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Multisymplectic geometry: generic and exceptional

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Abstract

We shall discuss various examples of multisymplectic manifolds, both generic and exceptional, from the point of view of their groups of diffeomorphisms properties.

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1 Introduction: first examples

Multisymplectic geometry is a broad and so far not very well defined area of mathematics. Starting with the definition itself of a multisymplectic manifold, it changes substantially depending on the initial interests of the people working in the field. Thus, if we were using forms of higher degree in developping a geometrical background for variational calculus, the emphasis would be on the additional geometrical structures present on the problem and we would be led to a restrictive notion of multisymplectic structure. However there is another approach, let us call it the minimal approach, that simply asks for the properties common to a class of geometrical structures characterized simply by the existence of a closed form on a manifold M. This is the starting point for symplectic geometry/topology where we are interested in the study of manifolds equipped with a closed nondegenerate 2–form. From a more fundamental perspective a closed form emerges as a representative of a cohomology class of the manifold, and such cohomology class usually represents an invariant of another structure. For instance an integer closed 2-form represents a line bundle, whose sections help us to understand the topology of the manifold itself. From this perspective it makes sense to ask what can be said on the geometry of higher order closed forms having as a reference the already mature field of symplectic geometry. Some of the ideas and results collected in this direction are the scope of these review notes.

We should acknowledge a number of practicioners of the field that though mainly interested in the geometrical background of the calculus of variations have laid the ground for most of the discussion here. Without pretending being exhaustive we should mention Takens, Anderson, Kijowski, Tulczyjew, P. L. García, Goldschmidt, Sternberg, Krupka, Aldaya, de Azcárraga, Kostant, Binz, Saunders, Gotay, Isenberg, Marsden, Montgomery, Crampin, Cariñena, Ibort, Martin, Gunther, Byrnes, Cantrijn, de León, Echeverría, Muñoz-Lecanda, Román-Roy, Shardanasvily, Schkoller, ... and apologize to any other person that not being listed here has contributed to the field.

The minimal attitude taken here with respect to the notion of a multisymplectic structure motivates the following definition.

Definition 1 *A multisymplectic manifold is a manifold M equipped with a closed k–form $2 \leq k \leq \dim M$, such that the map $\hat{\Omega}\colon TM \to \Lambda^{k-1}(T^*M)$ defined by $\hat{\Omega}(v)(u_1, \ldots, u_{k-1}) = \Omega(v, u_1, \ldots, u_{k-1})$, $v, u_1, \ldots, u_{k-1} \in TM$, is injective or, in other words, it is nondegenerate.*

The image of the bundle map $\hat{\Omega}$ is a subbundle of the bundle of $(k-1)$–forms on M and will be denoted by E. Notice that $\mathrm{rank} E = k$.

We should exhibit at this point some examples to show the scope and interest of this notion.

The first example which is deeply related to the geometrical formulation of the calculus of variations is that of exterior bundles.

1.1 A first example: multicotangent bundles

Let Q be a smooht manifold and T^*Q its cotangent bundle. We shall denote by $\Lambda^k(T^*Q)$ the kth exterior power of T^*Q and by $\pi_k\colon \Lambda^k(T^*Q) \to Q$ the canonical projection. There is a canonical k–form Θ on $\Lambda^k(T^*Q)$ defined as follows:

$$\Theta_\alpha(U_1, \ldots, U_k) = \langle \alpha, (\pi_k)_* U_1 \wedge \cdots \wedge (\pi_k)_* U_k \rangle, \qquad \forall U_1, \ldots, U_k \in T_\alpha \Lambda^k(T^*Q).$$

The form $\Omega = d\Theta$ is a closed nondegenerate $(k+1)$–form on $\Lambda^k(T^*Q)$. In local coordinates we have

$$\Theta = p_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

and

$$\Omega = dp_{i_1 \ldots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where x^i denote local coordinates on Q and $p_{i_1\ldots i_k}$ are canonical coordinates of k-covectors in the basis $dx^{i_1}\wedge\cdots\wedge dx^{i_k}$.

The space $\Lambda^k(T^*Q)$ together with the multisymplectic form Ω of degree $k+1$ will be called a multicotangent bundle. Notice that the cotangent bundle with its canonical symplectic structure corresponds to the case $k=1$.

A multisymplectic submanifold is a submanifold $i\colon S\to\Lambda^k(T^*Q)$ such that $i^*\Omega$ is nondegenerate. Simple examples of multisymplectic submanifolds are constructed as follows. Let $\pi\colon Q\to B$ be a locally trivial fibration and $\Lambda^k_r(\pi)$ the space of r–semibasic k–forms on Q, this is,

$$\begin{aligned}\Lambda^k_r(\pi) = \{\, \alpha\in\Lambda^k(T^*Q) \mid \ & \langle \alpha, V_1\wedge\cdots\wedge V_{r+1}\wedge U_{r+2}\wedge\cdots U_k\rangle = 0,\\ & \pi_* V_i = 0,\ i=1,\ldots,r+1\,\}.\end{aligned}$$

There is a natural embedding $i_{k,r}\colon\Lambda^k_r(Q)\to\Lambda^k(T^*Q)$ and $\Omega_{k,r} = i^*_{k,r}\Omega$ is multisymplectic.

Given a Lagrangian density $\mathcal{L}$ on $J^1(\pi)$ we can define the Legendre transformation $\mathcal{F}\colon J^1(\pi)\to\Lambda^n_1(Q)$, $n=\dim B$ and $\Omega_{\mathcal{L}}=\mathcal{F}^*\Omega_{n,1}$ is the Poincaré–Cartan form of the theory and the central object in the geometrical description of the calculus of variations defined by $\mathcal{L}$. Notice that $\Omega_{\mathcal{L}}$ does not need to be nondegenerate.

A diffeomorphism $\varphi\colon M\to M$, such that $\varphi^*\Omega=\Omega$ will be called a multisymplectic diffeomorphism. The group of all multisymplectic diffeomorphisms will be denoted by $G(M,\Omega)$.

The following theorem states a few well-known facts for the group of symplectic diffeomorphisms (see [16], [1]) of a cotangent bundle that extends naturally to multicotangent bundles.

Theorem 1 *Let M be the multicotangent bundle $\Lambda^k(T^*Q)$. The group $G(M,\Omega)$ is an infinite dimensional Lie group extension of Diff(Q). The group $G(M,\Omega)$ acts ω–transitively on M and is local.*

From this result a natural question arises: to what extent the group of multisymplectic diffeomorphisms of a multisymplectic manifold shares the properties of the group of multisymplectic diffeomorphisms of a multicotangent bundle? To answer such question we first need to understand better the problem of understanding the local structure of multisymplectic manifolds.

1.2 Other natural examples

To end this introduction we just list here some multisymplectic structures that have appeared in other branches of mathematics that are important by themselves and that show that multisymplectic structures are not only related to problems of the calculus of variations.

- Semisimple Lie groups. It is well-known that if G is a semisimple Lie group with Lie algebra $\mathfrak{g}$, then the left-invariant 3-form defined by

  $$\Omega(X, Y, Z) = \langle X, [Y, Z] \rangle + cyclic$$

  for every $X, Y, Z \in \mathfrak{g}$ is closed and nondegenerate, thus a multisymplectic form of degree 3.

- Cosymplectic manifolds.

- (Almost)-Quaternionic manifolds.

- Calabi-Yau manifolds. These class can be extended to the setting of symplectic geometry as follows. Let (M, ω) be a symplectic manifold and let J be a compatible almost complex structure on M. Consider the natural decomposition of the bundle of 1-forms with complex values in its holomorphic and anti-holomorphic part,

  $$\Lambda^1(T^*M, \mathbb{C}) = T^{(1,0)}M \oplus T^{(0,1)}M$$

  If $\dim M = 2n$, then the nth exterior power of $T^{(1,0)}M$ is a complex line bundle lover M. We shall denote it by $\Lambda^{n,0} = \Lambda^n(T^{(1,0)}M)$. If Ω is a section of $\Lambda^{n,0}$ which is closed and $\Omega \wedge \bar{\Omega} \neq 0$ we will call (M, ω, Ω) an almost-Calabi-Yau manifold. We will recover the usual notion when (M, ω) is Kähler.

2 The local classification problem

For the degrees $k = 2, n$ all multisymplectic structures are locally isomorphic because they correspond to the symplectic and volume cases.

The existence of a local model for all symplectic structures is a consequence of Darboux' theorem which is a combination of two facts: i) Symplectic structures a locally constant (they define a flat G-structure) and, ii) there is a normal form for skewsymmetric bilinear forms.

The situation changes dramatically for $2 < k < n - 1$ where there is no a Darboux's theorem. We shall examine briefly why (see also [14]).

Let V be a linear space of dimension n. The space of k-forms on V denoted by $\Lambda^k(V^*)$ has dimension C_k^n. The group of linear isomorphisms $GL(V)$ acts on $\Lambda^k(V^*)$ and the quotient space $\Sigma_k^n(V)$ is the space of normal forms for k-forms on V. The space $\Sigma_k^n(V)$ is an stratified manifold.

We denote by $\Lambda^k_r(V^*)$ the space of k-forms of rank r. Linear multisymplectic structures correspond to elements of $\Lambda^n_n(V^*)$ which is a $GL(V)$-invariant open dense subset of $\Lambda^n(V^*)$. The space of normal forms for linear multisymplectic structures of degree k is thus the quotient space

$$\Sigma^n_{k,n} = \Lambda^n_n(V^*)/GL(V)$$

and has dimension bigger than 1 if $n \geq 9$ and $k \geq 3$. Consequently it is not possible to expect a Darboux' like theorem for multisymplectic structures of degrees bigger or equal than 3 in manifolds of dimension bigger or equal than 9. Moreover, there is no yet a description of the set $\Sigma^n_{k,n}$ for $k > 3$, $n > 7$.

3 An example of an exceptional multisymplectic structure

However, the lack of a normal linear form for higher order forms as discussed in the preceeding section is not the worst thing that can happen, because the existence of families of normal forms would not be a disastreous thing if the topological properties of all of them would be the same.

Things are as bad as possible as the following example of a multisymplectic structure shows. Let M be the linear space $\mathbb{R}^7$. We shall denote by $\theta^1, \ldots, \theta^7$ a linear basis of the dual space and

$$\theta^{ij} = \theta^i \wedge \theta^j, \qquad \theta^{ijk} = \theta^i \wedge \theta^j \wedge \theta^k, \ldots$$

Then we define the 3-form

$$\Omega = \theta^{123} + \theta^{145} + \theta^{167} + \theta^{246} - \theta^{257} - \theta^{347} - \theta^{356}.$$

which is a constant multisymplectic form. Let $G(V, \Omega)$ be the group of multisymplectic diffeomorphisms of Ω, i.e., $\phi^*\Omega = \Omega$.

Then, we have the following characterization for $G = G(\mathbb{R}^7, \Omega)$.

Theorem 2 *[6] The group G is a compact, connected, simple, simply connected Lie group of dimension 14. In fact, $G \cong G_2$.*

This example of multisymplectic structure is the first of a family of exceptional ones characterized by the fact that their automorphisms groups are "exceptional" like in this case G_2.

4 Kleinian geometries

The failure of the group of automorphisms of a multisymplectic structure to resemble the group of symplectic diffeomorphisms of a symplectic manifold can be consider to point out the exceptional or non generic multisymplectic structures.

Where should we look for generic multisymplectic structures? One possible way to address this question would be to call Klein's erlangen programme to help in this search. It is well-known that the classical geometries which are characterized by their group of automorphisms are symplectic, volume and contact geometries. Are there other classes of multisymplectic geometries possessing this property? As we shall see immediately the answer to this question is positive. There is a class of multisymplectic structures possessing a local homogeneity property that is also characterized by its group of automorphisms.

Definition 2 *A local Liouville vector field in the neighborhood of a point x in M is a vector field Δ_x such that its support, supp$\Delta_x = \overline{\{ y \mid \Delta_x(y) \neq 0 \}}$, contains just one fixed point in its interior which is the accumulation point of all the integral curves of Δ_x.*

Let M be a manifold and $x \in M$. Let x^i be local coordinates in a neighborhood of x and $\Delta = x^1\partial/\partial x^1 + \cdots + x^n\partial/\partial x^n$. Let ρ_x be a bump function centered in x whose support is small enough to be contained in the local chart above, then $\Delta_x = \rho_x\Delta$ is a local Liouville vector field.

Definition 3 *A multisymplectic manifold (M, Ω) is locally homogeneous if for every $x \in M$ there exist a local Liouville vector field Δ_x such that the vector bundle E image of Ω is invariant with respect to its local flow.*

It is clear that if for any $x \in M$ there exists a local Liouville vector field such that

$$\mathcal{L}_{\Delta_x}\Omega = c\Omega$$

then (M, Ω) is locally homogeneous.

A symplectic manifold (M, ω) is locally homogeneous because the bundle E, the image of ω, is all T^*M, hence automatically invariant with respect to any flow. Volume manifolds are also locally homogeneous. Choosing local coordinates x^i such that $\Omega = dx^1 \wedge \cdots \wedge dx^n$, and the local dilation vector field $\Delta = x^1\partial/\partial x^1 + \cdots + x^n\partial/\partial x^n$, we have $\mathcal{L}_\Delta\Omega = n\Omega$. Moreover in radial coordinates we have $\Omega = r^{n-1}dr \wedge \Theta$, with $i(\Delta)\Theta = 0$ and $\mathcal{L}_\Delta\Theta = 0$. Then,

$i(\Delta)\Omega = r^n\Theta$. Choosing a bump function ρ centered around x and denoting by $\Delta_x = \rho\Delta$, a simple computation shows

$$\mathcal{L}_{\Delta_x}\Omega = d(i(\Delta_x)\Omega) = d(\rho i(\Delta)\Omega) = d\rho \wedge r^n\Theta + n\rho\Omega = (r\rho' + n\rho)\Omega.$$

The main result on locally homogeneous multisymplectic manifolds is the content of the following theorem, which states that they constitute another instance of Kleinian geometries.

Theorem 3 *Let (M_a, Ω_a), $a = 1, 2$ be two locally homogeneous multisymplectic manifolds and $G_a = G(M_a, \Omega_a)$ their corresponding groups of multisymplectic diffeomorphisms. Let $\Phi: G_1 \to G_2$ be a group isomorphism which his also a homeomorphism with respect to the point-open topology. Then there exists a C^∞-diffeomorphism $\varphi: M_1 \to M_2$ such that $\Phi(f) = \varphi \circ f \circ \varphi^{-1}$ for every $f \in G_1$ and $\varphi_*: TM_1 \to TM_2$ maps locally hamiltonian vector fields of (M_1, Ω_1) into locally hamiltonian vector fields of (M_2, Ω_2). In addition, if φ_* maps the graded Lie algebra of infinitesimal automorphisms of (M_1, Ω_1) into the graded Lie algebra of infinitesimal automorphisms of (M_2, Ω_2) then there exists a constant $c \neq 0$ such that $\varphi_*\Omega_2 = c\Omega_1$.*

Sketch of the proof (see [13] for details).

1st step. The key step is to show that locally hamiltonian vector fields are localizable. This is shown using a variation of Moser deformation argument. If $\mathcal{L}_X\Omega = 0$, then $i(X)\Omega = \eta \in E$ is a closed $(k-1)$–form. Then the $(k-1)$–form,

$$\eta' = -d\int_0^1 \phi_t^*(i(\Delta_{x,t}\eta)dt$$

where Δ_x is a local Liouville vector field leaving invariant E is another closed $(k-1)$–form lying in E, thus there exists X' such that $i(X')\Omega = \eta'$ which is the localized locally hamiltonian vector field we are looking for.

2nd step. Prove that locally hamiltonian vector fields span the tangent bundle.

3rd step. Show that the group $G(M, \Omega)$ acts transitively on M ([5]).

4th step. There exists a bijection $\varphi: M_1 \to M_2$ such that $\Phi(f) = \varphi \circ f \circ \varphi^{-1}$ (slight adaptation of [21]).

5th step. The map φ a homeomorphism (adaptation of a theorem by Takens [20]).

6th step. The map φ is smooth (adaptation of a theorem by Banyaga [1]).

7th step. A multisymplectic extension of Lee-Hwa-Chung theorem [12]. If α is a smooth form such that $\mathcal{L}_U\alpha = 0$ for all multivector hamiltonian fields, then $\alpha = c\Omega^p$ and $c \neq 0$ if k divides $|\alpha|$.

8th step. Show that $\mathcal{L}_X\varphi^*\Omega_2 = 0$ for all X implies that $\varphi^*\Omega_2 = c\Omega_1$. □

5 Reduction of multisymplectic structures: an application

One of the most effective tools to construct symplectic manifolds is that of symplectic reduction. A slightly generalized technique can be used to construct non trivial multisymplectic manifolds.

Let us describe first such generalized multisymplectic reduction and later we will construct a family of examples. Let (M,Ω) be a multisymplectic manifold and $i\colon S \to M$ a submanifold. Let $\mathcal{F}$ be a regular foliation of S and we shall denote by R the quotient space $S/\mathcal{F}$. Let π denote the canonical submersion $\pi\colon S \to R$. Let us denote by Ω_S the pull-back $i^*\Omega$ of Ω to S, then we will assume that $\ker\Omega_S = T\mathcal{F}$. We will say that the foliated submanifold $(S,\mathcal{F})$ is compatible with Ω. Under such conditions there exists a closed form Ω_R on R such that

$$\pi^*\Omega_R = \Omega_S.$$

When Ω is a nondegenerate 2–form, i.e., a symplectic form on M, we will call the previous process a generalized symplectic reduction of M by S and $\mathcal{F}$. The fact that Ω_R is a symplectic structure again on the quotient space R is the content of the symplectic reduction theorem.

A well-known example of such process is given by the following example. Let Σ a compact oriented riemannian surface, and $\mathcal{A}$ the space of irreducible $SU(N)$–connections on Σ. We can define a symplectic form on such space by means of

$$\Omega_A(U,V) = \int_\Sigma \mathrm{Tr}(U\wedge V), \qquad U,V \in T_A\mathcal{A}.$$

A submanifold S of $\mathcal{A}$ is given by the set of flat connections,

$$S = \{\, A \in \mathcal{A} \mid F_A = 0 \,\}$$

A foliation of S is given by the action of the group of gauge transformations $\mathcal{G}$. Because $F_{A^g} = Ad_g F_A$, clearly $\mathcal{G}$ leaves invariant S and the quotient space $\mathcal{R}$ is the moduli space of flat connections on Σ. The foliation defined by the action of $\mathcal{G}$ on S is compatible with Ω_S because it is contained in $\ker\Omega_S$. Hence, the moduli space of flat bundles $\mathcal{R}$ inherits a symplectic structure.

This example can be generalized to higher dimensions as follows.

Let V be an n-dimensional riemannian closed manifold and $\mathcal{A}$ the space of $SU(N)$-connections on it. Such space carries a multisymplectic structure Ω defined by

$$\Omega(U_1, \ldots, U_n) = \int_V \mathrm{Tr}(U_1 \wedge \cdots \wedge U_n)$$

for all $U_1, \ldots, U_n \in T_A\mathcal{A} = \Omega^1(V; su(n))$. Again, let S be the space of flat connections and Ω_S the restriction of Ω to S. A simple computations shows that $\ker \Omega_S$ is spanned by the vectors of the form $d_A\xi$, $\xi \in Lie(\mathcal{G})$, thus the foliation defined by the action of the group of gauge transformations is compatible with Ω_S and the moduli space carries a canonical multisymplectic structure of degree n. We conclude with the following statement.

Theorem 4 *Let V be a riemannian manifold of dimension n. Then the moduli space of flat hermitian bundles of rank N has a natural multisymplectic structure Ω of degree n.*

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