θ' has height exactly n. Let f'' be the trivial deformation of f; we wish to show that there is an isomorphism of f' with f'' which reduces to the identity modulo x.

Since f' and f'' are formal groups of height exactly n over $k[x]/(x^2)$, the collection of isomorphisms of f' and f'' is classified by a $k[x]/(x^2)$-algebra R which is an inductive limit of finite etale extensions of $k[x]/(x^2)$. It follows that $k[x]/(x^2)$-algebra homomorphism $R \to k$ lifts uniquely to a $k[x]/(x^2)$-algebra homomorphism $R \to k[x]/(x^2)$: in particular, the identity automorphism f extends uniquely to an isomorphism of f with f'. This completes the proof of Theorem 5.

Remark 8. Let A be a complete Noetherian local ring with residue field k and maximal ideal $\mathfrak{m}_A$. Then each $A/\mathfrak{m}_A^a$ is an infinitesimal thickening of k, and $A \simeq \varprojlim A/\mathfrak{m}_A^a$. It follows that Theorem 5 is also true for A: giving a deformation of the formal group f over A is equivalent to giving a ring homomorphism $W(k)[[v_1, \ldots, v_{n-1}]] \to A$ which is the identity on the common residue field k.

In particular, we see that $W(k)[[v_1, \ldots, v_{n-1}]]$ is characterized uniquely by Theorem 5. As such, it depends functorially on the residue field k together with the choice of formal group of height n over k.

In particular, if we take $k = \overline{\mathbb{F}}_{\mathrm{p}}$, then the Morava stabilizer group $\mathbb{G}$ acts on $W(\overline{\mathbb{F}}_{\mathrm{p}})[[v_1, \ldots, v_{n-1}]]$.

Remark 9. Let k and $R = W(k)[[v_1, \ldots, v_{n-1}]]$ be as above. Then the formal group law over R is Landweber exact: the sequence $v_0 = p, v_1, \ldots, v_{n-1}$ is regular by construction, and v_n has invertible image in $R/(v_0, v_1, \ldots, v_{n-1}) \simeq k$ by virtue of our assumption that the original formal group law f has height n.

Using results of previous lectures, we can construct an even periodic spectrum $E(n)$ with $\pi_* E(n) \simeq W(k)[[v_1, \ldots, v_{n-1}]][\beta^{\pm 1}]$, where β has degree 2. The cohomology theory $E(n)$ (which really depends not only on n, but on a choice of field k and a formal group of height n over k) is called *Morava E-theory.* It is also sometimes called *Lubin-Tate theory* or *completed Johnson-Wilson theory.*

Morava E-Theory and Morava K-Theory (Lecture 22)

April 27, 2010

Let k be a perfect field of characteristic p, and suppose we are given a formal group law f of height n over k. In the last lecture, we saw that the universal deformation of f is classified by the Lubin-Tate ring $R = W(k)[[v_1, \ldots, v_{n-1}]$. We note that this deformation of f over R is Landweber-exact: the sequence $v_0 = p, v_1, \ldots, v_{n-1}$ is regular by construction, and v_n has invertible image in $R/(v_0, v_1, \ldots, v_{n-1}) \simeq k$ by virtue of our assumption that the original formal group law f has height n.

Using results of previous lectures, we can construct an even periodic spectrum $E(n)$ with $\pi_* E(n) \simeq W(k)[[v_1, \ldots, v_{n-1}]][\beta^{\pm 1}]$, where β has degree 2. The cohomology theory $E(n)$ (which really depends not only on n, but on a choice of field k and a formal group of height n over k) is called *Morava E-theory.* It is also sometimes called *Lubin-Tate theory* or *completed Johnson-Wilson theory.*

Associated to $E(n)$ is a Bousfield localization functor $L_{E(n)}$. Note that a spectrum X satisfies $L_{E(n)} X = 0$ if and only if X is $E(n)$-acyclic: that is, the homology groups $E(n)_*(X) \simeq \mathrm{MP}_*(X) \otimes_L R$ vanish. We can associate to X a quasi-coherent sheaf $\mathcal{F}_X$ on $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$. The vanishing of $E(n)_*(X)$ is equivalent to the requirement that both $\mathcal{F}_X$ and $\mathcal{F}_{\Sigma(X)}$ (and therefore $\mathcal{F}_{\Sigma^k X}$ for every integer k) are supported on the closed substack $\mathcal{M}_{\mathrm{FG}}^{\geq n+1} \subseteq \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$. This is one sense in which $L_{E(n)}$ "behaves like" restriction to the open substack $\mathcal{M}_{\mathrm{FG}}^{\leq n} \subseteq \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$. This suggests that $L_{E(n)}$ should be a smashing localization. This is indeed the case:

Theorem 1 (Smash Product Theorem)**.** *The localization $L_{E(n)}$ is smashing: that is, it preserves direct sums.*

We will prove Theorem 1 later in this course.

Our next goal is to introduce a homotopy theoretic counterpart to the mechanism of restricting to the *closed* substack $\mathcal{M}_{\mathrm{FG}}^{n} \subseteq \mathcal{M}_{\mathrm{FG}}^{\leq n}$. This is more subtle, since $\mathcal{M}_{\mathrm{FG}}^{n}$ is not flat over $\mathcal{M}_{\mathrm{FG}}$, so we cannot proceed via Landweber's theorem.

Fix a prime p, and consider the p-local complex bordism spectrum $\mathrm{MU}_{(p)}$. This complex bordism spectrum has the structure of an E_∞*-ring.* In particular, there is a good theory of (structured) $\mathrm{MU}_{(p)}$-modules, and a relative smash product $(M, N) \mapsto M \otimes_{\mathrm{MU}_{(p)}} N$.

We have $\pi_* \mathrm{MU}_{(p)} \simeq L_{(p)} \simeq \mathbf{Z}_{(p)}[t_1, t_2, \ldots]$, where we may assume that $v_i = t^{p^i - 1}$ for each $i > 0$. By convention, we set $t_0 = p \in \pi_0 \mathrm{MU}_{(p)}$.

For each integer k, let $M(k)$ denote the cofiber of the map $\Sigma^{2k} \mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$ given by multiplication by t_k.

Lemma 2. *Each $M(k)$ admits a unital and homotopy associative multiplication (in the category of $\mathrm{MU}_{(p)}$-module spectra).*

Proof. We fix the unit of $M(k)$ to be the evident map $u : \mathrm{MU}_{(p)} \to M(k)$. The smash product $M(k) \otimes_{\mathrm{MU}_{(p)}}$

$M(k)$ can be realized as the total homotopy cofiber of the commutative diagram

$$\begin{tikzcd}
\Sigma^{4k} \mathrm{MU}_{(p)} \arrow[r, "t_k"] \arrow[d, "t_k"] & \Sigma^{2k} \mathrm{MU}_{(p)} \arrow[d, "t_k"] \\
\Sigma^{2k} \mathrm{MU}_{(p)} \arrow[r, "t_k"] & \mathrm{MU}_{(p)} .
\end{tikzcd}$$

We let K denote the total cofiber of an analogous diagram, where we replace the upper left hand corner with the zero spectrum. In other words, K is the cofiber of the map

$$(\Sigma^{2k} M(k))^2 \stackrel{(t_k,t_k)}{\rightarrow} M(k).$$

There is an evident map $\alpha : K \to M(k)$. To define an $\mathrm{MU}_{(p)}$-linear multiplication on $M(k)$ (having u as a unit) is equivalent to factoring u as a composition

$$K \xrightarrow{\beta} M(k) \otimes_{\mathrm{MU}_{(p)}} M(k) \xrightarrow{\gamma} M(k)$$

in the setting of $\mathrm{MU}_{(p)}$-modules.

To produce such a factorization, it suffices to show that the composition

$$\ker(\beta) \to K \xrightarrow{\alpha} M(k)$$

is nullhomotopic. Note that $\ker(\beta)$ can be identified with the desuspension of the total cofiber of the square

$$\begin{tikzcd}
\Sigma^{4k} M(k) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & 0;
\end{tikzcd}$$

that is, we have $\ker(\beta) \simeq \Sigma^{4k+1} M(k)$, and the relevant obstruction lives in $\pi_{4k+1} M(k) \simeq (L_p/t_k)_{4k+1} \simeq 0$.

We now show that the multiplication γ is homotopy associative (in the setting of $\mathrm{MU}_{(p)}$-modules). We have two natural multiplication maps

$$f, g : X = M(k) \otimes_{\mathrm{MU}_{(p)}} M(k) \otimes_{\mathrm{MU}_{(p)}} M(k) \to M(k).$$

We wish to prove that the difference $f - g$ is nullhomotopic. Note that X can be described as the total cofiber of a cube

$$\begin{tikzcd}
\Sigma^{6k} \mathrm{MU}_{(p)} \arrow[rr] \arrow[dd] \arrow[dr] & & \Sigma^{4k} \mathrm{MU}_{(p)} \arrow[dd] \arrow[dr] & \\
& \Sigma^{4k} \mathrm{MU}_{(p)} \arrow[rr] \arrow[dd] & & \Sigma^{2k} \mathrm{MU}_{(p)} \arrow[dd] \\
\Sigma^{4k} \mathrm{MU}_{(p)} \arrow[rr] \arrow[dr] & & \Sigma^{2k} \mathrm{MU}_{(p)} \arrow[dr] & \\
& \Sigma^{2k} \mathrm{MU}_{(p)} \arrow[rr] & & \mathrm{MU}_{(p)} .
\end{tikzcd}$$

Let Y be the total cofiber of an analogous diagram obtained by replacing the upper left corner by zero. By construction, the difference $f - g$ is nullhomotopic on Y, so that $f - g$ factors as a composition

$$X \to X/Y \simeq \Sigma^{6k+3} \mathrm{MU}_{(p)} \to M(k).$$

Since $\pi_{6k+3} M(k) \simeq (L_{(p)}/t_k)_{6k+3} \simeq 0$, the second map is nullhomotopic so that $f \simeq g$ as desired. $\square$

Remark 3. The multiplication on $M(k)$ constructed above is not unique: the same argument shows that the collection of such multiplications forms a torsor P for the group $\pi_{4k+2}M(k)$.

The torsor P has a canonical action of the permutation group Σ_2 (which acts on the smash product $M(k) \otimes_{\mathrm{MU}_{(p)}} M(k)$. If $p \neq 2$, then $\mathrm{H}^1(\Sigma_2; \pi_{2k+2}M(k)) \simeq 0$ so that P has a Σ_2-fixed point. This means that we can choose the multiplication on $M(k)$ to be homotopy commutative when $p \neq 2$.

Remark 4. By continuing the analysis of Lemma 2, one can show that $M(k)$ admits the structure of an A_∞-algebra over $\mathrm{MU}_{(p)}$.

Definition 5. Fix a prime number p and an integer $n > 0$. We let $K(n)$ denote the smash product (over $\mathrm{MU}_{(p)}$) of $\mathrm{MU}_{(p)}[v_n^{-1}]$ with $\bigotimes_{k \neq p^n-1} M(k)$. The spectrum $K(n)$ is called *Morava K-theory*.

Using Lemma 2, we see that $K(n)$ has the structure of a homotopy associative $\mathrm{MU}_{(p)}$-algebra; if $p \neq 2$, we can even assume that K is homotopy commutative.

A simple calculation shows that the homotopy groups of $K(n)$ are given by

$$\pi_* K(n) \simeq (\pi_* \mathrm{MU}_{(p)})[v_n^{-1}]/(t_0, t_1, \ldots, t_{p^n-2}, t_{p^n}, \ldots) \simeq \mathbf{F}_p[v_n^{\pm 1}],$$

where v_n has degree $2(p^n - 1)$.

We have a map of ring spectrum $\mathrm{MU}_{(p)} \to K(n)$, giving a complex orientation on $K(n)$. This determines a formal group law over the ring $\pi_* K(n) \simeq \mathbf{F}_p[v_n^{\pm 1}]$, which has height exactly n.

Warning 6. When $p = 2$, the Morava K-theory spectra generally do not admit homotopy commutative ring structures. Nevertheless, the theory of complex orientations makes sense in this setting: though $K(n)$ itself is not homotopy commutative, the cohomology rings $K(n)^*(X)$ are commutative for many important spaces (like $\mathbf{C}\mathrm{P}^\infty$, $BU(n)$, and so forth) since they are given by $\mathrm{MU}^*(X) \otimes_L \mathbf{F}_p[v_n^{\pm 1}]$.

Warning 7. Our construction of the ring spectra $M(k)$ (and therefore the Morava K-theories $K(n)$) involve a number of arbitrary choices. We will later see that, as a spectrum, $K(n)$ does not depend on these choices.

We let $L_{K(n)}$ denote the localization with respect to the Morava K-theory $K(n)$. We will later see that $L_{K(n)}$ behaves like completion along the locally closed substack $\mathcal{M}_{\mathrm{FG}}^n \subseteq \mathcal{M}_{\mathrm{FG}}$.

The Bousfield Classes of $E(n)$ and $K(n)$ (Lecture 23)

April 27, 2010

Let E and E' be homology theories. We say that E and E' are *Bousfield equivalent* if, for every spectrum X, the homology groups $E_*(X)$ vanish if and only if the homology groups $E'_*(X)$ vanish. Bousfield equivalence is an equivalence relation on spectra, and the equivalence classes are called *Bousfield classes*.

Example 1. Let E be a p-local complex oriented cohomology theory and suppose that the associated map $\operatorname{Spec}\pi_* E \to \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec}\mathbf{Z}_{(p)}$ is a flat covering of the open substack $\mathcal{M}_{\mathrm{FG}}^{\leq n}$. Then E is Bousfield equivalent to Morava E-theory $E(n)$. Indeed, for every spectrum X, the vanishing of $E_*(X)$ is equivalent to the requirement that, for each k, the localization $(\mathcal{F}_{\Sigma^k})_{(p)}$ of the quasi-coherent sheaf $\mathcal{F}_{\Sigma^k X}$ on $\mathcal{M}_{\mathrm{FG}}$ is supported on the closed substack $\mathcal{M}_{\mathrm{FG}}^{\geq n+1}$, which (by the same argument) is equivalent to the vanishing of $E(n)_*(X)$.

Let p be a prime number and an integer $n > 0$. Our main goal is to prove the following:

Proposition 2. *The spectrum $E(n)$ is Bousfield equivalent to $E(n-1) \times K(n)$. Here we agree by convention that $E(0) \simeq H\,\mathbf{Q}[\beta^{\pm 1}]$, which is Bousfield equivalent to $H\,\mathbf{Q}$.*

In other words, we claim that a spectrum X is $E(n)$-acyclic if and only if it is both $E(n-1)$-acyclic and $K(n)$-acyclic. To prove this, it will be convenient to replace introduce a different representative for the Bousfield class of $E(n)$.

Construction 3. *Recall that there exists an isomorphism $\pi_* \mathrm{MU}_{(p)} \simeq L_{(p)} \simeq \mathbf{Z}_{(p)}[t_1, t_2, \ldots]$ with $t_{p^n-1} = v_n$ for $n > 0$, and by convention we have $t_0 = v_0 = p$. For each $k \geq 0$, we let $M(k)$ denote the cofiber of the map $t_k : \Sigma^{2k}\mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$. In the last lecture, we saw that $M(k)$ admits the structure of a homotopy associative algebra in the category of $\mathrm{MU}_{(p)}$-modules.*

For $m \leq n$, we let $Z(m)$ denote the smash product (over $\mathrm{MU}_{(p)}$) of $\mathrm{MU}_p[v_n^{-1}]$ with $M(k)$, where k ranges over all nonnegative integers not of the form $p^{m'} - 1$ for $m \leq m' \leq n$.

By construction, $Z(m)$ is a complex-oriented ring spectrum with $\pi_* Z(m) = \mathbf{Z}_p[v_1, \ldots, v_{n-1}, v_n^{\pm 1}]/(v_0, v_1, \ldots, v_{m-1})$. We have $Z(n) \simeq K(n)$, and Example 1 shows that $Z(0)$ is Bousfield equivalent to $E(n)$.

Let us now prove Proposition 2. Suppose first that X is an $E(n)$-acyclic spectrum. Then each $(\mathcal{F}_{\Sigma^k X})_{(p)}$ is supported on the closed substack $\mathcal{M}_{\mathrm{FG}}^{\geq n+1} \subseteq \mathcal{M}_{\mathrm{FG}}$. Since $\mathcal{M}_{\mathrm{FG}}^{\geq n+1} \subseteq \mathcal{M}_{\mathrm{FG}}^{\geq n}$, we deduce immediately that X is $E(n-1)$-local. Since $Z(0)$ is Bousfield equivalent to $E(n)$, we have $X \otimes Z(0) \simeq 0$. Since $X \otimes K(n) \simeq X \otimes Z(n)$ is obtained from $X \otimes Z(0)$ by smashing (over $\mathrm{MU}_{(p)}$) with $M(p^k - 1)$ for $0 \leq k < n$, we conclude that $X \otimes K(n) \simeq 0$: that is, X is $K(n)$-acyclic.

Now suppose that X is $K(n)$-acyclic and $E(n-1)$-acyclic; we wish to prove that X is $E(n)$-acyclic. It will suffice to show that X is $Z(0)$-acyclic. We prove by descending induction on i that X is $Z(i)$-acyclic for each $i \leq n$. The case $i = n$ follows from our assumption that X is $K(n)$-acyclic (since $K(n) \simeq Z(n)$). Suppose therefore that $i < n$ and X is $Z(i+1)$-acyclic. We have a cofiber sequence

$$\Sigma^{2(p^i-1)} Z(i) \xrightarrow{v_i} Z(i) \to Z(i+1).$$

It follows that multiplication by v_i acts invertibly on $Z(i) \otimes X$, so that $Z(i) \otimes X \simeq Z(i)[v_i^{-1}] \otimes X$.It will therefore suffice to show that X is $Z(i)[v_i^{-1}]$-acyclic. Since $Z(i)$ is the smash product (over $\mathrm{MU}_{(p)}$) of $Z(0)$

with $M(p^j - 1)$ for $0 \leq j < i$, it will suffice to show that X is $Z(0)[v_i^{-1}]$-acyclic. Using Example 1, we see that $Z(0)[v_i^{-1}]$ is Bousfield equivalent to $E(i)$: since X is $E(n-1)$-acyclic and $i < n$, the first part of the proof shows that X is $E(i)$-acyclic and therefore $Z(0)[v_i^{-1}]$-acyclic as desired.

We now discussion the relationship between $E(n)$-localization and $K(n)$-localization. As a prototype, suppose that M is a finitely generated abelian group and we wish to describe its localization $M_{(p)}$ at a prime p. We can recover this localization as a fiber product

$$\begin{tikzcd} M_{(p)} \arrow[r] \arrow[d] & \widehat{M} \arrow[d] \\ M_{\mathbf{Q}} \arrow[r] & \widehat{M}_{\mathbf{Q}} \end{tikzcd}$$

where $\widehat{M}$ denotes the p-adic completion of M. To obtain a similar picture in our setting, we will need the following nontrivial fact:

Theorem 4 (Smash Product Theorem). *The localization functor $L_{E(n)}$ is smashing.*

Fix a spectrum X, and form a pullback diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & L_{K(n)}X \arrow[d] \\ L_{E(n-1)}X \arrow[r] & L_{E(n-1)}L_{K(n)}X. \end{tikzcd}$$

There is an evident map $\alpha : X \to X'$.

Proposition 5. *The map α exhibits X' as an E_n-localization of X.*

Since every $E(n)$-acyclic spectrum is $E(n-1)$-acyclic, every $E(n-1)$-local spectrum is $E(n)$-local; similarly, every $K(n)$-local spectrum is $E(n)$-local. Since the collection of $E(n)$-local spectra is stable under fiber products, we conclude immediately that X' is $E(n)$-local. To complete the proof, it will suffice to show that the map α is an $E(n)$-equivalence. By Proposition 2, it suffices to show that α induces both a $K(n)$-equivalence and an $E(n-1)$-equivalence. In other words, we must show that the diagrams

$$\begin{tikzcd} X \otimes K(n) \arrow[r] \arrow[d] & (L_{K(n)}X) \otimes K(n) \arrow[d] \\ (L_{E(n-1)}X) \otimes K(n) \arrow[r] & (L_{E(n-1)}L_{K(n)}X) \otimes K(n) \end{tikzcd} \qquad \begin{tikzcd} X \otimes E(n-1) \arrow[r] \arrow[d] & (L_{K(n)}X) \otimes E(n-1) \arrow[d] \\ (L_{E(n-1)}X) \otimes E(n-1) \arrow[r] & (L_{E(n-1)}L_{K(n)}X) \otimes E(n-1) \end{tikzcd}$$

are homotopy pullback squares. For the square on the right, this is obvious, since the vertical maps are both homotopy equivalences. For the square on the left, the upper horizontal map is a homotopy equivalence; we are therefore reduced to proving that the map $L_{E(n-1)}X \otimes K(n) \to (L_{E(n-1)}L_{K(n)}X) \otimes K(n)$ is an equivalence. This is a consequence of the following more general statement:

Lemma 6. *Let X be any spectrum. Then $L_{E(n-1)}X$ is $K(n)$-acyclic.*

Proof. Since $L_{E(n-1)}$ is smashing, we have $(L_{E(n-1)}X) \otimes K(n) \simeq X \otimes L_{E(n-1)}K(n)$. It therefore suffices to show that $L_{E(n-1)}K(n) \simeq 0$: in other words, that $E(n-1) \otimes K(n) \simeq 0$. This follows from the observation that $E(n-1) \otimes K(n)$ is complex orientable, and the associated formal group must have height $\leq n-1$ and exactly n. $\square$

Remark 7. According to Proposition 5, if X is an $E(n)$-local spectrum, then X can be recovered as the homotopy fiber product $L_{E(n-1)}X \times_{L_{E(n-1)}L_{K(n)}X} L_{K(n)}X$. Conversely, suppose that we are given an arbitrary $K(n)$-local spectrum Y and $E(n-1)$-local spectrum Z, together with a map $\alpha : Z \to L_{E(n-1)}Y$. Form a pullback diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & L_{E(n-1)}Y. \end{tikzcd}$$

Then X is $E(n)$-local. Moreover, since the lower horizontal map is an $K(n)$-equivalence (since the $K(n)$-homology of both sides vanishes), we deduce that $X \to Y$ is a $K(n)$-equivalence: that is, Y can be identified with $L_{K(n)}X$. Similarly, since the right vertical map is an $E(n-1)$-equivalence, we conclude that $X \to Z$ is an $E(n-1)$-equivalence so that Z can be identified with $L_{E(n-1)}X$. It follows that the $E(n)$-local stable homotopy category can be recovered as the homotopy category of triples $(Y, Z, \alpha : Z \to L_{E(n-1)}Y)$, where Y is $K(n)$-local, Z is $E(n-1)$-local, and α is a map of $E(n-1)$-local spectra.

In other words, the $E(n)$-local stable homotopy category admits a "semi-orthogonal" decomposition into the $E(n-1)$-local and $K(n)$-local stable homotopy categories.

Uniqueness of Morava K-Theory (Lecture 24)

April 27, 2010

Fix a prime number p and an integer $0 < n < \infty$. In Lecture 22, we introduced the Morava K-theory spectrum $K(n)$: a homotopy associative (and commutative if $p > 2$) ring spectrum with $\pi_* K(n) \simeq \mathbf{F}_p[v_n^{\pm 1}]$, $v_n \in \pi_{2(p^n-1)} K(n)$. However, our construction involved a number of arbitrary choices. Our goal in this lecture is to show that the underlying spectrum of $K(n)$ is independent of these choices (though its ring structure is not).

We begin with the following:

Definition 1. Let R be a commutative evenly graded ring. We will say that R is a *graded field* if every nonzero homogeneous element of R is invertible. Equivalently, R is a graded field if either R is either a field k concentrated in degree zero, or has the form $k[\beta^{\pm 1}]$ for some element β of positive even degree.

Remark 2. If R is a graded field, then every graded R-module M admits a free basis of homogeneous elements. This is clear if R is a field. If $R \simeq k[\beta^{\pm 1}]$ where β has degree $d > 0$, then any k-basis for $\bigoplus_{0 \leq i < d} M_i$ is an R-basis for M.

Definition 3. We will say that a homotopy associative ring spectrum E is a *field* if $\pi_* E$ is a graded field.

If E is as in Definition 3, then every E-module spectrum M is free: that is, it has the form $\bigoplus_\alpha \Sigma^{k_\alpha} E$ for some integers k_α. To see this, choose a homogeneous basis of $\pi_* M$ as a $\pi_* E$-module. Such a basis determines a map of E-module spectra $\alpha : \bigoplus \Sigma_{k_\alpha} E \to M$, which is obviously a homotopy equivalence.

Example 4. For every prime number p and every integer n, the Morava K-theory spectrum $K(n)$ is a field.

Example 5. For every field k (in the usual algebraic sense), the Eilenberg-MacLane spectrum Hk is a field in the sense of Definition 3. In particular, $H\mathbf{Q}$ and $H\mathbf{F}_p$ are fields. It is convenient to view these as special cases of the above: note that the definition of $K(n)$ makes sense when $n = 0$ and yields the Eilenberg-MacLane spectrum $K(0) \simeq H\mathbf{Q}$, and we agree to the convention that $K(\infty) = H\mathbf{F}_p$.

Remark 6. Let E be a field and let X and Y be spectra. Since E is a field, we can write $E \otimes X = \bigoplus \Sigma^{k_\alpha} E$; in particular, $E_* X = \pi_*(E \otimes X)$ is a free $\pi_* E$-module on generators of degree k_α. We have

$$E_*(X \otimes Y) \simeq \pi_*(E \otimes X \otimes Y) \simeq \pi_*(\bigoplus \Sigma^{k_\alpha} E \otimes Y) \simeq \bigoplus E_{*-k_\alpha}(Y) \simeq E_*(X) \otimes_{\pi_* E} E_*(Y).$$

In other words, for every field there is a Kunneth formula for computing the homology of a smash product of spectra (and therefore a Kunneth formulat for computing the homology of a product of spaces).

Lemma 7. *Let $f : X \to Y$ be a map of spectra. Suppose that X and Y each admit the structure of a $K(n)$-module, so that $\pi_* X$ and $\pi_* Y$ are modules over $\pi_* K(n) \simeq \mathbf{F}_p[v_n^{\pm}]$. Then the pushforward map $f_* : \pi_* X \to \pi_* Y$ is $\mathbf{F}_p[v_n^{\pm 1}]$-linear.*

Proof. We can factor f as a composition

$$X \to K(n) \otimes X \to K(n) \otimes Y \to Y.$$

Here we regard $K(n) \otimes X$ and $K(n) \otimes Y$ as $K(n)$-module spectra via the left action of $K(n)$ on itself. Each of the maps in this diagram is a $K(n)$-module map except for the first. It therefore suffices to treat the case $Y = K(n) \otimes X$. Since $K(n)$ is a field, the module X is free, so we may reduce to the case $X = K(n)$. In this case, we are required to prove that the two evident maps

$$f, g : K(n) \to K(n) \otimes K(n)$$

induce the same map on homotopy groups. In other words, we must show that $f_*(v_n) = g_*(v_n)$.

Note that $K(n) \otimes K(n)$ comes equipped with two complex orientations, determining two formal group laws over $R = \pi_{\mathrm{even}}(K(n) \otimes K(n))$. These formal group laws have p-series $[p](t) \equiv f_*(v_n) t^{p^n} \mod (t)^{p^n+1}$ and $[p]'(t) \equiv g_*(v_n) t^{p^n} \mod (t)^{p^n+1}$. Since these formal group laws differ by a coordinate change of the form $t \mapsto t + b_1 t^2 + b_2 t^3 + \ldots$, we conclude that $f_* v_n = g_* v_n$. □

Proposition 8. *Let X be a spectrum which admits the structure of a $K(n)$-module, and let Y be a retract of X. Then Y admits the structure of a $K(n)$-module.*

Proof. We have maps

$$Y \xrightarrow{i} X \xrightarrow{r} Y$$

whose composition is homotopic to id_Y. The composition $f = i \circ r$ is a map from X to itself. It follows from the Lemma that f_* is an $\mathbf{F}_p[v_n^{\pm 1}]$-module map from $\pi_* X$ to itself. In particular, the image of f_* is a graded $\mathbf{F}_p[v_n^{\pm 1}]$-submodule of $\pi_* X$. This image is automatically free, and so has a basis of classes $\eta_\alpha \in \pi_{k_\alpha} X$. This basis determines a map $\alpha : \bigoplus \Sigma^{k_\alpha} K(n) \to X$. By construction, the map $r \circ \alpha$ induces an isomorphism on homotopy groups and therefore determines an equivalence $Y \simeq \bigoplus \Sigma^{k_\alpha} K(n)$. □

Proposition 9. *Let E be any field and suppose that $E \otimes K(n)$ is nonzero. Then E admits the structure of a $K(n)$-module.*

Proof. If $E \otimes K(n)$ is nonzero, then the unit map $E \to E \otimes K(n)$ is nonzero. Using the assumption that E is a field, we deduce that E is a direct summand of $E \otimes K(n)$, and so admits a $K(n)$-module structure by Proposition 8. □

Proposition 10. *Let E be a complex-oriented ring spectrum whose associated formal group has height exactly n. If $E \neq 0$, then $E \otimes K(n) \neq 0$.*

Note that $E \otimes E(n-1)$ is a complex oriented ring spectrum whose formal group is both of height $\leq n-1$ and exactly n; it follows that $E \otimes E(n-1) \simeq 0$. If $E \otimes K(n) \simeq 0$, then it follows from the last lecture that $E \otimes E(n) \simeq 0$. But $\pi_*(E \otimes E(n))$ is the pullback of the quasi-coherent sheaf $\mathcal{F}_E$ on $\mathcal{M}_{\mathrm{FG}}$ along the flat map $\operatorname{Spec} W(k)[[v_1, \ldots, v_{n-1}]] \to \mathcal{M}_{\mathrm{FG}}$. It follows that $\mathcal{F}_E$ vanishes when restricted to the open substack $\mathcal{M}_{\mathrm{FG}}^{\leq n}$. Since the formal group of E has height $\leq n$, we conclude that $E \simeq 0$.

Proposition 11. *Let E be any complex-oriented ring spectrum whose formal group has height exactly n, and whose homotopy groups are given by $\pi_* E \simeq \mathbf{F}_p[v_n^{\pm 1}]$. Then there is a homotopy equivalence of spectra $E \simeq K(n)$.*

Proof. Since $E \neq 0$ and the formal group of E has height n, we conclude that $E \otimes K(n) \neq 0$ (Proposition 10). It follows from Proposition 9 that E admits the structure of a $K(n)$-module. This module is automatically free (since $K(n)$ is a field); it follows by inspecting homotopy groups that E must be free of rank 1: that is, $E \simeq K(n)$. □

It follows from Proposition 11 that when $n = 1$, the Morava K-theory $K(n)$ reduces to something familiar. Let K denote the complex K-theory spectrum. Then K has a canonical complex orientation, whose associated formal group law is given by $f(x, y) = x + y + \beta xy$, where $\beta \in \pi_2 K$ denotes the Bott element. Fix a prime number p, and let $\widehat{K}$ denote the p-adic completion of K. Then $\pi_0 \widehat{K} \simeq \mathbf{Z}_p$, and the formal group law over $\mathbf{Z}_p$ deforms the multiplicative formal group law (of height 1) over $\mathbf{F}_p$. We deduce:

Proposition 12. *Let f be the multiplicative formal group law over the field $\mathbf{F}_p$. Then the associated Morava E-theory is given by $E(1) = \widehat{K}$.*

We have seen that, as a homology theory, $E(1)$ is functorial with respect to automorphisms of the underlying formal group. In particular, the automorphism group of the formal multiplicative group acts on $\widehat{K}$. We have seen that this group can be identified with the group of units $\mathbf{Z}_p^\times$. For every p-adic unit λ, we let ψ^λ denote the corresponding map from $\widehat{K}$ to itself (these are given by the classical *Adams operations*).

The group $\mathbf{Z}_p^\times$ contains a finite subgroup μ_{p-1}, consisting of $(p-1)$st roots of unity. This finite group acts on the associated homology theory $\widehat{K}_*$. We let $\widehat{K}_*^{Ad}$ denote the μ_{p-1}-invariants in $\widehat{K}_*$. Since $\widehat{K}_*$ takes values in $\mathbf{Z}_{(p)}$-modules, passage to invariants is an exact functor so that $\widehat{K}_*^{Ad}$ is a homology theory, represented by a spectrum $\widehat{K}^{Ad}$. This is called the *Adams summand* of $\widehat{K}$; we have

$$\pi_* \widehat{K}^{Ad} \simeq \mathbf{Z}_p[\beta^{\pm(p-1)}] \subseteq \mathbf{Z}_p[\beta^{\pm 1}] \simeq \pi_* \widehat{K}.$$

With a bit more effort, one can show that $\widehat{K}^{Ad}$ has the structure of a ring spectrum, and that the cofiber of the multiplication by p map $p : \widehat{K}^{Ad} \to \widehat{K}^{Ad}$ inherits the structure of a ring spectrum. We denote this cofiber by $\widehat{K}^{Ad}/p$. A simple calculation gives $\pi_*(\widehat{K}^{Ad}/p) \simeq \mathbf{F}_p[\beta^{\pm(p-1)}]$. Moreover, $\widehat{K}^{Ad}/p$ has a canonical complex orientation, so we get a class $v_1 \in \pi_{2p-2}(\widehat{K}^{Ad}/p)$ which coincides with β^{p-1} (in fact, the p-series for $f(x,y) = x + y + \beta xy$ is given by $[p](t) = \frac{(1+\beta t)^p - 1}{\beta} \equiv \beta^{p-1} \mod p$). It follows from Proposition 11 that $K(1) \simeq \widehat{K}^{Ad}/p$: that is the 1st Morava K-group is given by the Adams summand of p-adic K-theory, reduced modulo p.

Remark 13. We will later see that every field satisfies the hypotheses of Proposition 9 for some $0 \leq n \leq \infty$. In other words, the Morava K-theories $K(n)$ are essentially the only examples of fields in the stable homotopy category (provided that we allow the cases $n = 0$ and $n = \infty$).

The Nilpotence Theorem (Lecture 25)

April 27, 2010

In the last lecture, we defined a ring spectrum E to be a *field* if $\pi_* E$ is a graded field. Every Morava K-theory is a field. Conversely, if E is any field, then we claim that E has the structure of a $K(n)$-module for some $0 \leq n \leq \infty$ (and some prime number p, if $n > 0$). Equivalently, we claim that $E \otimes K(n)$ is nonzero for some n.

Remark 1. The integer n is uniquely determined: the cohomology theory E is complex oriented and n can be characterized as the height of the associated formal group. (Similarly, the prime number p is uniquely determined: it is the characteristic of the field $\pi_0 E$).

For the remainder of this lecture, we will fix a prime number p.

Proposition 2. *Let $\{E^\alpha\}$ be a collection of ring spectra. The following conditions are equivalent:*

(1) *Let R be a p-local ring spectrum. If $x \in \pi_m R$ is a homotopy class whose image in $E^\alpha_0(R)$ is zero for all α, then x is nilpotent in $\pi_* R$.*

(2) *Let R be a p-local ring spectrum. If $x \in \pi_0 R$ is a homotopy class whose image in $E^\alpha_0(R)$ is zero for all α, then x is nilpotent in $\pi_0 R$.*

(3) *Let X be an arbitrary p-local spectrum. If $x \in \pi_0 X$ has trivial image under the Hurewicz map $\pi_0 X \to E^\alpha_0(X)$ for each α, then the induced class $x^{\otimes n} \in \pi_0 X^{\otimes n}$ is zero for $n \gg 0$.*

(4) *Let X be an arbitrary p-local spectrum, and let F be a finite spectrum. If $f : F \to X$ is such that each composite map $F \to X \to X \otimes E^\alpha_0$ is nullhomotopic, then $f^{\otimes n} : F^{\otimes n} \to X^{\otimes n}$ is nullhomotopic for $n \gg 0$.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious, and (2) $\Rightarrow$ (3) follows by taking R to be the ring spectum $\bigoplus_n X^{\otimes n}_{(p)}$. The implication (3) $\Rightarrow$ (4) follows by replacing X by the function spectrum X^F. Suppose now that (4) is satisfied, and let $x \in \pi_m R$ be a class whose image vanishes in $E^\alpha_n(R)$ for all α. Let us identify x with a map $S^m \to R$. Then x^n can be identified with the composition

$$S^{mn} \xrightarrow{x^{\otimes n}} R^{\otimes n} \to R,$$

where the second map is given by the multiplication on n. Since $x^{\otimes n}$ is nullhomotopic for $n \gg 0$ by (4), we conclude that x is nilpotent. $\square$

We say that a collection of ring spectra $\{E^\alpha\}$ *detects nilpotence* if the equivalent conditions of Proposition 2 are satisfied.

The following fundamental result was proven by Devinatz, Hopkins, and Smith:

Theorem 3 (Nilpotence Theorem)**.** *For any ring spectrum R, the kernel of the map $\pi_* R \to \mathrm{MU}_*(R)$ consists of nilpotent elements. In particular, the single cohomology theory* MU *detects nilpotence.*

Corollary 4 (Nishida)**.** *For $n > 0$, every element of $\pi_n S$ is nilpotent.*

Proof. Let $x \in \pi_n S$. Then x is torsion, so the image of x in $\mathrm{MU}_*(S) = \pi_* \mathrm{MU} \simeq L$ is torsion. Since L is torsion free, we conclude that the image of x is zero so that x is nilpotent by Theorem 3. □

We will use Theorem 3 to deduce the following:

Theorem 5. *The spectra $\{K(n)\}_{0 \leq n \leq \infty}$ detect nilpotence.*

We will prove that the spectra $\{K(n)\}_{0 \leq n \leq \infty}$ satisfy condition (3) of Proposition 2. Let T denote the homotopy colimit of the spectra

$$S \xrightarrow{x} X \xrightarrow{x} X^{\otimes 2} \xrightarrow{x} X^{\otimes 3} \to \cdots .$$

Lemma 6. *Let $x \in \pi_0 X$ and T be defined as above, and let E be any ring spectrum. The following conditions are equivalent:*

(1) *The spectrum T is E-acyclic.*

(2) *The image of $x^{\otimes n}$ in $E_0(X^{\otimes n})$ vanishes for $n \gg 0$.*

Proof. If (1) is satisfied, then the canonical map $S \to T \to T \otimes E$ is nullhomotopic. It follows that the map $S \xrightarrow{x^{\otimes n}} X^{\otimes n} \to X^{\otimes n} \otimes E$ is nullhomotopic for $n \gg 0$, so that (2) is satisfied. For the converse, we note that $T \otimes E$ can be identified with the homotopy colimit of the sequence

$$E \to X^{\otimes n} \otimes E \to X^{\otimes 2n} \otimes E \to \cdots$$

If (2) is satisfied, then each of the maps in this system is nullhomotopic, so the colimit is trivial. □

We now turn to the proof of Theorem 5. Fix $x \in \pi_0 X$ whose image in each $K(n)_0(X)$ is zero. We wish to prove that some smash power $x^{\otimes n}$ is trivial. By the nilpotence theorem, it will suffice to show that the image of x in $\mathrm{MU}_0(X)$ is nilpotent. By the Lemma, this is equivalent to showing that $\mathrm{MU}_*(T) \simeq 0$: that is, the quasi-coherent sheaf $\mathcal{F}_{\Sigma^k T}$ on $\mathcal{M}_{\mathrm{FG}}$ vanishes for $k \in \mathbf{Z}$.

Choose cofiber sequences

$$\Sigma^{2k} \mathrm{MU}_{(p)} \xrightarrow{t_k} \mathrm{MU}_{(p)} \to M(k)$$

as in the previous lectures. For $n \geq 0$, let $P(n)$ denote the smash product (taken over $\mathrm{MU}_{(p)}$) of the spectra $\{M(k)\}_{k \neq p^m - 1}$ and $\{M(p^m - 1)\}_{m < n}$, so that $P(n)$ is a ring spectrum with

$$\pi_* P(n) \simeq \mathbf{Z}_{(p)}[v_1, v_2, \ldots]/(v_0, v_1, \ldots, v_{n-1}).$$

In particular, $P(0)$ is the ring spectrum BP; we have seen that $P(0)$ is Landweber exact and that the map $\pi_* P(0) \to \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$ is faithfully flat. Then $P(0)_*(X)$ is the pullback of the quasi-coherent sheaf $\mathcal{F}_{\Sigma^{-*} X}$ on $\mathcal{M}_{\mathrm{FG}}$. It therefore suffices to show that $P(0)_*(T) \simeq 0$.

Let $P(\infty) \simeq \varinjlim P(n)$, so that $P(\infty) \simeq H\mathbf{F}_p$. By assumption, the image of x in $P(\infty)_0(X) \simeq \varinjlim P(n)_0(X)$ is zero. It follows that the image of x in $P(n)_*(X)$ vanishes for some $n < \infty$. By the lemma, we deduce that $P(n)_*(T) \simeq 0$.

We now prove that $P(m)_*(T) \simeq 0$ for all m, using descending induction on m. Assume that $P(m+1)_*(T) \simeq 0$. We have a cofiber sequence

$$\Sigma^{2(p^m - 1)} P(m) \xrightarrow{v_m} P(m) \to P(m+1).$$

It follows that multiplication by v_m is invertible on $P(m)_*(T)$, so that $P(m)_*(T) \simeq P(m)[v_m^{-1}]_* T$. Since $P(m)[v_m^{-1}]$ is a module over $\mathrm{MU}_{(p)}[v_m^{-1}]$, it will suffice to prove that T is $\mathrm{MU}_{(p)}[v_m^{-1}]$-acyclic. Note that $\mathrm{MU}_{(p)}[v_m^{-1}]$ is a Landweber-exact theory whose associated formal group has height $\leq m$ everywhere; it therefore suffices to show that T is $E(m)$-acyclic.

We now prove using ascending induction on $k \leq m$ that T is $E(k)$-acyclic. By the main result of Lecture 23, the inductive step is equivalent to showing that T is $K(k)$-acyclic. This follows from our lemma, since the image of x in $K(k)_0(X)$ vanishes by assumption.

Remark 7. Since $K(m)$ is a field, for each $n \geq 0$ the homology $K(m)_*(X^{\otimes n})$ is the nth (algebraic) tensor power of $K(m)_*(X)$ over $\pi_* K(m) \simeq \mathbf{F}_p[v_m^{\pm 1}]$. It follows that $x^{\otimes n}$ has trivial image in $K(m)_*(X^{\otimes n})$ if and only if x has trivial image in $K(m)_*(X)$ Consequently, we have the following slightly more precise result for a homotopy class $x \in \pi_0 X$ for a p-local spectrum X:

($*$) The class $x^{\otimes n} \in \pi_0 X^{\otimes n}$ is zero for $n \gg 0$ if and only if the image of x in $K(m)_0(X)$ vanishes for all m.

Remark 8. We can drop the requirement that X is p-local if we impose the same condition at all Morava K-theories (for all primes).

Corollary 9. *Let E be a nonzero p-local ring spectrum. Then $E \otimes K(n)$ is nonzero for some $0 \leq n \leq \infty$.*

Proof. If $K(n)_* E \simeq 0$ for all n, then Theorem 5 shows that every element of $\pi_0 E$ is nilpotent. In particular, the unit element $1 \in \pi_0 E$ is nilpotent, so that $E \simeq 0$. $\square$

Combining this with the results of the previous lecture, we deduce:

Corollary 10. *Let E be a ring spectrum such that $\pi_* E$ is a graded field. Then E has the structure of a $K(n)$-module for some n (and some prime number p).*

Thick Subcategories (Lecture 26)

April 27, 2010

Let p be a prime number, fixed throughout this lecture.

Let $\mathcal{C}$ be a full subcategory of the category of p-local spectra which is stable under homotopy colimits and desuspension, and which is generated under homotopy colimits by a small subcategory. The theory of Bousfield localization allows us to associate to every p-local spectrum X a canonical fiber sequence

$$C(X) \to X \to L(X),$$

where $C(X) \in \mathcal{C}$ and $L(X)$ is $\mathcal{C}$-local (that is, every map from an object of $\mathcal{C}$ into $L(X)$ is nullhomotopic).

Let $\mathcal{C}_0$ be the collection of all finite p-local spectra contained in $\mathcal{C}$. If $\mathcal{C}_0$ generates $\mathcal{C}$ under homotopy colimits, then the localization functor L is smashing. In this case, $\mathcal{C}_0$ determines $\mathcal{C}$ and vice versa. The following definition axiomatizes the expected properties of $\mathcal{C}_0$:

Definition 1. Let $\mathcal{T}$ be a full subcategory of the homotopy category of finite p-local spectra. We say that $\mathcal{T}$ is *thick* if it contains 0, is closed under the formation of fibers and cofibers, and if every retract of a spectrum belonging to $\mathcal{T}$ also belongs to $\mathcal{T}$.

Remark 2. Let $\mathcal{T}$ be a thick subcategory of finite p-local spectra. If $X \in \mathcal{T}$ and Y is any finite p-local spectrum, then $X \otimes Y \in \mathcal{T}$. Indeed, the collection of p-local finite spectra Y which for which $X \otimes Y \in \mathcal{T}$ is itself thick. Since it contains the p-local sphere $S_{(p)}$, it contains all finite p-local spectra (every finite p-local spectrum admits a finite cell decomposition).

Remark 3. Let $\mathcal{T}$ be any thick subcategory of the category of finite p-local spectra, and let $\mathcal{C}$ be the collection of p-local spectra generated by $\mathcal{T}$ under homotopy colimits. Every object $X \in \mathcal{C}$ can be written as a filtered colimit of objects $X_\alpha \in \mathcal{T}$. In particular, if X is a finite p-local spectrum, then the identity map $X \to \varinjlim X_\alpha$ factors through some X_α. Thus X is a retract of X_α and so $X \in \mathcal{T}$. Consequently, the construction $\mathcal{T} \mapsto \mathcal{C}$ determines a bijection between thick subcategories of finite p-local spectra and subcategories $\mathcal{C}$ of the category of all p-local spectra, which are stable under desuspension and generated by p-local finite spectra under homotopy colimits.

Our next goal is to describe some thick subcategories. We begin with the following observation:

Lemma 4. *Let X be a finite p-local spectrum. Suppose that $K(n)_*(X) \simeq 0$ for some $n > 0$. Then $K(n-1)_*(X) \simeq 0$.*

To prove this, we let R denote the ring spectrum obtained by smashing $\mathrm{MU}_{(p)}[v_n^{-1}]$ over $\mathrm{MU}_{(p)}$ with the spectra $\{M(k)\}_{k \neq p^n-1, p^{n-1}-1}$. For simplicity, let us assume $n > 1$ (the proof in the case $n = 1$ is essentially the same, but the notation changes). Then R is a ring spectrum with $\pi_* R \simeq \mathbf{F}_p[v_{n-1}, v_n^{\pm 1}]$. In particular, $\pi_0 R$ is equivalent to the polynomial ring $\mathbf{F}_p[v_{n-1}^a v_n^{-b}] = \mathbf{F}_p[t]$ where (a, b) is the minimal solution to $a(p^{n-1} - 1) - b(p^n - 1) = 0$. Note that for every integer k, $R_k(X)$ is a finitely generated module over $\pi_0 R$. We have a cofiber sequence

$$\Sigma^{2(p^{n-1}-1)} R \xrightarrow{v_{n-1}} R \to K(n).$$

Since $K(n)_* X \simeq 0$, we conclude that multiplication by v_{n-1} and hence multiplication by t acts invertibly on each $R_k(X)$. It follows that each $R_k(X)$ is a torsion module over $\mathbf{F}_p[t]$, and is therefore annihilated by almost every irreducible polynomial in $\mathbf{F}_p[t]$. In particular, we can choose a nonzero polynomial $f(t)$ which annihilates each $R_k(X)$ for $0 \leq k < 2(p^n - 1)$ and therefore for all values if k (since $\pi_* R$ is periodic with period $2(p^n - 1)$). Without loss of generality, $f(t)$ is divisible by t. For $k \gg 0$, the product $f(t)v_n^k$ can be written as a polynomial in v_{n-1} and v_n, and therefore comes from $\pi_* \mathrm{MU}$. We can therefore localize R to obtain a new ring spectrum $R[f(t)^{-1}]$ with $R[f(t)^{-1}]_* X \simeq R_* X[f(t)^{-1}] \simeq 0$.

By construction, $R[f(t)^{-1}]$ has a complex orientation and the associated formal group has height exactly $n-1$ (since $f(t)$ is divisible by t, so v_{n-1} is invertible in $\pi_* R[f(t)^{-1}]$). It follows that $R[f(t)^{-1}] \otimes K(m)$ vanishes for $m \neq n-1$. Since $R[f(t)^{-1}] \neq 0$, $R[f(t)^{-1}] \otimes K(n-1) \neq 0$ and therefore contains $K(n-1)$ as a retract. Since $X \otimes R[f(t)^{-1}] \simeq 0$, we conclude that $X \otimes R[f(t)^{-1}] \otimes K(n-1) \simeq 0$ so that $X \otimes K(n-1) \simeq 0$, as desired.

Remark 5. Let X be a finite p-local spectrum. Then $\mathrm{H}_*(X; \mathbf{F}_p) \simeq 0$ if and only if $X \simeq 0$. Moreover, $\mathrm{H}_k(X; \mathbf{F}_p)$ vanishes for almost all values of k. For $n \gg 0$, the Atiyah-Hirzebruch spectral sequence for $K(n)_*(X)$ degenerates to give $K(n)_*(X) \simeq \mathrm{H}_*(X; \mathbf{F}_p)[v_n^{\pm 1}]$. It follows that if $X \neq 0$, then $K(n)_*(X) \neq 0$ for $n \gg 0$.

Definition 6. We say that a p-local finite spectrum X has *type* n if $K(n)_*(X) \neq 0$ but $K(m)_*(X) \simeq 0$ for $m < n$. For example, X has *type* 0 if $\mathrm{H}_*(X; \mathbf{Q}) \simeq 0$, or equivalently if $\mathrm{H}_*(X; \mathbf{Z})$ is not a torsion group.

Every nonzero finite p-local spectrum X has type n for some unique n. By convention, we will say that the spectrum 0 has type ∞.

Definition 7. Let $\mathcal{C}_{\geq n}$ be the collection of finite p-local spectra which have type $\geq n$. In other words, $X \in \mathcal{C}_{\geq n}$ if and only if $K(m)_*(X) \simeq 0$ for $m < n$.

Using the long exact sequence in $K(m)$-homology, we see that if we are given a cofiber sequence

$$X' \to X \to X'',$$

and any two of X', X, and X'' has type $\geq n$, then so does the third. Moreover, it is clear that any retract of a spectrum of type $\geq n$ is also of type $\geq n$. Consequently, $\mathcal{C}_{\geq n}$ is a thick subcategory of the category of finite p-local spectra.

The main result of this lecture is the following:

Theorem 8 (Thick Subcategory Theorem)**.** *Let $\mathcal{T}$ be a thick subcategory of finite p-local spectra. Then $\mathcal{T} = \mathcal{C}_{\geq n}$ for some $0 \leq n \leq \infty$.*

In other words, the $\mathcal{C}_{\geq n}$ are exactly the thick subcategories of finite p-local spectra.

Remark 9. It is not yet clear that the classes $\mathcal{C}_{\geq n}$ are different for distinct n. This is equivalent to the following assertion: for every nonnegative integer n, there exists a finite p-local spectrum of type n. We will discuss the proof of this theorem in the next lecture.

Let $\mathcal{T}$ be as in Theorem 8. If $\mathcal{T}$ contains only the zero spectrum, then we can take $n = \infty$. Otherwise, there exists a nonzero spectrum $X \in \mathcal{T}$ having type n for $n < \infty$. Choose X so that n is minimal; we wish to prove that $\mathcal{T} = \mathcal{C}_{\geq n}$. The inclusion $\mathcal{T} \subseteq \mathcal{C}_{\geq n}$ is clear (otherwise, $\mathcal{T}$ would contain a spectrum of type $< n$, contradicting minimality). Theorem 8 can therefore be reformulated as follows:

Proposition 10. *Let $\mathcal{T}$ be a thick subcategory containing a type n spectrum X. If Y is a spectrum of type $\geq n$, then $Y \in \mathcal{T}$.*

To prove this, let DX denote the (p-local) Spanier-Whitehead dual of X. The identity map $X \to X$ is classified by a map $e : S_{(p)} \to X \otimes DX$. Since X has type n, we note that e induces an injection $K(m)_*(S_{(p)}) \to K(m)_*(X \otimes DX) \simeq K(m)_*(X) \otimes_{\mathbf{F}_p[v_m^{\pm 1}]} K(m)_*(X)^\vee$ for $m \geq n$. Form a fiber sequence

$$F \xrightarrow{f} S_{(p)} \to X \otimes DX.$$

It follows that the map $K(m)_*F \to K(m)_*(S_{(p)})$ is zero for $m \geq n$. Consider the composite map

$$g : F \xrightarrow{f} S_{(p)} \to Y \otimes DY.$$

Then g induces the zero map $K(m)_*F \to K(m)_*(Y \otimes DY)$ for $m \geq n$ (since f has the same property) and also for $m < n$ (since Y has type $\geq n$, so that $K(m)_*(Y \otimes DY) \simeq 0$). By the nilpotence theorem, we conclude that some smash power $F^{\otimes k} \to (Y \otimes DY)^{\otimes k}$ is nullhomotopic. Composing with the multiplication on $Y \otimes DY$, we get a nullhomotopic map $F^{\otimes k} \to Y \otimes DY$, which corresponds to the composition

$$F^{\otimes k} \otimes Y \xrightarrow{f} F^{\otimes k-1} \otimes Y \xrightarrow{f} \cdots \to Y.$$

It follows that Y is a retract of the cofiber $Y/(F^{\otimes k} \otimes Y)$. Consequently, to show that $Y \in \mathcal{T}$, it will suffice to show that $Y/(F^{\otimes k} \otimes Y) \in \mathcal{T}$.

The spectrum $Y/F^{\otimes k} \otimes Y$ admits a finite filtration by spectra of the form $(F^{\otimes a} \otimes Y)/(F^{\otimes a+1} \otimes Y)$. Since $\mathcal{T}$ is thick, it will suffice to show that each of these belongs to $\mathcal{T}$. Each of these spectra has the form

$$F^{\otimes a} \otimes Y \otimes (S_{(p)}/F) \simeq F^{\otimes a} \otimes Y \otimes DX \otimes X,$$

and therefore belongs to $\mathcal{T}$ since $X \in \mathcal{T}$ (Remark 2).

The Periodicity Theorem (Lecture 27)

April 27, 2010

Let p be a prime number, fixed throughout this lecture. In the last lecture, we asserted that for every integer $n \geq 0$, there exists a finite p-local spectrum X of type n. If $n = 0$, this just means that the rational homology $\mathrm{H}_*(X; \mathbf{Q})$ is nonzero. We can achieve this by taking X to be the p-local sphere $S_{(p)}$.

When $n = 1$, we can define X to be the mod p Moore spectrum, which is defined by the cofiber sequence

$$S \xrightarrow{p} S \to X.$$

This has no rational homology. However, since multiplication by p annihilates $K(1)_*(S) \simeq \mathbf{F}_p[v_1^{\pm 1}]$, the map $K(1)_*(S) \to K(1)_*X$ is injective. In particular, $K(1)_*(X) \neq 0$, so that X has type 1.

For $n > 1$, it is somewhat harder to construct spectra of type n. We can try to mimic the previous construction. Namely, suppose that we are given a spectrum X of type n. We might try to find a self map $f : \Sigma^k X \to X$ so that we can form a cofiber sequence

$$\Sigma^k X \xrightarrow{f} X \to X/f,$$

and hope that X/f has type $n+1$. It is clear that X/f has type $\geq n$. To guarantee that X/f has type exactly $n+1$, we need to know two things:

(1) The $K(n)$-homology of X/f vanishes: in other words, f induces an isomorphism from $K(n)_*X$ to itself.

(2) The $K(n+1)$-homology of X/f does not vanish: that is, f does *not* induce an isomorphism from $K(n+1)_*X$ to itself.

This motivates the following definition:

Definition 1. Let X be a p-local finite spectrum, and let $n \geq 1$. A *v_n-self map* is a map $f : \Sigma^k X \to X$ with the following properties:

(a) f induces an isomorphism $K(n)_*X \to K(n)_*X$.

(b) For $m \neq n$, the induced map $K(m)_*X \to K(m)_*X$ is nilpotent.

Remark 2. If X is a p-local finite spectrum which admits a v_n-self map f, then X must have type $\geq n$. For if X has type $m < n$, then X/f has nonvanishing $K(m)$-homology (since f is not an isomorphism on $K(m)_*X$) but vanishing $K(n)$-homology.

Example 3. If X is a spectrum of type $> n$, then $K(n)_*X$ vanishes: it follows that the zero map $0 : X \to X$ is a v_n-self map.

The crucial case to consider is where X has type n. In this case, a v_n-self map $f : \Sigma^k X \to X$ will satisfy conditions (1) and (2) above, so that X/f will be a spectrum of type $n+1$. Consequently, to verify the existence of spectra of type n for every n, it will suffice to prove the following:

Theorem 4 (Periodicity Theorem)**.** *Let X be a finite p-local spectrum of type $\geq n$. Then X admits a v_n-self map.*

It will be useful to reformulate the notion of a v_n-self map. If X is a finite p-local spectrum, then $R = X \otimes DX$ has the structure of a ring spectrum. Moreover, giving a self map $\Sigma^k X \to X$ is equivalent to giving an element of $\pi_k R$. The condition of being a v_n-self map translates as follows:

Definition 5. Let R be a p-local ring spectrum. An element $x \in \pi_k R$ is a *v_n-element* if the image of x in $K(m)_* R$ is nilpotent for $m \neq n$, and invertible for $m = n$.

This is equivalent to saying that left multiplication by x induces a v_n-self map from R to itself. In particular, it implies that R has type $\geq n > 0$, so that the homotopy groups $\pi_* R$ consist of p-power torsion.

Lemma 6. *Let R be a finite p-local ring spectrum and let $x \in \pi_k R$ be a v_n-element. After raising x to a suitable power, we may assume that $x \mapsto v_n^a \in K(n)_*(R)$ and $x \mapsto 0 \in K(m)_* R$ for $m \neq n$.*

Proof. Recall that $K(m)_* R \simeq \mathrm{H}_*(R; \mathbf{F}_p)[v_m^{\pm 1}]$ for $m \gg 0$. It follows that x is nilpotent in $\mathrm{H}_*(R; \mathbf{F}_p)$. Replacing x by a suitable power, we may assume that $x \mapsto 0 \in \mathrm{H}_*(R; \mathbf{F}_p)$ and therefore $x \mapsto 0 \in K(m)_* R$ for $m \gg 0$. Consequently, there are only finitely many integers $m \neq n$ for which the image of x does not vanish in $K(m)_*(R)$. Each of these images is nilpotent; raising x to a power, we may assume that $x \mapsto 0 \in K(m)_* R$ for $m \neq n$.

Note that $K(n)_* R$ is a finite module over $\pi_* K(n) \simeq \mathbf{F}_p[v_n^{\pm 1}$. It follows that $(K(n)_* R)/(v_n - 1)$ is finite. The image of $x \in (K(n)_* R)/(v_n - 1)$ is a unit, so after raising x to a power we may assume that $x \mapsto 1 \in (K(n)_* R)/(v_n - 1) \simeq \bigoplus_{0 \leq i < 2(p^n - 1)} K(n)_i R$. It follows that $x \mapsto v_n^a \in K(n)_* R$ for some a. $\square$

Lemma 7. *Let R be a $\mathbf{Z}_{(p)}$ algebra, and let $x, y \in R$ be commuting elements such that $x - y$ is torsion and nilpotent. Then $x^{p^k} = y^{p^k}$ for $k \gg 0$.*

Proof. We have

$$x^{p^k} = (y + (x - y))^{p^k} = y^{p^k} + \sum_{0 < i \leq p^k} \binom{p^k}{i} y^{p^k - i} (x - y)^i.$$

If $(x - y)^{p^a} = 0$, we can rewrite the right hand side as

$$y^{p^k} + \sum_{0 < i < p^a} \frac{p^k}{i} \binom{p^k - 1}{i - 1} y^{p^k - i} (x - y)^i.$$

Each expression $\frac{p^k}{i}$ is divisible by p^{k-a}, and therefore annihilates $x - y$ if $k \gg 0$. $\square$

Lemma 8. *Let R be a finite p-local ring spectrum and let $x \in \pi_k R$ be a v_n-element. After raising x to a suitable power, we may assume that x is central in $\pi_* R$.*

Proof. Without loss of generality we may assume that x satisfies the conclusions of Lemma 6. Let $A = R \otimes DR$, and let $a, b \in \pi_k A$ be given by the self-maps of R given by left and right multiplication by x. Then a and b commute. Since A has type > 0, $\pi_* A$ is torsion, so $a - b \in \pi_k A$ is torsion. We claim that $a - b$ is nilpotent. To prove this, it suffices to show that the image of $a - b$ vanishes in $K(m)_* A$ for every integer m: in other words, the composite maps

$$R \to K(m) \otimes R \xrightarrow{x\times} K(m) \otimes R$$

$$R \to K(m) \otimes R \xrightarrow{\times x} K(m) \otimes R$$

agree. If $m \neq n$, this is clear (since $x \mapsto 0 \in K(m)_* R$. For $m = n$, we are reduced to proving that left and right multiplication by v_n^j induce the same self-map of $K(m) \otimes R$. This is clear, since $K(n)$ is a homotopy associative ring spectrum in the category of $\mathrm{MU}_{(p)}$-modules and v_n lies in the image of $\pi_* \mathrm{MU}_{(p)} \to \pi_* K(n)$.

Lemma 7 gives $a^{p^j} = b^{p^j}$ for $j \gg 0$. Replacing x by x^{p^j}, we can assume that $a = b$, so that left and right multiplication by x agree. $\square$

Lemma 9. *Let R be a finite p-local ring spectrum and $x, y \in \pi_* R$ two v_n-elements. Then $x^a = y^b$ for suitable $a, b > 0$.*

Proof. Raising x and y to suitable powers, we may assume that $x, y \mapsto 0 \in K(m)_* R$ for $m \neq n$ and $x, y \mapsto v_n^j \in K(n)_* R$. Raising to a further power we may assume that x and y commute. Since R is of type > 0, $\pi_* R$ is torsion so that $x - y$ is a torsion element of $\pi_* R$. Then $x - y \mapsto 0 \in K(m)_* R$ for all m, so $x - y$ is nilpotent. Using Lemma 7 we conclude that $x^{p^j} = y^{p^j}$ for $j \gg 0$. □

Lemma 10. *Let X and Y be spectra which admit v_n-self maps $f : \Sigma^a X \to X$ and $g : \Sigma^b Y \to Y$. Let $h : X \to Y$ be any map. Then, replacing f and g by suitable powers, we may assume that $a = b$ and that the diagram*

$$\begin{tikzcd} \Sigma^a X \arrow[r, "h"] \arrow[d, "f"] & \Sigma^a Y \arrow[d, "g"] \\ X \arrow[r, "h"] & Y \end{tikzcd}$$

commutes up to homotopy.

Proof. We can view the map h as given by $e : S \to DX \otimes Y$. The commutativity of the diagram then amounts to a homotopy $(Df \otimes \mathrm{id}_Y) \circ e \simeq (\mathrm{id}_{DX} \otimes g) \circ e$. Since $Df \otimes \mathrm{id}_Y$ and $\mathrm{id}_{DX} \otimes g$ are two v_n-self maps of $DX \otimes Y$, this identity will hold after replacing f and g by appropriate powers (Lemma 9). □

Proposition 11. *Let $\mathcal{T}$ be the collection of p-local finite spectra which admit a v_n-self map. Then $\mathcal{T}$ is thick.*

Proof. It is clear that $0 \in \mathcal{T}$ and that $\mathcal{T}$ is closed under suspension and desuspension. We next show that $\mathcal{T}$ is closed under taking cofibers. Let $h : X \to Y$ be a map of p-local finite spectra which admit v_n-self maps f and g. By virtue of Lemma 10, we may assume that the diagram

$$\begin{tikzcd} \Sigma^k X \arrow[r, "h"] \arrow[d, "f"] & \Sigma^k Y \arrow[d, "g"] \\ X \arrow[r, "h"] & Y \end{tikzcd}$$

commutes up to homotopy. We may further assume that f and g are zero on $K(m)$-homology for $m \neq n$.

A choice of homotopy induces a map of cofibers $x : \Sigma^k (Y/X) \to Y/X$. We claim that x is a v_n-self map. Since f and g induce an isomorphism on $K(n)$-homology, the associated long exact sequence in homology shows that x induces an isomorphism on $K(n)$-homology. For $m \neq n$, we have a map of exact sequences

$$\begin{tikzcd} K(m)_{*-k} Y \arrow[r] \arrow[d, "0"] & K(m)_{*-k}(Y/X) \arrow[r] \arrow[d, "x"] & K(m)_{*-k-1} X \arrow[d, "0"] \\ K(m)_* Y \arrow[r, "\phi"] & K(m)_* Y/X \arrow[r] & K(m)_{*-1} X. \end{tikzcd}$$

It follows that x carries $K(m)_{*-k}(Y/X)$ into the image of ϕ and that multiplication by x is trivial on the image of ϕ, so that x^2 is trivial on $K(m)_* Y/X$.

It remains to prove that $\mathcal{T}$ is stable under retracts. Let X and Y be p-local spectra and assume that $X \oplus Y$ admits a v_n-self map f. Raising f to a power, we may assume that f vanishes on $K(m)_*(X \oplus Y)$ for $m \neq n$ and is given by multiplication by v_n^j on $K(n)_*(X \oplus Y)$. Then the composite map

$$\Sigma^k X \to \Sigma^k (X \oplus Y) \xrightarrow{f} X \oplus Y \to X$$

has the same properties, and is therefore a v_n self map. □

Let $\mathcal{T}$ be the thick subcategory of Proposition 11. The periodicity theorem can be restated as follows: $\mathcal{T}$ contains every spectrum of type $\geq n$. By the thick subcategory theorem, this is equivalent to the following result, which we assert without proof:

Proposition 12. *For every integer $n \geq 0$, there exists a finite p-local spectrum X of type n, and a v_n-self map $f : \Sigma^k X \to X$.*

Telescopic Localization (Lecture 28)

April 12, 2010

Let p be a prime number, fixed throughout this lecture.

Let X be a p-local finite spectrum of type $\geq n$. In the last lecture, we saw that X admits a v_n-self map $f : \Sigma^k X \to X$. Moreover, such a map is asymptotically unique: if $f' : \Sigma^{k'} X \to X$ is another v_n-self map, then $f^i \simeq f'^j$ for some integers $i, j > 0$. It follows that the colimit of the sequence

$$X \xrightarrow{f} \Sigma^{-k} X \xrightarrow{f} \Sigma^{-2k} X \to \cdots$$

is independent of f. Let us denote this colimit by $X[f^{-1}]$.

We can describe $X[f^{-1}]$ more intrinsically as follows. Let $\mathcal{C}_{\geq n+1}$ denote the collection of all p-local finite spectra of type $> n$. Then $\mathcal{C}_{\geq n+1}$ determines a localization of the category of p-local spectra: that is, for every p-local spectrum X there is a canonical cofiber sequence

$$C(X) \to X \to L_n^t(X),$$

where $C(X)$ can be written as a filtered colimit of objects in $\mathcal{C}_{\geq n+1}$, and $L_n^t(X)$ is local with respect to $\mathcal{C}_{\geq n+1}$: in other words, if Y is a finite p-local spectrum of type $> n$, then every map $e : Y \to L_n^t(X)$ is nullhomotopic.

Proposition 1. *Let X be a finite p-local spectrum of type $\geq n$, and let f be a v_n-self map of X. Then $L_n^t(X) \simeq X[f^{-1}]$.*

More precisely, the canonical map $u : X \to X[f^{-1}]$ exhibits $X[f^{-1}]$ as a $\mathcal{C}_{\geq n+1}$-localization of X. To see this, we must verify two things:

(1) The fiber of the map $u : X \to X[f^{-1}]$ is a filtered colimit of objects of $\mathcal{C}_{\geq n+1}$. This is clear: the cofiber of u can be identified with the colimit of the sequence

$$0 \to \Sigma^{-k} X/X \to \Sigma^{-2k} X/X \to \ldots$$

Each $\Sigma^{-bk} X/X$ is (up to a shift) the cofiber of the v_n-self map f^b on X, which has type $> n$.

(2) The object $X[f^{-1}]$ is $\mathcal{C}_{\geq n+1}$-local. In other words, if Y is a finite spectrum of type $> n$, then every map $e : Y \to X[f^{-1}]$ is nullhomotopic. To see this, it suffices to show that $DY \otimes X[f^{-1}]$ is nullhomotopic. Without loss of generality, we may suppose that f induces the zero map on $K(m)_* X$ for $m \neq n$. It follows that $\mathrm{id}_{DY} \otimes f$ induces the zero map on $K(m)_*(DY \otimes X)$ for all integers m: here we use the assumption that Y is of type $> n$ and the Kunneth formula to see that $K(n)_*(DY \otimes X) \simeq 0$. By the nilpotence theorem, we conclude that $\mathrm{id}_{DY} \otimes f^a$ is nilpotent for $a \gg 0$. Replacing f by f^a, we may assume that $\mathrm{id}_{DY} \otimes f$ is nullhomotopic, so that $DY \otimes X[f^{-1}]$ is the colimit of a sequence of nullhomotopic maps

$$DY \otimes X \to DY \otimes \Sigma^{-k} X \to \cdots$$

and therefore contractible.

Remark 2. The functor L_n^t is sometimes referred to as *telescopic localization*. This is essentially a reference to Proposition 1, which gives an explicit construction of L_n^t (for type n-spectra) as a telescope: that is, as the homotopy colimit of a sequence of spectra.

We can view the theory of v_n-self maps as providing an explicit description of the effect of the localization functor L_n^t on finite p-local spectra of type $\geq n$. By applying this reasoning iteratively, we can understand L_n^t on arbitrary p-local finite spectra. To see this, let us begin with a p-local finite spectrum X. By convention, we can think of multiplication by p as a v_0-self map of X. That is, we can form the colimit $X[p^{-1}]$ of the sequence

$$X \xrightarrow{p} X \xrightarrow{p} X \xrightarrow{p} X \to \cdots$$

The above reasoning shows that $X[p^{-1}]$ can be identified with $L_0^t(X)$. We therefore have a cofiber sequence

$$\varinjlim_k \Sigma^{-1} X/p^k \to X \to L_0^t(X)$$

where X/p^k denotes the cofiber of multiplication by p^k on X. Applying the functor L_1^t, we get a commutative diagram

$$\begin{tikzcd}
\varinjlim_k \Sigma^{-1} X/p^k \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & L_0^t(X) \arrow[d] \\
L_1^t \varinjlim \Sigma^{-1} X/p^k \arrow[r] & L_1^t(X) \arrow[r] & L_1^t L_0^t(X)
\end{tikzcd}$$

The vertical map on the right is an equivalence, since $L_0^t(X)$ is already local with respect to $\mathcal{C}_{\geq 2}$. It follows that the fiber F of the map $X \to L_1^t X$ can be identified with the fiber of the map

$$\varinjlim_k \Sigma^{-1} X/p^k \to L_1^t \varinjlim \Sigma^{-1} X/p^k$$

Since L_1^t is a smashing localization, it commutes with filtered colimits and we can therefore write F as the filtered colimit of the fibers of the maps

$$q : \Sigma^{-1} X/p^k \to \Sigma^{-1} L_1^t X/p^k.$$

Since each X/p^k is a finite p-local spectrum of type ≥ 1, Proposition **??** implies that $L_1^t X/p^k$ can be identified with $X/p^k[f_k^{-1}]$, where f_k is a v_1-self map of X/p^k. It follows that the fiber of q can be identified with the direct limit $\varinjlim \Sigma^{-2}(X/p^k)/f_k^l$. Thus F can be identified with the colimit $\varinjlim_k \varinjlim_l \Sigma^{-2}(X/p^k)/f_k^l$.

Here it is convenient to ignore the fact that f_k depends on k, and to denote all v_n-self maps by by the symbol v_n (so that $v_0 = p$). We can summarize our analysis informally as follows: we have a cofiber sequence

$$\varinjlim_{k_0, k_1} \Sigma^{-2} X/(v_0^{k_0}, v_1^{k_1}) \to X \to L_1^t(X).$$

This provides a somewhat explicit description of $L_1^t(X)$ as the cofiber of a map from a colimit of type ≥ 2-spectra into X.

Applying this argument repeatedly, we arrive at an "explicit" description of $L_n^t(X)$: it sits in a fiber sequence

$$\varinjlim_{k_0, \ldots, k_n} \Sigma^{-n} X/(v_0^{k_0}, \ldots, v_n^{k_n}) \to X \to L_n^t(X).$$

Since L_n^t is a smashing localization, it is in some sense determined by what it does to the (p-local) sphere spectrum. We have a cofiber sequence

$$\varinjlim_{k_0, \ldots, k_n} S^{-n}/(v_0^{k_0}, \ldots, v_n^{k_n}) \to S_{(p)} \to L_n^t S_{(p)}.$$

Smashing this cofiber sequence with X, we recover the sequence given above. However, there is another construction available in this context: instead of smashing with X, we can consider function spectra of maps into X. We get a fiber sequence

$$X^{L_n^t S_{(p)}} \to X \to \varprojlim X^{S^{-n}/(v_0^{k_0},\ldots,v_n^{k_n})}.$$

Unwinding the notation, we see that the function spectra on the right have a more direct description as the smash product of X with $S/(v_0^{k_0},\ldots,v_n^{k_n})$, which we will denote by $X/(v_0^{k_0},\ldots,v_n^{k_n})$. We can therefore think of the homotopy inverse limit on the right as a kind of completion of X.

Remark 3. Let $\mathcal{D}$ be the collection of all $\mathcal{C}_{\geq n+1}$-local spectra: that is, p-local spectra X such that every map $Y \to X$ is nullhomotopic if Y is a finite p-local spectrum of type $> n$. Then $\mathcal{D}$ is closed under shifts and homotopy colimits, and therefore determines another Bousfield localization functor R. That is, for every p-local spectrum X, there is a canonical cofiber sequence

$$D(X) \to X \to R(X)$$

where $D(X) \in \mathcal{D}$ and $R(X)$ is $\mathcal{D}$-local: that is, every map $g : Y \to R(X)$ is nullhomotopic if $Y \in \mathcal{D}$.

Proposition 4. *Let X be a p-local spectrum. Then the fiber sequence*

$$X^{L_n^t S_{(p)}} \to X \to \varprojlim X/(v_0^{k_0},\ldots,v_n^{k_n})$$

can be identified with the fiber sequence of Remark 3.

In other words, the functor R of Remark 3 can be described as a "completion" with respect to the ideal $v_0,\ldots,v_n$, given by $X \mapsto \varprojlim X/(v_0^{k_0},\ldots,v_n^{k_n})$.

As with Proposition 1, there are two things to prove:

(1) The function spectrum $X^{L_n^t S_{(p)}}$ belongs to $\mathcal{D}$. Let Y be a finite p-local spectrum of type $> n$; we wish to show that every map $u : Y \to X^{L_n^t S_{(p)}}$ is nullhomotopic. We can identify u with a map $Y \otimes L_n^t S_{(p)} \to X$. Such a map is automatically nullhomotopic, since $Y \otimes L_n^t S_{(p)} \simeq L_n^t Y$ vanishes by virtue of our assumption that Y has type $> n$.

(2) The homotopy inverse limit $\varprojlim X/(v_0^{k_0},\ldots,v_n^{k_n})$ is $\mathcal{D}$-local. Since the collection of $\mathcal{D}$-local spectra is stable under homotopy inverse limits, it suffices to show that each term in the system is $\mathcal{D}$-local. Each of these terms has the form X^K, where K is a finite p-local spectrum of type $> n$. Let $Y \in \mathcal{D}$ and suppose we are given a map $u : Y \to X^K$; we wish to show that u is nullhomotopic. We can identify u with a map $Y \otimes K \to X$. To see that such a map is nullhomotopic, it suffices to show that $Y \otimes K \simeq 0$. This is clear, since $Y \in \mathcal{D}$ implies that $Y \simeq L_n^t Y$, so that

$$Y \otimes K \simeq L_n^t Y \otimes K \simeq Y \otimes L_n^t K \simeq 0,$$

by virtue of the fact that K has type $> n$.

Telescopic vs. E_n-Localization (Lecture 29)

April 13, 2010

Let p be a prime number, fixed throughout this lecture. Let L be a Bousfield localization functor on p-local spectra. Our goal in this lecture is to obtain a structure theorem for L, under the assumption that L is smashing.

Let us begin by fixing a bit of terminology. We say a spectrum X is *L-local* if the map $X \to LX$ is an equivalence.

Lemma 1. *Let L be a localization functor. For $0 \leq n \leq \infty$, we have either $LK(n) \simeq 0$ or $LK(n) \simeq K(n)$.*

Proof. We have a map of ring spectra $K(n) \to LK(n)$. Consequently, $LK(n)$ has the structure of a $K(n)$-module. If $LK(n) \neq 0$, then $LK(n)$ contains $K(n)$ (possibly shifted) as a retract. Since $LK(n)$ is L-local, we conclude that $K(n)$ is L-local so that $K(n) \simeq LK(n)$. □

Lemma 2. *Let L be a smashing localization functor and let E be a nonzero complex-oriented cohomology theory whose formal group has height exactly n. Then $LE \simeq 0$ if and only if $LK(n) \simeq 0$.*

Proof. If $LE \simeq 0$, then $0 \simeq K(n) \otimes LE \simeq LK(n) \otimes E$. Since $K(n) \otimes E \neq 0$, we conclude that $LK(n) \simeq 0$ (Lemma 1). Conversely, suppose that $LK(n) \simeq 0$. Then $0 \simeq LK(n) \otimes E \simeq K(n) \otimes LE$. On the other hand, $LE \otimes K(m) \simeq 0$ for $m \neq n$, since it is a complex oriented ring spectrum whose formal group has height exactly m and exactly n. It follows from the nilpotence theorem that $LE \simeq 0$. □

Lemma 3. *Let L be a smashing localization functor. If $LK(m) \simeq 0$, then $LK(n) \simeq 0$ for $n > m$.*

Proof. For $k \geq 0$, let $M(k)$ denote the cofiber of the map $t_k : \Sigma^{2k} \mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$, and let R be the ring spectrum obtained by smashing (over $\mathrm{MU}_{(p)}$) the spectra $\{M(k)\}_{k \neq p^{m-1}, p^{n-1}}$ with $\mathrm{MU}_{(p)}[v_n^{-1}]$. For notational simplicity we will assume that $0 < m < n < \infty$, so that $\pi_* R \simeq \mathbf{F}_p[v_m, v_n^{\pm 1}]$. Note that $R[v_m^{-1}]$ is a ring spectrum whose associated formal group has height exactly m. It follows from Lemma 2 that $LR[v_m^{-1}] \simeq 0$. Since L is smashing, we can identify $LR[v_m^{-1}]$ with the colimit of the sequence

$$LR \xrightarrow{v_m} \Sigma^{-2(p^m-1)} LR \xrightarrow{v_m} \Sigma^{-4(p^m-1)} LR \to \cdots$$

It follows that $1 \in \pi_0 LR$ vanishes in $\pi_0 \Sigma^{-2k(p^m-1)} R$ for $k \gg 0$: in other words, the image of v_m^k vanishes in $\pi_* LR$. Let R' denote the cofiber of the map $v_m^{k+1} : \Sigma^{2(k+1)(p^m-1)} R \to R$, so that v_m^k vanishes in $\pi_* LR'$. Since $\pi_* R' \simeq \mathbf{F}_p[v_m, v_n^{\pm 1}]/(v_m^{k+1})$, we conclude that the map $\pi_* R' \to \pi_* LR'$ is not injective. In particular, R' is not L-local. Note that R' can be obtained as a successive extension of $k+1$ copies of $R/v_m \simeq K(n)$. It follows that $K(n)$ is not L-local. According to Lemma 1, this means that $LK(n) \simeq 0$. □

If L is any localization functor, let us denote by $\ker(L)$ the collection of all L-acyclic spectra: that is, spectra X such that $LX \simeq 0$.

Lemma 4. *Let L be a smashing localization functor, and let $n \geq 0$ be an integer. The following conditions are equivalent:*

(1) $LK(n) \simeq 0$.

(2) $LK(m) \simeq 0$ *for* $n \leq m \leq \infty$.

(3) *Every finite* p*-local spectrum* X *of type* $\geq n$ *belongs to* $\ker(L)$.

(4) *There exists a finite* p*-local spectrum* X *of type* n *in* $\ker(L)$.

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 3. The implication (3) $\Rightarrow$ (4) is clear (since there exists a finite p-local spectrum of type n). To prove that (4) $\Rightarrow$ (1), we note that $LX \simeq 0$ implies $LX \otimes K(n) \simeq X \otimes LK(n) \simeq 0$. If $LK(n) \neq 0$, then $LK(n) \simeq K(n)$ so that $X \otimes LK(n) \neq 0$, since X has type n.

It remains to prove that (2) $\Rightarrow$ (3). Let X be a p-local finite spectrum of type $\geq n$. We wish to prove that $LX \simeq 0$. Let $R = X \otimes DX$; since LX is an LR-module, it will suffice to show that $LR \simeq 0$. Since LR is a ring spectrum, by the nilpotence theorem it will suffice to show that $LR \otimes K(m) \simeq 0$ for every m. If $m < n$, we have $LR \otimes K(m) \simeq L(R \otimes K(m)) \simeq 0$ since R has type $\geq n > m$. If $m \geq n$, then $LR \otimes K(m) \simeq R \otimes LK(m) \simeq 0$ because $LK(m) \simeq 0$ by assumption (2). $\square$

(A) We have $LK(n) \simeq 0$ for all $0 \leq n < \infty$.

(B) We have $LK(n) \simeq K(n)$ for all $0 \leq n < \infty$.

(C) There exists an integer $n \geq 0$ such that $LK(n) \simeq K(n)$ but $LK(n+1) \simeq 0$.

In case (A), Lemma 2 guarantees that L annihilates every finite p-local spectrum of type ≥ 0. In particular, for every X we have

$$LX \simeq X \otimes LS_{(p)} \simeq X \otimes 0 \simeq 0 :$$

that is, L is the zero functor.

Let us now analyze case (C). Fix n such that $LK(n) \simeq K(n)$ but $LK(n+1) \simeq 0$. Lemma 4 implies that $\ker(L)$ contains every finite spectrum of type $> n$. Conversely, if X is a finite p-local spectrum such that $LX \simeq 0$, we have

$$0 \simeq K(n) \otimes LX \simeq LK(n) \otimes X \simeq K(n) \otimes X$$

so that X must have type $> n$. In other words, the finite p-local spectra belonging to $\ker(f)$ are *precisely* the spectra of type $> n$: that is, the spectra which are $E(n)$-acyclic. Conversely, we have the following:

Proposition 5. *Let* L *be a smashing localization, and suppose that* $LK(n) \simeq K(n)$*. Then every spectrum which belongs to* $\ker(L)$ *is* $E(n)$*-acyclic.*

Remark 6. An equivalent formulation is the following: if L is a smashing localization with $LK(n) \simeq K(n)$, then every $E(n)$-local spectrum is L-local.

Proof. Let $X \in \ker(L)$. We wish to show that X is $E(n)$-acyclic. Since $E(n)$ is Bousfield equivalent to $K(0) \oplus \cdots \oplus K(n)$, it suffices to show that X is $K(m)$-acyclic for $m \leq n$. This follows from

$$K(m) \otimes X \simeq LK(m) \otimes X \simeq K(m) \otimes LX \simeq 0,$$

since L is smashing and $LK(m) \simeq K(m)$ for $m \leq n$ (Lemma 3). $\square$

Let us now return to case (C). If L is a smashing localization with $LK(n) \simeq K(n)$ and $LK(n+1) \simeq 0$, then we conclude that $\ker(L)$ consists of $E(n)$-acyclic spectra, and contains all finite E_n-acyclic spectra. In other words, we have

$$\ker(L_n^t) \subseteq \ker(L) \subseteq \ker(L_{E(n)}).$$

The following conjecture of Ravenel is the main open problem left in the subject (though it is generally believed to be false):

Conjecture 7 (Telescope Conjecture)**.** The localization functors L_n^t and $L_{E(n)}$ coincide. In particular, every smashing localization L satisfying (C) above has the form L_n^t for some $n \geq 0$.

It remains to treat the case (B): suppose that L is a smashing localization with $LK(n) \simeq K(n)$ for $n \geq 0$. According to Remark 6, if X is an $E(n)$-local spectrum for any X, then X is L-local. In particular, the *chromatic tower*

$$\cdots \to L_{E(2)}S_{(p)} \to L_{E(1)}S_{(p)} \to L_{E(0)}S_{(p)}$$

consists of L-local spectra, so that homotopy inverse limit of this tower is L-local. Next week we will prove the following:

Theorem 8 (Chromatic Convergence Theorem)**.** *The homotopy inverse limit of the chromatic tower is $S_{(p)}$.*

Corollary 9. *Let L be a smashing localization such that $LK(n) \simeq K(n)$ for $0 \leq n < \infty$. Then L is equivalent to the identity functor.*

Proof. Using the chromatic convergence theorem and Remark 6, we deduce that $S_{(p)}$ is L-local. Then, for any p-local spectrum X, we have

$$LX \simeq X \otimes LS_{(p)} \simeq X \otimes S_{(p)} \simeq X.$$

□

Localizations and the Adams-Novikov Spectral Sequence (Lecture 30)

April 17, 2010

Throughout this lecture, we fix a ring spectrum E. We will assume for simplicity that E is a structured ring spectrum. To any spectrum X, we can associate the cosimplicial ring spectrum $[n] \mapsto X \otimes E^{\otimes n+1}$, which we will denote by $X^\bullet$. The homotopy inverse limit of $X^\bullet$ is called its totalization and denoted $\mathrm{Tot}(X^\bullet)$. It is given as an inverse limit of partial totalizations

$$\cdots \to \mathrm{Tot}^2(X^\bullet) \to \mathrm{Tot}^1(X^\bullet) \to \mathrm{Tot}^0(X^\bullet) \simeq X \otimes E,$$

called the *Adams tower* for X with respect to E. There is a canonical map $X \to \mathrm{Tot}(X^\bullet)$. We ask how closely this map approximates a homotopy equivalence.

The first observation is that $X^\bullet$ depends only on the localization $L_E X$: any E-homology equivalence $X \to Y$ induces a homotopy equivalence of cosimplicial spectra $X^\bullet \to Y^\bullet$. On the other hand, $\mathrm{Tot}\, X^\bullet$ is a homotopy inverse limit of E-modules, and is therefore automatically E-local. The best possible situation, then, is that $\mathrm{Tot}\, X^\bullet$ is an E-localization of X: equivalently, the map $X \to \mathrm{Tot}\, X^\bullet$ induces an isomorphism in E-homology. This is equivalent to the assertion that $E \otimes X \to E \otimes (\mathrm{Tot}\, X^\bullet)$ is a homotopy equivalence. The right hand side also admits a map to $\mathrm{Tot}(E \otimes X^\bullet)$. The augmented cosimplicial object $[n] \mapsto E \otimes X \otimes (E^{\otimes(n+1)})$ is *split*: that is, it admits an extra codegeneracy map. It follows formally that the composite map

$$E \otimes X \to E \otimes \mathrm{Tot}\, X^\bullet \to \mathrm{Tot}(E \otimes X^\bullet)$$

is a homotopy equivalence. Consequently, we obtain the following:

Proposition 1. *Let E be a structured ring spectrum and X a spectrum. Then the canonical map $X \to \mathrm{Tot}\, X^\bullet$ exhibits $\mathrm{Tot}\, X^\bullet$ as an E-localization of X if and only if $E \otimes \mathrm{Tot}\, X^\bullet \simeq \mathrm{Tot}(E \otimes X^\bullet)$.*

Note that $\mathrm{Tot}(E \otimes X^\bullet) \simeq \varprojlim \mathrm{Tot}^n(E \otimes X^\bullet)$. Each partial totalization Tot^n is given by a finite homotopy inverse limit, and therefore commutes with smash products. It follows that $\mathrm{Tot}(E \otimes X^\bullet)$ can be identified with $\varprojlim E \otimes \mathrm{Tot}^n(X^\bullet)$. Consequently, the condition of Proposition 1 can be restated as follows: the canonical map

$$E \otimes \varprojlim \mathrm{Tot}^n(X^\bullet) \to \varprojlim E \otimes \mathrm{Tot}^n(X^\bullet)$$

is a homotopy equivalence.

To understand this condition better, it is convenient to work in the setting of *pro-spectra*. A pro-spectrum is a formal inverse limit "$\varprojlim X_\alpha$" of a filtered diagram of spectra (for our needs, it will be sufficient to consider inverse limits of towers). Morphism spaces are computed by the formula

$$\mathrm{Map}(``\varprojlim X_\alpha", ``\varprojlim Y_\beta") = \varprojlim_\beta \varinjlim_\alpha \mathrm{Map}(X_\alpha, Y_\beta).$$

The collection of all pro-spectra form a homotopy theory, which we will denote by $\mathrm{Pro}(\mathrm{Sp})$. There is a forgetful functor $U : \mathrm{Pro}(\mathrm{Sp}) \to \mathrm{Sp}$, which carries a diagram "$\varprojlim X_\alpha$" to its homotopy inverse limit $\varprojlim X_\alpha$. We say that a pro-spectrum "$\varprojlim X_\alpha$" is *constant* if, in $\mathrm{Pro}(\mathrm{Sp})$, it is homotopy equivalent to a constant tower

$$\cdots X \to X \to X.$$

In this case, we have a canonical equivalence $\varprojlim X_\alpha \simeq X$.

If "$\varprojlim X''_\alpha$ is a pro-spectrum and E is any spectrum, then we can define a new prospectrum $E \otimes$"$\varprojlim X''_\alpha =$ "$\varprojlim E \otimes X''_\alpha$. We then have a natural map $E \otimes U(''\varprojlim X''_\alpha) \to U(E \otimes$ "$\varprojlim X''_\alpha)$. This map is not always an equivalence, but it is obviously an equivalence when "$\varprojlim X''_\alpha$ is constant. Applying this to our situation, we obtain the following:

Proposition 2. *The equivalent conditions of Proposition 1 are satisfied whenever the tower*

$$\cdots \to \mathrm{Tot}^2 X^\bullet \to \mathrm{Tot}^1 X^\bullet \to \mathrm{Tot}^0 X^\bullet$$

is constant as a pro-spectrum.

Consequently, it is of interest for us to have a criterion for determining when a tower of spectra

$$\cdots \to Y(2) \to Y(1) \to Y(0)$$

is constant as a pro-spectrum. Recall that any such tower determines a spectral sequence $\{E_r^{p,q}, d_r\}$, which (in good cases) converges to $\pi_q \varprojlim Y(n)$. Our goal is to establish the following criterion (a very imprecise version of a criterion of Bousfield):

Proposition 3 (Bousfield)**.** *Let $\cdots \to Y(2) \to Y(1) \to Y(0)$ be a tower of spectra. Suppose that there exists an integer $s \geq 1$ with the following property: for every finite spectrum F, if $\{E_r^{p,q}, d_r\}$ is the spectral sequence associated to the tower*

$$\cdots \to F \otimes Y(2) \to F \otimes Y(1) \to F \otimes Y(0),$$

then the groups $E_s^{p,q}$ vanish for $p \geq s$. Then the tower $\cdots \to Y(2) \to Y(1) \to Y(0)$ is constant as a pro-object.

To prove Proposition 3, we begin by fixing a tower of spectra

$$\cdots Y(2) \to Y(1) \to Y(0)$$

and assume that the associated spectral sequence $\{E_r^{p,q}\}$ satisfies $E_s^{p,q} \simeq 0$ for $p \geq s$. To exploit this hypothesis, we need to recall the details of the definition of the spectral sequence $\{E_r^{p,q}, d_r\}$. For $m \leq n$ let $F(m,n)$ denote the homotopy fiber of the map $Y(n) \to Y(m)$ (here we adopt the convention that $Y(m) \simeq 0$ for $m < 0$). Then $E_r^{p,q}$ is defined as the image of the map $\pi_q F(p+r-1, p-1) \to \pi_q F(p, p-r)$, and the differential d_r carries $E_r^{p,q}$ into $E_r^{p+r,q-1}$. If $p < 0$, then $F(p, p-r)$ is contractible so that $E_r^{p,q}$ automatically vanishes. If $p \geq s$, then $E_r^{p,q}$ vanishes for $r \geq s$ by assumption. It follows that if $r \geq s$, then at least one of the groups $E_r^{p,q}$ and $E_r^{p+r,q-1}$ vanishes, so that the differential d_r is identically zero. This proves:

- $(*)$ The groups $E_r^{p,q}$ are independent of r for $r \geq s$. That is, the spectral sequence $\{E_r^{p,q}, d_r\}$ collapses at the s-page.

Now suppose $r > p$. Since $F(p, p-r) \simeq Y(p)$, we have $\pi_q F(p, p-r) \simeq \pi_q Y(p)$. In this case, $E_r^{p,q}$ is the image of the composite map

$$\pi_q F(p+r-1, p-1) \to \pi_q Y(p+r-1) \to \pi_q Y(p).$$

The image of the first map is the kernel of the map $\pi_q Y(p+r-1) \to \pi_q Y(p-1)$. We therefore have:

- $(*')$ For $r > p$, the group $E_r^{p,q}$ is the intersection $\mathrm{Im}(\pi_q Y(p+r-1) \to \pi_q Y(p)) \cap \ker(\pi_q Y(p) \to \pi_q Y(p-1))$.

Combining $(*)$ and $(*')$, we deduce:

- $(*'')$ The intersection $\mathrm{Im}(\pi_q Y(p+r) \to \pi_q Y(p)) \cap \ker(\pi_q Y(p) \to \pi_q Y(p-1)$ is independent of r, provided that $r \geq p, s$.

Lemma 4. *For every integer $k \geq 0$, the intersection* $\mathrm{Im}(\pi_q Y(p+r) \to \pi_q Y(p)) \cap \ker(\pi_q Y(p) \to \pi_q Y(p-k))$ *is independent of r, provided that $r \geq p, s$.*

Proof. We use induction on k. The case $k = 0$ is trivial, so assume that $k > 0$. Suppose that $r \geq p, s$, and that $x \in \pi_q Y(p+r)$ has trivial image in $\pi_q Y(p-k)$. Let $y \in \pi_q Y(p)$ be the image of x; we wish to show that y lifts to $\pi_q Y(p+r+1)$. Let y' denote the image of y in $\pi_q Y(p-1)$. Then y' belongs the kernel of the map $\pi_q Y(p-1) \to \pi_q Y(p-k)$. Since y' lifts to $\pi_q Y(p+r)$, the inductive hypothesis implies that y' can be lifted to an element $x' \in \pi_q Y(p+r+1)$. Subtracting the image of x' from x, we can reduce to the case $y' = 0$. Then $y \in \ker(\pi_q Y(p) \to \pi_q Y(p-1))$, and the desired result follows from $(*'')$. $\square$

Taking $k = p+1$ in Lemma 4, we deduce that the image of the map $\pi_* Y(p+r) \to \pi_* Y(p)$ is independent of r, so long as $r \geq p, s$. Let us denote this image by $A(p)_*$. By construction, we have a sequence of surjections

$$\cdots A(3)_* \to A(2)_* \to A(1)_* \to A(0)_*.$$

By construction, each of these surjections fits into a short exact sequence

$$0 \to E_\infty^{p,*} \to A(p)_* \to A(p-1)_* \to 0$$

By assumption, the groups $E_\infty^{p,*}$ vanish for $p \geq s$. We deduce:

$(*''')$ The maps $A(p)_* \to A(p')_*$ are isomorphisms for $p \geq p' \geq s$.

Let us now consider the tower of graded abelian groups

$$\cdots \to \pi_* Y(4s) \xrightarrow{\theta_2} \pi_* Y(2s) \xrightarrow{\theta_1} \pi_* Y(s).$$

For $m \geq 0$, let $K(m)_* \subseteq \pi_* Y(2^m s)$ be the kernel of the map θ_m. Note that $K(m)_* \cap A(2^m s)_* = 0$, since each θ_m induces an isomorphism $A(2^m s)_* \to A(2^{m-1}s)_*$. For any class $x \in \pi_* Y(2^m s)$, the image $\theta_m(x) \in A(2^{m-1}s)_*$, so that $\theta_m(x) = \theta_m(x')$ for some $x' \in A(2^m s)_*$. It follows that $x = x' + x''$, where $x' \in A(2^m s)_*$ and $x'' \in K(m)_*$. In other words, for $m \geq 1$ we have a direct sum decomposition

$$\pi_* Y(2^m s) \simeq A(2^m s)_* \oplus K(m)_*.$$

It follows that, as a pro-object in graded abelian groups, the tower $\{\pi_* Y(2^m s)\}$ is equivalent to the constant group $A(s)_*$.

Let $Y = \varprojlim Y(p) \simeq \varprojlim_m Y(2^m s)$. The Milnor exact sequence

$$0 \to \varprojlim{}^1 \pi_{*+1} Y(p) \to \pi_* Y \to \varprojlim \pi_* Y(p) \to 0$$

gives $\pi_* Y \simeq A(s)_*$. For each integer $p \geq 0$, let $Y(p)/Y$ denote the cofiber of the canonical map $Y \to Y(p)$. It follows that the maps $\pi_* Y(2^m s) \to \pi_* Y(2^m s)/Y$ induce a composite isomorphism

$$K(m)_* \subseteq \pi_* Y(2^m s) \to \pi_* Y(2^m s)/Y.$$

We conclude that the tower of spectra

$$\cdots \to Y(4s)/Y \to Y(2s)/Y \to Y(s)/Y$$

has the following property: each map in the tower is trivial on all homotopy groups.

Let us now return to the setting of Proposition 3: that is, we assume that the spectral sequence $\{E_r^{p,q}, d_r\}$ has vanishing $E_s^{p,q}$ for $p \geq s$ not only for the tower $\{Y(p)\}$, but also for $\{Y(p) \otimes F\}$ for every finite spectrum F. The same reasoning shows that the maps

$$\cdots \to (Y(4s)/Y)_* \to (Y(2s)/Y)_* F \to (Y(s)/Y)_* F$$

are zero. In other words, each of the maps $Y(2^m s)/Y \to Y(2^{m-1}s)/Y$ is a phantom.

Lemma 5. *A composition of two phantom maps is zero.*

Proof. Fix a spectrum X, and consider a map $u : \bigoplus F_\alpha \to X$, where the sum ranges over all homotopy equivalence classes of maps from finite spectra into X. Using the argument given in Lecture 17, we see that the homotopy fiber X' of u is equivalent to a retract of a sum of finite spectra. Now suppose we are given phantom maps $f : X \to Y$ and $g : Y \to Z$. Since f is a phantom, $f \circ u \simeq 0$ and therefore f is equivalent to a composition $X \to \Sigma X' \to Y$. Consequently, $g \circ f$ factors through the composition $\Sigma X' \xrightarrow{v} Y \xrightarrow{g} Z$. Since g is a phantom and $\Sigma X'$ is a retract of a sum of finite spectra, the composition $g \circ v$ is nullhomotopic and therefore $g \circ f \simeq 0$. □

Applying this to our situation, we deduce that the maps

$$\cdots \to Y(16s)/Y \to Y(4s)/Y \to Y(s)/Y$$

are nullhomotopic, so that the pro-spectrum $\{Y(p)/Y\}$ is trivial. This proves that the tower $\{Y(p)\}$ is equivalent (as a pro-spectrum) to the constant spectrum Y.

The Smash Product Theorem (Lecture 31)

April 22, 2010

In this lecture, we will apply the results of Lecture 30 to prove (part of) the smash product theorem. We begin by summarizing the results of the previous lecture. Fix a ring spectrum E (which we take to be a structured ring spectrum for simplicity). For every specturm X, we let $X^\bullet$ denote the cosimplicial spectrum $[n] \mapsto E^{\otimes n+1} \otimes X$.

Proposition 1. *Let E be a ring spectrum and X an arbitrary spectrum. Suppose that there exists an integer $s \geq 1$ such that, for every finite spectrum F, the E-based Adams spectral sequence $\{E_r^{p,q}, d_r\}$ for $X \otimes F$ has $E_s^{p,q} \simeq 0$ for $p \geq s$. Then the Adams tower $\{\mathrm{Tot}^n X^\bullet\}$ is equivalent, as a pro-object, to a constant tower.*

In the situation of Proposition 1, we have an equivalence of pro-spectra $\{\mathrm{Tot}^n X^\bullet\} \simeq X'$ for some spectrum X'. We saw in the last lecture that $X' \simeq \mathrm{Tot}(X^\bullet)$ can be identified with the localization $L_E X$. Note that if the hypotheses of Proposition 1 are satisfied, then for every other spectrum Y, the tower $\{\mathrm{Tot}^n (X \otimes Y)^\bullet\} \simeq \{\mathrm{Tot}^n X^\bullet \otimes Y\}$ is pro-equivalent to the constant tower with value $X' \otimes Y$. It follows that the canonical map

$$(L_E X) \otimes Y \to L_E(X \otimes Y)$$

is a homotopy equivalence.

Proposition 2. *Let E be a p-local ring spectrum, and suppose that there exists a finite p-local spectrum X of type 0 which satisfies the hypotheses of Proposition 1. Then L_E is a smashing localization.*

Proof. Let $\mathcal{T}$ be the collection of p-local finite spectra X such that, for every p-local spectrum Y, the canonical map $(L_E X) \otimes Y \to L_E(X \otimes Y)$ is an equivalence. It is clear that $\mathcal{T}$ is thick. If $\mathcal{T}$ contains a finite p-local spectrum of type 0, then the thick subcategory theorem implies that $\mathcal{T}$ contains every finite p-local spectrum; in particular, $S_{(p)} \in \mathcal{T}$ so that $L_E S_{(p)} \otimes Y \simeq L_E(Y)$ for all Y and therefore L_E is smashing. $\square$

Let us now specialize to the case where E is Morava E-theory $E(n)$. The spectrum $E(n)$ is complex-oriented, and the map $\mathrm{Spec}\, \pi_* E \to \mathcal{M}_{\mathrm{FG}} \times \mathrm{Spec}\, \mathbf{Z}_{(p)}$ is a faithfully flat cover of the open substack $\mathcal{M}_{\mathrm{FG}}^{\leq n}$ classifying formal groups of height $\leq n$. For every spectrum X, let $\mathcal{F}_X$ denote the associated quasi-coherent sheaf on $\mathcal{M}_{\mathrm{FG}}$. The E_2-term of the $E(n)$-based Adams-Novikov spectral sequence for X is given by the cohomology of the chain complex

$$E(n)_*(X) \to (E(n) \otimes E(n))_* X \to (E(n) \otimes E(n) \otimes E(n))_* X \to \cdots,$$

which computes the cohomology of $\mathcal{M}_{\mathrm{FG}}^{\leq n}$ with coefficients in the quasi-coherent sheaves $\mathcal{F}_{\Sigma^k X}$ for varying k. Consequently, we obtain the following:

Proposition 3. *Suppose that there exists a finite p-local spectrum X of type 0 and an integer $s_0 \geq 1$ with the following property: for every finite spectrum F, the cohomology groups $\mathrm{H}^s(\mathcal{M}_{\mathrm{FG}}^{\leq n}; \mathcal{F}_{X \otimes F})$ vanish for $s \geq s_0$. Then the localization $L_{E(n)}$ is smashing.*

To make things more concrete, let us assume that X is an *even* finite p-local spectrum: that is, a finite p-local spectrum whose homology groups $\mathrm{H}_*(X; \mathbf{Z}_{(p)})$ are free $\mathbf{Z}_{(p)}$-modules concentrated in even degrees. This is equivalent to saying that X admits a finite cell decomposition, where each cell is an even suspension of $S_{(p)}$. Such a spectrum is always of type 0, provided that it is nonzero. For such a spectrum, the Atiyah-Hirzebrucn spectral sequence for computing $(\mathrm{MU}_{(p)})_*(X)$ degenerates: that is, $\mathrm{MU}_{(p)} \otimes X$ is a free module over $\mathrm{MU}_{(p)}$ (on generators corresponding to some basis for $\mathrm{H}_*(X; \mathbf{Z}_{(p)})$). It follows that $\mathcal{F}_X$ is a vector bundle on $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$, and that for any other spectrum F we have

$$(\mathrm{MU}_p)_*(X \otimes F) \simeq \pi_*((\mathrm{MU}_{(p)} \otimes X) \otimes_{\mathrm{MU}_{(p)}} (\mathrm{MU}_{(p)} \otimes F)) \simeq (\mathrm{MU}_{(p)})_*(X) \otimes_{\mathrm{MU}_{(p)}} (\mathrm{MU}_{(p)})_* F.$$

On the moduli stack $\mathcal{M}_{\mathrm{FG}}$, we deduce that the canonical map of quasi-coherent sheaves

$$\mathcal{F}_X \otimes \mathcal{F}_F \to \mathcal{F}_{X \otimes F}$$

is an isomorphism after localization at p. We conclude the following:

Proposition 4. *Suppose that there exists a nonzero finite even p-local spectrum X and an integer s_0 with the following property: for every quasi-coherent sheaf $\mathcal{G}$ on $\mathcal{M}_{\mathrm{FG}}^{\leq n}$, the cohomology groups $\mathrm{H}^s(\mathcal{M}_{\mathrm{FG}}^{\leq n}; (\mathcal{F}_X \,|\, \mathcal{M}_{\mathrm{FG}}^{\leq n}) \otimes \mathcal{G})$ vanish for $s \geq s_0$. Then the localization functor $L_{E(n)}$ is smashing.*

We can attack this problem using the filtration of $\mathcal{M}_{\mathrm{FG}}^{\leq n}$ by height. Suppose we have chosen an even finite p-local spectrum X. For each $k \leq n$, let $\mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}$ denote the closed substack of $\mathcal{M}_{\mathrm{FG}}^{\leq n}$ classifying formal groups which have height $\geq k$. Let us attempt to prove that, for every quasi-coherent sheaf $\mathcal{G}$ on $\mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}$, the groups $\mathrm{H}^s(\mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}; (\mathcal{F}_X \,|\, \mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}) \otimes \mathcal{G})$ vanish for large s. The idea is to use descending induction on k. Note that $\mathcal{M}_{\mathrm{FG}}^{\geq k+1, \leq n}$ can be regarded as a closed substack of $\mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}$: it is the zero locus of v_k, which we regard as a section of $\omega^{p^k - 1}$ (here ω is the line bundle on $\mathcal{M}_{\mathrm{FG}}$ given by assigning to each formal group the dual of its Lie algebra). In particular, multiplication by v_n induces a map of sheaves

$$\mathcal{G} \to \mathcal{G} \otimes \omega^{p^k - 1}$$

whose kernel and cokernel are supported on the closed substack $\mathcal{M}_{\mathrm{FG}}^{\geq k+1, \leq n}$. We may therefore assume, by our inductive hypothesis, that v_n induces an isomorphism

$$\mathrm{H}^s(\mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}; (\mathcal{F}_X \,|\, \mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}) \otimes \mathcal{G}) \to \mathrm{H}^s(\mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}; (\mathcal{F}_X \,|\, \mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}) \otimes \mathcal{G} \otimes \omega^{p^k - 1})$$

for sufficiently large s. It follows that for large s, we have an isomorphism

$$\mathrm{H}^s(\mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}; (\mathcal{F}_X \,|\, \mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}) \otimes \mathcal{G}) \simeq \varinjlim_m \mathrm{H}^s(\mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}; (\mathcal{F}_X \,|\, \mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}) \otimes \mathcal{G} \otimes \omega^{(p^k - 1)m});$$

here the latter group can be identified with the cohomology of $\mathcal{F}_X \otimes \mathcal{G}$ on the open substack of $\mathcal{M}_{\mathrm{FG}}^{\geq k, \leq n}$ complementary to $\mathcal{M}_{\mathrm{FG}}^{\geq k+1, \leq n}$: this is the moduli stack of formal groups of height exactly k. We are therefore reduced to the following:

Proposition 5. *Suppose that there exists a nonzero finite even p-local spectrum X and an integer s_0 with the following property: for $0 \leq k \leq n$ and every quasi-coherent sheaf $\mathcal{G}$ on $\mathcal{M}_{\mathrm{FG}}^k$, the cohomology groups $\mathrm{H}^s(\mathcal{M}_{\mathrm{FG}}^k; (\mathcal{F}_X \,|\, \mathcal{M}_{\mathrm{FG}}^k) \otimes \mathcal{G})$ vanish for $s \geq s_0$.*

The vanishing condition appearing in Proposition 6 is automatic when $k = 0$, since quasi-coherent sheaves on $\mathcal{M}_{\mathrm{FG}}^0 \simeq B\mathbb{G}_m$ have no higher cohomology. Assume that $k > 0$, and choose a formal group law $f(x, y)$ of height k over $\overline{\mathbf{F}}_p$. Let G_k denote the automorphism group of f (as a formal group law over $\overline{\mathbf{F}}_p$), regarded as a profinite group. We have a pullback diagram of algebraic stacks

$$\begin{tikzcd}
BG_k \times \operatorname{Spec} \overline{\mathbf{F}}_p \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{FG}}^k \arrow[d] \\
\operatorname{Spec} \overline{\mathbf{F}}_p \arrow[r] & \operatorname{Spec} \mathbf{F}_p.
\end{tikzcd}$$

This implies:

(a) Every quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{M}^k_{\mathrm{FG}}$ determines an $\overline{\mathbf{F}}_p$-vector space V equipped with a continuous action of G_k.

(b) We have a canonical isomorphism $\mathrm{H}^*(\mathcal{M}^k_{\mathrm{FG}}; \mathcal{F}) \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p \simeq \mathrm{H}^*(G_k; V)$.

Consequently, we are reduced to proving the following:

Proposition 6. *Suppose that there exists a nonzero finite even p-local spectrum X and an integer s_0 with the following property: for $1 \leq k \leq n$, if we let V denote the representation of the profinite group G_k associated to $\mathcal{F}_X$, then $\mathrm{H}^s(G_k; V \otimes W) \simeq 0$ for $s \geq s_0$ and any continuous representation W of G_k. Then $L_{E(n)}$ is a smashing localization.*

Let us now indicate briefly why it is plausible that the hypothesis of Proposition 6 should be satisfied. Fix $1 \leq k \leq n$. Recall that G_k can be described as the group of units in the ring $\mathrm{End}(f)$, which is a noncommutative valuation ring of rank k^2 over $\mathbf{Z}_p$. In particular, it is a p-adic Lie group. Consequently, on a sufficiently small open subgroup of G_k, the group structure on G_k closely approximates the (commutative) group structure on $\mathbf{Z}_p^{k^2}$. If M is a discrete p-torsion module over $\mathbf{Z}_p^{k^2}$, then the profinite group cohomology $\mathbf{Z}_p^{k^2}$ with coefficients in M agrees with the ordinary group cohomology of $\mathbf{Z}^{k^2}$ with coefficients in M, and therefore vanishes in degrees larger than k^2. Using this, Lazard shows that there is an open subgroup $U \subseteq G_k$ such that $\mathrm{H}^s(U; M)$ vanishes for $s > k^2$ and any G_k-module M. The same result does not necessarily hold when $U = G_k$. However, an argument of Serre shows that if G_k is not of finite cohomological dimension, then it must contain an element of order p. However, elements of order p are well-understood: these are pth roots of unity in the division algebra $D = \mathrm{End}(f)[p^{-1}]$, and they exist only when the rank k of D is divisible by the degree $(p-1)$ of the field extension $\mathbf{Q}_p(\zeta_p)$. In particular, if $p > n+1$, then the profinite groups $\{G_k\}_{1 \leq k \leq n}$ have finite cohomological dimension and the hypotheses of Proposition 6 are satisfied for $X = S_{(p)}$.

When $p \leq n$, we need to work harder. In this case, some of the groups G_k do contain elements of order p. However, each G_k contains at most a single conjugacy class of subgroups V having order p: this follows from the Skolem-Noether theorem. In this case, the subgroup V can be regarded as the "obstruction" to G_k being of finite cohomological dimension: one can show that the cohomology groups $\mathrm{H}^*(G_k; M)$ are bounded if the subgroup V acts freely on M. It therefore suffices to choose a spectrum X such that the associated representation V of G_k is free over V. This requires some representation-theoretic constructions which we will not pursue further.

The Chromatic Convergence Theorem (Lecture 32)

April 20, 2010

Fix a prime number p. For any p-local spectrum X, one can arrange its $E(n)$-localizations into the *chromatic tower*

$$\cdots \to L_{E(2)}X \to L_{E(1)}X \to L_{E(0)}X.$$

Our goal in this lecture and the next is to prove the following result:

Theorem 1 (Chromatic Convergence)**.** *If X is a finite p-local spectrum, then X is a homotopy limit of its chromatic tower.*

Remark 2. The collection of p-local spectra which satisfy the conclusion of Theorem 1 is obviously thick. It therefore suffices to prove Theorem 1 for a single p-local spectrum of type 0: for example, the p-local sphere).

For every spectrum X, let $C_n(X)$ denote the homotopy fiber of the map $X \to L_{E(n)}X$. Then $\varprojlim C_n(X)$ is the homotopy fiber of the map $X \to \varprojlim L_{E(n)}X$. The chromatic convergence theorem is therefore equivalent to the following:

Theorem 3. *The homotopy limit of the tower $\{C_n(S_{(p)})\}$ is trivial. Even better: for every integer m, the tower of abelian groups $\{\pi_m C_n(S_{(p)})\}$ is trivial (as a pro-abelian group).*

The starting point for Theorem 3 is the following result, which we will prove in the next lecture:

Proposition 4. *Each of the maps $C_n(S_{(p)}) \to C_{n-1}(S_{(p)})$ induces the zero map $\mathrm{MU}_*(C_n(S_{(p)})) \to \mathrm{MU}_*(C_{n-1}(S_{(p)})$.*

Let us assume Proposition 4 and see how it leads to a proof of Theorem 3. To this end, we recall the definition of the *Adams-Novikov filtration* on the homotopy groups $\pi_* X$ of a spectrum X. Let I denote the homotopy fiber of the unit map $S \to \mathrm{MU}$. There is an evident map $I \to S$, which induces a map $I^{\otimes m} \to S$ for each m. We say that an element $x \in \pi_n X$ has *Adams-Novikov filtration* $\geq m$ if x lies in the image of the map $\pi_n(I^{\otimes m} \otimes X) \to \pi_n X$.

Lemma 5. *Let $f : X \to Y$ be a map of spectra such that f induces the zero map $\theta : \mathrm{MU}_*(X) \to \mathrm{MU}_*(Y)$. Then f increases Adams-Novikov filtration. That is, if $x \in \pi_n X$ has Adams-Novikov filtration $\geq m$, then $f(x) \in \pi_n Y$ has Adams-Novikov filtration $\geq m+1$.*

Proof. Lift x to a class $\overline{x} \in \pi_n(I^{\otimes m} \otimes X)$. We then obtain $f(\overline{x}) \in \pi_n(I^{\otimes m} \otimes Y)$ lifting y. To lift y to $\pi_n(I^{\otimes m+1} \otimes Y)$, it suffices to show that the image of $\overline{y}$ vanishes in $I^{\otimes m} \otimes Y \otimes \mathrm{MU}$. Consequently, it will suffice to show that f induces the zero map

$$\theta_m : \mathrm{MU}_*(I^{\otimes m} \otimes X) \to \mathrm{MU}_*(I^{\otimes m} \otimes Y).$$

Recall that $\mathrm{MU}_*(\mathrm{MU}) \simeq (\pi_* \mathrm{MU})[b_1, b_2, \ldots]$ is a free $\pi_* \mathrm{MU}$-module on a basis consisting of monomials in the b_i. It follows that $\mathrm{MU}_*(\Sigma I)$ is a free $\pi_* \mathrm{MU}$-module on a basis consisting of monomials of positive length in the b_i. In particular, $\mathrm{MU} \otimes I$ is a free module over MU, so we have Kunneth decompositions

$$\mathrm{MU}_*(I^{\otimes m} \otimes X) = \mathrm{MU}_*(I)^{\otimes m} \otimes_{\pi_* \mathrm{MU}} \mathrm{MU}_*(X)$$

$$\mathrm{MU}_*(I^{\otimes m} \otimes Y) = \mathrm{MU}_*(I)^{\otimes m} \otimes_{\pi_* \mathrm{MU}} \mathrm{MU}_*(Y)$$

Since $\theta = 0$, it follows that $\theta_m = 0$. □

Combining Lemma 5 with Proposition 4, we deduce:

Proposition 6. *For all m, n, and s, the image of the map*

$$\pi_n C_{m+s} S_{(p)} \to \pi_n C_m S_{(p)}$$

consists of elements having Adams-Novikov filtration $\geq s$.

To complete the proof of Theorem 3, it will suffice to show the following:

Proposition 7. *For every pair of integers m and n, the Adams-Novikov filtration on $\pi_n C_m(S_{(p)})$ is finite. That is, there exists an integer s such that every element $x \in \pi_n C_m(S_{(p)})$ of Adams-Novikov filtration $\geq s$ is trivial.*

Let us now introduce some terminology which will be useful for proving Proposition 7.

Definition 8. Let $f : X \to Y$ be a map of spectra. We say that f is *phantom below dimension n* if the following condition is satisfied: for every finite spectrum F of dimension $\leq n$ and every map $u : F \to X$, the composition $f \circ u$ is nullhomotopic.

Remark 9. The map f is phantom if and only if it is phantom below dimension n, for every integer n.

Definition 10. A spectrum X is MU-*convergent* if, for every integer n, there exists s such that the map $I^{\otimes s} \otimes X \to X$ is phantom below dimension n.

If X is MU-convergent and n, s are as in Definition 10, then the map $I^{\otimes s} \otimes X \to X$ is trivial on π_n and so every element of $\pi_n X$ having Adams-Novikov filtration $\geq s$ is zero. Proposition 7 is therefore a consequence of the following:

Proposition 11. *Let X be any connective spectrum. Then $C_m(X)$ is* MU-*convergent for each $m \geq 0$.*

We need a few preliminary observations.

Lemma 12. *Let $f : X \to Y$ phantom below dimension n, and let W be a connective spectrum. Then the induced map $X \otimes W \to Y \otimes W$ is phantom below dimension n.*

Proof. Let F be a finite spectrum of dimension $\leq n$ and consider a map $u : F \to X \otimes W$. We wish to prove that $(f \otimes \mathrm{id}_W) \circ u$ is nullhomotopic. We can write W as a filtered colimit of finite connective spectra W_α. Since F is finite, u factors through $X \otimes W_\alpha$ for some α. Replacing W by W_α, we may assume that W is finite. In this case, we can identify u with a map $v : DW \otimes F \to X$. Since W is connective, $DW \otimes F$ has dimension $\leq n$; it follows that $f \circ v$ is nullhomotopic so that $(f \otimes \mathrm{id}_W) \circ u$ is nullhomotopic. □

Lemma 13. *Suppose we are given a fiber sequence of spectra*

$$X \to Y \to Z.$$

If X and Z are MU-*convergent, then Y is* MU-*convergent.*

Proof. Fix an integer n, and choose s such that the maps $I^{\otimes s} \otimes X \to X$ and $K^{\otimes s} \otimes Z \to Z$ are phantom below n. We will show that the map $I^{\otimes 2s} \otimes Y \to Y$ is phantom below n. Let F be a finite spectrum of dimension $\leq n$ with a map $u : F \to I^{\otimes 2s} \otimes Y$. Since $I^{\otimes 2s} \otimes Z \to I^{\otimes s} \otimes Z$ is phantom below n (Lemma 12), the composite map

$$F \to I^{\otimes 2s} \otimes Y \to I^{\otimes 2s} \otimes Z \to I^{\otimes s} \otimes Z$$

is nullhomotopic. It follows that the composition

$$F \otimes I^{\otimes 2s} \otimes Y \to I^{\otimes s} \otimes Y$$

factors through some map $v : F \to I^{\otimes s} \otimes X$. Then the composition

$$F \xrightarrow{u} I^{\otimes 2s} \otimes Y \to Y$$

is given by

$$F \xrightarrow{v} I^{\otimes s} \otimes X \to X \to Y$$

and is therefore nullhomotopic. □

Lemma 14. *Let X be an* MU*-module spectrum. Then X is* MU*-convergent.*

Proof. The unit map $X \to \mathrm{MU} \otimes X$ admits a section, given by the action of $\mathrm{MU}_{(p)}$ on X. This is equivalent to the statement that the map $I \otimes X \to X$ is nullhomotopic (and hence phantom below n, for any n). □

Lemma 15. *Let X be any spectrum. For each $n \geq 0$, the spectrum $L_{E(n)}X$ is* MU*-convergent.*

Proof. Let $X^\bullet = E(n)^{\otimes(\bullet+1)} \otimes X$ and let $\{\mathrm{Tot}^m X^\bullet\}$ be the $E(n)$-based Adams tower of X. The proof of the smash product theorem shows that $\{\mathrm{Tot}^m X^\bullet\}$ is equivalent to the constant tower with value $L_{E(n)}X$. It follows that $L_{E(n)}X$ is a retract of $\mathrm{Tot}^m X^\bullet$ for some m. It therefore suffices to show that each $\mathrm{Tot}^m X^\bullet$ is MU-convergent. Each $\mathrm{Tot}^m X^\bullet$ is a finite homotopy inverse limit of the spectra X^k; by Lemma 13 it suffices to show that each X^k is MU-convergent. But $X^k \simeq E(n)^{\otimes k+1} \otimes X$ has the structure of an $E(n)$-module spectrum. Since $E(n)$ is complex orientable, there is a map of ring spectra $\mathrm{MU} \to E(n)$ so that X^k admits an MU-module structure; the desired result now follows from Lemma 14. □

Lemma 16. *Let X be a connective spectrum. Then X is* MU*-convergent.*

Proof. We claim that for any finite CW complex F of dimension $\leq n$ and any map $u : F \to I^{\otimes n+1} \otimes X$, the composite map $u : F \to I^{\otimes n+1} \otimes X \to X$ is nullhomotopic. In fact, u itself is nullhomotopic, because $I^{\otimes n+1} \otimes X$ is n-connected. To check this, we note that since X is connective it suffices to show that K is connected: that is, we have $\pi_i K \simeq 0$ for $i \leq 0$. This follows from the long exact sequence associated to the fiber sequence

$$I \to S \to \mathrm{MU},$$

since the map $\pi_i S \to \mathrm{MU}$ is bijective for $i \leq 0$ and surjective when $i = 1$. □

Proof of Proposition 11. Let X be a connective spectrum. We have a fiber sequence

$$C_n(X) \to X \to L_{E(n)}X$$

where X is MU-convergent by Lemma 16 and $L_{E(n)}(X)$ is MU-convergent by Lemma 15. It follows from Lemma 13 that $C_n(X)$ is MU-convergent. □

Complex Bordism and $E(n)$-Localization (Lecture 33)

April 22, 2010

Our first goal in this lecture is to complete the proof of chromatic convergence theorem by verifying the following:

Proposition 1. *For each n, let $C_n(S_{(p)})$ denote the homotopy fiber of the localization map $S_{(p)} \to L_{E(n)} S_{(p)}$. Then the maps $\mathrm{MU}_* C_n(S_{(p)}) \to \mathrm{MU}_* C_{n-1}(S_{(p)})$ are equal to zero.*

We will prove this result by explicitly computing the complex bordism of each $C_n S_{(p)}$. First, let us establish a bit of notation. Let $L = \pi_* \mathrm{MU}$ be the Lazard ring. For each $n \geq 0$, let $I(n)$ denote the ideal generated by $(v_0, v_1, \ldots, v_{n-1}, v_n)$ in L. We will say that an L-module M is *$I(n)$-torsion* if every every element $x \in M$ is annihilated by some power of the ideal $I(n)$. The basis for our calculation is the following.

Proposition 2. *Let X be an MU-module spectrum whose homotopy groups $\pi_* X$ are an $I(n-1)$-torsion module, and let $n > 0$. Let $X[v_n^{-1}]$ be the spectrum obtained by inverting $v_n \in \pi_{2(p^n-1)} \mathrm{MU}$. Then the map $X \to X[v_n^{-1}]$ exhibits $X[v_n^{-1}]$ as an $E(n)$-localization of X.*

Lemma 3. *Let Y be an MU-module spectrum such that $\pi_* Y$ is an $I(n)$-torsion L-module. Then Y is $E(n)$-acyclic.*

Proof. We have $\mathrm{MU}_* Y \simeq (\mathrm{MU}_* \mathrm{MU}) \otimes_{\pi_* \mathrm{MU}} \pi_* Y$. Note that the two maps $\phi_1, \phi_2 : L \to \mathrm{MU}_* \mathrm{MU}$ carry $I(n)$ to the same ideal, since the condition that a formal group be of height $> n$ does not depend on the choice of coordinate. It follows that $\mathrm{MU}_* Y$ is an $I(n)$-torsion L-module. Since $E(n)$ is Landweber exact, we get $E(n)_* Y \simeq \pi_* E(n) \otimes_L \mathrm{MU}_* Y$. Since $I(n)$ generates the unit ideal in $E(n)$ (the formal group associated to $E(n)$ has height $\leq n$), we conclude that every element of $E(n)_* Y$ is generated by a power of the unit ideal in $\pi_* E(n)$: that is, $E(n)_* Y \simeq 0$. $\square$

Proof of Proposition . We must show two things:

(1) The spectrum $X[v_n^{-1}]$ is $E(n)$-local.

(2) The map $X \to X[v_n^{-1}]$ is an equivalence in $E(n)$-homology.

To prove (1), we observe that $X[v_n^{-1}]$ is a module spectrum for $\mathrm{MU}_{(p)}[v_n^{-1}]$, and therefore $E(n)$-local since $E(n)$ is Bousfield equivalent to $\mathrm{MU}_{(p)}[v_n^{-1}]$.

To prove (2), it suffices to show that the homotopy fiber of the map $X \to X[v_n^{-1}]$ is $E(n)$-acyclic. This homotopy fiber is a filtered colimit of the cofibers of maps

$$X \xrightarrow{v_n^k} \Sigma^{-2k(p^n-1)} X.$$

It therefore suffices to show that the homotopy fibers of each of these maps is $E(n)$-acyclic. Denote such a homotopy fiber by Y; then Y is an MU-module such that $\pi_* Y$ is $I(n)$-torsion, so that Y is $E(n)$-acyclic by Lemma 3. $\square$

Proposition 4. *For each $n \geq 0$, let M_n be the L-module given by the quotient of $L_{(p)}[v_0^{-1}, \ldots, v_n^{-1}]$ by the submodules $\{L_{(p)}[v_0^{-1}, \ldots, v_{i-1}^{-1}, v_{i+1}^{-1}, \ldots, v_n^{-1}]\}_{0 \leq i \leq n}$. Then there are canonical isomorphisms $M(n) \simeq \mathrm{MU}_* C_n S_{(p)}$.*

Remark 5. The L-modules $M(n)$ can be described by recursion: we have $M(-1) \simeq L_{(p)}$, and for $n \geq 0$ there is an isomorphism $M(n) \simeq M(n-1)[v_n^{-1}]/M(n-1)$.

Proof. We use induction on n, beginning with the case $n = 0$. Note that $C_0 S_{(p)}$ is the fiber of the map $S_{(p)} \to L_{E(0)} S_{(p)} = S_{\mathbf{Q}}$. It follows that $\mathrm{MU} \otimes C_0 S_{(p)}$ is the fiber of the map $\mathrm{MU}_{(p)} \to \mathrm{MU}_{\mathbf{Q}}$. This map is injective on homotopy, so we get a short exact sequence

$$0 \to \pi_* \mathrm{MU}_{(p)} \to \pi_* \mathrm{MU}_{\mathbf{Q}} \to \mathrm{MU}_* C_0 S_{(p)} \to 0,$$

giving the isomorphism $\mathrm{MU}_* C_0 S_{(p)} \simeq L_{(p)}[p^{-1}]/L_{(p)}$.

The general case is similar. We have a fiber sequence

$$C_{n-1} S_{(p)} \to S_{(p)} \to L_{E(n-1)} S_{(p)}.$$

Note that $L_{E(n-1)} S_{(p)}$ is already $E(n)$-local, so that $C_n(L_{E(n-1)} S_{(p)}) \simeq 0$. Applying C_n, we deduce that the map $C_n C_{n-1} S_{(p)} \to C_n S_{(p)}$ is an equivalence. In other words, we have a fiber sequence

$$C_n S_{(p)} \to C_{n-1} S_{(p)} \to L_{E(n)} C_{n-1} S_{(p)}.$$

The inductive hypothesis implies that $\mathrm{MU}_*(C_{n-1} S_{(p)})$ is an $I(n-1)$-torsion L-module. It follows from Proposition that

$$\mathrm{MU}_* L_{E(n)} C_{n-1} S_{(p)} \simeq MU_*(C_{n-1} S_{(p)})[v_n^{-1}] \simeq M(n-1)[v_n^{-1}].$$

We observe that the map $M(n-1) \to M(n-1)[v_n^{-1}]$ is injective. This implies that the map $\mathrm{MU}_* C_n S_{(p)} \to \mathrm{MU}_* C_{n-1} S_{(p)}$ is zero (thereby proving Theorem 1), and shows that we have a short exact sequence

$$0 \to M(n-1) \to M(n-1)[v_n^{-1}] \to \mathrm{MU}_* C_n S_{(p)} \to 0,$$

giving the isomorphism $\mathrm{MU}_* C_n S_{(p)} \simeq M(n)$. ☐

Proposition can also be used to get a bound on the discrepancy between $L_{E(n)}$ and the telescopic localization functor L_n^t.

Proposition 6. *Let X be any spectrum. Then the canonical map $L_n^t X \to L_{E(n)} X$ induces an isomorphism after smash product with* MU.

Proof. We work by induction on n. We wish to prove that the map

$$\mathrm{MU} \otimes L_n^t X \to \mathrm{MU} \otimes L_{E(n)} X \simeq L_{E(n)}(\mathrm{MU} \otimes X)$$

is an equivalence: that is, the map $\phi : \mathrm{MU} \otimes X \to \mathrm{MU} \otimes L_n^t X$ exhibits $\mathrm{MU} \otimes L_n^t X$ as an $E(n)$-localization of X. Since ϕ is obviously an $E(n)$-equivalence, it suffices to show that $\mathrm{MU} \otimes L_n^t X$ is $E(n)$-local. We have a cofiber sequence

$$Y \to X \to L_{n-1}^t X.$$

The inductive hypothesis implies that $L_n^t(\mathrm{MU} \otimes L_{n-1}^t X) \simeq \mathrm{MU} \otimes L_{n-1}^t X$ is $E(n-1)$-local, and therefore $E(n)$-local. It therefore suffices to show that $\mathrm{MU} \otimes L_n^t Y$ is $E(n)$-local By construction, Y is a direct limit of finite p-local spectra Y_α of type $\geq n$; since $L_{E(n)}$ is smashing, it suffices to show that each $\mathrm{MU} \otimes L_n^t Y_\alpha$ is MU-local. Since Y_α has type $\geq n$, $\mathrm{MU}_* Y_\alpha$ is an $I(n-1)$-torsion L-module, so that $\mathrm{MU}_* L_{E(n)} Y_\alpha \simeq (\mathrm{MU}_* Y_\alpha)[v_n^{-1}]$ by Proposition . On the other hand, we have seen that $L_n^t Y_\alpha \simeq Y_\alpha[f^{-1}]$, where f^{-1} is a v_n-self map of Y. To conclude that $\mathrm{MU}_* L_n^t Y_\alpha \simeq (\mathrm{MU}_* Y_\alpha)[v_n^{-1}]$, it suffices to prove the following:

Lemma 7. *Let Z be a finite p-local spectrum of type $\geq n$, and let $f : \Sigma^k Z \to Z$ be a v_n-self map of Z. Then, replacing f by a suitable power, we may assume that f induces the map v_n^i on $\mathrm{MU}_* Z$ (for some i).*

Proof. Let $R = Z \otimes DZ$, and regard f as an element of $\pi_* R$. Raising f to a suitable power, we may assume that $f \mapsto 0 \in K(m)_* R$ for $m \neq n$ and $f \mapsto v_n^i \in K(n)_* R$. We claim that $f^{p^k} = v_n^{ip^k}$ in $\mathrm{MU}_*(R)$ for $k \gg 0$. Since v_n^i and f commute in $\mathrm{MU}_* R$ (v_n being central) and the difference $v_n^i - f$ is p-power torsion (since $\pi_* R$ is p-power torsion), it suffices to show that $v_n^i - f$ is nilpotent. By the nilpotence theorem, it suffices to show that the image of $(v_n^i - f) \mapsto 0 \in K(m)_*(\mathrm{MU} \otimes R)$ for all m. This is clear for $m < n$ (since $\mathrm{MU} \otimes R$ is $K(m)$-acyclic). For $m \geq n$, we note that $v_n \in \pi_* \mathrm{MU}$ maps to 0 in $K(m)_* \mathrm{MU}$ for $m > n$ (since the formal group law of $K(m) \otimes \mathrm{MU}$ has height $> n$), so the statement holds since $f \mapsto 0 \in K(m)_* R$. In the case $m = n$, we are reduced to proving that the two images of v_n in $K(m)_* \mathrm{MU}$ coincide. This is clear: since $K(n) \otimes \mathrm{MU}$ is a cohomology theory with two complex orientations, the associated formal group laws (each of which has height $\geq n$) differ by a change of coordinates of the form $f(t) = t + b_1 t^2 + b_2 t^3 + \cdots$, so that the first nonvanishing coefficient of the p-series $[p](t)$ are the same. $\square$

$\square$

Monochromatic Layers (Lecture 34)

April 27, 2010

Fix a prime number p. To any spectrum X, we can associate its chromatic tower

$$\cdots \to L_{E(2)}X \to L_{E(1)}X \to L_{E(0)}X.$$

If X is a finite p-local spectrum, then the chromatic convergence theorem tells us that the homotopy limit of this tower is X. In particular, we can associate to X the *chromatic spectral sequence* $\{E_r^{p,q}, d_r\}$, where $E_1^{p,*}$ is given by the homotopy groups of the homotopy fiber of the map $L_{E(p)}X \to L_{E(p-1)}X$. (In fact, the proof of the chromatic convergence theorem tells us that this spectral sequence converges in a strong sense: for example, the chromatic filtration on each homotopy group $\pi_n X$ is finite). This motivates the following:

Definition 1. For each spectrum X, we let $M_n(X)$ denote the homotopy fiber of the map $L_{E(n)}X \to L_{E(n-1)}X$. We will refer to $M_n(X)$ as the *nth monochromatic layer of* X.

The essential features of $M_n(X)$ are captured by the following definition:

Definition 2. A spectrum X is *monochromatic of height* n if it is $E(n)$-local and $E(n-1)$-acyclic.

Remark 3. For any spectrum X, we have a map of $E(n)$-local spectra $L_{E(n)}X \to L_{E(n-1)}X$ which induces an isomorphism on $E(n-1)$-homology. It follows that the fiber $M_n(X)$ is monochromatic of height n. Conversely, if X is monochromatic of height n, then $L_{E(n)}X \simeq X$ and $L_{E(n-1)}X \simeq 0$, so that $X \simeq M_n(X)$.

Example 4. Let X be a finite p-local spectrum of type $\geq n$. Then $L_{E(n)}X$ is monochromatic of height n. To see this, it suffices to observe that $E(n-1)_* L_{E(n)}X \simeq E(n-1)_* X \simeq 0$.

Notation 5. Let $\mathcal{M}_n$ denote the collection of all spectra which are monochromatic of height n. Since $L_{E(n)}$ is a smashing localization, we see that $\mathcal{M}_n$ is closed under homotopy colimits. We say that that an object $X \in \mathcal{M}_n$ is *compact* if, for every filtered diagram $\{Y_\alpha\}$ of objects of $\mathcal{M}_n$, the induced map

$$\varinjlim \mathrm{Map}(X, Y_\alpha) \to \mathrm{Map}(X, \varinjlim Y_\alpha)$$

is a homotopy equivalence.

Example 6. Let X be a finite p-local spectrum of type $\geq n$. Then $L_{E(n)}X$ is a compact object of $\mathcal{M}_n$. To see this, we note that if $\{Y_\alpha\}$ is a filtered diagram in $\mathcal{M}_n$, then we have

$$\mathrm{Map}(L_{E(n)}X, \varinjlim Y_\alpha) \simeq \mathrm{Map}(X, \varinjlim Y_\alpha) \simeq \varinjlim \mathrm{Map}(X, Y_\alpha) \simeq \varinjlim \mathrm{Map}(L_{E(n)}X, Y_\alpha).$$

Our next goal is to establish a converse to Example 6. The essential observation is the following:

Proposition 7. *Let X be a spectrum which is monochromatic of height n. Then X can be written as a filtered colimit $\varinjlim X_\alpha$, where each X_α is the $E(n)$-localization of a finite spectrum of type $\geq n$.*

Proof. We have a cofiber sequence

$$X' \to X \to L_{n-1}^t X,$$

where X' is a filtered colimit of p-local finite spectra of type $\geq n$. This induces a cofiber sequence

$$L_{E(n)} X' \to L_{E(n)} X \to L_{E(n)} L_{n-1}^t X.$$

Since $X \in \mathcal{M}_n$ we have $L_{E(n)} X \simeq 0$, and since $L_{E(n)}$ is smashing we conclude that $L_{E(n)} X'$ is a filtered colimit of $E(n)$-localizations of finite p-local spectra of type $\geq n$. It will therefore suffice to show that $L_{E(n)} L_{n-1}^t X \simeq 0$; that is, that $L_{n-1}^t X$ is $E(n)$-acyclic. Since $E(n)$ is Landweber exact, it will suffice to show that $L_{n-1}^t X$ is MU-acyclic. In the last lecture, we saw that

$$\mathrm{MU}_* \, L_{n-1}^t X \simeq \mathrm{MU}_* \, L_{E(n-1)} X,$$

and the right hand side vanishes since X is assumed to be $E(n-1)$-acyclic. □

Corollary 8. *An object $X \in \mathcal{M}_n$ is compact if and only if it is a retract of $L_{E(n)} Y$ for some finite spectrum Y of type $\geq n$.*

Proof. Write X as a filtered colimit of spectra X_α of the form $L_{E(n)} Y_\alpha$. Since X is compact, the identity map $X \to \varinjlim X_\alpha$ factors through some X_α, so that X is a retract of $L_{E(n)} Y_\alpha$. □

Corollary 9. *The homotopy theory $\mathcal{M}_n$ is compactly generated: that is, every object of $\mathcal{M}_n$ can be obtained as a filtered colimit of compact objects of $\mathcal{M}_n$.*

We want to draw attention to a crucial features of the compact objects of $\mathcal{M}_n$. First, we state a slightly stronger version of the periodicity theorem of Lecture 27:

Theorem 10. *Let X be a finite p-local spectrum of type $\geq n$. Then there exists a v_n-self map $f : \Sigma^k X \to X$ where $k = 2(p^n - 1)p^N$ for $N \gg 0$, which acts by multiplication by $v_n^{p^N}$ on $K(n)_* X$.*

Corollary 11. *Let X be a compact object of $\mathcal{M}_n$. Then X is periodic. More precisely, for $N \gg 0$, there is a homotopy equivalence $X \simeq \Sigma^{2p^N(p^n-1)} X$.*

Proof. According to Corollary 8, we can assume that X is a retract of $L_{E(n)} Y$ for some finite p-local spectrum Y of type $\geq n$. Let $f : \Sigma^k Y \to Y$ be the v_n-self map of Theorem 10, where $k = 2p^N(p^n - 1)$. Then the action of f on $K(n)_* L_{E(n)} Y \simeq K(n)_* Y$ is given by $v_n^{p^N}$. It follows that the composite map

$$f' : \Sigma^k X \to \Sigma^k L_{E(n)} Y \xrightarrow{f} L_{E(n)} Y \to X$$

induces multiplication by $v_n^{p^N}$ on $K(n)_* X$; in particular, it is bijective. Since f' is also bijective on $K(m)_* X$ for $m < n$ (since these groups vanish), we conclude that the homotopy fiber of f' is $K(m)$-acyclic for $m \leq n$ and therefore $E(n)$-acyclic. Since X is $E(n)$-local, the homotopy fiber of f' is also $E(n)$-local and therefore trivial; this proves that f' is an equivalence $\Sigma^k X \simeq X$. □

If X is a general monochromatic spectrum of height n, then X is a filtered colimit of compact objects X_α, each of which is periodic of some period $2(p^n - 1)p^{N_\alpha}$. The exponent N_α generally depends on α, so that X itself is not periodic. Nevertheless, elements of the homotopy of X are organized into "periodic families": that is, any class $x \in \pi_k X$ is given by an element in some $\pi_k X_\alpha$, which in turn determines elements of $\pi_{k+2m(p^n-1)p^{N_\alpha}} X$ for all $m \in \mathbf{Z}$. This is the motivation for the term "chromatic homotopy theory": the chromatic tower of a spectrum X is like a prism, which separates X into "monochromatic layers" $M_n(X)$ each of which exhibit a sort of generalized $2(p^n - 1)$-fold periodicity.

We conclude with a few remarks relating the monochromatic category $\mathcal{M}_n$ with the $K(n)$-local homotopy category.

Proposition 12. *The constructions*

$$X \mapsto L_{K(n)}X$$
$$Y \mapsto M_n(Y)$$

determine mutually inverse equivalences between the homotopy category of monochromatic spectra of height n and the homotopy category of $K(n)$-local spectra.

We first recall a fact we proved earlier:

Lemma 13. *Let X be an $E(n-1)$-local spectrum. Then $K(n)_*X \simeq 0$.*

Proof. Since $L_{E(n-1)}$ is smashing, $K(n) \otimes X$ is $E(n-1)$-local. It will therefore suffice to show that $K(n) \otimes X$ is $E(n-1)$-acyclic; that is, that $E(n-1) \otimes K(n) \otimes X \simeq 0$. This is clear, since $E(n-1) \otimes K(n)$ is a complex orientable spectrum whose formal group has height $< n$ and exactly n, and therefore $E(n-1) \otimes K(n) \simeq 0$. □

Proof of Proposition 12. We argue that both composite functors are the identity. First, fix a monochromatic spectrum X of height n. We wish to show that $X \simeq M_n(L_{K(n)}X)$. Since $L_{K(n)}X$ is $K(n)$-local, it is $E(n)$-local; thus $M_n(L_{K(n)}X)$ can be identified with the homotopy fiber F of the map $L_{K(n)}X \to L_{E(n-1)}L_{K(n)}X$. Since X is monochromatic, $L_{E(n-1)}X \simeq 0$ so there is a canonical map $\alpha : X \to F$. We claim that α is an equivalence. Since X and F are both $E(n)$-local, it will suffice to show that α induces an isomorphism $K(m)_*X \to K(m)_*F$ for $m \leq n$. If $m < n$, then both groups vanish. If $m = n$, we are reduced to proving that

$$K(n) \otimes X \to K(n) \otimes L_{K(n)}X \to L_{E(n-1)}L_{K(n)}X$$

is a fiber sequence. This follows from the observation that the first map is an equivalence and the third term vanishes (Lemma **??**).

Now let Y be a $K(n)$-local spectrum. Then Y is $E(n)$-local, so that $M_n(Y)$ is the homotopy fiber of the map $Y \to L_{E(n-1)}Y$. We wish to prove that the map $M_n(Y) \to Y$ exhibits Y as a $K(n)$-localization of $M_n(Y)$. Since Y is $K(n)$-local, it suffices to show that this map is a $K(n)$-equivalence; that is, that $K(n)_*L_{E(n-1)}Y \simeq 0$; this also follows from Lemma **??**. □

Corollary 14. *The $K(n)$-local stable homotopy category is compactly generated; its compact objects are precisely the retracts of spectra of the form $L_{K(n)}X$, where X is a finite spectrum of type $\geq n$.*

Warning 15. For a general finite spectrum X, the localization $L_{K(n)}X$ is not a compact object of the $K(n)$-local stable homotopy category. For example, if $n > 0$, then the $K(n)$-local sphere $L_{K(n)}S$ is not a compact object of the $K(n)$-local stable homotopy category.

The Image of J (Lecture 35)

April 27, 2010

The chromatic convergence theorem implies that the homotopy groups of the p-local sphere spectrum $S_{(p)}$ can be recovered as the inverse limit of the tower

$$\cdots \rightarrow \pi_* L_{E(2)} S \rightarrow \pi_* L_{E(1)} S \rightarrow \pi_* L_{E(0)} S.$$

The bottom of this tower is easy to understand: it is the rational sphere $S_{\mathbf{Q}}$, which is homotopy equivalent to the Eilenberg-MacLane spectrum $H\,\mathbf{Q}$. Our goal in this lecture is to understand the next step up in the tower, $L_{E(1)}S$. For simplicity, we will assume that $p > 2$.

Our first step is to describe the $K(1)$-local sphere. Our starting point is the following:

Lemma 1. *For each n, the spectrum $E(n)$ is $K(n)$-local.*

Proof. Recall that $E(n)$ is the even periodic Landweber exact spectrum associated to the Lubin-Tate ring $R = W(k)[[u_1, \ldots, u_{n-1}]]$ associated to a formal group of height n over a perfect field k of characteristic p.

Choose a cofiber sequence

$$X \rightarrow S_{(p)} \rightarrow L^t_{n-1} S_{(p)}$$

where X is a filtered colimit of p-local finite spectra DF_α of type $\geq n$. The dual DX is given by the homotopy inverse limit of a pro-spectrum $\{F_\alpha\}$. Taking MU-homology, we get a pro-system of $\pi_* \mathrm{MU}$-modules $\mathrm{MU}_*\{F_\alpha\}$; the theory of v_n-self maps shows that this pro-system can be identified with $\{\pi_* \mathrm{MU}/(v_0^N, v_1^N, \ldots, v_{n-1}^N\}_{N \geq 0}$. Since $E(n)$ is Landweber exact, we conclude that the pro-system $E(n)_*\{F_\alpha\}$ is equivalent to $\{\pi_* E(n)/(v_0^N, \ldots, v_{n-1}^N\}$. Since R is complete with respect to its maximal ideal, we conclude that the natural map $E(n) \rightarrow \varprojlim E(n)^{F_\alpha}$ is a homotopy equivalence. To prove that $E(n)$ is $K(n)$-local, it therefore suffices to show that each $E(n)^{F_\alpha}$ is $K(n)$-local. Let Y be a $K(n)$-acyclic spectrum; we wish to show that every map $Y \rightarrow E(n)^{F_\alpha}$ is nullhomotopic. This map is adjoint to a map $Y \otimes F_\alpha \rightarrow E(n)$. To show that such a map is nullhomotopic, it suffices to show that $E(n)_*(Y \otimes F_\alpha) \simeq 0$. This is equivalent to the statement that $K(m)_*(Y \otimes F_\alpha) \simeq 0$ for $m \leq n$. If $m < n$, this follows from the fact that F_α has type $\geq n$; if $m = n$, it follows from our assumption that Y is $K(n)$-acyclic. $\square$

Let us now fix our notation a bit more precisely: choose a formal group f of height n over $\mathbf{F}_{p^n}$ such that all endomorphisms of f are defined over $\mathbf{F}_{p^n}$, and let $E(n)$ be the variant of Morava E-theory associated to this formal group. Then, in the homotopy category of spectra, $E(n)$ is acted on by a group G which fits into an exact sequence

$$0 \rightarrow \mathrm{End}(f)^\times \rightarrow G \rightarrow \mathrm{Gal}(\mathbf{F}_{p^n}/\mathbf{F}_p) \rightarrow 0.$$

In fact, the situation turns out to be even better than this: one can promote the "action of G on $E(n)$ up to homotopy" to a "homotopy coherent" action of G, which is continuous (with respect to the profinite topology on G). In this context, one can extract a continuous homotopy fixed point spectrum $E(n)^G$, which one can prove is equivalent to $L_{K(n)}S$.

All of this requires technology beyond the scope of this course. However, when $n = 1$ and p is odd, there is a lowbrow alternative. In this case, we can identify $E(n)$ with the p-adically completed K-theory spectrum $\widehat{K}$. The group G can be identified with the group $\mathbf{Z}_p^\times$ of p-adic units, which breaks up as a product

$$\mu_{p-1} \times (1 + p\mathbf{Z}_p)^\times$$

where the first factor is the finite group of $(p-1)$st roots of unity and the second is a pro-p group. When $p > 2$, the second group is actually the cyclic pro-p group: it is generated, for example, by the element $1 + p \in 1 + p\mathbf{Z}_p$.

Remark 2. It is easy to describe the induced action on $\pi_* \widehat{K}$. For any complex orientable cohomology theory E, we can identify $\pi_2 E$ with the dual of the Lie algebra of the associated formal group. Note that the action of $\mathbf{Z}_p^\times$ on $\widehat{K}$ is induced by its action on the multiplicative formal group $f(x, y) = x + y + xy$. The action of $\mathbf{Z}_p^\times$ on $\pi_2 \widehat{K}$ is therefore given by differentiating the action of $\mathbf{Z}_p^\times$ on the formal group itself: that is, it is given by the identity character of $\mathbf{Z}_p^\times$. Since $\pi_* \widehat{K} \simeq \mathbf{Z}_p[\beta^{\pm 1}]$ and $\mathbf{Z}_p^\times$ acts by ring homomorphisms, we conclude that $\mathbf{Z}_p^\times$ acts by the nth power of the identity character on $\pi_{2n} \mathbf{Z}_p$.

If $n \in \mathbf{Z}_p^\times$, we will denote the corresponding map $\widehat{K} \to \widehat{K}$ by ψ^n. One can show that these operations agree with the classical *Adams operations* in complex K-theory (which provides another proof of Remark 2).

If p is odd, then the group $\mathbf{Z}_p^\times$ is topologically cyclic: it has a generator given by $g = (\zeta, p+1)$, where ζ is any primitive $(p-1)$st root of unity. Consequently, we should expect taking continuous $\mathbf{Z}_p^\times$ homotopy fixed points to be easy: they should be given by the homotopy fiber of the map

$$\widehat{K} \overset{1-\psi^g}{\to} \widehat{K}.$$

Let us denote this homotopy fiber by F.

Proposition 3. *The map $\alpha : S \to F$ induces an isomorphism on $K(1)$-homology.*

Proof. Recall that $K(1)$ can be realized as a summand of $\widehat{K}/p$. It will therefore suffice to show that α induces an equivalence in $\widehat{K}/p$-homology. Since $\widehat{K}$ is Landweber exact, we have

$$\widehat{K}_0(\widehat{K}/p) \simeq \pi_0 \widehat{K} \otimes_L (\mathrm{MP}_0\, \mathrm{MP}) \otimes_L \pi_0 \widehat{K}/p$$

(moreover, the homologies in all even degrees are the same by periodicity, and the homologies in odd degrees vanish). This is the $\mathbf{F}_p$-algebra which classifies isomorphisms of the multiplicative formal group with itself: that is, the algebra A of continuous $\mathbf{F}_p$-valued functions on the profinite group $\mathbf{Z}_p^\times$. In terms of this identification, the operation ψ^g is given by translation by g. We observe that $1 - \psi^g$ is a surjective map from A to itself, and its kernel is the one-dimensional $\mathbf{F}_p$-vector space of constant functions on $\mathbf{Z}_p^\times$. Using the long exact sequence

$$(\widehat{K}/p)_* F \to (\widehat{K}/p)_* \widehat{K} \overset{1-\psi^g}{\to} (\widehat{K}/p)_* \widehat{K},$$

we conclude that $(\widehat{K}/p)_* F \simeq \mathbf{F}_p[\beta^{\pm 1}] \simeq (\widehat{K}/p)_* S$. $\square$

Since $\widehat{K}$ is $K(1)$-local, the spectrum F is also $K(1)$-local. It follows that:

Corollary 4. *The map $S \to F$ exhibits F as the $K(1)$-localization of S. In other words, the $K(1)$-local sphere $L_{K(1)} S$ is given by the homotopy fiber of the map $1 - \psi^g : \widehat{K} \to \widehat{K}$*

It follows that we have a long exact sequence

$$\pi_n \widehat{K} \overset{1-\psi^g}{\to} \pi_n \widehat{K} \to \pi_{n-1} L_{K(1)} S \to \pi_{n-1} \widehat{K}$$

which we can use to compute the homotopy groups of $L_{K(1)} S$. We note that ψ^g is the identity on $\pi_0 \widehat{K} \simeq \mathbf{Z}_p$, so that $1 - \psi^g$ vanishes on π_0 and we get isomorphisms

$$\pi_0 L_{K(1)} S \simeq \pi_{-1} L_{K(1)} S \simeq \mathbf{Z}_p.$$

The groups $\pi_n \widehat{K}$ vanish if n is odd. On $\pi_{2m} \widehat{K}$, the map $1 - \psi^g$ is given by $1 - g^m$ (Remark 2), and is therefore always injective for $m \neq 0$. Using the long exact sequence, we see that the even homotopy groups of $L_{K(1)} S$ vanish (except in degree zero), and we have an isomorphism $\pi_{2m-1} L_{K(1)} S \simeq \mathbf{Z}_p/(1 - g^m)$.

The cardinality of this group depends on m. If m is not divisible by $p-1$, then $g^m - 1$ is a unit modulo p so that $\pi_{2m-1} L_{K(1)} S$ vanishes. If $m = (p-1)m'$, then $g^m = (g^{p-1})^{m'}$ where g^{p-1} is a generator for the topologically cyclic pro-p-group $(1 + p\mathbf{Z}_p)^\times$. If we write $m' = p^k m''$, where m'' is prime to p, then g^m generates the cyclic subgroup $(1 + p^{k+1}\mathbf{Z}_p)^\times$, so that $1 - g^m$ is a generator for $p^{k+1}\mathbf{Z}_p \subseteq \mathbf{Z}_p$. We conclude:

Theorem 5. *The homotopy groups of $L_{K(1)}S$ are given as follows:*

$$\pi_n L_{K(1)} S \simeq \begin{cases} \mathbf{Z}_p & \text{if } n = 0, -1 \\ \mathbf{Z}/p^{k+1}\mathbf{Z} & \text{if } n+1 = (p-1)p^k m, m \not\equiv 0 \mod p \\ 0 & \text{otherwise.} \end{cases}$$

From Theorem 5 it is easy to describe the $E(1)$-local sphere. Recall that we have a homotopy pullback square

$$\begin{tikzcd} L_{E(1)}S \arrow[r] \arrow[d] & L_{K(1)}S \arrow[d] \\ L_{E(0)}S \arrow[r] & L_{E(0)}L_{K(1)}S. \end{tikzcd}$$

The localization $L_{E(0)}S$ is just the Eilenberg-MacLane spectrum $H\,\mathbf{Q}$. Theorem 5 implies that $\pi_n L_{E(0)} L_{K(1)} S \simeq \mathbf{Q}_p$ for $n = 0, 1$ and vanishes otherwise. Using the long exact sequence

$$\cdots \to \pi_{n+1} L_{E(0)} L_{K(1)} S \to \pi_n L_{E(1)} S \to \pi_n L_{K(1)} S \oplus \pi_n L_{E(0)} S \to\to \pi_n L_{E(0)} L_{K(1)} S \to \cdots,$$

we conclude that $\pi_n L_{E(1)} S \simeq \pi_n L_{K(1)} S$ unless $n \in \{0, -1, -2\}$. In these degrees, we have an exact sequence

$$0 \to \pi_0 L_{E(1)} S \to \mathbf{Z}_p \oplus \mathbf{Q} \to \mathbf{Q}_p \to \pi_{-1} L_{E(1)} S \to \mathbf{Z}_p \to \mathbf{Q}_p \to \pi_{-2} L_{E(1)} S \to 0.$$

Collecting these facts together, we obtain:

Theorem 6. *The homotopy groups of $L_{E(1)}S$ are given as follows:*

$$\pi_n L_{K(1)} S \simeq \begin{cases} \mathbf{Z} & \text{if } n = 0 \\ \mathbf{Q}_p/\mathbf{Z}_p & \text{if } n = -2 \\ \mathbf{Z}/p^{k+1}\mathbf{Z} & \text{if } n+1 = (p-1)p^k m, m \not\equiv 0 \mod p \\ 0 & \text{otherwise.} \end{cases}$$

There is an evident map $\pi_* S_{(p)} \to \pi_* L_{E(1)} S$, whose kernel is the second step in the chromatic filtration of $\pi_* S_{(p)}$. This map is obviously not surjective, since $\pi_* S_{(p)}$ is concentrated in positive degrees, while $\pi_* L_{E(1)} S$ is not. However, this turns out to be the only obstruction: the map $\pi_n S_{(p)} \to \pi_n L_{E(1)} S$ is surjective for $n \geq 0$. In other words, if $n > 0$, then every class in $\pi_n L_{E(1)} S \simeq \pi_n M_n(S)$ survives the chromatic spectral sequence. This is a result of Adams; let us briefly describe (without proof) the ideas involved.

Let $O(k)$ denote the orthogonal group of a k-dimensional vector space. Then $O(k)$ acts on the 1-point compactification of $\mathbb{R}^k$, fixing the point at infinity; this compactification can be identified with S^k. In particular, given a pointed map $X \to O(k)$ for any space X, we get a map $X \wedge S^k \to S^k$. Taking X to be a sphere, we get a map $\pi_n O(k) \to [S^{n+k}, S^k]$. Taking the limit as $k \mapsto \infty$, we get a homomorphism $\pi_n O \to \pi_n S$, where O denotes the infinite orthogonal group and S the sphere spectrum. This map is called the *J-homomorphism.*

The relationship between the J-homomorphism and the first chromatic layer can be stated as follows:

Theorem 7. *Let $\Im(J)_n$ denote the image of the J-homomorphism $\pi_n O \to \pi_n S \to \pi_n S_{(p)}$. For $n > 0$, the map $S_{(p)} \to L_{E(1)} S$ induces an isomorphism $\theta : \Im(J)_n \to \pi_n L_{E(1)} S$. In particular, the map $\pi_n S_{(p)} \to \pi_n L_{E(1)} S$ is surjective.*

The proof consists of two parts: proving that θ is injective and proving that θ is surjective. The surjectivity is not far from what we have done in class: we already know that each $\pi_n L_{E(1)} S$ is a cyclic group, so it suffices to show that θ hits a generator of the group; this can be proven by an explicit calculation.

Remark 8. The description of the image of the J-homomorphism was an important precursor to the development of the chromatic picture of homotopy theory: many of the ideas we have discussed had their origins in attempting to explain (and generalize) the "periodic behavior" exhibited by the image of the J homomorphism.