Monoidal Functors Generated by Adjunctions, with Applications to Transport of Structure

G.M. Kelly
School of Mathematics and Statistics
University of Sydney
NSW 2006, Australia
maxk@maths.usyd.edu.au

Stephen Lack
School of Mathematics and Statistics
University of Sydney NSW 2006
Australia
and
School of Quantitative Methods and Mathematical Sciences
University of Western Sydney
Locked Bag 1797 DC South Penrith 1797 NSW
Australia
s.lack@uws.edu.au

Abstract. Bénabou pointed out in 1963 that a pair $f \dashv u : A \to B$ of adjoint functors induces a monoidal functor $[f, u] : [A, A] \to [B, B]$ between the (strict) monoidal categories of endofunctors. We show that this result about adjunctions in the monoidal 2-category **Cat** extends to adjunctions in any right-closed monoidal 2-category $\mathcal{V}$, or more generally in any 2-category $\mathcal{A}$ with an action $*$ of a monoidal 2-category $\mathcal{V}$ admitting an adjunction $\mathcal{A}(T * A, B) \cong \mathcal{V}(T, \langle A, B\rangle)$; certainly such an adjunction exists when $*$ is the canonical action of $[\mathcal{A}, \mathcal{A}]$ on $\mathcal{A}$, provided that $\mathcal{A}$ is complete and locally small. This result allows a concise and general treatment of the transport of algebraic structure along an equivalence.

1 Introduction

We suppose given a *monoidal 2-category* $\mathcal{V}$: that is, a 2-category $\mathcal{V}$ along with a monoidal structure $(\otimes, I, a, l, r)$ for which $\otimes$ is a 2-functor and a, l, r are 2-natural. We further suppose given a 2-category $\mathcal{A}$ and an *action* of $\mathcal{V}$ on $\mathcal{A}$: that is, a 2-functor $* : \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ together with 2-natural isomorphisms $\alpha : (X \otimes Y) * A \cong X * (Y * A)$ and $\lambda : I * A \cong A$ satisfying the usual two coherence axioms. Finally

---

2000 *Mathematics Subject Classification.* Primary 18D05; Secondary 18A40, 18D10.
The support of the Australian Research Council is gratefully acknowledged.

we suppose each 2-functor $- * A : \mathcal{V} \to \mathcal{A}$ to have a right adjoint $((A, -))$, so that we have a 2-natural isomorphism

$$\Phi : \mathcal{A}(X * A, B) \cong \mathcal{V}(X, ((A, B))). \tag{1.1}$$

A first example is that where $\mathcal{A}$ is $\mathcal{V}$ itself, with $\otimes$ for $*$ and a, l for α, λ; then $((A, B))$ is an "internal hom", more commonly denoted by $[A, B]$, whose existence makes of $\mathcal{V}$ a *right-closed* monoidal 2-category. A second example is that where $\mathcal{A}$ is any 2-category which is locally small and complete, while $\mathcal{V}$ is the monoidal 2-category $[\mathcal{A}, \mathcal{A}]$ of endofunctors of $\mathcal{A}$ (meaning of course *endo-2-functors*, since $\mathcal{A}$ is a 2-category), with composition for its tensor product and the identity functor $1 = 1_{\mathcal{A}}$ for its unit object. The action $[\mathcal{A}, \mathcal{A}] \times \mathcal{A} \to \mathcal{A}$ we intend here is that given by evaluation, sending (T, A) to TA and similarly defined on morphisms. Now (1.1) takes the form

$$\Phi : \mathcal{A}(XA, B) \cong [\mathcal{A}, \mathcal{A}](X, (A, B)), \tag{1.2}$$

where (A, B) is the right Kan extension of $B : 1 \to \mathcal{A}$ along $A : 1 \to \mathcal{A}$ given by $(A, B)C = B^{\mathcal{A}(C,A)}$.

There is a sense in which the second example is "extremal". For in the context of a general example as in (1.1), we can still apply (1.2) (provided $\mathcal{A}$ is locally small and complete) to get

$$\mathcal{A}(X * A, B) \cong [\mathcal{A}, \mathcal{A}](X * -, (A, B)),$$

so that we have a natural isomorphism

$$\mathcal{V}(X, ((A, B))) \cong [\mathcal{A}, \mathcal{A}](X * -, (A, B)). \tag{1.3}$$

Moreover, as was discussed in [6], it is common in examples of such actions for the 2-functor $\mathcal{V} \to [\mathcal{A}, \mathcal{A}]$ sending X to $X * -$ to have a right adjoint $\theta : [\mathcal{A}, \mathcal{A}] \to \mathcal{V}$; and when this is so, (1.3) gives a natural isomorphism $((A, B)) \cong \theta(A, B)$. In these circumstances, our main results below for the general case (1.1) are consequences of those for the extremal case (1.2).

However, it in fact costs nothing to consider the general case throughout, especially if we use the coherence to simplify the notation as follows. Forget for the moment that $\mathcal{V}$ and $\mathcal{A}$ are 2-categories. To give a monoidal category $\mathcal{V}$ and an action $* : \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ of $\mathcal{V}$ on $\mathcal{A}$ is equally to give a bicategory $\mathbb{B}$ with just two object 0 and 1, having

$$\mathbb{B}(0,0) = \mathcal{V},\ \mathbb{B}(1,0) = \mathcal{A}\ , \mathbb{B}(1,1) = 1,\ \mathbb{B}(0,1) = 0,$$

where the last 0 denotes the empty category. As shown by Mac Lane and Paré [15] — for a more elegant alternative proof attributed to Gordon and Power see also [7] — we can replace $\mathbb{B}$ by an equivalent bicategory $\mathbb{C}$, with the same objects 0 and 1, in which composition is strictly associative. (Recall that this is indeed an equivalence, and not merely a biequivalence: there are homomorphisms $\mathbb{B} \to \mathbb{C}$ and $\mathbb{C} \to \mathbb{B}$, each of whose composites is *isomorphic* to the identity via (invertible) strong transformations.) When $\mathcal{V}$ and $\mathcal{A}$ are in fact 2-categories as above, the 2-cells of $\mathcal{V}$ and of $\mathcal{A}$, which are 3-cells in $\mathbb{B}$, just go along for the ride in the equivalence. Accordingly, so long as we deal with properties stable under such an equivalence, we may simplify by supposing henceforth that both $\otimes$ and $*$ are strictly associative — which allows us to write XY for $X \otimes Y$ in $\mathcal{V}$ and XA for $X * A$ in $\mathcal{A}$, with 1 for I.

Moreover, because of the importance of the extremal case, we shall henceforth write (A, B) rather than $((A, B))$ in the general case, so that (1.1) becomes

$$\Phi : \mathcal{A}(XA, B) \cong \mathcal{V}(X, (A, B)); \tag{1.4}$$

and we shall henceforth use Φ without further explanation to denote this isomorphism.

Of course $(-, -)$ admits a unique structure of 2-functor $\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{V}$ for which Φ is 2-natural in each variable. Let us write

$$e = e_{A,B} : (A, B)A \to B \tag{1.5}$$

for the 2-natural counit of the adjunction, and recall that we have a multiplication

$$M = M^B_{A,C} : (B, C)(A, B) \to (A, C) \tag{1.6}$$

determined, using the adjunction, by the commutativity of

$$\begin{array}{ccc} (B,C)(A,B)A & \xrightarrow{MA} & (A,C)A \\ {\scriptstyle (B,C)e}\downarrow & & \downarrow{\scriptstyle e} \\ (B,C)B & \xrightarrow[e]{} & C, \end{array} \tag{1.7}$$

as well as a "unit map" $J = J_A : 1 \to (A, A)$ which is the mate under Φ of $\lambda : 1A \to A$ (here given by the identity), so that we have

$$1A \xrightarrow{JA} (A, A)A \xrightarrow{e} A$$

equal to the identity. As is well known — see for example [6] — M and J provide the composition and the unit for a $\mathcal{V}$-category $\mathbb{A}$, whose underlying ordinary category is $\mathcal{A}$ and whose $\mathcal{V}$-valued hom $\mathbb{A}(A, B)$ is (A, B).

For each $A \in \mathcal{A}$ we have on (A, A) the structure of a monoid $((A, A), i, m)$, where $m : (A, A)(A, A) \to (A, A)$ is M^A_{AA} and $i : 1 \to (A, A)$ is J_A. For a second object B of $\mathcal{A}$, let us write $((B, B), j, n)$ for the monoid structure; in the extremal case where $\mathcal{V} = [\mathcal{A}, \mathcal{A}]$, these monoids are of course *monads on* $\mathcal{A}$ (meaning 2-monads, since $\mathcal{A}$ is a 2-category).

Our central result concerns an adjunction

$$\eta, \varepsilon : f \dashv f^* : A \to B \tag{1.8}$$

in the 2-category $\mathcal{A}$. Write w for $(f, f^*) : (A, A) \to (B, B)$, noting that it is the image under Φ of the composite

$$(A, A)B \xrightarrow{(A,A)f} (A, A)A \xrightarrow{e} A \xrightarrow{f^*} B,$$

which we shall denote more briefly by $t : (A, A)B \to B$.

In the very simple case where $\mathcal{A} = \mathcal{V} = \mathbf{Cat}$ with its cartesian monoidal structure, A is a category and $(A, A) = [A, A]$ is the strict monoidal category of endofunctors of A. Now an adjunction $f \dashv f^* : A \to B$ in $\mathcal{A}$ is just an adjunction in the original sense of the word; and Bénabou [1] observed that here $w = (f, f^*)$ is part of a *monoidal functor* $(w, w^\circ, \widetilde{w})$. Indeed w sends $u : A \to A$ to f^*uf, and we have only to take $w^\circ : 1 \to f^*1f$ to be $\eta : 1 \to f^*f$, and to take $\widetilde{w}_{u,v} : f^*uff^*vf \to f^*uvf$ to be $f^*u\varepsilon vf$. Our central aim is to prove a similar result in the general case, providing for w the structure of a *lax map of monoids in* $\mathcal{V}$. Doing so is equivalent to providing for $t : (A, A)B \to B$ the structure of a *lax action on* B

of the monoid (A, A); and this observation allows us to enrich the central result as follows. The evaluation $e : (A, A)A \to A$ is itself a strict action of (A, A) on A, and we show $f^* : A \to B$ to admit the structure of a lax map of lax (A, A)-algebras, while $f : B \to A$ becomes a colax map of such algebras. Under further hypotheses on the adjunction $f \dashv f^*$, which are certainly satisfied when it is an equivalence (that is, when η and ε are invertible), the whole adjunction enriches to one in the 2-category Ps-(A, A)-Alg of pseudo (A, A)-algebras. When A has the structure of a T-algebra, the corresponding map $T \to (A, A)$ of monoids provides a 2-functor from Ps-(A, A)-Alg to Ps-T-Alg carrying the adjunction to one in Ps-T-Alg, which can be seen as a rule for transporting pseudo T-algebra structures along an equivalence. Finally, the 2-functor from Ps-T-Alg to T-Alg, which we have in the case of a *flexible* monoid T, carries the adjunction in Ps-T-Alg to one in T-Alg, giving a rule for transporting (strict) T-algebra structures along an equivalence when T is flexible.

We provide below the detailed statements of these and related results, along with their proofs. First, we recall in the next section the definitions of lax maps of monoids, of lax algebras, and of lax morphisms of lax algebras.

It is a pleasure to thank Ross Street for several helpful comments on the contents of this paper.

2 The definitions

The 2-category Colax$[\mathbf{2}, \mathcal{V}]$ has for objects the arrows $f : X \to Y$ of $\mathcal{V}$, for arrows $f \to f'$ the triples (u, ρ, v) of the form

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ {\scriptstyle f}\downarrow & \overset{\rho}{\Rightarrow} & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow[v]{} & Y', \end{array}$$

and for 2-cells $(u, \rho, v) \to (\bar{u}, \bar{\rho}, \bar{v})$ the pairs (α, β) where $\alpha : u \to \bar{u}$ and $\beta : v \to \bar{v}$ satisfy the obvious coherence condition [10, p.221]. This 2-category has an evident monoidal structure in which the tensor product of $f : X \to Y$ and $g : W \to Z$ is $fg : XW \to YZ$. For monoids $T = (T, i, m)$ and $S = (S, j, n)$ in $\mathcal{V}$ (recall that these are 2-monads in the case $\mathcal{V} = [\mathcal{A}, \mathcal{A}]$), a *lax map of monoids* or *lax monoid map* $w = (w, w^\circ, \widetilde{w}) : T \to S$ consists of a map $w : T \to S$ in $\mathcal{V}$ along with 2-cells

$$\begin{array}{ccc} & & T \\ & {}^{i}\nearrow & \\ 1 & \overset{w^\circ}{\Rightarrow} & \downarrow{\scriptstyle w} \\ & {}_{j}\searrow & \\ & & S, \end{array} \qquad \begin{array}{ccc} TT & \xrightarrow{m} & T \\ {\scriptstyle ww}\downarrow & \overset{\widetilde{w}}{\Rightarrow} & \downarrow{\scriptstyle w} \\ SS & \xrightarrow[n]{} & S, \end{array}$$

satisfying the three equations [9, (4.2-4.4)] which make of $w : T \to S$ a monoid in Colax$[\mathbf{2}, \mathcal{V}]$. If now $z = (z, z^\circ, \widetilde{z})$ is another lax monoid map, a 2-cell $\theta : w \to z$ is said to be a *monoid 2-cell* if it makes commutative the diagrams

$$\begin{array}{ccc} & & wi \\ & \nearrow^{w^\circ} & \downarrow{\scriptstyle \theta i} \\ j & \searrow_{z^\circ} & zi, \end{array} \qquad \begin{array}{ccc} n.ww & \xrightarrow{\tilde{w}} & wm \\ {\scriptstyle n.\theta\theta}\downarrow & & \downarrow{\scriptstyle \theta m} \\ n.zz & \xrightarrow[\tilde{z}]{} & zm, \end{array}$$

wherein $\theta\theta$ denotes the common value of

$$S\theta.\theta T : Sw.wT \to Sz.zT \quad \text{and} \quad \theta S.T\theta : wS.Tw \to zS.Tz.$$

Thus we have a 2-category $\mathrm{Mon}_l\mathcal{V}$ of monoids in $\mathcal{V}$, lax maps of these, and monoid 2-cells. A *pseudo map of monoids* is a lax one for which w° and $\tilde{w}$ are invertible; with the same notion of 2-cell, these form a 2-category $\mathrm{Mon}_p\mathcal{V}$. And of course a *strict map of monoids*, or simply a *monoid map*, is just a lax one for which w° and $\tilde{w}$ are identities; with the same notion of 2-cell once again, these form a 2-category $\mathrm{Mon}\mathcal{V}$, which may also be called $\mathrm{Mon}_s\mathcal{V}$ if we wish to emphasize the strictness of the maps.

When, in the definition above of lax monoid map, (S, j, n) is the monoid $((B, B), j, n)$ described in Section 1, to give the arrow $w : T \to (B, B)$ is equally, by (1.4), to give an arrow $t : TB \to B$ in $\mathcal{A}$; similarly to give the 2-cells w° and $\tilde{w}$ is equally to give 2-cells

$$\begin{array}{ccc} & & TB \\ & \nearrow^{iB} & \\ B & \overset{\hat{t}}{\Rightarrow} & \downarrow{\scriptstyle t} \\ & \searrow_{1} & \\ & & B, \end{array} \qquad \begin{array}{ccc} T^2B & \xrightarrow{mB} & TB \\ {\scriptstyle Tt}\downarrow & \overset{\bar{t}}{\Rightarrow} & \downarrow{\scriptstyle t} \\ TB & \xrightarrow[t]{} & B, \end{array}$$

and the three axioms on $(w, w^\circ, \tilde{w})$ are easily converted to the three axioms [9, (4.6-4.8)] for $(t, \hat{t}, \bar{t})$ to be a *lax action* of the monoid T on the object B of $\mathcal{A}$. It is a *pseudo action* when $\hat{t}$ and $\bar{t}$ are invertible, and a *strict action* — or merely an *action* — when $\hat{t}$ and $\bar{t}$ are identities. Reversing the sense of $\hat{t}$ and $\bar{t}$ produces the notion of a *colax action* of T on B. When $(t, \hat{t}, \bar{t})$ is a lax action of T on B, we call the quadruple $(B, t, \hat{t}, \bar{t})$ a *lax T-algebra*; similarly for the notions of *pseudo T-algebra*, of *strict T-algebra* (or merely *T-algebra*), and of *colax T-algebra*.

If $(B, b, \hat{b}, \bar{b})$ and $(A, a, \hat{a}, \bar{a})$ are lax T-algebras, a *lax morphism* (or *lax map*) from $(B, b, \hat{b}, \bar{b})$ to $(A, a, \hat{a}, \bar{a})$ is a pair $(f, \bar{f})$ where $f : B \to A$ is a morphism in $\mathcal{A}$ while $\bar{f}$ is a 2-cell $a.Tf \to fb$ satisfying the two axioms [9, (4.10) and (4.11)]. (Note that we have explained lax monoid maps as monoids in a suitable monoidal 2-category, and explained lax actions as lax monoid maps $T \to \langle B, B \rangle$; the corresponding rationale for the definition of lax morphisms of lax algebras will be given a little later.)

The lax morphism is said to be a *pseudo morphism*, or just a *morphism*, of the lax T-algebras when $\bar{f}$ is invertible, and to be a *strict morphism* when $\bar{f}$ is the identity; while reversing the sense of $\bar{f}$ gives the notion of a *colax morphism*. In the case where $\hat{b}$, $\bar{b}$, $\hat{a}$, and $\bar{a}$ are identities, we recover the usual notions of lax, pseudo, strict, or colax morphisms of (strict) T-algebras; even for these, following the lead of [4], we use "morphism" without a modifier to mean "pseudo morphism". In the same way, when $\widehat{b}$, $\bar{b}$, $\widehat{a}$, and $\bar{a}$ are invertible, we call $(f, \bar{f})$ a lax morphism of pseudo T-algebras, and so on.

The notion of *algebra 2-cell* $\varphi : f \to g : (B, b, \widehat{b}, \bar{b}) \to (A, a, \widehat{a}, \bar{a})$ is the same for lax algebras, pseudo ones, or strict ones: namely a 2-cell $\varphi : f \to g$ in $\mathcal{A}$ satisfying the single obvious equation. So we have 2-categories and inclusions

$$\text{Lax-}T\text{-Alg}_s \to \text{Lax-}T\text{-Alg}_p = \text{Lax-}T\text{-Alg} \to \text{Lax-}T\text{-Alg}_l$$

of lax T-algebras with, respectively, strict morphisms, pseudo morphisms (often just called morphisms), and lax morphisms; as well as the 2-category Lax-T-Alg_c whose morphisms are the colax ones. Similarly, there are strings of inclusions with Lax-T-Alg_l replaced by Ps-T-Alg_l (when we restrict to the pseudo algebras) or by T-Alg_l (when we restrict to the strict ones).

We promised to give a "rationale" for the definition of lax morphism of lax T-algebras; in fact our needs below in proving the central result make it more convenient to work with colax morphisms. We therefore define a 2-category Lax[**2**, $\mathcal{A}$], analogous to Colax[**2**, $\mathcal{A}$], in which an object is once again a morphism $f : B \to A$ in $\mathcal{A}$, while an arrow $(b, r, a) : f \to g$ consists of morphisms b and a together with a 2-cell r as in

$$\begin{array}{ccc} B & \xrightarrow{\;b\;} & D \\ {\scriptstyle f}\downarrow & \Downarrow r & \downarrow{\scriptstyle g} \\ A & \xrightarrow[\;a\;]{} & C, \end{array}$$

and we have the obvious definition of 2-cell. There is an evident action of the monoidal 2-category $\mathcal{V}$ on Lax[**2**, $\mathcal{A}$], sending $(T, f : B \to A)$ to $Tf : TB \to TA$, and defined in the obvious way on morphisms and 2-cells. To give a map $(b, r, a) : Tf \to g$ in Lax[**2**, $\mathcal{A}$] is equally to give β, ρ, and α as in

$$\begin{array}{ccccc} & & \langle B, D\rangle & & \\ & {\scriptstyle\beta}\nearrow & & \searrow{\scriptstyle\langle B,g\rangle} & \\ T & & \Downarrow\rho & & \langle B, C\rangle, \\ & {\scriptstyle\alpha}\searrow & & \nearrow{\scriptstyle\langle f,C\rangle} & \\ & & \langle A, C\rangle & & \end{array}$$

where $\beta = \Phi b$, $\alpha = \Phi a$, and $\rho = \Phi r$; so that b, r, and a are recovered, using the evaluation e, as $b = e_{B,D}.\beta B$, $r = e_{B,C}.\rho B$, and $a = e_{A,C}.\alpha A$. If we now write

$$\begin{array}{ccccc} & & \langle B, D\rangle & & \\ & {\scriptstyle u}\nearrow & & \searrow{\scriptstyle\langle B,g\rangle} & \\ \{f, g\} & & \Downarrow\lambda & & \langle B, C\rangle \\ & {\scriptstyle v}\searrow & & \nearrow{\scriptstyle\langle f,C\rangle} & \\ & & \langle A, C\rangle & & \end{array}$$

for the comma object, to give (β, ρ, α) is equally to give a map $\gamma : T \to \{f, g\}$ in $\mathcal{V}$: namely the unique map for which $u\gamma = \beta$, $\lambda\gamma = \rho$, and $v\gamma = \alpha$. These bijections sending γ to $(u, \lambda, v)\gamma = (\beta, \rho, \alpha)$ and sending (β, ρ, α) to $(e.\beta B, e.\rho B, e.\alpha A) = (b, r, a)$ clearly extend to 2-cells and become isomorphisms of categories, their composite being a natural isomorphism

$$\mathcal{V}(T, \{f, g\}) \cong \text{Lax}[\mathbf{2}, \mathcal{A}](Tf, g) \tag{2.1}$$

exhibiting $\{f,-\}$ as the right adjoint of the 2-functor $\mathcal{V} \to \mathrm{Lax}[\mathbf{2},\mathcal{A}]$ sending T to Tf. The counit $E_{f,g} : \{f,g\}f \to g$ of the adjunction, which we may again call the *evaluation*, has the form

$$\begin{array}{ccc} \{f,g\}B & \xrightarrow{E^0_{f,g}} & D \\ {\scriptstyle \{f,g\}f}\downarrow & \Downarrow E^{\to}_{f,g} & \downarrow{\scriptstyle g} \\ \{f,g\}A & \xrightarrow[E^1_{f,g}]{} & C, \end{array}$$

and is obtained by setting $T = \{f,g\}$ and $\gamma = 1$, so that $E^0_{f,g} = e.uB$, $E^{\to}_{f,g} = e.\lambda B$, and $E^1_{f,g} = e.vA$. When $g = f$, the comma object $\{f,g\}$ becomes

$$\begin{array}{ccccc} & & \langle B,B\rangle & & \\ & {\scriptstyle u}\nearrow & & \searrow{\scriptstyle \langle B,f\rangle} & \\ \{f,f\} & & \Downarrow\lambda & & \langle B,A\rangle. \\ & {\scriptstyle v}\searrow & & \nearrow{\scriptstyle \langle f,A\rangle} & \\ & & \langle A,A\rangle & & \end{array}$$

By the general results on actions in Section 1, for any $f : B \to A$ the object $\{f,f\}$ admits a canonical structure $(\{f,f\},k,l)$ of monoid in $\mathcal{V}$, where $l : \{f,f\}\{f,f\} \to \{f,f\}$ is determined by the commutativity in $\mathrm{Lax}[\mathbf{2},\mathcal{A}]$ of

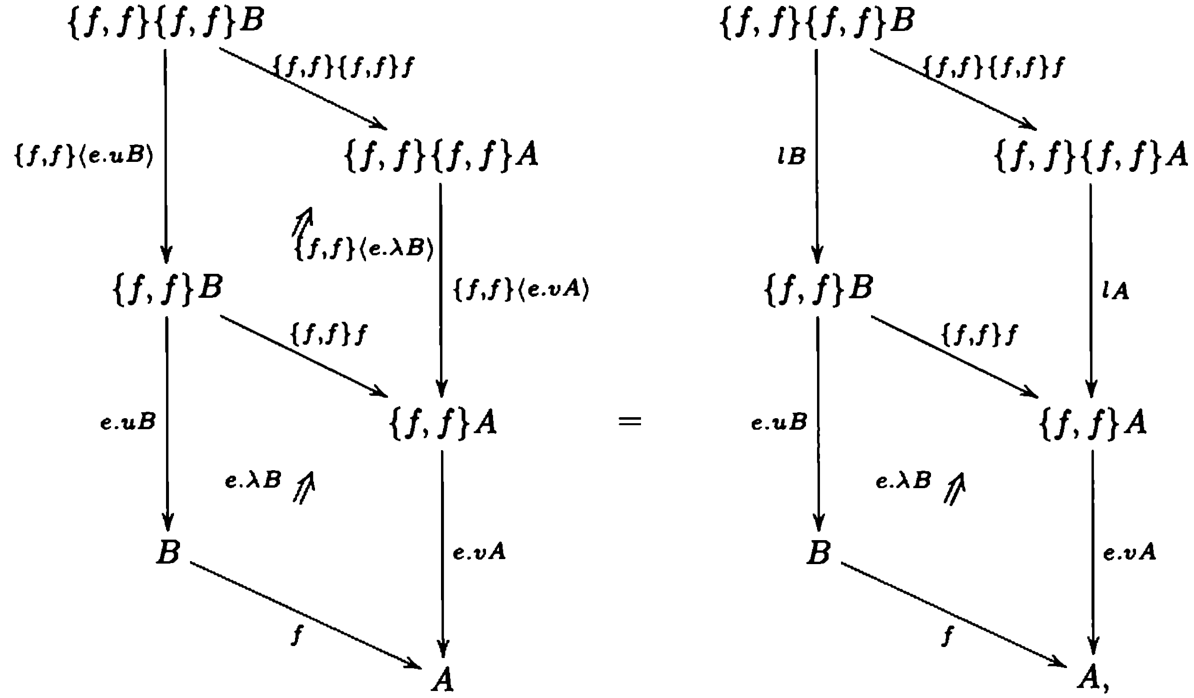
$$\begin{array}{ccc} \{f,f\}\{f,f\}f & \xrightarrow{lf} & \{f,f\}f \\ {\scriptstyle \{f,f\}E_{f,f}}\downarrow & & \downarrow{\scriptstyle E_{f,f}} \\ \{f,f\}f & \xrightarrow[E_{f,f}]{} & f, \end{array} \tag{2.2}$$

and similarly $k : 1 \to \{f,f\}$ is determined by the equation $E_{f,f}.kf = 1_f$ in $\mathrm{Lax}[\mathbf{2},\mathcal{A}]$. Moreover, for a monoid $T = (T,i,m)$ in $\mathcal{V}$, to give a lax monoid map $\gamma : T \to \{f,f\}$ is equivalently, by the earlier part of this section, to give (as its image under (2.1)) a lax action $(c,\widehat{c},\overline{c}) : Tf \to f$. Here c consists of maps $b : TB \to B$ and $a : TA \to A$ in $\mathcal{A}$ and a 2-cell $\overline{f} : fb \to a.Tf$, while $\widehat{c}$ is a pair $(\widehat{b},\widehat{a})$ of 2-cells $\widehat{b} : 1 \to b.iB$ and $\widehat{a} : 1 \to a.iA$, and $\overline{c}$ is a pair $(\overline{b},\overline{a})$ of 2-cells $\overline{b} : b.Tb \to b.mB$ and $\overline{a} : a.Ta \to a.mA$; all these data satisfying equations which assert precisely that $(b,\widehat{b},\overline{b})$ and $(a,\widehat{a},\overline{a})$ are lax actions of T on B and A and that $(f,\overline{f})$ is a colax morphism $(B,b,\widehat{b},\overline{b}) \to (A,a,\widehat{a},\overline{a})$ of lax T-algebras.

In particular, a strict monoid map $\gamma : T \to \{f,f\}$ corresponds to strict actions $b : TB \to B$ and $a : TA \to A$, along with an $\overline{f}$ making $(f,\overline{f}) : (B,b) \to (A,a)$ a colax morphism of T-algebras. The strict monoid maps $\beta : T \to \langle B,B\rangle$ and $\alpha : T \to \langle A,A\rangle$ corresponding to the strict actions b and a are the composites of γ with $u : \{f,f\} \to \langle B,B\rangle$ and $v : \{f,f\} \to \langle A,A\rangle$; since γ here may be the identity map of $\{f,f\}$, we conclude that *u and v are themselves strict monoid maps.*

We can be more explicit about the value of $k : 1 \to \{f,f\}$: it corresponds of course under (2.1) to the identity $1f \to f$, and hence is the unique k for which uk and vk are the units $j : 1 \to \langle B,B\rangle$ and $i : 1 \to \langle A,A\rangle$ of the monoids $\langle B,B\rangle$ and $\langle A,A\rangle$ while $\lambda k = id$.

The explicit description of the multiplication $l : \{f,f\}\{f,f\} \to \{f,f\}$ is slightly more complicated. First we unravel (2.2) to obtain



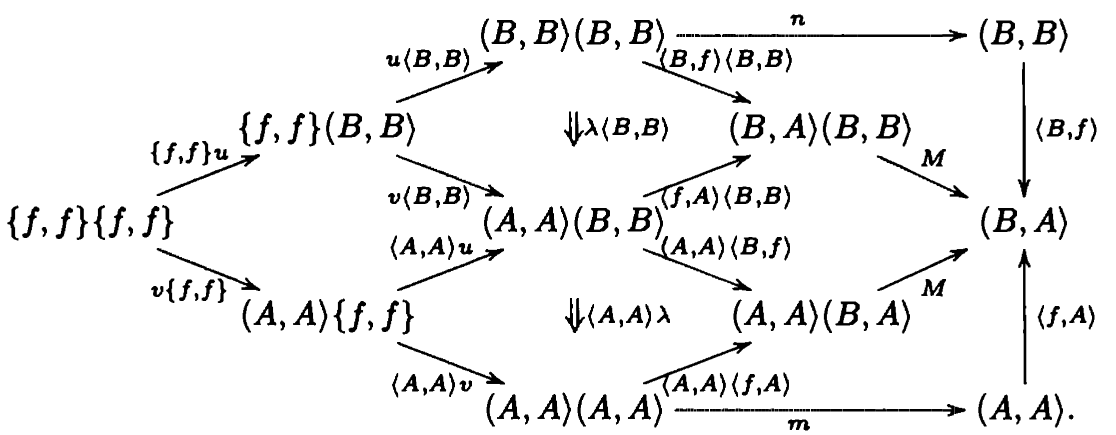
(2.3)

and then apply the isomorphism Φ to this equality. The resulting equality, at the level of 1-cells, asserts that ul and vl are the composites

$$\{f,f\}\{f,f\} \xrightarrow{uu} (B,B)(B,B) \xrightarrow{n} (B,B),$$

$$\{f,f\}\{f,f\} \xrightarrow{vv} (A,A)(A,A) \xrightarrow{m} (A,A),$$

(repeating our observation above that u and v are strict monoid maps); at the level of 2-cells, it reduces, as we indicate below, to the assertion that λl is given by



(2.4)

Since the image under Φ of the right side of (2.3) is the composite λl, we must exhibit (2.4) as the image under Φ of the left side of (2.3); and this left side is the "vertical" composite of $e.\lambda B.\{f,f\}e.\{f,f\}uB$ with $e.vA.\{f,f\}e.\{f,f\}\lambda B$. Because the action $* : \mathcal{V} \times \mathcal{A} \to \mathcal{A}$, denoted by juxtaposition, is a 2-functor, we have $\lambda B.\{f,f\}e = (B,A)e.\lambda(B,B)B$, so that

$$\begin{aligned} e.\lambda B.\{f,f\}e.\{f,f\}uB &= e.(B,A)e.\lambda(B,B)B.\{f,f\}uB \\ &= e.MB.\lambda(B,B)B.\{f,f\}uB, \end{aligned}$$

where the second step uses (1.7) to replace $e.(B,A)e$ by $e.MB$; and the image $M.\lambda(B,B).\{f,f\}u$ of $e.MB.\lambda(B,B)B.\{f,f\}uB$ under Φ is the top half of (2.4). Similar arguments justify the steps in

$$\begin{aligned} e.vA.\{f,f\}e.\{f,f\}\lambda B &= e.(A,A)e.v(B,A)B.\{f,f\}\lambda B \\ &= e.MB.(A,A)\lambda B.v\{f,f\}B; \end{aligned}$$

and the image $M.(A,A)\lambda.v\{f,f\}$ of this last under Φ is the bottom half of (2.4).

3 The central result

We begin with the following, due to Street [16, Proposition 5]:

Lemma 3.1 *Let*

$$\begin{array}{ccccc} & & B & & \\ & {}^{u}\nearrow & & \searrow^{f} & \\ D & & \Downarrow\lambda & & A \\ & {}_{v}\searrow & & \nearrow_{g} & \\ & & C & & \end{array}$$

be a comma object in the 2-category $\mathcal{A}$. If f has a right adjoint given by $\eta,\varepsilon : f \dashv f^* : A \to B$, then v has a right adjoint given by $\zeta, id : v \dashv v^* : C \to D$, where v^* is the unique map satisfying $uv^* = f^*g$, $\lambda v^* = \varepsilon g$, and $vv^* = 1$; while $\zeta : 1 \to v^*v$ is the unique 2-cell for which $v\zeta : v \to vv^*v$ is the identity on $v(= vv^*v)$ and $u\zeta : u \to uv^*v$ is the composite

$$u \xrightarrow{\eta u} f^*fu \xrightarrow{f^*\lambda} f^*gv = uv^*v.$$

Remark 3.2 There is of course a similar result where we replace the comma object by an iso-comma object, and the adjunction $f \dashv f^*$ by an equivalence; but we shall not need to refer to this below.

Returning now to the general situation of a monoidal 2-category $\mathcal{V}$ acting on a 2-category $\mathcal{A}$ with a right adjoint expressed by the 2-natural isomorphism Φ of (1.4), consider an arbitrary adjunction $\eta,\varepsilon : f \dashv f^* : A \to B$ in $\mathcal{A}$, and note that the 2-functor $\langle B, -\rangle : \mathcal{A} \to \mathcal{V}$ takes this adjunction into an adjunction

$$\langle B,\eta\rangle, \langle B,\varepsilon\rangle : \langle B,f\rangle \dashv \langle B,f^*\rangle : \langle B,A\rangle \to \langle B,B\rangle$$

in $\mathcal{V}$. Supposing henceforth $\mathcal{V}$ to admit comma objects, we can apply Lemma 3.1 to this adjunction and to the comma object

$$\begin{array}{ccccc} & & \langle B,B\rangle & & \\ & {}^{u}\nearrow & & \searrow^{\langle B,f\rangle} & \\ \{f,f\} & & \Downarrow\lambda & & \langle B,A\rangle \\ & {}_{v}\searrow & & \nearrow_{\langle f,A\rangle} & \\ & & \langle A,A\rangle & & \end{array}$$

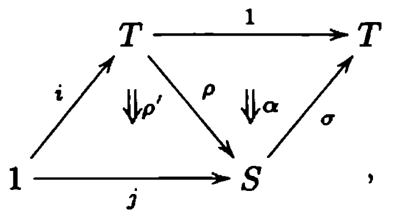
to get:

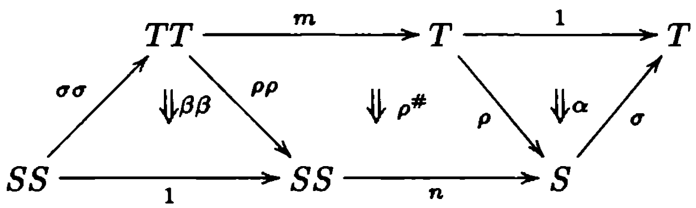
Proposition 3.3 *In the presence of the adjunction* $\eta, \varepsilon : f \dashv f^* : A \to B$, *the map* $v : \{f,f\} \to \langle A, A\rangle$ *has a right adjoint given by* $\zeta, id : v \dashv z : \langle A, A\rangle \to \{f,f\}$, *where* z *is the unique map satisfying* $uz = \langle f, f^*\rangle$, $\lambda z = \langle f, \varepsilon\rangle : \langle f, ff^*\rangle \to \langle f, A\rangle$, *and* $vz = 1$; *while* $\zeta : 1 \to zv$ *is the unique 2-cell for which* $v\zeta = id$ *and* $u\zeta$ *is the composite*

$$u \xrightarrow{\langle B,\eta\rangle u} \langle B, f^*f\rangle u \xrightarrow{\langle B,f^*\rangle \lambda} \langle f, f^*\rangle v \,.$$

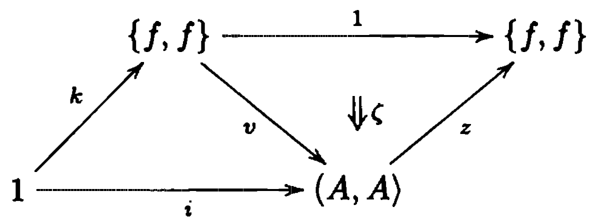
The result of the following lemma is very like that of [8, Theorem 1.2], of which it is not, however, a consequence; the situation is rather that the proof-techniques of that paper adapt so readily to the present lemma that we can safely leave the details to the reader.

Lemma 3.4 *Let* $\alpha, \beta : \rho \dashv \sigma : S \to T$ *be an adjunction in the monoidal 2-category* $\mathcal{V}$, *where* T *and* S *have monoid structures* (T, i, m) *and* (S, j, n). *Then there is a bijection between enrichments of* ρ *to a colax monoid map* $(\rho, \rho', \rho^{\#})$ *and enrichments of* σ *to a lax monoid map* $(\sigma, \sigma^\circ, \widetilde{\sigma})$, *where* σ° *and* $\widetilde{\sigma}$ *are given respectively by the pasting composites*

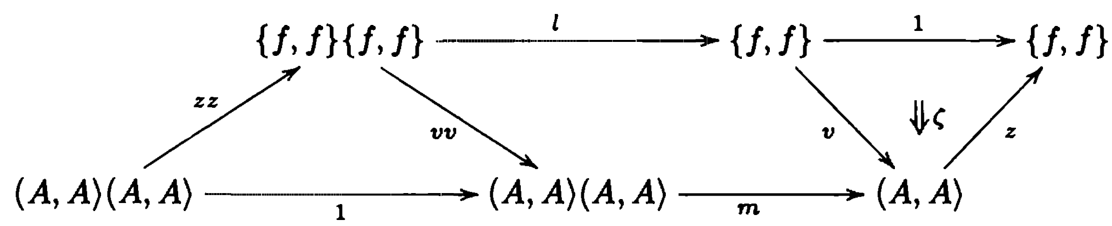


,


We now apply the lemma to the adjunction $\zeta, id : v \dashv z : \langle A, A\rangle \to \{f,f\}$. We saw in Section 2 that v is a strict monoid map, which we can see as a colax monoid map (v, id, id); it follows from the lemma that z admits an enrichment to a lax monoid map $(z, z^\circ, \widetilde{z})$ where z° is given by



or simply ζk, while $\widetilde{z}$ is given by



or simply $\zeta l.zz$. Note that, since $v\zeta = id$ by Proposition 3.3, we have $vz^\circ = id$ and $v\widetilde{z} = id$; thus the composite $(vz, vz^\circ, v\widetilde{z})$ of the strict monoid map v and the lax monoid map $z = (z, z^\circ, \widetilde{z})$ is the identity monoid map $1 = (1, id, id) : \langle A, A\rangle \to \langle A, A\rangle$.

Let us set $w = \langle f, f^*\rangle : \langle A, A\rangle \to \langle B, B\rangle$, as indicated in Section 1. Then $w = uz$ by the definition of z; and since u is a strict monoid map while $z = (z, z^\circ, \widetilde{z})$ is a lax monoid map, we have a lax monoid map

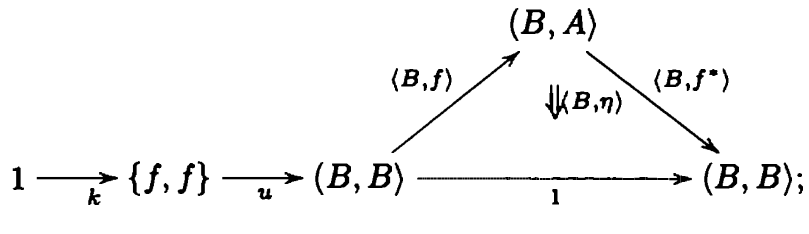
$$(w, w^\circ, \widetilde{w}) = (uz, uz^\circ, u\widetilde{z}) : \langle A, A\rangle \to \langle B, B\rangle.$$

To complete our central result, therefore, it remains to describe more explicitly w° and $\widetilde{w}$; or equivalently to describe the $\widehat{t}$ and the $\bar{t}$ which enrich the $t : \langle A, A\rangle B \to B$ of Section 1, given by

$$\langle A, A\rangle B \xrightarrow{\langle A,A\rangle f} \langle A, A\rangle A \xrightarrow{e} A \xrightarrow{f^*} B,$$

to the lax action $(t, \widehat{t}, \bar{t})$ of $\langle A, A\rangle$ on B corresponding to the lax monoid map $(w, w^\circ, \widetilde{w})$.

Now $w^\circ = uz^\circ = u\zeta k$, which, by Proposition 3.3 and the observation in Section 2 that $\lambda k = id$, is just the 2-cell $\langle B, \eta\rangle uk$ in



and since $uk = j$, this is just $\langle B, \eta\rangle j$. Moreover, applying Φ^{-1} to each side of the equality $w^\circ = \langle B, \eta\rangle j$ shows that $\widehat{t} : 1 \to t.iB$ is given by $\eta : 1 \to f^* f$; observe here, using the naturality of i, that $f^* f = f^* e.iA.f = f^* e.\langle A, A\rangle f.iB = t.iB$.

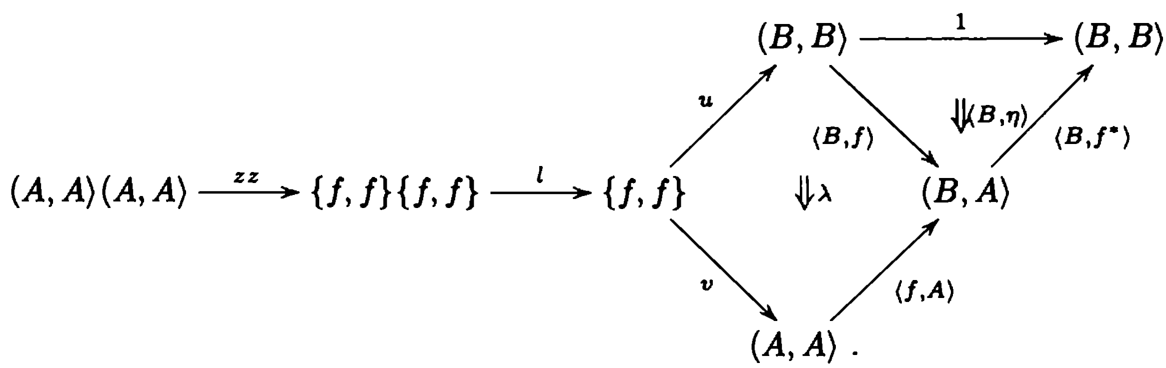
It remains to describe the 2-cell

$$\begin{array}{ccc} \langle A, A\rangle\langle A, A\rangle & \xrightarrow{m} & \langle A, A\rangle \\ {\scriptstyle ww}\downarrow & \overset{\widetilde{w}}{\Rightarrow} & \downarrow{\scriptstyle w} \\ \langle B, B\rangle\langle B, B\rangle & \xrightarrow[n]{} & \langle B, B\rangle, \end{array}$$

or equivalently the component

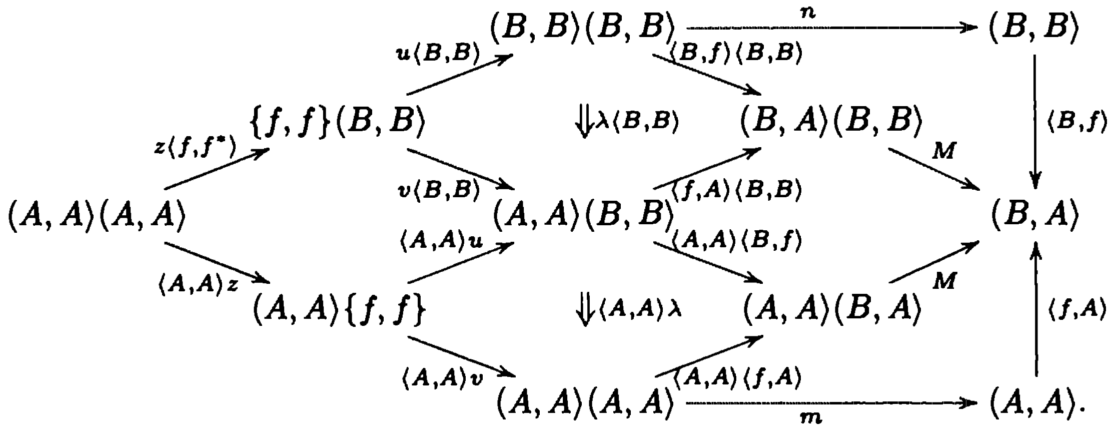
$$\begin{array}{ccc} \langle A, A\rangle\langle A, A\rangle B & \xrightarrow{mB} & \langle A, A\rangle B \\ {\scriptstyle \langle A,A\rangle t}\downarrow & \overset{\bar{t}}{\Rightarrow} & \downarrow{\scriptstyle t} \\ \langle A, A\rangle B & \xrightarrow[t]{} & B \end{array}$$

of the lax action $(t, \widehat{t}, \bar{t})$ of $\langle A, A\rangle$ on B. Now $\widetilde{w} = u\widetilde{z} = u\zeta l.zz$, so from the definition of $u\zeta$ in Proposition 3.3 it follows that $\widetilde{w}$ is given by the pasting composite



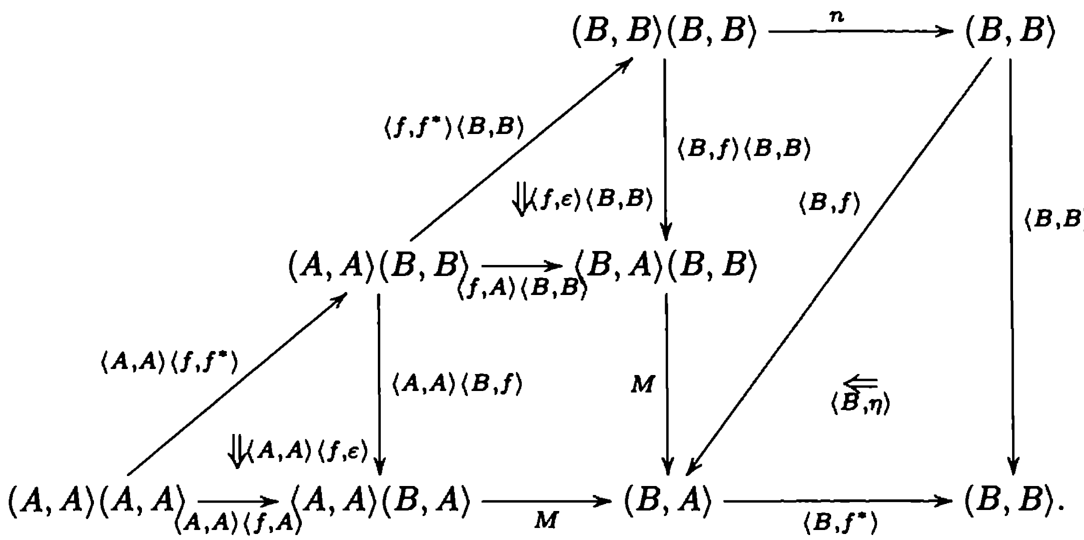
(3.1)

Using the description (2.4) of λl and the equations $vz = 1$ and $uz = (f, f^*)$, we see that $\lambda l.zz$ may be written as

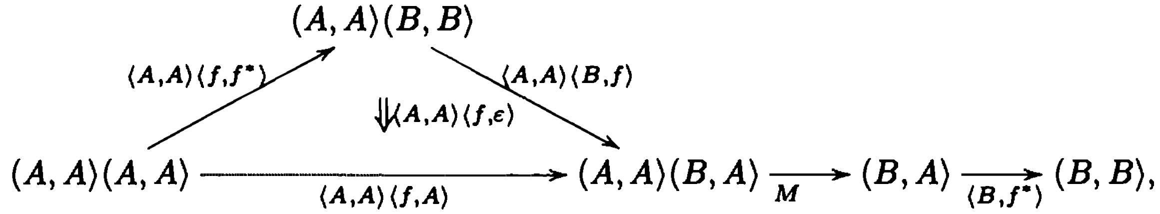


(3.2)

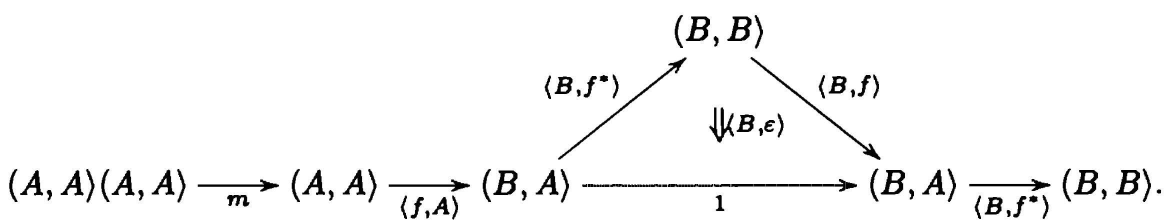
Using the equation $\lambda z = (f, \varepsilon)$ to simplify (3.2), and substituting the result into (3.1), we conclude that $\widetilde{w}$ is the composite:



By the "extraordinary" naturality of the M's and one of the triangular equations, this reduces to



and so, using extraordinary naturality once again, to



(3.3)

Finally $\bar{t}$ is obtained from $\widetilde{w}$ by applying $(\ )B$ and composing with the evaluation $e : (B,B)B \to B$; by the ordinary and the extraordinary 2-naturality of e this gives

$$\begin{array}{ccccccccccc} & & & & & & & & B & & \\ & & & & & & & {\scriptstyle f^*}\nearrow & \Downarrow\varepsilon & \searrow{\scriptstyle f} & \\ (A,A)(A,A)B & \xrightarrow[mB]{} & (A,A)B & \xrightarrow[(A,A)f]{} & (A,A)A & \xrightarrow[e]{} & A & & \xrightarrow[1]{} & & A \xrightarrow[f^*]{} B, \end{array}$$

which is perhaps more readily seen as a 2-cell $t.(A,A)t \to t.mB$ by using the 2-naturality of e once more, to display it in the form

$$\begin{array}{ccccccc} (A,A)(A,A)B & & & \xrightarrow{mB} & & & (A,A)B \\ \downarrow{\scriptstyle (A,A)(A,A)f} & & & & & & \downarrow{\scriptstyle (A,A)f} \\ (A,A)(A,A)A & & & \xrightarrow{mA} & & & (A,A)A \\ \downarrow{\scriptstyle (A,A)e} & & & & & & \downarrow{\scriptstyle e} \\ (A,A)A & & & \xrightarrow{e} & & & A \\ \downarrow{\scriptstyle (A,A)f^*} & \searrow{\scriptstyle 1} & \Uparrow{\scriptstyle (A,A)\varepsilon} & & & & \downarrow{\scriptstyle f^*} \\ (A,A)B & \xrightarrow[(A,A)f]{} & (A,A)A & \xrightarrow[e]{} & A & \xrightarrow[f^*]{} & B. \end{array} \qquad (3.4)$$

Summing up, we have as our central result:

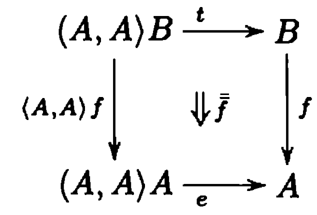
Theorem 3.5 *Let the monoidal 2-category $\mathcal{V}$ admit comma objects, and let it so act on the 2-category $\mathcal{A}$ that we have the adjunction $\Phi : \mathcal{A}(XA,B) \cong \mathcal{V}(X,(A,B))$. Then each adjunction $\eta, \varepsilon : f \dashv f^* : A \to B$ in $\mathcal{A}$ gives rise to a lax map of monoids $(w, w^\circ, \widetilde{w}) : (A,A) \to (B,B)$ in $\mathcal{V}$, where $w = (f, f^*)$ and w° is given by $(B,\eta)j$, while $\widetilde{w}$ is given by* (3.3)*. In fact to give a lax map $(w,w^\circ,\widetilde{w}) : (A,A) \to (B,B)$ of monoids is equally to give a lax action $(t,\widehat{t},\bar{t})$ of the monoid (A,A) on B; and here $t : (A,A)B \to B$ is the composite*

$$(A,A)B \xrightarrow{(A,A)f} (A,A)A \xrightarrow{e} A \xrightarrow{f^*} B,$$

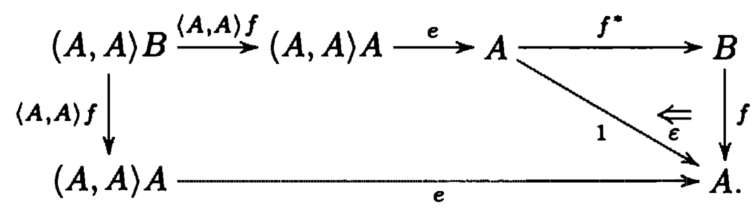
while $\widehat{t} = \Phi^{-1}(w^\circ)$ is given by η and $\bar{t} = \Phi^{-1}(\widetilde{w})$ is given by (3.4)*. When η and ε are invertible, so that the adjunction $f \dashv f^*$ is an equivalence, the 2-cells w°, $\widetilde{w}$, $\widehat{t}$, and $\bar{t}$ are invertible, so that $(w,w^\circ,\widetilde{w})$ is a pseudo map of monoids, while $(t,\widehat{t},\bar{t})$ is a pseudo action of (A,A) on B.*

4 The enrichments of f and f^*

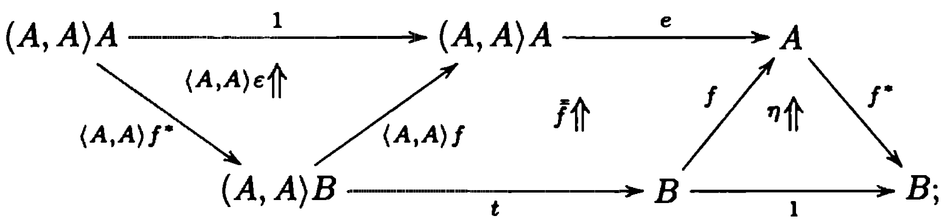
We continue to suppose satisfied the hypotheses of Theorem 3.5. As we saw in Section 3, the lax monoid map $(z,z^\circ,\widetilde{z}) : (A,A) \to \{f,f\}$ satisfies $u(z,z^\circ,\widetilde{z}) = (w,w^\circ,\widetilde{w})$ and $v(z,z^\circ,\widetilde{z}) = 1$ in $\mathrm{Mon}_l\,\mathcal{V}$. Since the lax monoid map $(w,w^\circ,\widetilde{w}) : (A,A) \to (B,B)$ corresponds to the lax action $(t,\widehat{t},\bar{t}) : (A,A)B \to B$ and the strict monoid map $1 : (A,A) \to (A,A)$ corresponds to the strict action $e : (A,A)A \to A$, it follows from our observations in Section 2 that we have a colax map $(f, \bar{\bar{f}}) : (B, t, \hat{t}, \bar{t}) \to (A, e)$ of lax $\langle A, A\rangle$-algebras, where the diagram



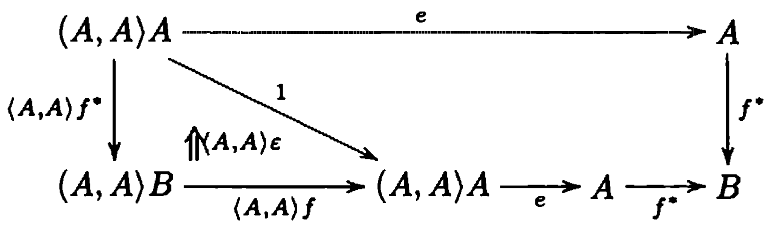
is the image under Φ^{-1} of λz. Since $\lambda z = \langle B, \varepsilon\rangle\langle f, A\rangle$ by Proposition 3.3, an easy calculation exhibits $\bar{\bar{f}}$ as the 2-cell



It now follows from [8, Theorem 1.2] that we have a lax map $(f^*, \overline{f^*}) : (A, e) \to (B, t, \hat{t}, \bar{t})$ of lax $\langle A, A\rangle$-algebras, where the 2-cell $\overline{f^*}$ is the composite



which, on substituting for $\bar{\bar{f}}$ its explicit value above and using one of the triangular equations, gives



as the value of $\overline{f^*}$.

When $\bar{\bar{f}} = \varepsilon e.\langle A, A\rangle f$ is invertible — and so in particular when ε itself is invertible — we have a lax map $(f, \bar{f})$ of lax $\langle A, A\rangle$-algebras, where $\bar{f} = \bar{\bar{f}}^{-1}$, and by [8, Proposition 1.3] we have an adjunction $(f, \bar{f}) \dashv (f^*, \overline{f^*})$ in Lax-$\langle A, A\rangle$-Alg_l. We state this formally only in the most important case where ε is itself invertible; then both $\bar{f}$ and $\overline{f^*}$ are invertible so that $(f, \bar{f})$ and $(f^*, \overline{f^*})$ become pseudomorphisms:

Theorem 4.1 *Let the counit $\varepsilon : ff^* \to 1$ of the adjunction $\eta, \varepsilon : f \dashv f^* : \mathcal{A} \to \mathcal{B}$ be invertible. Then we have an adjunction $\eta, \varepsilon : (f, \bar{f}) \dashv (f^*, \overline{f^*}) : (A, e) \to (B, t, \hat{t}, \bar{t})$ in* Lax-$\langle A, A\rangle$-Alg, *where $\bar{f}$ is given by $\varepsilon^{-1}e.\langle A, A\rangle f$ and $\overline{f^*}$ is given by $f^*e.\langle A, A\rangle\varepsilon$. When η too is invertible, so that the original adjunction is an equivalence in $\mathcal{A}$, both $\hat{t}$ and $\bar{t}$ are invertible, so that the adjunction $(f, \bar{f}) \dashv (f^*, \overline{f^*})$ becomes an equivalence in* Ps-$\langle A, A\rangle$-Alg.

A somewhat different case that has been useful historically, as the motivation for introducing the concept of *flexibility* for 2-monads, is that where we suppose the invertibility only of $\overline{f^*} = f^*e.\langle A,A\rangle\varepsilon$, which gives us an adjunction

$$\eta,\varepsilon : (f,\bar{\bar{f}}) \dashv (f^*,(\overline{f^*})^{-1}) : (A,e) \to (B,t,\widehat{t},\bar{t})$$

(with $\bar{t}$ too invertible) in the 2-category Lax-$\langle A,A\rangle$-Alg_c of lax $\langle A,A\rangle$-algebras and *colax* maps. The historical example supposed η too to be invertible — indeed, to be an identity — so that also $\widehat{t}$ was invertible, and we were dealing with pseudo $\langle A,A\rangle$-algebras. To regain lax maps instead of colax ones, we need only to pass to the dual case by supposing the original adjunction $\eta,\varepsilon : f \dashv f^* : A \to B$ to lie in $\mathcal{A}^{co}$ rather than $\mathcal{A}$. Leaving the reader to work through the simple dualizing process, we merely state the result (which essentially repeats [8, Theorem 3.2], itself a generalization of [5].)

Theorem 4.2 *With $\mathcal{V}$ and $\mathcal{A}$ as before, let $\eta,\varepsilon : f \dashv f^* : A \to B$ have ε invertible, and let $fe.\langle A,A\rangle\eta$ be invertible. Then we can enrich the adjunction $f \dashv f^*$ to an adjunction $\eta,\varepsilon : (f,\bar{f}) \dashv (f^*,\overline{f^*}) : (A,s,\widehat{s},\bar{s}) \to (B,e)$ in* Ps-$\langle B,B\rangle$-Alg_l.

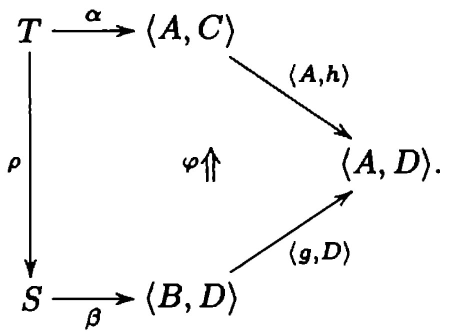
5 The monoid $\langle f,f^*\rangle : \langle A,A\rangle \to \langle B,B\rangle$ as an endo-object

Before turning to the applications of the above to transport of structure, we revisit our central results in Theorem 3.5 and in the prologue to Theorem 4.1, to cast a new light on them. The first of these asserts that, under the conditions of the theorem, the map $w = \langle f,f^*\rangle : \langle A,A\rangle \to \langle B,B\rangle$ underlies a lax monoid map $(w,w^\circ,\widetilde{w})$. By Section 2, such a lax monoid map is the same thing as a monoid in the monoidal 2-category Colax$[\mathbf{2},\mathcal{V}]$. Now probably the most direct way of showing an object of a monoidal 2-category to admit a monoid structure is to exhibit it as an "object $\langle\langle A,A\rangle\rangle$ of endomorphisms" in the context of an action admitting the adjunction Φ of (1.1); in this way we saw $\langle A,A\rangle$ to be a monoid in Section 1, and $\{f,f\}$ in Section 2: the latter involving the action of $\mathcal{V}$ on Lax$[\mathbf{2},\mathcal{V}]$ and the adjunction (2.1). Of course $\mathcal{V}$ also acts on Colax$[\mathbf{2},\mathcal{A}]$ in the dual fashion, with a right adjoint say $\{f,g\}'$, so that a strict monoid map $T \to \{f,f\}'$ corresponds to a lax map $(f,\bar{f})$ of T-algebras. These three are all examples of monoids in $\mathcal{V}$; and the question suggests itself whether the monoid $(w,w^\circ,\widetilde{w})$ in Colax$[\mathbf{2},\mathcal{V}]$, enriching $w = \langle f,f^*\rangle : \langle A,A\rangle \to \langle B,B\rangle$, is an object of endomorphisms for some suitable action.

To this end, we consider an action of Colax$[\mathbf{2},\mathcal{V}]$ on Colax$[\mathbf{2},\mathcal{A}]$ which extends the action above of $\mathcal{V}$ on Colax$[\mathbf{2},\mathcal{A}]$. The 2-functor $* :$ Colax$[\mathbf{2},\mathcal{V}] \times$ Colax$[\mathbf{2},\mathcal{A}] \to$ Colax$[\mathbf{2},\mathcal{A}]$ on objects sends $(\rho : T \to S, g : A \to B)$ to $\rho g : TA \to SB$; on morphisms it sends $((\alpha,\lambda,\beta),(a,\theta,b))$ to $(\alpha a,\lambda\theta,\beta b)$; and on 2-cells it sends $((\gamma,\delta),(\xi,\eta))$ to $(\gamma\xi,\delta\eta)$. That this is indeed an action is immediate. Consider now what it is to give a morphism $(a,\theta,b) : \rho g \to h$, as in

$$\begin{array}{ccc} TA & \xrightarrow{\;a\;} & C \\ {\scriptstyle \rho g}\downarrow & \overset{\theta}{\Rightarrow} & \downarrow{\scriptstyle h} \\ SB & \xrightarrow[\;b\;]{} & D. \end{array} \qquad (5.1)$$

It comes to giving the images (α, φ, β) of (a, θ, b) under the isomorphism Φ, as in

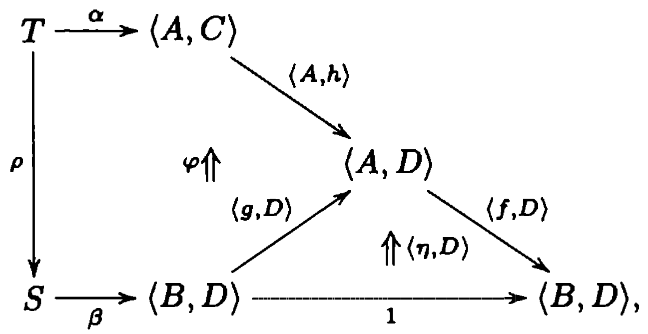


(5.2)

In general, this is not of the form $\rho \to \sigma$ for some object σ of Colax$[\mathbf{2}, \mathcal{V}]$: the present action does not admit a right adjoint like that in (1.1). Suppose however that the morphism g is a right adjoint — say $g = f^*$ where, as before, we have $\eta, \varepsilon : f \dashv f^* = g : A \to B$ in $\mathcal{A}$. These same data constitute, in $\mathcal{A}^{\mathrm{op}}$, an adjunction $\eta, \varepsilon : g \dashv f : A \to B$, which is sent by the 2-functor $\langle -, D\rangle : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ to the adjunction

$$\langle \eta, D\rangle, \langle \varepsilon, D\rangle : \langle g, D\rangle \dashv \langle f, D\rangle : \langle A, D\rangle \to \langle B, D\rangle$$

in $\mathcal{V}$. Accordingly, to give the $\varphi : \langle g, D\rangle\beta\rho \to \langle A, h\rangle\alpha$ of (5.2) is equally (see [13]) to give a 2-cell $\psi : \beta\rho \to \langle f, D\rangle\langle A, h\rangle\alpha = \langle f, h\rangle\alpha$, where ψ is given in terms of φ as the pasting composite



(5.3)

with a similar formula giving φ in terms of ψ. The passage from φ to ψ is clearly 2-natural in the ρ and in the h of (5.1), so that we have a 2-natural isomorphism

$$\mathrm{Colax}[\mathbf{2}, \mathcal{A}](\rho g, h) \cong \mathrm{Colax}[\mathbf{2}, \mathcal{V}](\rho, \langle f, h\rangle).$$

Thus, although we have for a general g no adjunction of the form

$$\mathrm{Colax}[\mathbf{2}, \mathcal{A}](\rho g, h) \cong \mathrm{Colax}[\mathbf{2}, \mathcal{V}](\rho, [g, h]),$$

yet we do have such a $[g, h]$ when g is of the form f^*, it being given by $[g, h] = \langle f, h\rangle$; more succinctly, we have $[f^*, h] = \langle f, h\rangle$. In particular, $\langle f, f^*\rangle : \langle A, A\rangle \to \langle B, B\rangle$ is the value of $[f^*, f^*]$, which is a monoid in Colax$[\mathbf{2}, \mathcal{V}]$ because it has the form of an object of endomorphisms.

We now turn to our second main result, namely the observation in Section 4 that $f : B \to A$ underlies a colax map of lax $\langle A, A\rangle$-algebras, or equivalently that $f^* : A \to B$ underlies a lax map of such algebras. We can approach the latter, too, in terms of the present action of Colax$[\mathbf{2}, \mathcal{V}]$ on Colax$[\mathbf{2}, \mathcal{A}]$.

We may identify an object T of $\mathcal{V}$ with the object $1_T : T \to T$ of Colax$[\mathbf{2}, \mathcal{V}]$; and a monad structure on T gives rise to one on 1_T, with the same notation. To give a [lax] action of such a monad on an object $g : A \to B$ of Colax$[\mathbf{2}, \mathcal{A}]$ is clearly to give [lax] actions of T on A and on B, along with a lax map $(g, \overline{g}) : A \to B$ of such [lax] T-algebras; and the same is true when we omit each "[lax]". Accordingly to enrich $f^* : A \to B$ to a lax map of lax $\langle A, A\rangle$-algebras, we have only to provide in Colax$[\mathbf{2}, \mathcal{A}]$ a lax action $\langle A, A\rangle f^* \to f^*$, or equivalently to provide in Colax$[\mathbf{2}, \mathcal{V}]$ a lax monoid map $\langle A, A\rangle \to [f^*, f^*]$. Recall that $\langle A, A\rangle$ here stands for $1 : \langle A, A\rangle \to \langle A, A\rangle$, while $[f^*, f^*] = \langle f, f^*\rangle : \langle A, A\rangle \to \langle B, B\rangle$. We simplify now by writing C for $\langle A, A\rangle$ and D for $\langle B, B\rangle$, with $k : C \to D$ for $\langle f, f^*\rangle$; recall that $C = \langle A, A\rangle$ is a monoid (C, i, m) in $\mathcal{V}$, while $D = \langle B, B\rangle$ is a monoid (D, j, n), and $k : C \to D$ is a monoid in Colax$[\mathbf{2}, \mathcal{V}]$, or equally a lax map $(k, k^\circ, \widetilde{k})$ of monoids in $\mathcal{V}$, where k° and $\widetilde{k}$ have the forms

$$\begin{array}{ccc} & & C \\ & \nearrow^{i} & \\ 1 & \overset{k^\circ}{\Rightarrow} & \downarrow k \\ & \searrow_{j} & \\ & & D, \end{array} \qquad \begin{array}{ccc} CC & \xrightarrow{m} & C \\ {\scriptstyle kk}\downarrow & \overset{\widetilde{k}}{\Rightarrow} & \downarrow{\scriptstyle k} \\ DD & \xrightarrow[n]{} & D. \end{array}$$

What we seek is a lax monoid map $(h, h^\circ, \widetilde{h}) : 1_C \to k$ in Colax$[\mathbf{2}, \mathcal{V}]$. For h we take the map $(1, id, k)$ as in

$$\begin{array}{ccc} C & \xrightarrow{1} & C \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow[k]{} & D. \end{array}$$

Next, h° has to be a 2-cell

$$\text{from}\quad \begin{array}{ccc} 1 & \xrightarrow{i} & C \\ {\scriptstyle 1}\downarrow & \overset{k^\circ}{\Rightarrow} & \downarrow{\scriptstyle k} \\ 1 & \xrightarrow[j]{} & D \end{array} \quad\text{to}\quad \begin{array}{ccccc} 1 & \xrightarrow{i} & C & \xrightarrow{1} & C \\ {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle k} \\ 1 & \xrightarrow[i]{} & C & \xrightarrow[k]{} & D, \end{array}$$

for which we take the pair (id, k°). Similarly the 2-cell $\widetilde{h}$

$$\text{from}\quad \begin{array}{ccccc} CC & \xrightarrow{11} & CC & \xrightarrow{m} & C \\ {\scriptstyle 11}\downarrow & & {\scriptstyle kk}\downarrow & \overset{\widetilde{k}}{\Rightarrow} & \downarrow{\scriptstyle k} \\ CC & \xrightarrow[kk]{} & DD & \xrightarrow[n]{} & D \end{array} \quad\text{to}\quad \begin{array}{ccccc} CC & \xrightarrow{m} & C & \xrightarrow{1} & C \\ {\scriptstyle 11}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle k} \\ CC & \xrightarrow[m]{} & C & \xrightarrow[k]{} & D \end{array}$$

is provided by the pair $(id, \widetilde{k})$. The easy verification that $(h, h^\circ, \widetilde{h})$ is indeed a lax map of monoids provides us with the desired enrichment $(f^*, \overline{f^*})$ of f^* to a lax map of lax $\langle A, A\rangle$-algebras.

We carry this analysis no further, since the calculations which give the explicit values of $\langle f, f^*\rangle : \langle A, A\rangle \to \langle B, B\rangle$ and of $(f^*, \overline{f^*}) : A \to B$ are no shorter if we begin with these present observations than were our calculations above based on the observations of Sections 3 and 4.

6 Transport of structure along an equivalence

We restrict ourselves here to the important case where the adjunction $\eta, \varepsilon : f \dashv f^* : A \to B$ is an *equivalence* in $\mathcal{A}$; the reader interested in the more general situations of the previous section will easily make the necessary extensions. We are used in universal algebra to the transport of structure along an isomorphism: if T is a monad on the mere category $\mathcal{A}$, if $a : TA \to A$ is an action of T on $A \in \mathcal{A}$, and if $f : B \to A$ is an isomorphism in $\mathcal{A}$ with inverse $f^* : A \to B$, there is a unique action $b : TB \to B$ of T on B for which f becomes an isomorphism of T-algebras — namely that given by $b = f^*a.Tf$. What replaces this result when $\mathcal{A}$ is a 2-category, the monad $T = (T, i, m)$ is a 2-monad, and the isomorphism $f : B \to A$ is replaced by the adjoint equivalence $f \dashv f^*$? We make use of the results above, taking for $\mathcal{V}$ the monoidal 2-category $[\mathcal{A}, \mathcal{A}]$, and supposing $\mathcal{A}$ complete and locally small, so that we have the adjunction $\Phi : \mathcal{A}(XA, B) \cong [\mathcal{A}, \mathcal{A}](X, \langle A, B\rangle)$. It is well known — see for instance [13] — that a strict map $\alpha : T \to S$ of monads on $\mathcal{A}$ (2-monads, of course, since $\mathcal{A}$ is a 2-category) induces a 2-functor $\alpha^* : S\text{-Alg} \to T\text{-Alg}$, commuting with the forgetful 2-functors to $\mathcal{A}$, and restricting to a 2-functor $S\text{-Alg}_s \to T\text{-Alg}_s$. We need a less strict analogue of this: we show that a pseudo map $\alpha = (\alpha, \alpha^\circ, \widetilde{\alpha}) : T \to S$ of monads induces a 2-functor $\alpha^* : \text{Ps-}S\text{-Alg} \to \text{Ps-}T\text{-Alg}$, again commuting with the forgetful 2-functors to $\mathcal{A}$, and again admitting a restriction $\alpha^*_s : \text{Ps-}S\text{-Alg}_s \to \text{Ps-}T\text{-Alg}_s$ to the *strict* maps of pseudo algebras. First, a pseudo action $s = (s, \widehat{s}, \overline{s})$ of S on B corresponds as in Section 2 to a pseudo map $\sigma = (\sigma, \sigma^\circ, \widetilde{\sigma}) : S \to \langle B, B\rangle$ of monads, which composes with $\alpha = (\alpha, \alpha^\circ, \widetilde{\alpha})$ to give a pseudo map $\rho = (\rho, \rho^\circ, \widetilde{\rho}) : T \to \langle B, B\rangle$, corresponding to a pseudo action $r = (r, \widehat{r}, \overline{r})$ of T on B; and $(B, r, \widehat{r}, \overline{r})$ is the image under α^* of $(B, s, \widehat{s}, \overline{s})$. Next, given a morphism $f : B \to B'$ where $(B, s, \widehat{s}, \overline{s})$ and $(B', s', \widehat{s}', \overline{s}')$ are pseudo S-algebras, to give f the structure of a morphism (that is, a pseudo morphism) of pseudo S-algebras is to give a pseudo monad map $\gamma : S \to \langle\langle f, f\rangle\rangle$ with $\chi\gamma = \sigma$ and $\chi'\gamma = \sigma'$, where

$$\begin{array}{ccccc} & & \langle B, B\rangle & & \\ & \overset{\chi}{\nearrow} & & \searrow^{\langle B, f\rangle} & \\ \langle\langle f, f\rangle\rangle & & \mu\Uparrow & & \langle B, B'\rangle \\ & \underset{\chi'}{\searrow} & & \nearrow_{\langle f, B'\rangle} & \\ & & \langle B', B'\rangle & & \end{array} \tag{6.1}$$

is the iso-comma object in $[\mathcal{A}, \mathcal{A}]$; and then the composite pseudo monad-map $\gamma\alpha : T \to \langle\langle f, f\rangle\rangle$ corresponds to a morphism $(f, \overline{\overline{f}}) : \alpha^*(B, s, \widehat{s}, \overline{s}) \to \alpha^*(B', s', \widehat{s}', \overline{s}')$ which is the desired $\alpha^*(f, \overline{f})$. The isomorphism $\overline{f}$ is an identity — that is, the morphism $(f, \overline{f})$ is strict — when $\gamma : S \to \langle\langle f, f\rangle\rangle$ factorizes through the canonical $\delta : (f, f) \to \langle\langle f, f\rangle\rangle$, where (f, f) is the pullback

$$\begin{array}{ccccc} & & \langle B, B\rangle & & \\ & \nearrow & & \searrow^{\langle B, f\rangle} & \\ (f, f) & & & & \langle B, B'\rangle \; ; \\ & \searrow & & \nearrow_{\langle f, B'\rangle} & \\ & & \langle B', B'\rangle & & \end{array}$$

in which case $\gamma\alpha$ factorizes through δ. Thus α^* does indeed send strict morphisms to strict morphisms, and we have established the 2-functors $\alpha^* : \text{Ps-}S\text{-Alg} \to \text{Ps-}T\text{-Alg}$ and $\alpha^*_s : \text{Ps-}S\text{-Alg}_s \to \text{Ps-}T\text{-Alg}_s$.

With these tools at hand, we return to the question of transporting structure: let us have the adjoint equivalence $\eta, \varepsilon : f \dashv f^* : A \to B$ in $\mathcal{A}$, and a pseudo action $(a, \widehat{a}, \overline{a})$ of T on A. This corresponds to a pseudo monad-map $(\alpha, \alpha^\circ, \widetilde{\alpha}) : T \to \langle A, A\rangle$, where $\alpha : T \to \langle A, A\rangle$ is the image under Φ of $a : TA \to A$. This $\alpha = (\alpha, \alpha^\circ, \widetilde{\alpha})$ induces a 2-functor $\alpha^* : \text{Ps-}\langle A, A\rangle\text{-Alg} \to \text{Ps-}T\text{-Alg}$ which carries the adjoint equivalence $(f, \overline{f}) \dashv (f^*, \overline{f^*})$ in $\text{Ps-}\langle A, A\rangle\text{-Alg}$ of Theorem 4.1 to an adjoint equivalence

$$\eta, \varepsilon : \alpha^*(f, \overline{f}) \dashv \alpha^*(f^*, \overline{f^*}) : \alpha^*(A, e) \to \alpha^*(B, t, \widehat{t}, \overline{t})$$

in Ps-T-Alg. Since the strict action $e : \langle A, A\rangle A \to A$ corresponds to the identity morphism $\langle A, A\rangle \to \langle A, A\rangle$, the pseudo T-algebra $\alpha^*(A, e)$ is the $(A, a, \widehat{a}, \overline{a})$ we started with. Calculating $\alpha^*(B, t, \widehat{t}, \overline{t})$ is also straightforward, but we do it explicitly below only for the important case where A is a strict T-algebra, with $\widehat{a}$ and $\overline{a}$ identities. In fact there is a theoretical sense in which it suffices to study this case: it is shown in [3] that, under modest conditions on T, a pseudo T-algebra is just a T'-algebra for another monad T'; we shall return to this observation below, in connection with *flexible* monads.

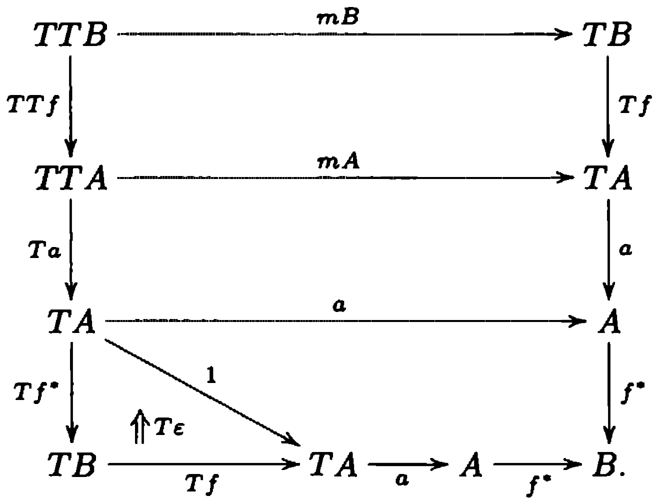
We take (A, a), then, to be a strict T-algebra, observing that $\alpha : T \to \langle A, A\rangle$, as the image under Φ of $a : TA \to A$, satisfies $e.\alpha A = a$. (Note that we have earlier used i for the unit and m for the multiplication not only of $\langle A, A\rangle$ but also of T; but continuing to do so will lead to no confusion.) Let us write $(B, b, \widehat{b}, \overline{b})$ for the T-algebra $\alpha^*(B, t, \widehat{t}, \overline{t})$. Since $b : TB \to B$ is the composite $t.\alpha B$, while t is given by the composite

$$\langle A, A\rangle B \xrightarrow{\langle A,A\rangle f} \langle A, A\rangle A \xrightarrow{e} A \xrightarrow{f^*} B\,,$$

the naturality of α along with the equation $e.\alpha A = a$ gives b as the composite

$$TB \xrightarrow{Tf} TA \xrightarrow{a} A \xrightarrow{f^*} B.$$

Similarly $\widehat{b}$ is simply $\eta : 1 \to f^*f = f^*a.Tf.iB$, while by the description (3.4) of $\overline{t}$ we see that $\overline{b}$ is given by



(6.2)

Similarly, $\alpha^*(f, \overline{f}) = (f, \overline{\overline{f}})$ and $\alpha^*(f^*, \overline{f^*}) = (f^*, \overline{\overline{f}}^*)$, where $\overline{\overline{f}}$ is given by $\varepsilon^{-1}a.Tf : a.Tf \to ff^*a.Tf (= fb)$, and $\overline{\overline{f}}^*$ by $f^*a.T\varepsilon : (b.Tf^* =) f^*a.Tf.Tf^* \to f^*a$. Summing up, we have:

Theorem 6.1 *Given the equivalence $\eta, \varepsilon : f \dashv f^* : A \to B$ in the complete and locally small 2-category $\mathcal{A}$, and an algebra (A, a) for the monad $T = (T, i, m)$ on $\mathcal{A}$, the equivalence enriches to an equivalence*

$$\eta, \varepsilon : (f, \overline{\overline{f}}) \dashv (f^*, \overline{\overline{f}}^*) : (A, a) \to (B, b, \widehat{b}, \overline{b})$$

in Ps-T-Alg, *where $\widehat{b} = \eta$ and $\overline{b}$ is given by $f^*a.T\varepsilon.Ta.T^2f$ as in* (6.2), *and where $\overline{\overline{f}} = \varepsilon^{-1}a.Tf$ and $\overline{\overline{f}}^* = f^*a.T\varepsilon$.*

Consider the case where $\mathcal{A} = \mathbf{Cat}$ and $T = (T, i, m)$ is the 2-monad whose algebras are the strict monoidal categories. A consequence of the coherence theorem for monoidal categories is that for any monoidal category B there is a strict monoidal category A — that is, a strict T-algebra (A, a) — and an equivalence $\eta, \varepsilon : f \dashv f^* : A \to B$ with f and f^* strong monoidal functors. Now suppose that B is a skeleton of the category of countable sets, equipped with the cartesian monoidal structure; then the monoidal structure on A is again the cartesian one, for some choice of binary products and terminal object. If the equivalence $\eta, \varepsilon : f \dashv f^*$ underlay an equivalence in T-Alg, then the monoidal structure on B would be both cartesian and strict. By an argument due to Isbell [14, p.160], however, this is impossible.

In general, then, it is not possible to enrich an adjoint equivalence to one in T-Alg. However such an enrichment does exist when the monad T is *flexible* — a notion, originally introduced in [8], which we now recall. First note that, in the present case where $\mathcal{V} = [\mathcal{A}, \mathcal{A}]$, so that a monoid in $\mathcal{V}$ is a monad on $\mathcal{A}$, the 2-categories $\mathrm{Mon}_l\,\mathcal{V}$, $\mathrm{Mon}_p\,\mathcal{V}$, and $\mathrm{Mon}\,\mathcal{V}$ of Section 2 are conveniently renamed $\mathrm{Mnd}_l\,\mathcal{A}$, $\mathrm{Mnd}_p\,\mathcal{A}$, and $\mathrm{Mnd}\,\mathcal{A}$. In particular we have the inclusion 2-functor $J : \mathrm{Mnd}\,\mathcal{A} \to \mathrm{Mnd}_p\,\mathcal{A}$; and it was shown in Blackwell's thesis [3] that a partial left adjoint to J is defined at the monad T if $\mathcal{A}$ is cocomplete and T has some rank. (An endofunctor T of $\mathcal{A}$ is said to have rank κ, where κ is a regular cardinal, if T preserves κ-filtered colimits.) To say that the partial adjoint is defined at T means, of course, that there is a pseudo map $p : T \to T'$ of monads on $\mathcal{A}$ such that, for any monad S on $\mathcal{A}$, the 2-functor $\mathrm{Mnd}\,\mathcal{A}(T', S) \to \mathrm{Mnd}_p\,\mathcal{A}(T, S)$ given by composition with p is an isomorphism of 2-categories. In more elementary terms, every pseudo map $g : T \to S$ is of the form hp for a unique strict map $h : T' \to S$, and every monad 2-cell $\alpha : hp \to h'p$, where the monad maps h and h' are strict, is βp for a unique monad 2-cell $\beta : h \to h'$.

In particular, there is a unique strict monad map $q : T' \to T$ for which $qp = 1_T$. Even before Blackwell's result, Kelly had shown in [9] that, whenever the partial left adjoint is defined at T, there is an invertible 2-cell $\rho : 1_{T'} \cong pq$ with $\rho p = id$ and $q\rho = id$, so that we have in $\mathrm{Mnd}_p\,\mathcal{A}$ the equivalence

$$\rho, id : q \dashv p : T \to T'.$$

Taking $S = \langle B, B \rangle$ in the universal property of $p : T \to T'$ shows that to give a pseudo action of T on B is just to give a strict action of T' on B. And taking for S the $\langle\!\langle f, f \rangle\!\rangle$ of (6.1) shows that enriching $f : B \to B'$ to a morphism $(f, \overline{f})$ of pseudo T-algebras is the same as enriching it to a morphism of T'-algebras. Accordingly we have an isomorphism of 2-categories

$$\text{Ps-}T\text{-Alg} \cong T'\text{-Alg}$$

which commutes with the underlying 2-functors to $\mathcal{A}$, and which restricts to an isomorphism of 2-categories

$$\text{Ps-}T\text{-Alg}_s \cong T'\text{-Alg}_s;$$

moreover, by a similar argument, it extends to an isomorphism

$$\text{Ps-}T\text{-Alg}_l \cong T'\text{-Alg}_l.$$

This is the intent of our earlier remark that a pseudo T-algebra is just a T'-algebra for a certain monad T'. The strict monad map $q : T' \to T$ induces a 2-functor $q^* : T\text{-Alg} \to T'\text{-Alg}$, restricting to $T\text{-Alg}_s \to T'\text{-Alg}_s$ and extending to

$T\text{-Alg}_l \to T'\text{-Alg}_l$. If we identify T'-Alg with Ps-T-Alg via the isomorphism above, $q^* : T\text{-Alg} \to T'\text{-Alg}$ is of course nothing but the inclusion $T\text{-Alg} \to \text{Ps-}T\text{-Alg}$.

The notion of flexibility was introduced by Kelly in [8] as a property of 2-monads, which is the case of interest here; later it was generalized to be a property of algebras for a 2-monad, of which a 2-monad itself is a special case — see [4]; in another special case introduced there, flexibility is a property of a *weight* for **Cat**-enriched limits, the corresponding *flexible limits* being studied in [2].

Supposing the monad T on $\mathcal{A}$ to be such that $p : T \to T'$ exists as above, with $q : T' \to T$ the unique strict map for which $qp = 1$, we say that T is *flexible* if there is some *strict* map $r : T \to T'$ for which $qr = 1$. Since we have $\rho : 1 \cong pq$ as above, we have $\rho r : r \cong pqr = p$; so that besides the equivalence $\rho, id : q \dashv p : T \to T'$ in $\text{Mnd}_p \mathcal{A}$, we now have an equivalence

$$\sigma, id : q \dashv r : T \to T' \tag{6.3}$$

in $\text{Mnd}\,\mathcal{A}$ itself. One easily sees that (supposing the left adjoints to exist) the monad T' is always flexible, and in fact a monad S is flexible precisely when it is a retract in $\text{Mnd}\,\mathcal{A}$ of some T'; the details can be found in [4]. For a flexible T, the equivalence of 2-categories

$$q^* \dashv r^* : T'\text{-Alg} \cong \text{Ps-}T\text{-Alg} \to T\text{-Alg}$$

induced by the equivalence (6.3) restricts of course to an equivalence

$$q^* \dashv r^* : T'\text{-Alg}_s \cong \text{Ps-}T\text{-Alg}_s \to T\text{-Alg}_s$$

between the Eilenberg-Moore 2-categories for the monads.

We can now give our main result on flexible monads:

Theorem 6.2 *Let $\eta, \varepsilon : f \dashv f^* : A \to B$ be an equivalence in the complete, cocomplete, and locally-small 2-category $\mathcal{A}$, let $T = (T, i, m)$ be a flexible monad on $\mathcal{A}$ having some rank, let $a : TA \to A$ be an action (meaning a strict one) of T on A, and let $qr = 1_T$, where $q : T' \to T$ is as above and r is a strict monad map. Then the given equivalence has an enrichment to an equivalence*

$$\eta, \varepsilon : (f, \check{f}) \dashv (f^*, \check{f}^*) : (A, a) \to (B, \check{b})$$

in T-Alg.

Proof Identifying Ps-T-Alg with the isomorphic T'-Alg, we find the desired equivalence as the image under $r^* : \text{Ps-}T\text{-Alg} \to T\text{-Alg}$ of the equivalence of Theorem 6.1. Note here that the (A, a) in the equivalence of Theorem 6.1 really denotes $q^*(A, a)$ — the T-algebra (A, a) seen as a pseudo T-algebra — and that $r^*q^*(A, a)$ is (A, a) itself, since $qr = 1$. □

Remark 6.3 That the scope of the theorem is extremely broad will be clear from the forthcoming article [11], where it is shown that a monad T on **Cat** is flexible if the structure of a T-algebra can be presented by operations and equations, in the sense of [12], in such a way that there are no equations between objects, only between maps; with similar results for many other 2-categories in place of **Cat**.

References

[1] J. Bénabou, *Catégories avec multiplication*, C.R.Acad.Sci.Paris **256** (1963), 1887–1890.

[2] G.J. Bird, G.M. Kelly, A.J. Power, and R.H. Street, *Flexible limits for 2-categories*, J. Pure Appl. Alg. **61** (1989), 1–27.

[3] R. Blackwell, *Some existence theorems in the theory of doctrines*, Ph.D. Thesis, Univ. New South Wales, 1976.
[4] R. Blackwell, G.M. Kelly, and A.J. Power, *Two dimensional monad theory*, J. Pure Appl. Alg. **59** (1989), 1–41.
[5] B.J. Day, *A reflection theorem for closed categories*, J. Pure Appl. Alg. **2** (1972), 1–11.
[6] G. Janelidze and G.M. Kelly, *A note on actions of a monoidal category*, Theory Appl. Categ. **9** (2001), 61–91.
[7] André Joyal and Ross Street, *Braided tensor categories*, Adv. Math. **102** (1993), 20–78.
[8] G.M. Kelly, Doctrinal adjunction, in *Sydney Category Seminar*, Lecture Notes Math. 420, pp. 257–280, Springer, 1974.
[9] G.M. Kelly, *Coherence theorems for lax algebras and for distributive laws* in Sydney Category Seminar, Lecture Notes Math. vol. 420, Springer, Berlin-Heidelberg, 1974, pp. 281–375.
[10] G.M. Kelly and S. Lack, *On property-like structures* Theory Appl. Categ. **3** (1997), 213–250.
[11] G.M. Kelly, S. Lack, and A.J. Power, *A criterion for flexibility of a 2-monad*, in preparation.
[12] G.M. Kelly and A.J. Power, *Adjunctions whose counits are coequalizers and presentations of finitary enriched monads*, J. Pure Appl. Alg. **89** (1993), 163–179.
[13] G.M. Kelly and R. Street, *Review of the elements of 2-categories* in Sydney Category Seminar, Lecture Notes Math. vol. 420, Springer, Berlin-Heidelberg, 1974, pp. 75–103.
[14] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics vol. 5, Springer, New York-Heidelberg-Berlin, 1971.
[15] S. Mac Lane and R. Paré, *Coherence for bicategories and indexed categories*, J. Pure Appl. Algebra **37** (1985), 59–80.
[16] R. Street, *Fibrations and Yoneda's lemma in a bicategory* in Sydney Category Seminar, Lecture Notes Math. vol. 420, Springer, Berlin-Heidelberg, 1974, pp. 104–133.

Received December 23, 2002; in revised form October 22, 2003