Vector bundles on Orbispaces

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June 2005

Orbispaces are spaces with extra structure. The main examples come from compact Lie group actions $X \circlearrowleft G$ and are denoted $[X/G]$, their underlying space being X/G. By definition, every orbispace is locally of the form $[X/G]$, but the group G might vary.

To be more precise, an orbispace is a topological stack which is locally equivalent to $[X/G]$ for G a Lie group and X a G-CW-complex [2] [4] [5]. However, if all the stabilizer groups are finite, there exists a more concrete alternative definition [1].

Definition 1 *An orbispace is a map $p : E \to Y$ satisfying the following condition: there exists an open cover $\{U_i\}$ of Y and finite group actions $X_i \circlearrowleft G_i$ such that $p^{-1}(U_i) \to U_i$ is fiberwise homotopy equivalent to $(X_i \times EG_i)/G_i \to X_i/G_i$. The space Y is called the coarse moduli space, and the space E is called the total space.*

A morphism of orbispaces $(E, Y) \to (E', Y')$ is a commutative diagram:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & E' \arrow[d] \\ Y \arrow[r, "g"] & Y' \end{tikzcd}$$

If $g = g'$, there may exist 2-morphisms $(E, Y) \overset{\Downarrow}{\rightrightarrows} (E', Y')$ between morphisms (f, g) and (f', g'). A 2-morphism is a homotopy $h : E \times [0, 1] \to E'$ such that $p' \circ h_t = g \circ p$ for all $t \in [0, 1]$:

$$\begin{tikzcd} E \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow h"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ Y \arrow[r, "g"] & Y' \end{tikzcd} \tag{1}$$

If $g \neq g'$, there are no 2-morphisms. Two 2-morphisms are considered to be the same if they are homotopic to each other relatively to the endpoints.

We have two notions of point: those of the total space, and those of the coarse moduli space. If the stabilizer groups are not finite, Definition 1 is no longer valid, but these two notions still exist. The points "of the total space" form the objects of a groupoid, and the points of the coarse moduli space are the isomorphism classes of objects. Given an object of this groupoid, its automorphism group corresponds to the stabilizer of the action.

An orbispace always comes with a map to its coarse moduli space. By a suborbispace $\mathfrak{X}' \subset \mathfrak{X}$, we shall mean an orbispace obtained by pulling back along a subspace of the coarse moduli space.

If $\mathfrak{X} = (E \to Y)$ is an orbispace as in Definition 1, a vector bundle V over $\mathfrak{X}$ is a vector bundle $V \to E$ equipped with a flat connection in the direction of the fibers of p. Otherwise, one needs to use the classical notion of vector bundles on stacks.

It is tempting to define K-theory as the Grothendieck group of vector bundles. But, as is shown in [3], this is not always a good idea. For example, one needs the following technical property in order to prove the excision axiom:

Definition 2 *An orbispace $\mathfrak{X}$ has enough vector bundles if for every suborbispace $\mathfrak{X}' \subset \mathfrak{X}$ and every (finite dimensional) vector bundle V on $\mathfrak{X}'$, there exists a vector bundle W on $\mathfrak{X}$ and a linear embedding $V \hookrightarrow W$.*

This condition is not always satisfied (Example 4, see also section 5 of [3]), but it is if $\mathfrak{X}$ is compact with finite stabilizer (Corollary 6). We first establish the following connection with global quotients:

Theorem 3 *Let $\mathfrak{X}$ be a compact orbispace (i.e. it's coarse moduli space is compact). Then the following are equivalent:*

1. *$\mathfrak{X}$ is a global quotient by a compact Lie group i.e. $\mathfrak{X} = [X/G]$ for some compact Lie group G acting on a compact space X.*
2. *$\mathfrak{X}$ has enough vector bundles.*
3. *There exists a vector bundle W on $\mathfrak{X}$ such that for every point x ("of the total space") the action of $\mathrm{Aut}(x)$ on W_x is faithful.*

Proof. 1. $\Rightarrow$ 2. Let $X \circlearrowleft G$ be such that $\mathfrak{X} \simeq [X/G]$, and let $X' \subset X$ be the G-invariant subspace corresponding to $\mathfrak{X}' \subset \mathfrak{X}$. Let V be a vector bundle on $\mathfrak{X}'$ and let $\tilde{V}$ be the corresponding G-equivariant vector bundle on X'. It is well known that any equivariant vector bundle $\tilde{V}$ on a compact space X' embeds in one of the form $X' \times M$, where M is a representation of G. Let W be the vector bundle on $\mathfrak{X}$ corresponding to $X \times M \to X$. Since $\tilde{V}$ embeds in $X \times M$, the bundle V embeds in W.

2. $\Rightarrow$ 3. Suppose that $\mathfrak{X}$ has enough vector bundles, and let $\{U_i\}$ be a finite cover of $\mathfrak{X}$ such that $U_i \simeq [X_i/G_i]$. Let M_i be faithful representations of G_i, and let V_i be the vector bundles on U_i corresponding to $X_i \times M_i \to X_i$. Since M_i is faithful, the stabilizer groups act faithfully on the fibers of V_i. Let W_i be vector bundles on $\mathfrak{X}$ such that $V_i \hookrightarrow W_i$, and let $W := \bigoplus W_i$. Clearly, the stabilizer groups act faithfully on the fibers of W.

3. $\Rightarrow$ 1. Let P be the total space of the frame bundle of W. The stabilizer groups act faithfully on the fibers of W, hence they act freely on the fibers of P. Having no stabilizer groups, P is a space (as opposed to an orbispace). We have $\mathfrak{X} = [P/O(n)]$ and so $\mathfrak{X}$ is a global quotient. □

Note that there exist compact orbispaces which are not global quotients.

Example 4 *Let $P \to S^3$ be the principal BS^1-bundle classified by*

$$1 \in [S^3, B(BS^1)] = \mathbb{Z}.$$

Then $\mathfrak{X} := [P/ES^1]$ is not a global quotient by a Lie group.

Indeed, suppose that $\mathfrak{X} = [X/G]$. Since $\mathfrak{X} \to S^3$ is homotopically non-trivial, the map $X \to S^3$ needs to be be a non-trivial G-fiber bundle with fiber $S^1 \backslash G \circlearrowleft G$. Let $H = \mathrm{Aut}_G(S^1\backslash G)$ be the structure group of that bundle. All compact Lie groups have trivial π_2, therefore $[S^3, BH] = \pi_3 BH = \pi_2 H = 0$. The bundle $X \to S^3$ is trivial, a contradiction.

The above example shows that more orbispaces are global quotients by topological groups, than by compact Lie groups. Actually, all orbispaces are global quotients by topological groups.

The following theorems should have been the main results of [1]. Unfortunately the proofs have some gaps, but we believe that we are now able too fill them.

Almost-Theorems 5 *1. There exists a topological group G_1 such that every orbispace is a global quotient by G_1.*

2. There exists an ind-Lie group G_2 such that every orbispace with finite stabilizers is a global quotient by G_2.

3. Every compact orbispace with finite stabilizers is the global quotient by some compact Lie group.

The group G_1 can be any group with the following property. First, it must contain all compact Lie groups as subgroups. Moreover, if $K, K' \subset G_1$ are compact Lie subgroups, and if $f : K \to K'$ is a monomorphism, then the space

$$\left\{ g \in G_1 \,\middle|\, Ad(g)|_K = f \right\}$$

must be contractible. In particular, the group G_1 itself has to be contractible.

The group G_2 be described more explicitly. It is the colimit of $U(n!)$ under the maps $U(n!) \to U((n+1)!) : A \mapsto A \otimes \mathbf{1}_{n+1}$.

Corollary 6 *If $\mathfrak{X}$ is a compact orbispace with finite stabilizer groups, then $\mathfrak{X}$ has enough vector bundles.*

Sketch a proof of Theorems 5. Theorem 5.3 is an easy corollary of Theorem 5.2, so we concentrate on the proofs of Theorems 5.1 and 5.2.

The coarse moduli space of an orbispace is stratified by the type of stabilizer group. Let P be the poset of isomorphism classes of groups (compact Lie groups for the proof of Theorem 5.1, finite groups for the proof of Theorem 5.2). For now on, all spaces will be stratified by P. Similarly, maps between stratified spaces will always send a stratum of the source to the corresponding stratum of the target.

Let an *orbispace structure* on a stratified space Y be an orbispace $\mathfrak{X}$ over Y, inducing the given stratification. One can then build a universal orbispace $\underline{\mathfrak{Orb}}$ with coarse moduli space $\underline{\mathsf{Orb}}$, such that for each stratified space Y we then have a bijection

$$\left\{ \begin{array}{c} \text{Homotopy classes of} \\ \text{stratified maps } Y \to \underline{\mathsf{Orb}} \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{orbispace structures on } Y \end{array} \right\}.$$

Let G $(= G_1$ or $G_2)$ be a topological group and Y a stratified space. A G-*quotient structure* on Y is a G-space X over Y inducing a stratified homeomorphism $X/G \simeq Y$, where the stratification on X/G is by stabilizer groups. Again, there exists a universal G-space $\underline{E}G$ with quotient $\underline{B}G = \underline{E}G/G$, such that for every stratified space Y we have a bijection

$$\left\{ \begin{array}{c} \text{Homotopy classes of} \\ \text{stratified maps } Y \to \underline{B}G \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ G\text{-quotient structures on } Y \end{array} \right\}.$$

The functor $(X \circlearrowleft G) \mapsto [X/G]$ is represented by a stratified map $\underline{B}G \to \underline{\mathsf{Orb}}$. The theorem is then equivalent to the existence of a section

$$\begin{tikzcd} \underline{B}G \arrow[r] & \underline{\mathsf{Orb}} \arrow[l, bend right, dotted] \end{tikzcd}.$$

The obstructions to the existence of such a section lie in $H^{k+1}(\underline{\mathsf{Orb}}; \pi_k(Fiber))$, which can be computed to be zero [1].

Note that the homotopy type of the fiber varies with the stratum, and so $\pi_k(Fiber)$ is not a local system but rather a constructible sheaf. The argument therefore needs a new type of obstruction theory. This is where the proof in [1] has gaps.

References

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