Eisenstein series
on Kac–Moody groups

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Based on joint work with Philipp Fleig

`[FK, JHEP` **1206** `(2012) 054, arXiv:1204.3043]`

Context and Plan

Hidden symmetries in supergravity `[Cremmer, Julia 1978; Julia 1980s; West 2001; Damour, Henneaux, Nicolai 2002;...]`

U-dualities constraining string scattering amplitudes `[Green, Gutperle 1997; Green, Miller, Russo, Vanhove 2010; Pioline 2010;...]`

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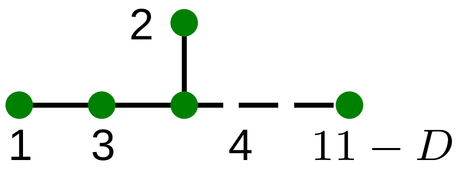
Plan

- Appearance of hidden symmetries and dualities
- Eisenstein series for Kac-Moody groups
- Perturbative terms and consistency checks
- Outlook

Hidden symmetries in supergravity

Space-time symmetries can lead to giobal symmetries. For maximal supergravity in D dimensions (T^{11-D})

D	Global symmetry $E_{11-D}(\mathbb{R})$
$10B$	$SL(2,\mathbb{R})$
$\vdots$	$\vdots$
6	$SO(5,5,\mathbb{R})$
5	$E_6(\mathbb{R})$
4	$E_7(\mathbb{R})$
3	$E_8(\mathbb{R})$
2	$E_9(\mathbb{R})$



Moduli fields

$$\Phi \in E_{11-D}/K(E_{11-D})$$

[Cremmer, Julia 1978]

[Nicolai 1987]

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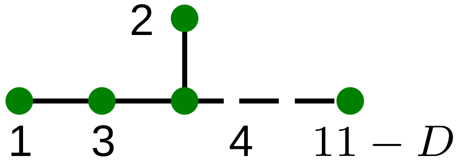

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1	$E_{10}(\mathbb{R})$
0	$E_{11}(\mathbb{R})$

KM ↓

[Cremmer, Julia 1978]

[Nicolai 1987]

[Julia 1982; Mizoguchi 1998; DHN 2002]

[West 2001]

Quantization of symmetries

Hidden symmetries get quantized when embedded in string theory (cf. [Font et al. 1992; Hull, Townsend 1994])

D	Global symmetry $E_{d+1}(\mathbb{R})$	U-duality symmetry
$10B$	$SL(2,\mathbb{R})$	$SL(2,\mathbb{Z})$
$\vdots$	$\vdots$	$\vdots$
6	$SO(5,5,\mathbb{R})$	$SO(5,5,\mathbb{Z})$
5	$E_6(\mathbb{R})$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$E_8(\mathbb{Z})$ (?)
2	$E_9(\mathbb{R})$	$E_9(\mathbb{Z})$?
1	$E_{10}(\mathbb{R})$	$E_{10}(\mathbb{Z})$?
0	$E_{11}(\mathbb{R})$	$E_{11}(\mathbb{Z})$?

Chevalley groups

Double cosets

Constraints from dualities

Physics should be invariant under U-duality.

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Ex.: D-dim'l four-graviton scattering (Einstein frame)

$$\ell_D^{D-2} S^{(D)} = \int d^D x \sqrt{-g}\Big(R + \ell_D^6 \mathcal{E}^{(D)}_{(0,0)}(\Phi) R^4 + \ell_D^{10} \mathcal{E}^{(D)}_{(1,0)}(\Phi) D^4 R^4 + \ldots\Big)$$

Planck length$\sim \alpha'$

Function of moduli $\Phi \in E_{11-D}/K(E_{11-D})$

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Function of moduli $\Phi \in E_{11-D}/K(E_{11-D})$

$\mathcal{E}^{(D)}_{(p,q)}(\Phi)$ must/should

- be invariant under U-duality $E_{11-D}(\mathbb{Z})$
- satisfy differential equations (max. susy)
- have a well-defined perturbative string expansion
- obey relations between various D

Example: type IIB in $D = 10$

U-duality $SL(2,\mathbb{Z})$. Differential eq'n for R^4 [Green, Sethi 1998]

$$\left(\Delta_{SL(2,\mathbb{R})/SO(2)} - \frac{3}{4}\right)\mathcal{E}^{(10)}_{(0,0)}(\Phi) = 0$$

and $\Phi = C_{(0)} + i/g_s$. Expansion for small g_s

$$g_s^{-1/2}\mathcal{E}^{(10)}_{(0,0)} = 2\zeta(3)g_s^{-2} + 4\zeta(2) + O(g_s^2) + \text{non-pert.}$$

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$SL(2,\mathbb{Z})$ invariant completion `[Green, Gutperle '97; Pioline '98]`

$$\mathcal{E}^{(10)}_{(0,0)}(\Phi) = 2\zeta(3)\sum_{\gamma\in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})}(\gamma\cdot g_s^{-1})^{3/2}$$

$B(\mathbb{Z})$ leaves g_s invariant. Example of Eisenstein series!

Eisenstein series

Eisenstein series for $G \equiv E_{11-D}$ parametrized by weight λ

$$E^G(\lambda, \Phi) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | \gamma \cdot \Phi \rangle}$$

Satisfies simple Laplace equation (and other diff. eq'ns).
$B(\mathbb{Z})$ stabilizer.

Of particular interest: $\lambda = 2s\Lambda_{i_*} - \rho$.

fund. weight of node i_* Weyl vector

Then maximal parabolic Eisenstein series

$$E^G_{i_*;s}(\Phi) = E^G(\lambda, \Phi)$$

Perturbative terms in $D \geq 3$ (I)

For finite-dimensional $G = E_{11-D}$ can compute

$$\int_{N(\mathbb{Z})\backslash N(\mathbb{R})} E^G(\lambda,\Phi)dn = \sum_{w\in\mathcal{W}} M(w,\lambda)e^{\langle w\lambda+\rho|\Phi\rangle}$$

integrate out 'axions'

Weyl group (finite)

num. coefficient

$$M(w,\lambda) = \prod_{\substack{\alpha>0\\ w\alpha<0}} \frac{\xi(\lambda\cdot\alpha)}{\xi(-\lambda\cdot\alpha)}$$

RHS is polynomial in Cartan subalgebra components of Φ. Above is Langlands' constant term formula.

In physical terms, polynomial in string coupling g_s and radii R_i/ℓ_D of compactifying torus $\Longrightarrow$ perturbative terms

Perturbative terms in $D \geq 3$ (II)

Laplace equations [Green, Russo, Vanhove 2010]

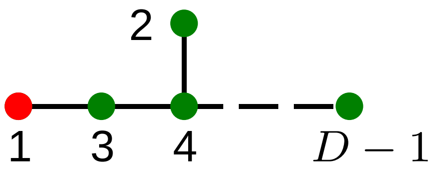
$$\left(\Delta^{(D)} - \frac{3(11-D)(D-8)}{D-2}\right)\mathcal{E}^{(D)}_{(0,0)} = 6\pi\delta_{D,8}$$

$$\left(\Delta^{(D)} - \frac{5(12-D)(D-7)}{D-2}\right)\mathcal{E}^{(D)}_{(1,0)} = 40\zeta(2)\delta_{D,7}$$

(Likely) solutions [Green, Vanhove, Russo, Pioline, Miller,...]

$$R^4 \;:\qquad \mathcal{E}^{(D)}_{(0,0)} = 2\zeta(3)E^{G}_{1;3/2}$$

$$D^4R^4 \;:\qquad \mathcal{E}^{(D)}_{(1,0)} = \zeta(5)E^{G}_{1;5/2}$$



Structure well-understood for R^4 and D^4R^4 in $D \geq 3$; passed many tests. [Green, Vanhove, Russo, Pioline, Miller,...]

Perturbative terms in $D < 3$ (I)

$$\int\limits_{N(\mathbb{Z})\backslash N(\mathbb{R})} E^G(\lambda, \Phi) dn = \sum_{w\in\mathcal{W}} M(w,\lambda) e^{\langle w\lambda+\rho|\Phi\rangle}$$

(Mathematical) issues for Kac–Moody case

- Weyl group $\mathcal{W}$ is infinite
- Set of roots $\alpha > 0$ is infinite
- Laplace eigenvalues appear ill-defined
- Theory of Eisenstein series not fully developed [see `[Garland]` for affine case]

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Result

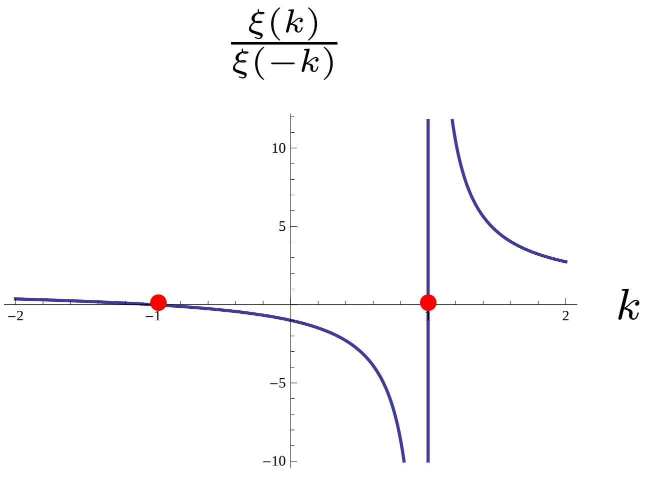
Issues can be overcome, requires apt choice of λ. `[FK]`

Perturbative terms in $D < 3$ (II)

Special properties depend on

$$M(w,\lambda) = \prod_{\alpha>0\,:\,w\alpha<0} \frac{\xi(\lambda\cdot\alpha)}{\xi(-\lambda\cdot\alpha)}$$

$$\xi(k) = \pi^{-k/2}\Gamma\left(\frac{k}{2}\right)\zeta(k)$$



Special things happen when the argument $\lambda\cdot\alpha = \pm 1$. This happens preferably for integral weights (as for $D \geq 3$).

For $\lambda = 2s\Lambda_1 - \rho$ with $s = 3/2$ and $s = 5/2$ the number of non-vanishing $M(w,\lambda)$ is finite and the perturbative terms are calculable down to $D = 0$!

Perturbative terms for $E_{11-D}(\mathbb{Z})$

Non-vanishing $M(w,\lambda)$ for $\lambda = 2s\Lambda_1 - \rho$

	$s=1/2$	$s=1$	$s=3/2$	$s=2$	$s=5/2$	$s=3$
E_7	2	126	8	14	35	56
E_8	2	2160	9	16	44	72
E_9	2	∞	10	18	54	90
E_{10}	2	∞	11	20	65	110
E_{11}	2	∞	12	22	77	132

‘Perturbative terms in maximal parabolic’ can also be evaluated (integrating out fewer axions, only one parameter becomes perturbative).

Examples of perturbative terms: E_{10}

For $s = 3/2$ (R^4 1/2-BPS) in string perturbation theory

$$\underset{\substack{\uparrow\\ \text{tree}}}{2\zeta(3)r^3} + \underset{\substack{\uparrow\\ \text{one-loop}}}{\frac{5\zeta(7)}{4\zeta(2)}r^{7/2}} E^{SO(9,9)}_{10;7/2}$$

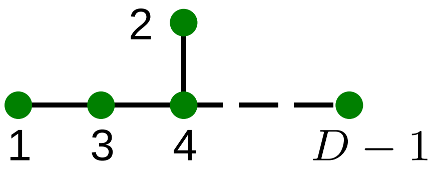
For $s = 5/2$ (D^4R^4 1/4-BPS)

$$\underset{\substack{\uparrow\\ \text{tree}}}{\zeta(5)r^5} + \underset{\substack{\uparrow\\ \text{one-loop}}}{\frac{7\zeta(11)}{16\zeta(2)}r^{11/2}} E^{SO(9,9)}_{10;11/2} + \frac{7\zeta(6)}{3\zeta(2)}r^6 \underset{\substack{\uparrow\\ \text{two-loop}}}{E^{SO(9,9)}_{3;2}}$$

$r \ll 1$ related to string coupling.

Results and comments

Perturbative terms of



$$\mathcal{E}^{(2)}_{(0,0)} = 2\zeta(3) v E^{E_9}_{1;3/2}, \qquad \mathcal{E}^{(2)}_{(1,0)} = \zeta(5) v E^{E_9}_{1;5/2}$$

$$\mathcal{E}^{(1)}_{(0,0)} = 2\zeta(3) E^{E_{10}}_{1;3/2}, \qquad \mathcal{E}^{(1)}_{(1,0)} = \zeta(5) E^{E_{10}}_{1;5/2}$$

$$\mathcal{E}^{(0)}_{(0,0)} = 2\zeta(3) E^{E_{11}}_{1;3/2}, \qquad \mathcal{E}^{(0)}_{(1,0)} = \zeta(5) E^{E_{11}}_{1;5/2}$$

pass all tests with flying colours! v is related to derivation of affine E_9. The correct $D = 2$ Laplace eigenvalue is [FK]

$$\left(\Delta^{(2)} + 150\right) \mathcal{E}^{(2)}_{(0,0)} = 0$$

Determined from careful analysis of physical scales. Perturbative terms develop $\log$ and $\log^2$.

Final comments

- Good bookkeeping device.
- Alternative interpretation with BKL limit [FK].
- Full Fourier decomposition (constant + abelian + non-abelian)?
- Instanton terms and contributing states? Physical meaning? Relation to lattice constructions?
- Small number of perturbative terms $\leftrightarrow$ BPS protection? $\leftrightarrow$ small automorphic representation [Ginzburg et al; Pioline; Green et al.] $\leftrightarrow$ nilpotent orbits?
- Other correction terms ($D^{2k}R^4$)? Other processes?
- Relevance for quantum gravity [Ganor; AK, Koehn, Nicolai]?

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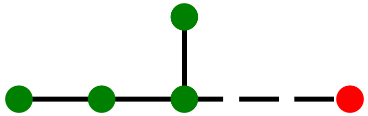
Thank you for your attention!

Consistency checks

Different limits (different cusps)

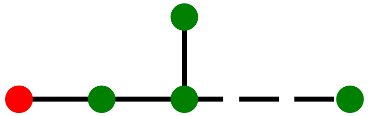
- Decompactification limit $R_{D+1}/\ell_D \gg 1$

$$E_{11-(D+1)} \subset E_{11-D}$$



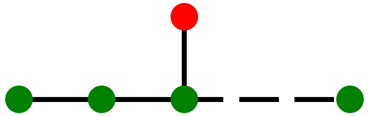
- String perturbation limit $g_D \ll 1$

$$SO(10-D,10-D) \subset E_{11-D}$$



- M-theory limit $\text{vol}(T^{11-D})/\ell_D^{11-D} \gg 1$

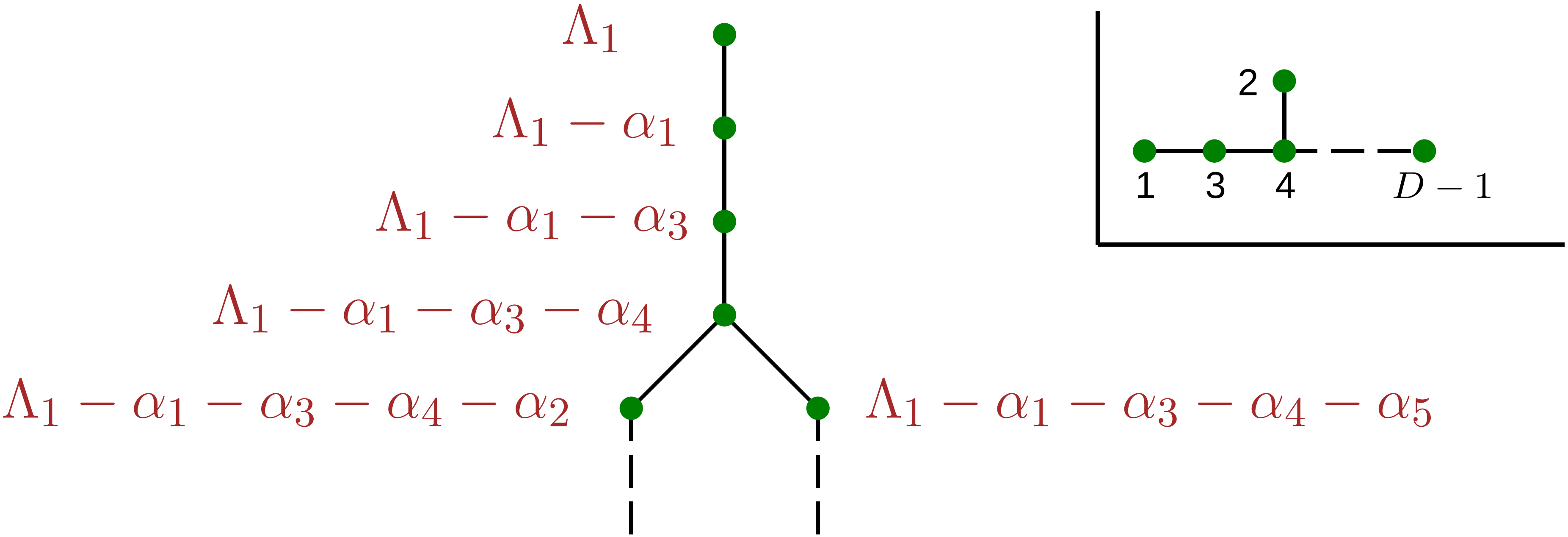
$$SL(11-D) \subset E_{11-D}$$



In all cases, behaviour of scattering amplitudes known ($D \geq 3$). Extended to $D < 3$. →back

Tree structure of perturbative terms

$\lambda = 2s\Lambda_{i_*} - \rho$; Weyl orbit of Λ_{i_*} is a rooted ‘tree’. E.g. $i_* = 1$:



Λ_1

$\Lambda_1 - \alpha_1$

$\Lambda_1 - \alpha_1 - \alpha_3$

$\Lambda_1 - \alpha_1 - \alpha_3 - \alpha_4$

$\Lambda_1 - \alpha_1 - \alpha_3 - \alpha_4 - \alpha_2$

$\Lambda_1 - \alpha_1 - \alpha_3 - \alpha_4 - \alpha_5$

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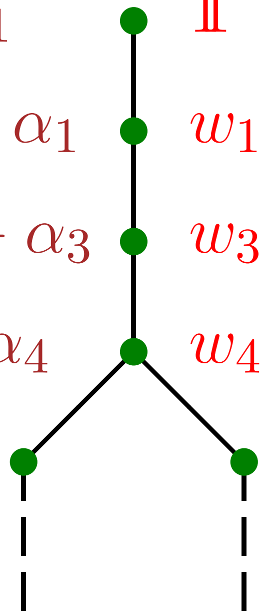


Λ_1 $\mathbb{1}$

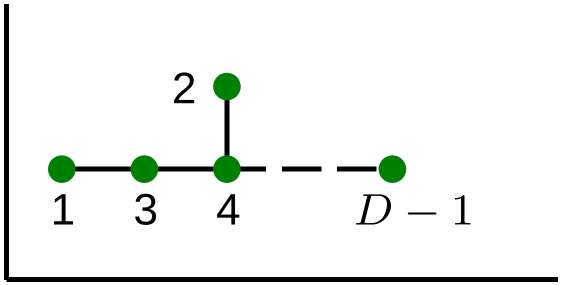
$\Lambda_1 - \alpha_1$ w_1

$\Lambda_1 - \alpha_1 - \alpha_3$ $w_3 w_1$

$\Lambda_1 - \alpha_1 - \alpha_3 - \alpha_4$ $w_4 w_3 w_1$

$\Lambda_1 - \alpha_1 - \alpha_3 - \alpha_4 - \alpha_2$ $w_2 w_4 w_3 w_1$

$\Lambda_1 - \alpha_1 - \alpha_3 - \alpha_4 - \alpha_5$ $w_5 w_4 w_3 w_1$



Element in Weyl orbit $\longleftrightarrow$ Weyl word

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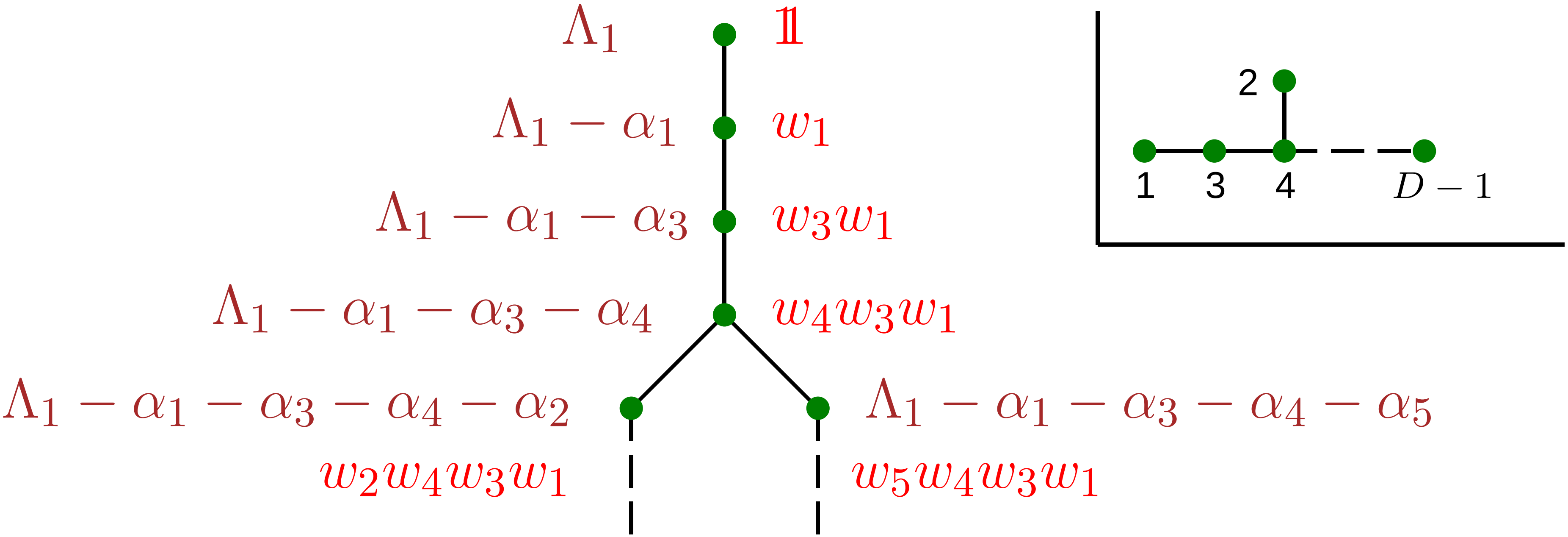

Λ_1 $\mathbb{1}$

$\Lambda_1 - \alpha_1$ w_1

$\Lambda_1 - \alpha_1 - \alpha_3$ $w_3 w_1$

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$\Lambda_1 - \alpha_1 - \alpha_3 - \alpha_4 - \alpha_5$ $w_5 w_4 w_3 w_1$

Element in Weyl orbit $\longleftrightarrow$ Weyl word

$\Rightarrow$ Compute coefficient $M(w,\lambda)$ along the tree. Due to

$$M(w\tilde{w},\lambda) = M(w,\tilde{w}\lambda)M(\tilde{w},\lambda)$$

can terminate a given branch once $M(w,\lambda) = 0$. →back