Line Bundles in String Model Building

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arXiv: 1810.0444, 1906.0363, 1906.08769, 1906.08730
in collaboration with: Callum Brodie, Andrei Constantin, Rehan Deen, Yang-Hui He

Outline

- Introduction
- Counting line bundle standard models
- Formulae for line bundle cohomology
- Machine learning line bundle cohomology
- Conclusion

Introduction: line bundle models

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N=1, D=4 GUT with gauge group $SU(5) \times S(U(1)^5)$ and matter in $\mathbf{10}, \bar{\mathbf{10}}, \bar{\mathbf{5}}, \mathbf{5}, \mathbf{1}$

- freely acting symmetry Γ on X, so $\hat{X} = X/\Gamma$ is smooth and non simply-connected

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standard-like model (hopefully) with gauge group $G_{\rm SM} \times S(U(1)^5)$

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* At this level: String models with the exact (MS)SM spectrum plus modes uncharged under the standard model group.

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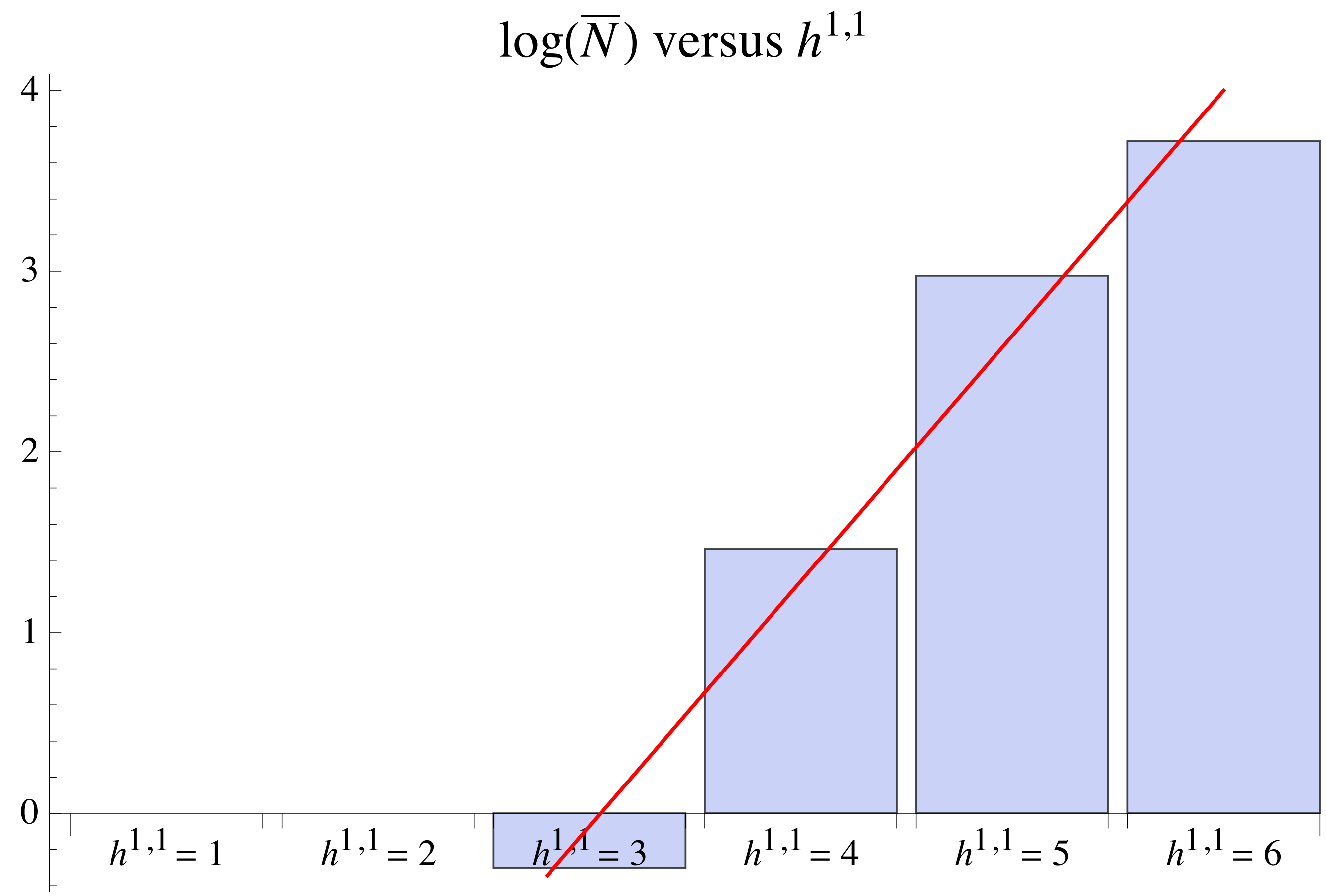
Q: For a given CY manifold, what is the number of consistent line bundle models $N = N(h, c_i, d_{ijk})$ with chiral asymmetry six?

A rough count

Consider average number of models per CY, $\bar{N} = \bar{N}(h)$, as a function of h only (neglect dependence on c_i , d_{ijk} for now):



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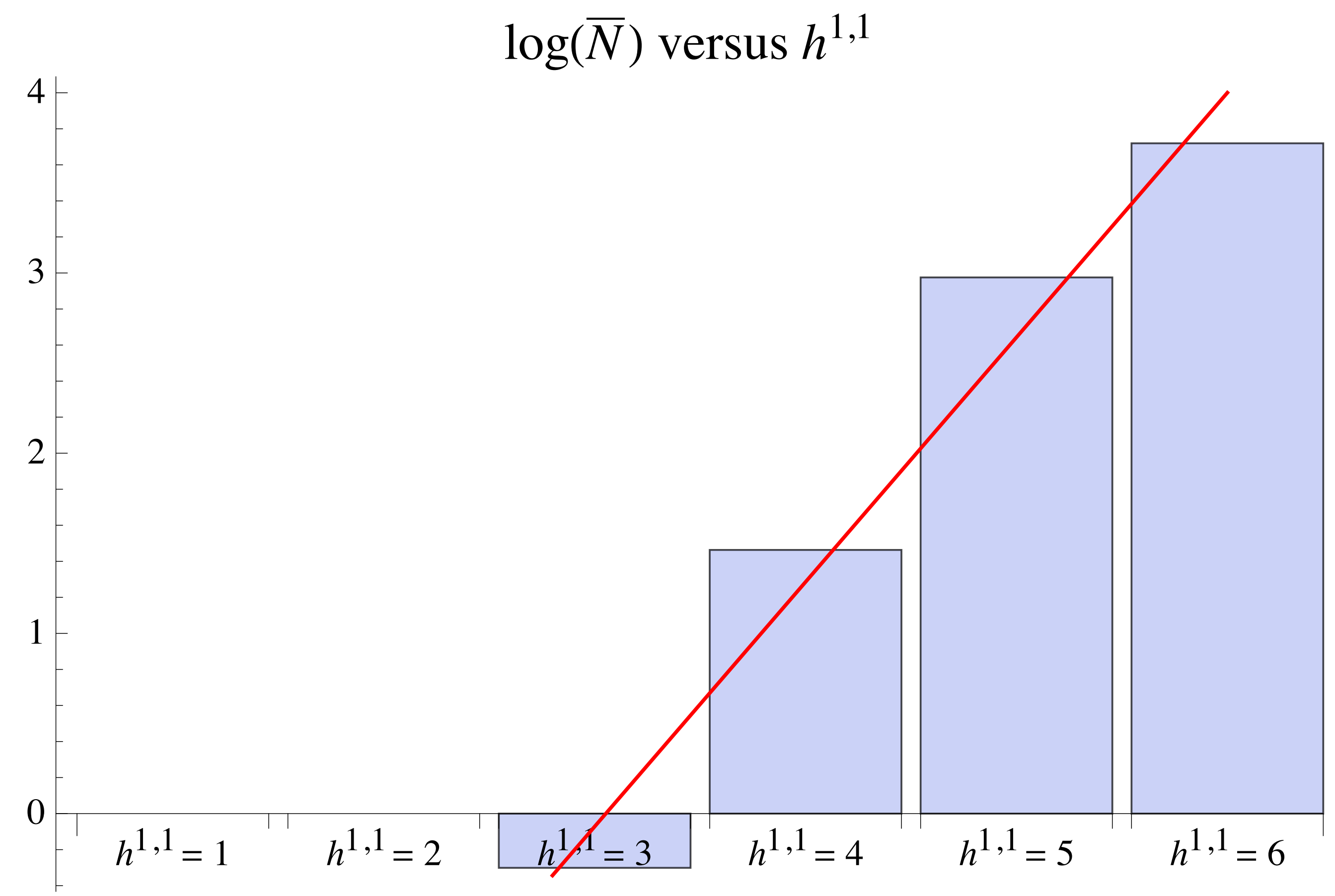

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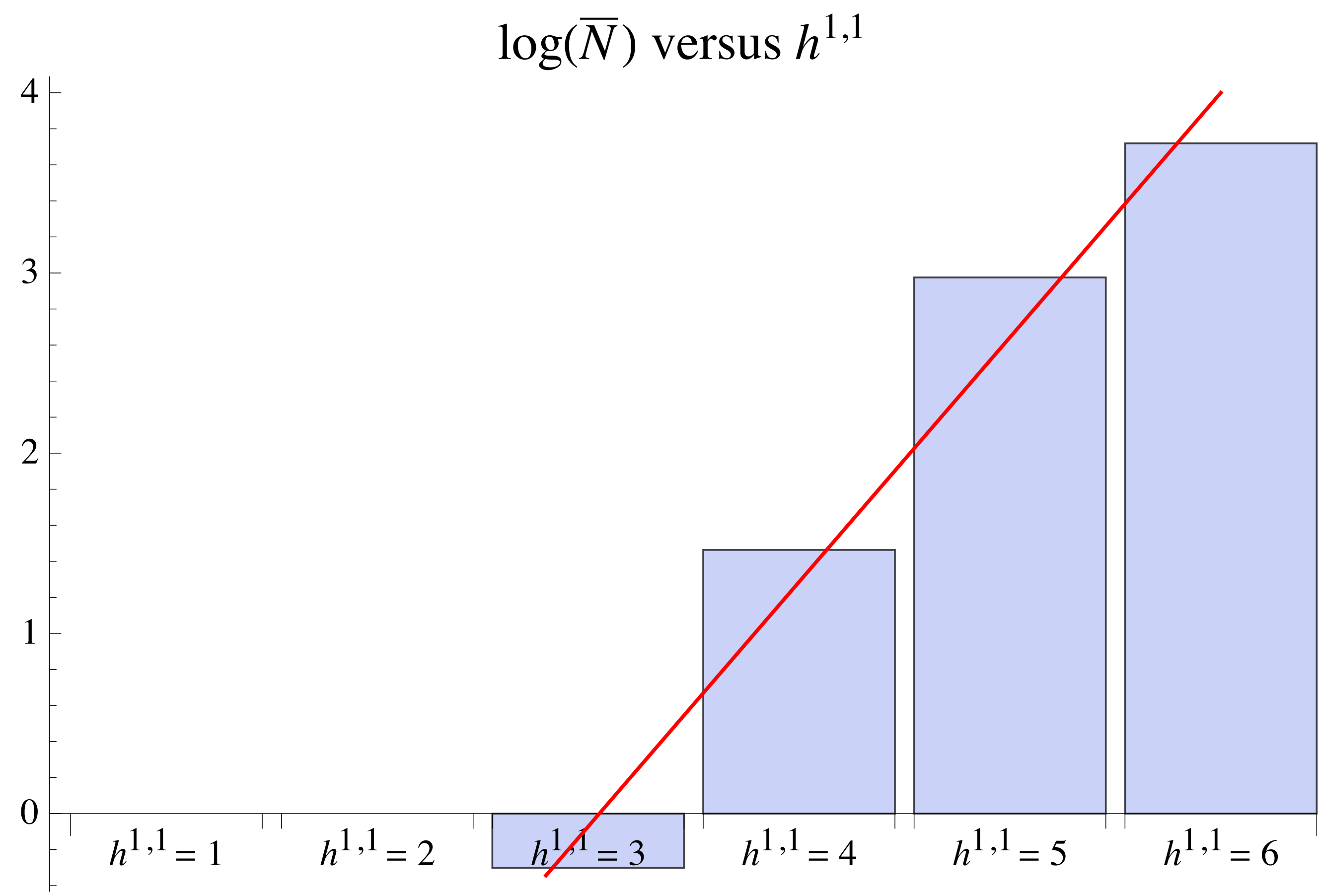

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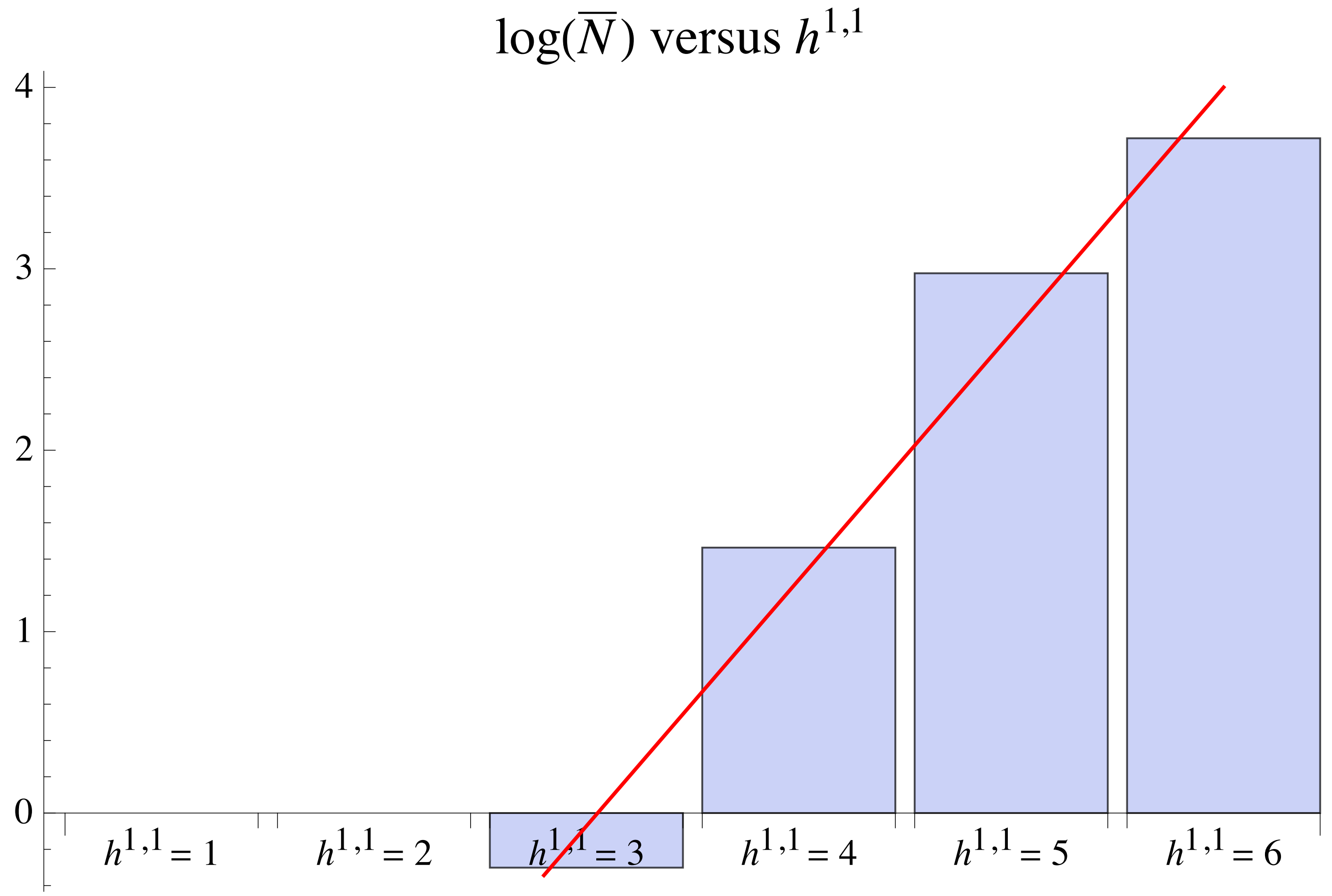

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For CICYs: $h_{\max} = 19$ $N(h_{\max}) \simeq 10^{23}$

All known CYs: $h_{\max} = 491$ $N(h_{\max}) \simeq 10^{662}$

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\begin{aligned}
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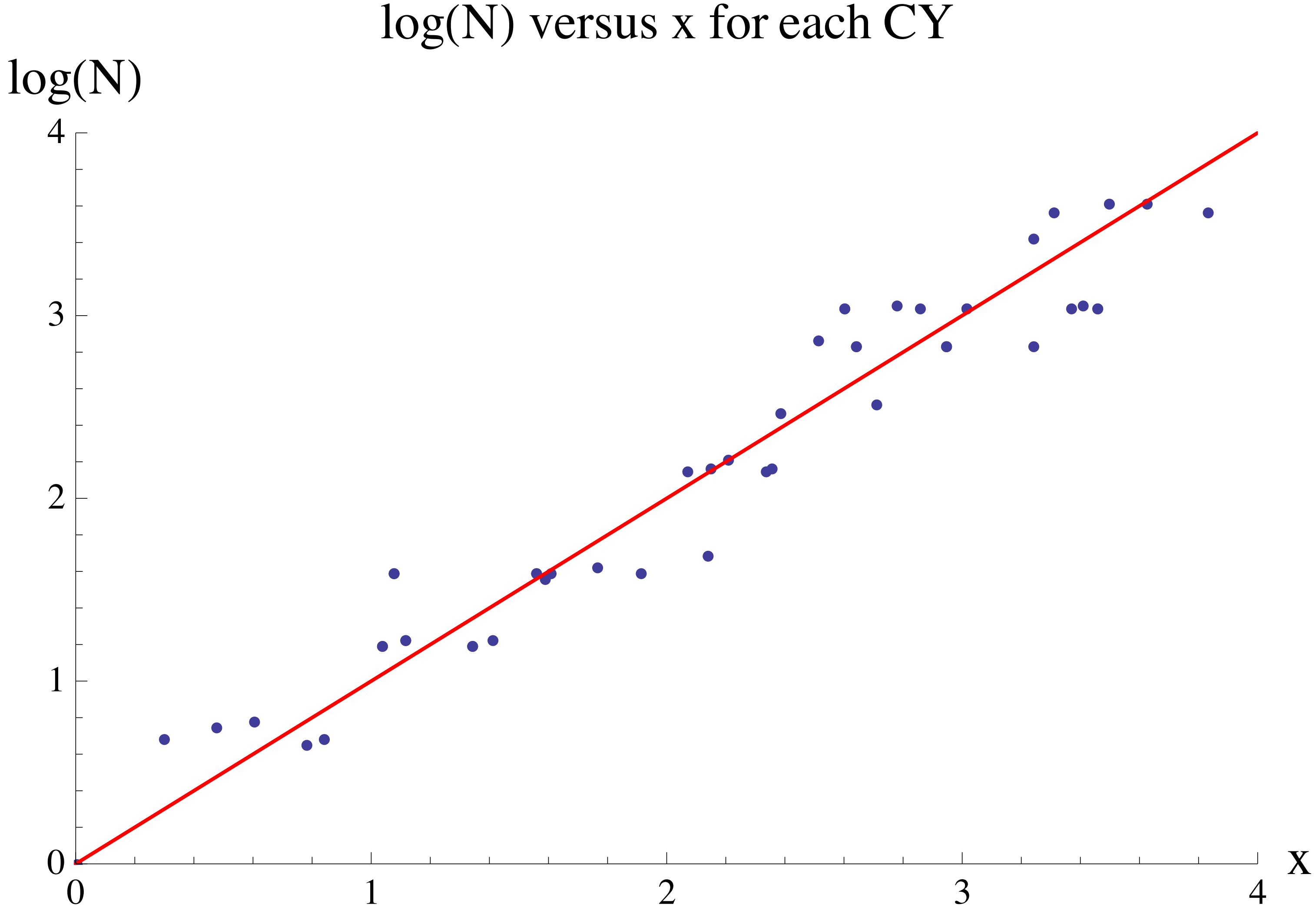
Ansatz for $\log(N) = x$:

$$x = A_0 + B_0\,h + \sum_{i=1}^{7}(A_i + B_i\,h)\log d_i$$

Fixing A_i and B_i by a fit to the data leads to:



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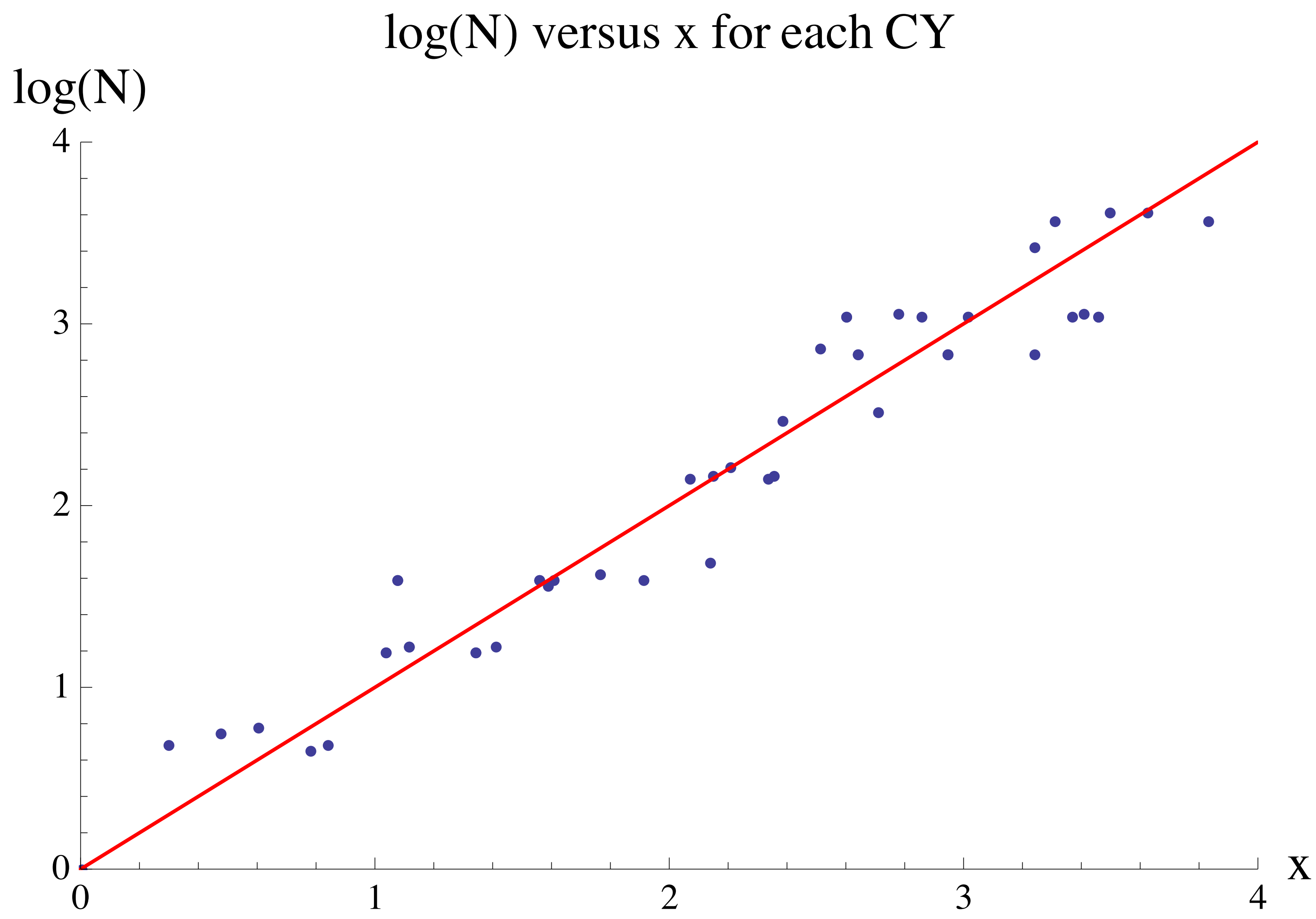

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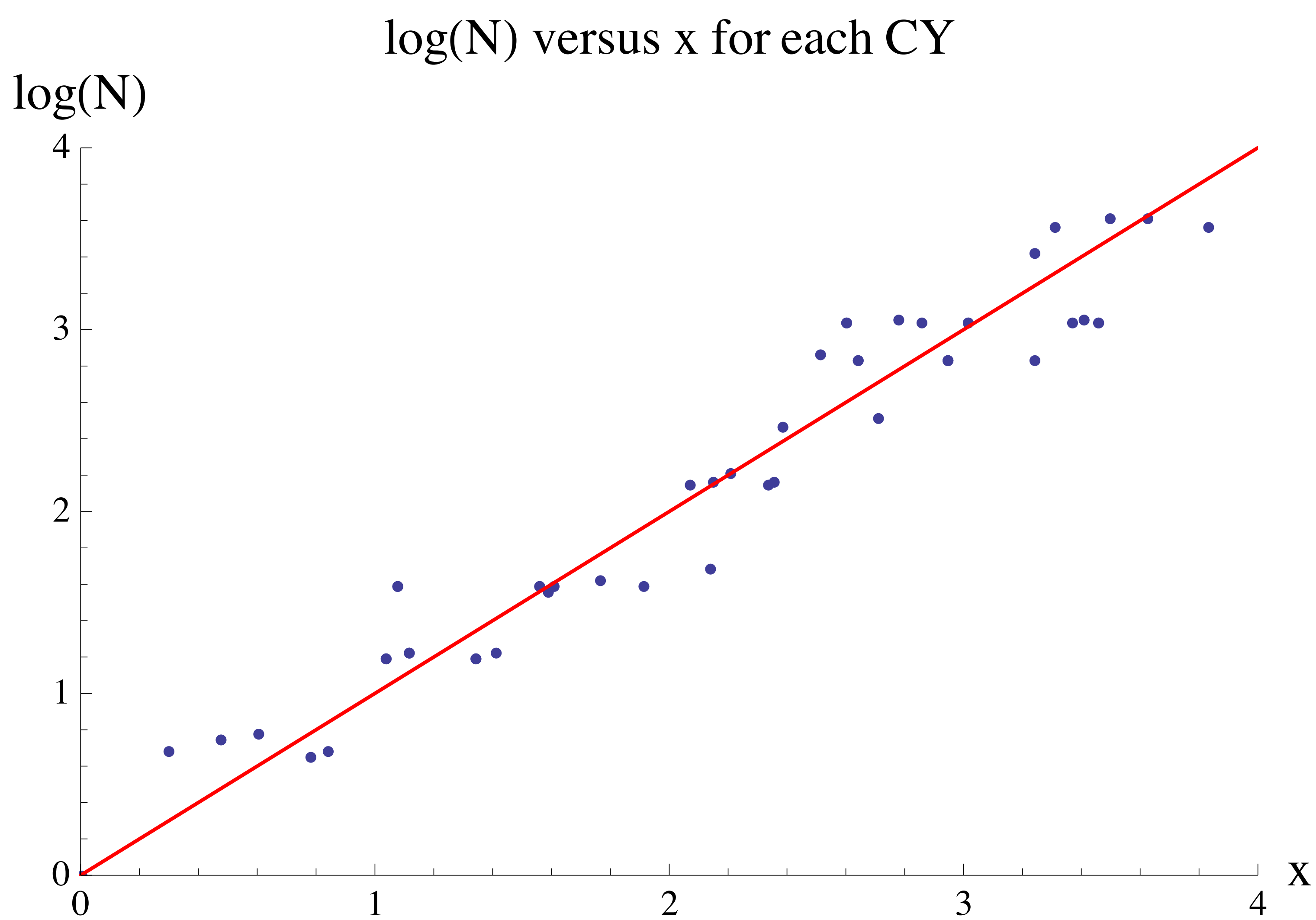

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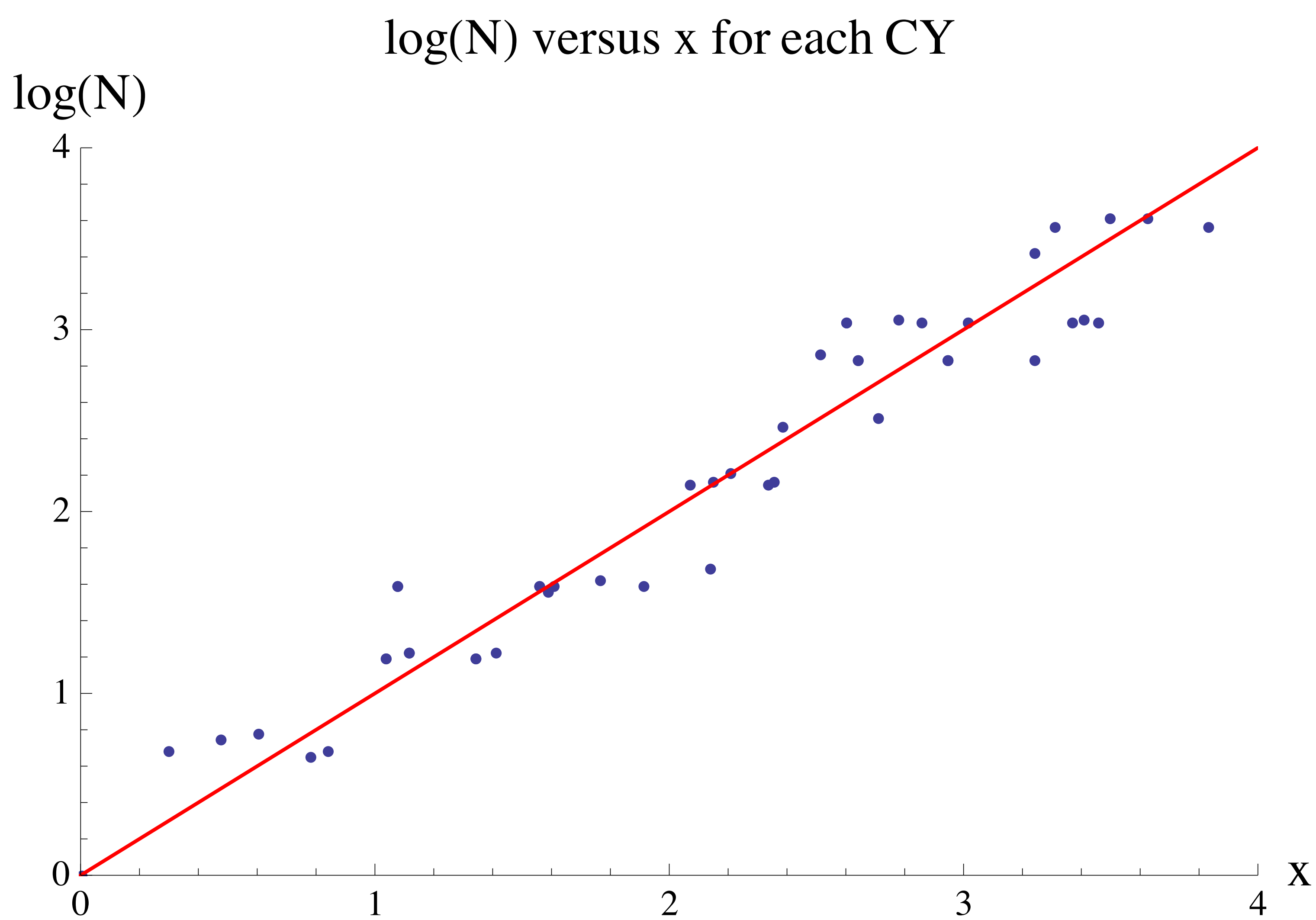

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For example . . .

bi-cubic in $\mathbb{P}^2 \times \mathbb{P}^2$:

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$$h^0(X,L) = \begin{cases} \frac{1}{2}(1+k_2)(2+k_2) \ , & k_1 = 0, \ k_2 \geq 0 \\ \mathrm{ind}(L) \ , & k_1, k_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$h^1(X,L) = \begin{cases} \frac{1}{2}(-1+k_2)(-2+k_2) \ , & k_1 = 0, \ k_2 > 0 \\ -\mathrm{ind}(L) \ , & k_1 < 0, \ k_2 > -k_1 \\ 0 & \text{otherwise} \ , \end{cases}$$

$$\mathrm{ind}(L) = \frac{3}{2}(k_1 + k_2)(2 + k_1 k_2).$$

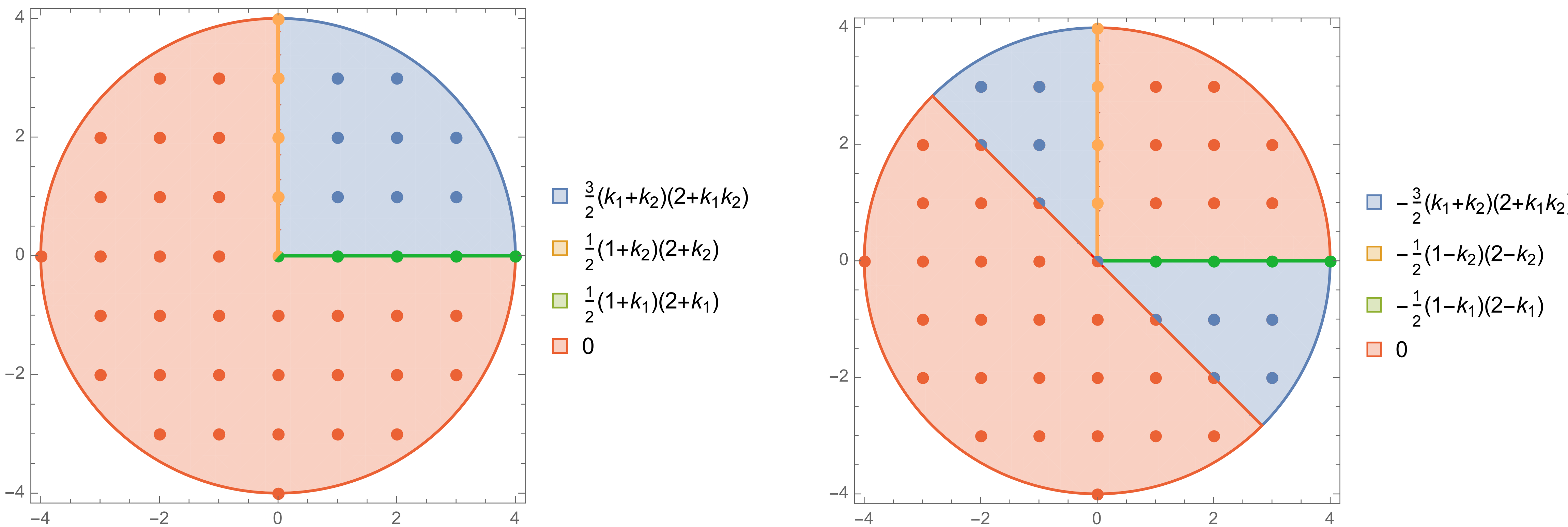

Figure 2: *Regions in k-space where $h^0(X,L)$ (left) and $h^1(X,L)$ (right) take different polynomial forms. In the blue regions $h^0(X,L) = \mathrm{ind}(L)$ and $h^1(X,L) = -\mathrm{ind}(L)$. By Serre duality, the plots for $h^2(X,L)$ and $h^3(X,L)$ are obtained from the plots for $h^1(X,L)$ and, respectively, $h^0(X,L)$ by reflection about the origin.*

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Such a master formula now exists for many complex surfaces.
(Brodie, Constantin, Deen, Lukas, 1906.08363, 1906.08769)

On a complex surface S, there is a map of (effective) divisors

$$D \to \tilde{D} = D - \sum_{C \in \mathcal{I}} \theta(-D \cdot C)\, \mathrm{ceil}\left(\frac{D \cdot C}{C^2}\right) C$$

(where $\mathcal{I}$ are the irreducible, negative self-intersection curves)

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(For more details see talks by Callum Brodie and Andrei Constantin)

Machine learning line bundle cohomology

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Q: Can we use machine learning to conjecture piecewise polynomial formulae for line bundle cohomology?

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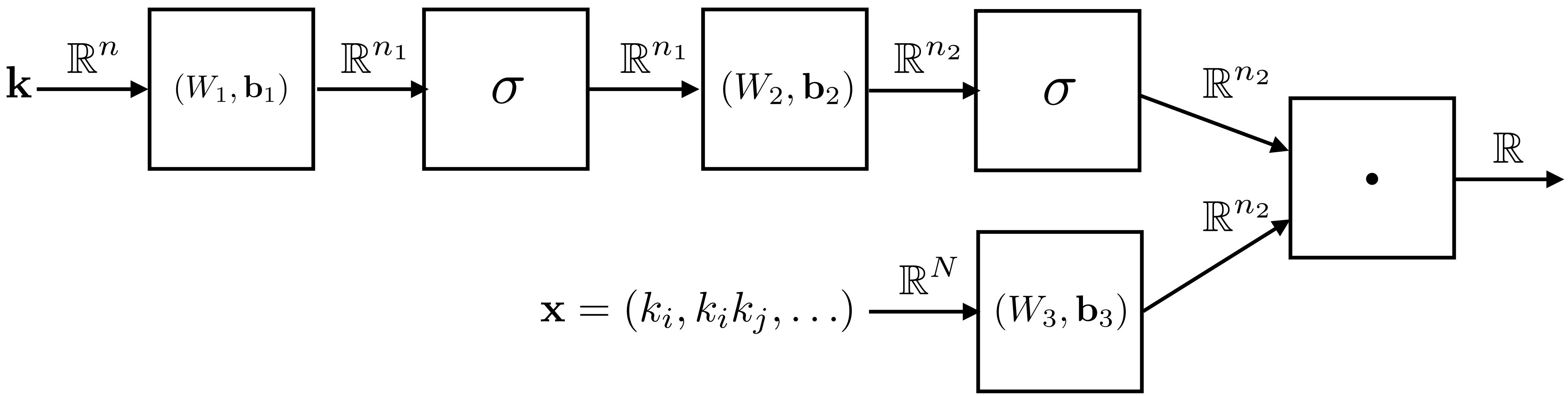

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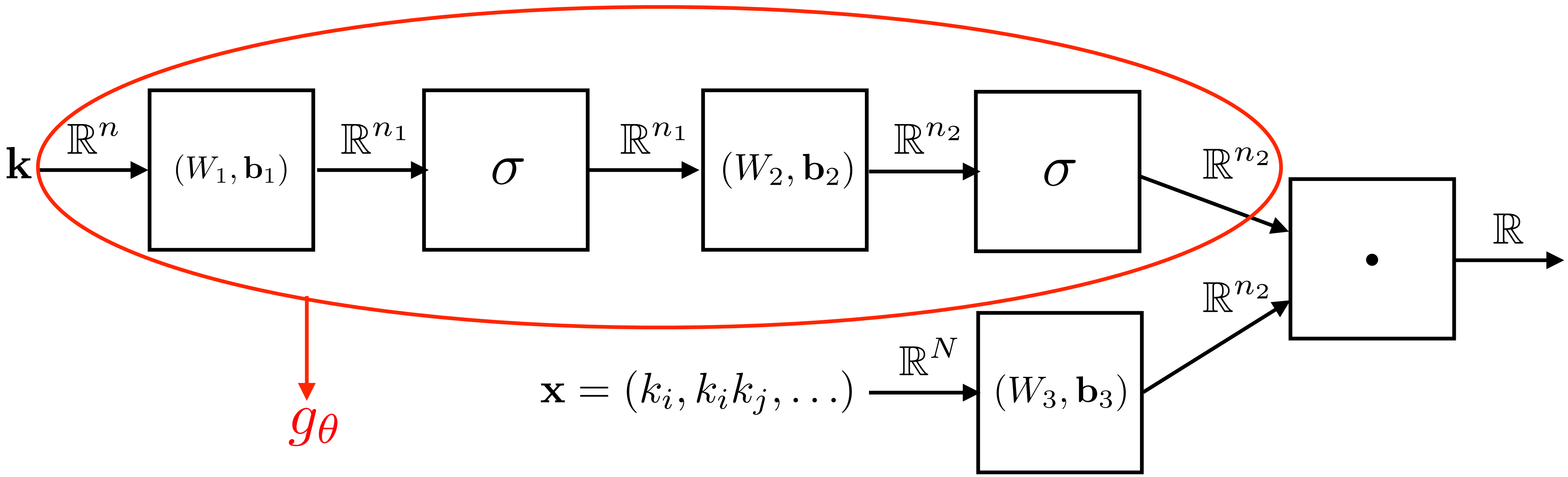

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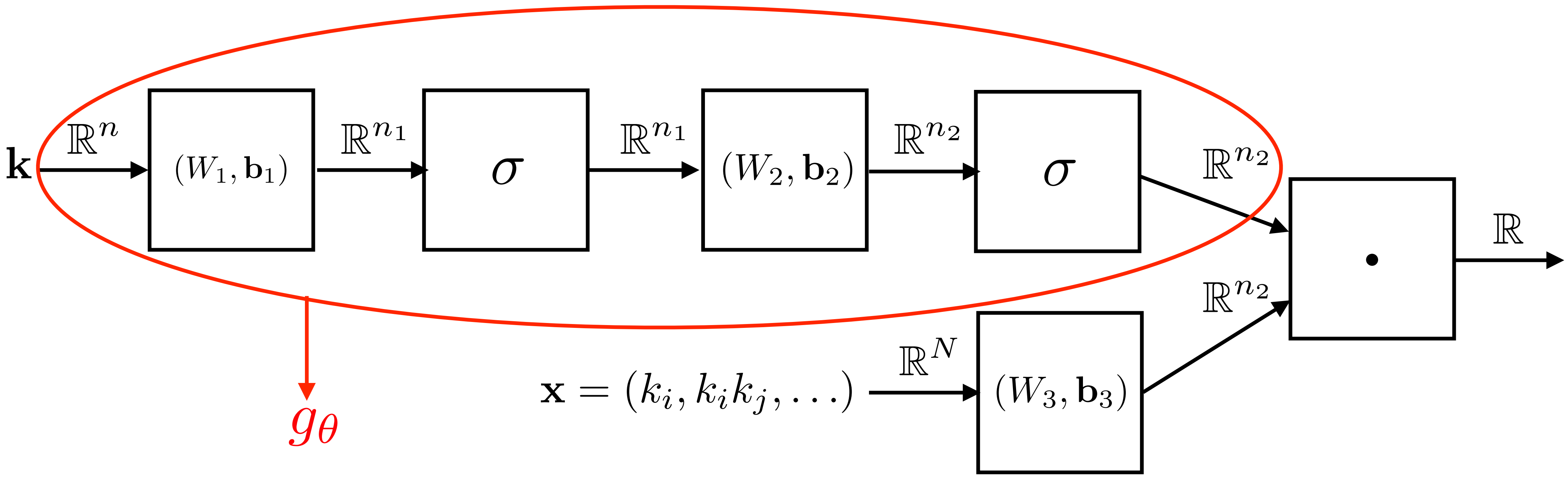

Training data:

$$(k_i, k_i k_j)_{i \le j} \longrightarrow h^q(\mathcal{O}_X(\mathbf{k})) \quad \text{for} \quad d = 2$$

$$(k_i, k_i k_j, k_i k_j k_l)_{i \le j \le l} \longrightarrow h^q(\mathcal{O}_X(\mathbf{k})) \quad \text{for} \quad d = 3$$

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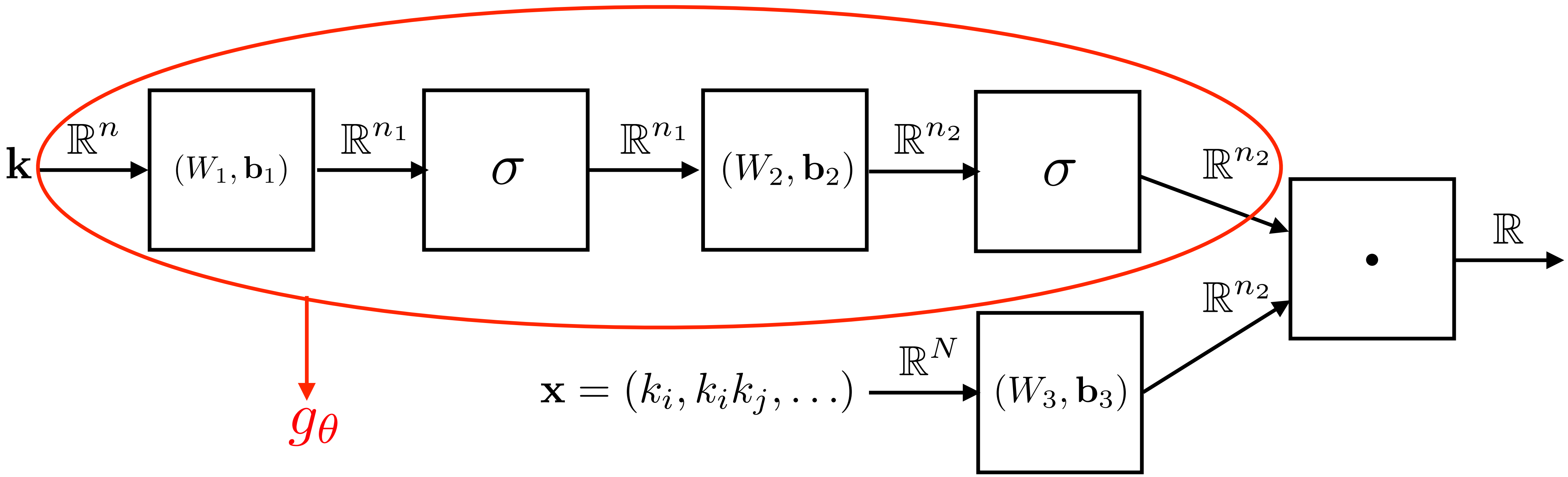

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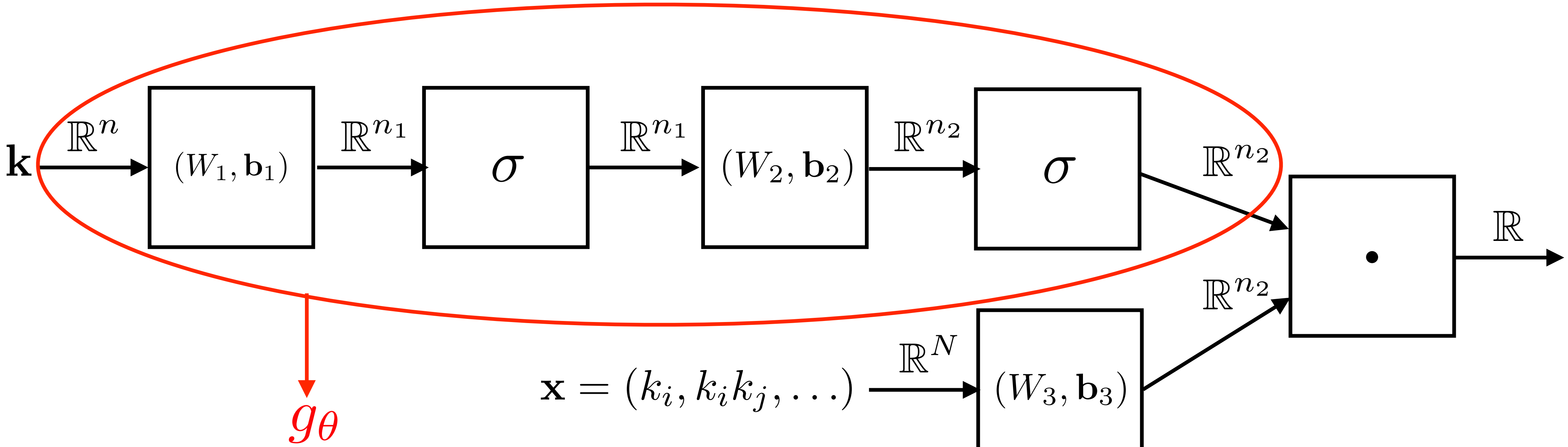

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$$\mathbf{a}(\mathbf{k}) := \left(g_{\bar{\theta}}(\mathbf{k}) \cdot \bar{\mathbf{b}}_3 \, , \; g_{\bar{\theta}}(\mathbf{k}) \cdot \bar{W}_3\right)$$

$$\mathbf{k}, \; \mathbf{k}' \text{ in the same region} \quad \leftrightarrow \quad |\mathbf{a}(\mathbf{k}) - \mathbf{a}(\mathbf{k}')| < \epsilon$$

Example: $h^1(\mathcal{O}_X(k_1, k_2))$ for bi-cubic

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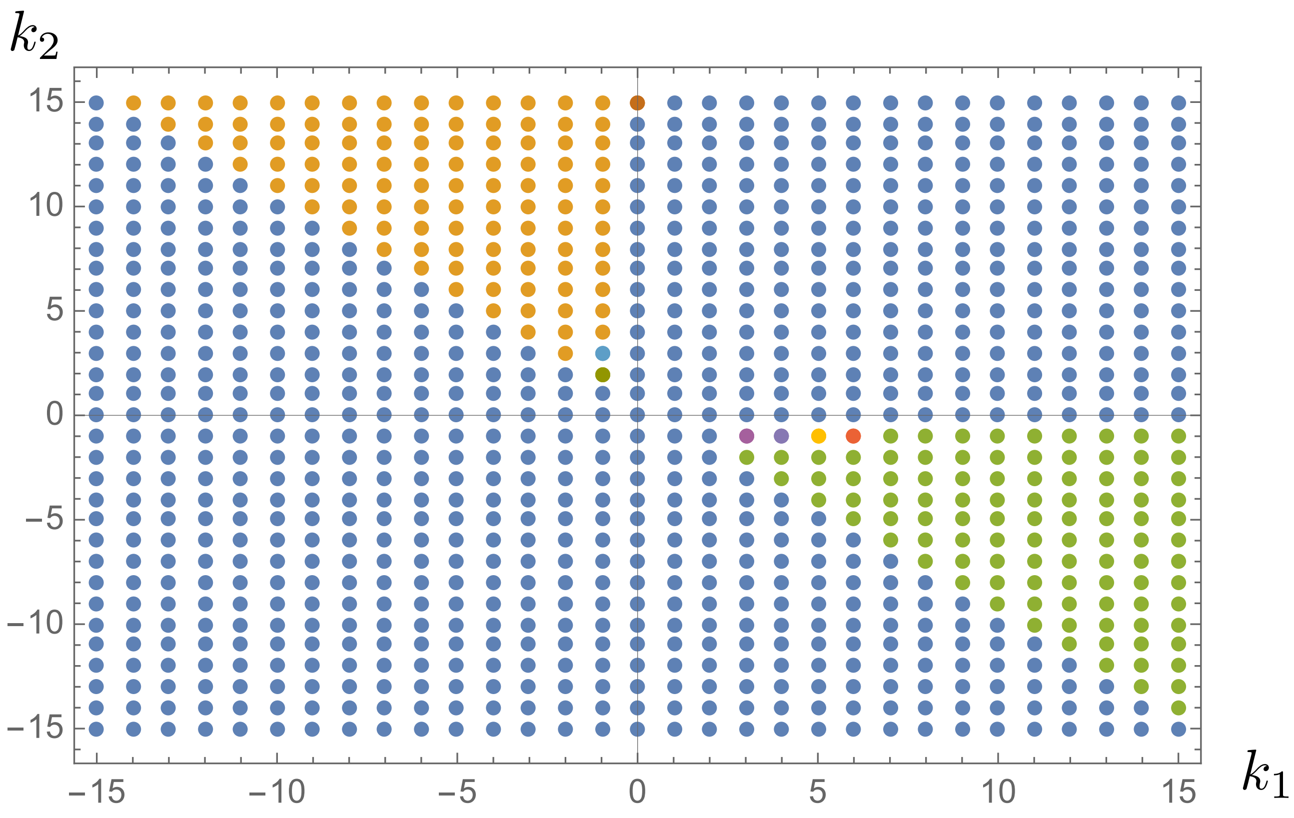

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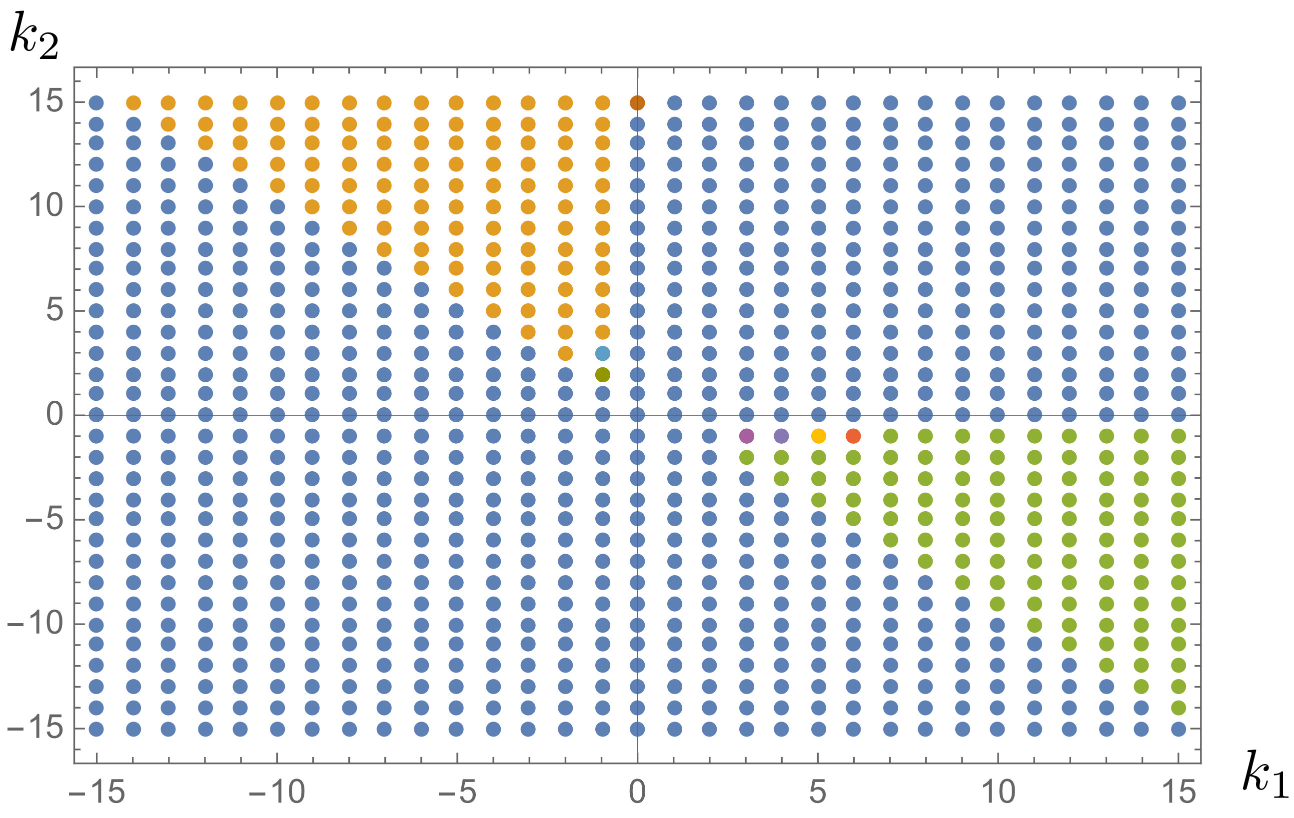

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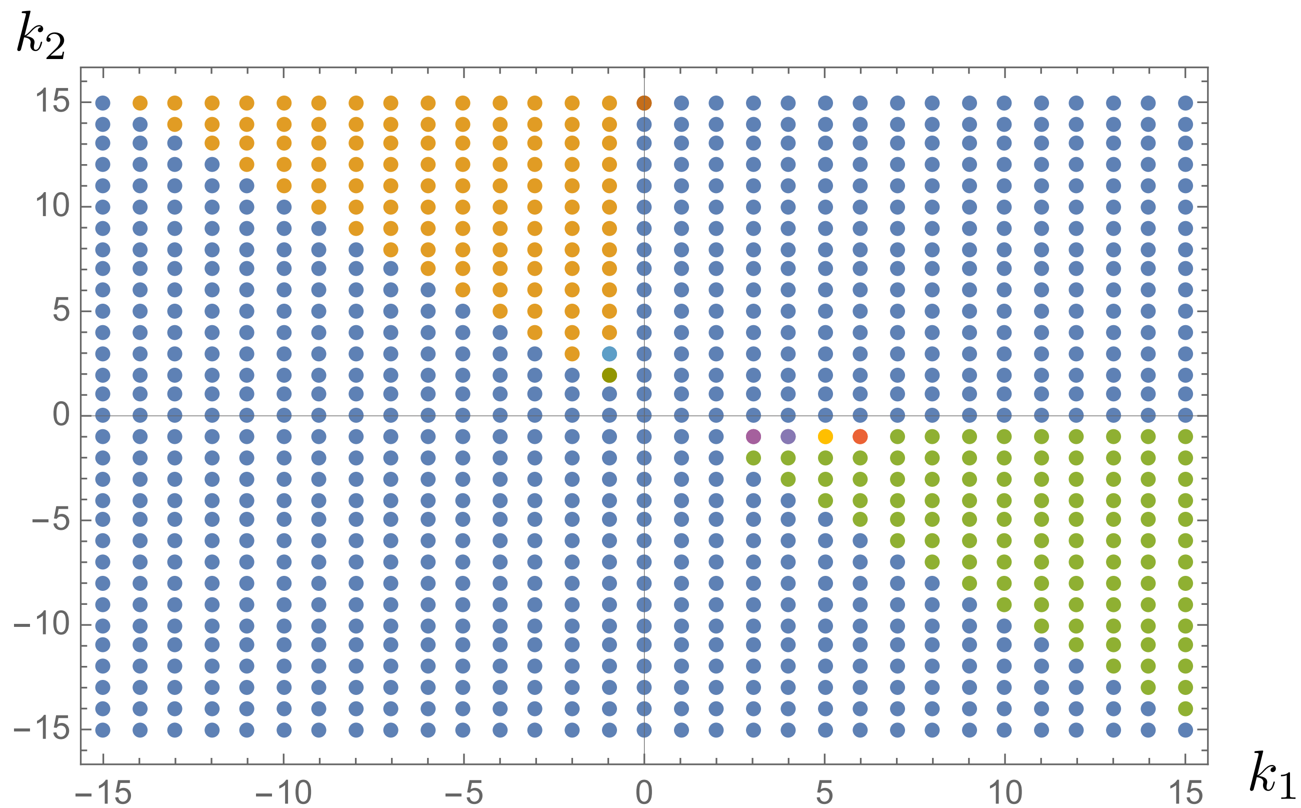

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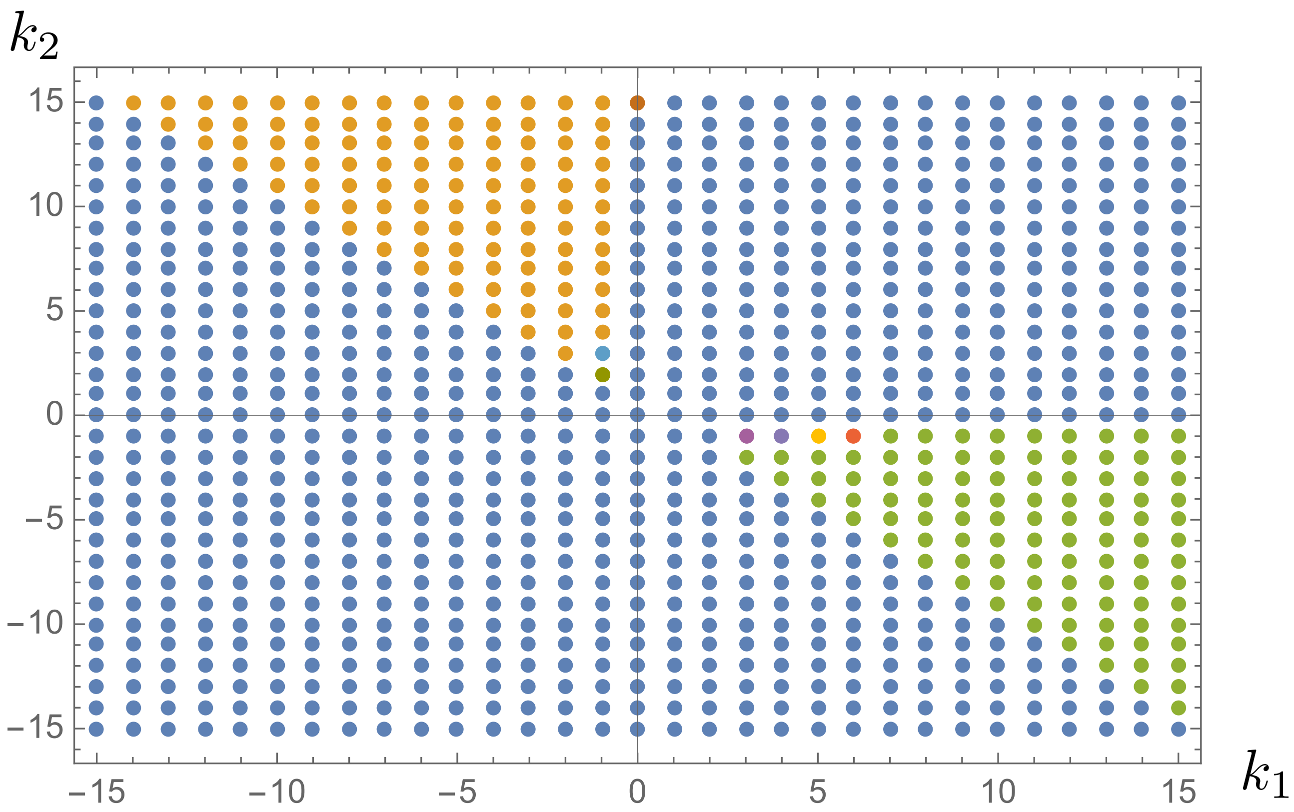

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yellow/green: $h^1(\mathcal{O}_X(k_1, k_2)) = -\frac{3}{2}(k_1 + k_2)(2 + k_1 k_2)$

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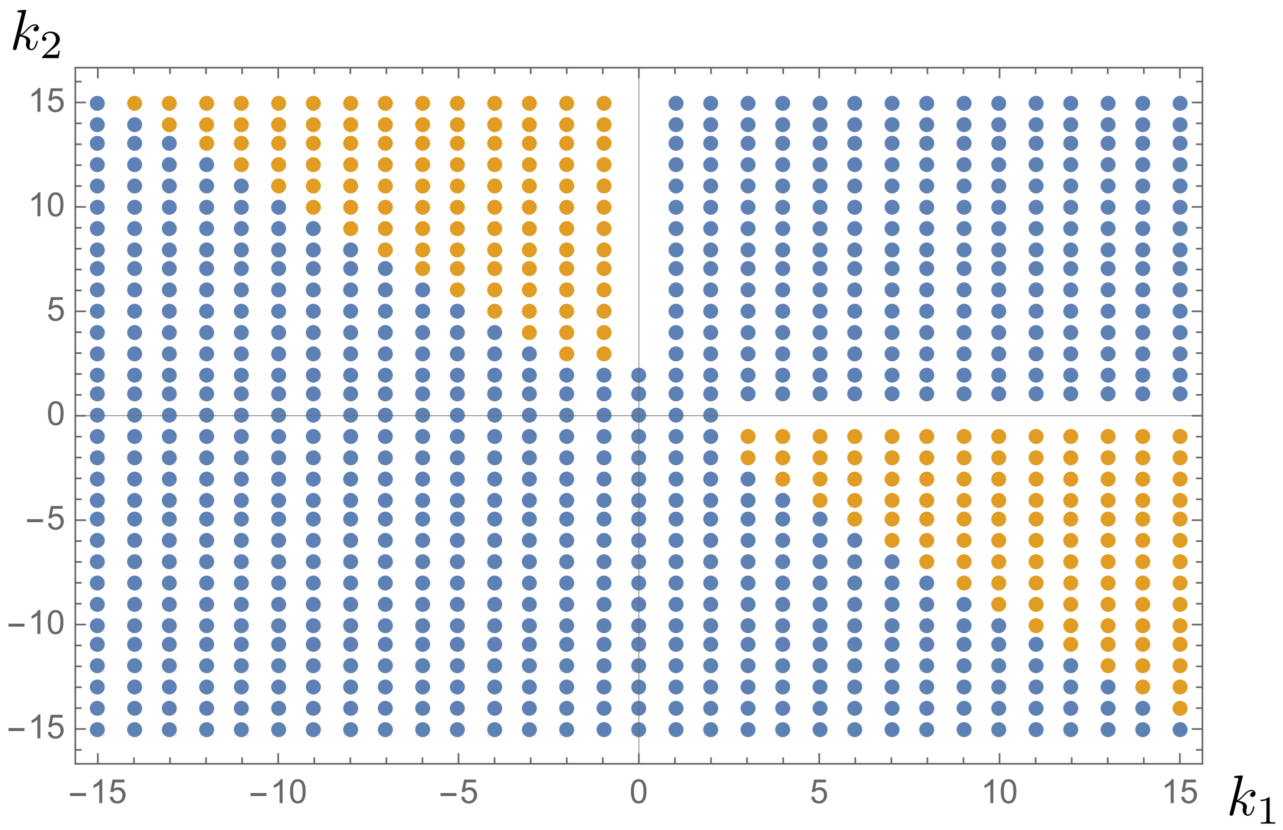

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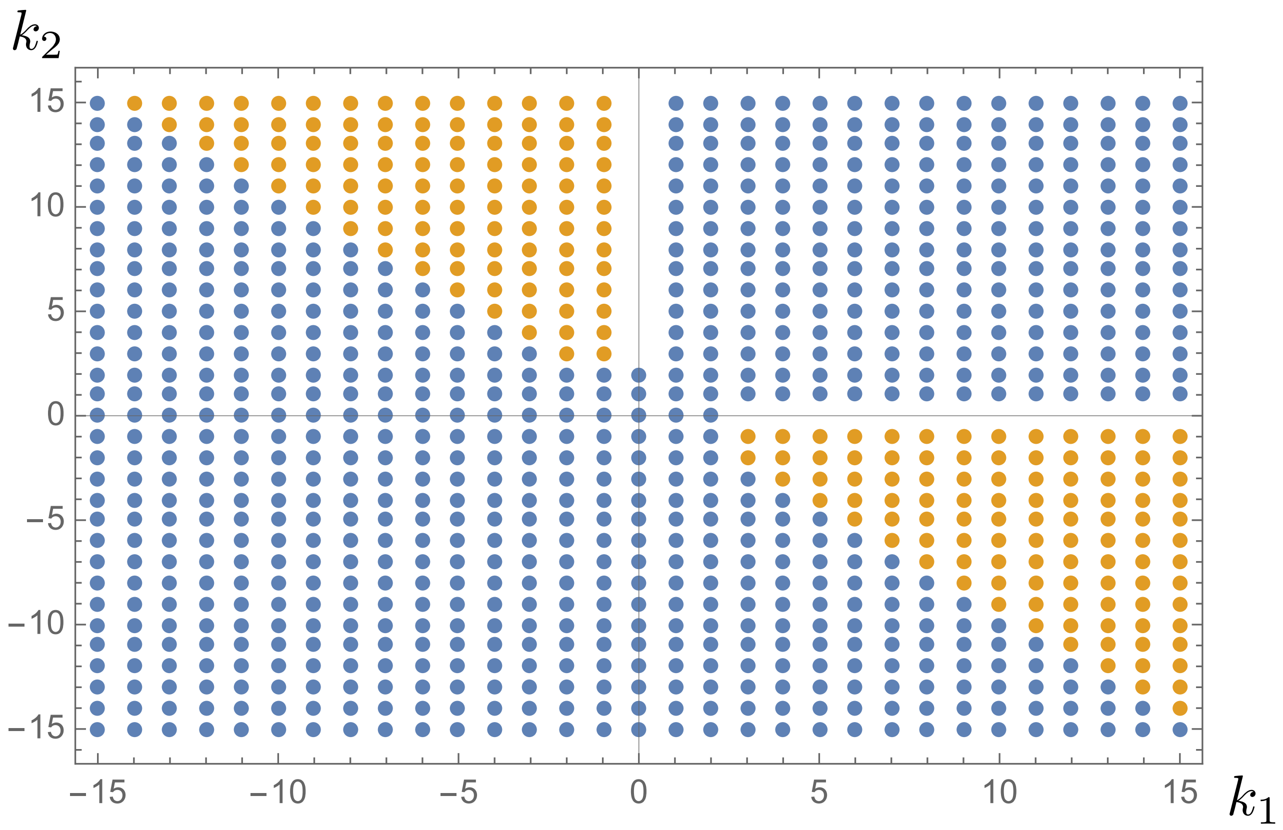

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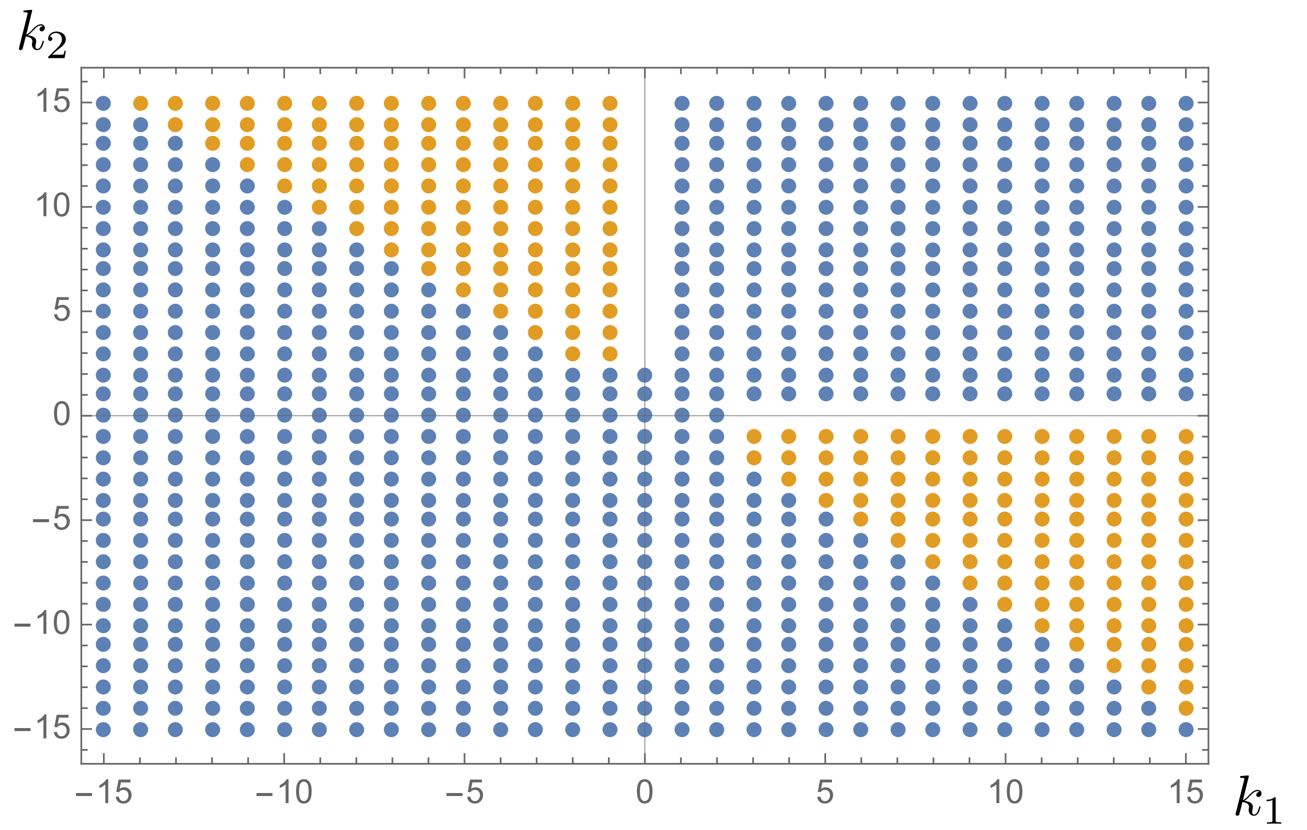

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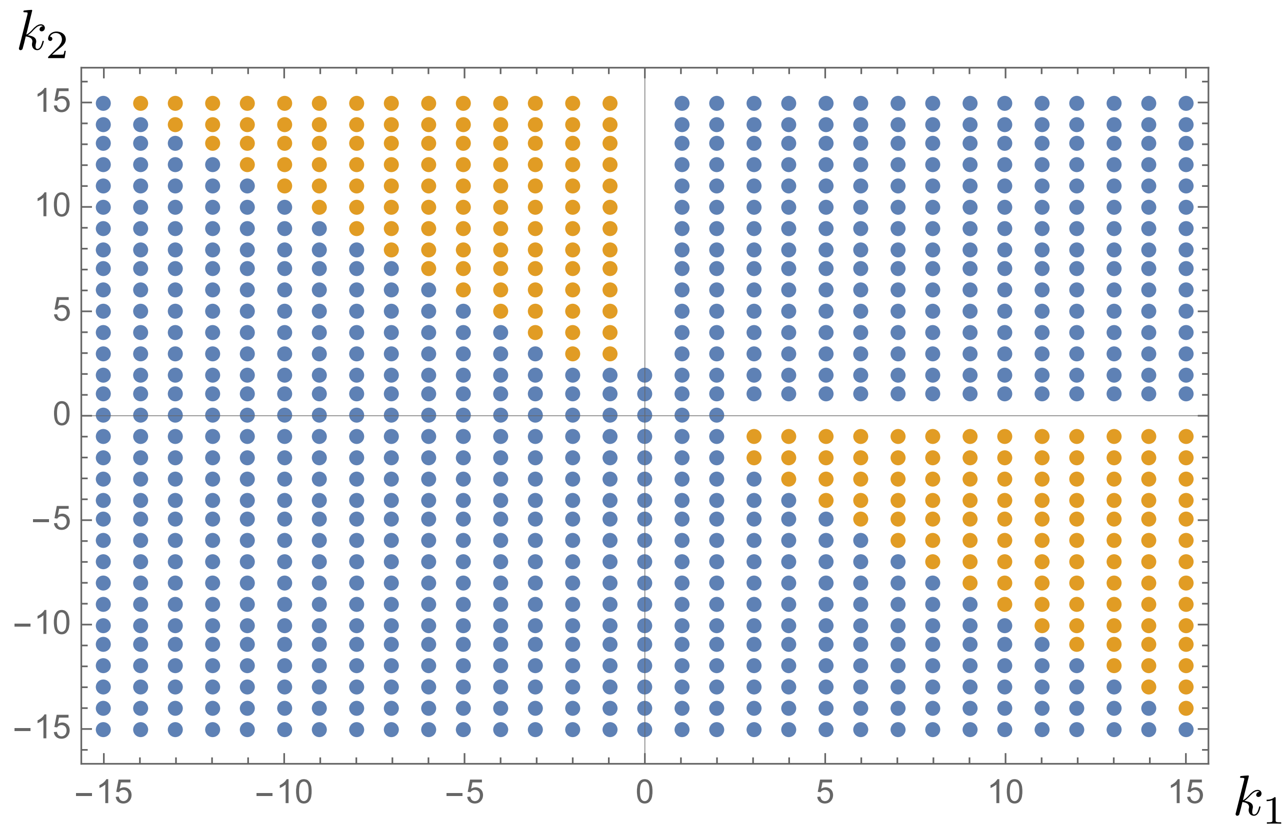

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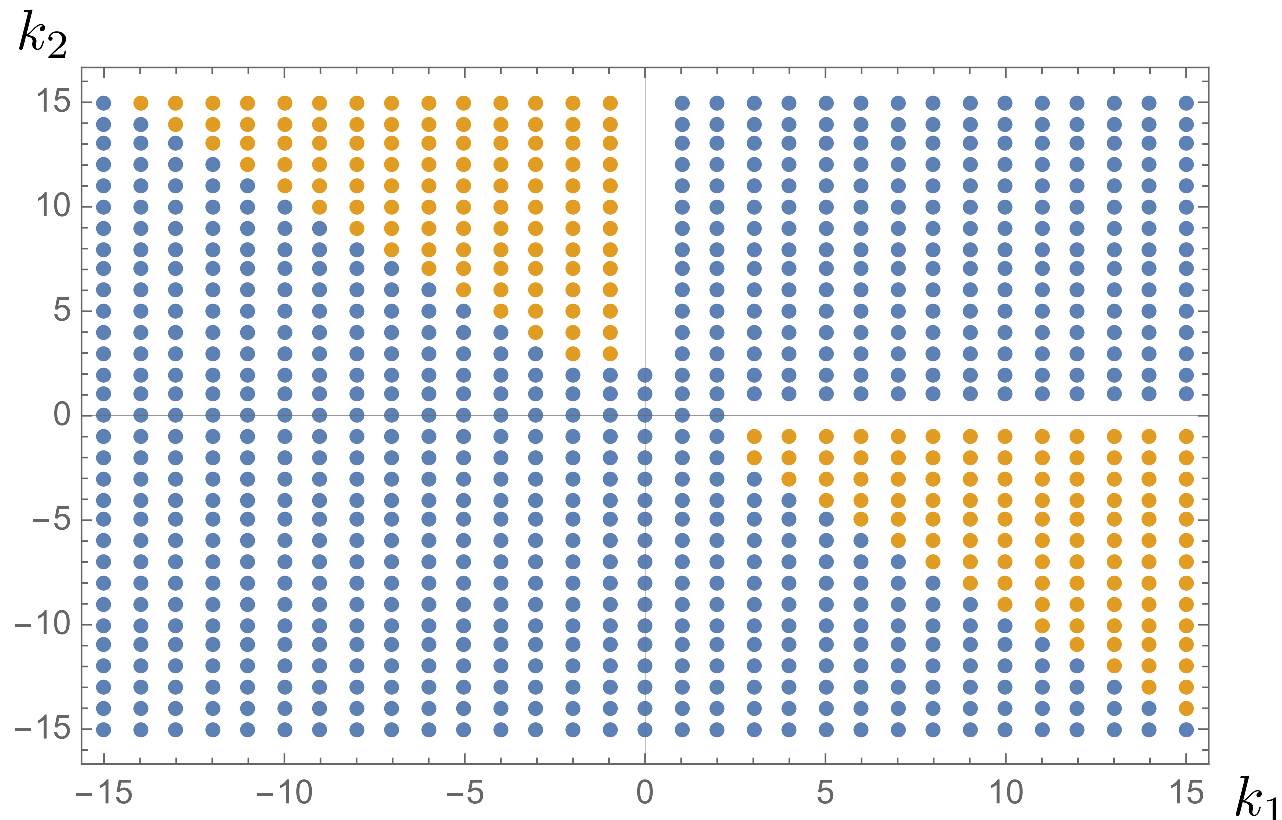

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This also works for surfaces and other CY three-folds.

Example: dP_2

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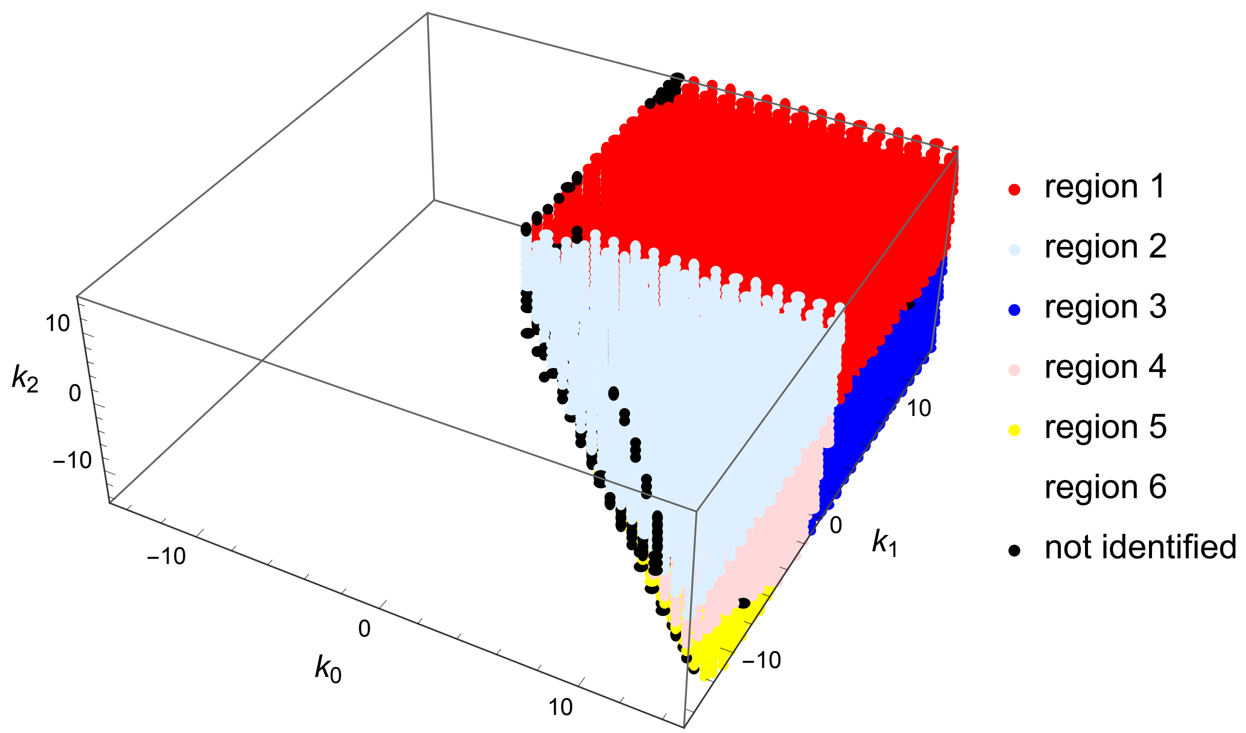

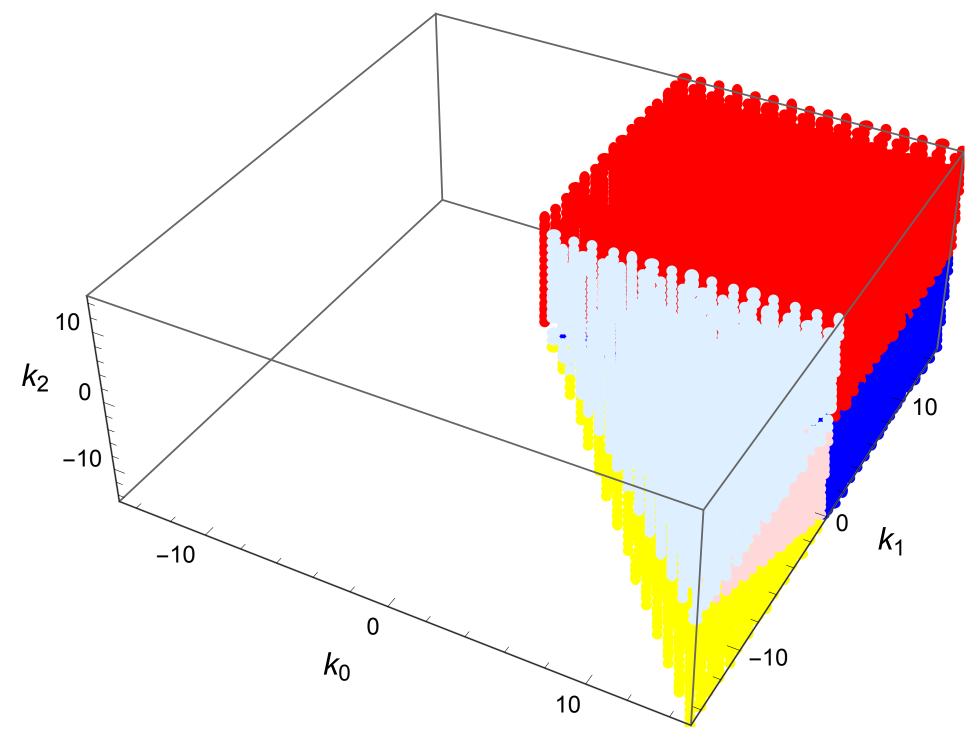

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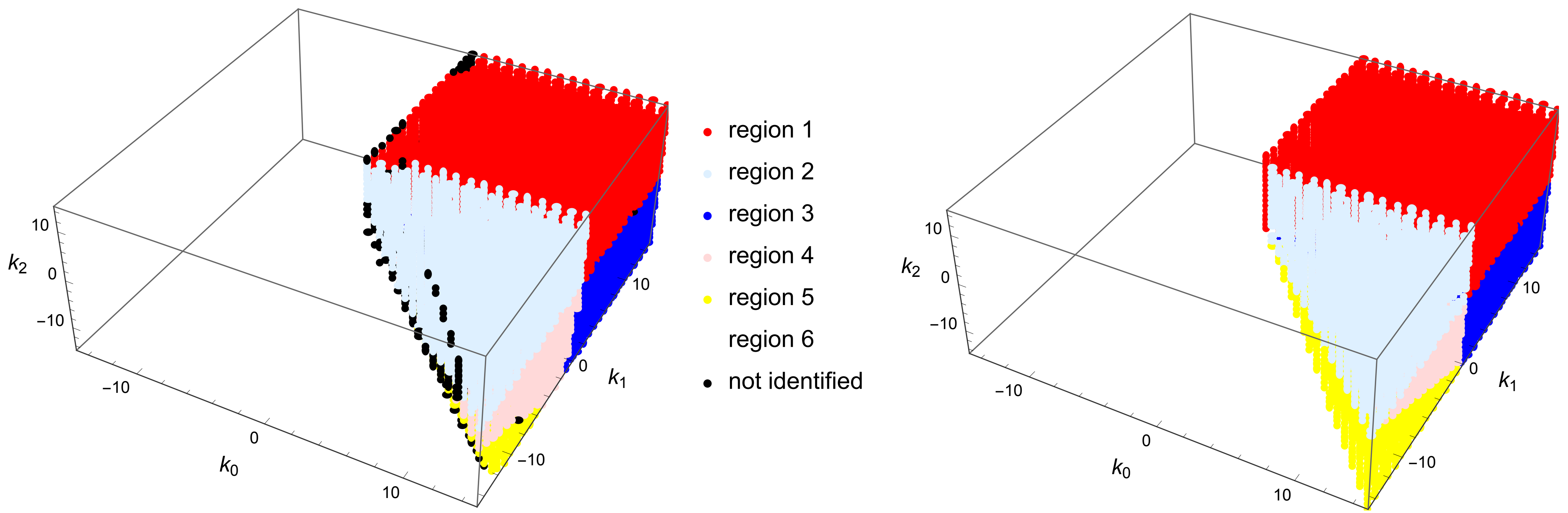

$$h^0(\mathcal{O}_{\mathrm{dP}_2}(\mathbf{k})) = \begin{cases} 1 + \frac{3}{2}k_0 + \frac{1}{2}k_0^2 + \frac{1}{2}k_1 - \frac{1}{2}k_1^2 + \frac{1}{2}k_2 - \frac{1}{2}k_2^2 & \text{in region } 1\,, \\ 1 + 2k_0 + k_0^2 + k_1 + k_0k_1 + k_2 + k_0k_2 + k_1k_2 & \text{in region } 2\,, \\ 1 + \frac{3}{2}k_0 + \frac{1}{2}k_0^2 + \frac{1}{2}k_2 - \frac{1}{2}k_2^2 & \text{in region } 3\,, \\ 1 + \frac{3}{2}k_0 + \frac{1}{2}k_0^2 + \frac{1}{2}k_1 - \frac{1}{2}k_1^2 & \text{in region } 4\,, \\ 1 + \frac{3}{2}k_0 + \frac{1}{2}k_0^2 & \text{in region } 5\,. \\ 0 & \text{in region } 6\,. \end{cases}$$

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Region 2:			$k_0 + k_1 + k_2 < 0$	$k_0 + k_1 \geq 0$	$k_0 + k_2 \geq 0$	
Region 3:	$-k_1 < 0$	$-k_2 \geq 0$			$k_0 + k_2 \geq 0$	
Region 4:	$-k_1 \geq 0$	$-k_2 < 0$			$k_0 + k_2 \geq 0$	
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This is indeed the formula which follows from the theorems for surfaces.

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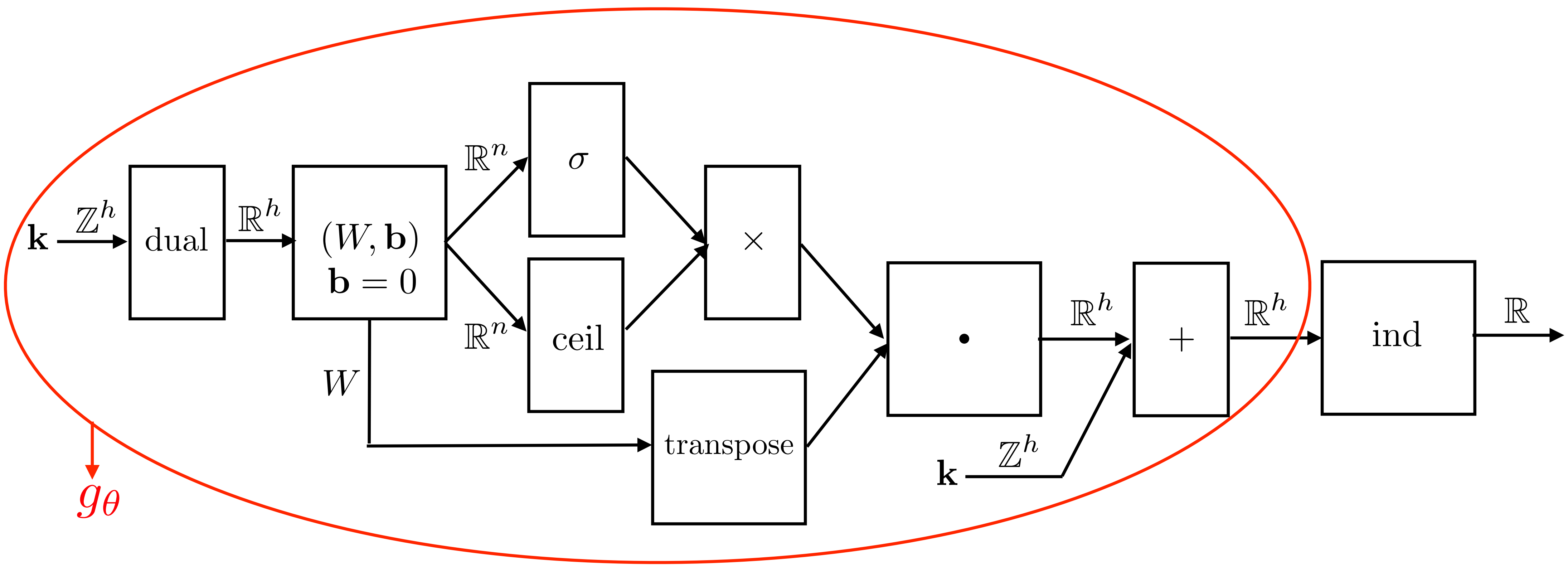
Design network accordingly:



$$g_\theta : D \to \tilde{D}$$

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This network finds the correct divisors for dP_n , $n = 1, 2, 3, 4$.

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Thanks