Mathematik

Madsen-Tillmann-Weiss Spectra
and
a Signature Problem for Manifolds

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Abstract

This thesis is concerned with Madsen-Tillmann-Weiss spectra and their homotopy groups. Since they are Thom spectra, their homotopy groups admit interpretations as certain bordism groups. For an arbitrary tangential structure there is an associated cofibre sequence of spectra, and after interpretting the induced maps on homotopy groups some explicit computations of these groups are made.

In the particular case where the tangential structure is Orientation, manifolds representing elements of these bordism groups are oriented and so their signatures are defined. This leads to a Signature Problem, which asks "what are the possible signatures of elements of these groups?" This problem is solved for certain small degrees.

This thesis also gathers basic results about the Euler class in the Appendix and proves a result which uses the Euler class to determine whether two stably isomorphic vector bundles are isomorphic or not.

Introduction

The Madsen-Tillmann-Weiss spectrum $\mathrm{MT}\,\theta_d$ associated to a tangential structure $\theta_d\colon B \to \mathrm{BO}(d)$ was introduced in [7] in order to provide an infinite loop space model for the classifying space of the cobordism category of θ_d-manifolds, $\mathcal{C}_{\theta_d}$. Namely they provided a weak homotopy equivalence

$$B\mathcal{C}_{\theta_d} \to \Omega^{\infty-1}\,\mathrm{MT}\,\theta_d$$

Previously, in the special case of Orientation $\mathrm{BSO}(2) \to \mathrm{BO}(2)$, Madsen and Weiss [21] had used the spectrum MTSO(2) (under a different name) to prove Mumford's Conjecture by appealing to a result of Tillmann [35] which related Segal's conformal cobordism category with the classifying space of the stable mapping class group (for a thorough discussion, see Tillmann's survey article [36]). In higher dimensions, Galatius and Randal-Williams [8] generalized Mumford's Conjecture by proving a homological stability result for diffeomorphism groups of high-dimensional manifolds by using $\mathrm{MT}\,\theta$ for a tangential structure θ which depends on the manifold. Although these results are of great importance and are a motivation for studying Madsen-Tillmann-Weiss spectra in their own right, this thesis will not discuss them further.

Much of this thesis is concerned primarily with the homotopy groups of the spectrum $\mathrm{MTSO}(d)$. They can be described as bordism groups of manifolds with special structure (see Section 1.2). In particular it is concerened with constructing concrete manifold representatives for elements of these groups and in determining these groups in some cases.

For any $(k, d) \in \mathbb{Z} \times \mathbb{N}$ there is a natural signature homomorphism which factors through the usual signature defined on the oriented bordism group:

$$\begin{array}{ccc} \pi_k\,\mathrm{MTSO}(d) & \xrightarrow{\ \sigma_{k,d}\ } & \mathbb{Z} \\ & \searrow \quad \nearrow_{\sigma} & \\ & \Omega_{k+d} & \end{array}$$

If $k + d$ is divisible by 4 then this homomorphism is potentially non-zero, and can be used to detect non-zero, torsion-free elements of $\pi_k\,\mathrm{MTSO}(d)$. Moreover, we can ask if representing an element of $\pi_k\,\mathrm{MTSO}(d)$ induces some restriction on the signatue.

The Signature Problem. *What is the image of $\sigma_{k,d}$?*

There are some trivial cases. For one, if $k + d$ is not divisible by 4 then any manifold of that dimension has signature 0, by definition. Secondly if $k + d = 0$

then a compact, oriented $(k+d)$-manifold is a finite set of signed points, and the signature is the sum of these signs; it follows that $\sigma_{-d,d}$ is surjective for each d. Some results are obtained for certain small values of d and k:

Theorem 2.0.10+2.0.11. *Suppose $(k,d) \in \mathbb{Z} \times \mathbb{N}$ with $k+d$ positive and divisible by 4. Then:*

If $d = 0$ or 1, then $\sigma_{k,d} = 0$.
If $k \leqslant 0$, then $\mathrm{Im}(\sigma_{k,d}) = \mathbb{Z}$.
If $k = 1$, then $\mathrm{Im}(\sigma_{k,d}) = 2\mathbb{Z}$.
If $k = 2$, then $\mathrm{Im}(\sigma_{k,d}) = 4\mathbb{Z}$.
If $k = 3$ and $k+d \geqslant 8$, then $\mathrm{Im}(\sigma_{k,d}) = 8\mathbb{Z}$.
If $k = 4$ and $k+d \geqslant 8$, then $\mathrm{Im}(\sigma_{k,d}) = 16\mathbb{Z}$.
If $k = 5$ and $k+d = 12$, then $\mathrm{Im}(\sigma_{k,d}) \supset 32\mathbb{Z}$. It is not known whether 16 is realizable or not in this case.
If $k = 5$ or 6 and $k+d \geqslant 16$, then $\mathrm{Im}(\sigma_{k,d}) = 16\mathbb{Z}$.

In all cases above, except for $d \in \{0,1\}$ and possibly $(k,d) = (5,7)$, the image of $\sigma_{k,d}$ is as large as possible.

By "as large as possible" what is meant is that Corollary 2.1.13 provides a lower bound for the index of $\mathrm{Im}(\sigma_{k,d})$ in $\mathbb{Z}$ in terms of k, and in these cases this lower bound is attained.

Parts of this theorem are proven by elementary means in Section 2.2, with basic examples constructed for $(k,d) = (1,3)$ and $(2,2)$, and other examples constructed using an external product structure on the groups $\pi_k \,\mathrm{MTSO}(d)$ as k and d vary (see Definition 1.2.9). For the final case where $k \in \{5,6\}$ and $k+d \geqslant 16$, a non-trivial result of Bökstedt, Dupont, and Svane [4, Theorem 1.2] is used to construct an element of $\pi_6 \,\mathrm{MTSO}(10)$ with signature 16 in Section 2.4.

Computing $\pi_k \,\mathrm{MTSO}(d)$ for arbitrary k and d is highly non-trivial: $\mathrm{MTSO}(1)$ has the homotopy type of $\mathbb{S}^{-1}$ and so is exactly as complicated as the stable homotopy groups of spheres. For $d = 2$ Rognes [27, Theorem 2.13] computed the 2-primary parts of $\pi_k \,\mathrm{MTSO}(2)$ for $k \leqslant 20$, but complete determination of these groups is still mostly unknown. However, the above theorem provides some torsion-free elements of $\pi_k \,\mathrm{MTSO}(d)$ for low values of k, and in Section 3.3 of this thesis the groups $\pi_k \,\mathrm{MTSO}(2)$ are fully computed for $k \leqslant 4$, and $\pi_k \,\mathrm{MTSO}(3)$ is obtained for $k \leqslant 1$.

Theorem 3.0.8. *The values of $\pi_k \,\mathrm{MTSO}(2)$ for $k \leqslant 4$ are given by*

k	$\leqslant -3$	-2	-1	0	1	2	3	4
$\pi_k \,\mathrm{MTSO}(2)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}/24$	$\mathbb{Z}$

The values of $\pi_k \,\mathrm{MTSO}(3)$ for $k \leqslant 1$ are

k	$\leqslant -4$	-3	-2	-1	0	1
$\pi_k \,\mathrm{MTSO}(3)$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Moreover, $\pi_1 \,\mathrm{MTSO}(3)$ is generated by a class with signature 2 and $\pi_2 \,\mathrm{MTSO}(2)$ is generated by a class with signature 4.

The values of π_k MTSO(2) for $k \leqslant 3$ had been known (cf. [20, Corollary 4.4]), but the details had not been published. The values for $k < 0$ and $k = 0$ follow from general principles (Corollary 1.2.5 and Proposition 1.2.7, respectively), so the interesting computations are when $k \geqslant 1$. In order to perform these computations, we derive in Section 3.1 a cofibre sequence of spectra of the form

$$\Sigma^{-1} \mathrm{MT}\,\theta_{d-1} \to \mathrm{MT}\,\theta_d \to \Sigma^\infty B(d)_+ \to \mathrm{MT}\,\theta_{d-1}$$

for a fibration $\theta_d \colon B(d) \to \mathrm{BO}(d)$ and its restriction θ_{d-1} to $\mathrm{BO}(d-1)$. This cofibre sequence was given in [7] when θ_d is the identity $\mathrm{BO}(d) \to \mathrm{BO}(d)$ or the universal covering $\mathrm{BSO}(d) \to \mathrm{BO}(d)$. Full details are given in Section 3.1 because they don't appear in the literature, and moreover the explicit descriptions given are used in Section 3.2 to provide bordism-level interpretations of the induced maps on homotopy groups. The resulting descriptions of these homomorphisms are then used to perform the computations of Section 3.3.

Finally, the appendix provides a result about oriented vector bundles which was not found in the literature and which is needed for the above proofs. Namely:

Theorem A2. *Let n be even, let X be CW complex of dimension n, and let V_0, V_1 be two oriented rank n vector bundles over X. If V_0 is stably isomorphic to V_1 and they have the same Euler class, then $V_0 \cong V_1$.*

The appendix presents standard properties of the Euler class for completion, and proves a basic lemma about homotopy fibres (Lemma A.2.1). These ingredients are used in the proof of Theorem A2, and are also used to prove the more standard fact Theorem A1, which asserts that the Euler class is the principle obstruction to finding a non-zero section of an oriented vector bundle.

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Contents

Chapter 1

Preliminaries

1.1 Fundamental Notions

This section does not present anything new, but gathers some necessary results and sets conventions. As such, the material does not attempt to follow a coherent narrative.

1.1.1 Homotopy

A decent source for the material in the subsection is Hatcher [11].

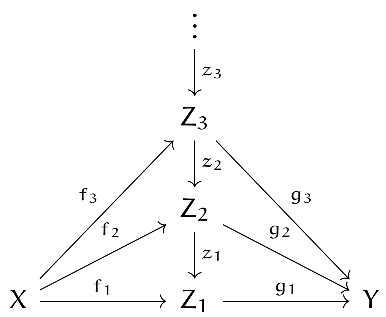
We will say that X is an n-**connected space** if $\pi_i(X) = 0$ for all $i \leqslant n$. We will say that f is an n-**connected map** if $\pi_i(f)$ is an isomorphism for all $i < n$ and a surjection for $i = n$; f is n-**coconnected** if $\pi_n(f)$ is an injection and $\pi_i(f)$ is an isomorphism for all $i > n$.

Let $f\colon (X, x_0) \to (Y, y_0)$ be a continuous function between pointed spaces. The **homotopy fibre** of f is defined as

$$\mathrm{hofib}(f) := \{(x, \gamma) \in X \times Y^I \mid \gamma(0) = f(x) \text{ and } \gamma(1) = y_0\}$$

This space is pointed by (x_0, y_0) where y_0 denotes the constant path at y_0.

Let $f\colon X \to Y$ be a continuous function between path-connected, pointed spaces. A **Moore-Postnikov tower** for f (see [11, Section 4.3]) is a commutative diagram of the form



so that for each $n \geqslant 1$

1. $g_n \circ f_n \sim f$,
2. $f_n \colon X \to Z_n$ is n-connected,
3. $g_n \colon Z_n \to Y$ is n-coconnected,
4. $z_n \colon Z_{n+1} \to Z_n$ is a fibration, with fibre $K(\pi_n \operatorname{hofib}(f), n)$.

The composition $g_n \circ f_n \colon X \to Z_n \to Y$ will be called the **n-th Moore-Postnikov stage** or **decomposition** of f. Then a map between connected CW complexes has a Moore-Postnikov tower [11, Theorem 4.71] , which is unique up to homotopy equivalence; moreover the fibrations z_n can be chosen as principal fibrations if $\pi_1(X)$ acts trivially on $\pi_n(\mathrm{Cyl}(f), X)$ for all n.

A map $f \colon Y \to X$ is an **n-connected cover** (cf [11, Example 4.20]) if Y is n-connected and $\pi_i(f)$ is an isomorphism for all $i > n$, i.e. f is n-coconnected since 0 certainly injects into $\pi_n X$. In other words, an n-connected cover is the n-th Moore-Postnikov stage of a map $* \to X$. If X is a connected CW complex then an n-connected cover exists and is unique up to homotopy, and we will denote any model by $X\langle n\rangle$.

1.1.2 Bundles

Relevant sources for bundle theory are Steenrod [31] and Husemoller [13].

If $E \to X$ is a fibre bundle, a typical element will be denoted by $(x; e)$ where $x \in X$ and e is in the fibre over x. The information to the left of the semi-colon is redundant, but it helps conceptually.

The trivial vector bundle $X \times \mathbb{R}^n$ will usually be denoted by ε^n, without reference to the base-space.

If $E \to X$ is a bundle and $f \colon Y \to X$ a continuous map, then the canonical map from the pull-back to V will be denoted $\tilde{f} \colon f^*V \to V$.

Vector bundles E and F of the same rank are said to be **stably isomorphic** if there is a $k \geqslant 0$ and an isomorphism $\psi \colon E \oplus \varepsilon^k \cong F \oplus \varepsilon^k$. ψ is called a **stable isomorphism**, and the relation will be denoted by $E \cong_s F$. If E and F are not of the same rank, we can write $E \cong_s F \oplus \varepsilon^r$ where $r = \mathrm{rank}(E) - \mathrm{rank}(F) \in \mathbb{Z}$; if $r < 0$ what is really meant is that $E \oplus \varepsilon^{-r} \cong_s F$, but for some purposes it is convenient to have a consistent notation.

If E is a vector bundle, the **Thom space** $\mathrm{Th}(E)$ can be described by one-point compactifying each fibre and then identifying all of the compactification points; if the base space is already compact, this is the same as the one-point compactification of the total space. If E has a metric, this is homeomorphic to $D(E)/S(E)$, the disk-bundle of E modulo the sphere-bundle. If $\varphi \colon E \to F$ is a bundle map, it induces a map $\mathrm{Th}(\varphi) \colon \mathrm{Th}(E) \to \mathrm{Th}(F)$ between Thom spaces.

Lemma 1.1.1. *If $E \to X$ is a vector bundle of rank $r \geqslant 2$ then $\pi_1 \mathrm{Th}(E) = 0$.*

Proof. Let $x_0 \in X$ and $s_0 \in S(E)$ be basepoints so that $p \colon S(E) \to X$ is a pointed map. Expressing the Thom space as a pushout results in a pushout of fundamental groups by the Seifert-van Kampen theorem:

$$\begin{array}{ccc} S(E) & \longrightarrow & C(S(E)) \\ \downarrow & & \downarrow \\ D(E) & \longrightarrow & \mathrm{Th}(E) \end{array} \qquad\qquad \begin{array}{ccc} \pi_1 S(E) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \pi_1 X & \longrightarrow & \pi_1 \mathrm{Th}(E) \end{array}$$

Then $\pi_1 \operatorname{Th}(E) = 0$ iff $\pi_1 S(E) \to \pi_1 X$ is surjective. But given a loop $\gamma\colon I \to X$ at x_0, the path lifting property gives a lift $\tilde{\gamma}\colon I \to S(E)$ such that $\tilde{\gamma}(0) = s_0$ and $\tilde{\gamma}(1) \in p^{-1}(x_0)$. But so long as $r \geqslant 2$ the fibre over x_0 will be path connected, establishing surjectivity. □

Lemma 1.1.2. *Let $V \to Y$ be a vector bundle of rank $r \geqslant 2$ with Y connected and let $f\colon X \to Y$ be an m-connected continuous function. Then the map*

$$\operatorname{Th}(\tilde{f})\colon \operatorname{Th}(f^*V) \to \operatorname{Th}(V)$$

is $(r+m)$-connected.

In the particular case that V is trivial we see that $\Sigma^r f\colon \Sigma^r X_+ \to \Sigma^r Y_+$ is $(r+m)$-connected for any m-connected map $f\colon X \to Y$.

Proof. Since f is m-connected, $f_*\colon H_k(X; f^*\mathcal{A}) \to H_k(Y;\mathcal{A})$ is an isomorphism for any coefficient system $\mathcal{A}$ on Y and $k < m$, and is surjective for $k = m$. The bundle $V \to Y$ induces an orientation character $\omega\colon \pi_1 Y \to \mathbb{Z}/2$ and coefficient system $\mathbb{Z}^\omega$; the pullback f^*V has orientation character $f^*\omega$ and coefficient system $\mathbb{Z}^{f^*\omega} \cong f^*(\mathbb{Z}^\omega)$ for X. Then using the Thom isomorphism with twisted coefficients (e.g. [19, Theorem 3.31]) there is the commutative diagram:

$$\begin{array}{ccc} \tilde{H}_k(\operatorname{Th}(f^*V);\mathbb{Z}) & \xrightarrow{\operatorname{Th}(\tilde{f})_*} & \tilde{H}_k(\operatorname{Th}(V);\mathbb{Z}) \\ \downarrow\cong & & \downarrow\cong \\ H_{k-r}(X; f^*\mathbb{Z}^\omega) & \xrightarrow{f_*} & H_{k-r}(Y;\mathbb{Z}^\omega) \end{array}$$

The lower map is an isomorphism for all $k < r+m$ and surjective for $k = r+m$, and so the same holds for the upper map. If $r \geqslant 2$ then the Thom spaces are simply-connected and therefore $\operatorname{Th}(\tilde{f})$ is $(r+m)$-connected by [33, Theorem 10.28]. □

We will also use the following fact about normal bundles:

Lemma 1.1.3 ([16, IV.1.4]). *Let $f\colon X \to Y$ be a smooth map of manifolds, and Z a submanifold of Y such that $f \pitchfork Z$; let $M = f^{-1}(Z)$. Then*

$$\nu_M^X \cong (f^*\nu_Z^Y)|_M$$

In particular, $\dim(M) = \dim(X) + \dim(Z) - \dim(Y)$ *assuming this number is non-negative, and M is empty otherwise.*

In the special case that X is the total space of a vector bundle $\xi\colon X \to Y$, and Z is the zero-section, then $\nu_M^X \cong f^*\xi|_M$ and it follows that $\dim(M) = \dim(X) - \operatorname{rank}(\xi)$.

1.1.3 Stiefel and Grassmann Manifolds

A lot of our constructions will involve Grassmannians so it pays to have some familiarity. For a reference on Stiefel and Grassmanian manifolds, see Husemoller [13, Chapter 7].

In order to prove Lemma 3.1.6 we will need to know the connectivities of two families of maps; Corollary 1.1.6 and Lemma 1.1.7 provide these connectivities, and are presented here to help the readability of Section 3.1.

Let $sh_+ \colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ be the map sending the vector $\sum_i a_i e_i$ to $\sum_i a_i e_{i+1}$; on the orthogonal complement of e_1 this map has an inverse $sh_- \colon \langle e_2, \ldots, e_{n+1} \rangle \to \mathbb{R}^n$.

For any $n \geqslant d \geqslant 0$ denote by $St_d(\mathbb{R}^n)$ the Stiefel manifold of orthonormal d-frames in $\mathbb{R}^n$, pointed by $(e_1, \ldots, e_d)$ and topologized as the homogenous space $O(n)/O(n-d)'$, where $O(n-d)'$ is the subgroup of matrices which are concentrated in the lower right $(n-d)\times(n-d)$ block. Note that $St_d(\mathbb{R}^d) = O(d)$ and $St_1(\mathbb{R}^n) = S^{n-1}$.

There are two "stabilization" maps we want to have: one which increases the ambient dimension, and one which increases ambient dimension as well as the dimension of the planes. If we demand that both the maps be pointed, the natural choice for

$$a_n \colon St_d(\mathbb{R}^n) \to St_d(\mathbb{R}^{n+1})$$

sends a d-frame in $\mathbb{R}^n$ to itself, as a subset of $\mathbb{R}^{n+1}$. The other stabilization map is trickier to define: one does not simply concatenate e_{n+1} to the end of any d-frame in $\mathbb{R}^n$ because this map is not pointed. Instead define the pointed map

$$s_n \colon St_d(\mathbb{R}^n) \to St_{d+1}(\mathbb{R}^{n+1})$$

by sending $(v_1, \ldots, v_d)$ to $(e_1, sh_+(v_1), \ldots, sh_+(v_d))$.

Aside: if we had chosen to use $(e_{n-d+1}, \ldots, e_n)$ as the basepoint of $St_d(\mathbb{R}^n)$ then we could have defined s_n by concatenating with e_{n+1}, but we still would have had to use sh_+ in order to make a_n pointed. A convention must be chosen and we chose the above one.

Lemma 1.1.4. *s_n is $(n-1)$-connected*

Proof. The map $St_{d+1}(\mathbb{R}^{n+1}) \to St_1(\mathbb{R}^{n+1})$ sending $(v_1, \ldots, v_{d+1})$ to v_1 is a fibre bundle (follows from [13, Theorem 7.3.8]). The fibre over the basepoint e_1 is the set of all $(e_1, v_2, \ldots, v_{d+1})$ where $(v_2, \ldots, v_{d+1})$ is a d-frame in $\langle e_2, \ldots, e_{d+1} \rangle$. Thus the map sh_- identifies the fibre and its inclusion map with $St_d(\mathbb{R}^n)$ and s_n. The connectivity follows from the long exact sequence for this fibre bundle and the fact that $St_1(\mathbb{R}^{n+1}) \cong S^n$. □

In particular the map $O(d) \to O(d+1)$ sending a matrix A to $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ is $(d-1)$-connected.

The manifold $St_d(\mathbb{R}^n)$ has a free $O(d)$ action: if a d-frame is considered as an $n \times d$ matrix, then $d \times d$ matrices can multiply from the right. The quotient manifold is $Gr_d(\mathbb{R}^n)$, the Grassmannian of d-dimensional subspaces of $\mathbb{R}^n$. That is, the map

$$\langle - \rangle \colon St_d(\mathbb{R}^n) \to Gr_d(\mathbb{R}^n)$$

sending a d-frame to the subspace it spans is a principal $O(d)$ bundle, and $Gr_d(\mathbb{R}^n)$ is identified with $O(n)/\big(O(d) \times O(n-d)\big)$. Define

$$\psi_n \colon Gr_d(\mathbb{R}^n) \to Gr_{d+1}(\mathbb{R}^{n+1})$$

by $\psi_n(P) = \langle e_1 \rangle \oplus sh_+(P)$.

Lemma 1.1.5. ψ_n *is* $(d-1)$*-connected.*

Proof. There is a map of fibre-bundles

$$\begin{array}{ccccc} O(d) & \longrightarrow & St_d(\mathbb{R}^n) & \longrightarrow & Gr_d(\mathbb{R}^n) \\ \downarrow & & \downarrow s_n & & \downarrow \psi_n \\ O(d+1) & \longrightarrow & St_{d+1}(\mathbb{R}^{n+1}) & \longrightarrow & Gr_{d+1}(\mathbb{R}^{n+1}) \end{array}$$

The map $O(d) \to O(d+1)$ is $(d-1)$-connected and s_n is $(n-1)$ connected, and $n \geqslant d$ so in particular s_n is also $(d-1)$ connected. Then the long exact sequences of these bundles and the Five-lemma yield the desired connectivity. □

There is an inclusion map $\iota_n\colon Gr_d(\mathbb{R}^n) \to Gr_d(\mathbb{R}^{n+1})$ induced by the standard inclusion $\mathbb{R}^n \subset \mathbb{R}^{n+1}$. The colimit over these inclusion maps is naturally homeomorphic to $Gr_d(\mathbb{R}^\infty)$, the Grassmannian of d-dimensional subspaces of $\bigoplus_{i=0}^{\infty} \mathbb{R}$.

Corollary 1.1.6. ι_n *is* $(n-d-1)$*-connected. In particular the map into the colimit* $Gr_d(\mathbb{R}^n) \to Gr_d(\mathbb{R}^\infty)$ *is* $(n-d-1)$*-connected.*

Proof. The map $\perp\colon Gr_d(\mathbb{R}^n) \to Gr_{n-d}(\mathbb{R}^n)$, sending a d-plane to its orthogonal complement, is a diffeomorphism. Then the map ι_n is equivalent to

$$-\oplus \langle e_{n+1}\rangle\colon Gr_{n-d}(\mathbb{R}^n) \to Gr_{n-d+1}(\mathbb{R}^{n+1})$$

which is shown to be $(n-d-1)$-connected by using the argument in Lemma 1.1.5 with different basepoints. □

$Gr_d(\mathbb{R}^\infty)$ is a model for $BO(d)$, with tautological vector bundle γ_d; let $U_{d,n}$ be the restriction of γ_d to $Gr_d(\mathbb{R}^n)$. Since n is finite, $U_{d,n}$ has a finite-dimensional orthogonal complement $U_{d,n}^\perp$, an $(n-d)$-plane bundle. It is immediate that $\iota_n^* U_{d,n+1}$ is canonically isomorphic to $U_{d,n}$ and that the map

$$\phi_n\colon U_{d,n}^\perp \oplus \varepsilon \cong \iota_n^* U_{d,n+1}^\perp$$

defined by $\phi_n(P; w, t) = (P; w + te_{n+1})$ is also an isomorphism.

The pull back $\psi_n^* U_{d+1,n+1}^\perp$ is the set of $(P, (P'; w)) \in Gr_d(\mathbb{R}^n) \times U_{d+1,n+1}^\perp$ such that $P' = \langle e_1\rangle \oplus sh_+(P)$. Since $w \perp P'$, w is in the domain of sh_- and $sh_-(w) \perp P$. Hence there is an isomorphism

$$\psi_n^* U_{d+1,n+1}^\perp \cong U_{d,n}^\perp$$

sending $(P, (P'; w))$ to $(P; sh_-(w))$.

Lemma 1.1.7. *The map* $f_n\colon Gr_d(\mathbb{R}^n) \to S(U_{d+1,n+1})$ *sending* P *to* $(\psi_n(P); e_1)$ *is* $(n-1)$*-connected.*

Proof. The sphere bundle can be described using the Borel construction as follows:

$$\begin{aligned} S(U_{d+1,n+1}) &\cong St_{d+1}(\mathbb{R}^{n+1}) \times_{O(d+1)} S^d \\ &\cong O(n+1)/O(n-d)' \times_{O(d+1)} O(d+1)/O(d) \\ &\cong O(n+1)/(O(d) \times O(n-d))' \end{aligned}$$

Then the map of the lemma can be identified with

$$\frac{O(n)}{O(d)\times O(n-d)} \to \frac{O(n+1)}{O(d)\times O(n-d)}$$

so there is a fibre bundle

$$\mathrm{Gr}_d(\mathbb{R}^n) \to S(U_{d+1,n+1}) \to O(n+1)/O(n) \cong S^n$$

Now apply the long exact sequence of homotopy groups. □

1.1.4 The Vertical Tangent Bundle of a Vector Bundle

This is another result which will be used in exactly one place (Lemma 3.2.4) but whose proof would further clutter the section where it appears. It is also sort of interesting on its own.

Let $\pi\colon E \to X$ be a fibre bundle where the fibres are modelled by a smooth manifold M and the structure group is Diff(M). Let P denote the underlying principal Diff(M) bundle.

Definition 1.1.8. *Define the **vertical tangent bundle** of E as*

$$T_v E := P \times_{\mathrm{Diff}(M)} TM$$

If X is a smooth manifold and π is differentiable, then $T_v E \cong \ker T\pi$.

Consider the specific case where the model fibre is the smooth manifold $\mathbb{R}^n$ and the structure group is $O(n)$, i.e. E is a vector bundle with metric. Let $p\colon S(E) \to X$ denote the projection of the sphere bundle, an S^{n-1} bundle with structure group $O(n)$. Then the bundle p^*E admits a canonical non-zero section $\sigma\colon S(E) \to p^*E$ defined by

$$\sigma(x;u) = (x,u;u)$$

and which induces a decomposition $p^*E = \langle\sigma\rangle^\perp \oplus \langle\sigma\rangle$.

Proposition 1.1.9. *If $E \to X$ is a vector bundle with metric and $p\colon S(E) \to X$ is the sphere bundle, then there is a bundle-isometry $T_vS(E) \cong \langle\sigma\rangle^\perp$, and hence an isomorphism*

$$p^*E \cong T_vS(E) \oplus \varepsilon$$

of bundles over $S(E)$.

The proof goes as follows: for $(x;v) \in E$ there is a canonical identification $i\colon T_{(x,v)}E_x \cong E_x$ since E_x is a vector space, and hence an embedding

$$I\colon \bigcup_x E_x \times E_x \cong \bigcup_x T(E_x) \to TE$$

Say that a smooth path $\gamma\colon \mathbb{R} \to E$ is **vertical** if $\dot\gamma_0 \in \mathrm{Im}\, I$; it follows that T_vE is isomorphic to the bundle of equivalence classes of vertical paths in TE.

If we have a vertical path γ which is based at $(x;\nu)$ then $\dot{\gamma}_0 \in E_x$, hence we can define an evaluation map

$$\begin{aligned} ev\colon T_\nu E &\to \pi^* E \\ [\gamma] &\to (\gamma_0; \dot{\gamma}_0) \end{aligned}$$

This map is a bundle isomorphism, and if the vector bundle E is given a metric it induces a metric on $T_\nu E$ so that ev is an isometry.

If we restrict the picture to $S(E)$ we get an isomorphism

$$ev|_{S(E)}\colon (T_\nu E)|_{S(E)} \cong p^* E$$

$T_\nu S(E)$ embeds into $(T_\nu E)|_{S(E)}$ as a subbundle; denote by ν its orthogonal complement. The bundle p^*E has a canonical section $\sigma(x;u) = (x, u; u)$ as above, with image in $S(p^*E)$.

Lemma 1.1.10. $ev|_{S(E)}(\nu) = \langle\sigma\rangle$

Proof. Let $(x;u) \in S(E)$, and define $\gamma_u\colon \mathbb{R} \to E$ by $\gamma_u(t) = (1+t)\cdot u$. Then γ_u is a vertical path since it is entirely contained in E_x, and $\dot{\gamma}_u(0) \perp S(E_x)$ so $[\gamma_u] \in \nu$. Since ν is one-dimensional we have $\nu|_{(x;u)} = \langle[\gamma_u]\rangle$. But

$$ev([\gamma_u]) = (\gamma_u(0); \dot{\gamma}_u(0)) = (x, u; u) = \sigma(x; u)$$

□

Then since ev is an isometry, it restricts to an isomorphism

$$ev|_{S(E)}\colon T_\nu S(E) \cong \langle\sigma\rangle^\perp$$

proving the proposition.

1.1.5 The Signature

In this section we recall the definition of the signature of a closed, oriented manifold, and state Hirzebruch's Signature Formula.

Let M be a closed, oriented manifold of dimension $n = 4k$. Since it is closed and oriented, it has a fundamental class $[M] \in H_n(M;\mathbb{Z})$. Recall its **interestion form**

$$I(-,-)\colon H^{2k}(M;\mathbb{Z}) \times H^{2k}(M;\mathbb{Z}) \to \mathbb{Z}$$

is given by $I(\alpha,\beta) = \langle\alpha \cup \beta, [M]\rangle$. Then I is a symetric, unimodular bilinear form, and as such can be diagonalized over $\mathbb{Q}$: let b^+ be the number of positive entries on the diagonal, and b^- the number of nagative entries. The numbers b^+ and b^- do not depend on the diagonalization.

Definition 1.1.11. *Let M be a closed, oriented manifold of dimension n.*
*If $n \in 4\mathbb{N}$ then the **signature** of M, denoted $\sigma(M)$, is defined to be $b^+ - b^-$.*
Otherwise, $\sigma(M)$ is defined to be 0.

Hirzebruch's Signature formula (see for example [24, 19.4]) relates the signature of a manifold to its Pontryagin classes. Very roughly it can be stated as follows:

Theorem 1.1.12 (Hirzebruch). *Let $x_1, x_2, x_3, \dots$ be a set of variables where x_i has degree i. Then for each $k \geqslant 0$ there is a rational polynomial $L_k \in \mathbb{Q}[x_1, \dots, x_k]$ which is homogeneous of degree k so that for any closed, smooth, oriented manifold M^{4k}*

$$\langle L_k\big(p_1(TM), \dots, p_k(TM)\big), [M]\rangle = \sigma(M)$$

The first two polynomials are given by

$$L_1(x_1) = \frac{1}{3}x_1$$
$$L_2(x_1, x_2) = \frac{1}{45}(7x_2 - x_1^2)$$

An immediate corollary of this theorem is that if M is stably parallelizable then $\sigma(M) = 0$. We will not need the explicit formulas for any other L_k.

1.2 The Spectrum $\mathrm{MT}\,\theta_d$

This section recalls the definition of the central objects of interest for this thesis: namely for each integer $d \geqslant 0$ and any "tangential structure" $\theta_d \colon B(d) \to BO(d)$ there is an associated Madsen-Tillmann-Weiss spectrum $\mathrm{MT}\,\theta_d$.

The homotopy groups of these spectra admit interpretations as bordism groups (see Proposition 1.2.3) with objects of the form (M, E, φ), where M is a closed manifold, E is a bundle over M with "θ_d-structure", and φ is a stable isomorphism between TM and E; this will be made precise in Subsection 1.2.2. This bordism interpretation is a starting point for the Signature Problem of Chapter 2, and is fundamental to the computations of Chapter 3.

1.2.1 Definition of Madsen-Tillmann-Weiss Spectra

The notion of stable vector bundle is covered in [29, IV.5.12], where they are refered to as "stable $\mathcal{F}$-objects". For our purposes, a **stable vector bundle** $\mathcal{V}$ will mean a sequence of spaces

$$X_k \xrightarrow{i_k} X_{k+1} \xrightarrow{i_{k+1}} X_{k+2} \xrightarrow{i_{k+2}} \dots$$

for some $k \in \mathbb{Z}$, with a set of vector bundles $\{p_n \colon V_n \to X_n\}_{n \geqslant k}$, and for each n an isomorphism $\phi_n \colon V_n \oplus \varepsilon \cong i_n^* V_{n+1}$. The isomorphism ϕ_n ensures that $\mathrm{rank}(V_{n+1}) = \mathrm{rank}(V_n) + 1$ and so $\mathrm{rank}(\mathcal{V}) := \mathrm{rank}(V_n) - n \in \mathbb{Z}$ is well-defined. The data of a stable vector bundle is precisely enough to produce a **Thom spectrum** $\mathrm{Th}(\mathcal{V})$, whose n-th space is $\mathrm{Th}(V_n)$ and whose n-th structure map is induced by $\mathrm{Th}(\phi_n)$. Then the spectrum $\mathrm{Th}(\mathcal{V})$ is $(\mathrm{rank}(\mathcal{V})-1)$-connected.

The bundles $U_{d,n}^{\perp} \to \mathrm{Gr}_d(\mathbb{R}^n)$, along with the maps ι_n and ϕ_n, form a stable vector bundle of rank $-d$, denoted here by $-\gamma_d$.

Definition 1.2.1. *The **unstructured Madsen-Tillmann-Weiss spectrum** $\mathrm{MTO}(d)$ is defined as the Thom spectrum $\mathrm{Th}(-\gamma_d)$.*

One typically wants to consider bundles with extra structure, for example an orientation. For this purpose a d-**dimensional tangential structure** is simply a fibration $\theta_d \colon B \to BO(d)$; for a rank d vector bundle $V \colon X \to BO(d)$ a

θ_d-**structure** is a bundle map $l\colon V \to \theta_d^*\gamma_d$. The prototypical examples are $BSO(d)$, $BSpin(d)$, and other connected covers $BO(d)\langle n\rangle \to BO(d)$ (indeed $BSO(d) = BO(d)\langle 1\rangle$ and $BSpin(d) = BO(d)\langle 2\rangle$).

A **stable tangential structure** is a fibration $\theta\colon B \to BO$, and a stable tangential structure can be restricted

$$\theta_d\colon B(d) := \theta^{-1}(BO(d)) \to BO(d)$$

to a d-dimensional tangential structure along the map into the colimit $BO(d) \to BO$. Note that if θ is a *pointed* map then any trivial bundle, classified by the constant map to $BO(d)$, has a preferred θ_d-structure.

Suppose $\theta_d\colon B(d) \to BO(d)$ is a d-dimensional tangential structure, and let $\theta_{d-1}\colon B(d-1) \to BO(d-1)$ be the pullback along $s_{d-1}\colon BO(d-1) \to BO(d)$. Now suppose E is a θ_{d-1}-bundle over a space X classified by a map $f\colon X \to B(d-1)$, that is $E \cong f^*\theta_{d-1}^*\gamma_{d-1}$; then the bundle $\varepsilon \oplus E$ naturally has a θ_d-structure by commutativity of

$$\begin{array}{ccccc} X & \xrightarrow{f} & B(d-1) & \xrightarrow{\tilde{s}_{d-1}} & B(d) \\ & & \downarrow{\scriptstyle \theta_{d-1}} & & \downarrow{\scriptstyle \theta_d} \\ & & BO(d-1) & \xrightarrow{s_{d-1}} & BO(d) \end{array}$$

In particular, if $\theta\colon B \to BO$ is stable then the stabilization of any finite-rank θ-bundle has a natural θ-structure.

Given a tangential structure $\theta_d\colon B \to BO(d)$, a stable vector bundle is constructed analogously to $-\gamma_d$ as follows: $BO(d)$ is filtered by the finite-dimensional Grassmannians $\iota_n\colon Gr_d(\mathbb{R}^n) \subset Gr_d(\mathbb{R}^{n+1})$, so B is also filtered by $B_n = \theta_d^{-1} Gr_d(\mathbb{R}^n)$; let $\theta_{d,n}$ denote the restriction $\theta_d|_{B_n}$. Then the inclusion $\lambda_n\colon B_n \to B_{n+1}$ covering ι_n is itself covered by a bundle isomorphism/map

$$\begin{array}{ccc} \theta_{d,n}^* U_{d,n}^\perp \oplus \mathbb{R} \cong \lambda_n^*\theta_{d,n+1}^* U_{d,n+1}^\perp & \xrightarrow{\tilde{\lambda}_n \circ \theta_{d,n}^* \phi_n} & \theta_{d,n+1}^* U_{d,n+1}^\perp \\ \downarrow & & \downarrow \\ B_n & \xrightarrow{\lambda_n} & B_{n+1} \end{array}$$

Thus the set $\{\theta_{d,n}^* U_{d,n}^\perp\}$ forms a stable vector bundle over $\{B_n\}$, denoted $\theta_d^*(-\gamma_d)$. For brevity, let $U_{d,n}^\theta = \theta_{d,n}^* U_{d,n}$ and $U_{d,n}^{\theta,\perp} = \theta_{d,n}^* U_{d,n}^\perp$.

Definition 1.2.2. *The* ***Madsen-Tillmann-Weiss spectrum with*** d***-dimensional tangential structure*** θ_d *is defined as* $MT\,\theta_d = Th(\theta_d^*(-\gamma_d))$.

1.2.2 Bordism Interpretation of Homotopy Groups

Since $\mathrm{MT}\,\theta_d$ is a Thom spectrum, its homotopy groups can be described using the Pontryagin-Thom correspondence (see for example [32, Chapter 2]).

Proposition 1.2.3. *Let* $\theta_d \colon B \to \mathrm{BO}(d)$ *be a tangential structure. Then the group* $\pi_k \mathrm{MT}\,\theta$ *is isomorphic to the bordism group of triples* $[M, E, \varphi]$ *where*

- M *is a closed, smooth manifold of dimension* $k + d$*,*
- E *is a rank* d *vector bundle over* M *with* θ_d*-structure,*
- φ *is a stable isomorphism* $TM \cong_s E \oplus \varepsilon^k$*.*

A triple is null-bordant if there is a $(k + d + 1)$*-manifold* W*, a rank* d θ*-bundle* F *and a stable isomorphism* $\psi \colon TW \cong_s F \oplus \varepsilon^{k+1}$ *which restricts to the given data on the boundary.*

Proof sketch. Here we will give an indication of where this data comes from; for more complete details see [32].

By definition $\pi_k \mathrm{MT}\,\theta_d = \mathrm{colim}_n\, \pi_{n+k}\, \mathrm{Th}(U^{\theta,\perp}_{d,n})$, so choose a pointed map

$$\Phi \colon S^{n+k} \to \mathrm{Th}(U^{\theta,\perp}_{d,n})$$

for $n \gg d$. Without loss of generality the composition of Φ with $\mathrm{Th}(\tilde{\theta}) \colon \mathrm{Th}(U^{\theta,\perp}_{d,n}) \to \mathrm{Th}(U^{\perp}_{d,n})$ is transverse to the zero section $\mathrm{Gr}_d(\mathbb{R}^n)$. Let

$$M = (\Phi \circ \mathrm{Th}(\tilde{\theta}))^{-1}(\mathrm{Gr}_d(\mathbb{R}^n)) \subset \mathbb{R}^{n+k}$$

and let f be the restriction of this composition to M. Then M is a closed manifold of dimension $n + k - (n - d) = k + d$, and $\nu_M^{S^{n+k}} \cong f^* U^{\perp}_{d,n}$. Then since M is a proper subset of S^{n+k} we have

$$\varepsilon^{n+k} \cong TS^{n+k}|_M \cong TM \oplus f^* U^{\perp}_{d,n}$$

By adding $f^* U_{d,n}$ to both sides we get an isomorphism

$$\varphi \colon TM \oplus \varepsilon^n \cong \varepsilon^{n+k} \oplus f^* U_{d,n}$$

Note that since the bundle map $U^{\theta}_{d,n} \to U_{d,n}$ is an isomorphism in each fibre, f in fact factors through B_n so $f^* U_{d,n}$ comes with a θ_d-structure. □

Note that if $k < 0$ then $\dim(M) < \mathrm{rank}(E)$, and so $E \cong TM \oplus \varepsilon^{-k}$ by obstruction theory.

Definition 1.2.4. *For a tangential structure* $\theta_d \colon B(d) \to \mathrm{BO}(d)$ *and a natural number* $n \leqslant d$*, define the* n***-th*** θ_d ***bordism group*** $\Omega_n^{\theta_d}$ *as the bordism group of closed* n*-manifolds* M *with a* θ_d *structure on* $TM \oplus \varepsilon^{d-n}$*.*

Corollary 1.2.5. *If* $k < 0$ *then* $\pi_k \mathrm{MT}\,\theta_d \cong \Omega^{\theta_d}_{d+k}$

Now let $k = 0$, and consider a connected, stable tangential structure

$$\theta \colon B \to \mathrm{BSO} \to \mathrm{BO}$$

factoring through Orientation; then the restriction $\theta_d \colon B(d) \to \mathrm{BO}(d)$, for any $d \geqslant 0$, also factors through $\mathrm{BSO}(d)$.

Definition 1.2.6. *Let $\mathrm{Eul}_n^\theta \subset \mathbb{Z}$ be the set of all $\langle e(V), [M]\rangle$ such that M is a closed n-dimensional manifold, and $V \to M$ is a rank n vector bundle which is stably isomorphic to TM and which admits a θ_n structure.*

Denote by Eul_n the set corresponding to $B = \mathrm{BSO}$ and θ the identity.

It follows that Eul_n is the set of all $\chi(M)$ where M is an oriented closed manifold. As stated in [6], Eul_n is given by

$$\mathrm{Eul}_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \mathbb{Z} & \text{if } n \equiv 0 \mod 4 \\ 2\mathbb{Z} & \text{if } n \equiv 2 \mod 4 \end{cases}$$

In general, it will follow from Lemma 3.2.3 that Eul_n^θ is a subgroup of $\mathbb{Z}$.

Proposition 1.2.7. *Let $\theta\colon B \to \mathrm{BSO}$ be a stable tangential structure, and let θ_d, θ_{d+1} be its restriction to $\mathrm{BSO}(d)$ and $\mathrm{BSO}(d+1)$ respectively, and suppose $B(d+1) = \theta^{-1}\,\mathrm{BSO}(d+1)$ is connected. Then there is a short-exact sequence*

$$0 \to \mathbb{Z}/\,\mathrm{Eul}_{d+1}^\theta \to \pi_0\,\mathrm{MT}\,\theta_d \to \Omega_d^{\theta_{d+1}} \to 0$$

Moreover this sequence is split, except possibly for the case where $d+1 \in 4\mathbb{N}$. If $B = \mathrm{BSO}$ and θ is the identity then it always splits.

Proof. A proof for $B(d+1) = \mathrm{BSO}(d+1)$ is given in Appendix A of [6], and the proof given in this thesis is essentially the same. Since the proof will require the interpretations of Section 3.2, it will be presented there. □

Higher homotopy groups are harder to describe explicitly.

By taking the bordism interpretation of these groups, one can attempt to define an external product. Given a stable tangential structure $\theta\colon B \to \mathrm{BO}$ and elements $[M, E, \varphi] \in \pi_k\,\mathrm{MT}\,\theta_d$ and $[N, F, \psi] \in \pi_l\,\mathrm{MT}\,\theta_e$, the tuple

$$(M \times N, E \times F, \varphi \times \psi)$$

defines an element of $\pi_{k+l}\,\mathrm{MT}\,\theta_{d+e}$ provided that $E \times F$ is equipped with a θ_{d+e} structure. E and F being θ bundles means there are lifts $\lambda_E\colon M \to B(d)$ and $\lambda_F\colon N \to B(e)$ of their classifying maps, giving a diagram

$$\begin{array}{ccccc} & & B(d) \times B(e) & \overset{\mu}{\dashrightarrow} & B(d+e) \\ & \overset{\lambda_E \times \lambda_F}{\nearrow} & \downarrow & & \downarrow \\ M \times N & \xrightarrow{E \times F} & \mathrm{BO}(d) \times \mathrm{BO}(e) & \xrightarrow{B\oplus} & \mathrm{BO}(d+e) \end{array}$$

where $\oplus\colon O(d) \times O(e) \to O(d+e)$ is the block-sum. If there were a map μ covering $B\oplus$, then $\mu \circ (\lambda_E \times \lambda_F)$ would be a θ_{d+e} structure on $E \times F$.

Definition 1.2.8. *A stable tangential structure $\theta\colon B \to \mathrm{BO}$ is **multiplicative** if there is given, for all $d, e \geqslant 0$, a map $\mu_{d,e}\colon B(d) \times B(e) \to B(d+e)$ covering $B\oplus$.*

Examples are BSO, BSpin; more generally if $h\colon H \to \mathrm{BO}$ is an H-space homomorphism then h is a multiplicative tangential structure.

Definition 1.2.9. *If* $\theta\colon B \to BO$ *is a multiplicative tangential structure, then define the product*

$$\pi_k \operatorname{MT}\theta_d \times \pi_l \operatorname{MT}\theta_e \to \pi_{k+l} \operatorname{MT}\theta_{d+e}$$

by taking the product coordinate-wise, and using the map $\mu_{k,d}$ *to put a* θ *structure on the product of bundles.*

Chapter 2

The Signature Problem

Consider an element

$$[M, E, \varphi] \in \pi_k \operatorname{MTSO}(d)$$

for some $d \geqslant 0$ and $k \in \mathbb{Z}$ i.e. the bordism class of a triple where M is closed and $(k+d)$-dimensional, E is an oriented rank d-bundle over M, and $\varphi\colon TM \cong_s E \oplus \varepsilon^k$ is a stable isomorphism. Note that the orientation on E induces one on TM via φ. In the particular case that $k + d \in 4\mathbb{N}$ the signature $\sigma(M)$ becomes a meaningful topological invariant, and moreover it is an invariant of the $\operatorname{MTSO}(d)$-bordism class. This induces a homomorphism

$$\sigma_{k,d}\colon \pi_k \operatorname{MTSO}(d) \to \Omega_{k+d} \to \mathbb{Z}$$

which factors through the usual oriented bordism group of $(k+d)$-manifolds via the map which forgets E and φ.

If $k \geqslant 1$ and $k+d = 4n$ then this forgetful map is never surjective. Indeed the bordism class of $\mathbb{CP}^{2n}$ cannot be in the image: since $\binom{2n+1}{2n} = 2n+1$ is always odd it follows that $w_{2n}(T\mathbb{CP}^{2n}) \neq 0$, so if M is oriented-bordant to $\mathbb{CP}^{2n}$ then $w_{2n}(TM) \neq 0$ as well because Stiefel-Whitney numbers are bordism invariants; now by the Whitney product formula it follows that M cannot represent an element of $\pi_k \operatorname{MTSO}(d)$ for any $k \geqslant 1$. We might then expect the composition $\sigma_{k,d}$ to not be surjective either, and ask the following:

Question (The Signature Problem). *For $(k, d) \in \mathbb{Z} \times \mathbb{N}$, which integers can be realized as the signature of an element of $\pi_k \operatorname{MTSO}(d)$?*

Equivalently, what is the subgroup $\operatorname{Im}(\sigma_{k,d}) \subset \mathbb{Z}$?

The purpose of this chapter is to provide partial solutions to the Signature Problem, by finding theoretical lower bounds and constructing examples which either attain or are close to these bounds.

In Section 2.2 we obtain results for very small values of k and d using elementary methods, and in particular all cases where $k + d = 4$. Two basic examples are constructed, and are later used for computations in Section 3.3: an element $g_2 \in \pi_1 \operatorname{MTSO}(3)$ with signature 2 is given in Proposition 2.2.1, and an element $g_4 \in \pi_2 \operatorname{MTSO}(2)$ with signature 4 is given in Theorem 2.2.3. The outcome is as follows:

Theorem 2.0.10. *Suppose* $(k, d) \in \mathbb{Z} \times \mathbb{N}$ *with* $k + d$ *positive and divisible by* 4. *Then:*
If $d = 0$ *or* 1, *then* $\mathrm{Im}(\sigma_{k,d}) = 0$.
If $k \leqslant 0$, *then* $\mathrm{Im}(\sigma_{k,d}) = \mathbb{Z}$.
If $k = 1$, *then* $\mathrm{Im}(\sigma_{k,d}) = 2\mathbb{Z}$.
If $k = 2$, *then* $\mathrm{Im}(\sigma_{k,d}) = 4\mathbb{Z}$.

The methods used in Section 2.2 break down in higher dimensions when $k \geqslant 3$, so we have to appeal to pre-existing theory. If we further restrict the stable isomorphism condition to the case where there is an *un*stable isomorphism $\varphi'\colon TM \cong E \oplus \varepsilon^k$, this is the same as asking for k linearly independent vector fields (we will define the "span" of a manifold M is the largest k so that M admits k linearly independent vector fields). This is already a classical and heavily studied problem, the so-called "Vector Field Problem"; a very brief survey of the Vector Field problem is given in Section 2.3. In particular, there are already general divisibility results for the signature of a manifold with span k. Namely there is the following:

Theorem (Atiyah, Mayer, Frank)**.** *If* M *admits* k *linearly independent vector fields then its signature is divisible by a number* r_k, *which is characterized by the following table*

k	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
r_k	*2*	*4*	*8*	*16*	*16*	*16*	*16*	*32*

plus the relation $r_{k+8} = 16r_k$

The number r_k (which is related to the rank of irreducible Clifford modules) and this Theorem are used as black-boxes.

In Section 2.1 the Signature Problem for manifolds with a stable isomorphism $TM \cong_s E \oplus \varepsilon^k$ is reduced to the case where the manifold admits an unstable isomorphism $TM \cong E \oplus \varepsilon^k$, in other words to the context of the Vector Field Problem, via the following useful result:

Proposition 2.1.1. *Let* $[M, E, \phi] \in \pi_k \,\mathrm{MTSO}(d)$, *where* $k \geqslant 1$, *and where* $k + d$ *is even and at least* 4. *Then there is a stably parallelizable manifold* N *such that* M#N *admits* k *linearly independent vector fields.*

Then since $\sigma(N) = 0$, Corollary 2.1.13 concludes that if $[M, E, \phi] \in \pi_k \,\mathrm{MTSO}(d)$ then r_k divides $\sigma(M)$, establishing a lower bound for our Signature Problem.

(It should be remarked that Section 2.1 appears first mainly because Proposition 2.1.1 is also used to construct the example g_2 in Section 2.2.)

Section 2.4 continues the program of trying to realize r_k as the signature of an element of $\pi_k \,\mathrm{MTSO}(d)$ for some d by using known results from Section 2.3, and obtains some results for $k \leqslant 6$. The "hand-made" examples g_2 and g_4 from Section 2.2 can be used with the product structure from Definition 1.2.9 to produce a few examples, but their signatures are only optimal if $k \leqslant 4$: for example one can check by case analysis that any combination of g_2 and g_4 in $\pi_5 \,\mathrm{MTSO}(d)$ will have signature at least 32, but $r_5 = 16$. Further examples are given using the obstruction results of [4] as discussed in Section 2.1; however in order to apply their results one must add the assumption that $k < d$. The summary of results is:

Theorem 2.0.11. *Suppose* $(k, d) \in \mathbb{Z} \times \mathbb{N}$ *with* $k + d$ *positive and divisible by* 4. *Then:*

If $k = 3$ *and* $d \geqslant 8$, *then* $\mathrm{Im}(\sigma_{k,d}) = 8\mathbb{Z}$.

If $k = 4$ *and* $d \geqslant 8$, *then* $\mathrm{Im}(\sigma_{k,d}) = 16\mathbb{Z}$.

If $k = 5$ *and* $d = 12$, *then* $\mathrm{Im}(\sigma_{k,d}) \supset 32\mathbb{Z}$. *It is not known wether* 16 *is realizable or not in this case.*

If $k = 5$ *or* 6 *and* $d \geqslant 16$, *then* $\mathrm{Im}(\sigma_{k,d}) = 16\mathbb{Z}$.

It is important to emphasize that in all cases listed, except for possibly $(k, d) = (5, 12)$, the subgroup $\mathrm{Im}(\sigma_{k,d})$ in $\mathbb{Z}$ is generated by the number r_k.

One set of cases which is not considered at all in this thesis is when $2 \leqslant d < k$. It is likely that further restrictions would be added to the signature, due to the fact that if $d < k$ then Pontryagin classes begin to disappear, with the extreme case being $d = 0, 1$ where all manifolds have vanishing signature. Some of the values of $\pi_k \,\mathrm{MTSO}(2)$ are given by Theorem 3.0.8, but the only k where the group has interesting signatures is $k = 2$.

2.1 Stable Span Versus Span

In this section, the problem of determining the minimal signature of elements of $\pi_k \,\mathrm{MTSO}(d)$ is reduced to determining the minimal signature of manifolds with "span" k. After presenting the notions of span and "stable span", the following is proven:

Proposition 2.1.1. *Let* $[M, E, \phi] \in \pi_k \,\mathrm{MTSO}(d)$, *and suppose that* $k \geqslant 1$, *that* $k + d \geqslant 4$ *and is even, and that* M *is connected. Then there is a stably parallelizable manifold* N *such that* $M\#N$ *admits* k *linearly independent vector fields.*

Therefore an element of $\pi_k \,\mathrm{MTSO}(d)$ cannot achieve a smaller signature than a manifold whose tangent bundle *un*stably reduces to a rank d vector bundle.

Before proving this, first some definitions and basic properties of span and stable span:

Definition 2.1.2. *Let* $E \to X$ *be a vector bundle.*

Define the ***span*** *of* E, *denoted* $\mathrm{span}(E)$, *to be the maximum number of linearly independent sections of* E.

Define the ***stable span***, $\widetilde{\mathrm{span}}(E)$, *to be the maximum of* $\mathrm{span}(E \oplus \varepsilon^n) - n$ *over all* $n \geqslant 0$.

For M *a smooth manifold,* $\mathrm{span}(M)$ *and* $\widetilde{\mathrm{span}}(M)$ *will denote the respective functions applied to* TM.

Then $\widetilde{\mathrm{span}}(M) \geqslant k$ iff M represents an element of $\pi_k \,\mathrm{MTO}(\dim(M) - k)$, and if M is also oriented then $\widetilde{\mathrm{span}}(M) \geqslant k$ iff M represents an element of $\pi_k \,\mathrm{MTSO}(\dim(M) - k)$.

The first observation is that $\widetilde{\mathrm{span}}(E) \geqslant \mathrm{span}(E)$. Furthermore, span behaves very poorly with respect to stabilization: $\mathrm{span}(S^{2n}) = 0$ since $\chi(S^{2n}) \neq 0$, but $\mathrm{span}(TS^{2n} \oplus \varepsilon) = 2n + 1$, and in fact $\widetilde{\mathrm{span}}(S^{2n}) = 2n$. However, it is indeed the case that for any vector bundle $\widetilde{\mathrm{span}}(E \oplus \varepsilon) = \widetilde{\mathrm{span}}(E) + 1$.

Lemma 2.1.3. *Let* $E \to X$ *be a rank* r *vector bundle over a finite CW complex of dimension* n.

If $r > n$ *then* $\widetilde{\mathrm{span}}(E) = \mathrm{span}(E) \geqslant r - n$.

If $r \leqslant n$ *then* $\widetilde{\mathrm{span}}(E) = \mathrm{span}(E \oplus \varepsilon^{n-r+1}) - (n - r + 1)$

Proof. This is a well-known result and standard exercise in obstruction theory (cf. Exercise II.6.10 of [15]). □

Corollary 2.1.4. *If* E *and* E' *are vector bundles over* X *which are stably isomorphic, then* $\widetilde{\mathrm{span}}(E) = \widetilde{\mathrm{span}}(E')$. *In other words, stable span is a stable isomorphism invariant.*

Proof. Add enough trivial bundles so that they become isomorphic and the rank exceeds the dimension of the base space. □

The following is apparent:

Lemma 2.1.5. *If* $E \to X$ *and* $E' \to X'$ *are bundles, then*

$$\mathrm{span}(E \times E') \geqslant \mathrm{span}(E) + \mathrm{span}(E')$$

If $X = X'$ *then moreover* $\mathrm{span}(E \oplus E') \geqslant \mathrm{span}(E) + \mathrm{span}(E')$.

To prove Proposition 2.1.1 we will have to consider the span/stable span of the connected sum of two vector bundles. For M and N two connected manifolds of the same dimension n and $E \to M$, $F \to N$ two vector bundles of the same rank r, we will define the bundle $E\#_f F \to M\#N$, the connected sum of E and F clutched by a function $f\colon S^{n-1} \to O(r)$.

First, we recall an explicit construction for the connected sum of two smooth manifolds. Let M and N be connected, closed smooth manifolds of dimension n, and let

$$\phi_M\colon D^n \to M \quad \phi_N\colon D^n \to N$$

be smooth embeddings of the standard disk. Let

$$M_0 = M - \phi_M(0) \quad N_0 = N - \phi_N(0) \quad D_0 = D^n - (S^{n-1} \cup 0)$$

Define a smooth automorphism $\alpha\colon D_0 \to D_0$ by $\alpha(x) = (1 - \|x\|) \cdot x$. Then the smooth manifold $M\#N$ is defined to be the pushout

$$\begin{array}{ccc} D_0 & \xrightarrow{\phi_M} & M_0 \\ \big\downarrow{\scriptstyle \phi_N \circ \alpha} & & \big\downarrow{\scriptstyle \iota_M} \\ N_0 & \xrightarrow{\iota_N} & M\#N \end{array}$$

The tangent bundle of $M\#N$ is also a pushout. Since ϕ_M and ϕ_N were smooth they induce trivializations

$$T\phi_M\colon D^n \times \mathbb{R}^n \to TM \quad T\phi_N\colon D^n \times \mathbb{R}^n \to TN$$

and we get the following pushout diagram:

$$\begin{array}{ccc} D_0 \times \mathbb{R}^n & \xrightarrow{T\phi_M} & TM_0 \\ \big\downarrow{\scriptstyle T(\phi_N \circ \alpha)} & & \big\downarrow{\scriptstyle T\iota_M} \\ TN_0 & \xrightarrow{T\iota_N} & T(M\#N) \end{array}$$

In terms of orientations, notice that α is orientation reversing, and so to induce an orientation on $T(M\#N)$ from given orientations on TM and TN it is required that ϕ_M and ϕ_N have different orientation parities.

Now suppose $E \to M$ and $F \to N$ are arbitrary vector bundles of rank m. Now the smooth charts ϕ_M and ϕ_N no longer *induce* trivializations, and we have to choose

$$\psi_E \colon D^n \times \mathbb{R}^n \to E \quad \psi_F \colon D^n \times \mathbb{R}^n \to F$$

covering them. Moreover, we are also not bound to use $T\alpha$ as the bundle map covering α anymore. Instead, choose any continuous function

$$f \colon S^{n-1} \to O(m)$$

and use the same symbol for the map $f \colon D_0 \to O(m)$ which is f pre-composed with the the norm map $D_0 \to S^{n-1}$. Then define $\alpha_f \colon D^n \times \mathbb{R}^n \to D^n \times \mathbb{R}^n$ by

$$\alpha_f(x, v) = (\alpha(x), f_x(v))$$

(note that $f_x = f_{\alpha(x)}$).

Definition 2.1.6. *The bundle* $E\#_f F$*, called the* ***connected sum of*** E ***and*** F***, clutched by*** f*, is defined as the pushout*

$$\begin{array}{ccc} D_0 \times \mathbb{R}^n & \xrightarrow{\psi_E} & E|_{M_0} \\ \downarrow{\scriptstyle \psi_F \circ \alpha_f} & & \downarrow{\scriptstyle \iota_E} \\ F|_{N_0} & \xrightarrow{\iota_F} & E\#_f F \end{array}$$

Denote by $E\#_* F$ *the connected sum clutched by a constant map.*

If E and F are oriented, to get an orientation on $E\#_f F$ you could either choose orientation-preserving trivializations and a map $f \colon S^{n-1} \to SO(m)$, or choose $f \colon S^{n-1} \to A \cdot SO(m)$ for $\det(A) = -1$ and take trivializations with different orientation parity. We will take the first convention.

The map $T\alpha$ is a map $D_0 \times \mathbb{R}^n \to D_0 \times \mathbb{R}^n$ whose second coordinate doesn't depend on the norm of the first coordinate, so it can be considered as a map

$$T\alpha \colon S^{n-1} \to O(n)$$

Note that the image of $T\alpha$ is in $A \cdot SO(n)$; let $T\alpha^+(x) = A^{-1} \cdot T\alpha(x)$.

Lemma 2.1.7. $T(M\#N) \cong TM\#_{T\alpha^+}TN$

It is easy to see explicitly that $T\alpha^+$ is stably null-homotopic. Moreover $T\alpha^+$ clutches the tangent bundle of S^n, so since the correspondances

$$\pi_{n-1}SO(m) \cong \pi_n BSO(m) \text{ (for every } m)$$

send a stabilized clutching function to the stabilized bundle, this abstractly shows $T\alpha^+$ is stably null-homotopic.

Lemma 2.1.8. *If* $f \sim g$ *then* $E\#_f F \cong E\#_g F$.

Lemma 2.1.9. $(E\#_f F) \oplus V \cong (E \oplus V)\#_{f\oplus \mathrm{Id}_V}(F \oplus V)$.

Corollary 2.1.10. $T(M\#N) \cong_s TM\#_* TN$.

Having set up the relevant basics, the proof of Proposition 2.1.1, as suggested to the author by Oscar Randal-Williams, primarily consists of the following two Lemmata:

Lemma 2.1.11. *Let M be a connected oriented manifold of even dimension n, let $k \geqslant 1$, and suppose $\chi(M) = 0$. Then $\widetilde{\mathrm{span}}(M) \geqslant k$ implies $\mathrm{span}(M) \geqslant k$.*

Proof. If $\widetilde{\mathrm{span}}(M) \geqslant k$ then $\widetilde{\mathrm{span}}(TM \oplus \varepsilon) \geqslant k+1$, so by Lemma 2.1.3 in fact $\mathrm{span}(TM \oplus \varepsilon) \geqslant k+1$, so there is a rank $n-k$ bundle E over M so that $TM \oplus \varepsilon \cong E \oplus \varepsilon^{k+1}$. Moreover $e(TM) = 0$ by assumption and $e(E \oplus \varepsilon^k) = 0$ since $k \geqslant 1$, so $TM \cong E \oplus \varepsilon^k$ by Theorem A2. □

Lemma 2.1.12. *Let M be a connected, oriented manifold of even dimension $n \geqslant 4$, and suppose $\chi(M)$ is even. Then there is a stably parallelizable manifold N such that $\chi(M\#N) = 0$.*

Proof. Note that $\chi(M_1\#M_2) = \chi(M_1) + \chi(M_2) - 2$ when $\dim(M_i)$ is even. Let $m = \frac{1}{2}n$ and $c = \frac{1}{2}\chi(M)$. Then the manifold

$$N = \begin{cases} S^n & \text{if } \chi(M) = 0 \\ \#^c S^m \times S^m & \text{if } \chi(M) > 0 \text{ and } m \text{ odd, or } \chi(M) < 0 \text{ and } m \text{ even} \\ \#^c S^{m-1} \times S^{m+1} & \text{otherwise} \end{cases}$$

has the desired property. □

Proof of Proposition 2.1.1. Let $[M, E, \varphi] \in \pi_k \mathrm{MTSO}(d)$, with $k \geqslant 1$ and $k+d$ even. Since $k \geqslant 1$ then $w_{k+d}(M) = 0$ so $\chi(M)$ is even, so we find a stably parallelizable N such that $\chi(M\#N) = 0$. Then

$$T(M\#N) \oplus \varepsilon \cong (TM\#_{T\alpha^+}TN) \oplus \varepsilon \cong (TM \oplus \varepsilon)\#_{T\alpha^+ \oplus 1}(TN \oplus \varepsilon)$$

and this is isomorphic to

$$(E \oplus \varepsilon^{k+1})\#_* \varepsilon^{n+1} \cong (E\#_* \varepsilon^{n-k}) \oplus \varepsilon^{k+1}$$

i.e. $\widetilde{\mathrm{span}}(M\#N) \geqslant k$. Since its Euler characteristic vanishes, it follows that $\mathrm{span}(M\#N) \geqslant k$. □

We will typically denote $M\#N$ by M_0. As a corollary of this and Theorem 2.3.5, we have

Corollary 2.1.13. *Suppose M represents an element of $\pi_k \mathrm{MTSO}(d)$. Then r_k divides $\sigma(M)$.*

Proof. $\sigma(N) = 0$ for any stably parallelizable N. □

2.2 The Hands-On Cases

Some immediate observations about $\pi_k \operatorname{MTSO}(d)$ can be made for small values of k and d. If $d = 0$ or 1 then any element of $\pi_k \operatorname{MTSO}(d)$ will be stably parallelizable, and hence $\sigma_{k,d} = 0$; thus the interesting case is when $d \geqslant 2$.

As for the variable k, if $k \leqslant 0$ then any signature is possible: letting $M = \mathbb{CP}^{\frac{k+d}{2}}$, take the element $[M, TM, \mathrm{Id}]$. Therefore we will tend to restrict to the case $k \geqslant 1$.

Proposition 2.2.1. *For all $n \geqslant 1$ there is an element of $\pi_1 \operatorname{MTSO}(4n-1)$ with signature 2, and this is the minimum positive signature.*

Proof. If k is equal to 1, an element $[M, E, \varphi] \in \pi_1 \operatorname{MTSO}(4n-1)$ will have $w_{4n}(TM) = 0$ which means that $\chi(M) \equiv 0 \mod 2$. Since $\sigma(M)$ and $\chi(M)$ have the same parity, then the signature must also be even. For $4n \in 4\mathbb{Z}$ an element of signature 2 can be described as follows. Begin with the manifold $\#^2\mathbb{CP}^{2n}$: since this manifold has even dimension and Euler characteristic, Proposition 2.1.1 produces a stably parallelizable manifold N so that

$$(\#^2\mathbb{CP}^{2n})_0 := (\#^2\mathbb{CP}^{2n})\#N$$

has vanishing Euler characteristic and signature 2. Then, by the classical Poincare-Hopf theorem, $T(\#^2\mathbb{CP}^{2n})_0$ admits a non-zero section s and hence a decomposition as $\langle s\rangle \oplus \langle s\rangle^{\perp}$. Therefore

$$[(\#^2\mathbb{CP}^{2n})_0, \langle s\rangle^{\perp}, \mathrm{Id}] \in \pi_1 \operatorname{MTSO}(4n-1)$$

□

Definition 2.2.2. *Let $g_2 = [(\#^2\mathbb{CP}^2)_0, \langle s\rangle^{\perp}, \mathrm{Id}] \in \pi_1 \operatorname{MTSO}(3)$*

The rest of the section will prove the following:

Theorem 2.2.3. *There is an element $g_4 \in \pi_2 \operatorname{MTSO}(2)$ with signature 4, and this is the minimum positive signature.*

This is proven with fairly elementary methods, exploiting the close relationship between the intersection form of a 4-manifold and its characteristic classes.

First recall Wu classes (see [24]). Let M be a closed manifold of any dimension n, let $[M] \in H_n(M; \mathbb{Z}/2)$ be its mod 2 fundamental class, and for $0 \leqslant i \leqslant n$ let

$$I_i(-,-)\colon H^i(M; \mathbb{Z}/2) \times H^{n-i}(M; \mathbb{Z}/2) \to \mathbb{Z}/2$$

be its mod 2 cup-pairing, i.e. $I_i(x, y) = \langle x \cup y, [M]\rangle_{\mathbb{Z}/2}$ where $\langle -,-\rangle_{\mathbb{Z}/2}$ is the mod 2 Kronecker pairing. Then $I_i(-,-)$ is non-degenerate, which implies that for each i there is a unique class $\nu_i \in H^i(M; \mathbb{Z}/2)$ such that for all $x \in H^{n-i}(M; \mathbb{Z}/2)$

$$\langle x \cup \nu_i, [M]\rangle_{\mathbb{Z}/2} = \langle \mathrm{Sq}^i\, x, [M]\rangle_{\mathbb{Z}/2}$$

where Sq^i is the i-th mod 2 Steenrod operation. The classes ν_i are called the **Wu classes**. In particular $\nu_0 = 1$ and $\nu_i = 0$ if $2i > n$.

Theorem 2.2.4 (Wu's Formula). *Let* M *be a closed manifold, let* $V = 1+\nu_1+\nu_2+\dots$ *be its total Wu class,* $W = 1 + w_1 + w_2 + \dots$ *its total Stiefel-Whitney class, and* $\mathrm{Sq} = 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \dots$ *the total Steenrod square. Then*

$$W = \mathrm{Sq}\, V$$

A surprising corollary of this theorem is that, since V and Sq only depend on the cup-product structure of the cohomology of M, the Stiefel-Whitney classes of M's tangent bundle are invariant under homotopy equivalences. However we will only use a much simpler corollary:

Corollary 2.2.5. *If* M *is a closed manifold then* $\nu_1 = w_1$*, and if* M *is orientable then* $\nu_2 = w_2$.

Proof. Immediately from Wu's formula we have $\nu_1 = w_1$ for any manifold, so $\nu_1 = 0$ if it is orientable. Then

$$w_2 = \nu_2 + \mathrm{Sq}^1 \nu_1 + \mathrm{Sq}^2 1 = \nu_2$$

□

Now we recall some basic things about bilinear forms. Let A be a finitely-generated abelian group. For a bilinear form $q\colon A \times A \to \mathbb{Z}$ over A, say that $c \in A$ is a **characteristic element of** q if for all $x \in A$

$$q(x, c) \equiv q(x, x) \mod 2$$

Immediately from the definition we see that if M is a closed manifold of dimension $2n$ then a characteristic element of its intersection form is the same as an integral lift of ν_n. If q is also unimodular, then the following holds:

Lemma 2.2.6 (van der Blij). *Let* $q\colon A \times A \to \mathbb{Z}$ *be a symmetric unimodular bilinear form. Then* q *has a (not necessarily unique) characteristic element* c*, and*

$$q(c, c) \equiv \mathrm{sign}(q) \mod 8$$

Proof. Originally derived in [38] by an argument using Gaussian sums, though it has a more algebraic proof as Lemma II.5.2 in [25]. □

Now we can easily prove the following:

Proposition 2.2.7. *If* $[M, E, \phi] \in \pi_2\, \mathrm{MTSO}(2)$ *then* $\sigma(M)$ *is divisible by* 4.

Proof. Suppose that $[M, E, \phi] \in \pi_2\, \mathrm{MTSO}(2)$, and let $I(-,-)$ denote its intersection form. Since M is orientable, by the above corollary we have $w_2(M) = \nu_2(M)$ and so a characteristic element of I is any integral lift of $w_2(M)$. The stable isomorphism $TM \cong_s E \oplus \varepsilon^2$ implies $w_2(M) = w_2(E)$, hence the class $e(E)$ is characteristic for the intersection form of M, so by van der Blij's Lemma

$$I(e(E), e(E)) = \langle e(E)^2, [M] \rangle \equiv \sigma(M) \mod 8$$

But $e(E)^2 = p_1(E) = p_1(M)$, and so by Hirzebruch's signature formula $3\sigma(M) = p_1[M] \equiv \sigma(M) \mod 8$, and therefore $2\sigma(M)$ is divisible by 8. □

Since 4 is a lower bound, the ideal example would have signature 4. Let $M_4 = \#^4\mathbb{CP}^2$ be the connected sum of 4 copies of $\mathbb{CP}^2$ with the canonical orientation, so that $\sigma(M_4) = 4$. For notation's sake, let $[M_4] \in H_4(M_4;\mathbb{Z})$ denote the fundamental class induced by the orientation, and let $\mu_4 \in H^4(M_4;\mathbb{Z})$ be its dual. Let τ_4 be the tangent bundle of M_4.

In order for M_4 to represent an element of $\pi_2\,\mathrm{MTSO}(2)$, recall that this requires an oriented rank 2 bundle E over M_4 and a stable isomorphism $\tau_4 \cong_s E \oplus \varepsilon^2$. Since oriented rank 2 bundles are parametrized by $H^2(M_4;\mathbb{Z})$ because $\mathrm{BSO}(2) \simeq K(\mathbb{Z}, 2)$, producing candidates for E is easy; therefore it is pertinent to have a way of detecting wether two given bundles over M_4 are stably isomorphic. For this we employ K-theory (see for example [17]).

Lemma 2.2.8. *Let X be a 7-dimensional CW complex with $H^3(X;\mathbb{Z}/2) = 0$. If V and W are oriented bundles over X with $w_2(V) = w_2(W)$ and $p_1(V) = p_1(W)$, then V and W are stably isomorphic.*

Proof. Consider V and W as pointed maps $X \to \mathrm{BO}$, which factor through BSO since they are orientable. The proof is by obstruction theory and employs the following two facts:

1. $\pi_2\,\mathrm{BSO} \cong \mathbb{Z}/2$ and $w_2\colon \mathrm{BSO} \to K(\mathbb{Z}/2, 2)$ induces an isomorphism on π_2.
2. $\pi_4\,\mathrm{BSO} \cong \mathbb{Z}$ and $p_1\colon \mathrm{BSO} \to K(\mathbb{Z}, 4)$ induces multiplication by ± 2 on π_4.

The identification of the groups $\pi_2\,\mathrm{BSO}$ and $\pi_4\,\mathrm{BSO}$ is part of Bott periodicity in the real case [17, I.9.21]. That w_2 is an isomorphism on π_2 follows from Corollary A.3.2 and the fact that S^1 is parallelizable, alternatively, the tautological complex line bundle $L \to \mathbb{CP}^1 \cong S^2$ has $w_2 \neq 0$ when considered as an oriented 2-plane bundle.

To show that p_1 induces multiplication by ± 2, first note that by definition it is the composition of two maps:

$$\begin{array}{ccc} \mathrm{BSO} & \xrightarrow{\quad p_1 \quad} & K(\mathbb{Z},4) \\ & {\scriptstyle\kappa}\searrow \quad \nearrow{\scriptstyle -c_2} & \\ & \mathrm{BU} & \end{array}$$

where κ is induced by complexification $SO(n) \to U(2n)$, and c_2 represents the second Chern class. Recall that, for spheres, the reduced Chern character

$$\mathrm{ch}\colon \widetilde{KU}(S^{2n}) \cong \pi_{2n}\,\mathrm{BU} \to H^{2n}(S^{2n};\mathbb{Q})$$

factors through an isomorphism onto $H^{2n}(S^{2n};\mathbb{Z}) \cong \pi_{2n}K(\mathbb{Z}, 2n)$ for all n [15, V.3.25]. When $n = 2$, c_1 always vanishes and the Chern character reduces to the second component $\mathrm{ch}_2 = \frac{1}{2}(-2c_2 + c_1^2) = -c_2$ [26, IV.4.18], therefore $-c_2$ induces an isomorphism on π_4. As for κ, there is a long exact sequence of KO and KU groups [1]:

$$\cdots \longrightarrow KO^{-3} \xrightarrow{\times\eta} KO^{-4} \xrightarrow{\kappa} KU^{-4} \xrightarrow{\rho\circ b^{-1}} KO^{-2} \xrightarrow{\times\eta} KO^{-3} \longrightarrow \cdots$$

where $b\colon KU^{-n} \cong KU^{-n-2}$ is the Bott isomorphism, $\eta \in KO^{-1}$ is the generator, and $\rho\colon KU^{-n} \to KO^{-n}$ is realification. Now, $KO^{-3} = 0$, $KO^{-2} \cong \mathbb{Z}/2$ and $KO^{-4} \cong \mathbb{Z} \cong KU^{-4}$, so by exactness κ is multiplication by ± 2.

For the sake of notation let $K = K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}, 4)$, and let $f\colon BSO \to K$ be the product $w_2 \times p_1$. By assumption, the compositions $f \circ V$ and $f \circ W$ are homotopic as pointed maps $X \to K$, so to show the bundles are stably isomorphic we want to lift the homotopy to BSO. The problem of lifting the homotopy to BSO is described by the following diagram

$$\begin{array}{ccc} (\{0,1\} \times X) \cup ([0,1] \times x_0) & \longrightarrow & BSO \\ \downarrow & \nearrow & \downarrow f \\ [0,1] \times X & \longrightarrow & K \end{array}$$

where x_0 is the basepoint of X. The obstruction groups are then $\tilde{H}^r(X; \pi_r F)$ where F is the homotopy fibre of f. From the long exact sequence of homotopy groups for $F \to BSO \to K$ it follows that $\pi_k F = 0$ for $k \leqslant 7$ and $\neq 3$, and $\pi_3 F = \mathbb{Z}/2$. Since X has dimension 7 and $H^3(X; \mathbb{Z}/2) = 0$ by assumption, all of the obstruction groups vanish and therefore $V \cong_s W$. □

Now we construct a bundle with the same w_2 and p_1 as M_4:

Lemma 2.2.9. *There is an oriented rank 2 real vector bundle $V \to M_4$ with $w_2(V) = w_2(\tau_4)$ and $p_1(V) = p_1(\tau_4)$.*

Proof. As stated above, isomorphism classes of oriented rank 2 vector bundles over M_4 are in bijective correspondance with $H^2(M_4; \mathbb{Z}) \cong \mathbb{Z}^4$, where a vector bundle V corresponds to its Euler class $e(V)$. Moreover, for such a bundle V we have $p_1(V) = e(V)^2$ and $w_2(V) = \rho_2 e(V)$ where $\rho_2\colon H^*(-; \mathbb{Z}) \to H^*(-; \mathbb{Z}/2)$ is reduction modulo 2.

By the formula $w_{2i}(\mathbb{CP}^n) = \rho_2(\binom{n}{i} u^i)$, where u is the standard generator of $H^2(\mathbb{CP}^n; \mathbb{Z})$, one obtains $w_2(\tau_4) = (1,1,1,1) \in H^2(M_4; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^4$, and by Hirzebruch's signature formula $p_1(\tau_4) = 12\mu_4$, where μ_4 was dual to the fundamental class of M_4.

Now let V be the bundle with Euler class $(3,1,1,1)$. Then V has $w_2(V) = \rho_2 e(V) = (1,1,1,1)$, and because the intersection form of M_4 is the standard scalar product on $\mathbb{Z}^4$ we have $p_1(V) = (3,1,1,1)^2 = 12\mu_4$. □

Corollary 2.2.10. *There is an oriented rank 2 real vector bundle $V_4 \to M_4$ such that $V_4 \cong_s \tau_4$.*

Proof. Take the bundle V_4 corresponding to $(3,1,1,1)$ as above. Then since w_2, p_1 agree on V_4 and τ_4 it follows that there is a stable isomorphism $\varphi_4\colon \tau_4 \oplus \varepsilon \cong V_4 \oplus \varepsilon^3$. □

Hence $(\#^4\mathbb{CP}^2, V_4, \varphi_4)$ represents an element $g_4 \in \pi_2\, MTSO(2)$. Moreover, it has the minimal positive signature by the above Lemma, so it is an indivisible element. This element will be used in Proposition 3.3.13 to show $\pi_2\, MTSO(2) \cong \mathbb{Z}$, and is in fact a generator.

2.3 Span Versus Signature

Much of the basic material discussed in this section can be found in Emery Thomas' extensive expository paper [34]. Other results cited here are found in Atiyah-Dupont [2], Bökstedt-Dupont-Svane [4], and Lawson-Michelsohn [17].

This section discusses some known results about the span of an oriented smooth manifold. Fix M^n a smooth, compact, oriented manifold with metric, and let k be a positive integer. k will be the number of linearly independent vector fields we want be able to find.

The criteria for determining if $\text{span}(M) \geqslant 1$ is a classical result of Hopf. Considering TM as a manifold of dimension $2n$, by transversality a generic section s will intersect the zero-section at finitely many points $\{x_1, \ldots x_m\}$. By choosing small disks D_i around each x_i and trivializations $TM|_{D_i} \cong D_i \times \mathbb{R}^n$, s determines a non-zero section of

$$TM|_{\bigsqcup_i \partial D_i} \cong \bigsqcup_i S^{n-1} \times \mathbb{R}^n$$

Normalizing s over this union of spheres by using the metric on M, s determines an element

$$\text{Ind}(s) := \sum_{i=0}^{m} s|_{\partial D_i} \in \pi_{n-1} S^{n-1} \cong \mathbb{Z}$$

called the **index** of s. Then, if M is connected, s can be replaced by a non-zero vector field iff $\text{Ind}(s) = 0$ (see the proof of Theorem 2.10 in [12]).

Many choices were made in the definition of $\text{Ind}(s)$, but the remarkable result is that they didn't matter for the reason that the index is actually equal to a topological invariant of the manifold:

Theorem 2.3.1 (Poincare-Hopf). *In the situation outlined above,* $\text{Ind}(s) = \chi(M)$.

Corollary 2.3.2. *If* M *is connected then* $\text{span}(M) \geqslant 1$ *iff* $\chi(M) = 0$.

In particular $\text{span}(S^{2n}) = 0$ since $\chi(S^{2n}) = 2$, and $\text{span}(M) \geqslant 1$ for every odd-dimensional manifold. Furthermore, if $\text{span}(M) \geqslant 1$ then $\sigma(M)$ is even, since $\sigma(M) \equiv \chi(M) \mod 2$.

The case $k > 1$ is drastically more delicate. Even if M admits non-zero vector fields, if they are transverse then the set of points where they fail to be linearly independent may be a submanifold of positive dimension. In order to implement a similar strategy to the one above, one can impose an extra assumption on the manifolds under consideration.

Definition 2.3.3. *A set of vector fields* $s = \{s_1, \ldots, s_k\} \in \Gamma M^k$ *has* ***finite singularities*** *if they are only linearly dependent at a finite set of points. Such a set will be called a* ***finitely-singular k-field****.*

For example, spin a globe and take the velocity vector field: this is a section $s = \{s_1\}$ with finite singularities since it only has two zeros. If all vectors in this field are rotated southward by an angle $0 < \theta < \pi$ this produces another section which is linearly independent from the first everywhere except for the same two zeros. Thus, S^2 admits two vector fields with finite singularites, even though it doesn't admit a single non-zero vector field.

In general, the assumption that M admits a finitely-singular k-field is non-trivial, but there is a convenient necessary condition:

Proposition 2.3.4 ([34]). *M admits a finitely singular 2-field, except possibly when* $n \equiv 1 \mod 4$ *and* $w_{n-1}(M) \neq 0$.

For general k, *suppose* M *is* $(k-2)$*-connected, and either* $w_{n-k+1}(M) = 0$ *if* $n-k$ *is odd, or* $\beta w_{n-k}(M) = 0$ *if* $n-k$ *is even, where* β *is the Bockstein homomorphism for the sequence* $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2$. *Then* M *admits a finitely-singular* k*-field.*

In the case M does admit a finitely-singular k-field $s = \{s_1, \ldots, s_k\}$, the definition of $\mathrm{Ind}(s)$ generalizes from the case $k = 1$. Say s is linearly dependent at the points $x_1, \ldots x_m$, and trivialize the tangent bundle of M over disjoint oriented disks D_i around the singular points. Since M was assumed to have a metric, the Gram-Schmidt procedure naturally orthogonalizes s over the complement of the singular points, so in particular s defines an element

$$\mathrm{Ind}(s) = \sum_{i=1}^{m} s|_{\partial D_i} \in \pi_{n-1} \mathrm{St}_k(\mathbb{R}^n)$$

where $\mathrm{St}_k(V)$ is the Stiefel manifold of k-frames in V for any inner-product space V. Then, again, M admits k linearly independent sections iff $\mathrm{Ind}(s) = 0$.

In contrast to the $k = 1$ case, $\mathrm{Ind}(s)$ can sometimes depend on s and so as an element of $\pi_{n-1} \mathrm{St}_k(\mathbb{R}^n)$ it can't always be interpreted as an invariant of the manifold. It was not until much later, with the advent of Index Theory in the 1960s, that further progress was made in this direction.

One of the earliest applications of Index Theory to the vector field problem provided a necessary condition for $\mathrm{span}(M) \geqslant k$ in terms of a topological invariant of M. Recall from [17, IV.2] the number $2a_k$, defined as the rank of an irreducible $\mathbb{Z}/2$-graded $Cl(\mathbb{R}^k)$ module. A complete description of the values of a_k is given by the first eight values

k	1	2	3	4	5	6	7	8
a_k	1	2	4	4	8	8	8	8

and the relation $a_{k+8} = 16a_k$. For notation's sake, define $r_k = 2a_k$ when k is not divisible by 4, and $r_{4l} = 4a_{4l}$.

Theorem 2.3.5 ([17, IV.2.7]). *Let M be a smooth, closed, orientable manifold. If* $\mathrm{span}(M) \geqslant k$ *then* $r_k | \sigma(M)$.

This theorem is usually attributed to Mayer [22], though the result above is not explicitly stated in the referenced paper. Atiyah [3] around the same time proved slightly simpler statements using more elementary methods, but nonetheless still Index Theoretic in nature.

Note that the divisibility in Theorem 2.3.5 only depends on the number of linearly independent vector fields, and not on the dimension of M. In his expository paper from 1969, Thomas made the following conjecture:

Conjecture 1 ([34, Conjecture 4]). *If* $n \in 4\mathbb{Z}$ *and* $n > 4$, *and* M *is a connected* n*-manifold admitting a finitely singular* k*-field, then* $\mathrm{span}(M) \geqslant k$ *iff* $\chi(M) = 0$ *and* $r_k | \sigma(M)$.

Shortly after, this conjecture was verified for $k = 2$ and 3 by Atiyah and Dupont [2] via a different application of index theory. They produced a natural homomorphism from $\pi_{n-1}\operatorname{St}_k(\mathbb{R}^n)$ to a certain KR group, and then interpretted the image of $\operatorname{Ind}(s)$ as an invariant of M using the Index Theorem for KR-Theory. Their result is the following

Theorem 2.3.6. *Suppose M^n admits a finitely-singular k-field s, where $k = 2, 3$. Then $\operatorname{Ind}(s)$ can be interpreted as:*

$n \mod 4$	$\operatorname{Ind}(s)$
0	$\chi(M) \oplus \frac{1}{2}(\chi(M) + \sigma(M)) \in \mathbb{Z} \oplus \mathbb{Z}/b_k$
1	$\operatorname{Kerv}(M) \in \mathbb{Z}/2$
2	$\chi(M) \in \mathbb{Z}$
3	0

where b_k happens to be $\frac{r_k}{2}$, and

$$\operatorname{Kerv}(M) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \operatorname{rank}_{\mathbb{Z}/2} H^i(M; \mathbb{Z}/2)$$

is the Kervaire semi-characteristic.

Corollary 2.3.7. *Suppose $n \in 4\mathbb{Z}$. Then $\operatorname{span}(M) \geqslant 2$ iff $\chi(M) = 0$ and $4|\sigma(M)$. If M admits a finitely singular 3-field then $\operatorname{span}(M) \geqslant 3$ iff $\chi(M) = 0$ and $8|\sigma(M)$.*

Interpretations of $\operatorname{Ind}(s)$ for higher values of k didn't arrive until very recently, in a 2014 publication by Böckstedt, Dupont, and Svane [4], using a construction involving Madsen-Tillman-Weiss spectra and a very involved Adams spectral sequence computation. They were able to interpret the index in nice cases when $k = 4, 5$, or 6. In particular, they proved the following:

Theorem 2.3.8. *Suppose n is even, $k < \frac{n}{2}$ and $k = 4, 5$, or 6. Suppose M admits finitely-singular k-field s.*

If $n \equiv 2 \mod 4$ then $\operatorname{Ind}(s) = \chi(M) \in \mathbb{Z}$.

If $n \equiv 0 \mod 4$ then $\operatorname{Ind}(s) = \chi(M) \oplus \frac{1}{2}(\chi(M) + \sigma(M)) \in \mathbb{Z} \oplus \mathbb{Z}/8$.

Corollary 2.3.9. *Suppose M admits k vector fields with finite singularities, with $n \in 4\mathbb{Z}$. If*

$k = 4, 5$ *and* $n \geqslant 12$*, or*

$k = 6$ *and* $n \geqslant 16$

then $\operatorname{span}(M) \geqslant k$ iff $\chi(M) = 0$ and $16|\sigma(M)$.

Note that again $16 = r_4 = r_5 = r_6$. This suggets that the following conjecture may be more likely to be true than Conjecture 1:

Conjecture 2. *Suppose M^n admits k sections with finite singularities, where $2k < n \in 4\mathbb{Z}$. Then $\operatorname{span}(M) \geqslant k$ iff $\chi(M) = 0$ and $r_k|\sigma(M)$.*

2.4 Examples and Results

The aim of this section is to produce, for a given $k > 0$, closed manifolds M^{4n} for any $n > 0$ with $\widetilde{\text{span}}(M) = k$ and $\sigma(M) = r_k$, using the results from the previous sections. All examples here have $k \leqslant 2n$ and in many cases $k < 2n$.

In Proposition 2.2.1 an element $g_2 = [(\#^2\mathbb{CP}^2)_0, \langle s\rangle^\perp, \mathrm{Id}] \in \pi_1 \mathrm{MTSO}(3)$ with signature 2 was given, and in Proposition 3.3.13 it will be shown to be a generator. Furthermore, taking products with complex projective spaces produces elements in $\pi_1 \mathrm{MTSO}(3 + 4n)$ for every $n \geqslant 1$ as well. Specifically:

Lemma 2.4.1. *Suppose $k + d$ is divisible by 4, and suppose there is an element $[M] := [M, E, \varphi] \in \pi_k \mathrm{MTSO}(d)$ with signature $\sigma \neq 0$. Then for every $n \geqslant 0$ there are at least $\pi(n)$ linearly independant elements of $\pi_k \mathrm{MTSO}(d + 4n)$ with signature σ, where $\pi(n)$ is the number of partitions of n.*

What this Lemma means for this problem is that if the Signature Problem has been solved for some k and d, then it has also been solved for that same k and all higher values of d.

Proof. Let $I = (a_1, \dots, a_l)$ be a partition of $n \geqslant 1$. Then for each $1 \leqslant i \leqslant l$ the manifold $\mathbb{CP}^{2a_i}$ determines an element $[\mathbb{CP}^{2a_i}] := [\mathbb{CP}^{2a_i}, T\mathbb{CP}^{2a_i}, \mathrm{Id}] \in \pi_0 \mathrm{MTSO}(4a_i)$. Now let

$$[\mathbb{CP}^{2I}] := \times_{i=1}^{l} [\mathbb{CP}^{2a_i}] \in \pi_0 \mathrm{MTSO}(4n)$$

Then for each I the element $[M] \times [\mathbb{CP}^{2I}]$ is an element of $\pi_k \mathrm{MTSO}(d + 4n)$ with signature σ. In order to see that these are linearly independent as I varies over partitions of n, consider their image under the homomorphism

$$\pi_k \mathrm{MTSO}(d + 4n) \to \Omega_{k+d+4n}$$

Since $\sigma(M) \neq 0$, $[M]$ remains non-zero after projecting to the torsion-free part. But the torsion-free part of the oriented cobordism ring is a polynomial algebra generated by the complex projective spaces $\{\mathbb{CP}^{2m}\}$, so the elements $[M] \times [\mathbb{CP}^{2I}]$, as I varies over all partitions of n, become linearly independent in Ω_{k+d+4n}. Therefore they are also independent in $\pi_k \mathrm{MTSO}(d + 4n)$. ◻

In fact Proposition 2.2.1 gave for every $n \geqslant 1$ an element of $\pi_1 \mathrm{MTSO}(4n-1)$ with signature $2 = r_1$. In light of the above Lemma, we could have just taken the generator $g_2 \in \pi_1 \mathrm{MTSO}(3)$ and taken products with complex projective spaces to produce more elements.

For $k = 2$, Theorem 2.2.3 gave the element g_4 generating $\pi_2 \mathrm{MTSO}(2)$ with $\sigma(g_4) = 4 = r_2$. Again, taking products with $[\mathbb{CP}^{2I}]$ as I ranges over partitions of n gives linearly independent elements of $\pi_2 \mathrm{MTSO}(4n + 2)$ for all $n \geqslant 1$. I.e.

Proposition 2.4.2. *There are at least $\pi(n)$ linearly independent elements of $\pi_2 \mathrm{MTSO}(2 + 4n)$ with signature $4 = r_2$ for any $n \geqslant 0$.*

In the case of $k = 3$, since $\mathrm{MTSO}(1) \simeq \mathbb{S}^{-1}$ it follows that $\pi_3 \mathrm{MTSO}(1) = 0$, so the search begins with 8-manifolds. The product $g_2 \times g_4$ is indeed an element of $\pi_3 \mathrm{MTSO}(5)$ and its signature is $8 = r_3$. Thus

Proposition 2.4.3. *There are at least* $\pi(n)$ *linearly independent elements of* π_3 MTSO$(5+4n)$ *with signature* $8 = r_3$ *for any* $n \geqslant 0$.

For $k = 4$ again an example can be constructed with the external product, namely $g_4 \times g_4 \in \pi_4$ MTSO(4), having signature $16 = r_4$.

Proposition 2.4.4. *There are at least* $\pi(n)$ *linearly independent elements of* π_4 MTSO$(4+4n)$ *with signature* $16 = r_4$ *for any* $n \geqslant 0$.

For the case $k = 5$ however, products fail to produce an element of π_5 MTSO(7) with signature $16 = r_5$, though the element $g_2 \times g_4^2$ has signature 32. We do have the following:

Proposition 2.4.5. *There are possibly two different elements of* π_5 MTSO(7) *with signature* $32 = 2r_5$.

There are at least $\pi(n)$ *linearly independent elements of* π_5 MTSO$(11+4n)$ *with signature* $16 = r_5$ *for any* $n \geqslant 0$.

Proof. As stated above, $g_2 \times g_4^2$ is an element of π_5 MTSO(7) with signature 32. Another example can be constructed using the obstruction of Böckstedt-Dupont-Svane by producing a 12-manifold which is 3-connected, has $w_8 = 0$, $\chi = 0$, and has signature divisible by 16; such a manifold will then have span $\geqslant 5$.

Begin with $K = K_3 \times (\mathbb{HP}^2 \# \mathbb{HP}^2)$, where K_3 is one of the spin, signature 16 complex surfaces named for Kähler, Kodaira, and Kummer, all of which are diffeomorphic (see for example [30, II.3.3]). Then since K is spin any embedded 1-, 2-, or 3-sphere will have trivial normal bundle, so it can be surgered into a 3-connected manifold K' [23]. After doing these surgeries, construct K'_0 by adding copies of $S^5 \times S^7$ to eliminate the Euler characteristic, as per Lemma 2.1.12; then K'_0 is 3-connected by construction. Now, $w_8(\mathbb{HP}^2) \neq 0$ but $w_8(\mathbb{HP}^2 \# \mathbb{HP}^2) = 0$, and it follows that $w_8(TK'_0) = 0$ so K'_0 admits a finitely singular 5-field by Proposition 2.3.4. Then since $\sigma(K'_0) = 32$ is divisible by 16 and $\chi(K'_0) = 0$ this manifold admits 5 linearly independent vector fields by Corollary 2.3.9. This is not ideal since the signature of K'_0 is $2r_5$, but having two copies of $\mathbb{HP}^2$ in the construction ensures that $w_8 = 0$ so that we can apply the necessary condition for the existence of a finitely singular 5-field.

For higher dimensions, consider the manifold $(\#^{16}\mathbb{OP}^2)_0$. Then this manifold is 6-connected since copies of $S^7 \times S^9$ were added to eliminate the Euler characteristic, $w_{10} = 0$, and its signature is 16. Then the results of [4] tell us again that $\mathrm{span}((\#^{16}\mathbb{OP}^2)_0) \geqslant 6$, so in particular it is at least 5. □

Corollary 2.4.6. *There are at least* $\pi(n)$ *linearly independent elements of* π_6 MTSO$(10+4n)$ *with signature* $16 = r_6$ *for any* $n \geqslant 0$.

At the time of writing an element of π_5 MTSO(7) with signature 16 remains elusive, and π_5 MTSO(3) remains unconsidered.

Chapter 3

A Cofibre Sequence and Some Computations

In the landmark four-author paper by Galatius, Madsen, Tillmann, and Weiss [7], one of their smaller propositions, Proposition 3.1, provides a cofibre sequence of spectra

$$\Sigma^{-1}\,\mathrm{MTO}(d-1) \to \mathrm{MTO}(d) \to \Sigma^{\infty}\mathrm{BO}(d)_{+} \to \mathrm{MTO}(d-1)$$

as well as the analogue for $SO(d)$. The details of the proof were sparse to say the least. Section 3.1 provides extensive details for a generalization to arbitrary tangential structures; namely it proves the following:

Proposition 3.0.7. *Let $\theta_d : B(d) \to \mathrm{BO}(d)$ be a d-dimensional tangential structure for $d \geqslant 1$, and let θ_{d-1} be its restriction to $\mathrm{BO}(d-1)$. Then there are maps (described in Section 3.1) giving a cofibre sequence of spectra*

$$\Sigma^{-1}\,\mathrm{MT}\,\theta_{d-1} \xrightarrow{\tilde{p}} \mathrm{MT}\,\theta_d \xrightarrow{i} \Sigma^{\infty}B(d)_{+} \xrightarrow{PT} \mathrm{MT}\,\theta_{d-1}$$

Specifically, we will construct a cofiber sequence of spectra of the form

$$\mathbb{P} \to \mathrm{MT}\,\theta_d \to \mathbb{G} \to S^1 \wedge \mathbb{P}$$

as well as homotopy equivalences $\mathbb{G} \to \Sigma^{\infty}B(d)_{+}$ and $\Sigma^{-1}\,\mathrm{MT}\,\theta_{d-1} \to \mathbb{P}$.

The chapter culminates by using these basic results to prove the following:

Theorem 3.0.8. *The first four positive homotopy groups of* MTSO(2) *are given by*

k	*1*	2	3	4
$\pi_k\,\mathrm{MTSO}(2)$	*0*	$\mathbb{Z}$	$\mathbb{Z}/24$	$\mathbb{Z}$

The group $\pi_2\,\mathrm{MTSO}(2)$ is generated by a class with signature 4*.*

Furthermore, the group $\pi_1\,\mathrm{MTSO}(3)$ is isomorphic to $\mathbb{Z}$ and is generated by a class with signature 2.

The values of the first three homotopy groups of MTSO(2) were remarked in Corollary 4.4 of [20] (where, at the time, MTSO(2) was referred to as $\mathbb{CP}^{\infty}_{-1}$)

and the idea they had in mind likely parallels the proof given here. However they do not state a result for π_4 MTSO(2) or π_1 MTSO(3), and at the time of writing this thesis computations of these groups are not found in the literature. The computations given here use the cofibre sequence of Proposition 3.0.7 and the interpretation of homotopy groups given in Proposition 1.2.3, and are contained in Section 3.3.

The first two homotopy groups of MTSO(2) can also be computed via highly non-trivial means by appealing to the Madsen-Weiss theorem [21] and homological stability for mapping class groups (cf [10], [14], [5]). If M is a manifold, let $\mathrm{Diff}_\partial(M)$ denote the topological group of diffeomorphisms which fix (a collar of) ∂M. If $\Sigma_{g,b}$ denotes a genus g surface with b boundary components, then the **mapping class group of** $\Sigma_{g,b}$ is defined as

$$\Gamma_{g,b} := \pi_0 \,\mathrm{Diff}_\partial(\Sigma_{g,b})$$

For the purposes of this discussion fix $b = 1$. Let H be the 2-torus with two disks removed and boundary components labelled B_- and B_+, and let $\Sigma_{g+1,1}$ be the result of attaching H to $\Sigma_{g,1}$ along B_-. Then there is an induced homomorphism $\Gamma_{g,1} \to \Gamma_{g+1,1}$ which extends a given diffeomorphism by Id_H; in particular it induces a map on homology

$$H_*(\Gamma_{g,1}) \to H_*(\Gamma_{g+1,1})$$

The results that Harer derived in [10] show that this map (which is really a composition of two more basic maps) is an isomorphism for $2 \leqslant * \leqslant \frac{g+2}{3}$, and if $* = 1$ it is an isomorphism given that $g \geqslant 3$. (Ivanov [14] improved the homological-stability range to $* \leqslant \frac{g-1}{2}$, and Boldsen [5] further improved this to $* \leqslant \frac{2g-2}{3}$.) Therefore if we continue the process of attaching H to $\Sigma_{g+n,1}$ along B_-, the homology of the mapping class groups $\Gamma_{g+n,1}$ stabilizes to the homology of the **stable mapping class group** $\Gamma_\infty := \mathrm{colim}_n\, \Gamma_{g+n,1}$. Then the homology of MTSO(2) comes into play using the homology equivalence

$$\mathbb{Z} \times B\Gamma_\infty \to \Omega^\infty \,\mathrm{MTSO}(2)$$

given by the Madsen-Weiss theorem [21, Theorem 1.1]. If $g \geqslant 3$ then $\Gamma_{g,1}$ is perfect, so $H_1(\Omega^\infty \mathrm{MTSO}(2)) \cong H_1(\Gamma_\infty) = 0$; but infinite loop spaces have abelian fundamental group, so $\pi_1 \,\mathrm{MTSO}(2) \cong \pi_1(\Omega^\infty \mathrm{MTSO}(2)) = 0$. In an earlier paper [9] Harer also computed that $H_2(\Gamma_{g,1}) \cong \mathbb{Z}$ for $g \geqslant 5$, so using the Hurewicz theorem we can deduce that $\pi_2 \,\mathrm{MTSO}(2) \cong \mathbb{Z}$.

The methods used in this thesis are somewhat more elementary. Section 3.2 studies the maps in the cofibre sequence of Proposition 3.0.7, namely it interprets their induced homomorphisms at the level of bordism groups. Section 3.3 uses these interpretations to help compute the groups in Theorem 3.0.8, and gives explicit generators $g_2 \in \pi_1 \,\mathrm{MTSO}(3)$ and $g_4 \in \pi_2 \,\mathrm{MTSO}(2)$.

3.1 The Cofibre Sequence

Let X be a compact topological space, and let E and F be finite-dimensional vector bundles with metric over X. Let $p\colon S(F) \to X$ be the projection of the sphere bundle of F to X. If E is pulled back to $S(F)$ along p then there is a tautological bundle map

$$\tilde{p} \colon p^*E \to E$$

Furthermore, E naturally imbeds into $F \oplus E$; let

$$j \colon E \cong 0 \oplus E \subset F \oplus E$$

denote this embedding. As they are bundle maps, $\tilde{p}$ and j induce maps $\mathrm{Th}(\tilde{p})$, $\mathrm{Th}(j)$ on Thom spaces.

The total space of $\varepsilon \oplus p^*E$ can be embedded as an open subspace of $F \oplus E$: for $x \in X$, u a unit vector of F_x, $r \in \mathbb{R}$ and $w \in E_x$, the embedding is given by $((x, u); r, w) \mapsto (x; e^r \cdot u, w)$. In fact $\varepsilon \oplus p^*E$ is isomorphic to the normal bundle of $S(F)$ in $F \oplus E$. Define

$$\mathrm{PT} \colon \mathrm{Th}(F \oplus E) \to \mathrm{Th}(\varepsilon \oplus p^*E)$$

by $\mathrm{PT}(\infty) = \infty$ and for an element $(x; v, w) \in F \oplus E$ it is given by

$$\mathrm{PT}(x; v, w) = \begin{cases} \infty & v = 0 \\ \left((x, \frac{v}{\|v\|}); \ln \|v\|, w\right) & \text{otherwise} \end{cases} \tag{3.1.1}$$

Then PT could be though of as the Pontryagin-Thom collapse map of the embedding $S(F) \to F \oplus E$.

Lemma 3.1.1. *The situation described above produces a cofiber sequence of spaces:*

$$\mathrm{Th}(p^*E) \xrightarrow{\mathrm{Th}(\tilde{p})} \mathrm{Th}(E) \xrightarrow{\mathrm{Th}(j)} \mathrm{Th}(F \oplus E) \xrightarrow{\mathrm{PT}} \mathrm{Th}(\varepsilon \oplus p^*E)$$

Proof. $\mathrm{Th}(j)$ is a closed embedding with image $(0 \oplus E)^+$. PT is a topological embedding outside of the subspace $\mathrm{PT}^{-1}(\infty) = \mathrm{Im}\,\mathrm{Th}(j)$. It follows that

$$\mathrm{Th}(\varepsilon \oplus p^*E) \cong \mathrm{Th}(F \oplus E)/\operatorname{Im}\mathrm{Th}(j) \simeq \mathrm{Cone}(\mathrm{Th}(j))$$

Hence the last three spaces form a cofiber sequence.

In order to show that the first three spaces form a cofiber sequence, replace $\mathrm{Th}(E)$ with the mapping cylinder, and consider the mapping cone. Explicitly, let $\mathrm{Cone}(\mathrm{Th}(\tilde{p}))$ be the space

$$(\mathrm{Th}(p^*E) \times I) \bigcup \mathrm{Th}(E)$$

modulo the subspace $\mathrm{Th}(p^*E) \times 0$ and the relation $(b, u; v, 1) \sim (b; v)$ for $b \in X$, $u \in S(F_x)$, and $v \in E_x$. Then we can define a homeomorphism

$$\mathrm{Cone}(\mathrm{Th}(\tilde{p})) \to \frac{D(F) \oplus D(E)}{(D(F) \oplus S(E)) \cup (S(F) \oplus D(E))}$$

by sending $(b; v) \in \mathrm{Th}(E)$ to $(b; 0, v)$, and sending $(b, u; v, t) \in \mathrm{Th}(p^*E) \times I$ to $(b; (1-t)u, v)$. Then this is indeed well-defined, a bijection, and continuous in both directions. Finally, the space on the right is homeomorphic to $\mathrm{Th}(F \oplus E)$. □

Consider the situation where

$$X_n = B(d)_n, \; E_n = \theta^*_{d,n} U^{\perp}_{d,n}, \; F_n = \theta^*_{d,n} U_{d,n}$$

and label the projection map $p_n \colon S(F_n) \to X_n$. For the sake of notation let $U^{\theta}_{d,n} = \theta^*_{d,n} U_{d,n}$ and $U^{\theta,\perp}_{d,n} = \theta^*_{d,n} U^{\perp}_{d,n}$. Then for each n Lemma 3.1.1 gives a cofiber sequence

$$\mathrm{Th}(p_n^* U^{\theta,\perp}_{d,n}) \xrightarrow{\mathrm{Th}(\tilde{p}_n)} \mathrm{Th}(U^{\theta,\perp}_{d,n}) \xrightarrow{\mathrm{Th}(i_n)} \mathrm{Th}(U^{\theta}_{d,n} \oplus U^{\theta,\perp}_{d,n}) \xrightarrow{PT_n} \mathrm{Th}(\varepsilon \oplus p_n^* U^{\theta,\perp}_{d,n})$$

Although $U^{\theta}_{d,n} \oplus U^{\theta,\perp}_{d,n} \cong \varepsilon^n$, it helps to use this orthogonal decomposition to define bundle maps.

Recall that as n varies, the bundles $U^{\theta,\perp}_{d,n}$ form a stable vector bundle, whose Thom spectrum is $MT\,\theta_d$.

Lemma 3.1.2. *The sets* $\{U^{\theta}_{d,n} \oplus U^{\theta,\perp}_{d,n} \to B(d)_n\}_{n \geqslant d}$ *and* $\{p_n^* U^{\theta,\perp}_{d,n} \to S(U^{\theta}_{d,n})\}_{n \geqslant d}$ *form stable vector bundles, and the sets* $\{\tilde{p}_n\}_{n \geqslant d}$ *and* $\{i_n\}_{n \geqslant d}$ *are maps of stable vector bundles.*

Proof. Remember that for each $b \in B(d)_n$, $\theta(b)$ is a plane in $\mathrm{Gr}_d(\mathbb{R}^n)$. Given $b \in B(d)_n$, $v \in \theta(b)$, $w \perp \theta(b)$ and $t \in \mathbb{R}$, define a bundle map

$$\beta_n \colon U^{\theta}_{d,n} \oplus U^{\theta,\perp}_{d,n} \oplus \varepsilon \to U^{\theta}_{d,n+1} \oplus U^{\theta,\perp}_{d,n+1} \tag{3.1.2}$$

by $\beta_n(b; v, w, t) = (b; v, w + te_{n+1})$. Then this map covers the inclusion map $\lambda_n \colon B(d)_n \to B(d)_{n+1}$ and is an isomorphism in each fibre, so induces the required isomorphism for a stable vector bundle.

An element of $S(U^{\theta}_{d,n})$ has the form (b, u) where $b \in B(d)_n$ and $u \in \theta(b)$ is a unit vector; an element of $p_n^* U^{\theta,\perp}_{d,n}$ then has the form $((b,u); w)$ for $(b,u) \in S(U^{\theta}_{d,n})$ and $w \perp \theta(b)$. The inclusion $\mathbb{R}^n \to \mathbb{R}^{n+1}$ induces an inclusion $\tilde{\lambda}_n \colon S(U^{\theta}_{d,n}) \to S(U^{\theta}_{d,n+1})$ which covers λ_n:

$$\begin{array}{ccc} S(U^{\theta}_{d,n}) & \xrightarrow{\tilde{\lambda}_n} & S(U^{\theta}_{d,n+1}) \\ \downarrow p_n & & \downarrow p_{n+1} \\ B(d)_n & \xrightarrow{\lambda_n} & B(d)_{n+1} \end{array}$$

Then $\theta^*_{d,n}\phi_n \colon U^{\theta,\perp}_{d,n} \oplus \varepsilon \cong \lambda_n^* U^{\theta,\perp}_{d,n+1}$, where ϕ_n was the isomorphism $U^{\perp}_{d,\oplus}\varepsilon \cong \iota_n^* U^{\perp}_{d,n+1}$, induces an isomorphism/bundle map

$$p_n^* \theta^*_{d,n} \phi_n \colon p_n^* U^{\theta,\perp}_{d,n} \oplus \varepsilon \cong p_n^* \lambda_n^* U^{\theta,\perp}_{d,n+1} = \tilde{\lambda}_n^* p_{n+1}^* U^{\theta,\perp}_{d,n+1} \to p_{n+1}^* U^{\theta,\perp}_{d,n+1} \tag{3.1.3}$$

sending $((b,u); w, t)$ to $((b,u); w + te_{n+1})$, and covering $\tilde{\lambda}$.

$\{\tilde{p}_n\}$ and $\{i_n\}$ induce maps of stable vector bundles because the following diagram commutes:

$$\begin{array}{ccccc} p_n^* U^{\theta,\perp}_{d,n} \oplus \varepsilon & \xrightarrow{\tilde{p}_n \oplus \mathrm{Id}} & U^{\theta,\perp}_{d,n} \oplus \varepsilon & \xrightarrow{i_n \oplus \mathrm{Id}_\varepsilon} & U^{\theta}_{d,n} \oplus U^{\theta,\perp}_{d,n} \oplus \varepsilon \\ \downarrow p_n^* \theta(d)_n^* \phi_n & & \downarrow \theta(d)_n^* \phi_n & & \downarrow \beta_n \\ p_{n+1}^* U^{\theta,\perp}_{d,n+1} & \xrightarrow{\tilde{p}_{n+1}} & U^{\theta,\perp}_{d,n+1} & \xrightarrow{i_{n+1}} & U^{\theta}_{d,n+1} \oplus U^{\theta,\perp}_{d,n+1} \end{array}$$

□

Now we can define the spectra $\mathbb{G}$ and $\mathbb{P}$ alluded to in the introduction:

Definition 3.1.3. *Let $\mathbb{P}$ be the Thom spectrum of the stable vector bundle $\{p_n^* U_{d,n}^{\theta,\perp}\}$, and let $\mathbb{G}$ be the Thom spectrum of the stable vector bundle $\{U_{d,n}^{\theta} \oplus U_{d,n}^{\theta,\perp}\}$.*

It should be remarked that as n varies the bundle isomorphisms $U_{d,n}^{\theta} \oplus U_{d,n}^{\theta,\perp} \cong_n B(d)_n \times \mathbb{R}^n$ induce homeomorphisms $\mathbb{G}_{(n)} \cong (B(d)_n)_+ \wedge S^n$ making the following diagram commute:

$$\begin{array}{ccc} Th(U_{d,n}^{\theta} \oplus U_{d,n}^{\theta,\perp}) \wedge S^1 & \xrightarrow{\cong_n \wedge Id} & (B(d)_n)_+ \wedge S^n \wedge S^1 \\ \downarrow{\scriptstyle Th(\beta_n)} & & \downarrow{\scriptstyle \lambda_n \wedge h_n} \\ Th(U_{d,n+1}^{\theta} \oplus U_{d,n+1}^{\theta,\perp}) & \xrightarrow{\cong_{n+1}} & (B(d)_{n+1})_+ \wedge S^{n+1} \end{array}$$

where h_n is the standard homeomorphism $S^n \wedge S^1 \cong S^{n+1}$. However, we will continue using the $U_{d,n}^{\theta} \oplus U_{d,n}^{\theta,\perp}$ description so that there are less identifications to keep track of.

Lemma 3.1.4. *The collections of maps $\{\tilde{p}_n\}$, $\{i_n\}$ and $\{PT_n\}$ induce maps of spectra.*

$$\tilde{p}\colon \mathbb{P} \to MT\,\theta_d,\ i\colon\ MT\,\theta_d \to \mathbb{G} \text{ and } PT\colon \mathbb{G} \to S^1 \wedge \mathbb{P}$$

Proof. By Lemma 3.1.2, the sets $\{\tilde{p}_n\}$, $\{i_n\}$ are maps of stable vector bundles. Applying the Thom space functor induces maps $\tilde{p} := Th(\{\tilde{p}_n\})$ and $i := Th(\{i_n\})$.

The maps $\{PT_n\}$ are not given by a bundle maps, so showing they induce a spectrum map needs to be done explicitly. Recall that the n-th structure map of $S^1 \wedge \mathbb{P}$ is $Id_{S^1} \wedge Th(p_n^* \theta(d)_n^* \phi_n)$; then the following diagram must commute:

$$\begin{array}{ccc} Th(U_{d,n}^{\theta} \oplus U_{d,n}^{\theta,\perp} \oplus \varepsilon) & \xrightarrow{PT_n \wedge Id_{S^1}} & Th(\varepsilon \oplus p_n^* U_{d,n}^{\theta,\perp} \oplus \varepsilon) \\ \downarrow{\scriptstyle Th(\beta_n)} & & \downarrow{\scriptstyle Id_{S^1} \wedge Th(p_n^* \theta(d)_n^* \phi_n)} \\ Th(U_{d,n+1}^{\theta} \oplus U_{d,n+1}^{\theta,\perp}) & \xrightarrow{PT_{n+1}} & Th(\varepsilon \oplus p_{n+1}^* U_{d,n+1}^{\theta,\perp}) \end{array}$$

Let $b \in B(d)_n$, $v \in \theta(b)$, $w \perp \theta(b)$, and $t \in \mathbb{R}$. If $v = 0$ then both compositions send $(b; v, w, t)$ to ∞, and if $v \neq 0$ then

$$\begin{aligned} &(Id_{S^1} \wedge Th(p_n^* \theta(d)_n^* \phi_n)) \circ (PT_n \wedge Id_{S^1})(b; v, w, t) = \\ &Id_{S^1} \wedge Th(p_n^* \theta(d)_n^* \phi_n) \left((b, \frac{v}{\|v\|}); \ln\|v\|, w, t \right) = \left((b, \frac{v}{\|v\|}); \ln\|v\|, w + te_{n+1} \right) \end{aligned} \tag{3.1.4}$$

and

$$PT_{n+1} \circ Th(\beta_n)(b; v, w, t) = PT_{n+1}(b; v, w + te_{n+1}) = \left((b, \frac{v}{\|v\|}); \ln\|v\|, w + te_{n+1} \right)$$

□

Corollary 3.1.5. $\mathbb{P} \xrightarrow{\tilde{p}} MT\,\theta_d \xrightarrow{i} \mathbb{G} \xrightarrow{PT} S^1 \wedge \mathbb{P}$ *is a cofiber sequence of spectra.*

Proof. By Lemma 3.1.1 this sequence is a level-wise cofibre sequence. □

Therefore we have identified *some* cofibre sequence of spectra, but the spectra $\mathbb{P}$ and $\mathbb{G}$ aren't precisely the spectra $\Sigma^{-1}\,\mathrm{MT}\,\theta_{d-1}$ and $\Sigma^\infty B(d)_+$ that we want. Thankfully they are of the same homotopy type.

Lemma 3.1.6. *There are weak homotopy equivalences of Thom specra*

$$\iota\colon \mathbb{G} \to \Sigma^\infty B(d)_+,\ \ \psi\colon \Sigma^{-1}\,\mathrm{MT}\,\theta_{d-1} \to \mathbb{P}$$

Proof. The maps in question here are induced by pulling back stable vector bundles.

For each n, $B(d)_n \subset B(d)$ and the trivial bundle pulls back to $U^\theta_{d,n} \oplus U^{\theta,\perp}_{d,n}$ in a way which is compatible with stabilization, so there is a map from the stable vector bundle underlying $\mathbb{G}$ to the one whose n-th space is $B(d) \times \mathbb{R}^n$, inducing a map of Thom spectra $\iota\colon \mathbb{G} \to \Sigma^\infty B(d)_+$. The connectivity of the map $B(d)_n \to B(d)$ is the same as the map $\mathrm{Gr}_d(\mathbb{R}^n) \to \mathrm{Gr}_d(\mathbb{R}^\infty)$, which is $(n-d-1)$-connected by Corollary 1.1.6. Then the map $\Sigma^n(B(d)_n)_+ \to \Sigma^n B(d)_+$ is $(2n-d-1)$ connected by Lemma 1.1.2, i.e. it induces an isomorphism on π_{n+k} as long as $n > k+d+1$. Therefore the induced map of spectra is a weak homotopy equivalence.

There are bundle maps

$$\begin{array}{ccccc}
U^\perp_{d-1,n-1} & \longrightarrow & p_n^* U^\perp_{d,n} & \longrightarrow & U^\perp_{d,n} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Gr}_{d-1}(\mathbb{R}^{n-1}) & \xrightarrow{f_{n-1}} & S(U_{d,n}) & \xrightarrow{p_n} & \mathrm{Gr}_d(\mathbb{R}^n)
\end{array}$$

where $f_{n-1}(P) = (\langle e_1\rangle \oplus \mathrm{sh}\,P; e_1)$. In other words, the stable vector bundle whose Thom spectrum is $\Sigma^{-1}\,\mathrm{MTO}(d-1)$ is the pullback of $\{p_n^* U^\perp_{d,n}\}$ along $\{f_{n-1}\}$. Then map f_{n-1} is $(n-2)$-connected by Lemma 1.1.7.

Moreover, we can define $g_{n-1}\colon B(d-1)_{n-1} \to S(U^\theta_{d,n})$ by $g_{n-1}(b) = (b; e_1)$ since $\theta_d(b) = \langle e_1\rangle \oplus \theta_{d-1}(b)$. With tangential structure included, there is the diagram

$$\begin{array}{ccccc}
 & & p_n^* U^{\theta,\perp}_{d,n} & \longrightarrow & U^{\theta,\perp}_{d,n} = \theta^*_{d,n} U^\perp_{d,n} \\
 & & \downarrow & & \downarrow \\
B(d-1)_{n-1} & \xrightarrow{g_{n-1}} & S(U^\theta_{d,n}) & \xrightarrow{p_n} & B(d)_n \\
\downarrow{\scriptstyle \theta_{d-1,n-1}} & & \downarrow & & \downarrow{\scriptstyle \theta_{d,n}} \\
\mathrm{Gr}_{d-1}(\mathbb{R}^{n-1}) & \xrightarrow{f_{n-1}} & S(U_{d,n}) & \longrightarrow & \mathrm{Gr}_d(\mathbb{R}^n)
\end{array}$$

Then by commutativity of this diagram, $g^*_{n-1} p^*_n U^{\theta,\perp}_{d,n} \cong \theta^*_{d-1,n-1} U^\perp_{d-1,n-1}$, and so there is an induced map $\psi\colon \Sigma^{-1}\,\mathrm{MT}\,\theta_{d-1} \to \mathbb{P}$.

The map $g_{n-1}\colon B(d-1)_{n-1} \to S(U^\theta_{d,n})$ has the same connectivity as f_{n-1}. Then, so long as $n-d \geqslant 2$, the induced map on Thom spaces has connectivity $(2n-d-1)$ by Lemma 1.1.2, so again it induces an isomorphism on π_{n+k} when $n > k+d+1$, hence ψ is a weak-homotopy equavalence.

□

All of the above is summarized in the following, which proves Proposition 3.0.7:

Corollary 3.1.7. *There is the following diagram of spectra, where the horizontal sequence is a cofibration and the vertical arrows are homotopy equivalences:*

$$\begin{array}{ccccccc}
\mathbb{P} & \xrightarrow{\tilde{p}} & \mathrm{MT}\,\theta_d & \xrightarrow{i} & \mathbb{G} & \xrightarrow{PT} & S^1 \wedge \mathbb{P} \\
{\scriptstyle\psi}\uparrow & & & & \downarrow{\scriptstyle\iota} & & {\scriptstyle \mathrm{Id}\wedge\psi}\uparrow \\
\Sigma^{-1}\,\mathrm{MT}\,\theta_{d-1} & & & & \Sigma^{\infty}B(d)_+ & & S^1 \wedge (\Sigma^{-1}\,\mathrm{MT}\,\theta_{d-1}) \\
 & & & & & & \downarrow{\scriptstyle\simeq} \\
 & & & & & & \mathrm{MT}\,\theta_{d-1}
\end{array}$$

3.2 Interpretting Induced Maps on Homotopy Groups

In light of Proposition 1.2.3, the maps between spectra in the cofibre sequence of Proposition 3.0.7 induce maps of bordism groups, whose interpretations are given here.

The basic template is as follows: suppose $\mathbb{E}$ and $\mathbb{E}'$ are Thom spectra with n-th spaces $\mathrm{Th}(V_n)$ and $\mathrm{Th}(V_n')$ respectfully, where both V_n and V_n' are smooth bundles over manifolds, and let $f\colon \mathbb{E} \to \mathbb{E}'$ be any map. Given a map $\Phi\colon S^{n+k} \to \mathrm{Th}(V_n)$ representing an element of $\pi_k\mathbb{E}$, we arrange (by replacing with homotopic maps) that Φ and $f \circ \Phi$ are transverse to the zero sections of V_n and V_n'. Then, via the Pontryagin-Thom correspondence, Φ produces a manifold M with some bundle data D, $f \circ \Phi$ produces (M', D') in a similar way, and

$$\pi_k(f)[M, D] = [M', D']$$

Then, a procedure is given which explicitly turns the data (M, D) into something bordant to (M', D'). As long as this procedure is bordism invariant it will give an explicit formula for $\pi_k(f)$, since any element of $\pi_k\mathbb{E}$ can be represented in this way.

In practice, a bundle V_n will be over something like $B(d)_n$ which is not a smooth manifold, but it will be pulled-back from a bundle U_n over something like $\mathrm{Gr}_d(\mathbb{R}^n)$. Then composing Φ with the map $\mathrm{Th}(V_n) \to \mathrm{Th}(U_n)$ can be made transverse to $\mathrm{Gr}_d(\mathbb{R}^n)$ to give the manifold and bundle data as above, and the "lift" $\Phi|_M\colon M \to B(d)_n$ gives the tangential structure.

Since $\mathbb{P} \simeq \Sigma^{-1}\,\mathrm{MT}\,\theta_{d-1}$ we can describe its homotopy groups using Proposition 1.2.3, as bordism classes of triples (M, E, φ) where M has dimension $(k+1) + (d-1) = k + d$ and E is a θ_{d-1}-bundle. However we will use the $\mathbb{P}$ model of this spectrum in the proof of the following.

Lemma 3.2.1. *The homomorphism $\pi_k(\tilde{p})\colon \pi_{k+1}\,\mathrm{MT}\,\theta_{d-1} \to \pi_k\,\mathrm{MT}\,\theta_d$ sends the class $[M, E, \varphi]$ to $[M, \varepsilon \oplus E, \varphi]$, where $\varepsilon \oplus E$ has the natural θ_d-structure.*

Proof. Use the model $\mathbb{P}$ for $\Sigma^{-1}\,\mathrm{MT}\,\theta_{d-1}$. Let $q_n\colon S(U_{d,n}^{\perp}) \to \mathrm{Gr}_d(\mathbb{R}^n)$, $p_n\colon S(U_{d,n}^{\theta}) \to B(d)_n$, and let $\tilde{\theta}\colon S(U_{d,n}^{\theta}) \to S(U_{d,n})$ be the obvious map covering θ.

Take a map $\Phi\colon S^{n+k} \to \mathrm{Th}(p_n^* U_{d,n}^{\theta,\perp})$ so that the composition with

$$S^{n+k} \xrightarrow{\Phi} \mathrm{Th}(p_n^* U_{d,n}^{\theta,\perp}) \xrightarrow{\mathrm{Th}(\tilde\theta)} \mathrm{Th}(q_n^* U_{d,n}^{\perp})$$

is transverse to the zero-section $S(U_{d,n})$; let $M = (\mathrm{Th}(\tilde\theta) \circ \Phi)^{-1}(S(U_{d,n}))$ and $\bar f = (\mathrm{Th}(\tilde\theta)\circ\Phi)|_M$. Note that $\bar f$ factors through a map $f\colon M \to \mathrm{Th}(p_n^* U_{d,n}^{\theta,\perp})$ because $\tilde\theta$ is an isomorphism in each fibre. Then indeed M has dimension $k+d$, and the embedding $M \subset S^{n+k}$ induces an isomorphism $TM \oplus \nu_M^{S^{n+k}} \cong \varepsilon^{n+k}$. Since $\nu_M^{S^{n+k}} \cong f^* p_n^* U_{d,n}^{\theta,\perp}$, adding $f^* p_n^* U_{d,n}^{\theta}$ to both sides gives an isomorphism

$$\varphi\colon TM \oplus \varepsilon^n \cong \varepsilon^{n+k} \oplus f^* p_n^* U_{d,n}^{\theta}$$

Unlike $U_{d,n}^{\theta}$ the bundle $p_n^* U_{d,n}^{\theta} \to S(U_{d,n}^{\theta})$ has a canonical non-zero section, namely $\sigma(b;u) = (b,u;u)$, which induces an isomorphism $p_n^* U_{d,n}^{\theta} \cong \varepsilon \oplus \langle\sigma\rangle^\perp$. Then Φ defines the element $[M, f^*\langle\sigma\rangle^\perp, \varphi] \in \pi_k \mathbb{P}$.

Now consider the composition

$$\mathrm{Th}(\tilde p_n)\circ\Phi\colon S^{n+k} \to \mathrm{Th}(p_n^* U_{d,n}^{\theta,\perp}) \to \mathrm{Th}(U_{d,n}^{\theta,\perp})$$

Then composition of this map with the one to $\mathrm{Th}(U_{d,n}^{\perp})$ is transverse to $\mathrm{Gr}_d(\mathbb{R}^n)$. If $M' = (\mathrm{Th}(\tilde p_n)\circ\Phi)^{-1}(\mathrm{Gr}_d(\mathbb{R}^n))$ and $f' = \mathrm{Th}(\tilde{p_n})\circ\Phi|_{M'}$, then just as above there is an induced isomorphism

$$\varphi'\colon TM' \oplus \varepsilon^n \cong \varepsilon^{n+k} \oplus f'^* U_{d,n}^{\theta}$$

Then $\mathrm{Th}(\tilde p_n)\circ\Phi$ defines the element $[M', f'^* U_{d,n}^{\theta}, \varphi'] \in \pi_k \mathrm{MT}\,\theta_d$ and

$$\pi_k(\tilde p)([M, f^*\langle\sigma\rangle^\perp, \varphi]) = [M', f'^* U_{d,n}^{\theta}, \varphi']$$

However $M' = (\mathrm{Th}(\tilde p_n)\circ\Phi)^{-1}(\mathrm{Gr}_d(\mathbb{R}^n)) = \Phi^{-1}(S(U_{d,n}^{\theta})) = M$. Thus $f' = \tilde p_n \circ f$ so $f'^* U_{d,n}^{\theta} = \varepsilon \oplus f^*\langle\sigma\rangle^\perp$ and $\varphi' = \varphi$. □

Now consider the spectrum $\mathbb{G} \simeq \Sigma^\infty B(d)_+$. An element of $\pi_k\mathbb{G}$ is represented by a map

$$\Gamma\colon S^{n+k} \to \mathrm{Th}(U_{d,n}^{\theta} \oplus U_{d,n}^{\theta,\perp})$$

such that $\mathrm{Th}(\tilde\theta)\circ\Gamma \pitchfork \mathrm{Gr}_d(\mathbb{R}^n)$; let $N = (\mathrm{Th}(\tilde\theta)\circ\Gamma)^{-1}(\mathrm{Gr}_d(\mathbb{R}^n))$ and $g = (\mathrm{Th}(\tilde\theta)\circ\Gamma)|_N$. Then $\nu_N^{S^{n+k}} \cong g^*\varepsilon^n$ and hence we get an isomorphism

$$\psi\colon TN \oplus \varepsilon^n \cong TS^{n+k}|_N \cong \varepsilon^{n+k}$$

We also naturally have a θ_d-bundle over N, namely $g^* U_{d,n}^{\theta}$.

Lemma 3.2.2. *$\pi_k\mathbb{G}$ is the bordism group of triples $[N, \psi, E]$ where (N,ψ) is a framed manifold and E is a map $N \to B(d)$.*

This can also be seen by noting that $\Sigma^\infty B(d)_+ \simeq B(d)_+ \wedge \mathbb{S}$, so its k-th homotopy group is the (un-reduced) k-th homology group associated to the spectrum $\mathbb{S}$ applied to $B(d)$, that is $\Omega_k^{fr}(B(d))$.

Now we interpret the homomorphism $\pi_k(i)\colon \pi_k \mathrm{MT}\,\theta_d \to \pi_k\Sigma^\infty B(d)_+$. Suppose $[M, E, \varphi]$ is an element of $\pi_k \mathrm{MT}\,\theta_d$. Let s be a section of E transverse to the zero-section, and let $S = s^{-1}(0)$, a smooth manifold of dimension k with normal bundle $\nu_S^M \cong E|_S$. Denote by $j\colon S \hookrightarrow M$ the inclusion. If we choose some complement $E^\perp$ of rank r, then the map $\tilde\varphi$ defined by

$$\begin{array}{ccc} \tilde\varphi\colon TS \oplus E|_S \oplus \varepsilon^n \oplus E|_S^\perp & \longrightarrow & \varepsilon^{d+n+k+r} \\ \downarrow\cong & & \uparrow\cong \\ TM|_S \oplus \varepsilon^n \oplus E|_S^\perp & \xrightarrow{\varphi|_S \oplus \mathrm{Id}_{E^\perp}} & E|_S \oplus \varepsilon^{n+k} \oplus E|_S^\perp \end{array}$$

is a stable framing of S.

Lemma 3.2.3. *The map* $\pi_k(i)\colon \pi_k \operatorname{MT}\theta_d \to \pi_k\Sigma^\infty B(d)_+$ *is given by*

$$\pi_k(i)[M, E, \varphi] = [S, \tilde\varphi, E|_S]$$

Proof. Let $\bar s$ be a section of $U_{d,n}$, transverse to the zero-section; this induces a section $s\colon B(d)_n \to U^\theta_{d,n}$ because $U^\theta_{d,n}$ a pull-back of $U_{d,n}$. Note that the map

$$s \oplus \mathrm{Id}\colon U^{\theta,\perp}_{d,n} \to U^\theta_{d,n} \oplus U^{\theta,\perp}_{d,n}$$

defined by $(s \oplus \mathrm{Id})(b; w) = (b; s(b), w)$ is not a bundle map so the Thom space functor doesn't apply, but it still induces a map $(s \oplus \mathrm{Id})^+$ on the one-point compactifications. Since $s \oplus \mathrm{Id} \sim 0 \oplus \mathrm{Id}$, the map $\pi_k(i)$ can be described by composing with $(s \oplus \mathrm{Id})^+ \sim \mathrm{Th}(0 \oplus \mathrm{Id})$.

Now let $\Phi\colon S^{n+k} \to \mathrm{Th}(U^{\theta,\perp}_{d,n})$ so that $\mathrm{Th}(\tilde\theta) \circ \Phi \pitchfork \mathrm{Gr}_d(\mathbb{R}^n)$, and let $\Gamma = (s \oplus \mathrm{Id})^+ \circ \Phi$. Then the composition

$$S^{n+k} \xrightarrow{\Phi} \mathrm{Th}(U^{\theta,\perp}_{d,n}) \xrightarrow{(s\oplus\mathrm{Id})^+} \mathrm{Th}(U^\theta_{d,n} \oplus U^{\theta,\perp}_{d,n}) \xrightarrow{\mathrm{Th}\,\tilde\theta} \mathrm{Th}(U_{d,n} \oplus U^\perp_{d,n})$$

is again transverse to $\mathrm{Gr}_d(\mathbb{R}^n)$. Then Φ and Γ simultaneously define triples (M^{k+d}, E, φ) and (N, ψ, E') respectively, where $E = \Phi|_M^* U^\theta_{d,n}$ and $E' = \Gamma|_N^* U^\theta_{d,n}$, and

$$\pi_k(i)[M, E, \varphi] = [N, \psi, E']$$

Let $f = \Phi|_M,\ g = \Gamma|_N$. Note that $s \circ f$ defines a section of $f^* U^\theta_{d,n}$ which is transverse to M. Then

$$(s \circ f)^{-1}(M) = \{x \in M \mid sf(x) = (f(x); 0)\} = \mathrm{Th}(\tilde\theta) \circ \Gamma^{-1}(\mathrm{Gr}_d(\mathbb{R}^n))$$

That is N is the manifold S described before the statement of the Lemma. Moreover $g^* U^\theta_{d,n} \cong f^* U^\theta_{d,n}|_N$, again since $s \sim 0$, i.e. $E' = E|_N$.

Finally, the framing of N given by the above is of the form $\psi\colon N \oplus \varepsilon^n \cong \varepsilon^{n+k}$, but the framing $\tilde\varphi$ can be interpreted as $\psi \oplus \mathrm{Id}_{\varepsilon^{d+r}}$, and these two framings are equivalent under the bordism relation. I.e.

$$(N, \psi, E') \sim (S, \tilde\varphi, E|_S)$$

□

We want to now describe $\pi_k \operatorname{PT}\colon \pi_k\Sigma^\infty B(d)_+ \to \pi_k \operatorname{MT}\theta_{d-1}$, so consider an element $[N, \psi, E] \in \pi_k\Sigma^\infty B(d)_+$. The total space of the sphere bundle $p\colon S(E) \to N$ is a closed manifold of dimension $k+d-1$, and choosing a metric on E induces an isomorphism $\delta\colon TS(E) \cong T_v S(E) \oplus p^*TN$. Then let $\tilde\psi$ be the composition

$$(\mathrm{Id} \oplus \psi) \circ (\delta \oplus \mathrm{Id})\colon TS(E) \oplus \varepsilon \cong T_v S(E) \oplus p^*TN \oplus \varepsilon \cong T_v S(E) \oplus \varepsilon^{k+1}$$

Note that $T_v S(E)$ is a rank $d-1$ bundle over $S(E)$. By Proposition 1.1.9, $\varepsilon \oplus T_v S(E)$ is naturally isomorphic to p^*E where $p\colon E \to N$ is the bundle projection, and p^*E does have a $\theta(d)$ structure, say $\lambda\colon S(E) \to B(d)$. Then the decomposition $p^*E \cong \langle\sigma\rangle^\perp \oplus \varepsilon$ induces a homotopy of its classifying map into $BO(d-1) \hookrightarrow BO(d)$, which lifts to a homotopy of λ into $B(d-1) \hookrightarrow B(d)$ by the homotopy lifting property; this induces a θ_{d-1} structure on $T_v S(E)$.

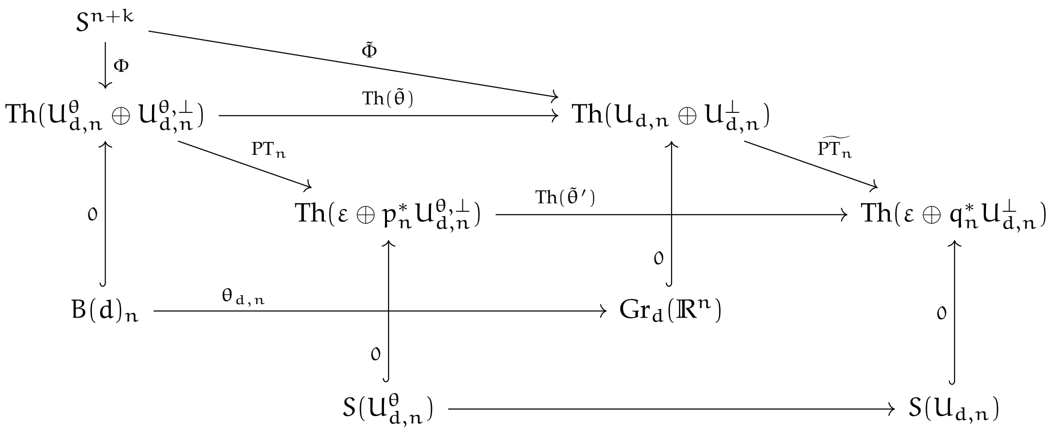
Lemma 3.2.4. *The map $\pi_k \, PT\colon \pi_k \Sigma^\infty B(d)_+ \to \pi_k \, MT\,\theta_{d-1}$ is given by*

$$\pi_k \, PT[N, \psi, E] = [S(E), T_v S(E), \phi]$$

where ϕ is described in the proof.

Proof. We continue to use the $S^1 \wedge \mathbb{P}$ model for $MT\,\theta d - 1$.

Let $\Phi\colon S^{n+k} \to Th(U^\theta_{d,n} \oplus U^{\theta,\perp}_{d,n})$ be continuous, let $Th(\tilde\theta)$ denote the map $Th(U^\theta_{d,n} \oplus U^{\theta,\perp}_{d,n}) \to Th(U_{d,n} \oplus U^\perp_{d,n})$ induced by pulling-back along θ, and let $\tilde\Phi = Th(\tilde\theta) \circ \Phi$. Let $M = \tilde\Phi^{-1}(S(U_{d,n} \oplus 0))$, $f = \tilde\Phi|_M$. A diagram might help:



Let $D = 2 \cdot D(U_{d,n} \oplus U^\perp_{d,n}) \subset Th(U_{d,n} \oplus U^\perp_{d,n})$. Wlog $\tilde\Phi^{-1}(D)$ is a tubular nhd of $M \subset S^{n+k}$, say τ. Then in particular, $\tilde\Phi$ is transverse to the submanifolds $Gr_d(\mathbb{R}^n)$, $S(U_{d,n} \oplus 0)$, and $D(U_{d,n} \oplus 0)$. Let $W = \tilde\Phi^{-1}(D(U_{d,n} \oplus 0))$, let $F = \tilde\Phi|_W$, let $N = \tilde\Phi^{-1}(Gr_d(\mathbb{R}^n))$ and $f = \tilde\Phi|_N$: then W is a d-disk bundle over N with projection π, say, and M is its sphere bundle with projection $p = \pi|_M$. Moreover the normal bundle of N in W is isomorphic to $f^* U_{d,n}$.

Note that the normal bundle of $D(U_{d,n} \oplus 0)$ in $U_{d,n} \oplus U^\perp_{d,n}$ is the pullback of $U^\perp_{d,n}$ along the projection to $Gr_d(\mathbb{R}^n)$; then the normal bundle of W in S^{n+k} is $\pi^* f^* U^\perp_{d,n}$ since the following commutes

$$\begin{array}{ccc} W & \xrightarrow{F} & U_{d,n} \oplus 0 \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ N & \xrightarrow{f} & Gr_d(\mathbb{R}^n) \end{array}$$

Then there is an isomorphism

$$\xi\colon \varepsilon^{n+k} = TS^{n+k}|_W \cong TW \oplus \pi^* f^* U^\perp_{d,n} \cong \pi^* TN \oplus \pi^* f^* U_{d,n} \oplus \pi^* f^* U^\perp_{d,n}$$

Restricting this to N gives the usual isomorphism

$$\psi\colon TN \oplus \varepsilon^n \cong TN \oplus f^*U_{d,n} \oplus f^*U_{d,n}^{\perp} \cong TS^{n+k}|_N \cong \varepsilon^{n+k}$$

Therefore $\tilde{\Phi}$ defines the element $[N, \psi, f^*U_{d,n}] \in \pi_k\Sigma^\infty BO(d)_+$. By definition F factors through $\mathrm{Th}(U_{d,n}^\theta \oplus U_{d,n}^{\theta,\perp})$ via $\Phi|_W$, mapping W to $D(U_{d,n}^\theta \oplus 0)$ and N to $B(d)_n$; in particular $f^*U_{d,n}$ comes with a $\theta(d)$-structure. Thus Φ (without a tilde) defines an element $[N, \psi, f^*U_{d,n}] \in \pi_k\Sigma^\infty B(d)_+$. Abbreviate $E = f^*U_{d,n}$.

Now consider Φ composed with $PT_n\colon \mathrm{Th}(U_{d,n}^\theta \oplus U_{d,n}^{\theta,\perp}) \to \mathrm{Th}(\varepsilon \oplus p_n^* U_{d,n}^{\theta,\perp})$. Then the care taken to arrange Φ ensures that $\mathrm{Th}(\tilde{\theta}') \circ PT_n \circ \Phi$ is transverse to the zero-section $S(U_{d,n})$ of $\varepsilon \oplus p_n^* U_{d,n}^{\perp}$. Then

$$(\mathrm{Th}(\tilde{\theta}') \circ PT_n \circ \Phi)^{-1}(S(U_{d,n})) = \Phi^{-1}(S(U_{d,n}^\theta \oplus 0)) = M$$

If $g = (\mathrm{Th}(\tilde{\theta}') \circ PT_n \circ \Phi)|_M$ we get $\varepsilon^{n+k} \cong TS|_M \cong TM \oplus \varepsilon \oplus g^* q_n^* U_{d,n}^{\perp}$ and in particular

$$\varphi\colon TM \oplus \varepsilon^{n+1} \cong \varepsilon^{n+k+1} \oplus g^* p_n^* U_{d,n} \cong \varepsilon^{n+k+2} \oplus g^* \langle\sigma\rangle^{\perp}$$

Regarding M as a sphere bundle over N, g is a map of sphere bundles and $\langle\sigma\rangle^{\perp} \cong T_\nu U_{d,n}$ so $g^*\langle\sigma\rangle^{\perp} \cong T_\nu M$. Therefore $\widetilde{PT_n} \circ \tilde{\Phi}$ determines the element $[S(E), T_\nu S(E), \varphi] \in \pi_k\, MTO(d-1)$, which we wish to upgrade to an element of $\pi_k\, MT\,\theta(d)$ by showing that $T_\nu S(f^*U_{d,n})$ has a natural $\theta(d-1)$-structure. Then

$$\pi_k\, PT[N, \psi, E] = [S(E), T_\nu S(E), \varphi]$$

It remains to describe the relation between ψ and φ. If the isomorphism ξ is restricted to $M = S(E)$ it becomes

$$\varepsilon^{n+k} \cong p^*TN \oplus p^*f^*U_{d,n} \oplus p^*f^*U_{d,n}^{\perp} \cong p^*TN \oplus \varepsilon \oplus T_\nu S(E) \oplus p^*f^*U_{d,n}^{\perp}$$

Adding $p^*f^*U_{d,n}$ to both sides results in φ, and it is seen from this that $\varphi = (\mathrm{Id} \oplus \psi) \circ (\delta \oplus \mathrm{Id})$. Intuitively the point is that ξ is supposed to interpolate between φ and ψ.

□

3.3 Some Computations

Using the above interpretations, some computations can be made for small values of k and d. First, we will finally give a proof of

Proposition 1.2.7. *Let* $\theta\colon B \to BSO$ *be a stable tangential structure, and let* θ_d, θ_{d+1} *be its restriction to* $BSO(d)$ *and* $BSO(d+1)$ *respectively, and suppose* $B(d+1) = \theta^{-1}\, BSO(d+1)$ *is connected. Then there is a short-exact sequence*

$$0 \to \mathbb{Z}/\,\mathrm{Eul}_{d+1}^{\theta} \to \pi_0\, MT\,\theta_d \to \Omega_d^{\theta_{d+1}} \to 0$$

Moreover, this sequence is split except possibly for the case where $d+1 \in 4\mathbb{N}$. *If* $B = BSO$ *and* θ *is the identity then it always splits.*

Note that by Lemma 3.2.3, $\mathrm{Eul}^{\theta}_{d+1} = \mathrm{Im}(\pi_0 i\colon \pi_0 \mathrm{MT}\,\theta_d \to \pi_0\Sigma^\infty B(d+1)_+)$ so $\mathrm{Eul}^{\theta}_{d+1}$ is indeed a subgroup of $\mathbb{Z}$.

Recall that

$$\mathrm{Eul}_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \mathbb{Z} & \text{if } n \equiv 0 \mod 4 \\ 2\mathbb{Z} & \text{if } n \equiv 2 \mod 4 \end{cases}$$

Lemma 3.3.1. *If* $n \not\equiv 0 \mod 4$ *then* $\mathrm{Eul}^{\theta}_n = \mathrm{Eul}_n$ *for any* $\theta\colon B \to BSO$.

Proof. If $n \equiv 1$ or $3 \mod 4$ then $\langle e(V), [M]\rangle = 0$ for *any* rank n bundle, since the Euler class of an odd-rank bundle is 2-torsion and a stable isomorphism $V \cong_s TM$ would mean M is orientable. Therefore $\mathrm{Eul}^{\theta}_n = \mathrm{Eul}_n = 0$.

Since $\theta_n\colon B(n) \to BSO(n)$ is pulled back from a tangential structure over $BSO(n+1)$, it follows from Lemma 5.6 of [8] that in particular TS^n admits a θ_n structure, and if n is even then $\langle e(TS^n), [S^n]\rangle = \chi(S^n) = 2$: it follows that $\mathrm{Eul}^{\theta}_n \supset 2\mathbb{Z}$ when n is even. When $n \equiv 2 \mod 4$ then $\mathrm{Eul}_n = 2\mathbb{Z}$ as above, and since the bundle V is required to be stably isomorphic to TM these bundles have the same w_n and hence their Euler numbers have the same parity, i.e. $\langle e(V), [M]\rangle$ cannot be odd and so $\mathrm{Eul}^{\theta}_n = 2\mathbb{Z} = \mathrm{Eul}_n$. □

Note that when $n \equiv 0 \mod 4$ then there is a disparity: a classical theorem of Rohlin [28] states that if M is a smooth, closed, spin 4-manifold then $\sigma(M)$ is divisible by 16, and since the signature and Euler characteristic of a manifold are congruent modulo 2 it follows that $\mathrm{Eul}^{spin}_4 \neq \mathbb{Z}$; however $\mathrm{Eul}_4 = \mathbb{Z}$ since $\chi(\mathbb{CP}^2\#(S^1 \times S^3)) = 1$.

Proof of Proposition 1.2.7. The tangential structure $\theta_{d+1}\colon B(d+1) \to BSO(d+1)$ leads to a cofibre sequence of spectra, as in Proposition 3.0.7:

$$\mathrm{MT}\,\theta_{d+1} \xrightarrow{i} \Sigma^\infty B(d+1)_+ \xrightarrow{PT} \mathrm{MT}\,\theta_d$$

Focusing on the π_0 region yields

$$\pi_0\,\mathrm{MT}\,\theta_{d+1} \xrightarrow{\pi_0(i)} \pi_0\Sigma^\infty B(d+1)_+ \xrightarrow{\pi_0 PT} \pi_0\,\mathrm{MT}\,\theta_d \longrightarrow \pi_{-1}\,\mathrm{MT}\,\theta_{d+1} \longrightarrow 0$$

According to Lemma 3.2.3, the map $\pi_0(i)$ takes a tuple $[M^{d+1}, E^{d+1}, \phi]$ to $\langle e(E), [M]\rangle$, and it follows that $\mathrm{Im}\,\pi_0(i) = \mathrm{Eul}^{\theta}_{d+1}$. Then the identifications $\pi_{-1}\,\mathrm{MT}\,\theta_{d+1} \cong \Omega^{\theta_{d+1}}_d$ and $\pi_0\Sigma^\infty B(d+1)_+ \cong \mathbb{Z}$ yeild the short exact sequence

$$0 \to \mathbb{Z}/\,\mathrm{Eul}^{\theta}_{d+1} \to \pi_0\,\mathrm{MT}\,\theta_d \to \Omega^{\theta_{d+1}}_d \to 0$$

The homomorphism $\pi_0 PT\colon \pi_0\Sigma^\infty B(d+1)_+ \to \pi_0\,\mathrm{MT}\,\theta_d$ takes a framed 0-manifold with rank $d+1$ θ_{d+1}-bundle $[M, \psi, E]$ to $[S(E), T_v S(E), \varphi]$ for a particular stable isomorphism φ, but since M is a compact 0-manifold the sphere bundle $S(E)$ is really a finite disjoint union of spheres, $T_v S(E) = TS(E)$, and ϕ can be taken to be the identity. In particular, a generator of $\pi_0\Sigma^\infty B(d+1)_+$ is given by $g = [*, \mathbb{R}^{d+1}, *]$ where $\mathbb{R}^{d+1}$ is given a θ_{d+1} structure covering the usual orientation, and

$$\pi_0 PT(g) = [S^d, TS^d, \mathrm{Id}]$$

Therefore, to give the splittings $s_d : \pi_0 \operatorname{MT}\theta_d \to \mathbb{Z}/\operatorname{Eul}^{\theta}_{d+1}$ it suffices to give a homomorphism taking the class $[S^d, TS^d, \mathrm{Id}]$ to 1. These are easy to describe using familiar topological invariants, and take the same form as the splittings given in Appendix A of [6]. Of course, the splitting depends on the residue class of $d \mod 4$.

If $d \equiv 0 \mod 4$, then $\mathbb{Z}/\operatorname{Eul}^{\theta}_{d+1} = \mathbb{Z}$ and for any oriented d-manifold M it holds that $\sigma(M) \equiv \chi(M) \mod 2$. Then the splitting can be given by $[M, E, \varphi] \mapsto \frac{1}{2}(\chi(M) - \sigma(M))$, and indeed $s_d[S^d, TS^d, \mathrm{Id}] = 1$.

If $d \equiv 2 \mod 4$, then again $\mathbb{Z}/\operatorname{Eul}^{\theta}_{d+1} = \mathbb{Z}$, and $\chi(M)$ is always even and for any rank d bundle $E \to M$ with $E \cong_s TM$ it follows that its Euler number is even as well. Then the splitting can be given by $s_d[M, E, \varphi] = \frac{1}{2}\langle e(E), [M]\rangle$, and again $s_d[S^d, TS^d, \mathrm{Id}] = 1$.

If $d \equiv 1 \mod 4$ then $\mathbb{Z}/\operatorname{Eul}^{\theta}_{d+1} = \mathbb{Z}/2$ and the splitting can be given by $M \mapsto \operatorname{Kerv}(M)$, where $\operatorname{Kerv}(M)$ is the Kervaire semi-characteristic.

If $d \equiv 3 \mod 4$ then $\operatorname{Eul}^{\theta}_{d+1}$ isn't known for arbitrary θ, but for the case of orientation $\mathbb{Z}/\operatorname{Eul}_{d+1} = 0$ so there is nothing to split and $\pi_0 \operatorname{MTSO}(4k-1) \cong \Omega_{4k-1}$.

Note that there is a non-trivial issue of whether $\langle e(E), [M]\rangle$ and $\operatorname{Kerv}(M)$ are well-defined with respect to the bordism relation in $\pi_0 \operatorname{MT}\theta_d$. However, in Appendix A of [6] it is shown that this is the case for $\pi_0 \operatorname{MTSO}(d)$, and since there is a natural forgetful homomorphism $\pi_0 \operatorname{MT}\theta_d \to \pi_0 \operatorname{MTSO}(d)$ it follows that these invariants are well-defined for $\pi_0 \operatorname{MT}\theta_d$ as well. □

Now we restrict to the case of the tangential structure $\mathrm{BSO}(d) \to \mathrm{BO}(d)$ and $d = 2$. The terms in the cofiber sequence given in Proposition 3.0.7 are then $\mathrm{MTSO}(2)$, $\Sigma^\infty \mathrm{BSO}(2)_+$, and $\mathrm{MTSO}(1) \simeq \mathbb{S}^{-1}$.

Much is known about the homotopy groups of $\mathbb{S}$, especially in low degrees. In particular

Lemma 3.3.2. *For all k, $\pi_k\mathbb{S} \cong \Omega^{fr}_k$, the k-th framed bordism group.*
Moreover (see for example [27, p.15])

1. $\pi_0\mathbb{S} \cong \mathbb{Z}$.
2. $\pi_1\mathbb{S} \cong \mathbb{Z}/2$, *generated by $\eta = [T, \mathcal{L}]$, where T is the circle-group and $\mathcal{L}$ is its Lie-group framing. The trivialization $\mathcal{L}$ is same as the trivialization induced by being an oriented 1-manifold.*
3. $\pi_2\mathbb{S} \cong \mathbb{Z}/2$, *generated by η^2.*
4. $\pi_3\mathbb{S} = \mathbb{Z}/24$, *generated by $\nu = [S^3, \mathcal{L}_3]$ where $\mathcal{L}_3$ is the Lie group framing on S^3, and $\eta^3 \neq 0$ (in particular it is the unique element of order 2).*
5. $\pi_4\mathbb{S} = \pi_5\mathbb{S} = 0$.
6. *Finally, $\pi_6\mathbb{S} \cong \mathbb{Z}/2$, where the non-trivial element is ν^2.*

As for $\Sigma^\infty \mathrm{BSO}(2)_+$, note that for any pointed space X there is a natural splitting $\pi_k\Sigma^\infty X_+ \cong \pi_k\Sigma^\infty X \oplus \pi_k\mathbb{S}$. Liulevicius [18] proved the following:

Lemma 3.3.3. *For* $k \leqslant 8$ *the values of* $\pi_k \Sigma^\infty \mathrm{BSO}(2)$ *are given by*

k	1	2	3	4	5	6	7	8
$\pi_k \Sigma^\infty \mathrm{BSO}(2)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/2$

An element $[M, L, \varphi] \in \pi_k \mathrm{MTSO}(1)$ is a $(k+1)$-manifold M, an oriented line bundle L, and an isomorphism $\phi\colon TM \oplus \varepsilon \cong L \oplus \varepsilon^{k+1}$. The orientation on L induces a trivialization, say ω, and so TM is stably framed via the isomorphism $(\omega \oplus \mathrm{Id}) \circ \varphi$. In order to help with computations we will want the following:

Lemma 3.3.4. *The isomorphism* $\pi_k \mathrm{MTSO}(1) \cong \pi_k \mathbb{S}^{-1} \cong \Omega^{fr}_{k+1}$ *sends the tuple* $[M, L, \varphi]$ *to* $[M, (\omega \oplus \mathrm{Id}) \circ \varphi]$.

Then we get

Corollary 3.3.5. *On the subgroup* $0 \oplus \Omega^{fr}_k \subset \pi_k \Sigma^\infty \mathrm{BSO}(2)_+$*, the map*

$$\pi_k \mathrm{PT}\colon 0 \oplus \Omega^{fr}_k \to \pi_k \mathrm{MTSO}(1) \cong \Omega^{fr}_{k+1}$$

agrees with $(-) \times \eta$.

Proof. An element in this subgroup has the form $x = [N, \varphi, \varepsilon^2]$, so $\pi_k \mathrm{PT}(x) = [N \times S^1, \phi \times \omega] = x \times [S^1, \omega]$. Depending on orientation conventions, ω agrees with $\mathcal{L}$ up to sign, and it follows that $[S^1, \omega] = \eta$. □

Finally recall from Proposition 1.2.7 that $\pi_0 \mathrm{MTSO}(2) \cong \Omega_2 \oplus \mathbb{Z} \cong \mathbb{Z}$.

Now apply the long exact sequence of homotopy groups to the cofibre sequence of spectra given by Proposition 3.0.7. The portion of the sequence which is relevant to $k = 1$ and 2 is then:

$$0 \longrightarrow \pi_2 \mathrm{MTSO}(2) \xrightarrow{\pi_2(i)} \pi_2 \Sigma^\infty \mathrm{BSO}(2)_+ \xrightarrow{\pi_2(\mathrm{PT})} \pi_2 \mathrm{MTSO}(1) \xrightarrow{\partial_2}$$
$$\pi_1 \mathrm{MTSO}(2) \xrightarrow{\pi_1(i)} \pi_1 \Sigma^\infty \mathrm{BSO}(2)_+ \xrightarrow{\pi_1(\mathrm{PT})} \pi_1 \mathrm{MTSO}(1) \xrightarrow{\partial_1} \pi_0 \mathrm{MTSO}(2)$$

where the isomorphism $\pi_i \mathrm{MTSO}(1) \cong \pi_{i-1} \Sigma^{-1} \mathrm{MTSO}(1)$ identifies the boundary map ∂_i with $\pi_{i-1}(\tilde{p})$. Substituting the known values of these groups yields:

$$0 \longrightarrow \pi_2 \mathrm{MTSO}(2) \xrightarrow{\pi_2(i)} \mathbb{Z} \oplus \mathbb{Z}/2 \xrightarrow{\pi_2(\mathrm{PT})} \mathbb{Z}/24 \xrightarrow{\partial}$$
$$\pi_1 \mathrm{MTSO}(2) \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}$$

Immediately it is seen that $\pi_2 \mathrm{MTSO}(2)$ is a subgroup of $\mathbb{Z} \oplus \mathbb{Z}/2$. By Theorem 2.2.3 the manifold $\#^4 \mathbb{CP}^2$ represents an element of $\pi_2 \mathrm{MTSO}(2)$ with non-zero signature, so since the signature is a homomorphism to $\mathbb{Z}$ it follows that $\pi_2 \mathrm{MTSO}(2) \cong \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$.

Proposition 3.3.6. $\pi_2 \mathrm{MTSO}(2) \cong \mathbb{Z}$*, generated by* $g_4 = [\#^4 \mathbb{CP}^2, V_{(3,1,1,1)}, \varphi_4]$.

Proof. Suppose $\pi_2\,\mathrm{MTSO}(2) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, and consider

$$\pi_2(i)\colon \mathbb{Z} \oplus \mathbb{Z}/2 \hookrightarrow \mathbb{Z} \oplus \mathbb{Z}/2$$

There is a unique element of $\mathbb{Z} \oplus \mathbb{Z}/2$ of order 2, namely $(0, 1)$, and so it must be the case that $\pi_2(i)(0, 1) = (0, \eta^2)$. Then by exactness $\pi_2(\mathrm{PT})(0, \eta^2)$ would be 0; however $\pi_2\,\mathrm{PT}(0, \eta^2) = \eta^3 \neq 0$ by Corollary 3.3.5.

g_4 is a generator since it has signature 4 and by Proposition 2.2.7 all elements of $\pi_2\,\mathrm{MTSO}(2)$ have signature divisible by 4. □

From the right side of the above sequence it follows by exactness that the map $\pi_1\,\mathrm{MTSO}(2) \to \mathbb{Z}/2$ is 0, and so $\pi_1\,\mathrm{MTSO}(2) = 0$ iff $\pi_2(\mathrm{PT})$ is surjective.

Lemma 3.3.7. *The kernel of* $\pi_2\,\mathrm{PT}$ *is generated by* $(12, 1)$.

Proof. Since $\pi_2\,\mathrm{MTSO}(2) = \langle g_4 \rangle$ is suffices to show

$$\pi_2(i)(g_4) = (12, 1)$$

Consider the homomorphism

$$\begin{aligned} e\colon \pi_2\Sigma^\infty \mathrm{BSO}(2)_+ &\to \mathbb{Z} \\ [N, f, E] &\mapsto \langle e(E), [N] \rangle \end{aligned}$$

This is split-surjective: send the integer 1 to the triple $[S^2, \psi, \gamma_1]$ where ψ is the stable framing of S^2 and γ_1 is the tautological line bundle over $S^2 \cong \mathbb{CP}^1$. Then indeed $\langle e(\gamma_1), [S^2] \rangle = 1$.

For any $[M, E, \varphi] \in \pi_2\,\mathrm{MTSO}(2)$, let S be the zero-locus of a section which is transverse to the zero-section of E, and $j\colon S \hookrightarrow M$ its inclusion. Then by Lemma 3.2.3 we have

$$e\circ\pi_2(i)[M, E, \varphi] = \langle e(E|_S), [S] \rangle = \langle e(E), j_*[S] \rangle = \langle e(E), e(E)\cap[M] \rangle = \langle e(E)^2, [M] \rangle$$

where the second-to-last equality uses the fact that $j_*[S]$ is Poincare-dual to $e(E)$, and the last equality uses the duality between cup- and cap- product. In particular

$$e \circ \pi_2(i)(g_4) = \langle (3, 1, 1, 1)^2, [\#^4\mathbb{CP}^4] \rangle = 12$$

and hence $\pi_2(i)(g_4) = (12, x)$ for some $x \in \mathbb{Z}/2$. If $x = 0$ then $\mathrm{Coker}(\pi_2(i)) \cong \mathbb{Z}/12 \oplus \mathbb{Z}/2$, which cannot inject into $\mathbb{Z}/24$ because it contains too many elements of order 2; therefore $x = 1$. □

Proposition 3.3.8. $\pi_1\,\mathrm{MTSO}(2) = 0$.

Proof. Consider $(1, 0) \in \pi_2\Sigma^\infty \mathrm{BSO}(d)_+$. Since the kernel of $\pi_2(\mathrm{PT})$ is generated by $(12, 1)$ the class $n \cdot (1, 0)$ is in the kernel iff 24 divides n. Hence $\pi_2(\mathrm{PT})(1, 0)$ has order 24, so $\pi_2(\mathrm{PT})$ is surjective. □

One extra outcome of this computation is two families of representatives for elements of Ω_3^{fr}. If Σ_g denotes the oriented surface of genus g, then Σ_g with the stable framing induced by embedding into $\mathbb{R}^3$ represents the trivial element of Ω_2^{fr}. Because we have a surjection

$$\begin{aligned} \kappa\colon \mathbb{Z} &\to \Omega_3^{fr} \cong \mathbb{Z}/24 \\ n &\mapsto \pi_2(\mathrm{PT})(n, 0) \end{aligned}$$

in particular we have

Corollary 3.3.9. *Let* $\gamma_S = \kappa(1) \in \Omega_3^{fr}$. *Then* $n \cdot \gamma_S$ *is represented by the circle bundle* $S_n \to \Sigma_g$ *of Euler number* n, *framed with the orientation of* $T_v S_n$ *and taking the trivial stable framing of* Σ_g. *In particular every element of* Ω_3^{fr} *can be represented by a circle bundle over* Σ_g *for any* g.

Observe that γ_S is represented by the manifold S^3 (by setting $g = 0$), framed by viewing it as the total space of the Hopf fibration and framing its tangent bundle with the orientation on the vertical tangent bundle and taking the trivial stable framing on S^2. I'm not sure if this is the same (up to framed bordism) as the Lie group framing. More generally, $\kappa(n)$ is a generator for any n coprime to 24.

Note that since $\ker(\pi_2 \, PT)$ is generated by $(12, 1)$ we have

$$\pi_2 \, PT(n, x) = \pi_2 \, PT(n + 12, x + 1)$$

for any $n \in \mathbb{Z}$ and $x \in \mathbb{Z}/2$. In particular every class in Ω_3^{fr} can be represented by a circle bundle over $(T^2, \mathcal{L}^2)$, since $\kappa(n) = \pi_2 \, PT(n + 12, 1)$.

One more weird corollary along this vein:

Corollary 3.3.10. *Consider the circle bundle* $S_{12} \to \Sigma_g$ *of Euler number* 12, *and framed as above. Then this is framed bordant to* $(T^3, \mathcal{L}^3)$.

Proof. $\pi_2 \, PT(12, 0) = \pi_2 \, PT(24, 1) = \pi_2 \, PT(0, 1) = \eta^3$. □

Return attention to MTSO(2). Using the result of Liulevicius we can attempt to continue climbing the long exact sequence.

Proposition 3.3.11. $\pi_3 \, MTSO(2) \cong \mathbb{Z}/24$.

Proof. Consider the portion

$$\pi_4 \, MTSO(1) \to \pi_3 \, MTSO(2) \to \pi_3 \Sigma^\infty BSO(2)_+ \to \pi_3 \, MTSO(1)$$

Since $\pi_4 \mathbb{S} = \pi_5 \mathbb{S} = 0$ then $\pi_3 \, MTSO(2) \cong \pi_3^{st} \, BSO(2) \oplus \pi_3 \mathbb{S} \cong \mathbb{Z}/24$. □

Proposition 3.3.12. $\pi_4 \, MTSO(2) \cong \mathbb{Z}$.

Proof. Consider

$$\pi_5 \Sigma^\infty BSO(2)_+ \to \pi_5 \, MTSO(1) \to \pi_4 \, MTSO(2) \to \pi_4 \Sigma^\infty BSO(2)_+ \to \pi_4 \, MTSO(1)$$

The last term is 0 and $\pi_4 \Sigma^\infty BSO(2)_+ \cong \mathbb{Z}$ so $\pi_4 \, MTSO(2)$ surjects onto $\mathbb{Z}$, and the first two terms are both $\mathbb{Z}/2$. Hence the proposition is equivalent to the claim that

$$\pi_5 \, PT \colon \pi_5 \Sigma^\infty BSO(2)_+ \to \pi_5 \, MTSO(1) \cong \Omega_6^{fr}$$

is an isomorphism. The non-trivial element of Ω_6^{fr} is $[S^3 \times S^3, \mathcal{L}_3 \times \mathcal{L}_3]$, so we seek a pre-image.

Consider the element $[S^3 \times S^2, \mathcal{L}_3 \times \psi_2, 0 \times \gamma_1] \in \pi_5 \Sigma^\infty BSO(2)_+$ where ψ_2 is the usual framing of S^2 and $p \colon \gamma_1 \to S^2$ is the tautological line bundle. Note $0 \times \gamma_1$ is isomorphic to the pull-back of γ_1 along the projection to S^2:

$$\begin{array}{ccc} 0 \times \gamma_1 & \longrightarrow & \gamma_1 \\ \big\downarrow{\scriptstyle \bar{p}} & & \big\downarrow{\scriptstyle p} \\ S^3 \times S^2 & \xrightarrow{pr} & S^2 \end{array}$$

Then applying π_5 PT results in $[S(0\times\gamma_1), T_v S(0\times\gamma_1), \mathrm{Id}\oplus\tilde{p}^*(\mathcal{L}_3\times\psi_2)]$. Note that $S(0\times\gamma_1)$ is diffeomorphic to $S^3\times S^3$ and the vertical tangent bundle has the form $0\times T_v S(\gamma_1)$. The framing is coming from

$$TS(0\times\gamma_1)\cong T_v S(0\times\gamma_1)\oplus\tilde{p}^*T(S^3\times S^2)\cong_s T_v S(0\times\gamma_1)\oplus\varepsilon^5$$

and with respect to these identifications it takes the form

$$\mathcal{L}_3\times(\mathrm{Id}\oplus p^*\psi_2)\colon TS^3\times TS^3\cong\varepsilon^3\times(T_v S(\gamma_1)\oplus\varepsilon^2)$$

Therefore as a framed manifold it is equal to $W := (S^3\times S^3, \mathcal{L}_3\times\omega\circ(\mathrm{Id}\oplus p^*\psi_2))$ where ω is the trivialization of $T_v S(\gamma_1)$ induced by being an oriented line bundle.

It now only remains to show that W is in the same class as $(S^3\times S^3, \mathcal{L}_3\times\mathcal{L}_3)$. The collection of framed bordism groups forms a graded ring and in particular there is a surjective homomorphism

$$[S^3,\mathcal{L}_3]\times(-)\colon\Omega_3^{fr}\to\Omega_6^{fr}\cong\mathbb{Z}/2$$

It follows that every odd multiple of $[S^3,\mathcal{L}_3]$ is mapped to the non-zero element, and since $[S^3,\omega\circ(\mathrm{Id}\oplus p^*\psi_2)] = \pi_2\,\mathrm{PT}(1,0)$ is a generator of Ω_3^{fr} by Corollary 3.3.9 it follows that $[W] = [S^3\times S^3,\mathcal{L}_3\times\mathcal{L}_3]$. Therefore π_5 PT is an isomorphism and $\pi_4\,\mathrm{MTSO}(2)\cong\mathbb{Z}$. □

Finally, consider MTSO(3). In Proposition 2.2.1 we constructed an element $g_2\in\pi_1\,\mathrm{MTSO}(3)$ which was represented by $(\mathbb{CP}^2\#\mathbb{CP}^2)_0$.

Proposition 3.3.13. *The group* $\pi_1\,\mathrm{MTSO}(3)$ *is isomorphic to* $\mathbb{Z}$*, generated by the element* g_2*. Moreover,* $\pi_1\tilde{p}(g_4) = 2g_2$*, since they have the same signature.*

Proof. Using the cofibre sequence with $d = 3$ we see

$$\pi_2\,\mathrm{MTSO}(2)\xrightarrow{\pi_1\tilde{p}}\pi_1\,\mathrm{MTSO}(3)\longrightarrow\pi_1\Sigma^\infty\mathrm{BSO}(3)_+\longrightarrow\pi_1\,\mathrm{MTSO}(2)$$

From the above computation we have $\pi_1\,\mathrm{MTSO}(2) = 0$ and $\pi_2\,\mathrm{MTSO}(2) = \langle g_4\rangle$; $\pi_1^{st}\,\mathrm{BSO}(3)$ vanishes since the space BSO(3) is simply connected, so the third term in this sequence is $\Omega_1^{fr}\cong\mathbb{Z}/2$.

The homomorphism $\pi_1\tilde{p}$ preserves the signature so $\pi_1\tilde{p}(g_4)\neq 0$, hence this homomorphism is injective. Therefore $\pi_1\,\mathrm{MTSO}(3)$ is a $\mathbb{Z}$ extension of $\mathbb{Z}/2$. Proposition 2.2.1 gives an element $g_2 = [\#^2\mathbb{CP}^2, \langle s\rangle^\perp, \mathrm{Id}]\in\pi_1\,\mathrm{MTSO}(3)$ which is indivisible and non-torsion, but since $\sigma_{1,3}(g_2) = 2$ it cannot be in the image of $\pi_1\tilde{p}$, so the extension takes the form $\mathbb{Z}\to\mathbb{Z}\to\mathbb{Z}/2$. □

Appendix A

The Euler Class

The purpose of this Appendix is to provide a proof of the following Theorem:

Theorem A2. *Let* n *be even, let* X *be finite CW complex of dimension* n*, and let* V_0, V_1 *be two oriented rank* n *vector bundles over* X*. If* $V_0 \oplus \varepsilon \cong V_1 \oplus \varepsilon$ *and* $e(V_0) = e(V_1)$*, then* $V_0 \cong V_1$.

This theorem is apparently well-known, though the author of this thesis could not find a reference in the literature. Observe that Theorem A2 is false when n is odd: all spheres are stably parallelizable and all odd-dimensional spheres have vanishing Euler class, but most odd-dimensional spheres are not parallelizable.

This Appendix will derive basic properties of the Euler class by elementary means. The proof of Theorem A2 will employ a lemma about homotopy fibres, Lemma A.2.1, which produces a $\mathbb{Z} \times \mathbb{Z}$ array of exact sequences of homotopy groups out of a commutative square of pointed spaces. As an other application of this lemma, we will prove

Theorem A1. *Let* X *be a finite CW complex of any dimension* $n \geqslant 0$*, and let* $V \to X$ *be an oriented rank* n *vector bundle. Then* V *admits a non-zero section iff* $e(V) = 0$.

This theorem is certainly well-known, and we present a proof for the sake of completion.

A.1 Basics

It will be tacitly assumed that all spaces are CW complexes in order to have a nice theory of bundles.

Recall that an orientation class (or Thom class) of a rank n vector bundle $V \to B$ is a cohomology class $u \in H^n(V, V_0; \mathbb{Z})$ such that for each fibre $(\mathbb{R}, \mathbb{R} - 0) \cong (F, F - 0) \hookrightarrow (V, V_0)$ the restriction of u to $(F, F - 0)$ is a generator of $H^n(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z}) \cong \mathbb{Z}$. V is defined to be orientable if it admits and orientation class, and (V, u) is called an oriented bundle (u is often omitted from the notation).

If (V, u) and (V', u') are two oriented bundles of the same rank (possibly over different spaces), say a bundle map $\phi\colon V \to V'$ is orientation preserving if $\phi^* u' = u$, and orientation reversing otherwise. Since a bundle map is an isomorphism when restricted to each fibre over the domain, it follows that over each component it either preserves or reverses orientation.

Definition A.1.1. *Let* (V, u) *be an oriented bundle of rank* n *over* B*, with zero section* $z\colon (B, \emptyset) \to (V, V_0)$*. The Euler class is defined as*

$$e(V) = z^* u \in H^n(B; \mathbb{Z})$$

Immediately it follows that if $\phi\colon V \to V'$ is an orientation preserving bundle map covering $f\colon B \to B'$ then $f^* e(V') = e(V)$, so in particular it is natural with respect to pullbacks. Hence if

$$e = e_n \in H^n(BSO(n); \mathbb{Z})$$

denotes the Euler class of the universal oriented n-plane bundle, then for any oriented bundle (V, u) and classifying map $f\colon B \to BSO(n)$, it is the case that $e(V) = f^* e$. Let $e_n\colon BSO(n) \to K(\mathbb{Z}, n)$ represent the cohomology class.

Lemma A.1.2. *If* V *admits a non-zero section, then* $e(V) = 0$*. Hence the composition* $e_n \circ s_{n-1}\colon BSO(n-1) \to BSO(n) \to K(\mathbb{Z}, n)$ *is null-homotopic.*

Proof. Let $s\colon (B, \emptyset) \to (V, V_0)$ be a non-zero section. Then s factors through (V_0, V_0) so there is the commutative diagram

$$\begin{array}{ccc} H^n(V, V_0) & \xrightarrow{\quad s^* \quad} & H^n(B) \\ & \searrow \quad \nearrow & \\ & H^n(V_0, V_0) = 0 & \end{array}$$

hence $s^* = 0$. But all sections are homotopic (as maps of pairs $(B, \emptyset) \to (V, V_0)$) so $z^* = s^*$ and in particular $e(V) = 0$.

The second assertion follows from the fact that if X is a CW complex and $f\colon X \to K(G, n)$ is a map inducing 0 on all homotopy groups, then f is null-homotopic. Since a rank n bundle admits a non-zero section iff its classifying map lifts to $BSO(n-1)$, the first part of the lemma implies that for all CW complexes B the map

$$(e_n s_{n-1})_*\colon [B, BSO(n-1)] \to [B, K(\mathbb{Z}, n)]$$

is the 0 map, so in particular it vanishes for homotopy groups. □

Using Grassmannian models for classifying spaces the standard stabilization maps are cofibrations, so using the homotopy extension property e_n can be homotoped so that $e_n s_{n-1}$ is constant.

In the particular case that V is the tangent bundle of an oriented manifold $(M, [M])$, the Euler class can be computed in terms of ranks of homology groups, or by counting cells in a cell structure:

Lemma A.1.3. $\chi(M) = \langle e(M), [M] \rangle$

Proof. This is proven in [24, Corollary 11.12] using mostly algebraic computations in cohomology. □

For example $\langle e(S^n), [S^n]\rangle = \chi(S^n) = 2$ if n is even and 0 if n is odd, since S^n has a CW structure with one 0-cell and one n-cell. If μ_n is the dual of $[S^n]$ then it follows that $e(S^{2n}) = 2\mu_{2n}$ and $e(S^{2n+1}) = 0$.

Lemma A.1.4. *For all n there is a fibration $S^n \to \mathrm{BSO}(n) \to \mathrm{BSO}(n+1)$.*

Proof. $\mathrm{SO}(n+1)$ acts smoothly on S^n by $(A, v) \mapsto Av$. The isotropy subgroup of e_{n+1} is $\mathrm{SO}(n) \oplus [1]$, and so the map $\mathrm{SO}(n+1) \to S^n$ sending A to Ae_{n+1} is a fibre bundle with fibre $\mathrm{SO}(n)$. In particular $S^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$. Then, for any topological group G and subgroup H there is a fibration

$$G/H \to BG \to BH$$

□

Corollary A.1.5. *The stabilization map $s_n \colon \mathrm{BSO}(n) \to \mathrm{BSO}(n+1)$ is n-connected. I.e. $\pi_i(s_n)$ is an isomorphism for $i < n$ and surjective when $i = n$.*

Proposition A.1.6. *The maps in the long exact sequence associated to this fibration are interpretted as follows:*

1. *For every k the map $\pi_k \mathrm{BSO}(n) \to \pi_k \mathrm{BSO}(n+1)$ takes a rank n bundle $E \to S^k$ to $E \oplus \varepsilon$.*

2. *The map $\pi_n S^n \to \pi_n \mathrm{BSO}(n)$ takes the homotopy class of Id_{S^n} to the isomorphism class of TS^n.*

3. *The boundary map $\pi_{n+1} \mathrm{BSO}(n+1) \to \pi_n S^n$ takes a bundle to its Euler number.*

Proof. The first item is clear. For the second item it is traditional to cite Steenrod [31, Section 23].

For the third item, the case where n is odd is covered in the proof of Proposition A.3.1. When n is even, every rank $n+1$ bundle over S^{n+1} has trivial Euler number, since the cohomology of S^{n+1} contains no 2-torsion. Moreover TS^n is non-trivial since $\chi(S^n) = 2 \neq 0$, so the next map is injective: therefore the boundary map is null in this case. □

Corollary A.1.7. *$\pi_n S^n \to \pi_n \mathrm{BSO}(n)$ is injective when n is even.*

Proof. This map sends $[\mathrm{Id}_S^{n-1}]$ to TS^{n-1}. To show TS^{n-1} is not torsion, note that

$$e_* \colon \pi_{n-1} \mathrm{BSO}(n-1) \to \pi_{n-1} K(\mathbb{Z}, n-1) \cong H^{n-1}(S^{n-1}; \mathbb{Z})$$

is a homomorphism, so $e(k \cdot TS^{n-1}) = k \cdot e(TS^{n-1}) = 2k\mu_{n-1}$ is non-zero for $k \neq 0$, and hence $k \cdot TS^{n-1}$ is non-zero for all non-zero k. □

Aside: this map is not injective for n odd, because in that case $\pi_n \mathrm{BSO}(n)$ is in fact 2-torsion. The question of whether this map is 0 or not is famously known to be answered by "Yes if $n = 1, 2, 3$ and No otherwise" but the proof is not possible by elementary means so it is not discussed here.

Corollary A.1.8. *If $n > 1$ is odd then every rank n vector bundle over S^n admits a non-zero section.*

Proof. In other words, the lemma asserts that $\pi_n \operatorname{BSO}(n-1) \to \pi_n \operatorname{BSO}(n)$ is surjective when n is odd. Consider the long exact sequence

$$\cdots \to \pi_n \operatorname{BSO}(n-1) \to \pi_n \operatorname{BSO}(n) \to \pi_{n-1} S^{n-1} \to \pi_{n-1} \operatorname{BSO}(n-1) \to \ldots$$

To prove the lemma it suffices to know that $\pi_{n-1} S^{n-1} \to \pi_{n-1} \operatorname{BSO}(n-1)$ is injective, which follows since $n-1$ is even. □

A.2 Lemma About Homotopy Fibres

The following fact is very useful.

Lemma A.2.1. *Assume we are given a commutative diagram of pointed spaces:*

$$\begin{array}{ccc} C & \xrightarrow{h_1} & D \\ {\scriptstyle \nu_0}\uparrow & & \uparrow{\scriptstyle \nu_1} \\ A & \xrightarrow{h_0} & B \end{array}$$

Then, after taking the standard homotopy fibres of all the maps, there are obvious continuous functions between homotopy fibres

$$\begin{array}{ccccc} \operatorname{hofib}(h_1) & \longrightarrow & C & \longrightarrow & D \\ {\scriptstyle \nu}\uparrow & & \uparrow & & \uparrow \\ \operatorname{hofib}(h_0) & \longrightarrow & A & \longrightarrow & B \\ & & \uparrow & & \uparrow \\ & & \operatorname{hofib}(\nu_0) & \xrightarrow{h} & \operatorname{hofib}(\nu_1) \end{array}$$

and a natural homeomorphism $\operatorname{hofib}(h) \cong \operatorname{hofib}(\nu)$.

Proof. Explicitly, define:

$$\operatorname{hofib}(h_0) = \{(a, \rho) \in A \times B^I \mid \rho(0) = h_0(a) \text{ and } \rho(1) = b_0\}$$

$$\operatorname{hofib}(h_1) = \{(c, \gamma) \in C \times D^I \mid \gamma(0) = h_1(c) \text{ and } \gamma(1) = d_0\}$$

$$\operatorname{hofib}(\nu_0) = \{(a, \nu) \in A \times C^I \mid \nu(0) = \nu_0(a) \text{ and } \nu(1) = c_0\}$$

$$\operatorname{hofib}(\nu_1) = \{(b, \gamma) \in B \times D^I \mid \gamma(0) = \nu_1(a) \text{ and } \gamma(1) = d_0\}$$

Each of these homotopy fibres are given the subspace topology, and are naturally pointed with the base point of the domain and the constant path at the base point in the codomain. (Here constant paths will be denoted by their value.)

Then define the pointed maps $h\colon \operatorname{hofib}(\nu_0) \to \operatorname{hofib}(\nu_1)$ and $\nu\colon \operatorname{hofib}(h_0) \to \operatorname{hofib}(h_1)$ by

$$h(a, \nu) = (h_0(a), h_1 \circ \nu) \text{ and } \nu(a, \rho) = (\nu_0(a), \nu_1 \circ \rho)$$

h and ν have the correct range since the initial maps are strictly commutative and pointed: explicitly, $h_1 \nu$ is a path in D from $\nu_1 h_0(a)$ to d_0 because $h_1 \nu(0) =$

$h_1\nu_0(a) = \nu_1 h_0(a)$ and $h_1\nu(1) = h_1(c_0) = d_0$; analogously for ν. They are continuous because they are coordinate-wise continuous.

Then write

$$\mathrm{hofib}(h) = \{\big((a,\nu),\eta\big) \in \mathrm{hofib}(\nu_0)\times\mathrm{hofib}(\nu_1)^I \,|\, \eta(0) = h(a,\nu) \text{ and } \eta(1) = (b_0,d_0)\}$$

$$\mathrm{hofib}(\nu) = \{\big((a,\rho),\delta\big) \in \mathrm{hofib}(h_0)\times\mathrm{hofib}(h_1)^I \,|\, \delta(0) = \nu(a,\rho) \text{ and } \delta(1) = (c_0,d_0)\}$$

Since $(X \times Y)^Z \cong X^Z \times Y^Z$, η decomposes as (η_B, η_D) where $\eta_B \in B^I$ and $\eta_D \in (D^I)^I$; anologously write $\delta = (\delta_C, \delta_D)$. Let $\mathfrak{s}\colon I^2 \cong I^2$ swap the two coordinates. Now define $f\colon \mathrm{hofib}(\nu) \to \mathrm{hofib}(h)$ by

$$f\big((a,\rho),(\delta_C,\delta_D)\big) = \big((a,\delta_C),(\rho,\delta_D\mathfrak{s})\big)$$

It must be verified that f has the correct codomain. First, $(a,\delta_C) \in \mathrm{hofib}(\nu_0)$ because by definition $\delta_C(0) = \nu_0(a)$ and $\delta_C(1) = c_0$. Next, $(\rho, \delta_D\mathfrak{s}) \in \mathrm{hofib}(\nu_1)^I$ because for all s the function $\delta_D\mathfrak{s}(\_,s) = \delta_D(s,\_)$, which is a path from $\nu_1\rho(s)$ to d_0. Lastly, $(\rho,\delta_D\mathfrak{s})$ is a path from $(h_0(a), h_1 \circ \delta_C)$ to (b_0,d_0), because for each s the function $\delta_D(\_,s)$ is a path from $h_1\delta_C(s)$ to d_0, so in particular $\delta_D\mathfrak{s}(s,0) = \delta_D(0,s) = h_1\delta_C(s)$.

f is continuous because again all of its coordinates are. Finally, f is a homeomorphism because its continuous inverse is given by the formula

$$f^{-1}\big((a,\nu),(\eta_B,\eta_D)\big) = \big((a,\eta_B),(\nu,\eta_D\mathfrak{s})\big)$$

□

A.3 Proof of Theorem A1

Theorem A1 is proven by showing that the n-th Moore-Postnikov stage of the stabilization map $s_{n-1}\colon BSO(n-1) \to BSO(n)$ is given by the homotopy fibre of the map $e\colon BSO(n) \to K(\mathbb{Z}, n)$. For suppose F_n is the n-th stage and is classified by e.

Since the map $s_{n-1}\colon BSO(n-1) \to BSO(n)$ is already $(n-1)$-connected, $BSO(n)$ is already the i-th Moore-Postnikov stage for $i < n$ and s_{n-1} is its own lift, so F_n would be the first non-trivial stage. $F_n \to BSO(n)$ is a principal $K(\mathbb{Z}, n-1)$ fibration classified by e, hence for any map $V\colon X \to BSO(n)$ the cohomology class $e(V)$ is the obstruction to lifting V to F_n. If X is n-dimensional then there are no higher obstructions, so the Euler class is the obstruction to lifting V all the way up the Moore-Postnikov tower, and hence to $BSO(n-1)$.

Let $F_n = \mathrm{hofib}(e\colon BSO(n) \to K(\mathbb{Z}, n))$. Then $F_n \to BSO(n)$ is the pull-back of the path-loop fibration of $K(\mathbb{Z}, n)$, so is a principal $K(\mathbb{Z}, n-1)$ fibration classified by e. Since $e \circ s_{n-1}$ can be made constant by Lemma A.1.2, s_{n-1} admits a lift

$$\begin{array}{ccccc} & & F_n & \longrightarrow & * \\ & \overset{g}{\nearrow} & \Big\downarrow f & & \Big\downarrow \\ BSO(n-1) & \xrightarrow{s_{n-1}} & BSO(n) & \xrightarrow{e} & K(\mathbb{Z}, n) \end{array}$$

Proposition A.3.1. *For all n, F_n is the n-th Moore-Postnikov stage of s_{n-1}.*

Proof. Specifically, the following must be proven:

1. $\pi_i(g)$ is an isomorphism for $i < n$ and surjective for $i = n$, and
2. $\pi_i(f)$ is an isomorphism for $i > n$ and injective for $i = n$.

The second item follows immediately from the long exact sequence for $F_n \to BSO(n) \to K(\mathbb{Z}, n)$.

Consider in general a continuous map $i\colon A \to X$, and let $F(i)$ be the homotopy fibre. Suppose S^{k-1} and D^k are based at the vector e_k. Given a pointed map of pairs

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{\tilde{f}} & A \\ \downarrow & & \downarrow \\ D^k & \xrightarrow{f} & X \end{array}$$

one can construct a kind of adjoint

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{\hat{f}} & P_\bullet X \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \end{array}$$

where $P_\bullet X$ is the space of paths in X ending at the basepoint, by

$$\hat{f}_x(t) = f\big((1-t)x + te_k\big)$$

In fact, for all $x \in S^{k-1}$, $\hat{f}_x(0) = i \circ \tilde{f}(x)$ and $\hat{f}_x(1) = *$, so $\hat{f}$ can also describe a map $S^{k-1} \to F(i)$ by

$$\hat{f}(x) = \big(\tilde{f}(x), \big[t \mapsto f\big((1-t)x + te_k\big)\big]\big)$$

This construction defines a well-defined isomorphism

$$\psi\colon \pi_k(X, A) \to \pi_{k-1}F(i)$$

by [37, 6.1.3]. Moreoever, if $A = *$ then this map agrees with the boundary map $\partial\colon \pi_k X \cong \pi_{k-1}\Omega X$ from the long exact sequence of the path-loop fibration of X.

Now consider an oriented rank n vector bundle $\pi\colon V \to B$ with metric. Then there is a Thom class $\tau \in H^n(DV, SV; \mathbb{Z})$, with the property that for any point $b \in B$ the restriction $\tau_b := \tau|_b$ is a generator of $H^n(DV_b, SV_b) \cong \mathbb{Z}$. The Thom class can be represented by a map of pairs

$$\begin{array}{ccc} SV & \longrightarrow & * \\ \downarrow & & \downarrow \\ DV & \xrightarrow{e} & K(\mathbb{Z}, n) \end{array}$$

where e represents the Euler class under the homotopy equivalence $DV \simeq B$. Choosing a point $b \in B$ and a basepoint $b' \in SV_b$, we apply the above construction to get

$$
\begin{array}{ccccc}
 & & & & \Omega K(\mathbb{Z}, n) \\
 & & & \nearrow & \downarrow \\
SV_b & \longrightarrow & SV & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
b & \longrightarrow & DV & \xrightarrow{e} & K(\mathbb{Z}, n)
\end{array}
$$

The the claim is that the map $SV_b \to \Omega K(\mathbb{Z}, n)$ is n-connected; for this it suffices to show it is an isomorphism on π_{n-1}. This follows because this map is $\psi(\tau|_b)$, and by definition τ_b is a generator of $H^n(DV_b, SV_b; \mathbb{Z}) \cong \pi_n K(\mathbb{Z}, n)$, and furthermore ψ is an isomorphism.

Now apply this to the universal bundle $\gamma_n \to BSO(n)$. It follows from Lemma 1.1.7 that there is a homotopy equivalence $S(\gamma_n) \simeq BSO(n-1)$, and the universal Thom class is identified with the universal "relative Euler class"

$$
\begin{array}{ccc}
BSO(n-1) & \longrightarrow & * \\
\downarrow & & \downarrow \\
BSO(n) & \xrightarrow{e} & K(\mathbb{Z}, n)
\end{array}
$$

Then applying Lemma A.2.1 we get the square of homotopy fibre sequences:

$$
\begin{array}{ccccc}
G_n & \longrightarrow & S^{n-1} & \longrightarrow & K(\mathbb{Z}, n-1) \\
\downarrow & & \downarrow & & \downarrow \\
BSO(n-1) & \longrightarrow & BSO(n-1) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
F_n & \longrightarrow & BSO(n) & \xrightarrow{e} & K(\mathbb{Z}, n)
\end{array}
$$

Then the map $S^{n-1} \to K(\mathbb{Z}, n-1)$ is n-connected, and hence the map $G_n \to S^{n-1}$ is a model for the $(n-1)$-connected cover. It then follows that the map $BSO(n-1) \to F_n$ is n-connected, completing the proof. $\square$

Some corollaries can be derived from this. Consider the diagram

$$
\begin{array}{ccc}
\pi_{n-1} BSO(n-1) & & \\
\uparrow {\scriptstyle c_{n-1}} & & \\
\pi_{n-1} S^{n-1} & \xrightarrow[\cong]{\phi} & \pi_{n-1} K(\mathbb{Z}, n-1) \\
\uparrow {\scriptstyle \partial} & & \uparrow {\scriptstyle \cong} \\
\pi_n BSO(n) & \xrightarrow{e_*} & \pi_n K(\mathbb{Z}, n)
\end{array}
$$

First, since ϕ is an isomorphism it follows that for every oriented rank n bundle V over S^n, $\partial(V) = \pm e(V)$. Then, since $c_{n-1}([\mathrm{Id}]) = TS^{n-1}$ by Proposition A.1.6 it follows moreover that there is an oriented rank n bundle over S^n with Euler number ± 1 iff S^{n-1} is paralellizable.

If we discussed Stiefel-Whitney classes and the relation $e(V^n) \cong w_n \mod 2$ when we would get the following corollary:

Corollary A.3.2. *Consider the map* $w_n \colon \mathrm{BSO} \to K(\mathbb{Z}/2, n)$. *Then* $\pi_n(w_n)$ *is surjective iff* S^{n-1} *is parallelizable.*

Proof. There are the following commutative diagrams, the first inducing the second:

$$\begin{array}{ccc} \mathrm{BSO}(n) & \xrightarrow{\;e\;} & K(\mathbb{Z}, n) \\ \downarrow{\scriptstyle s} & \searrow^{\tilde{w}_n} & \downarrow{\scriptstyle r_2} \\ \mathrm{BSO} & \xrightarrow{\;w_n\;} & K(\mathbb{Z}/2, n) \end{array} \qquad \begin{array}{ccc} \pi_n \mathrm{BSO}(n) & \xrightarrow{\;\pi_n e\;} & \pi_n K(\mathbb{Z}, n) \\ \downarrow{\scriptstyle \pi_n s} & \searrow^{\pi_n \tilde{w}_n} & \downarrow{\scriptstyle \pi_n r_2} \\ \pi_n \mathrm{BSO} & \xrightarrow{\;\pi_n w_n\;} & \pi_n K(\mathbb{Z}/2, n) \end{array}$$

where $\tilde{w}_n$ represents the class $w_n \in H^n(\mathrm{BSO}(n); \mathbb{Z}/2)$. It follows that $\pi_n(w_n)$ is surjective iff $\pi_n(e)$ is.

□

A.4 Proof of Theorem A2

The proof of Theorem A2 is more involved. In the following it will be always be assumed that n is even.

For brevity, let $s_* \colon \pi_n \mathrm{BSO}(n) \to \pi_n \mathrm{BSO}(n+1)$ denote the homomorphism induced by s_n, and let $e_* \colon \pi_n \mathrm{BSO}(n) \to H^n(S^n; \mathbb{Z})$ send a bundle V to its Euler class.

Note that the set $[S^n, K(\mathbb{Z}, n)]$ has two group operations: one coming from point-wise multiplication using an H-space structure on $K(\mathbb{Z}, n)$, giving $H^n(S^n; \mathbb{Z})$; and the other coming from the co-group structure on S^n, giving $\pi_n K(\mathbb{Z}, n)$. These two operations satisfy the interchange law and so by the general Eckmann-Hilton principle they agree.

Lemma A.4.1. *The function* e_* *is a homomorphism.*

Proof. Explicitly, the function e_* takes a homotopy class $[f] \in \pi_n \mathrm{BSO}(n)$ and returns $[e \circ f] \in \pi_n K(\mathbb{Z}, n) \cong H^n(S^n; \mathbb{Z})$. Given continuous maps $f, g \colon S^n \to \mathrm{BSO}(n)$, the element $e_*([f] + [g])$ is represented by

$$S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{f \vee g} \mathrm{BSO}(n) \xrightarrow{\;e\;} K(\mathbb{Z}, n)$$

Then the equation $e_*([f] + [g]) = e_*([f]) + e_*([g])$ corresponds to the equation $e \circ (f \vee g) = (e \circ f) \vee (e \circ g)$.

For a class $[f] \in \pi_n X$ for any X, we have $-[f] = [f \circ r]$ where $r \colon S^n \to S^n$ is any map of degree -1. Thus for $[f] \in \pi_n \mathrm{BSO}(n)$ we have $e_*(-[f]) = [e \circ f \circ r] = -e_*([f])$ by associativity of function composition. □

Let $w_n \colon \mathrm{BSO}(n + k) \to K(\mathbb{Z}/2, n)$ for $k \geqslant 0$ represent the n-th Stiefel-Whitney class; let w_n also denote $\pi_n(w_n)$. Let $r_2 \colon K(\mathbb{Z}, n) \to K(\mathbb{Z}/2, n)$ represent the surjective homomorphism $r_2 \colon \mathbb{Z} \to \mathbb{Z}/2$.

Lemma A.4.2. *The diagram*

$$\begin{array}{ccc} BSO(n) & \xrightarrow{s_n} & BSO(n+1) \\ \downarrow e & \searrow^{w_n} & \downarrow w_n \\ K(\mathbb{Z},n) & \xrightarrow{r_2} & K(\mathbb{Z}/2,n) \end{array}$$

commutes up to homotopy.

Proof sketch. The upper triangle is due to the Whitney sum formula; the bottom is expressing the relation $e \equiv w_2 \mod 2$. ◻

Lemma A.4.3. *For n even the following sequence is exact:*

$$0 \longrightarrow \pi_n \, BSO(n) \xrightarrow{s_* \oplus e_*} \pi_n \, BSO(n+1) \oplus \mathbb{Z} \xrightarrow{w_n - r_2} \mathbb{Z}/2 \longrightarrow 0$$

Proof. First we show injectivity of $s_* \oplus e_*$. Proposition A.1.6 gives us a fibration $S^n \to BSO(n) \to BSO(n+1)$ so examine its long exact sequence:

$$\pi_{n+1} \, BSO(n+1) \longrightarrow \pi_n S^n \xrightarrow{c_n} \pi_n \, BSO(n) \xrightarrow{s_*} \pi_n \, BSO(n+1) \longrightarrow 0$$

We know that c_n sends [Id] to TS^n and $e(TS^n) \neq 0$ for n even (Proposition A.1.6 and Lemma A.1.3, respectively); since $H^n(S^n; \mathbb{Z})$ is torsion-free and e_* is a homomorphism it follows that $k \cdot TS^n$ is non-trivial for every non-zero k, so c_n is injective. By exactness $\ker(s_*) = \operatorname{Im}(c_n) = \langle TS^n \rangle$, in other words if $V \in \pi_n \, BSO(n)$ is stably-trivial then $V = \#^k TS^n$ for some k and so $e(V) = 2k$. We see that the kernel of the homomorphism $s_* \oplus e_*$ only contains the trivial bundle.

Lemma A.4.2 implies $\operatorname{Im}(s_* \oplus e_*) \subset \ker(w_n - r_2)$. To show the opposite inclusion, let $(V, k) \in \pi_n \, BSO(n+1) \oplus \mathbb{Z}$ with $k \equiv w_n(V) \mod 2$. The map s is n-connected so in particular s_* is surjective, so choose $V' \in s_*^{-1}(V)$. Lemma A.4.2 then says $e(V') \equiv w_n(V') = w_n(V) \equiv k \mod 2$; say $k = e(V') + 2l$. Now let $V'' = V' \# (l \cdot TS^n)$, so that $s_*(V'') = V$ and $e(V'') = k$. Therefore $(s_* \oplus e_*)(V'') = (V, k)$.

To see $w_n - r_2$ is surjective, take $V \in \pi_n \, BSO(n+1)$ and let $k \in \mathbb{Z}$ be incongruent to $w_n(V)$ modulo 2.

◻

Having shown that the homomorphism

$$s_* \oplus e_* \colon \pi_n \, BSO(n) \to \pi_n \, BSO(n+1) \oplus H^n(S^n; \mathbb{Z})$$

sending V to $(V \oplus \varepsilon, e(V))$ is injective, Theorem A2 has been verified for $X = S^n$. The general case is more difficult because of course $[X, BSO(n)]$ is in general just a set. In the proof of Theorem A1 we managed to give an n-factorization of the map s_{n-1}; knowing the relation $e(V^{n-1} \oplus \varepsilon) = 0$ we tried considering the homotopy fibre of e, and we were lucky enough that it suited our purpose. In this case, we need to use a relation between s_n and e, and in light of Lemma A.4.3 we consider the homotopy fibre

$$F \xrightarrow{\iota} BSO(n+1) \times K(\mathbb{Z}, n) \xrightarrow{w_n - r_2} K(\mathbb{Z}/2, n)$$

where "$w_n - r_2$" is defined using the infinite loop-space structure on $K(\mathbb{Z}/2, n)$. Lemma A.4.2 implies that $(w_n - r_2) \circ (s \times e)$ is null-homotopic, so if $s \times e$ is replaced with a cofibration then $w_n - r_2$ can be homotoped to make the composition constant. Then there is a lift of $s \times e$ along ι, denoted

$$se\colon \mathrm{BSO}(n) \to F$$

Proposition A.4.4. *$\pi_i(\iota)$ is an isomorphism if $i \neq n$ and injective if $i = n$.*
$\pi_i(se)$ is an isomorphism if $i < n+1$ and surjective if $i = n+1$.

Proof. Apply Lemma A.2.1 to the commutative square.

$$\begin{array}{ccc} \mathrm{BSO}(n+1) \times K(\mathbb{Z}, n) & \xrightarrow{w_n - r_2} & K(\mathbb{Z}/2, n) \\ {\scriptstyle s\times e}\uparrow & & \uparrow \\ \mathrm{BSO}(n) & \longrightarrow & * \end{array}$$

to obtain

$$\begin{array}{ccccc} F & \xrightarrow{\iota} & \mathrm{BSO}(n+1) \times K(\mathbb{Z}, n) & \xrightarrow{w_n - r_2} & K(\mathbb{Z}/2, n) \\ {\scriptstyle se}\uparrow & & {\scriptstyle s\times e}\uparrow & & \uparrow \\ \mathrm{BSO}(n) & \xrightarrow{=} & \mathrm{BSO}(n) & \longrightarrow & * \\ \uparrow & & \uparrow & & \uparrow \\ G & \longrightarrow & H & \longrightarrow & K(\mathbb{Z}/2, n-1) \end{array}$$

Taking long exact sequences of homotopy groups gives a commutative diagram of groups indexed by $\mathbb{Z} \times \mathbb{Z}$, a "fundamental domain" of which is shown in Figure A.1 on page 58 (some of the periodicity has been emphasized). Some obvious deductions about injectivity and surjectivity have already been made; moreover $\pi_{n+1}(s)\colon \pi_{n+1}\,\mathrm{BSO}(n) \to \pi_{n+1}\,\mathrm{BSO}(n+1)$ is surjective by Corollary A.1.8, and Lemma A.4.3 says that $e_* \oplus s_*$ is injective and that $\pi_n(w_n - r_2)$ is surjective. This is enough to deduce the first assertion of the Proposition.

To obtain the second, begin analysis of Figure A.1 at the bottom. $\pi_i(s)$ and $\pi_i(\iota)$ are isomorphisms for $i \leqslant n-1$ and hence $\pi_i(se)$ is an isomorphism for $i \leqslant n-1$; it follows that $\pi_{n-2}G = 0$ and $\pi_{n-1}H$ surjects onto $\mathbb{Z}/2$. Since $\pi_{n-1}(s)$ is in particular injective, it follows that $\pi_{n-1}H \cong \mathrm{Coker}(s_* \oplus e_*) \cong \mathbb{Z}/2$ thus $\pi_{n-1}G = 0$. Therefore $\pi_n(se)$ is surjective, and it is also injective because it is the first map in an injective composition. Finally, $\pi_{n+1}(se)$ is surjective because $\pi_{n+1}(s)$ is. □

Corollary A.4.5. *$\iota \circ se$ is the n-th and $(n+1)$-st Moore-Postnikov stages of $s \times e$.*

Proof. Observe that $\pi_i(s \times e)$ is an isomorphism for $i < n$ but not surjective when $i = n$, so $\mathrm{Id} \circ (s \times e)$ is the $(n-1)$-st factorization. By definition $\iota\colon F \to \mathrm{BSO}(n+1) \times K(\mathbb{Z}, n)$ is a principal fibration classified by the map $w_n - r_2$. Proposition A.4.4 gives the relevant homotopical information. □

Proof of Theorem A2. Suppose we are given two bundles $V = V_0 \sqcup V_1 : X \times \partial I \to \mathrm{BSO}(n)$ with the same stable class and Euler class; that is, suppose $(s \times e) \circ V$ extends to $X \times I$. We want to show the composition

$$X \times I \xrightarrow{\bar{V}} \mathrm{BSO}(n+1) \times K(\mathbb{Z}, n) \xrightarrow{w_n - r_2} K(\mathbb{Z}/2, n)$$

is null-homotopic, so that we get a lift

$$\begin{array}{ccc} & & F \\ & \overset{\tilde{V}}{\nearrow} & \big\downarrow \iota \\ X \times I & \xrightarrow{\bar{V}} & BSO(n+1) \times K(\mathbb{Z}, n) \end{array}$$

We know that $(w_n - r_2) \circ (s \times e) \circ V_0$ is null-homotopic, so choose a null-homotopy H. Now define a null-homotopy $\bar{H}\colon X \times I \times I \to K(\mathbb{Z}/2, n)$ of $(w_n - r_2) \circ \bar{V}$ by

$$\bar{H}(x, t, s) = \begin{cases} (w_n - r_2) \circ \bar{V}(x, t(1-2s)) & \text{if } s \in [0, \frac{1}{2}]; \\ H(x, 2s-1) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

Hence there is no obstruction to extending the map $se \circ V$. Now consider the obstructions to extending the map V to $X \times I$; they live in the groups $H^r(X; \pi_r \, BSO(n))$. Since se is $(n+1)$-connected it induces isomorphisms

$$H^r(X; \pi_r \, BSO(n)) \cong H^r(X; \pi_r F)$$

for $r \leqslant n$ and we know that the obstruction to extending $se \circ V$ vanishes. Since X is at most n-dimensional and since F is also the $(n+1)$-st Moore-Postnikov stage it follows that are no higher obstructions. Hence V can be extended to $X \times I$ and so $V_0 \cong V_1$. □

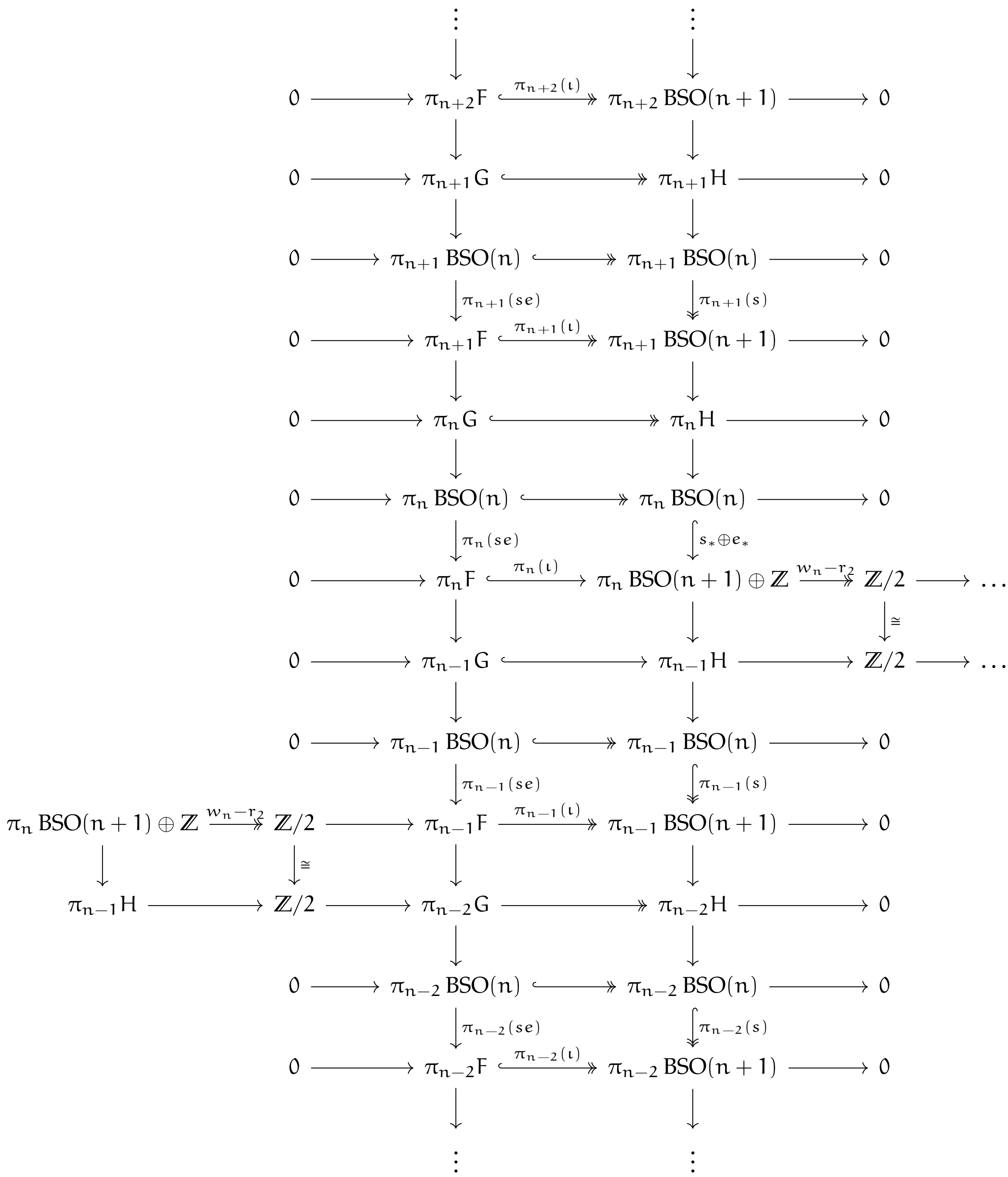


Figure A.1: The diagram to be chased.

Notation

This is a collection of essential notions used in this thesis with their meaning and the page where they are defined. This list is not complete but tries to cover the most common notations. If there is no page entry it is because the definition doesn't appear in this thesis.

Symbol	Meaning	Page number
$\mathbb{R}^n$	the standard n-dimensional vector space with standard basis $\{e_1, \dots, e_n\}$	
sh_+	the linear "shifting" map $\mathbb{R}^n \to \mathbb{R}^{n+1}$	4
$\mathbb{S}$	the sphere spectrum	
$X\langle n \rangle$	the n-connected cover of a space X	2
$\sigma(M)$	the signature of an oriented manifold	7
$\chi(X)$	the euler characteristic of a finite CW complex	
ε^n	the tivial rank n real vector bundle over any non-empty space	
$\cong_s$	stable isomorphism relation	2
$\mathrm{Th}(V)$	the Thom space of a vector bundle $V \to X$	2
$T_v E$	the vertical tangent bundle of a bundle of smooth manifolds	6
$\mathrm{St}_d(V)$	the Stiefel manifold of d-frames in V, for V an inner-product space	4
$\mathrm{Gr}_d(V)$	the Grassmannian manifold of d-dimensional subspaces of V, for V a vector space	4
ι_n	the stabilization map $\mathrm{Gr}_d(\mathbb{R}^n) \to \mathrm{Gr}_d(\mathbb{R}^{n+1})$	5
$U_{d,n}, U_{d,n}^\perp$	the tautological d-plane bundle over $\mathrm{Gr}_d(\mathbb{R}^n)$ and its orthogonal complement	5
ϕ_n	a bundle isomorphism $U_{d,n}^\perp \oplus \varepsilon \cong \iota_n^* U_{d,n+1}^\perp$	5
$BO(d),\ \gamma_d$	the classifying space of rank d bundles, and the universal rank d bundle	5
$\theta,\ \theta_d$	a tangential structure $B \to BO$ or $B(d) \to BO(d)$, respectively	8
$B(d)_n, \theta_{d,n}$	$\theta_d^{-1}(\mathrm{Gr}_d(\mathbb{R}^n))$ and $\theta_d\vert_{B(d)_n}$, respectively	9
λ_n	the inclusion map $B(d)_n \to B(d)_{n+1}$	9

Symbol	Meaning	Page number
$U^{\theta}_{d,n}$, $U^{\theta,\perp}_{d,n}$	$U_{d,n}$, respectively $U^{\perp}_{d,n}$, pulled back along $\theta_{d.n}\colon B(d)_n \to Gr_d(\mathbb{R}^n)$	9
$\mathrm{MTO}(d)$, $\mathrm{MT}\,\theta$	the Thom spectrum of $\{U^{\perp}_{d,n}\}_{n\geqslant d}$ and $\{U^{\theta,\perp}_{d,n}\}_{n\geqslant d}$, respectively	8, 9
Ω_d, Ω^{θ}_d	the oriented bordism group of smooth d-manifolds, and the bordism group of smooth d-manifolds with θ-structure	10
Eul_n	the subset of $\mathbb{Z}$ consiting of all Euler characteristics of closed, oriented n-dimensional manifolds	48
$\sigma_{k,d}$	the signature homomorphism $\pi_k\,\mathrm{MTSO}(d) \to \mathbb{Z}$	13
$\mathrm{span}(E)$, $\widetilde{\mathrm{span}}(E)$	the span and stable-span of a vector bundle	15
$E\#_f F$	the connected sum of two vector bundles, clutched by the function f	17
M_0	the result of eliminating an even-dimensional, closed, connected, oriented manidold's Euler characteristic via connected sum with a stably paralellizable manifold	18
g_2	the generator of $\pi_1\,\mathrm{MTSO}(3)$ with signature 2	19
g_4	the generator of $\pi_2\,\mathrm{MTSO}(2)$ with signature 4	22
$\mathrm{Ind}(s)$	the index of a finitely-singular k-field	24
a_k	half the rank of an irreducible $\mathbb{Z}/2$-graded $\mathscr{C}l_k$ module	24
r_k	$2a_k$ if 4 does not divide k, and $4a_k$ if it does	24
$\mathrm{Kerv}(M)$	the Kervaire semi-characteristic	25
$\mathbb{P}$, $\mathbb{G}$	alternate models of $\Sigma^{-1}\,\mathrm{MT}\,\theta_{d-1}$ and $\Sigma^{\infty}B(d)_+$	33
$\tilde{p}$, i, PT	maps in the cofibre sequence of spectra in Proposition 3.0.7	33
η, ν	the non-trivial element of of $\pi_1\mathbb{S}$ represented by the circle with its Lie-group framing, and the generator of $\pi_3\mathbb{S}$ represented by S^3 with its Lie-group framing	41
ω	a trivialization of an orientable line bundle induced by a chosen orientation	41
$e(V)$, e_n	the Euler class of a vector bundle, and the universal Euler class	48
$w_k(V)$, $p_k(V)$	Stiefel-Whitney and Pontryagin classes of a vector bundle	

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