2

Reformulation of Electromagnetism with Differential Forms

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1. Introduction

Classical electromagnetism is a well-established discipline. However, there remains some confusions and misunderstandings with respect to its basic structures and interpretations. For example, there is a long-lasting controversy on the choice of unit systems. There are also the intricate disputes over the so-called EH or EB formulations. In some textbooks, the authors respect the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ as fundamental quantities and understate $\boldsymbol{D}$ and $\boldsymbol{H}$ as auxiliary quantities. Sometimes the roles of $\boldsymbol{D}$ and $\boldsymbol{H}$ in a vacuum are totally neglected.

These confusions mainly come from the conventional formalism of electromagnetism and also from the use of the old unit systems, in which distinction between $\boldsymbol{E}$ and $\boldsymbol{D}$, or $\boldsymbol{B}$ and $\boldsymbol{H}$ is blurred, especially in vacuum. The standard scalar-vector formalism, mainly due to Heaviside, greatly simplifies the electromagnetic (EM) theory compared with the original formalism developed by Maxwell. There, the field quantities are classified according to the number of components: vectors with three components and scalars with single component. But this classification is rather superficial. From a modern mathematical point of view, the field quantities must be classified according to the tensorial order. The field quantities $\boldsymbol{D}$ and $\boldsymbol{B}$ are the 2nd-order tensors (or 2 forms), while $\boldsymbol{E}$ and $\boldsymbol{H}$ are the 1st-order tensors (1 forms). (The anti-symmetric tensors of order n are called n-forms.)

The constitutive relations are usually considered as simple proportional relations between $\boldsymbol{E}$ and $\boldsymbol{D}$, and between $\boldsymbol{B}$ and $\boldsymbol{H}$. But in terms of differential forms, they associate the conversion of tensorial order, which is known as the Hodge dual operation. In spite of the simple appearance, the constitutive relations, even for the case of vacuum, are the non-trivial part of the EM theory. By introducing relativistic field variables and the vacuum impedance, the constitutive relation can be unified into a single equation.

The EM theory has the symmetry with respect to the space inversion, therefore, each field quantity has a definite parity, even or odd. In the conventional scalar-vector notation, the parity is assigned rather by hand not from the first principle: the odd vectors $\boldsymbol{E}$ and $\boldsymbol{D}$ are named the polar vectors and the even vectors $\boldsymbol{B}$ and $\boldsymbol{H}$ are named the axial vectors. With respect to differential forms, the parity is determined by the tensorial order and the pseudoness (twisted or untwisted). The pseudoness is flipped under the Hodge dual operation. The way of parity assignment in the framework of differential forms is quite natural in geometrical point of view.

It is well understood that the Maxwell equations can be formulated more naturally in the four dimensional spatio-temporal (Minkowski) space. However, the conventional expression with tensor components (with superscripts or subscripts) is somewhat abstract and hard to read out its geometrical or physical meaning. Here it will be shown that the four-dimensional differential forms are the most suitable method for expressing the structure of the EM theory. We introduce two fundamental, relativistic 2-forms, which are related by the four-dimensional Hodge's dual operation and the vacuum impedance.

In this book chapter, we reformulate the EM theory with the differential forms by taking care of physical perspective, the unit systems (physical dimensions), and geometric aspects, and thereby provide a unified and clear view of the solid and beautiful theory.

Here we introduce notation for dimensional consideration. When the ratio of two quantities X and Y is dimensionless (just a pure number), we write $X \overset{\mathrm{SI}}{\sim} Y$ and read "X and Y are dimensionally equivalent (in SI)." For example, we have $c_0 t \overset{\mathrm{SI}}{\sim} x$. If a quantity X can be measured in a unit u, we can write $X \overset{\mathrm{SI}}{\sim} \mathrm{u}$. For example, for $d = 2.5\,\mathrm{m}$ we can write $d \overset{\mathrm{SI}}{\sim} \mathrm{m}$.

2. The vacuum impedance as a fundamental constant

The vacuum impedance was first introduced explicitly in late 1930's (Schelkunoff (1938)) in the study of EM wave propagation. It is defined as the amplitude ratio of the electric and magnetic fields of plane waves in vacuum, $Z_0 = E/H$, which has the dimension of electrical resistance.

It is also called the characteristic impedance of vacuum or the wave resistance of vacuum. Due to the historical reasons, it has been recognized as a special parameter for engineers rather than a universal physical constant. Compared with the famous formula (Maxwell (1865)) representing the velocity of light c_0 in terms of the vacuum permittivity ε_0 and the vacuum permeability μ_0,

$$c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}, \tag{1}$$

the expression for the vacuum impedance

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}, \tag{2}$$

is used far less often. In fact the term is rarely found in index pages of textbooks on electromagnetism.

As we will see, the pair of constants (c_0, Z_0) can be conveniently used in stead of the pair (ε_0, μ_0) for many cases. However, conventionally the asymmetric pairs (c_0, μ_0) or (c_0, ε_0) are often used and SI equations become less memorable.

In this section, we reexamine the structure of electromagnetism in view of the SI system (The International System of Units) and find that Z_0 plays very important roles as a universal constant.

Recent development of new type of media called metamaterials demands the reconsideration of wave impedance. In metamaterials (Pendry & Smith (2004)), both permittivity ε and permeability μ can be varied from values for vacuum and thereby the phase velocity $v_{\rm ph} = 1/\sqrt{\varepsilon\mu}$ and the wave impedance $Z = \sqrt{\mu/\epsilon}$ can be adjusted independently. With the control of wave impedance the reflection at the interfaces of media can be reduced or suppressed.

2.1 Roles of the vacuum impedance

In this section, we show some examples for which Z_0 plays important roles (Kitano (2009)). The impedance (resistance) is a physical quantity by which voltage and current are related. In the SI system, the unit for voltage is $\mathrm{V}(=\mathrm{J/C})$ (volt) and the unit for current is $\mathrm{A}(=\mathrm{C/s})$ (ampere). We should note that the latter is proportional to and the former is inversely proportional to the unit of charge, C (coulomb). Basic quantities in electromagnetism can be classified into two categories as

$$\begin{aligned} &\phi,\quad \boldsymbol{A},\quad \boldsymbol{E},\quad \boldsymbol{B} && \text{Force quantities} && \propto \mathrm{V},\\ &\boldsymbol{D},\quad \boldsymbol{H},\quad \boldsymbol{P},\quad \boldsymbol{M},\quad \varrho,\quad \boldsymbol{J} && \text{Source quantities} && \propto \mathrm{A}. \end{aligned} \tag{3}$$

The quantities in the former categories contain V in their units and are related to electromagnetic forces. On the other hand, the quantities in the latter contain A and are related to electromagnetic sources. The vacuum impedance Z_0 (or the vacuum admittance $Y_0 = 1/Z_0$) plays the role to connect the quantities of the two categories.

2.1.1 Constitutive relation

The constitutive relations for vacuum, $\boldsymbol{D} = \varepsilon_0\boldsymbol{E}$ and $\boldsymbol{H} = \mu_0^{-1}\boldsymbol{B}$, can be simplified by using the relativistic pairs of variables as

$$\begin{bmatrix} \boldsymbol{E} \\ c_0\boldsymbol{B} \end{bmatrix} = Z_0 \begin{bmatrix} c_0\boldsymbol{D} \\ \boldsymbol{H} \end{bmatrix}. \tag{4}$$

The electric relation and magnetic relation are united under the sole parameter Z_0.

2.1.2 Source-field relation

We know that the scalar potential $\Delta\phi$ induced by a charge $\Delta q = \varrho\Delta v$ is

$$\Delta\phi = \frac{1}{4\pi\varepsilon_0}\frac{\varrho\Delta v}{r}, \tag{5}$$

where r is the distance between the source and the point of observation. The charge is presented as a product of charge density ϱ and a small volume Δv. Similarly a current moment (current times length) $\boldsymbol{J}\Delta v$ generates the vector potential

$$\Delta\boldsymbol{A} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{J}\Delta v}{r}. \tag{6}$$

The relations (5) and (6) are unified as

$$\Delta \begin{bmatrix} \phi \\ c_0\boldsymbol{A} \end{bmatrix} = \frac{Z_0}{4\pi r} \begin{bmatrix} c_0\varrho \\ \boldsymbol{J} \end{bmatrix} \Delta v. \tag{7}$$

We see that the vacuum impedance Z_0 plays the role to relate the source $(\boldsymbol{J}, c_0\varrho)\Delta v$ and the resultant fields $\Delta(\phi, c_0\boldsymbol{A})$ in a unified manner.

2.1.3 Plane waves

We know that for linearly polarized plane waves propagating in one direction in vacuum, a simple relation $E = c_0 B$ holds. If we introduce $H\ (= \mu_0^{-1} B)$ instead of B, we have $E = Z_0 H$. This relation was introduced by Schelkunoff (Schelkunoff (1938)). The reason why H is used instead of B is as follows. A dispersive medium is characterized by its permittivity ε and and permeability μ. The monochromatic plane wave solution satisfies $E = vB$, $H = vD$, and $E/H = B/D = Z$, where $v = 1/\sqrt{\varepsilon\mu}$ and $Z = \sqrt{\mu/\varepsilon}$. The boundary conditions for magnetic fields at the interface of media 1 and 2 are $H_{1\text{t}} = H_{2\text{t}}$ (tangential) and $B_{1\text{n}} = B_{2\text{n}}$ (normal). For the case of normal incidence, which is most important practically, the latter condition becomes trivial and cannot be used. Therefore the pair of E and H is used more conveniently. The energy flow is easily derived from E and H with the Poynting vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$. In the problems of EM waves, the mixed use of the quantities (E and H) of the force and source quantities invites Z_0.

2.1.4 Magnetic monopole

Let us compare the force between electric charges q ($\overset{\text{SI}}{\sim}$ A s = C) and that between magnetic monopoles g ($\overset{\text{SI}}{\sim}$ V s = Wb). If these forces are the same for equal distance, r, i.e., $q^2/(4\pi\varepsilon_0 r^2) = g^2/(4\pi\mu_0 r^2)$, we have the relation $g = Z_0 q$. This means that a charge of 1 C corresponds to a magnetic charge of $Z_0 \times 1\,\text{C} \sim 377\,\text{Wb}$.

With this relation in mind, the Dirac monopole g_0 (Sakurai (1993)), whose quantization condition is $g_0 e = h$, can be beautifully expressed in terms of the elementary charge e as

$$g_0 = \frac{h}{e} = \frac{h}{Z_0 e^2}(Z_0 e) = \frac{Z_0 e}{2\alpha}, \tag{8}$$

where $h = 2\pi\hbar$ is Planck's constant. The dimensionless parameter $\alpha = Z_0 e^2/2h = e^2/4\pi\varepsilon_0\hbar c_0 \sim 1/137$ is called the fine-structure constant, whose value is independent of unit systems and characterizes the strength of electromagnetic interaction.

2.1.5 The fine-structure constant

We have seen that the fine-structure constant itself can be represented more simply with the use of Z_0. Further, by introducing the von Klitzing constant (the quantized Hall resistance) (Klitzing et al. (1980)) $R_\text{K} = h/e^2$, the fine-structure constant can be expressed as $\alpha = Z_0/2R_\text{K}$ (Hehl & Obukhov (2005)). We have learned that the use of Z_0 helps to keep SI-formulae in simple forms.

3. Dual space and differential forms

3.1 Covector and dual space

We represent a (tangential) vector at position $\boldsymbol{r}$ as

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 \overset{\text{SI}}{\sim} \text{m}, \tag{9}$$

which represents a small spatial displacement from $\boldsymbol{r}$ to $\boldsymbol{r}+\boldsymbol{x}$. We have chosen an arbitrary orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ with inner products $(\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{10}$$

is Kronecker's delta. We note that $x_i \overset{\text{SI}}{\sim} \text{m}$ and $\boldsymbol{e}_i \overset{\text{SI}}{\sim} 1$.

Such vectors form a linear space which is called a tangential space at $\boldsymbol{r}$. The inner product of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is $(\boldsymbol{x}, \boldsymbol{y}) = x_1 y_1 + x_2 y_2 + x_3 y_3 \overset{\text{SI}}{\sim} \text{m}^2$.

We consider a linear function $\phi(\boldsymbol{x})$ on the tangential space. For any $c_1, c_2 \in \mathbb{R}$, and any vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, $\phi(c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2) = c_1\phi(\boldsymbol{x}_1) + c_1\phi(\boldsymbol{x}_2)$ is satisfied. Such linear functions form a linear space, because the (weighted) sum of two functions $d_1\phi_1 + d_2\phi_2$ with $d_1, d_2 \in \mathbb{R}$ defined with

$$(d_1\phi_1 + d_2\phi_2)(\boldsymbol{x}) = d_1\phi_1(\boldsymbol{x}) + d_2\phi_2(\boldsymbol{x}) \tag{11}$$

is also a linear function. This linear space is called a dual space. The dimension of the dual space is three. In general, the dimension of dual space is the same that for the original linear space. We can introduce a basis $\{\nu_1, \nu_2, \nu_3\}$, satisfying $\nu_i(\boldsymbol{e}_j) = \delta_{ij}$. Such a basis, which is dependent on the choice of the original basis, is called a dual basis. Using the dual basis, the action of a dual vector $\phi(\,) = a_1\nu_1(\,) + a_2\nu_2(\,) + a_3\nu_3(\,)$, $a_1, a_2, a_3 \in \mathbb{R}$ can be written simply as

$$\phi(\boldsymbol{x}) = (a_1\nu_1 + a_2\nu_2 + a_3\nu_3)(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3) \tag{12}$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3} a_i x_j \nu_i(\boldsymbol{e}_j) = x_1 a_1 + x_2 a_2 + x_3 a_3. \tag{13}$$

Here we designate an element of dual space with vector notation as $\boldsymbol{a}$ rather as a function $\phi(\,)$ in order to emphasize its vectorial nature, i.e.,

$$\boldsymbol{a} \cdot \boldsymbol{x} = \phi(\boldsymbol{x}). \tag{14}$$

We call $\boldsymbol{a}$ as a dual vector or a *covector*. The dual basis $\{\nu_1, \nu_2, \nu_3\}$ are rewritten as $\{\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3\}$ with $\boldsymbol{n}_i \cdot \boldsymbol{e}_j = \delta_{ij}$. The dot product $\boldsymbol{a} \cdot \boldsymbol{x}$ and the inner product $(\boldsymbol{x}, \boldsymbol{y})$ should be distinguished. Here bold-face letters $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, and $\boldsymbol{e}$ represent tangential vectors and other bold-face letters represent covectors.

A covector $\boldsymbol{a}$ can be related to a vector $\boldsymbol{z}$ uniquely using the relation, $\boldsymbol{a} \cdot \boldsymbol{x} = (\boldsymbol{z}, \boldsymbol{x})$ for any $\boldsymbol{x}$. The vector $\boldsymbol{z}$ and the covector $\boldsymbol{a}$ are called conjugate each other and we write $\boldsymbol{z} = \boldsymbol{a}^\top$ and $\boldsymbol{a} = \boldsymbol{z}^\top$. In terms of components, namely for $\boldsymbol{a} = \sum_i a_i \boldsymbol{n}_i$ and $\boldsymbol{z} = \sum_i z_i \boldsymbol{e}_i$, $a_i = z_i$ $(i = 1, 2, 3)$ are satisfied

For the case of orthonormal basis, we note that $\boldsymbol{n}_i^\top = \boldsymbol{e}_i$, $\boldsymbol{e}_i^\top = \boldsymbol{n}_i$. Due to these incidental relations, we tend to identify $\boldsymbol{n}_i$ with $\boldsymbol{e}_i$. Thus a covector $\boldsymbol{a}$ is identified with its conjugate $\boldsymbol{a}^\top$ mostly. However, we should distinguish a covector as a different object from vectors since it bears different functions and geometrical presentation (Weinreich (1998)).

The inner product for covectors are defined with conjugates as $(\boldsymbol{a}, \boldsymbol{b}) = (\boldsymbol{a}^\top, \boldsymbol{b}^\top)$. We note the dual basis is also orthonormal, since $(\boldsymbol{n}_i, \boldsymbol{n}_j) = (\boldsymbol{n}_i^\top, \boldsymbol{n}_j^\top) = (\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta_{ij}$.

A good example of covector is the electric field at a point $\boldsymbol{r}$. The electric field is determined through the gained work W when an electric test charge q at $\boldsymbol{r}$ is displaced by $\boldsymbol{x}$. A function $\phi : \boldsymbol{x} \mapsto W/q$ is linear with respect to $\boldsymbol{x}$ if $|\boldsymbol{x}|$ is small enough. Therefore $\phi(\,)$ is considered as a covector and normally written as $\boldsymbol{E}$, i.e., $\phi(\boldsymbol{x}) = \boldsymbol{E} \cdot \boldsymbol{x} \overset{\mathrm{SI}}{\sim} \mathrm{V}$. Thus the electric field vector can be understood as a covector rather than a vector. It should be expanded with the dual basis as

$$\boldsymbol{E} = E_1\boldsymbol{n}_1 + E_2\boldsymbol{n}_2 + E_3\boldsymbol{n}_3 \overset{\mathrm{SI}}{\sim} \mathrm{V/m}. \tag{15}$$

The norm is given as $||\boldsymbol{E}|| = (\boldsymbol{E}, \boldsymbol{E})^{1/2} = \sqrt{E_1^2 + E_2^2 + E_3^2}$. We note that $\boldsymbol{n}_i \overset{\mathrm{SI}}{\sim} 1$ and $E_i \overset{\mathrm{SI}}{\sim} \mathrm{V/m}$.

3.2 Higher order tensors

Now we introduce a tensor product of two covectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as $\mathsf{T} = \boldsymbol{ab}$, which acts on two vectors and yield a scalar as

$$\mathsf{T} : \boldsymbol{xy} = (\boldsymbol{ab}) : \boldsymbol{xy} = (\boldsymbol{a} \cdot \boldsymbol{x})(\boldsymbol{b} \cdot \boldsymbol{y}). \tag{16}$$

It can be considered as a bi-linear functions of vectors, i.e., $\mathsf{T} : \boldsymbol{xy} = \Phi(\boldsymbol{x}, \boldsymbol{y})$ with

$$\begin{aligned} \Phi(c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2, \boldsymbol{y}) &= c_1\Phi(\boldsymbol{x}_1, \boldsymbol{y}) + c_1\Phi(\boldsymbol{x}_2, \boldsymbol{y}), \\ \Phi(\boldsymbol{x}, c_1\boldsymbol{y} + c_2\boldsymbol{y}_2) &= c_1\Phi(\boldsymbol{x}, \boldsymbol{y}_1) + c_1\Phi(\boldsymbol{x}, \boldsymbol{y}_2), \end{aligned} \tag{17}$$

where $c_1, c_2 \in \mathbb{R}$. We call it a bi-covector.

We can define a weighted sum of bi-covectors $\mathsf{T} = d_1\mathsf{T}_1 + d_2\mathsf{T}_2$, $d_1, d_2 \in \mathbb{R}$, which is not necessarily written as a tensor product of two covectors but can be written as a sum of tensor products. Especially, it can be represented with the dual basis as

$$\mathsf{T} = \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}\boldsymbol{n}_i\boldsymbol{n}_j, \tag{18}$$

where $T_{ij} = \mathsf{T} : \boldsymbol{e}_i\boldsymbol{e}_j$ is the (i, j)-component of T.

Similarly we can construct a tensor product of three covectors as $\mathcal{T} = \boldsymbol{abc}$, which acts on three vectors linearly as $\mathcal{T} : \boldsymbol{xyz}$. Weighted sums of such products form a linear space, an element of which is called a tri-covector. Using a tensor product of n covectors, a multi-covector or an n-covector is defined.

3.3 Anti-symmetric multi-covectors — n-forms

If a bicovector T satisfies $\mathsf{T} : \boldsymbol{yx} = -\mathsf{T} : \boldsymbol{xy}$ for any vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, then it is called antisymmetric. Anti-symmetric bicovectors form a subspace of the bicovector space. Namely, a weighted sum of anti-symmetric bicovector is anti-symmetric. It contains an anti-symmetrized tensor product, $\boldsymbol{a} \wedge \boldsymbol{b} := \boldsymbol{ab} - \boldsymbol{ba}$, which is called a *wedge* product. In terms of basis, we have

$$\boldsymbol{a} \wedge \boldsymbol{b} = \sum_{i=1}^{3} a_i\boldsymbol{n}_i \wedge \sum_{j=1}^{3} b_j\boldsymbol{n}_j = \sum_{(i,j)} (a_ib_j - a_jb_i)\boldsymbol{n}_i \wedge \boldsymbol{n}_j, \tag{19}$$

where the last sum is taken for $(i,j) = (1,2),(2,3),(3,1)$. A general anti-symmetric bicovector can be written as

$$\mathsf{T} = \sum_{(i,j)} T_{ij}\boldsymbol{n}_i \wedge \boldsymbol{n}_j. \tag{20}$$

We see that the 2-form has three independent components; $T_{12} = -T_{21}$, $T_{23} = -T_{32}$, $T_{31} = -T_{13}$, and others are zero. The norm of T is $||\mathsf{T}|| = (\mathsf{T}, \mathsf{T})^{1/2} = \sum_{(i,j)} T_{ij}T_{ij}$.

If a bicovector T satisfies $\mathsf{T} : \boldsymbol{xx} = 0$ for any $\boldsymbol{x}$, then it is anti-symmetric. It is easily seen from the relation: $0 = \mathsf{T} : (\boldsymbol{x}+\boldsymbol{y})(\boldsymbol{x}+\boldsymbol{y}) = \mathsf{T} : \boldsymbol{xx} + \mathsf{T} : \boldsymbol{xy} + \mathsf{T} : \boldsymbol{yx} + \mathsf{T} : \boldsymbol{yy}$.

An anti-symmetric multi-covector of order n are often called an n-form. A scalar and a covector are called a 0-form and a 1-form, respectively. The order n is bounded by the dimension of the vector space, $d = 3$, in our case. An n-form with $n > d$ vanishes due to the anti-symmetries.

Geometrical interpretations of n-forms are given in the articles (Misner et al. (1973); Weinreich (1998)).

3.4 Field quantities as n-forms

Field quantities in electromagnetism can be naturally represented as differential forms (Burke (1985); Deschamps (1981); Flanders (1989); Frankel (2004); Hehl & Obukhov (2003)). A good example of 2-form is the current density. Let us consider a distribution of current that flows through a parallelogram spanned by two tangential vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ at $\boldsymbol{r}$. The current $I(\boldsymbol{x},\boldsymbol{y})$ is bilinearly dependent on $\boldsymbol{x}$ and $\boldsymbol{y}$. The antisymmetric relation $I(\boldsymbol{y},\boldsymbol{x}) = -I(\boldsymbol{x},\boldsymbol{y})$ can understood naturally considering the orientation of parallelograms with respect to the current flow. Thus the current density can be represented by a 2-form J as

$$\mathsf{J} : \boldsymbol{xy} = I(\boldsymbol{x},\boldsymbol{y}) \overset{\text{SI}}{\sim} \text{A}, \quad \mathsf{J} = \sum_{(i,j)} J_{ij}\boldsymbol{n}_i \wedge \boldsymbol{n}_j \overset{\text{SI}}{\sim} \text{A}/\text{m}^2. \tag{21}$$

The charge density can be represented by a 3-form $\mathcal{R}$. The charge Q contained in a parallelepipedon spanned by three tangential vectors $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$:

$$\mathcal{R} : \boldsymbol{xyz} = Q(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \overset{\text{SI}}{\sim} \text{C}, \quad \mathcal{R} = R_{123}\boldsymbol{n}_1 \wedge \boldsymbol{n}_2 \wedge \boldsymbol{n}_3 \overset{\text{SI}}{\sim} \text{C}/\text{m}^3. \tag{22}$$

Thus electromagnetic field quantities are represented as n-forms ($n = 0,1,2,3$) as shown in Table 1, while in the conventional formalism they are classified into two categories, scalars and vectors, according to the number of components. We notice that a quantity that is represented n-form contains physical dimension with m^{-n} in SI. An n-form takes n tangential vectors, each of which has dimension of length and is measured in m (meters).

In this article, 1-forms are represented by bold-face letters, 2-forms sans-serif letters, and 3-forms calligraphic letters as shown in Table 1.

3.5 Exterior derivative

The nabla operator $\boldsymbol{\nabla}$ can be considered as a kind of covector because a directional derivative $\boldsymbol{\nabla} \cdot \boldsymbol{u}$, which is a scalar, is derived with a tangential vector $\boldsymbol{u}$. It acts as a differential operator

rank	quantities (unit)	scalar/vector
0-form	ϕ (V)	scalar
1-form	$\boldsymbol{A}$ (Wb/m), $\boldsymbol{E}$ (V/m), $\boldsymbol{H}$ (A/m), $\boldsymbol{M}$ (A/m)	vector
2-form	B (Wb/m^2), D (C/m^2), P (C/m^2), J (A/m^2)	vector
3-form	$\mathcal{R}$ (C/m^3)	scalar

Table 1. Electromagnetic field quantities as n-forms

and also as a covector. Therefore it can be written as

$$\boldsymbol{\nabla} = \boldsymbol{n}_1 \frac{\partial}{\partial x_1} + \boldsymbol{n}_2 \frac{\partial}{\partial x_2} + \boldsymbol{n}_3 \frac{\partial}{\partial x_3} \overset{\text{SI}}{\sim} 1/\text{m}. \tag{23}$$

The wedge product of the nabla $\boldsymbol{\nabla}$ and a 1-form $\boldsymbol{E}$ yields a 2-form;

$$\boldsymbol{\nabla} \wedge \boldsymbol{E} = \sum_{(i,j)} \left(\frac{\partial E_j}{\partial x_i} - \frac{\partial E_i}{\partial x_j} \right) \boldsymbol{n}_i \wedge \boldsymbol{n}_j, \tag{24}$$

which corresponds to $\boldsymbol{\nabla} \times \boldsymbol{E}$ in the scalar-vector formalism. Similarly a 2-form J are transformed into a 3-form as

$$\boldsymbol{\nabla} \wedge J = \left(\frac{\partial J_{23}}{\partial x_1} + \frac{\partial J_{31}}{\partial x_2} + \frac{\partial J_{12}}{\partial x_3} \right) \boldsymbol{n}_1 \wedge \boldsymbol{n}_2 \wedge \boldsymbol{n}_3, \tag{25}$$

which corresponds to $\boldsymbol{\nabla} \cdot \boldsymbol{J}$.

3.6 Volume form and Hodge duality

We introduce a 3-form, called the volume form, as

$$\mathcal{E} = \boldsymbol{n}_1 \wedge \boldsymbol{n}_2 \wedge \boldsymbol{n}_3 = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} \boldsymbol{n}_i \boldsymbol{n}_j \boldsymbol{n}_k \overset{\text{SI}}{\sim} 1, \tag{26}$$

where

$$\epsilon_{ijk} = \begin{cases} 1 & (i, j, k : \text{cyclic}) \\ -1 & (\text{anti-cyclic}) \\ 0 & (\text{others}) \end{cases} . \tag{27}$$

It gives the volume of parallelepipedon spanned by $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$;

$$V(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \mathcal{E} \vdots \boldsymbol{xyz} \quad \overset{\text{SI}}{\sim} \text{m}^3. \tag{28}$$

Using the volume form we can define a relation between n-forms and $(d-n)$-forms, which is call the Hodge dual relation. First we note that n-forms and $(d-n)$-forms have the same degrees of freedom (the number of independent components), ${}_dC_n = {}_dC_{d-n}$, and there could be a one-to-one correspondence between them. In our case of $d = 3$, there are two cases: $(n, d-n) = (0, 3)$ and $(1, 2)$. We consider the latter case. With a 1-form $\boldsymbol{E}$ and a 2-form D, we can make a 3-form $\boldsymbol{E} \wedge D = f(\boldsymbol{E}, D)\mathcal{E}$. The scalar factor $f(\boldsymbol{E}, D)$ is bilinearly dependent on $\boldsymbol{E}$ and D. Therefore, we can find a covector (a 1-form) $\boldsymbol{D}$ that satisfies $(\boldsymbol{E}, \boldsymbol{D}) = f(\boldsymbol{E}, D)$ for any $\boldsymbol{E}$. Then, $\boldsymbol{D}$ is called the Hodge dual of D and we write $\boldsymbol{D} = *D$ or $D = *\boldsymbol{D}$ using a unary

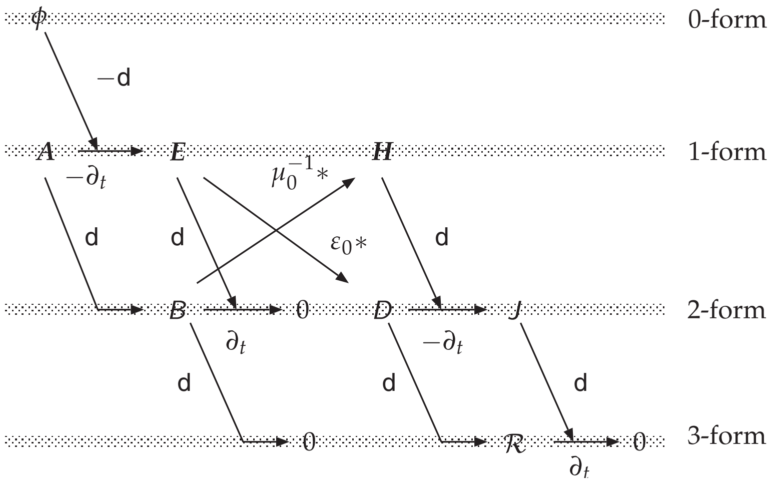

Fig. 1. Relations of electromagnetic field forms in three dimension

operator "$*$" called the Hodge star operator. In terms of components, $D_1 = D_{23}$, $D_2 = D_{31}$, $D_3 = D_{12}$ for $\boldsymbol{D} = *D$ with $\boldsymbol{D} = \sum_i D_i \boldsymbol{n}_i$ and $D = \sum_{(i,j)} D_{ij} \boldsymbol{n}_i \wedge \boldsymbol{n}_j$.

Physically, $(\boldsymbol{E}, \boldsymbol{D}) \overset{\text{SI}}{\sim} \mathrm{J/m^3}$ corresponds to the energy density and can be represented by the 3-form $\mathcal{U} = \frac{1}{2}\boldsymbol{E} \wedge D = \frac{1}{2}(\boldsymbol{E}, \boldsymbol{D})\mathcal{E}$, because $\mathcal{U} : \boldsymbol{xyz}$ is the energy contained in the parallelepipedon spanned by $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$.

The charge density form can be written as $\mathcal{R} = \varrho \boldsymbol{n}_1 \wedge \boldsymbol{n}_2 \wedge \boldsymbol{n}_3 = \varrho\mathcal{E}$ with the conventional scalar charge density ϱ. The relation can be expressed as $\mathcal{R} = *\varrho$ or $\varrho = *\mathcal{R}$. Similarly, we have $\mathcal{E} = *1$ and $1 = *\mathcal{E}$.

Equations (24) and (25) are related to the conventional notations; $*(\boldsymbol{\nabla} \wedge \boldsymbol{E}) = \boldsymbol{\nabla} \times \boldsymbol{E}$ and $*(\boldsymbol{\nabla} \wedge J) = \boldsymbol{\nabla} \cdot \boldsymbol{J}$, respectively.

3.7 The Hodge duality and the constitutive equation

In electromagnetism, the Hodge duality and the constitutive relations are closely related. We know that the electric field $\boldsymbol{E}$ and the electric flux density D are proportional. However we cannot compare them directly because they have different tensorial orders. Therefore we utilize the Hodge dual and write $D = \varepsilon_0(*\boldsymbol{E})$. Similarly, the magnetic relation can be written as $\boldsymbol{H} = \mu_0^{-1}(*B)$. Generally speaking, the constitutive relations in vacuum are considered to be trivial relations just describing proportionality. Especially in the Gaussian unit system, they tend to be considered redundant relations. But in the light of differential forms we understand that they are the keystones in electromagnetism.

With the polarization P and the magnetization $\boldsymbol{M}$, the constitutive relations in a medium are expressed as follows:

$$D = \varepsilon_0(*\boldsymbol{E}) + P, \quad \boldsymbol{H} = \mu_0^{-1}(*B) - \boldsymbol{M}. \tag{29}$$

4. The Maxwell equations in the differential forms

With differential forms, we can rewrite the Maxwell equations and the constitutive relations as,

$$\begin{aligned} &\boldsymbol{\nabla} \wedge B = 0, \quad \boldsymbol{\nabla} \wedge \boldsymbol{E} + \frac{\partial B}{\partial t} = 0, \\ &\boldsymbol{\nabla} \wedge D = \mathcal{R}, \quad \boldsymbol{\nabla} \wedge \boldsymbol{H} - \frac{\partial D}{\partial t} = J, \\ &D = \varepsilon_0 \mathcal{E} \cdot \boldsymbol{E} + P, \quad \boldsymbol{H} = \tfrac{1}{2}\mu_0^{-1}\mathcal{E} : B - \boldsymbol{M}. \end{aligned} \tag{30}$$

In the formalism of differential forms, the spatial derivative $\boldsymbol{\nabla} \wedge \llcorner\!\lrcorner$ is simply denoted as $\mathsf{d}\llcorner\!\lrcorner$. Together with the Hodge operator "$*$", Eq. (30) is written in simpler forms;

$$\begin{aligned} &\mathsf{d}B = 0, \quad \mathsf{d}\boldsymbol{E} + \partial_t B = 0, \\ &\mathsf{d}D = \mathcal{R}, \quad \mathsf{d}\boldsymbol{H} - \partial_t D = J, \\ &D = \varepsilon_0(*\boldsymbol{E}) + P, \quad \boldsymbol{H} = \mu_0^{-1}(*B) - \boldsymbol{M}, \end{aligned} \tag{31}$$

where $\partial_t = \partial/\partial t$.

In Fig. 1, we show a diagram corresponding Eq. (31) and related equations (Deschamps (1981)). The field quantities are arranged according to their tensor order. The exterior derivative "d" connects a pair of quantities by increasing the tensor order by one, while time derivative ∂_t conserves the tensor order. $\boldsymbol{E}$ (B) is related to D ($\boldsymbol{H}$) with the Hodge star operator and the constant ε_0 (μ_0). The definitions of potentials and the charge conservation law

$$\boldsymbol{E} = -\mathsf{d}\phi - \partial_t \boldsymbol{A}, \quad B = \mathsf{d}\boldsymbol{A}, \quad \mathsf{d}J + \partial_t \mathcal{R} = 0 \tag{32}$$

are also shown in Fig. 1. We can see a well-organized, perfect structure. We will see the relativistic version later (Fig. 2).

5. Twisted forms and parity

5.1 Twist of volume form

We consider two bases $\Sigma = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and $\Sigma' = \{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3\}$. They can be related as $\boldsymbol{e}'_i = \sum_j R_{ij}\boldsymbol{e}_j$ by a matrix $R = [R_{ij}]$ with $R_{ij} = (\boldsymbol{e}'_i, \boldsymbol{e}_j)$. It is orthonormal and therefore $\det R = \pm 1$. In the case of $\det R = 1$, the two bases have the same orientation and they can be overlapped by a continuous transformation. On the other hand, for the case of $\det R = -1$, they have opposite orientation and an operation of reversal, for example, a diagonal matrix $\mathrm{diag}(-1, 1, 1)$ is needed to make them overlapped with rotations. Thus we can classify all the bases according to the orientation. We denote the two classes by C and C', each of which contains all the bases with the same orientation. The two classes are symmetric and there are no *a priori* order of precedence, like for i and $-$i.

We consider a basis $\Sigma = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\} \in C$ and the reversed basis $\Sigma' = \{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3\} = \{-\boldsymbol{e}_1, -\boldsymbol{e}_2, -\boldsymbol{e}_3\}$, which belongs to C'. The volume form $\mathcal{E}$ in Σ is defined so as to satisfy $\mathcal{E} \,\vdots\, \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 = +1$, i.e., the volume of the cube defined by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ should be $+1$. Similarly, the volume form $\mathcal{E}'$ in Σ' is defined so as to satisfy $\mathcal{E}' \,\vdots\, \boldsymbol{e}'_1\boldsymbol{e}'_2\boldsymbol{e}'_3 = +1$. We note that $\mathcal{E}' \,\vdots\, \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 = -\mathcal{E}' \,\vdots\, \boldsymbol{e}'_1\boldsymbol{e}'_2\boldsymbol{e}'_3 = -1$, namely, $\mathcal{E}' = -\mathcal{E}$. Thus we have two kinds of volume forms $\mathcal{E}$ and $\mathcal{E}'(= -\mathcal{E})$ and use either of them depending on the orientation of basis.

Assume that Alice adopts the basis $\Sigma \in C$ and Bob adopts $\Sigma' \in C'$. When we pose a parallelepipedon by specifying an ordered triple of vectors $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ and ask each of them to measure its volume, their answers will always be different in the sign. It seems inconvenient but there is no principle to choose one over the other. It is only a customary practice to use the right-handed basis to avoid the confusion. Fleming's left-hand rule (or right-hand rule) seems to break the symmetry but it implicitly relies upon the use of the right-handed basis.

5.2 Twisted forms

Tensors (or forms) are independent of the choice of basis. For example, a second order tensor can be expressed in Σ and Σ' as

$$\mathsf{B} = \sum_i \sum_j B'_{ij} \boldsymbol{n}'_i \boldsymbol{n}'_j = \sum_k \sum_l B_{kl} \boldsymbol{n}_k \boldsymbol{n}_l, \tag{33}$$

with the change of components $B_{kl} = \sum_i \sum_j R_{ik} R_{jl} B'_{ij}$. We note the dual basis has been flipped as $\boldsymbol{n}'_i = -\boldsymbol{n}_i$.

Similarly, in the case of 3-forms, we have

$$\mathcal{T} = T'_{123} \boldsymbol{n}'_1 \wedge \boldsymbol{n}'_2 \wedge \boldsymbol{n}'_3 = T_{123} \boldsymbol{n}_1 \wedge \boldsymbol{n}_2 \wedge \boldsymbol{n}_3 \tag{34}$$

with $T_{123} = T'_{123}$. However, for the volume form the components must be changed as

$$\epsilon_{123} = (\det R) \epsilon'_{123}, \tag{35}$$

to have $\mathcal{E}' = -\mathcal{E}$ in the case of reverse of orientation. Therefore, the volume form is called a *pseudo* form in order to distinguish from an ordinary form. The pseudo (normal) form are also call a *twisted* (untwisted) form.

In electromagnetism, some quantities are defined in reference to the volume form or to the Hodge star operator. Therefore, they could be twisted or untwisted. First of all, ϕ, $\boldsymbol{A}$, $\boldsymbol{E}$, and B are not involved with the volume form, they are all untwisted forms. On the other hand, $\boldsymbol{H} = \mu_0^{-1}(*\mathsf{B})$ and $\mathsf{D} = \varepsilon_0(*\boldsymbol{E})$ are twisted forms. The Hodge operator transforms an untwisted (twisted) form to a twisted (untwisted) form.

The quantities $\boldsymbol{M}$, P, J, and $\mathcal{R}$ (charge density), which represent volume densities of electromagnetic sources, are also twisted as shown below. We have found that the force fields are untwisted while the source fields are twisted in general.

5.3 Source densities

Here we look into detail why the quantities representing source densities are represented by twisted forms. As examples, we deal with polarization and charge density. Other quantities can be treated in the same manner.

5.3.1 Polarization

We consider two tangential vectors $\mathrm{d}\boldsymbol{x}$, $\mathrm{d}\boldsymbol{y}$ at a point P. Together with the volume form $\mathcal{E}$, we can define

$$\mathrm{d}\boldsymbol{S} = \mathcal{E} : \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \quad \overset{\mathrm{SI}}{\sim} \mathrm{m}^2, \tag{36}$$

which is a pseudo 1-form. (Conventionally, it is written as $\mathrm{d}\boldsymbol{S} = \boldsymbol{x} \times \boldsymbol{y}$.) In fact, for a tangential vector $\boldsymbol{\zeta}$ at P, the volume $\mathrm{d}\boldsymbol{S} \cdot \boldsymbol{\zeta} = \mathcal{E} \vdots \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\,\boldsymbol{\zeta}$, spanned by the three vectors is a linear function of $\boldsymbol{\zeta}$. We choose $\mathrm{d}\boldsymbol{z}$ that is perpendicular to the plane spanned by $\mathrm{d}\boldsymbol{x}$ and $\mathrm{d}\boldsymbol{y}$, i.e., $(\mathrm{d}\boldsymbol{z}, \mathrm{d}\boldsymbol{x}) = (\mathrm{d}\boldsymbol{z}, \mathrm{d}\boldsymbol{y}) = 0$. We assume $|\mathrm{d}\boldsymbol{z}| \ll |\mathrm{d}\boldsymbol{x}|$ and $|\mathrm{d}\boldsymbol{z}| \ll |\mathrm{d}\boldsymbol{y}|$. $\mathrm{d}V = \mathrm{d}\boldsymbol{S} \cdot \mathrm{d}\boldsymbol{z}$ is the volume of thin parallelogram plate.

When a charge $+q$ is displaced by $\boldsymbol{l}$ from the other charge $-q$, they form an electric dipole $\boldsymbol{p} = q\boldsymbol{l}$. We consider an electric dipole moment at a point P in $\mathrm{d}V$. The displacement $\boldsymbol{l}$ can be considered as a tangential vector at P, to which $\mathrm{d}\boldsymbol{S}$ acts as $\mathrm{d}\boldsymbol{S} \cdot \boldsymbol{l} = q^{-1}\mathrm{d}\boldsymbol{S} \cdot \boldsymbol{p}$. Then

$$q' = \frac{\mathrm{d}\boldsymbol{S} \cdot \boldsymbol{p}}{\mathrm{d}\boldsymbol{S} \cdot \mathrm{d}\boldsymbol{z}} = q\frac{\mathrm{d}\boldsymbol{S} \cdot \boldsymbol{l}}{\mathrm{d}V} \tag{37}$$

is the surface charge that is contributed by $\boldsymbol{p}$. In the case of $\mathrm{d}\boldsymbol{S} \cdot \boldsymbol{p} = 0$, there are no surface charge associated with $\boldsymbol{p}$. If $\mathrm{d}\boldsymbol{z}$ and $\boldsymbol{p}$ are parallel, $q'\mathrm{d}\boldsymbol{z} = q\boldsymbol{l} = \boldsymbol{p}$ holds.

Next we consider the case where many electric dipoles $\boldsymbol{p}_i = q_i\boldsymbol{l}_i$ are spatially distributed. The total surface charge is given as

$$\begin{aligned} Q' &= \sum_{i\in \mathrm{d}V} q_i' = \sum_{i\in \mathrm{d}V} \frac{\mathrm{d}\boldsymbol{S} \cdot \boldsymbol{p}_i}{\mathrm{d}V} \\ &= (\mathrm{d}V)^{-1} \sum_{i\in \mathrm{d}V} \mathcal{E} \cdot \boldsymbol{p}_i : \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} = \mathsf{P} : \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}, \end{aligned} \tag{38}$$

where the sum is taken over the dipoles contained in $\mathrm{d}V$. The pseudo 2-form

$$\mathsf{P} := (\mathrm{d}V)^{-1} \sum_{i\in \mathrm{d}V} \mathcal{E} \cdot \boldsymbol{p}_i \quad \overset{\mathrm{SI}}{\sim} \mathrm{C/m^2} \tag{39}$$

corresponds to the polarization (the volume density of electric dipole moments).

5.3.2 Charge density

The volume $\mathrm{d}V$ spanned by tangential vectors $\mathrm{d}\boldsymbol{x}$, $\mathrm{d}\boldsymbol{y}$, $\mathrm{d}\boldsymbol{z}$ at P is

$$\mathrm{d}V = \mathcal{E} \vdots \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{z} \quad \overset{\mathrm{SI}}{\sim} \mathrm{m}^3. \tag{40}$$

For distributed charges q_i, the total charge in $\mathrm{d}V$ is given as

$$\begin{aligned} Q &= \sum_{i\in \mathrm{d}V} q_i = \sum_{i\in \mathrm{d}V} \frac{q_i \mathrm{d}V}{\mathrm{d}V} \\ &= (\mathrm{d}V)^{-1} \sum_{i\in \mathrm{d}V} q_i \mathcal{E} \vdots \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{z} = \mathcal{R} \vdots \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{z}. \end{aligned} \tag{41}$$

The pseudo 3-form

$$\mathcal{R} := (\mathrm{d}V)^{-1} \sum_{i\in \mathrm{d}V} q_i \mathcal{E} \quad \overset{\mathrm{SI}}{\sim} \mathrm{C/m^3} \tag{42}$$

gives the charge density.

untwist/twist	rank	quantities	parity	polar/axial	scalar/vector
untwisted	0-form	ϕ	even	–	scalar
untwisted	1-form	$\boldsymbol{A}$, $\boldsymbol{E}$	odd	polar	vector
untwisted	2-form	B	even	axial	vector
twisted	1-form	$\boldsymbol{H}$, $\boldsymbol{M}$	even	axial	vector
twisted	2-from	D, P, J	odd	polar	vector
twisted	3-form	$\mathcal{R}$	even	–	scalar

Table 2. Electromagnetic field quantities as twisted and untwisted n-forms

5.4 Parity

Parity is the eigenvalues for a spatial inversion transformation. It takes $p = \pm 1$ depending on the types of quantities. The quantity with eigenvalue of $+1$ (-1) is called having even (odd) parity. In the three dimensional case, the spatial inversion can be provided by simply flipping the basis vectors; $\mathcal{P}\boldsymbol{e}_i = -\boldsymbol{e}_i$ $(i = 1, 2, 3)$. The dual basis covectors are also flipped; $\mathcal{P}\boldsymbol{n}_j = -\boldsymbol{n}_j$ $(j = 1, 2, 3)$.

A scalar (0-form) ϕ is even because $\mathcal{P}\phi = \phi$. The electric field $\boldsymbol{E}$ is a 1-form and transforms as

$$\mathcal{P}\boldsymbol{E} = \mathcal{P}(\sum_i E_i \boldsymbol{n}_i) = \sum_i E_i \mathcal{P}\boldsymbol{n}_i = -\sum_i E_i \boldsymbol{n}_i = -\boldsymbol{E}, \tag{43}$$

and, therefore, it is odd. The magnetic flux density B is a 2-form and even since it transforms as

$$\mathcal{P}B = \mathcal{P}(\sum_{(i,j)} B_{ij} \boldsymbol{n}_i \wedge \boldsymbol{n}_j) = \sum_{(i,j)} B_{ij} \mathcal{P}\boldsymbol{n}_i \wedge \mathcal{P}\boldsymbol{n}_j = B. \tag{44}$$

It is easy to see that the parity of an n-forms is $p = (-1)^n$.

The volume form is transformed as

$$\mathcal{P}\mathcal{E} = \mathcal{P}(V_{123}\boldsymbol{n}_1 \wedge \boldsymbol{n}_2 \wedge \boldsymbol{n}_3) = -V_{123}\mathcal{P}\boldsymbol{n}_1 \wedge \mathcal{P}\boldsymbol{n}_2 \wedge \mathcal{P}\boldsymbol{n}_3 = \mathcal{E}. \tag{45}$$

The additional minus sign is due to the change in the orientation of basis. If $\Sigma \in C$, then $\mathcal{P}\Sigma \in C'$, and *vice versa*. The twisted 3-form has even parity. In general, the parity of a twisted n-form is $p = (-1)^{(n+1)}$.

In the conventional vector-scalar formalism, the parity is introduced rather empirically. We have found that 1-forms and twisted 2-forms are unified as polar vectors, 2-forms and twisted 1-forms as axial vectors, and 0-forms and twisted 3-forms as scalars. Thus we have unveiled the real shapes of electromagnetic quantities as twisted and untwisted n-forms.

6. Relativistic formulae

6.1 Metric tensor and dual basis

Combining a three dimensional orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and a unit vector $\boldsymbol{e}_0$ representing the time axis, we have a four-dimensional basis $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. With the basis, a four (tangential) vector can be written

$$\underline{\boldsymbol{x}} = (c_0 t)\boldsymbol{e}_0 + x\boldsymbol{e}_2 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3 = x^\alpha \boldsymbol{e}_\alpha, \tag{46}$$

where the summation operator $\sum_{\alpha=0}^{3}$ is omitted in the last expression according to the Einstein summation convention. The vector components are represented with variables with superscripts. The sum is taken with respect to the Greek index repeated once as superscript and once as subscript. With the four-dimensional basis, the Lorentz-type inner product can be represented as

$$(\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}) = -(c_0 t)^2 + x^2 + y^2 + z^2 = x^\alpha x^\beta (\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta) = x^\alpha g_{\alpha\beta} x^\beta = x_\beta x^\beta, \tag{47}$$

where we set $x_\beta = x^\alpha g_{\alpha\beta}$ and $(\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta) = g_{\alpha\beta}$ with $g_{\alpha\beta} = 0$ $(\alpha \neq \beta)$, $-g_{00} = g_{ii} = 1$ $(i = 1, 2, 3)$.

We introduce the corresponding dual basis as $\{\boldsymbol{e}^0, \boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3\}$ with $\boldsymbol{e}^\mu \cdot \boldsymbol{e}_\nu = \delta^\mu_\nu$, where $\delta^\mu_\nu = 0$ $(\mu \neq \nu)$, $\delta^0_0 = \delta^i_i = 1$ $(i = 1, 2, 3)$. The dual basis covector has a superscript, while the components have subscripts. A four covector can be expressed with the dual basis as

$$\underline{\boldsymbol{a}} = a_\alpha \boldsymbol{e}^\alpha. \tag{48}$$

Then the contraction (by dot product) can be expressed systematically as

$$\boldsymbol{a} \cdot \boldsymbol{x} = a_\alpha \boldsymbol{e}^\alpha \cdot x^\beta \boldsymbol{e}_\beta = a_\alpha x^\beta\, \boldsymbol{e}^\alpha \cdot \boldsymbol{e}_\beta = a_\alpha x^\beta \delta^\alpha_\beta = a_\alpha x^\alpha. \tag{49}$$

We note that the dual and the inner product (metric) are independent concepts. Especially the duality can be introduced without the help of metric.

Customary, tensors which are represented by components with superscripts (subscripts) are designated as contravariant (covariant) tensors. With this terminology, a vector (covector) is a contravariant (covariant) tensor.

The symmetric second order tensor $\boldsymbol{g} = g_{\alpha\beta} \boldsymbol{e}^\alpha \boldsymbol{e}^\beta$ is called a metric tensor. Its components are

$$g_{\alpha\beta} = \begin{cases} -1 & (\alpha = \beta = 0) \\ 1 & (\alpha = \beta \neq 0) \\ 0 & (\text{other cases}) \end{cases}. \tag{50}$$

For a fixed four vector $\underline{\boldsymbol{z}}$, we can find a four covector $\underline{\boldsymbol{a}} = a_\beta \boldsymbol{e}^\beta$ that satisfy

$$\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{x}} = (\underline{\boldsymbol{z}}, \underline{\boldsymbol{x}}) \tag{51}$$

for any $\underline{\boldsymbol{x}}$. The left and right hand sides can be written as

$$\begin{aligned} \underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{x}} &= a_\beta x^\alpha \boldsymbol{e}^\beta \cdot \boldsymbol{e}_\alpha = a_\beta x^\alpha \delta^\beta_\alpha = a_\beta x^\beta, \\ (\underline{\boldsymbol{z}}, \underline{\boldsymbol{x}}) &= z^\alpha x^\beta (\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta) = z^\alpha g_{\alpha\beta} x^\beta, \end{aligned} \tag{52}$$

respectively. By comparing these, we obtain $a_\beta = z^\alpha g_{\alpha\beta}$. We write this covector $\underline{\boldsymbol{a}}$ determined by $\underline{\boldsymbol{z}}$ as

$$\underline{\boldsymbol{a}} = \underline{\boldsymbol{z}}^\top = z^\alpha g_{\alpha\beta} \boldsymbol{e}^\beta = z_\beta \boldsymbol{e}^\beta, \tag{53}$$

which is called the conjugate of $\underline{\boldsymbol{z}}$. We see that $(\boldsymbol{e}_0)^\top = -\boldsymbol{e}^0$, $(\boldsymbol{e}_i)^\top = \boldsymbol{e}^i$ $(i = 1, 2, 3)$, i.e., $(\boldsymbol{e}_\alpha)^\top = g_{\alpha\beta} \boldsymbol{e}^\beta$ [1]. With $g^{\alpha\beta} = (\boldsymbol{e}^\alpha, \boldsymbol{e}^\beta)$, the conjugate of a covector $\underline{\boldsymbol{a}}$ can be defined similarly with $z^\alpha = g^{\alpha\beta} a_\beta$ as $\underline{\boldsymbol{z}} = \underline{\boldsymbol{a}}^\top$.

[1] An equation $\boldsymbol{e}_\alpha = g_{\alpha\beta} \boldsymbol{e}^\beta$, which we may write carelessly, is not correct.

We have $(\underline{z}^\top)^\top = \underline{z}$, $(\underline{a}^\top)^\top = \underline{a}$, namely, ${}^{\top\top} = 1$.

We introduce the four dimensional completely anti-symmetric tensor of order 4 as

$$\begin{aligned}\mathfrak{E} &= \boldsymbol{e}^0 \wedge \boldsymbol{e}^1 \wedge \boldsymbol{e}^2 \wedge \boldsymbol{e}^3 \\ &= \epsilon_{\alpha\beta\gamma\delta} \boldsymbol{e}^\alpha \boldsymbol{e}^\beta \boldsymbol{e}^\gamma \boldsymbol{e}^\delta. \end{aligned} \tag{54}$$

The components $\epsilon_{\alpha\beta\gamma\delta}$, which are called the Levi-Civita symbol[2] can be written explicitly as

$$\epsilon_{\alpha\beta\gamma\delta} = \begin{cases} 1 & (\alpha\beta\gamma\delta \text{ is an even permutation of } 0123) \\ -1 & (\text{an odd permutation}) \\ 0 & (\text{other cases}) \end{cases} . \tag{55}$$

We note that

$$\begin{aligned}\mathfrak{E}^\top &= (\boldsymbol{e}^0)^\top \wedge (\boldsymbol{e}^1)^\top \wedge (\boldsymbol{e}^2)^\top \wedge (\boldsymbol{e}^3)^\top \\ &= (-\boldsymbol{e}_0) \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \\ &= \epsilon_{\alpha\beta\gamma\delta} g^{\alpha\mu} g^{\beta\nu} g^{\gamma\sigma} g^{\delta\tau} \boldsymbol{e}_\mu \boldsymbol{e}_\nu \boldsymbol{e}_\sigma \boldsymbol{e}_\tau \\ &= \epsilon^{\mu\nu\sigma\tau} \boldsymbol{e}_\mu \boldsymbol{e}_\nu \boldsymbol{e}_\sigma \boldsymbol{e}_\tau, \end{aligned} \tag{56}$$

where we introduced, $\epsilon^{\mu\nu\sigma\tau} = \epsilon_{\alpha\beta\gamma\delta} g^{\alpha\mu} g^{\beta\nu} g^{\gamma\sigma} g^{\delta\tau}$.

The conjugate of the metric tensor is given by

$$\underline{\boldsymbol{g}}^\top = g_{\alpha\beta} (\boldsymbol{e}^\alpha)^\top (\boldsymbol{e}^\beta)^\top = g_{\alpha\beta} g^{\alpha\mu} g^{\beta\nu} \boldsymbol{e}_\mu \boldsymbol{e}_\nu = g^{\mu\nu} \boldsymbol{e}_\mu \boldsymbol{e}_\nu. \tag{57}$$

6.2 Levi-Civita symbol

Here we will confirm some properties of the completely anti-symmetric tensor of order 4. From the relation between covariant and contravariant components

$$\epsilon^{\alpha\beta\gamma\delta} = g^{\alpha\mu} g^{\beta\nu} g^{\gamma\sigma} g^{\delta\tau} \epsilon_{\mu\nu\sigma\tau}, \tag{58}$$

and $g^{00} = -1$, we see that $\epsilon^{0123} = -\epsilon_{0123}$ and similar relations hold for other components. Here we note $\epsilon^{0123} = -1$.

With respect to contraction, we have

$$\epsilon^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta\gamma\delta} = -24 \quad (= -4!) \tag{59}$$

$$\epsilon^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta\gamma\tau} = -6\delta^\delta_\tau \tag{60}$$

$$\epsilon^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta\sigma\tau} = -2(\delta^\gamma_\sigma \delta^\delta_\tau - \delta^\gamma_\tau \delta^\delta_\sigma) = -4\delta^\gamma_{[\sigma} \delta^\delta_{\tau]} \tag{61}$$

$$\epsilon^{\alpha\beta\gamma\delta} \epsilon_{\alpha\nu\sigma\tau} = -6\delta^\beta_{[\nu} \delta^\gamma_{\ \sigma} \delta^\delta_{\tau]}. \tag{62}$$

where $_{[]}$ in the subscript represents the anti-symmetrization with respect to the indices. For example, we have $A_{[\alpha\beta} B_{\gamma]} = (A_{\alpha\beta} B_\gamma + A_{\beta\gamma} B_\alpha + A_{\gamma\alpha} B_\beta - A_{\beta\alpha} B_\gamma - A_{\gamma\beta} B_\alpha - A_{\alpha\gamma} B_\beta)/6$. We note $\underline{\boldsymbol{A}} \wedge \underline{\boldsymbol{B}} = A_{\alpha\beta} B_\gamma \boldsymbol{e}^\alpha \wedge \boldsymbol{e}^\beta \wedge \boldsymbol{e}^\gamma = 6A_{[\alpha\beta} B_{\gamma]} \boldsymbol{e}^\alpha \boldsymbol{e}^\beta \boldsymbol{e}^\gamma$.

[2] In the case of three dimension, the parity of a permutation can simply be discriminated by the cyclic or anti-cyclic order. In the case of four dimension, the parity of $0ijk$ follows that of ijk and those of $i0jk$, $ij0k$, $ijk0$ is opposite to that of ijk.

6.3 Hodge dual of anti-symmetric 2nd-order tensors

The four-dimensional Hodge dual $(\underline{*}A)_{\alpha\beta}$ of a second order tensor $A_{\alpha\beta}$ is defined to satisfy

$$(\underline{*}A)_{[\alpha\beta}B_{\gamma\delta]} = \frac{1}{2}(A^{\mu\nu}B_{\mu\nu})\epsilon_{\alpha\beta\gamma\delta}, \tag{63}$$

for an arbitrary tensor $B_{\gamma\delta}$ of order $(d-2)$ (Flanders (1989)). This relation is independent of the basis [3].

Here, we will show that

$$(\underline{*}A)_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\mu\nu}A_{\mu\nu}. \tag{64}$$

Substituting into the left hand side of Eq. (63) and contracting with $\epsilon^{\alpha\beta\gamma\delta}$, we have

$$\epsilon^{\alpha\beta\gamma\delta}\frac{1}{2}A_{\mu\nu}\epsilon_{[\alpha\beta}{}^{\mu\nu}B_{\gamma\delta]} = 3\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta}{}^{\mu\nu}A_{\mu\nu}B_{\gamma\delta} = 3\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\mu\nu}A^{\mu\nu}B_{\gamma\delta} = -12A^{\gamma\delta}B_{\gamma\delta}. \tag{65}$$

With Eq. (59), the right hand side of Eq. (63) yields $-12A^{\mu\nu}B_{\mu\nu}$ with the same contraction. We also note

$$\begin{aligned}(\underline{**}A)_{\alpha\beta} &= \frac{1}{4}\epsilon_{\alpha\beta}{}^{\gamma\delta}\epsilon_{\gamma\delta}{}^{\mu\nu}A_{\mu\nu} = \frac{1}{4}\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\gamma\delta\mu\nu}A_{\mu\nu} \\ &= -\frac{1}{2}(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu})A_{\mu\nu} = -\frac{1}{2}(A_{\alpha\beta} - A_{\beta\alpha}) = -A_{\alpha\beta},\end{aligned} \tag{66}$$

i.e., $\underline{**} = -1$, which is different from the three dimensional case; $** = 1$.

7. Differential forms in Minkowski spacetime

7.1 Standard formulation

According to Jackson's textbook (Jackson (1998)), we rearrange the ordinary scalar-vector form of Maxwell's equation in three dimension into a relativistic expression. We use the SI system and pay attention to the dimensions. We start with the source equations

$$\boldsymbol{\nabla}\times\boldsymbol{H} - \frac{\partial}{\partial(c_0t)}(c_0\boldsymbol{D}) = \boldsymbol{J}, \quad \boldsymbol{\nabla}\cdot(c_0\boldsymbol{D}) = c_0\varrho. \tag{67}$$

Combining field quantities and differential operators as four-dimensional tensors and vectors as

$$(\tilde{G}^{\alpha\beta}) = \begin{bmatrix} 0 & c_0D_x & c_0D_y & c_0D_z \\ -c_0D_x & 0 & H_z & -H_y \\ -c_0D_y & -H_z & 0 & H_x \\ -c_0D_z & H_y & -H_x & 0 \end{bmatrix} \overset{\text{SI}}{\sim} \text{A/m}, \tag{68}$$

[3] With the four-dimensional volume form $\mathfrak{E} = \boldsymbol{e}^0\wedge\boldsymbol{e}^1\wedge\boldsymbol{e}^2\wedge\boldsymbol{e}^3$, the Hodge dual for p form ($p = 1,2,3$) can be defined as $(\underline{*}\underline{\boldsymbol{A}})\wedge\underline{\boldsymbol{B}} = (\underline{\boldsymbol{A}},\underline{\boldsymbol{B}})\mathfrak{E}$, $(\underline{*}\underline{A})\wedge\underline{B} = (\underline{A},\underline{B})\mathfrak{E}$, and $(\underline{*}\underline{\mathcal{A}})\wedge\underline{\mathcal{B}} = (\underline{\mathcal{A}},\underline{\mathcal{B}})\mathfrak{E}$. The inner product for p forms is defined as $(\boldsymbol{a}_1\wedge\cdots\wedge\boldsymbol{a}_p, \boldsymbol{b}_1\wedge\cdots\wedge\boldsymbol{b}_p) = \det(\boldsymbol{a}_i,\boldsymbol{b}_j)$. (Flanders (1989))

$$(\partial_\beta) = \begin{bmatrix} c_0^{-1}\partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \overset{\text{SI}}{\sim} 1/\text{m}, \quad (\tilde{J}^\alpha) = \begin{bmatrix} c_0\varrho \\ J_x \\ J_y \\ J_z \end{bmatrix} \overset{\text{SI}}{\sim} \text{A}/\text{m}^2, \tag{69}$$

we have a relativistic equation

$$\partial_\beta \tilde{G}^{\alpha\beta} = \tilde{G}^{\alpha\beta}{}_{,\beta} = \tilde{J}^\alpha. \tag{70}$$

We append "~", by the reason described later. The suffix 0 represents the time component, and the suffixes $1, 2, 3$ correspond to x, y, z-components. The commas in suffixes "$_,$" means the derivative with respect to the following spatial component, e.g., $H_{2,1} = (\partial/\partial x_1)H_2$.

On the other hand, the force equations

$$\nabla \times \boldsymbol{E} + \frac{\partial}{\partial(c_0 t)}(c_0\boldsymbol{B}) = 0, \quad \nabla \cdot (c_0\boldsymbol{B}) = 0, \tag{71}$$

are rearranged with

$$(\tilde{F}^{\alpha\beta}) = \begin{bmatrix} 0 & c_0B_x & c_0B_y & c_0B_z \\ -c_0B_x & 0 & -E_z & E_y \\ -c_0B_y & E_z & 0 & -E_x \\ -c_0B_z & -E_y & E_x & 0 \end{bmatrix} \overset{\text{SI}}{\sim} \text{V}/\text{m}, \tag{72}$$

as

$$\partial_\beta \tilde{F}^{\alpha\beta} = \tilde{F}^{\alpha\beta}{}_{,\beta} = 0. \tag{73}$$

Thus the four electromagnetic field quantities $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$, and $\boldsymbol{H}$ are aggregated into two second order, antisymmetric tensors $\tilde{F}^{\alpha\beta}$, $\tilde{G}^{\alpha\beta}$.

In vacuum, the constitutive relations $\boldsymbol{D} = \varepsilon_0\boldsymbol{E}$, $\boldsymbol{H} = \mu_0^{-1}\boldsymbol{B}$ hold, therefore, these tensors are related as

$$\tilde{G}^{\alpha\beta} = Y_0(\underline{*}\tilde{F})^{\alpha\beta}, \quad \text{or} \quad \tilde{F}^{\alpha\beta} = -Z_0(\underline{*}\tilde{G})^{\alpha\beta}, \tag{74}$$

where $Z_0 = 1/Y_0 = \sqrt{\mu_0/\varepsilon_0} \overset{\text{SI}}{\sim} \Omega$ is the vacuum impedance.

The operator $\underline{*}$ is the four-dimensional Hodge's star operator. From Eq. (64), the action for a 2nd-order tensor is written as

$$(\underline{*}A)^{ij} = A^{0k}, \quad (\underline{*}A)^{0i} = -A^{jk}, \tag{75}$$

where i, j, k $(i, j = 1, 2, 3)$ are cyclic. We note that $\underline{**} = -1$, i.e., $\underline{*}^{-1} = -\underline{*}$ holds.

Equation (74) is a relativistic version of constitutive relations of vacuum and carries two roles. First it connect dimensionally different tensors $\tilde{G}$ and $\tilde{F}$ with the vacuum impedance Z_0. Secondly it represents the Hodge's dual relation. The Hodge operator depends both on the handedness of the basis[4] and the metric.

[4] We note $\epsilon_{\alpha\beta\gamma\delta}$ is a pseudo form rather than a form. Therefore, the Hodge operator makes a form into a pseudo form, and a pseudo form into a normal form.

Finally, the Maxwell equations can be simply represented as

$$\partial_\beta \tilde{G}^{\alpha\beta} = \tilde{J}^\alpha, \quad \partial_\beta \tilde{F}^{\alpha\beta} = 0, \quad \tilde{G}^{\alpha\beta} = Y_0(\underline{*}\tilde{F})^{\alpha\beta}. \tag{76}$$

This representation, however, is quite unnatural in the view of two points. First of all, the components of field quantity should be covariant and should have lower indices. Despite of that, here, all quantities are contravariant and have upper indices in order to contract with the spatial differential operator ∂_α with a lower index. Furthermore, it is unnatural that in Eqs. (68) and (72), D and B, which are 2-forms with respect to space, have indices of time and space, and E and H have two spatial indices.

The main reason of this unnaturalness is that we have started with the conventional, scalar-vector form of Maxwell equations rather than from those in differential forms.

7.2 Bianchi identity

In general textbooks, the one of equations in Eq. (76) is further modified by introducing a covariant tensor $F_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\tilde{F}^{\gamma\delta}$. Solving it as $\tilde{F}^{\alpha\beta} = -\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$ and substituting into Eq. (73), we have

$$0 = \partial_\beta \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \epsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta}. \tag{77}$$

Considering α as a fixed parameter, we have six non-zero terms that are related as

$$\begin{aligned} 0 &= \partial_\beta(F_{\gamma\delta} - F_{\delta\gamma}) + \partial_\gamma(F_{\delta\beta} - F_{\beta\delta}) + \partial_\delta(F_{\beta\gamma} - F_{\gamma\beta}) \\ &= 2\left(\partial_\beta F_{\gamma\delta} + \partial_\gamma F_{\delta\beta} + \partial_\delta F_{\beta\gamma}\right) \quad (\beta, \gamma, \delta = 0, \ldots, 3). \end{aligned} \tag{78}$$

Although there are many combinations of indices, this represents substantially four equations. To be specific, we introduce the matrix representation of $F_{\alpha\beta}$ as

$$(F_{\alpha\beta}) = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & c_0 B_z & -c_0 B_y \\ E_y & -c_0 B_z & 0 & c_0 B_x \\ E_z & c_0 B_y & -c_0 B_x & 0 \end{bmatrix}. \tag{79}$$

Comparing this with $\tilde{G}^{\alpha\beta}$ in Eq. (68) and considering the constitutive relations, we find that the signs of components with indices for time "0" are reversed. Therefore, with the metric tensor, we have

$$\tilde{G}^{\alpha\beta} = Y_0 g^{\alpha\gamma} g^{\beta\delta} F_{\gamma\delta} = Y_0 F^{\alpha\beta}. \tag{80}$$

Substitution into Eq. (70) yields $Y_0\partial_\beta F^{\alpha\beta} = \tilde{J}^\alpha$. After all, relativistically, the Maxwell equations are written as

$$\partial_\beta F^{\alpha\beta} = Z_0 \tilde{J}^\alpha, \quad \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0. \tag{81}$$

Even though this common expression is simpler than that for the non relativistic version, symmetry is somewhat impaired. The covariant and contravariant field tensors are mixed. The reason is that the constitutive relations, which contains the Hodge operator, is eliminated.

7.3 Differential forms

Here we start with the Maxwell equations (31) in three-dimensional differential forms. We introduce a basis $\{e_0, e_1, e_2, e_3\}$, and the corresponding dual basis $\{e^0, e^1, e^2, e^3\}$, i.e., $e^\mu \cdot e_\nu = \delta^\mu_\nu$. With $\underline{G} = e^0 \wedge H + c_0 D$, $\underline{\mathcal{J}} = e^0 \wedge (-J) + c_0 \mathcal{R}$, and $\underline{\nabla} = c_0^{-1} \partial_t e^0 + \nabla$, the source equations

$$\nabla \wedge H - \frac{\partial}{\partial(c_0 t)}(c_0 D) = J, \quad \nabla \wedge (c_0 D) = \mathcal{R}, \tag{82}$$

are unified as

$$\underline{\nabla} \wedge \underline{G} = \underline{\mathcal{J}}, \tag{83}$$

where $\wedge$ represent the anti-symmetric tensor product or the wedge product. In components, Eq. (83) is

$$\partial_{[\gamma} G_{\alpha\beta]} = G_{[\alpha\beta,\gamma]} = J_{\alpha\beta\gamma}/3. \tag{84}$$

The tensor $G_{\alpha\beta}$ can be written as

$$(G_{\alpha\beta}) = (\underline{G} : e_\alpha e_\beta) = \begin{bmatrix} 0 & H_x & H_y & H_z \\ -H_x & 0 & c_0 D_z & -c_0 D_y \\ -H_y & -c_0 D_z & 0 & c_0 D_x \\ -H_z & c_0 D_y & -c_0 D_x & 0 \end{bmatrix}. \tag{85}$$

The covariant tensors (forms) $G_{\alpha\beta}$ and $J_{\alpha\beta\gamma}$ are related to $\tilde{G}^{\alpha\beta}$ and $\tilde{J}^\alpha$ in the previous subsection as

$$G_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\tilde{G}^{\gamma\delta}, \quad J_{\alpha\beta\gamma} = -\epsilon_{\alpha\beta\gamma\delta}\tilde{J}^\delta, \quad \text{or} \quad \tilde{G}^{\alpha\beta} = -\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}G_{\gamma\delta}, \quad \tilde{J}^\alpha = \frac{1}{6}\epsilon^{\alpha\beta\gamma\delta}J_{\beta\gamma\delta}. \tag{86}$$

Similarly,

$$\nabla \wedge E + \frac{\partial}{\partial(c_0 t)}(c_0 B) = 0, \quad \nabla \wedge (c_0 B) = 0, \tag{87}$$

can be written as

$$\underline{\nabla} \wedge \underline{F} = 0, \tag{88}$$

with $\underline{F} = e^0 \wedge (-E) + c_0 B$. In components,

$$\partial_{[\gamma} F_{\alpha\beta]} = F_{[\alpha\beta,\gamma]} = 0, \tag{89}$$

where

$$(F_{\alpha\beta}) = (\underline{F} : e_\alpha e_\beta) = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & c_0 B_z & -c_0 B_y \\ E_y & -c_0 B_z & 0 & c_0 B_x \\ E_z & c_0 B_y & -c_0 B_x & 0 \end{bmatrix}. \tag{90}$$

The covariant tensor (form) $F_{\alpha\beta}$ is related to $\tilde{F}^{\alpha\beta}$ in the previous subsection as

twisted/untwisted	order	quantities
untwisted	1-form	$\underline{V} = \phi e^0 + c_0(-\boldsymbol{A})$
untwisted	2-form	$\underline{F} = e^0 \wedge (-\boldsymbol{E}) + c_0 B$
twisted	2-form	$\underline{G} = e^0 \wedge \boldsymbol{H} + c_0 D$
twisted	2-form	$\underline{I} = e^0 \wedge (-\boldsymbol{M}) + c_0 P$
twisted	3-form	$\underline{\mathcal{J}} = e^0 \wedge (-J) + c_0 \mathcal{R}$

Table 3. Four dimensional electromagnetic field quantities as twisted and untwisted n-forms

$$F_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\tilde{F}^{\gamma\delta}, \quad \text{or,} \quad \tilde{F}^{\alpha\beta} = -\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}. \tag{91}$$

The Hodge operator acts on a four-dimensional two form as

$$\underline{*}(e^0 \wedge \boldsymbol{X} + Y) = e^0 \wedge (-(*Y)) + (*\boldsymbol{X}) = e^0 \wedge (-\boldsymbol{Y}) + X, \tag{92}$$

where $\boldsymbol{X}$ and Y are a three-dimensional 1-form and a three-dimensional 2-form, and $*$ is the three-dimensional Hodge operator[5]. Now the constitutive relations $D = \varepsilon_0(*\boldsymbol{E})$ and $\boldsymbol{H} = \mu_0^{-1}(*B)$ are four-dimensionally represented as

$$\underline{G} = -Y_0(\underline{*F}), \quad \text{or} \quad \underline{F} = Z_0(\underline{*G}). \tag{93}$$

With components, these are represented as

$$G_{\alpha\beta} = -Y_0(\underline{*}F)_{\alpha\beta}, \quad \text{or} \quad F_{\alpha\beta} = Z_0(\underline{*}G)_{\alpha\beta}, \tag{94}$$

with the action of Hodge's operator on anti-symmetric tensors of rank 2:

$$(\underline{*}A)_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\gamma\delta}A_{\gamma\delta} = \frac{1}{2}g_{\alpha\mu}g_{\beta\nu}\epsilon^{\mu\nu\gamma\delta}A_{\gamma\delta}. \tag{95}$$

Now we have the Maxwell equations in the four-dimensional forms with components:

$$\partial_{[\gamma}G_{\alpha\beta]} = J_{\alpha\beta\gamma}/3, \quad \partial_{[\gamma}F_{\alpha\beta]} = 0, \quad F_{\alpha\beta} = Z_0(\underline{*}G)_{\alpha\beta}, \tag{96}$$

and in basis-free representations:

$$\underline{\boldsymbol{\nabla}} \wedge \underline{G} = \underline{\mathcal{J}}, \quad \underline{\boldsymbol{\nabla}} \wedge \underline{F} = 0, \quad \underline{F} = Z_0(\underline{*G}), \tag{97}$$

or

$$\underline{\mathrm{d}G} = \underline{\mathcal{J}}, \quad \underline{\mathrm{d}F} = 0, \quad \underline{F} = Z_0(\underline{*G}). \tag{98}$$

with the four-dimensional exterior derivative $\underline{\mathrm{d}}_{\sqcup} = \underline{\boldsymbol{\nabla}} \wedge {}_{\sqcup}$. These are much more elegant and easier to remember compared with Eqs. (76) and (81). A similar type of reformulation has been given by Sommerfeld (Sommerfeld (1952)).

[5] We note the similarity with the calculation rule for complex numbers: $\mathrm{i}(X + \mathrm{i}Y) = -Y + \mathrm{i}X$. If we can formally set as $\boldsymbol{G} = \boldsymbol{H} + \mathrm{i}c_0\boldsymbol{D}$ and $\boldsymbol{F} = -\boldsymbol{E} + \mathrm{i}c_0\boldsymbol{B}$, we have $\boldsymbol{G} = -\mathrm{i}Y_0\boldsymbol{F}$ and $\boldsymbol{H} = \mathrm{i}Z_0\boldsymbol{G}$.

7.4 Potentials and the conservation of charge

We introduce a four-dimensional vector potential $\underline{\boldsymbol{V}} = \phi \boldsymbol{e}^0 + c_0(-\boldsymbol{A})$, i.e.,

$$(V_\alpha) = (\underline{\boldsymbol{V}} \cdot \boldsymbol{e}_\alpha)$$
$$= (\phi, -c_0 A_x, -c_0 A_y, -c_0 A_z). \tag{99}$$

Then we have $\underline{\boldsymbol{\nabla}} \wedge \underline{\boldsymbol{V}} = -\underline{F}$, or

$$\partial_{[\alpha} V_{\beta]} = -F_{\alpha\beta}/2, \tag{100}$$

which is a relation between the potential and the field strength. Utilizing the potential, the force equation becomes very trivial,

$$0 = \underline{\boldsymbol{\nabla}} \wedge (\underline{\boldsymbol{\nabla}} \wedge \underline{\boldsymbol{V}}) = \underline{\boldsymbol{\nabla}} \wedge \underline{F}, \tag{101}$$

since $\underline{\boldsymbol{\nabla}} \wedge \underline{\boldsymbol{\nabla}} = 0$ or $\underline{\mathsf{d}}\underline{\mathsf{d}} = 0$ holds.

The freedom of gauge transformation with a 0-form Λ can easily be understood; $\underline{\boldsymbol{V}}' = \underline{\boldsymbol{V}} + \underline{\mathsf{d}}\Lambda$ gives no difference in the force quantities, i.e., $\underline{F}' = \underline{F}$. A similar degree of freedom exist for the source fields (Hirst (1997)). With a 1-form $\underline{\boldsymbol{L}}$, we define the transformation $\underline{G}' = \underline{G} + \underline{\mathsf{d}}\underline{\boldsymbol{L}}$, which yields $\underline{\mathcal{J}}' = \underline{\mathcal{J}}$.

The conservation of charge is also straightforward;

$$0 = \underline{\boldsymbol{\nabla}} \wedge \underline{\boldsymbol{\nabla}} \wedge \underline{G} = \underline{\boldsymbol{\nabla}} \wedge \underline{\mathcal{J}}$$
$$= \boldsymbol{e}^0 \wedge (\partial_t \mathcal{R} + \boldsymbol{\nabla} \wedge J). \tag{102}$$

7.5 Relativistic representation of the Lorentz force

Changes in the energy E and momentum p of a charged particle moving at velocity $\boldsymbol{u}$ in an electromagnetic field are

$$\frac{\mathrm{d}E}{\mathrm{d}t} = q\boldsymbol{E} \cdot \boldsymbol{u},$$
$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}. \tag{103}$$

By introducing the four dimensional velocity $u^\alpha = \left[c_0\gamma, u_x, u_y, u_z\right]$, and the four dimensional momentum $p_\alpha = \left[-E/c_0, p_x, p_y, p_z\right]$, we have the equation of motion

$$\frac{\mathrm{d}p_\alpha}{\mathrm{d}\tau} = qF_{\alpha\beta}u^\beta, \tag{104}$$

where $\mathrm{d}\tau = \mathrm{d}t/\gamma$ the proper time of moving charge, and $\gamma = (1 - u^2/c_0^2)^{-1/2}$ is the Lorentz factor. The change in action ΔS can be written

$$\Delta S = p_\alpha \Delta x^\alpha = -E\Delta t + \boldsymbol{p} \cdot \Delta \boldsymbol{x}. \tag{105}$$

$$
\begin{array}{ccc}
\underline{\text{F quantities}} & & \underline{\text{S quantities}} \\
\underline{\boldsymbol{V}} = \phi \boldsymbol{e}^0 - c_0 \boldsymbol{A} & & \\
\downarrow \underline{\mathsf{d}} & Y_0 & \\
\underline{F} = -\boldsymbol{e}^0 \wedge \boldsymbol{E} + c_0 B & \leftarrow \underline{*} \rightarrow & \underline{G} = \boldsymbol{e}^0 \wedge \boldsymbol{H} + c_0 D \\
\downarrow \underline{\mathsf{d}} & Z_0 & \downarrow \underline{\mathsf{d}} \\
0 & & \underline{\mathcal{J}} = -\boldsymbol{e}^0 \wedge J + c_0 \mathcal{R} \\
 & & \downarrow \underline{\mathsf{d}} \\
 & & 0
\end{array}
$$

Fig. 2. Relations of electromagnetic field forms in four dimension

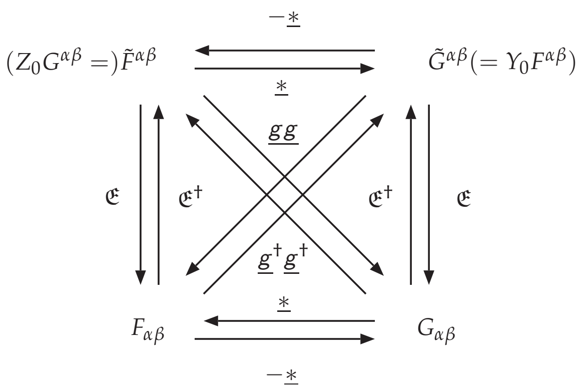

Fig. 3. Various kinds of tensors of order 2 used in the relativistic Maxwell equations

7.6 Summary for relativistic relations

In Fig. 2, the relativistic quantities are arranged as a diagram, the rows of which correspond to the orders of tensors ($n = 1, 2, 3, 4$). In the left column, the quantities related to the electromagnetic forces (F quantities), and in the right column, the quantities related to the electromagnetic sources (S quantities) are listed. The exterior derivative "$\underline{\mathsf{d}}$" connects a pair of quantities by increasing the tensor order by one. These differential relations correspond to the definition of (scalar and vector) potentials, the Maxwell's equations, and the charge conservation (See Fig. 1). Hodge's star operator "$\underline{*}$" connects two 2-forms: $\underline{F}$ and $\underline{G}$. This corresponds to the constitutive relations for vacuum and here appears the vacuum impedance $Z_0 = 1/Y_0$ as the proportional factor.

In Fig. 3, various kinds of tensors of order 2 in the relativistic Maxwell equations and their relations are shown. The left column corresponds to the source fields (D, H), the right column corresponds to the force fields (E, B). Though not explicitly written, due to the difference in dimension, the conversions associate the vacuum impedance (or admittance). "$\mathfrak{E}$" and "$\mathfrak{E}^\top$" represent the conversion by Levi-Civita (or by its conjugate), "$\underline{*}$" represents the conversion by Hodge's operator. Associated with the diagonal arrows, "$\underline{g}^\top \underline{g}^\top$", and "$\underline{g}\,\underline{g}$" represent raising and lowering of the indices with the metric tensors, respectively. The tensors in the upper row are derived from the scalar-vector formalism and those in the lower row are derived from the differential forms. In order to avoid the use of the Hodge operator, the diagonal pair $F_{\alpha\beta}$, $F^{\alpha\beta}$ $(= Z_0\tilde{G}^{\alpha\beta})$ are used conventionally and the symmetry is sacrificed.

8. Conclusion

In this book chapter, we have reformulated the electromagnetic theory. First we have confirmed the role of vacuum impedance Z_0 as a fundamental constant. It characterizes the electromagnetism as the gravitational constant G characterizes the theory of gravity. The velocity of light c_0 in vacuum is the constant associated with space-time, which is a framework in which electromagnetism and other theories are constructed. Then, Z_0 is a single parameter characterizing electromagnetism, and $\varepsilon_0 = 1/(Z_0 c_0)$ and $\mu_0 = Z_0/c_0$ are considered derived parameters.

Next, we have introduced anti-symmetric covariant tensors, or differential forms, in order to represent EM field quantities most naturally. It is a significant departure from the conventional scalar-vector formalism. But we have tried not to be too mathematical by carrying over the conventional notations as many as possible for continuous transition. In this formalism, the various field quantities are defined through the volume form, which is the machinery to calculate the volume of parallelepipedon spanned by three tangential vectors. To be precise, it is a pseudo (twisted) form, whose sign depends on the orientation of basis.

Even though the constitutive relation seems as a simple proportional relation, it associates the conversion by the Hodge dual operation and the change in physical dimensions by the vaccum impedance. We have found that this non-trivial relation is the keystone of the EM theory.

The EM theory has the symmetry with respect to the space inversion, therefore, each field quantity has a definite parity, even or odd. We have shown that the parity is determined by the tensorial order and the pseudoness (twisted or untwisted).

The Maxwell equations can be formulated most naturally in the four dimensional space-time. However, the conventional expression with tensor components (with superscripts or subscripts) is somewhat abstract and hard to read out its geometrical or physical meaning. Moreover, sometimes contravariant tensors are introduced in order to avoid the explicit use of the Hodge dual with sacrificing the beauty of equations. It has been shown that the four-dimensional differential forms (anti-symmetric covariant tensors) are the most suitable tools for expressing the structure of the EM theory.

The structured formulation helps us to advance electromagnetic theories to various areas. For example, the recent development of new type of media called metamaterials, for which we have to deal with electric and magnetic interactions simultaneously, confronts us to reexamine theoretical frameworks. It may also be helpful to resolve problems on the electromagnetic momentum within dielectric media.

9. Acknowledgment

I thank Yosuke Nakata for helpful discussions. This research was supported by a Grant-in-Aid for Scientific Research on Innovative Areas (No. 22109004) and the Global COE program "Photonics and Electronics Science and Engineering" at Kyoto University.

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