SOLVING CLASSICAL FIELD EQUATIONS

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Feynman graphs are often thought of as tools for computations in perturbative quantum field theories. However, there is nothing particularly quantum about them and, in fact, Feynman rules for tree diagrams also arise when one solves classical field equations of interacting theories (i.e. non-linear PDEs) perturbatively. I explained this in my lectures on "Introduction to Quantum Field Theory" and since I am not aware of a textbook treatment of this material (although all this is pretty standard and known to all practitioners) I decided to write up these lecture notes.

1. The Klein-Gordon equation for the free field

We start with the Klein-Gordon equation

$$(\Box + m^2)\phi = 0.$$

This arises from the action

$$\mathcal{S} = \int d^d x \, \phi(\Box + m^2)\phi.$$

The plane wave Ansatz $\phi(t, \vec{x}) = \exp(i(\omega t - \vec{k} \cdot \vec{x}))$ is a solution provided the dispersion relation

$$\omega^2 - \vec{k}^2 = m^2$$

holds. Upon the identification $\omega \mapsto E$ and $\vec{k} \mapsto \vec{p}$ this is nothing but the relativistic dispersion relation

$$E^2 = m^2 + \vec{p}^2$$

of a particle of rest mass m in our units where $c = \hbar = 1$. As the Klein-Gordon equation is linear there is a superposition principle and any sum or multiple of solutions yields a new solution. We use this when we write the general solution in terms of its Fourier modes which are plane wave solutions:

$$\phi(t, \vec{x}) = \int d\omega \int d\vec{k} \, \tilde{\phi}(\omega, \vec{k}) \delta(\omega^2 - \vec{k}^2 - m^2) e^{i(\omega t - \vec{k} \cdot \vec{x})}$$

$$= \int \frac{d\vec{k}}{2\omega(\vec{k})} \left(a(\vec{k}) e^{i(\omega(\vec{k})t - \vec{k} \cdot \vec{x})} + a^*(\vec{k}) e^{i(-\omega(\vec{k})t - \vec{k} \cdot \vec{x})} \right),$$

where we defined $\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$. The measure factor in the second line arises from the δ-function transformation $(\delta(f(x)) = \delta(x - x_0)/|f'(x_0)|$ for $f(x_0) = 0)$ which contributes at $\omega = \pm\omega(\vec{k})$.

2. The non-linear field equation

Ultimately, we would like to solve equations of interacting fields stemming from a general potential $V(\phi)$:

$$(\Box + m^2)\phi = V'(\phi)$$

For example, we are interested in monomials $V(\phi) = g\frac{1}{n}\phi^n$. For odd n, the potential is unbounded from below and one would be able to gain in infinite amount of energy by making ϕ very large (or very negative). Such a system would be unstable and thus we restrict our attention to even n. For $n = 0$, the right hand side vanishes. $n = 2$ gives just another mass term and we can deal with it by redefining m^2. $n = 4$ is the first interesting case and we will start out studying this "ϕ^4-theory".

The equation $(\Box + m^2)\phi = g\phi^3$ is non-linear and there is no longer a superposition principle. This makes it very hard (and in general impossible) to write down exact solutions to this equation beyond the trivial $\phi = 0$ (the appendix, however, discusses a special class of exact solutions, the "kinks" or "domain-walls").

Thus, in general, we will have to resort to an approximate procedure: For $g \ll 1$, we can treat the right hand side as a small perturbation of the free Klein-Gordon equation and obtain approximations to the true solution by perturbing solutions to the Klein-Gordon equation.

3. The inhomogeneous field equation and Green's functions

To this end, let us step back a bit and first analyse the inhomogeneous Klein-Gordon equation

$$(\Box + m^2)\phi(x) = u(x)$$

where in the right hand side we have an arbitrary function which we think of as given rather than the RHS of ϕ^4-theory which is a function of the unknown ϕ. Already this equation is not linear but the space of solutions is still affine: When we have two solutions to two such equations $(\Box + m^2)\phi_1 = u_1$ and $(\Box + m^2)\phi_2 = u_2$ we obtain a solution for the equation where the RHS's add:

$$(\Box + m^2)(\phi_1 + \phi_2) = (u_1 + u_2).$$

We can use this idea to solve this equation in the case where the RHS can be decomposed into a sum of "elementary" functions:

$$u(x) = \sum_i \alpha_i u_i(x)$$

If we can solve these elementary equations $(\Box + m^2)\phi_i = u_i$, we obtain a solution as

$$\phi = \sum_i \alpha_i \phi_i.$$

We can take this idea to the extreme by decomposing the function $u(x)$ into a "sum" of δ-functions:

$$u(x) = \int dy\, u(y)\delta(x - y)$$

2

Here, $u(y)$ plays the role of the coefficients α_i and the only x dependent function on the right hand side is the δ-function. So, once we have a solution to

$$(\Box + m^2)\phi_g(x) = \delta(x)$$

we have a solution

$$\phi(x) = \int dy\, u(y)\phi_g(x-y)$$

as we can directly compute:

$$
\begin{aligned}
(\Box + m^2)\phi &= (\Box + m^2)\int dy\, u(y)\phi_g(x-y) \\
&= \int dy\, u(y)(\Box + m^2)\phi_g(x-y) \\
&= \int dy\, u(y)\delta(x-y) \\
&= u(x).
\end{aligned}
$$

This trick to solve inhomogeneous equations is obviously not restricted to the Klein-Gordon equation. Such a solution ϕ_g which solves an equation with a δ-inhomogeneity is called "Green's function" or "propagator" (in physics circles) and "fundamental solution" (by mathematicians). Strictly speaking, ϕ_g is not really a function but in general a distribution (like the δ distribution) and and ϕ is obtained as a convolution which then yields a proper function if u is nice enough but we will not analyse this in more detail.

It remains to find a solution ϕ_g of $(\Box + m^2)\phi_g = \delta$. As the differential operator is translation invariant (it does not contain x dependent coefficients), this can be done using a Fourier decomposition as differentiation becomes a simple multiplication in momentum space. We take the Fourier transform on both sides of the equation

$$\int \frac{dk}{\sqrt{2\pi}^d}\left((\Box + m^2)\phi_g(x)\right)e^{ikx} = \int \frac{dk}{\sqrt{2\pi}^d}\,\delta(x)e^{ikx}$$

$$\int \frac{dk}{\sqrt{2\pi}^d}(-k^2 + m^2)\phi_g(x)e^{ikx} = \frac{1}{\sqrt{2\pi}^d}$$

This allows us to read off

$$\tilde{\phi}_g(k) = \frac{1}{\sqrt{2\pi}^d}\frac{1}{-k^2 + m^2}$$

for the Fourier transform $\tilde{\phi}_g$ of ϕ_g:

$$\phi_g(x) = \int \frac{dk}{\sqrt{2\pi}^d}\,\tilde{\phi}_g(k)e^{-ikx} = \int \frac{dk}{(2\pi)^d}\frac{e^{-ikx}}{-k^2 + m^2}.$$

3

4. Solving the interacting theory perturbatively

Armed with this ability to solve arbitrary inhomogeneous equations we now come back to the ϕ^4-equation

$$(\Box + m^2)\phi = g\phi^3.$$

We want to view this as a family of equations parametrised by the coupling constant g. Similarly, the solutions to all these equations will depend on g. Underlying the idea of perturbation theory is the idea that these solutions can be written as a power-series in g, i.e. that they are analytic in g around $g = 0$.

Unfortunately, this is not really the case as can be seen as follows: Power series (in the complex plane) have a radius of convergence (which can be zero or infinite): Everywhere inside a circle of this radius the power series converges and outside it diverges. Thus, if the power-series would converge for any $g > 0$ it would as well have to converge for some $g < 0$. But for $g < 0$, again, the potential is unbounded from below and the system is unstable: Solutions will be radically different from solutions of the free equation and not be small perturbations. In fact, as is shown in the appendix, the kink solutions have energy and action scaling like $1/g$ which has a singularity at $g = 0$. In a path integral (which in a stationary phase approximation reproduces the classical behavior), these solutions appear as saddle points contributing $e^{-S} \approx e^{-1/g}$. These contributions are exponentially small for small g. In fact, this function has an essential singularity at $g = 0$ and is invisible in a Taylor expansion around this point. Indeed, "solitoninc" solutions like the kink are believed to be what is missed by the perturbative treatment. Their contributions are exponentially small for small g and can thus be safely ignored if one is interested in solutions to a finite precision.

Nevertheless, we will just proceed and pretend that solutions to the ϕ^4-equation can be written as a power series

$$\phi = \sum_{n=0}^{\infty} \phi_n g^n$$

for some coefficient functions $\phi_n(x)$.

Now plug this Ansatz into the equation and collect powers of g:

$$\sum_{n=0}^{\infty}(\Box + m^2)\phi_n g^n = g\left(\sum_{n=0}^{\infty}\phi_n g^n\right)^3 = \sum_{n=0}^{\infty}\left(\sum_{\substack{k,l,m \\ k+l+m+1=n}} \phi_k \phi_l \phi_m\right)g^n$$

Comparing coefficients we find

$$(\Box + m^2)\phi_n = \sum_{\substack{k,l,m \\ k+l+m+1=n}} \phi_k \phi_l \phi_m.$$

This simple manipulation has helped us a lot: We can now work our ways up starting from $n = 0$ to larger n. The important observation here is that this is a differential equation for ϕ_n in terms of a right-hand side given in terms of ϕ_k, ϕ_l, and ϕ_m where all $k, l, m < n$. That is, when computing ϕ_n we already know these ϕ_k, ϕ_l, and ϕ_m!

Let's see how this works out for the first couple of n:

$$(\Box + m^2)\phi_0 = 0$$

Nothing to be done. We know the solution is given in terms of plane waves obeying the dispersion relation. Next is

$$(\Box + m^2)\phi_1 = \phi_0^3$$

That was simple. Using the Green's function, we can write down the solution:

$$\phi_1(x) = \int dy\, \phi_g(x - y)\phi_0(y)^3.$$

Now comes

$$(\Box + m^2)\phi_2 = 3\phi_0^2\phi_1.$$

The 3 arises as there are three possible assignments of two 0's and one 1 to (k, l, m). The solution is

$$\phi_2(x) = 3\int dy\, \phi_g(x - y)\phi_0(y)^2\phi_1(y)$$

$$= 3\int dy \int dy'\, \phi_g(x - y)\phi_0(y)^2\phi_g(y - y')\phi_0(y')^3$$

Now for $n = 3$:
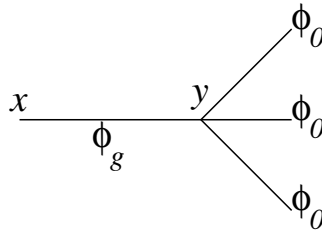
$$(\Box + m^2)\phi_3 = 3\phi_0^2\phi_2 + 3\phi_0\phi_1^2.$$

The iterated solution gets longer and longer:

$$\phi_3(x) = \int dy\, \phi_g(x - y)\left(3\phi_0(y)^2\phi_2(y) + 3\phi_0(y)\phi_1(y)\right)$$

$$= 9\int dy \int dy' \int dy''\, \phi_g(x - y)\phi_0(y)^2\phi_g(y - y')\phi_0(y')^2\phi_g(y' - y'')\phi_0(y'')^3$$

$$+ 3\int dy \int dy' \int dy''\, \phi_g(x - y)\phi_0(y)\phi_g(y - y')\phi_0(y')^3\phi_g(y - y'')\phi_0(y'')^3$$

## 5. Feynman graphs in position space

Obviously, continuing like this will be more and more cumbersome. However, we see a simple pattern of these terms emerging: We can represent the solution for ϕ_1 like this:

We obtain the solution by bringing together three ϕ_0's at one point y and then transport this to x using the Green's function ϕ_g. At higher orders, this pattern is iterated. For $n = 2$, we have
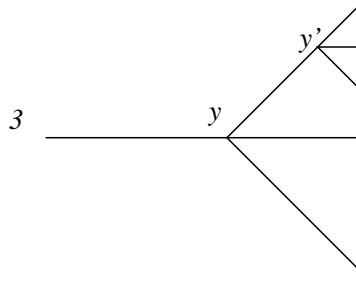
where the factor 3 arises because the graph for ϕ_1 can be substituted at any of the three legs. At level $n = 3$, there are two different graphs
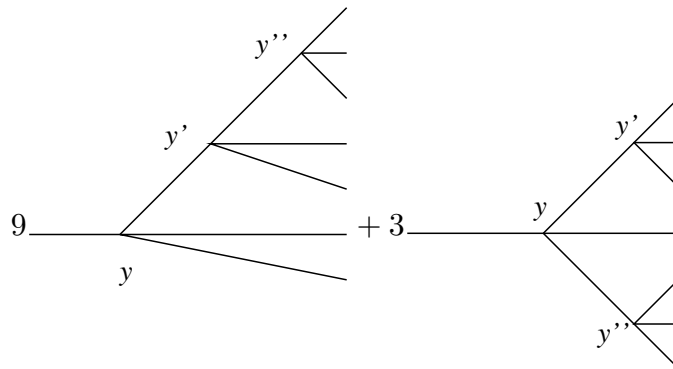
again with "symmetry factors" indicating the number of possibilities of obtaining these graphs.

In this graphical notation, it should be clear what we have to do to obtain the expression for ϕ_n: We have to draw all possible graphs according to these rules:

- Draw n vertices for the expression for ϕ_n at order g^n.
- Each vertex gets one in-going line at the left and three outgoing lines to the right.
- A line can either connect to the in-going port of another vertex or to the right-hand side of the diagram.
- Write down an integral for the point of each vertex.
- For a line connecting two vertices at points y_1 and y_2, write down a Green's function $\phi_g(y_1 - y_2)$.
- For a line ending on the right. write down a factor of ϕ_0 evaluated at the point of the vertex at the left of the line.
- Multiply by the number of permutations of outgoing lines at the vertices which yield different diagrams ("symmetry factor").

6

6. More general field equations

Looking back at how these rules came up, we can immediately guess the generalisation to other field equations: The fourth order potential $V(\phi) = \frac{g}{4}\phi^4$ resulted in a field equation with a cubic right-hand side. The cube in the field equation became $\phi_k\phi_l\phi_m$ with the constrained sum over k, l, and m in the equation for ϕ_n and eventually resulted in the rule that each vertex has to have three outgoing lines. This suggests that for a potential $V(\phi) = \frac{g'}{p}\phi^p$ we would derive similar rules but now with $(p-1)$ outgoing lines at each vertex.

If we have a potential which consists of a sum of more than one monomial there will be a separate coupling constant for each monomial.

$$V(\phi) = \sum_{i=1}^{I} \frac{g_i}{p_i}\phi^{p_i}$$

As a result, we would express ϕ in a multi-dimensional power series over all coupling constants:

$$\phi = \sum_{n_1\cdots n_I} \phi_{n_1\cdots n_I} g_1^{n_1} \cdots g_I^{n_I}.$$

Consequently, we now have I different types of vertices. To compute the solution for $\phi_{n_1\cdots n_I}$ we draw all diagrams with n_1 vertices of type 1 (which have $(p_1 - 1)$ outgoing legs), n_2 vertices of type 2 (with $(p_2 - 1)$ outgoing legs) and so on to n_I vertices of type I.

Of course, we can have also more than one type of field. In that case each field comes with its own equation of motion which we solve perturbatively: Each field has its Green's function and thus, in the graphical notation, we have different types of lines denoting the different fields and the vertices have "ports" connecting to the different types of lines. For example in Quantum electrodynamics, there is an electron field denoted by a straight line and a photon field denoted by a wavy line. There is a cubic term in the action which reads $e\bar{\psi}\gamma^\mu A_\mu\psi$. Here ψ is the electron field, A_μ is the photon field (which is a fancy name for the vector potential of electromagnetism), e is the charge of the electron playing the role of the coupling constant and γ^μ is some matrix needed to write down the Dirac equation which is the analogue of the Klein-Gordon equation for spin $1/2$ fermions like the electron. The bar indicates a conjugate (which is finally responsible for the difference between electrons and positrons) which results in the electron lines having a direction which is indicated by an arrow. This cubic term in the action ends up in quadratic right hand sides of the field equations: The field equation for the photon has a term quadratic in the electron on its right-hand side (namely the expression for the electromagnetic current) whereas the electron has a right-hand side which is bilinear in the electron and the photon. A typical

diagram then looks like this:
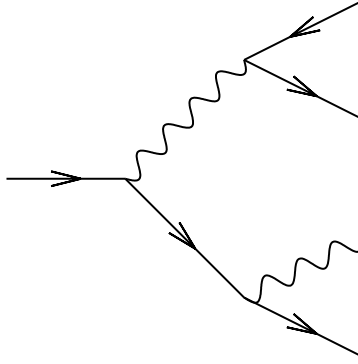
7. Relation to the quantum theory

Looking back at the "Feynman rules" above for the ϕ^4-theory which describe the allowed graphs we notice that these rule simply describe all graphs with 4-valent vertices (there are in total four lines at each vertex) that do not contain any loops. In quantum field theory, there is a similar perturbative expansion which similarly is most easily written down in a graphical notation in which each Feynman diagram corresponds to an integral expression. It turns out that the rules which translate graphs to integrals are identical to our rules above and the only difference between our classical theory here and the quantum theory is that we drop the distinction between in-going and outgoing lines at a vertex and only require that there are four lines at each vertex (for ϕ^4-theory), no matter if in- or out-going. Weakening this rule has the consequence that also graphs containing loops are allowed. This fact that the perturbative solution in classical field theory and in the quantum theory are so similar is then explained by the observation that each loop in a Feynman graph contributes a factor $\hbar$ and thus the contribution of the loop graphs vanish in the classical limit of $\hbar \to 0$.

8. Feynman rules in momentum space

For practical calculations (at least those in flat space), our Feynman rules which are based in position space are not the simplest ones possible. It is easy to see that in momentum space, we don't have to do any integrals! (This will, however, change in the quantum theory where one has to do one integral per loop). For simplicity, let us once more consider ϕ^4-theory. Assume we have drawn all our diagrams and translated them to an integral expression. Let us now zoom in to one particular vertex representing a space-time point y. It has one in-going and three out-going lines either representing ϕ_g or ϕ_0. We can write all these four functions in terms of their Fourier transform

$$\phi_g(x-y) = \int \frac{dk}{(2\pi)^d} \frac{e^{-ik(x-y)}}{-k^2 + m^2} \qquad \text{and} \qquad \phi_0(y) = \int dk\, \tilde{\phi}_0(k)\delta(k^2 - m^2)e^{iky}.$$

Let us denote the momentum of the in-going line at the vertex by k_1 and those of the three outgoing lines by k_2, k_3, and k_4. Then the only occurrence of the point y is in the combined exponential

$$e^{-iy(k_1 - k_2 - k_3 - k_4)}.$$

8

The integral over the position y now yields a δ-function via

$$\int dy \, e^{-iy(k_1 - k_2 - k_3 - k_4)} = (2\pi)^d \delta(k_1 - k_2 - k_3 - k_4).$$

This now trivialises the k_1 integration of the in-going line in terms of setting $k_1 \mapsto k_2 + k_3 + k_4$. This of course is nothing but momentum conservation. Once we have repeated this procedure for all vertices, all the momenta on the "internal" lines representing Green's functions ϕ_g are specified in terms of the momenta of the external lines representing ϕ_0. Thus we arrive at an algebraic expression without any integrals for the Fourier modes of ϕ_n. In momentum space, we found the new Feynman rules:

- Draw the same diagrams as in the position space representation including symmetry factors.
- Specify momenta satisfying the dispersion relation $k^2 = m^2$ on the lines extending to right end of the diagram.
- Use momentum conservation at the vertices to determine the momenta k_i of the internal lines connecting two vertices.
- write down a factor $\frac{1}{\sqrt{2\pi}^d} \frac{1}{-k_i^2 + m^2}$ for each internal line and a factor $(2\pi)^d$ for each vertex.

These rules then determine the Fourier component $\tilde{\phi}_n(p)$ where p is the sum of the momenta of all external lines extending to the right end of the diagram.

The method of Feynman diagrams suggests a new perspective on particle-wave duality: In the beginning, we set out to find solutions to classical field equations. But in that investigation we were lead to a language of diagrams which is much more natural in a particle interpretation: The lines of the Feynman diagrams are just the world lines of particles of mass m: The free, external solutions obey the dispersion relation for this rest mass and at the vertices the particles collide and interact observing conservation of momentum. As it turns out, in theories with additional symmetries also the Noether charges are conserved at the vertices. Thus it is natural to ask if we are really dealing with a field/wave-theory or rather with a particle theory. However, after a bit of contemplation one realises that this dichotomy is not real: This theory describes both: When looking at it in terms of a partial differential equation it looks more like a wave theory but the perturbative solutions in terms of Feynman diagrams are best understood in terms of interacting particles. This theory is really both, a wave- and a particle-theory even though it is classical and not quantum!

9. Appendix: The solitonic kink

In this appendix, we will discuss properties of a special class of non-perturbative solutions to ϕ^4 theory. Specifically, we want to consider the potential

$$V(\phi) = \frac{m^4}{16g} - \frac{m^2}{2}\phi^2 + \frac{g}{4}\phi^4 = \frac{g}{4}\left(\phi^2 - \frac{m^2}{4g}\right)^2.$$

Here, we have written the quadratic (mass) term as part of the potential. Note that it has an unusual sign but this is not a problem as the potential is unbounded from below and

9

in fact positive. The constant term does not influence the field equations but simplifies expressions in the following. Thus, we are looking for solutions to

$$\Box \phi = -V'(\phi) = m^2 \phi - g\phi^3.$$

The right-hand side vanishes at the extrema of the potential: The minima are at $\phi_{1/2} = \pm\frac{m}{\sqrt{g}}$ and the unstable maximum is at $\phi_3 = 0$. The two minima are the two degenerate ground-states of the system and ϕ sitting in one of these "vacua" is a trivial solution.

To have any solution of finite energy (and action) the field ϕ has to be in its ground-state at large distances. However, as we have a degenerate ground-state it can be in a different ground state in different directions. Specifically, we can consider the case $d = 1$ with only one spatial coordinate or, alternatively, solutions which only depend on one coordinate x. A solution which is in the ground-state $\phi_1 = -\frac{m}{\sqrt{g}}$ for $x \to -\infty$, then passes over the hill of the potential around the origin and goes to the other vacuum $\phi_2 = +\frac{m}{\sqrt{g}}$ for $x \to +\infty$ is called a kink. Here we will derive such a solution explicitly.

We are assuming that ϕ only depends on one spatial coordinate x and thus the field equation reads

$$\frac{d^2\phi}{dx^2} = \frac{dV}{d\phi}.$$

In analogy with conservation of energy in mechanics, we observe that if we solve the first order equation

$$\left(\frac{d\phi}{dx}\right)^2 = 2V$$

we automatically solve the second order equation of motion as can be seen by taking the x-derivative on both sides. Hence, we solve

$$\frac{d\phi}{dx} = \pm\sqrt{\frac{g}{2}}\left(\phi^2 - \frac{m^2}{g}\right)$$

by separation of variables

$$\int \frac{d\phi}{\phi^2 - m^2/g} = \pm\sqrt{\frac{g}{2}}\int dx.$$

Thus

$$\phi(x) = \pm\frac{m}{\sqrt{g}}\tanh\left(m(x - x_0)/\sqrt{2}\right).$$

The constant of integration x_0 determines where ϕ passes over the hill in the potential.

It is instructive to compute the action for this solution. As ϕ depends only on x the integrals over the other variables yield irrelevant (infinite) factors and it remains (using the first order equation $\phi' = \pm\sqrt{2V}$ several times)

$$\mathcal{S} = -\int dx \left[\left(\frac{d\phi}{dx}\right)^2 + V(\phi)\right] = -3\int dx \left(\frac{d\phi}{dx}\right)^2 = -3\int_{\phi_1}^{\phi_2} d\phi \frac{d\phi}{dx} = -3\int_{\phi_1}^{\phi_2} d\phi \sqrt{2V(\phi)} = \frac{8m^3}{g}.$$
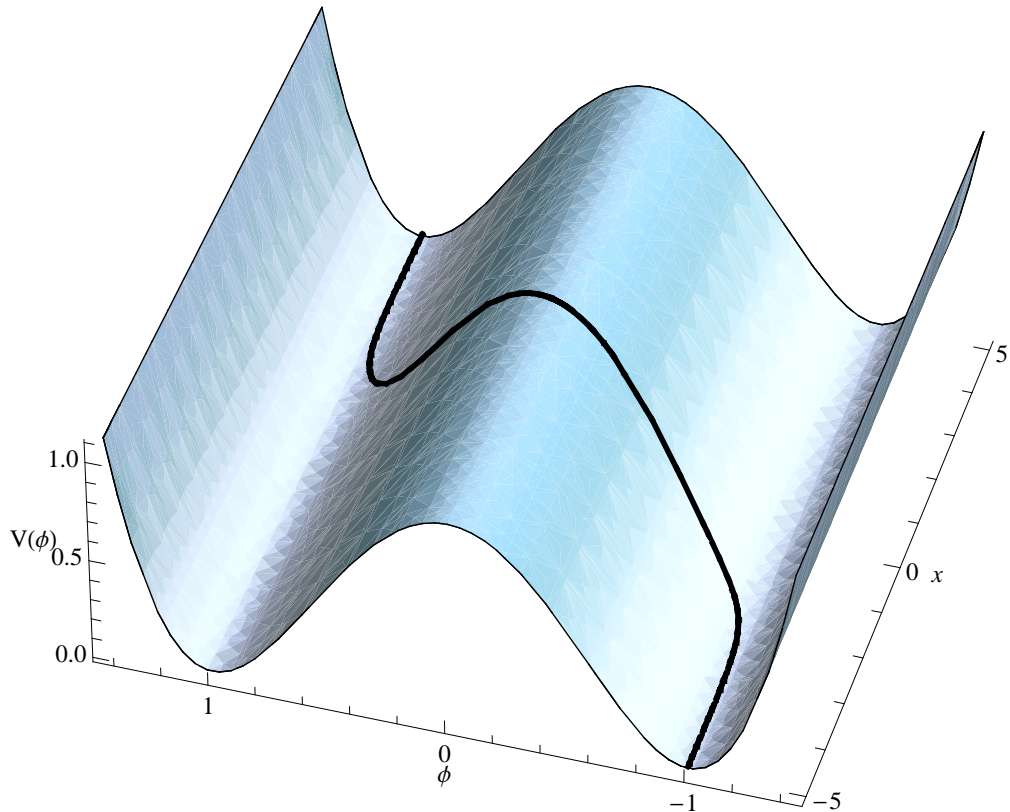
10

Fig. 1: The kink solution

We see that the action scales like $1/g$. Thus, in a path integral (which gives the classical theory in the $\hbar \to 0$ saddle point approximation), these solutions contribute with a weight $e^{-1/g}$ which is exponentially small for small g as we assume it in the perturbative analysis. In fact, due to the essential singularity of this expression at $g = 0$, all g-derivatives vanish at $g = 0$ and thus in a Taylor expansion (as we assume it for the perturbative Ansatz), this function is indistinguishable from 0. Thus, these "solitonic" solutions are missed by a perturbative expansion. It is believed that these missed solutions are the reason that the perturbative expansion cannot be convergent to the exact solution.

The kink solution above has a parameter or "modulus", the constant of integration x_0. This could have been expected from the very beginning as the equation of motion is translationally invariant but the solution is not. Thus there should be a whole "moduli-space" parametrising the kink solution. In this simple case, the moduli space is just the real line which we can think of as the configuration space of a "kink-particle". If this interpration of the kink as a particle holds it should also be possible for it to move. That is we can consider solutions where x_0 has a (slow) time dependence: $x_0(t)$.

Although this is not a solution of the equation of motion anymore (one should not confuse this with a kink to which a boost is applied, that is a solution in which x is substituted by $(x + vt)/\sqrt{1 - v^2}$ which is again a solution), we can work out the action for

$$\phi(x, t) = \pm \frac{m}{\sqrt{g}} \tanh\left(m(x - x_0(t))/\sqrt{2}\right) :$$

11

With the time dependent x_0, $\partial\phi/\partial t$ does not vanish anymore, but we have

$$\frac{\partial\phi}{\partial t} = -\frac{\partial\phi}{\partial x}\frac{dx_0}{dt}.$$

The action for this Ansatz gets a new term from expanding $\partial_\mu\phi\partial^\mu\phi$

$$\mathcal{S} = \int dt \int dx \left(\frac{\partial\phi}{\partial t}\right)^2 = \int dt \left(2\int dx\, V(\phi)\right)\dot{x}_0^2 = \int dt\, \frac{1}{2}\left(\frac{16m^3}{3g}\right)\dot{x}_0^2.$$

This is of course the action of a particle with mass $M = \frac{16m^3}{3g}$. This we should interpret as the the mass of the kink. It diverges in the weak coupling $g \to 0$ limit.