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Maxwell's equations and metrics [†]

By

Friedrich Kottler at Vienna

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In a previous work ([1]), the independence of pure field physics from any metric was established in the context of Newtonian laws of attraction. A distinction was made there between force flux and force; the former is independent of any given metric by way of the field, the latter is determined from it by way of the notion of work, into which the metric enters. The origin of the metric in Newtonian theory is thus energetic or dynamical in nature.

The independence of pure field physics from any metric is also true in a completely analogous way for Maxwell's electrodynamics. Here, in contrast to Newtonian field physics, one has two field vectors (six-vectors in the four-dimensional universe), which we shall distinguish in the sequel as the electromagnetic and magneto-electric field vectors; both of them are independent of the metric. Here, the metric will not, as in Newtonian mechanics, be introduced from the energetic-dynamical standpoint – since the field determines its own electromagnetic mechanics – but from the kinematical standpoint, namely, through the laws of field radiation (light). In order to show this, we must associate both fields by way of the so-called constitutive relations (dielectricity and permeability). The origin of the metric in Maxwell's electrodynamics, and thus, as a further consequence, its origin in the Lorentz-Minkowski-Einstein theory of relativity, lies in these relations.

1.

We begin with the representation of the two sets of four Maxwell differential equations in the form that they were given by Minkowski ([2]), and then write them in the form of two integral theorems, i.e., in the form that extends the idea of local effects into

[†] Translated by D.H. Delphenich.

[1] F. Kottler, Newton'sches Gesetz und Metrik, These Proceedings, v. 131 ([1922], denoted by N.).

[2] H. Minkowski, Die Grundgleichungen für die elektromagnetischen Vorgänger in bewegten Körper, Gött. Nachr. 1908, esp., § 7-8.

field physics (see N, no. 2). This form was first given by R. Hargreaves ([1]); H. Bateman ([2]) and the author ([3]) then devoted time to a detailed investigation of it.

As is well known, the Minkowskian representation relates to a four-dimensional manifold (the "Universe") with three spacelike coordinates and one timelike coordinate. The three-dimensional universe of Newtonian static field physics is, as is natural, replaced with a four-dimensional one in this dynamical field physics of time-varying fields, in which each point will also be denoted with the time at which it is found at each point. There is nothing more than Lagrange's geometry of four dimensions, a realm in which the aforementioned researchers already examined mechanics – although time does not enter into classical mechanics on an equal basis with the spacelike coordinates, but as an independent parameter, as opposed to Minkowski's theory – when the metric of this four-dimensional space of Minkowski does not enter into it. This metric is (pseudo-) Euclidian in Cartesian coordinates. In that way, we find that the Minkowski notation for Maxwell's differential equations is already strained, and it becomes our problem to free these differential equations from it.

To that end, we next remark that Minkowski's notation (loc. cit. § 7 (*A*) and (*B*) therein, and § 12, in which the dual six-vector was introduced into (*B*) with the generality of scope that was found in Laue's book) will be given unequivocal preference here, as A. Einstein later remarked from the standpoint of general relativity theory ([4]). The deeper grounds for this lies in the fact that in the first form the quadruple (*B*) already has the form of the coefficients of an integral form of third degree, hence, which would best correspond to the basic ideas of field physics, and which has a form that is independent of any metric. The advance that we shall make in this article consists of also bringing the quadruple (*A*) into such a form.

We next give the aforementioned Minkowskian notion. Let us set (in the notion of Weyl ([5])):

$$H_{23} = \mathfrak{H}_x, \qquad H_{31} = \mathfrak{H}_y, \qquad H_{12} = \mathfrak{H}_z, \tag{1}$$
$$H_{14} = -\mathfrak{D}_x\sqrt{-1}, \qquad H_{24} = -\mathfrak{D}_y\sqrt{-1}, \qquad H_{34} = -\mathfrak{D}_z\sqrt{-1},$$

in which we have written:

$$x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = ct\sqrt{-1};$$

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[1] R. Hargreaves, Integral forms and their connection with physical equations, Camb. Phil. Trans. **21** (1908), pp. 107.

[2] H. Bateman, The transformation of the electrodynamical equations. Lond. Math. Soc. Proceed. Ser. 2, vol. **8** (1910), pp. 223, et seq. – Of the numerous later publications of H. Bateman that must be given particular emphasis at this point in time, let us cite: His book "Electrical and Optical Wave Motion," Cambridge University Press, 1915, and his article: "Electromagnetic Vectors," Physical Review **12** (1918), pp. 459. – cf., also, the article of Bateman's student F. D. Murnaghan: "The absolute significance of Maxwell's Equations," Physical Review, **17** (1921), pp. 73, et seq.

[3] F. Kottler, Über die Raumzeitlinien der Minkowski'schen Weit. These Proceedings **121** (1912), esp. § 3.

[4] A. Einstein, Eine neue formale Deutung der Maxwell'schen Feldgleichungen der Elektrodynamik. Berl. Ber., 1916, 1, pp. 184, et seq.

[5] H. Weyl, Raum-Zeit-Materie, 4th ed., (1921), § 23, pp. 173, et seq.

furthermore:

$$S_1 = \frac{\mathbf{i}_x}{c}, \qquad S_2 = \frac{\mathbf{i}_y}{c}, \qquad S_3 = \frac{\mathbf{i}_z}{c}, \qquad S_4 = \rho\sqrt{-1}. \tag{2}$$

Here, H means a vector (alternating tensor) of degree 2 (six-vector), whose spacelike part is the magnetic field strength $\mathfrak{H}$, and whose timelike part is the electrical displacement $\mathfrak{D}$, and furthermore, S is a vector of degree 1 (four-vector) whose spacelike part is the electrical current density $\mathbf{i}$, and whose timelike part is the electrical charge density ρ; finally, c is the velocity of light, x, y, z, t are Cartesian coordinates and time. From Minkowski, one then has:

$$\sum_{k=1}^{4} \frac{\partial H_{ik}}{\partial x_k} = S_i\,, \qquad i = 1, 2, 3, 4 \tag{A}$$

for the four-dimensional form of the first Maxwell quadruple, which reads like:

$$\left.\begin{aligned} \operatorname{rot} \mathfrak{H} - \frac{1}{c}\frac{\partial \mathfrak{D}}{\partial t} = \frac{\mathbf{i}}{c}, \\ \operatorname{div} \mathfrak{D} = \rho, \end{aligned}\right\} \tag{A'}$$

in ordinary notation. The physical meaning of the first three of equations (A'), or, as Maxwell and Lorentz would write:

$$\operatorname{rot} \mathfrak{H} = \frac{1}{c}\frac{\partial \mathfrak{D}}{\partial t} + \frac{\mathbf{i}}{c},$$

is well known: The total electric current is the source of a magnetic field (Biot-Savart law). We rightfully speak of (A') as the electromagnetic law. The last equation in (A') states: The true electric charge is the source of the electrical displacement.

By comparison, formula (A) states: The displacement current and the magnetic rotation are coupled to each other, since they are nothing but the timelike (spacelike, resp.) parts of the spacelike components of the four-dimensional divergence of the six-vector H; the timelike component of this divergence is identical with the three-dimensional divergence of the electrical displacement. The source of this divergence of the six-vector H, the electromagnetic six-vector, is, as we would reasonably like to assume, the electrical four-current S, which combines the Galvanic current (convection current, resp.) with the electrical charge. We also see here a return to Maxwell, who emphasized the unity of the Galvanic and displacement current, because he proposed the magnetic equivalence of both types of current ([1]); from Minkowski's standpoint, this naturally eliminates any meaning to a union of space and time, i.e., of electricity and

[1] This idea of the equivalence with regard to magnetic effects is historically important because the development of modern field notions in empty space took a detour by way of the displacement current. One sees in this the presence of Hertzian notions regarding the extension of electrical forces. Today, this detour is naturally superfluous since the notion of the field is completely established.

magnetism. More precisely, the consideration of the electrical substance (current and charge) thus emerges in (*A*), which is perhaps a deficiency in this form of field physics.

Let us further set:

$$\left.\begin{array}{lll} F_{23} = \mathfrak{B}_x, & F_{31} = \mathfrak{B}_y, & F_{12} = \mathfrak{B}_z, \\ F_{14} = -\mathfrak{E}_x\sqrt{-1} & F_{24} = -\mathfrak{E}_y\sqrt{-1} & F_{34} = -\mathfrak{E}_z\sqrt{-1}. \end{array}\right\} \tag{3}$$

Here, F refers to a vector of degree 2, whose spacelike part is the magnetic induction $\mathfrak{B}$, and whose timelike part is the electrical field strength $\mathfrak{E}$. According to Minkowski, one then has:

$$\frac{\partial F_{lm}}{\partial x_k} + \frac{\partial F_{mk}}{\partial x_l} + \frac{\partial F_{kl}}{\partial x_m} = 0 \qquad (klm) \tag{B}$$

for each of the four combinations (klm) of the four indices 1234 taken three at a time for the four-dimensional form of the second Maxwell quadruple, which reads like:

$$\left.\begin{array}{r} \operatorname{rot}\mathfrak{E} + \dfrac{1}{c}\dfrac{\partial\mathfrak{B}}{\partial t} = 0, \\ \operatorname{div} B = 0 \end{array}\right\} \tag{B'}$$

in the ordinary Maxwell notation. The physical meaning of the first of the three equations in (B'), or, as Maxwell and Lorentz would write:

$$\operatorname{rot}\mathfrak{E} = -\frac{1}{c}\frac{\partial\mathfrak{B}}{\partial t}$$

is well known: The magnetic induction current is the source of an electrical field (Faraday's law of induction). We rightfully speak of (B') as the magneto-electric law. The last equation of (B') states: The source of the magnetic induction is zero.

In contrast to this, formula (*B*) states: The magnetic induction current and the electrical rotation belong together since they are nothing but the timelike (spacelike, resp.) parts of the spacelike components of the four-dimensional rotation of the six-vector *F*; the timelike components of this rotation are identical with the three-dimensional divergence of the magnetic induction. The source of this rotation of the six-vector *F*, as we would like to reasonably assume, is the magnetic four-current, when it is given. However, since the true magnetic current and the true magnetic charge are null in the usual Maxwellian formulation, the rotation of the six-vector *F* is equal to null.

What we have denoted here as the four-dimensional divergence (four-dimensional rotation, resp.) of a six-vector proves, on closer examination, although we shall say nothing further about this, to be the interior (exterior, resp.) product (in the Grassmann sense) of the so-called vector operator ∇, whose components are:

$$\nabla_1 = \frac{\partial}{\partial x_1}, \qquad \nabla_2 = \frac{\partial}{\partial x_2}, \qquad \nabla_3 = \frac{\partial}{\partial x_3}, \qquad \nabla_4 = \frac{\partial}{\partial x_4},$$

with the six-vector.

2.

We once more return to the notation that was used by Hargreaves that represents the eight Maxwell differential equations in the form of two integral laws whose true nature will be revealed and in terms of which the idea of local action in physics is most reasonable.

The quadruple (B) already has, as we mentioned, the desired form. Then:

$$\frac{\partial F_{lm}}{\partial x_k} + \frac{\partial F_{mk}}{\partial x_l} + \frac{\partial F_{kl}}{\partial x_m} = \mathbf{F}_{klm}$$

has the form of the coefficients of an integral form of third degree, as they appear in the four-dimensional general theorem of Gauss and Green ([1]):

$$\iint F_{23}\, dx_2\, dx_3 + F_{31}\, dx_3\, dx_1 + F_{12}\, dx_1\, dx_2 + F_{14}\, dx_1\, dx_4 + F_{24}\, dx_2\, dx_4 + F_{34}\, dx_3\, dx_4 =$$
$$= \iiint \mathbf{F}_{234}\, dx_2\, dx_3\, dx_4 + \mathbf{F}_{134}\, dx_1\, dx_3\, dx_4 + \mathbf{F}_{124}\, dx_1\, dx_2\, dx_4 + \mathbf{F}_{123}\, dx_1\, dx_2\, dx_3 \,. \tag{4}$$

In (4), the double integral ranges over a closed two-dimensional surface that bounds a three-dimensional space, over which the triple integral ranges. Accordingly, Hargreaves and Bateman replace the quadruple (B) with the integral theorem:

$$\iint \sum_{(ik)} F_{ik}\, dx_i\, dx_k = 0, \tag{II}$$

in which the sum is over all six combinations (ik) of the four indices 1234 taken two at a time.

For the quadruple (A), this formalism is not so simple. If one thus introduces six-vector $\overset{*}{H}$, which is complementary (dual) to H, (cf., N, no. 3), then one arrives at a formula that is analogous to (II). Since, for Minkowski, we have a Euclidian metric in Cartesian coordinates the covariant and contravariant coordinates are identical, and we may set:

$$H_{12} = \overset{*}{H}_{34}\,, \qquad H_{13} = \overset{*}{H}_{42}\,, \qquad H_{14} = \overset{*}{H}_{23}\,,$$
$$H_{34} = \overset{*}{H}_{12}\,, \qquad H_{24} = \overset{*}{H}_{31}\,, \qquad H_{23} = \overset{*}{H}_{14}\,.$$

Instead of $\overset{*}{H}$, we write good old E, and thus have:

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[1] On this, cf., F. Kottler, Über die Raumzeitlinien der Minkowski'schen Welt. These Proceedings, **121** (1912), § 2. – For the notion, cf. N, no. 2.

$$\begin{aligned}
\sum_k \frac{\partial H_{1k}}{\partial x_k} &= \frac{\partial}{\partial x_2} E_{34} + \frac{\partial}{\partial x_3} E_{42} + \frac{\partial}{\partial x_4} E_{23} &&= \mathbf{E}_{234} \,, \\
\sum_k \frac{\partial H_{2k}}{\partial x_k} &= -\frac{\partial}{\partial x_1} E_{34} - \frac{\partial}{\partial x_3} E_{41} - \frac{\partial}{\partial x_4} E_{13} &&= -\mathbf{E}_{134} \,, \\
\sum_k \frac{\partial H_{3k}}{\partial x_k} &= \frac{\partial}{\partial x_1} E_{24} + \frac{\partial}{\partial x_2} E_{41} + \frac{\partial}{\partial x_4} E_{12} &&= \mathbf{E}_{124} \,, \\
\sum_k \frac{\partial H_{4k}}{\partial x_k} &= -\frac{\partial}{\partial x_1} E_{23} - \frac{\partial}{\partial x_2} E_{31} - \frac{\partial}{\partial x_3} E_{12} &&= -\mathbf{E}_{123} \,.
\end{aligned}$$

Likewise, we introduce the vector $\overset{*}{S}$ of third degree that is complementary to the vector S of first degree ([1]); moreover, we have, in which we write $\mathbf{S}$ instead of $\overset{*}{S}$:

$$S_1 = \mathbf{S}_{234} \,, \qquad S_2 = -\mathbf{S}_{134} \,, \qquad S_3 = \mathbf{S}_{124} \,, \qquad S_4 = -\mathbf{S}_{123} \,,$$

in which one naturally has, e.g.:

$$\mathbf{S}_{123} = -\mathbf{S}_{132} = -\mathbf{S}_{213} = \mathbf{S}_{231} = \mathbf{S}_{312} = -\mathbf{S}_{321} \,.$$

This therefore means that:

$$\begin{aligned}
&E_{23} = -\mathfrak{D}_x \sqrt{-1}, &&E_{31} = -\mathfrak{D}_y \sqrt{-1}, &&E_{12} = -\mathfrak{D}_z \sqrt{-1}, \\
&E_{14} = \mathfrak{H}_x, &&E_{24} = \mathfrak{H}_y, &&E_{34} = \mathfrak{H}_z \,;
\end{aligned} \tag{5}$$

furthermore:

$$\mathbf{S}_{234} = \frac{\mathfrak{i}_x}{c}, \qquad \mathbf{S}_{134} = -\frac{\mathfrak{i}_y}{c}, \qquad \mathbf{S}_{124} = \frac{\mathfrak{i}_z}{c}, \qquad \mathbf{S}_{123} = \rho\sqrt{-1}. \tag{6}$$

We then come to the quadruple:

$$\mathbf{E}_{klm} = \mathbf{S}_{klm} \qquad (klm)$$

in place of (A), which can also take the form of an integral law:

$$\iint \sum_{(ik)} E_{ik}\, dx_i\, dx_k = \iiint \sum_{(ikl)} \mathbf{S}_{klm}\, dx_i\, dx_k\, dx_l \,. \tag{I}$$

The first sum goes over all six combinations (ik) of the four indices 1234 taken two at a time, and the second sum goes over all four combinations (ikl) of the four indices

[1] Here, one occasionally remarks that the complement of $\overset{*}{S}$ is not S again, but $-S$; then $\overset{*}{\mathbf{S}}_1 = -\mathbf{S}_{231} = -S_1$, etc. Cf., the remarks on pp. 8 of article N.

1234 taken two at a time. The double integral is taken over the closed two-dimensional bounding surface of a three-dimensional space, over which the triple integral is taken.

3.

We would like to once more focus our consideration of Maxwell's equations and their metric upon the formulas (I) and (II). One immediately sees that both integral theorems (I) and (II) are independent of any metric, and independent of the meaning that one ascribes to the coordinates x. They can then be generalized to the spacetime (dynamical) picture that Faraday envisioned, in which the spacelike (static) force effect of the field in any closed neighborhood can be replaced with one that takes place on the bounding surface. Hence, these integral theorems must be essential invariants under all transformations of the coordinate systems, so they certainly cannot be linked with any metric because one would prefer not to deal with such a thing in the evaluation of boundary surface integrals and the like. Rather, only the theorems of integral calculus are necessary; i.e., the manifolds over which one integrates must be made more precise in the sense of Analysis Situs (connectivity relations, singularities, etc.), not, however, in the sense of metric geometry ([1]).

Accordingly, we would like regard the x in (I) and (II) as any coordinates in any four-fold extended, simply connected, nowhere singular manifold, in which we, in order to preserve the reality conditions, would like to distinguish the x_1, x_2, x_3 as the spacelike coordinates and x_4 as the timelike one. The form of (I) and (II) naturally remains the same under any transformation to new coordinates x', as is already known from Jacobi's application of such integrals to the transformation of the Laplacian differential expressions. It thus transforms the E and the F like covariant vectors (alternating tensors) of second degree, hence, e.g.:

$$E_{ik} = \sum_{p,q} E'_{pq} \frac{\partial x'_p}{\partial x_i} \frac{\partial x'_q}{\partial x_k} \qquad (ik)$$

etc., and the **S** naturally transform like covariant vectors of third degree:

$$\mathbf{S}_{ikl} = \sum_{pqr} S'_{pqr} \frac{\partial x'_p}{\partial x_i} \frac{\partial x'_q}{\partial x_k} \frac{\partial x'_r}{\partial x_l}, \qquad (ikl).$$

In place of the integral theorems (I) and (II), we can also pose the eight differential equations:

$$\mathbf{E}_{klm} = \mathbf{S}_{klm}, \qquad (klm) \qquad \text{(I } a\text{)}$$

$$\mathbf{F}_{klm} = 0, \qquad (klm). \qquad \text{(II } a\text{)}$$

This is therefore the archetype for the form that the Maxwell equations become when they are expressed independently of any metric. In these equations, we mean, for future reference:

[1] On this, cf., H. Poincaré, Analysis situs. Journal de l'école polytechnique, 2. série, cah. 1 (1895), § 7.

$$\mathbf{E}_{klm} = \frac{\partial}{\partial x_k} E_{lm} + \frac{\partial}{\partial x_l} E_{mk} + \frac{\partial}{\partial x_m} E_{kl} \, ,$$
$$\mathbf{F}_{klm} = \frac{\partial}{\partial x_k} F_{lm} + \frac{\partial}{\partial x_l} F_{mk} + \frac{\partial}{\partial x_m} F_{kl} \, ,$$

and E is the electromagnetic six-vector, while F is the magneto-electric six-vector. Furthermore, **S** is the vector of third degree that yields the amount of electricity that flows through the "spatially" oriented surface element in the time element (is contained in the "spatially" oriented volume element, resp.): i.e., the electrical four-current. One easily finds that the next higher construction in the **E**:

$$\mathbf{E}_{1234} = \frac{\partial}{\partial x_1}\mathbf{E}_{234} - \frac{\partial}{\partial x_2}\mathbf{E}_{134} - \frac{\partial}{\partial x_3}\mathbf{E}_{124} - \frac{\partial}{\partial x_4}\mathbf{E}_{123} \equiv 0$$

vanishes identically. Hence, one has the continuity equation of the electrical current:

$$\mathbf{S}_{1234} = \frac{\partial}{\partial x_1}\mathbf{S}_{234} - \frac{\partial}{\partial x_2}\mathbf{S}_{134} - \frac{\partial}{\partial x_3}\mathbf{S}_{124} - \frac{\partial}{\partial x_4}\mathbf{S}_{123} \equiv 0 \, .$$

In order to understand the physical meaning of the Maxwell equations in the new form (I *a*) and (II *a*) we revert, for the moment, to the special Minkowski interpretation, hence, to equations (3) (equations (5), resp.) for the magneto-electric vector F (the electromagnetic vector E, resp.). There, the spacelike part of F is the magnetic induction $\mathfrak{B}$ and the timelike part of F is the electric field $\mathfrak{E}$; furthermore, the spacelike part of E is the electric induction $\mathfrak{D}$ (as we would like to say instead of "displacement," by way of analogy) and the timelike part of E is the magnetic field $\mathfrak{H}$. As is well known, in Maxwell's phenomenological picture only the "field" reigns in the ether, whereas in ponderable matter the field will increase due to the effects that the matter contributes, such as magnetization (polarization, resp.), in the form of "induction." Therefore, the induction and the field in a non-ferromagnetic (isotropic, resp.) medium are proportional to each other, and the proportionality factor is the magnetic permeability (the dielectric constant, resp.), which represent constitutive relations:

$$\mathfrak{B} = \mu\, \mathfrak{H}, \qquad \mathfrak{D} = \varepsilon\, \mathfrak{F} \, .$$

In the Minkowski picture, the distinction between space and time dissolves; therefore, the magnetic induction and the electric field combine into a single entity, the magneto-electric vector F, and likewise the electric induction and the magnetic field combine into the electromagnetic vector E. (One notes the strict analogy between induction and field in both cases, which was not true for Minkowski, since he coupled the complementary vector H to the vector F, which is strictly analogous to E, instead of E.) We will now examine both vectors E and F with regard to their four-dimensional rotation. This rotation is different from null only where a true electric (magnetic, resp.)

four-current exists. Since the true magnetic four-current always vanishes, the rotation of the magneto-electric vector is always null.

Physically speaking, one thus has in *E*, the "field" [combined magnetic field and electrical induction (displacement-) current] of a true electrical quantity (whether moving or at rest), and in *F*, the "field" (combined electric field and magnetic induction current) of a true magnetic quantity (when it is given); *E* represents the electromagnetic effect of the electricity and *F* represents the magneto-electric effect of the magnetism.

Corresponding to the phenomenological viewpoint of field physics, which only tries to describe collective phenomena, but not causal orientation, one can naturally exchange causes and effects in the previous statements and say: The electromagnetic field *E* produces singularities, viz., the true electric substance, and the magneto-electric field *F* produces singularities, viz., the true magnetic substance.

In this formulation, there are obviously no opposing reactions between the electromagnetic and magneto-electric fields. Then, from the well-known laws of induction (Lenz's rule) one has that, e.g., a magnetic field that is produced by an electric current tends to weaken the current that produced it; in this example, the electromagnetic effect thus has a magneto-electric effect as a consequence that tends to weaken it. This is therefore an obvious failing of field physics. In truth, this reaction already assumes the statement of constitutive relations of the type (7) between both vectors *E* and *F*, and it will be shown that these relations are based on the assumption of a metric, hence, on one of the quantitative features that are foreign to the original "qualitative ([1])" field theory.

In the following section, it will also be shown that this metric follows from the measurement of the velocity of light; in fact, as has been completely understood since the time of W. Weber and F. Kohlsrausch, the velocity of light plays an essential role in all of the relations that couple electricity and magnetism, and indeed in all practical measurements in the neighborhood of them. However, since the later development of special relativity has shown, a theory that is linked with difficult problems with the velocity of light, problems that A. Einstein gave a provisional resolution to with his postulate of the constancy of the velocity of light, we cannot set aside the cornerstone of pure field physics, viz., the qualitative description of the coupling relations between the electromagnetic field and the magneto-electric field, or ignore the metrically quantitative physics in the latter, hence, in a neighborhood in which the arbitrary convention plays an essential role. The absence of opposing reactions between the electromagnetic and the magneto-electric fields is therefore no failing of pure field physics. (On this, we remark that, as will be shown in no. 6, one can also not speak of an energetic viewpoint. The energetic viewpoint [the impossibility of the *perpetuum mobile*] can support Lenz's rule, but only when the dielectric constant and permeability are already known.)

Before we leave behind the prototype for Maxwell's equations (I *a*) ((II *a*, resp.), we need to make a formal remark. As was known in N, no. 3, the geometrical principle of duality also governs vectors as geometrical quantities. One can thus introduce the complementation of *E* and *F* only when one takes care to choose an **arbitrary** proportionality factor that one omits, and which proves to be a vector of fourth degree here:

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[1] For Poincaré, qualitative refers to the Analysis Situs, as opposed to the quantitative metric (despite the use of coordinates, hence, numbers introduced for the purpose of ordering). Cf., Poincaré, La valeur de la science. Chap III, § 2: "la géometrie qualitative."

$$\varepsilon_{1234},$$

and which naturally varies from position to position. In order to remain in agreement with the usual notation (*A*), (*B*) for Maxwell's equations, we confine ourselves to that convention, so as to introduce the complementation solely for *E* (and *S*), but not for *F*. One has:

$$E_{12} = \varepsilon_{1234}\overset{*}{E}{}^{34}, \qquad E_{13} = \varepsilon_{1234}\overset{*}{E}{}^{42}, \qquad E_{14} = \varepsilon_{1234}\overset{*}{E}{}^{23},$$
$$E_{34} = \varepsilon_{1234}\overset{*}{E}{}^{12}, \qquad E_{42} = \varepsilon_{1234}\overset{*}{E}{}^{13}, \qquad E_{23} = \varepsilon_{1234}\overset{*}{E}{}^{14},$$

in which $\overset{*}{E}$ is the contravariant vector of degree 2 that is complementary to E. Likewise:

$$\mathbf{S}_{234} = -\varepsilon_{1234}\overset{*}{\mathbf{S}}{}^{1}, \qquad \mathbf{S}_{134} = +\varepsilon_{1234}\overset{*}{\mathbf{S}}{}^{2}, \qquad \mathbf{S}_{124} = -\varepsilon_{1234}\overset{*}{\mathbf{S}}{}^{3},$$
$$\mathbf{S}_{123} = -\varepsilon_{1234}\overset{*}{\mathbf{S}}{}^{4},$$

and therefore, instead of (I *a*), we have:

$$\mathbf{E}_{234} = \sum_k \frac{\partial}{\partial x_k}(\varepsilon_{1234}\overset{*}{E}{}^{1k}) = \mathbf{S}_{234} = -\varepsilon_{1234}\overset{*}{\mathbf{S}}{}^{1},$$
$$\mathbf{E}_{134} = -\sum_k \frac{\partial}{\partial x_k}(\varepsilon_{1234}\overset{*}{E}{}^{2k}) = \mathbf{S}_{134} = +\varepsilon_{1234}\overset{*}{\mathbf{S}}{}^{2},$$
$$\mathbf{E}_{124} = \sum_k \frac{\partial}{\partial x_k}(\varepsilon_{1234}\overset{*}{E}{}^{3k}) = \mathbf{S}_{124} = -\varepsilon_{1234}\overset{*}{\mathbf{S}}{}^{3},$$
$$\mathbf{E}_{123} = -\sum_k \frac{\partial}{\partial x_k}(\varepsilon_{1234}\overset{*}{E}{}^{4k}) = \mathbf{S}_{123} = +\varepsilon_{1234}\overset{*}{\mathbf{S}}{}^{4},$$

or, when summarized for the Maxwellian quadruple:

$$\frac{1}{\varepsilon_{1234}}\sum_k \frac{\partial}{\partial x_k}(\varepsilon_{1234}\overset{*}{E}{}^{ik}) = -\overset{*}{\mathbf{S}}{}^{1}, \qquad i = 12, 34 \tag{I b}$$
$$\mathbf{F}_{klm} = \frac{\partial F_{lm}}{\partial x_k} + \frac{\partial F_{mk}}{\partial x_l} + \frac{\partial F_{kl}}{\partial x_m} = 0 \qquad (klm)\,. \tag{II b}$$

Thus, $\overset{*}{\mathbf{S}}$ satisfies the continuity equation:

$$\sum_i \frac{\partial}{\partial x_i}(\varepsilon_{1234}\overset{*}{\mathbf{S}}{}^{i}) = 0\,.$$

4.

After having presented Maxwell's equations in the previous section in a form (I *a*) ((II *a*), resp.) that was independent of any metric, we must ask how it is that a metric enters into the ordinary representation, a metric that modern relativity theory even presents as its result.

The basis for this is the constitutive relations between the two vectors *E* and *F*. Here, we would next like to discuss which viewpoint is necessary for their introduction. Up till now, we have regarded the electromagnetic field *E* and the magneto-electric field *F* as known and then computed the electric four-current **S** (the null magnetic four-current, resp.). However, in our experience, it is not the field that is given, since we possess no sense organ that could make us aware of the presence of an electromagnetic field. Rather, we base our experience primarily on our sense of sight, which recognizes motion, but not fields. We thus try to ensure that all of our observations will lead back to observations of a kinematical nature. Thus, we know only the coordinates x_1, x_2, x_3, x_4, which establish the position and time for an event; e.g., the ponderomotive effect of a field. Accordingly, instead of regarding the field quantities as the quintessential givens, as we did before, we must regard them as unknowns whose connection to the actual knowns x_1, x_2, x_3, x_4 is yet to be discovered, hence, their nature as dependent functions of these independent variables. The integral theorems I and II (their corollaries I *a* and II *a*, resp.), when applied to this way of thinking, are transformed into eight partial differential equations for the twelve unknowns *E* and *F*. One can regard the four quantities **S** as given since they are connected with a motion, namely, the current of electricity.

However, if one considers the eight differential equations (I *a*), (II *a*) for the twelve unknowns *E* and *F* from this new viewpoint then one sees that the problem of integrating them is underdetermined. One next sees that the four equations (I *a*):

$$\mathbf{E}_{klm} = \mathbf{S}_{klm} \qquad (klm) \qquad \text{(I } a\text{)}$$

are not independent of each other, since the identity:

$$\mathbf{E}_{1234} = \frac{\partial}{\partial x_1}\mathbf{E}_{234} - \frac{\partial}{\partial x_2}\mathbf{E}_{134} + \frac{\partial}{\partial x_3}\mathbf{E}_{124} - \frac{\partial}{\partial x_4}\mathbf{E}_{123} \equiv 0$$

exists. Likewise, the four equations (II *a*):

$$\mathbf{E}_{klm} = 0 \qquad (klm) \qquad \text{(II } a\text{)}$$

are not independent of each other, since the identity:

$$\mathbf{F}_{1234} = \frac{\partial}{\partial x_1}\mathbf{F}_{234} - \frac{\partial}{\partial x_2}\mathbf{F}_{134} + \frac{\partial}{\partial x_3}\mathbf{F}_{124} - \frac{\partial}{\partial x_4}\mathbf{F}_{123} \equiv 0$$

exists. We thus obtain only six independent differential equations for twelve unknowns, viz., six equations too few. Thus, if one would like to define a well-determined integration problem then one must add six algebraic equations to the six differential equations that one may naturally regard as arbitrary. In this ambiguity, the previously-mentioned conventional element, and likewise the dubious side of Maxwellian theory, such as the numerous difficulties in crystal optics, ferromagnetic theory, and not least of all, the theory of optical phenomena in moving media (relativity theory), has been sufficiently substantiated historically.

The aforementioned six equations that are to be appended are precisely the six constitutive relations that represent the six quantities E as functions of the six quantities F. One thus chooses ordinary linear relations, such as, for example, (7) or the relations for the anisotropic media of crystal optics; naturally, one must never forget that linear approximations are usually nothing but approximations, initial terms in a Taylor series development for a complicated function that has been truncated after the first power. (Confer the analogous case of Hooke's law in elasticity theory.) Thus, if, as is often asserted nowadays, Maxwell's equations cannot be extended to describe the phenomena inside of matter, which is entirely correct, then the basis for this must lie completely in the linearity of the Maxwellian constitutive relation as an approximation that is not satisfied for ponderable matter, and not in Maxwell's equations themselves.

We would now like to give a special example of how the constitutive equations between E and F lead to a metric as a consequence; indeed, we would like to take the case of matter-free space. We will thus obtain the Minkowski metric (the generalized Einstein metric when we consider gravitation, resp.) The mathematical process is thus similar to the introduction of the metric in N, sec. 4: there, the covariant vector of force was derived from the contravariant complement of force flux by means of a polar correlation (orthogonality). Likewise, we will use a polar correlation to derive a covariant vector from the contravariant complement of the magneto-electric vector F, which we would then like to identify with the vector E. The vector F and the polar reciprocal vector to its complement are then both solutions to the Maxwell equations for the case of matter-free space.

One thus has the constitutive equations:

$$E_{ik} = \sum_{p,q} a_{ip} a_{kq} \overset{*}{F}{}^{pq} \qquad (ik) \tag{8}$$

in which the form:

$$\sum_{i,k=1}^{4} a_{ik} \xi_i \xi_k \tag{9}$$

for the orthogonality has been assumed (cf., N, sec. 4). If the a_{ik} are constant and, in particular:

$$a_{ik} = \delta_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

then one has the Minkowski metric, whereas if they are variable functions of the chosen position x_1, x_2, x_3, x_4 then one has the Einstein metric (hence, gravitation). Furthermore, one now has, as in N, sec. 4:

$$\varepsilon_{1234} = \sqrt{a} \tag{10}$$

which makes:

$$F_{12} = \overset{*}{F}{}^{34}\sqrt{a}\,, \qquad F_{13} = \overset{*}{F}{}^{42}\sqrt{a}\,, \qquad F_{14} = \overset{*}{F}{}^{23}\sqrt{a}\,,$$
$$F_{34} = \overset{*}{F}{}^{12}\sqrt{a}\,, \qquad F_{42} = \overset{*}{F}{}^{13}\sqrt{a}\,, \qquad F_{23} = \overset{*}{F}{}^{14}\sqrt{a}\,.$$

If one chooses, as Minkowski did, $a_{ik} = \delta_{ik}$ then one has, in particular:

$$E_{12} = \overset{*}{F}{}^{12} = F_{34}, \qquad E_{13} = \overset{*}{F}{}^{13} = F_{42}, \text{ etc.},$$

or [cf., (3) and (5)]:

$$\mathfrak{D} = \mathfrak{E}, \qquad \mathfrak{B} = \mathfrak{H}.$$

In matter-free space with no gravitational field, field and induction are equal to each other. In general, according to Einstein, in matter-free space with a gravitational field one has, if we write g_{ik} instead of a_{ik},:

$$E_{ik} = \sum_{p,q} g_{ip}\, g_{kq} \cdot \frac{1}{\sqrt{g}} \cdot F_{p'q'} \qquad (ik)$$

in which $pqp'q'$ means a positive permutation of the indices 1234. If one further sets, in which x no longer refers to a Cartesian coordinate:

$$E_{23} = -\,\mathfrak{D}_1\sqrt{-1}\,, \qquad E_{31} = -\,\mathfrak{D}_2\sqrt{-1}\,,$$
$$E_{14} = \quad \mathfrak{H}_1\sqrt{-1}\,, \qquad E_{24} = \quad \mathfrak{H}_2\sqrt{-1}\,, \qquad \text{etc.},$$

and likewise:

$$F_{23} = \quad \mathfrak{B}_1, \qquad F_{31} = \quad \mathfrak{B}_2,$$
$$F_{14} = -\,\mathfrak{E}_1\sqrt{-1}\,, \qquad F_{24} = -\,\mathfrak{E}_2\sqrt{-1}\,, \qquad \text{etc.},$$

then one has, e.g.:

$$\begin{aligned} -\,\mathfrak{D}_1\sqrt{-1}\cdot\sqrt{g} \;=\; & (g_{21}\,g_{32} - g_{22}\,g_{31})\,(-\,\mathfrak{E}_3\sqrt{-1}) \\ & + (g_{23}\,g_{31} - g_{21}\,g_{33})\,(-\,\mathfrak{E}_2\sqrt{-1}) \\ & + (g_{22}\,g_{33} - g_{24}\,g_{32})\,(-\,\mathfrak{E}_1\sqrt{-1}) \\ & + (g_{21}\,g_{34} - g_{24}\,g_{31})\quad \mathfrak{B}_1 \\ & + (g_{22}\,g_{34} - g_{24}\,g_{32})\quad \mathfrak{B}_2 \\ & + (g_{23}\,g_{34} - g_{24}\,g_{33})\quad \mathfrak{B}_3, \qquad \text{etc.,} \end{aligned} \tag{12}$$

which again reduces to $\mathfrak{D}_1 = \mathfrak{E}_1$ if $g_{ik} = \delta_{ik}$.

If one considers the constitutive relations (8) then one may eliminate the vector E by way of F in the (I a) ((I b), resp.), and one obtains, e.g., in place of (I b) (II b), as a simple calculation shows ([1]):

$$\sum_{k,p,q} \frac{\partial}{\partial x_k}(\sqrt{a}\, a^{ip} a^{kq} F_{pq}) = S^i \sqrt{a} \qquad i = 1, 2, 3, 4 \tag{I c}$$

$$\frac{\partial F_{lm}}{\partial x_k} + \frac{\partial F_{mk}}{\partial x_l} + \frac{\partial F_{kl}}{\partial x_m} = 0 \qquad (klm). \tag{II c}$$

In this, we have set $\overset{*}{\mathbf{S}}{}^i = - S^i$. This is the usual preferred notation for Maxwell's equations (for matter-free space), which is already admittedly burdened with a well-defined metric. Thus, the problem of integrating (I c) and (II c) when S is given is completely determined.

For Minkowski's metric ($a_{ik} = \delta_{ik}$) one has (in matter-free space):

$$\sum_k \frac{\partial}{\partial x_k} F_{ik} = S^i \qquad i \tag{I d}$$

$$\frac{\partial F_{lm}}{\partial x_k} + \frac{\partial F_{mk}}{\partial x_l} + \frac{\partial F_{kl}}{\partial x_m} = 0 \qquad (klm), \tag{II d}$$

which one would like to equate with (A) and (B) in sec. 1 ([2]).

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[1] Cf., F. Kottler, Über die Raumzeitlinien der Minkowsi'schen Welt, these Proceedings **121** (1912), pp. 1688, in which one finds an easily recognizable typographical error.

[2] Here, two passing remarks are in order:

The first one concerns equations (II). From Volterra's generalization of complex variables in n-dimensional spaces, a connection such as the one that one finds here between the coefficients E and F of two integral forms of second degree in a four-dimensional space is governed by the theory of conjugate functions. Cf., V. Volterra, Linc. Rend. Ser. V, vol. 5, sem. 1 (1889), pp. 599, et seq., "sulle funzioni conjugate" or "Leçons sur l'integration des equations aux dérivées partielles professées à Stockholm," Paris 1912, lessons 5 and 6. Volterra operates with integrals of the type:

$$\iint \sum_{(lk)} F_{lk}\, dx^l dx^k \,,$$

which are taken over an open surface such that $\mathbf{F}_{klm} = 0$ depends only upon the boundary curve of this surface, and which are called functions of this curve (fonctions de ligne). The conjugate function to it is:

$$\iint \sum_{(ik)} \overset{*}{F}_{ik}\, dx^i dx^k \,,$$

in which $\overset{*}{F}_{12} = F_{34}$, etc.; hence, relations (11) or the Euclidian metric in Cartesian coordinates must apply. In any case, these conjugate functions must be a functions of the curve; i.e., $\overset{*}{\mathbf{F}}_{klm} \equiv 0$. Thus, the F satisfy equations of the form (I d) (with $S = 0$) and (II d) On such systems of differential equations cf., Volterra "Sulla integrazioni di un sistema di equazione differenziali a derivate parziali, etc.," Rend. Circ. Palermo **3** (1889), pp. 260, et seq. The term "conjugate function" is justified since the analogy with the Cauchy integral:

5.

The definition of the constitutive relations that we just carried out determines the dielectric and magnetic properties of the medium. Physically speaking, this has two senses: a static one and a dynamical one. However, it is well-known that one can regard the static case as a limiting case of the dynamical one, namely, one can obtain statics by considering electromagnetic waves with infinitely large wavelengths. Accordingly, being given the dielectric and magnetic properties that we briefly mentioned equivalent to being given the laws of radiation for electromagnetic fields inside the medium considered. So to speak, our metric thus amounts to the same thing as the measurement of the velocity of light.

The radiation of waves is known to be a chapter in the theory of partial differential equations that unfortunately has been developed essentially only for partial differential equations in one unknown up till now, whereas here we have, e.g., in (I *d*), (II *d*), a system of partial differential equations in six unknowns. We must thus refrain from

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$$\int f(z)dz$$

of classical function theory is valid when this integral is taken along an open curve the two-dimensional complex z-plane. When f is a univalent function of the complex variable z this integral depends, in any case, upon the boundary points (point function). If one sets $z = x_1 + ix_2$, $f(z) = u_1 + iu_2$ then one obtains two real integrals:

$$\int u_1\,dx_1 - u_2\,dx_2 \qquad \left(\int u_2\,dx_1 + u_1\,dx_2, \text{ resp.}\right),$$

which are point functions that are conjugate to each other. Thus, the u_1, u_2 satisfy the Cauchy-Riemann differential equations, which are thus the two-dimensional analog of the Maxwell differential equations (I *d*) (with $S = 0$) and (II *d*). Related facts are found in other places. Cf., also L. Hanni, "Über den Zusammenhang zwischen den Cauchy-Riemann'schen und den Maxwell'schen Differentialgleichungen," Tôhuku Math. Journal **5** (1914), pp. 142 to 175. – One sees in the foregoing the origin of the Euclidian metric in the complex plane of classical function theory: two line integrals are given that are merely point functions; this corresponds to the viewpoint of Analysis Situs. They become united as the conjugate functions to a single function of one complex variable with the help of constitutive equations of the form (11) between the coefficients of their integral forms of first degree on two-dimensional space. Through these constitutive equations, we introduce a metric, as in our text.

The second remark concerns equations (12). Such equations were already found for the establishment of Einstein's generalized metric by Bateman in the cited article "The transformation of the electrodynamical equations," loc. cit., pp. 259, et seq, in particular formula (2) Bateman then chose a form for the form (9) in our text that included dielectric and permeability constants in order to obtain Minkowski's completely understood relations for moving ponderable matter in this way. This is therefore fundamentally unacceptable, since special relativity theory must be true for the form (9) with $a_{ik} = \delta_{ik}$, as well for ponderable matter. The correct form of the constitutive equations for moving matter no longer depends, like (12), simply upon the fundamental metric form alone. In passing, these forms are coupled here by:

$$\mu E_{i'k'} = F_{ik} - (\varepsilon\mu - 1)\frac{S_i \cdot \sum_r F_{kr} S_r - S_k \cdot \sum_r F_{ir} S_r}{\sum_r S_r^2} \qquad (ik)$$

In this, $iki'k'$ is a positive permutation of 1234. For electricity at rest ($S_1 = S_2 = S_3 = 0$) this turns into the well-known constitutive equations (7) of our text for ponderable matter at rest.

treating the deeper connection between the metric in Maxwell's equations with aforementioned mathematical theory of wave fronts or characteristics ([1]).

In the following, we treat the problem of founding the metric of the field equations on the laws of radiation for the field in a somewhat different way. We restrict ourselves to the Minkowskian equations (I *d*) (II *d*) in electricity-free space ($S = 0$):

$$\sum_k \frac{\partial}{\partial x_k} F_{ik} = 0 \qquad i \tag{13}$$

$$\frac{\partial F_{lm}}{\partial x_k} + \frac{\partial F_{mk}}{\partial x_l} + \frac{\partial F_{kl}}{\partial x_m} = 0 \qquad (klm), \tag{14}$$

integrate them by way of the well-known integral in terms of planar light waves, and then show that, conversely, when the field vectors for these waves have the characteristic form then the associated field equations (I *a*) (II *a*) must exhibit the Minkowski metric, hence, the form (I *d*), (II *d*), or the constitutive equations (11). We remark that since all of optics is defined in terms of plane waves we will most certainly confine ourselves to that realm of wave radiation.

One obtains plane waves when one demands that all components of the field vectors must be proportional to one and the same function φ that is linear in its arguments:

$$\sum_{h=1}^{4} C_h x_h$$

(the phase). With this, one next obtains for the magneto-electric field vector F:

$$F_{ik} = A_{ik}\,\varphi\left(\sum_h C_h x_h\right) \tag{15}$$

in which $A_{ik} = -A_{ki}$, and similarly the C_h shall be constants. One obtains from (14):

$$A_{ik}\,C_k + A_{mk}\,C_l + A_{kl}\,C_m = 0 \qquad (klm),$$

equations from which one easily deduces:

$$A_{ik} = C_i\,D_k - C_k\,D_i \qquad (ik)$$

in which D is a constant vector. Thus, it follows from (13) that:

[1] J. Hadamard, Leçons sur la propagation des ondes. Paris 1903, Chap. VII. – R. d'Adhemer, Les équations aux dérivées partielles à caracteristiques réelles. Coll. Scientia, no. 29, Paris 1907. – In the theory of characteristics, the four-dimensional (*n*-dimensional, resp.) metric of relativity theory usually appears. Cf., the "conormal" of d'Adhemar. C.R., 11 February 1901. Cf., also Hadamard, Theorie du problème de Cauchy, Acta Math. **31** (1907), pp. 333 et seq., in particular, pp. 334 to 336.

$$C_i \sum_k D_k C_k - D_i \sum_k C_k^2 = 0 \qquad i$$

from which it necessarily follows that:

$$\sum C_k^2 = \sum C_k D_k = 0.$$

As is well-known, the first of these equations states the existence of radiation with the velocity of light. One sets:

$$C_1 = \mathfrak{u}_x, \qquad C_2 = \mathfrak{u}_y, \qquad C_3 = \mathfrak{u}_z,$$

in which $\mathfrak{u}$ is the unit vector to the three-dimensional wave normal ($\mathfrak{u}_x^2 + \mathfrak{u}_y^2 + \mathfrak{u}_z^2 = 1$). One then necessarily has: $C_4 = \pm\sqrt{-1}$. If one chooses the upper sign, which is permissible, then one has:

$$\sum_{h=1}^{4} C_h x_h = \mathfrak{u}_x x + \mathfrak{u}_y y + \mathfrak{u}_z z - ct,$$

from which the radiation of waves with light velocity along the direction of the wave normal $\mathfrak{u}$ becomes evident.

The second of the equations above asserts the transversality of the plane waves. Namely, one now has:

$$F_{ik} = (C_i D_k - C_k D_i)\,\varphi\left(\sum_k C_h x_h\right) \qquad (ik), \qquad (15\ a)$$

in which we have simply set:

$$D_1 = \mathfrak{a}_x, \qquad D_2 = \mathfrak{a}_y, \qquad D_3 = \mathfrak{a}_z, \qquad D_4 = 0,$$

in which $\mathfrak{a}$ is a three-dimensional vector (amplitude) that satisfies the transversality condition $\mathfrak{a}_x\mathfrak{u}_x + \mathfrak{a}_y\mathfrak{u}_y + \mathfrak{a}_z\mathfrak{u}_z = 0$, since $\sum_k C_h D_h = 0$, and (15 *a*) becomes equivalent to:

$$\begin{aligned} \mathfrak{B} &= [\mathfrak{u}\ \mathfrak{a}]\ \varphi\,(\mathfrak{u}_x x + \mathfrak{u}_y y + \mathfrak{u}_z z - ct) \\ \mathfrak{E} &= \quad \mathfrak{a}\ \varphi\,(\mathfrak{u}_x x + \mathfrak{u}_y y + \mathfrak{u}_z z - ct). \end{aligned}$$

If one sets:

$$[\mathfrak{u}\ \mathfrak{a}] = \mathfrak{b}$$

in which $\mathfrak{b}$ is another three-dimensional vector for which one obviously has:

$$\mathfrak{u}_x\mathfrak{b}_x + \mathfrak{u}_y\mathfrak{b}_y + \mathfrak{u}_z\mathfrak{b}_z = \mathfrak{a}_x\mathfrak{b}_x + \mathfrak{a}_y\mathfrak{b}_y + \mathfrak{a}_z\mathfrak{b}_z = 0$$

in which:

$$[\mathfrak{u}\ \mathfrak{b}] = \mathfrak{a}\ .$$

If one defines a four-vector:

$$B_1 = \mathfrak{b}_x\sqrt{-1}\,,\quad B_2 = \mathfrak{b}_y\sqrt{-1}\,,\quad B_3 = \mathfrak{b}_z\sqrt{-1}\,,\quad B_4 = 0,$$

then these relations satisfy:

$$\sum_k B_h C_h = \sum_k B_h D_h = 0$$

and, as one sees from $\mathfrak{B} = \mathfrak{H}$, $\mathfrak{E} = \mathfrak{D}$ one obviously sets the electromagnetic field vector E equal to:

$$E_{ik} = (C_i\,B_k - C_k\,B_i)\ \varphi\left(\sum_k C_h x_h\right) \qquad (ik) \tag{15 b}$$

or, from (5):

$$\mathfrak{H} = \mathfrak{b}\ \ \varphi\,(\mathfrak{u}_x\,x + \mathfrak{u}_y\,y + \mathfrak{u}_z\,z - ct)$$
$$\mathfrak{D} = [\mathfrak{b}\ \mathfrak{u}]\ \varphi\,(\mathfrak{u}_x\,x + \mathfrak{u}_y\,y + \mathfrak{u}_z\,z - ct)\ .$$

In summation, the plane wave in vacuo is characterized by the values of both field vectors:

$$E_{ik} = (C_i\,B_k - C_k\,B_i)\ \varphi\left(\sum_k C_h x_h\right) \qquad (ik) \tag{15 b}$$

$$F_{ik} = (C_i\,D_k - C_k\,D_i)\ \varphi\left(\sum_k C_h x_h\right) \qquad (ik), \tag{15 a}$$

in which one likewise has for the constants C, B, D:

$$\sum_h C_h^2 = \sum_h C_h D_h = \sum_h C_h B_h = \sum_h B_h D_h = 0. \tag{16}$$

From this, it follows conversely: When equations (15 *a*), (15 *b*) are valid, along with (16) then one has, as a simple calculation shows:

$$E_{i'k'} \sim F_{ik} \qquad (ik)$$

when $iki'k'$ is a positive permutation of 1234. Since the (naturally constant) proportionality factor does enter here, one can directly set:

$$E_{i'k'} = F_{ik} \qquad (ik)\ .$$

However, these are the Minkowskian constitutive equations (11), hence, a Minkowski metric and the form (I *d*) (II *d*) is valid for the solutions (15 *a*) (15 *b*) with the conditions (16). Q.E.D.

The relations (16) are essential for the previous proof of the appearance of the metric. Also, without this condition (15 *a*) (15 *b*) would be a solution of the Maxwell equations (I *a*) (II *a*), from which, all of the components of the two field vectors are proportional to one and the same function of a linear argument. However, the interaction between both fields would not be precise and the constitutive equations (11), which are equivalent to (10), would fail: physically speaking, Lenz's rule for any interaction would fail. These solutions, (15 *a*), (15 *b*), but not (16), can naturally be referred to as plane waves since the argument $\sum_k C_h x_h$, without the relation $\sum_h C_h^2 = 0$, says nothing about the velocity of light that the phase propagates at. The radiation laws of the field are first given by the metric or conversely.

With the foregoing, we have explained how the Minkowski metric is based on the laws of radiation of the field in vacuo, and the same task for the Einstein metric in a gravitational field grows more important. We thus have to seek an integral of equations (I *c*), (II *c*) with $S = 0$ that would be analogous to plane waves in the vacuum, and from which all of optics could be reproduced. Unfortunately, this problem is insoluble in full generality. One must therefore appeal to the aforementioned theory of characteristics for help here, which, when it is sufficiently well-defined for a system of partial differential equations, in any case where this is permissible, should take a detour around the problem of finding an actual integral of the differential equations and read off the laws of radiation directly. As long as this road is not passable, we must satisfy ourselves with the following consideration: The gravitational field of Einstein deviates from the Minkowski vacuum only in terms of higher order; this deviation first becomes significant in a large domain. If one restricts oneself to a small domain then one can regard the a_{ik} in (1 *c*) (II *c*) as constant ([1]), and one can then recall the proof above, in which one must merely replace $\sum_h C_h^2$, etc., with $\sum_{g,h} a^{gh} C_g C_h$, etc. The connection between local laws of radiation of the field with the Einstein metric is thereby likewise established.

6.

After the independence of the prototype form of the Maxwell equations (I *a*), (II *a*) ((I *b*), (II *b*), resp.) of any metric was established in sec. 3, in which both fields – the electromagnetic field (*E*), as well as the magneto-electric field (*F*) – were regarded as previously-given quantities in an experiment, the actual phenomenological calculations were carried out in sec. 4, in which the fields were regarded as unknown functions of the quantities *x* – position and time – that are actually experienced. This led to the necessity of introducing constitutive relations between *E* and *F*, which establish interaction between electromagnetic and magneto-electric phenomena in space and time. These constitutive relations give rise to the appearance of a metric in Maxwell's equations, as their new form shows, e.g., (I *d*) and (II *d*) in matter-free space without gravity, according to Minkowski (with gravity, (I *c*) and (II *c*), according to Einstein). In sec. 5, it was then

[1] More precisely: one can introduce a so-called "geodetic" coordinate system.

shown that for the Minkowskian (Einsteinian, resp.) vacuum the given of constitutive relations implies the given of the laws of radiation of the field in space and time, and conversely, the given of the latter implies the given of the former.

In contrast to the foregoing representation, one might perhaps be inclined to look for the origin of the metric and constitutive relations in the energetic behavior of the fields and their coupling relations instead. This goes back to over-emphasis on energetics in physics that is still often present, and which, it seems, we have not overcome. Thus, it is good to remember Poincaré's criticism ([1]): After he argued that the energy principle goes back to it, that the differential equations of mechanics can be expressed as integral equations of the form: "a certain sum is constant," that, however, the choice of which of these integrals actually represents the total energy is in now unique, he continued: "il y a quelque chose qui demeure constant. Sous cette form il se trouve à son tour hors des atteintes de l'expérience et se réduit à une sorte de tautologie. Il est clair que si le monde est gouverné par les lois, il a aura des quantités qui demeureront constants."

The truth of this skeptical statement remains unchanged in the face of the modern form of the energy principle, which, as one knows, mixes the energy principle and the impulse theorem into a unified four-dimensional statement, in which the first part is timelike and the last part is spacelike. The impulse-energy theorem of Minkowskian electrodynamics expresses the constancy of impulse and energy in the form of: The impulse (energy, resp.) that is included in a unit volume can only increase or decrease when force fluxes (energy currents, resp.) flow through the boundary surface of the volume. This recalls the notion of local field physics; accordingly, the impulse-energy theorem also takes the form of a divergence. However, when one goes over to the generalized Einstein metric, one loses this analogy with field physics. Whereas (N, no. 2), the analysis of field quantities, viz., alternating tensors (vectors), is actually independent of the metric, the analysis of general tensors, by which impulse, energy, surface and body stresses are represented, depends on the metric, which agrees with the latter analysis only for the Euclidian Minkowskian metric. Physically, this dependency, that the gravitational field also carries energy and momentum, originates in the variability of the metric. However, one finds oneself in an undesirable state of dependency on a special coordinate system that one bases ones considerations upon, since the gravitational field vanishes upon it. It is well-known that Einstein ([2]) sought to regard impulse and energy as stresses in the gravitational field. When one expresses energy and impulse in terms of closed system they therefore have only a universal, but not a local sense, so to speak. The impulse theorem then reduces once more to a sort of "tautology," as Poincaré had suggested.

One can thus only expect from these considerations that one can derive identities from the original field equations, just as one does from the equations of mechanics, whose generalization would represent the conservation of energy and impulse that we only known in one special case. The idea of energy currents and impulse currents (the

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[1] H. Poincaré, La science et l'hypothése, chap. VIII, energy and thermodynamics, pp. 153, et seq.

[2] A. Einstein, Der Energiesatz in der allgemeinen Relativitätstheorie. Berl. Ber., 1918, pp. 448.

latter due to Planck ([1])), through which one seeks to reduce the ambiguity in the aforementioned method (e.g., in which one sets the impulse proportional to the energy current [the law of work and energy] ([2])), breaks down, as is clear above, since the idea of field flux, viz., the connection between three-fold volume integrals and two-fold surface integrals can not be applied to energetic quantities in general. For example, if one takes the integral of impulse-energy currents throughout a three-fold extended domain then it does not reduce to a mere boundary surface integral.

Therefore, when one finds oneself working within the approximation of the impulse-energy theorem, it is easy to refute the search for a foundation of the metric on it. The identities that this theorem represents actually follow from the prototype for the Maxwell equations (I *a*), (II *a*), just as it does in the special case of Einstein (I *c*), (II *c*), and Minkowski (I *d*), (II *d*); in this way, one thus proves that the general form of the impulse-energy theorem is independent of any metric.

In order to prove this, we adopt, for the time being, the form (I *a*), (II *a*) that is equivalent to Maxwell's equations:

$$\sum_k \frac{\partial}{\partial x_k}(\varepsilon_{1234}\overset{*}{E}{}^{ik}) = \varepsilon_{1234}\, S^i \qquad i \qquad \text{(I } b\text{)}$$

$$\frac{\partial F_{lm}}{\partial x_k} + \frac{\partial F_{mk}}{\partial x_l} + \frac{\partial F_{kl}}{\partial x_m} = 0 \qquad (klm), \qquad \text{(II } b\text{)}$$

in which we have only set $\overset{*}{\mathbf{S}} = - S$. One multiplies the first quadruple by F_{hi} and sums over i ; by the use of (II *b*), it is transformed into:

$$\sum_k \frac{\partial}{\partial x_k}\left(-\varepsilon_{1234}\sum_i F_{hi}\overset{*}{E}{}^{ki} + \frac{1}{4}\varepsilon_{1234}\delta_h^k \sum_{ij} F_{ij}\overset{*}{E}{}^{ij}\right) + \frac{1}{4}\sum_{i,k}\left(\frac{\partial F_{ik}}{\partial x_k}\cdot \varepsilon_{1234}\overset{*}{E}{}^{ki} - F_{ik}\cdot\frac{\partial}{\partial x_h}[\varepsilon_{1234}\overset{*}{E}{}^{ik}]\right)$$
$$= \varepsilon_{1234}\sum_i F_{hi}\, S^i\,. \qquad (17)$$

In this expression, δ has the well-known meaning. If one sets ([3]):

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[1] M. Planck, Bemerkungen zum Prinzip der Aktion und Reaktion in der allgemeinen Dynamik, Phys. Zeit. **9** (1908), pp. 828.

[2] Cf., e.g., M. Laue, Zur Dynamik der Relativitätstheorie. Ann. d. Physik **35** (1911), in particular, pp. 520.

[3] Instead of (18 *a*), one can also, as one easily sees, set:

$$T_h^k = +\sum_i E_{hi}\overset{*}{F}{}^{ki} - \frac{1}{4}\delta_h^k\sum_{i,j} E_{ij}\overset{*}{F}{}^{ij} \qquad h, k$$

in such a way that neither of the fields is distinguished from the other.

$$T_h^k = -\sum_i F_{hi} \overset{*}{E}^{ki} + \frac{1}{4}\delta_h^k \sum_{ij} F_{ij} \overset{*}{E}^{ij} \quad h, k \tag{18 a}$$

$$P_h = \sum_i F_{hi} S^i, \tag{18 b}$$

then one easily recognizes in (18 *a*) the Maxwell-Faraday stress tensor, etc., and in (18 *b*), the expression for the ponderomotive electromagnetic Lorentz force. If one introduces the Einstein metric by means of the constitutive equations (12), hence, by means of:

$$\overset{*}{E}^{ik} = \sum_{p,q} g^{ip} g^{kq} F_{pq}, \qquad (ik)$$

then one actually obtains the well-known form ([1]) for that tensor and that force in general relativity theory. In this case, one obtains, instead of (17), the well-known form of the impulse-energy theorem in general relativity theory ([2]):

$$\sum_k \frac{\partial}{\partial x_k}\left(\sqrt{g}T_h^k\right) - \frac{1}{2}\sqrt{g}\sum_{ij} T^{ij}\frac{\partial g_{ij}}{\partial x_h} = \sqrt{g}\,P_h \qquad h.$$

The left-hand side, as we already pointed out, does not have the simple form of a divergence in field physics, but the complicated form of a covariant divergence of a general tensor.

In (17) and (18), we thus have before us a form of energetics that has been freed from any metric. We have thus produced our proof of independence of both from each other. With that, we conclude with a remark: In the foregoing work, the metric of Newtonian mechanics was based on energy, as one would like to infer from N, no. 4. There, we had a covariant vector of second degree – the force flux – whose complement was a contravariant vector of first degree; the definition of the force in terms of the work done requires a covariant vector of first degree that was derived from the latter only by the introduction of a metric, viz., a polar correlation. Here, we likewise have a covariant vector of third degree – the electrical charge flux – whose complement is a contravariant vector of first degree, $\overset{*}{\mathbf{S}}$ or *S*. Here, the four-force *P* is likewise a covariant vector of first degree, as one can show from relativistic mechanics without difficulty. The derivation of this covariant vector of first degree from a contravariant vector of first degree once again requires a correlation. However, it is no polar correlation (metric) that one must introduce here; rather, the matrix $\| F_{ik} \|$ that is constructed from the vector *F* already defines a correlation, and indeed, a null correlation, that will also actually be used in the definition (18 *b*) of the four-force, by way of the four-current *S*.

Again, this is only a contribution to all of what the previous considerations of this section succeeded in bringing to light: Electromagnetic mechanics is merely a consequence of laws of electromagnetic fields; it thus has no self-explanatory meaning.

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[1] See, e.g., H. Weyl, loc. cit., § 28, pp. 209, equation (11) and pp. 216, equation (28).

[2] Weyl, equation (28).

7.

Up till now, we have represented the field quantities as unknown functions of space and time quantities. These were themselves to be regarded as any four numbers x_1, x_2, x_3, x_4 , through which the timelike instant and the spacelike position of the event were described. A metric must be added to these four numbers when one desires to ascertain the field quantities.

Now, however, the phenomena of reality are actually very complicated: the four numbers x_1, x_2, x_3, x_4 are, in reality, not given at all, since we can deal with only a small part of space and time immediately; rather, we must first construct them. One merely need think of the organization of events on distant stars.

Now, we construct practical spacelike coordinates in reality by the measurement of distances and also determine practical times by the use of clocks. We thus make use of the properties of "rigid" bodies, hence, of mechanical aids. This is therefore a mechanical basis for the metric, assuming that there actually is the desired degree of precision and reproducibility in these rigid bodies. If one considers mechanics as something that is independent of electricity then we have a means of ordering events that is independent of the presence of an electromagnetic field that we intend to describe with the help of these coordinates and a suitably invented metric.

Now, however, the Michelson experiment has revealed the Lorentz contraction of rigid bodies in motion. Einstein carried out calculations regarding this fact in which he demanded that the metric of rigid bodies must obey the postulate that the velocity of light, or more generally, the laws of radiation of electromagnetic fields must remain the same in all (proper) systems of reference. In this, one finds the subordination of the mechanical metric to the laws of optics; Huntington ([1]) first gave a concise expression to this thought by determining not only the time (which Einstein 1905 and, incidentally, Poincaré 1900 had already done), but also the length by light signals. This is an optical basis for the metric.

If one accepts this – it rests implicitly on all of relativity theory (including the general theory ([2])) – then one comes to the difficulty of a circular argument. We desire that the electromagnetic field be represented by the coordinates we determine the coordinates by means of the field in practice. It is clear that in this way the metric form of the field equations steps into the foreground since the metric is indeed no longer additional, but given. (Principle of the constancy of the velocity of light or the invariance of the general ds^2.) In fact, in (special) relativity the Maxwell equations also led to the metric historically, and not the converse (in a certain form).

From the foregoing, it becomes clear that the practical necessity of constructing the coordinates with the assistance of the electromagnetic field, the Nature of Things, and the fundamental independence of pure field physics from any metric, were not touched upon. If one chooses a non-optical method of construction then one recognizes this

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[1] E. V. Huntington, A new approach to the theory of relativity, Weber-Festschrift, Leipzig 1912, pp. 147, et seq. – also Phil. Mag. **23** (1912), pp. 44 et seq. – The same thought recently appeared in H. Reichenbach, Bericht über eine Axiomatik der Einstein'schen Raum-Zeitlehre, Phys. Zeit. **22** (1921), pp. 683 et seq. ("Metrisches Axiom.")

[2] H. Poincaré, La théorie de Lorentz et le principe de la reaction (Lorentz-Festschrift). Archives Néederlanaises **5** (1900), in particular, pp. 272 et seq.

immediately. (Admittedly, the question of whether non-optical methods are even possible at all will not be dealt with here. One thinks of the great distances in interstellar space that one cannot possibly bridge except with light. One also thinks of the connection that obviously must exist between the properties of so-called rigid bodies and those of light. This connection will be addressed at another time.)

We have previously clarified the contradiction that we announced in the conclusion of N, sec. 5, that seems to reside in the fact that general relativity theory derives the "field" from the metric, but here we have demanded the independence of pure field physics from the metric. Perhaps we might add that the "field" of general relativity theory is certainly not the field that we treated here, which was defined as Faraday would have defined it. The field of general relativity theory is a gravitational field, i.e., it originates (according to Einstein) in the local variations of the metric. Thus, the laws of light first emerge. Special relativity theory teaches their invariance in a local domain, and general relativity, their variation from point to point due to the presence of matter. If one inverts this (experimentally well-founded) result then one has characterized the neighborhood of matter (its "field") by way of the (optical) metric. This inversion is the basic idea behind the explanation for gravitation in terms of general relativity theory. It is just for that reason, however, that the gravitational "field" reduces to a concomitant of light in our way of explaining things.