C O L L O Q U I U M M A T H E M A T I C U M

VOL. 73 1997 NO. 1

COMPONENTWISE INJECTIVE MODELS OF FUNCTORS TO DGAs

BY

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The aim of this paper is to present a starting point for proving existence of injective minimal models (cf. [8]) for some systems of complete differential graded algebras.

Sullivan [7] introduced the rational de Rham theory for connected simplicial complexes and applied it to show that the de Rham algebra A^*_X of differential forms (over the field of rationals $\mathbb{Q}$) on a simply connected complex X of finite type determines its rational homotopy type. The central results of Sullivan's theory have been generalized by Triantafillou [8] to equivariant context but under the assumption that X is a simplicial set of finite type with a finite group G action which is G-connected and nilpotent, i.e. the fixed point simplicial subsets X^H are nonempty, connected, and nilpotent for all subgroups $H \subseteq G$. In this case not only A^*_X with the induced G-action are considered but also the system of the de Rham algebras $A^*_{X^H}$ for all subgroups $H \subseteq G$. This means that a functor $\mathcal{A}^*_X$ on the category $\mathcal{O}(G)$ of canonical orbits is studied and its injectivity (as an $\mathcal{O}(G)$-module) is the key observation for the existence of an equivariant analogue of Sullivan's minimal models. In the case X is disconnected we have to work over the category $\mathcal{O}(G, X)$ with one object for each component of X^H for all subgroups $H \subseteq G$. In general, the category $\mathcal{O}(G, X)$ is not finite, and in the category of functors from this category to the category of finitely generated $\mathbb{Q}$-modules there are not sufficiently many injectives to give a description of the rational homotopy type of X. Thus we have to replace finitely generated $\mathbb{Q}$-modules by a neglected but very useful category of linearly compact $\mathbb{Q}$-modules considered already by Lefschetz in [5] and then we may omit the assumption on finite type of G-simplicial sets as well.

Now we give an outline of the paper. In Section 1 we investigate the category $k\mathbb{I}$-Mod of covariant functors (or $k\mathbb{I}$-modules) from a small category $\mathbb{I}$ to the category of k-modules over a field k. This approach is inspired by a category of functors on categories related to the orbit category $\mathcal{O}(G)$

1991 *Mathematics Subject Classification*: Primary 55N91, 55P62; Secondary 16W50, 18G05.

determined by a finite group G. For simplicity we replace these categories by an EI-category $\mathbb{I}$ (i.e. a small category such that all endomorphisms are isomorphisms). We introduce basic notions and present some properties of functors from $\mathbb{I}$ to the category of linearly complete (or compact) k-modules. In particular, we show (Proposition 1.5) that on the de Rham algebra A_X^* of rational polynomial forms on a simplicial set X there is a natural complete linear topology.

In Section 2 we show (Theorem 2.1) that for any complete $k\mathbb{I}$-algebra $\mathcal{A}$ there exists a complete and injective (as a $k\mathbb{I}$-module) $k\mathbb{I}$-algebra $\mathfrak{Q}(\mathcal{A})$ and a natural cohomology isomorphism $\mathcal{A} \to \mathfrak{Q}(\mathcal{A})$. Then we generalize the notion of an *injective minimal* system of k-differential graded algebras considered in [8] to such systems indexed by some EI-category. The results will be applied in the forthcoming paper to the category of G-simplicial sets, where G is a finite group.

The author wishes to express his indebtedness to Professor S. Balcerzyk for carefully reading the manuscript, very useful discussions and suggestions.

1. Preliminaries on systems of modules. Let k be a (discrete) field. The category of (left) k-modules is denoted by k-Mod. If $\mathbb{I}$ is a small category then a covariant functor $\mathbb{I} \to k$-Mod is called a *left* $k\mathbb{I}$*-module* (or a *system of* k*-modules*) and the category of left $k\mathbb{I}$-modules is denoted by $k\mathbb{I}$-Mod and called the *category of left* $k\mathbb{I}$-modules. We also have the category of contravariant functors $\mathbb{I} \to k$-Mod, alias *right* $k\mathbb{I}$*-modules* and denoted by Mod-$k\mathbb{I}$.

The notions of *submodule*, *quotient module*, *kernel*, *image* and *cokernel* for $k\mathbb{I}$-modules are defined object-wise. For each object $I \in \mathrm{Ob}(\mathbb{I})$ we have the right $k\mathbb{I}$-module

$$k\mathbb{I}(-, I) : \mathbb{I} \to k\text{-Mod}$$

determined by the Yoneda functor $\mathbb{I}(-, I)$ and similarly, the left $k\mathbb{I}$-module $k\mathbb{I}(I, -)$. *Projective* and *injective* $k\mathbb{I}$-modules are defined by usual lifting properties. Observe that the category of projective right $k\mathbb{I}$-modules is isomorphic to the category of all injectives in the category of all covariant functors from $\mathbb{I}$ to the category $k\text{-Mod}^{\mathrm{op}}$ dual to k-Mod.

In various categories considered in algebraic topology endomorphisms are isomorphisms. Therefore, let $\mathbb{I}$ be an *EI-category* which by definition, is a small category in which each endomorphism is an isomorphism. Following [6] we define a partial order (which is crucial for the sequel) on the set $\mathrm{Is}(\mathbb{I})$ of isomorphism classes $\bar{I}$ of objects $I \in \mathrm{Ob}(\mathbb{I})$ by

$$\bar{I} \le \bar{J} \quad \text{if} \quad \mathbb{I}(I, J) \neq \emptyset.$$

This induces a partial ordering on the set $\mathrm{Is}(\mathbb{I})$ of isomorphism classes of objects, since the EI-property ensures that $\bar{I} \leq \bar{J}$ and $\bar{J} \leq \bar{I}$ implies $\bar{I} = \bar{J}$. We write that $\bar{I} < \bar{J}$ if $\bar{I} \leq \bar{J}$ and $\bar{I} \neq \bar{J}$. As it was shown in [3] injective $k\mathbb{I}$-modules can be constructed from injective modules over group rings. If $I \in \mathrm{Ob}(\mathbb{I})$ with the automorphism group $\mathrm{Aut}(I)$, we let $k[I] = k\,\mathrm{Aut}(I)$ be the group ring of $\mathrm{Aut}(I)$ and write $k[I]$-Mod for the category of left $k[I]$-modules.

For a fixed $I \in \mathrm{Ob}(\mathbb{I})$ we introduce the following covariant functors.

The cosplitting functor $S_I : k\mathbb{I}\text{-Mod} \to k[I]\text{-Mod}$ is defined as follows. If M is a $k\mathbb{I}$-module, let $S_I(M)$ be the $k[I]$-submodule of $M(I)$ equal to the intersection of kernels of all k-homomorphisms $M(f) : M(I) \to M(J)$ induced by all non-isomorphisms $f : I \to J$ with I as a source. Each automorphism $g \in \mathrm{Aut}(I)$ induces a map $M(g) : M(I) \to M(I)$ which maps $S_I(M)$ into itself. Thus $S_I(M)$ becomes a left $k[I]$-module. It is clear how S_I is defined on morphisms.

The restriction functor $\mathrm{Res}_I : k\mathbb{I}\text{-Mod} \to k[I]\text{-Mod}$ sends M to $M(I)$.

The coextension functor $E_I : k[I]\text{-Mod} \to k\mathbb{I}\text{-Mod}$ sends N to $\mathrm{Hom}_{k[I]}(k\mathbb{I}(-, I), N)$.

The coinclusion functor $\mathrm{In}_I : k[I]\text{-Mod} \to k\mathbb{I}\text{-Mod}$ assigns to a $k[I]$-module N the $k\mathbb{I}$-module $\mathrm{In}_I(N)$ defined by

$$\mathrm{In}_I(N)(J) = \begin{cases} \mathrm{Hom}_{k[I]}(k\mathbb{I}(J, I), N) & \text{if } \bar{J} = \bar{I}, \\ 0 & \text{if } \bar{J} \neq \bar{I}. \end{cases}$$

We say a $k\mathbb{I}$-module M is of *type* T, for $T \subseteq \mathrm{Is}(\mathbb{I})$, if the set $\{\bar{I} \in \mathrm{Is}(\mathbb{I}) \mid M(I) \neq 0\}$ is contained in T. For any $\bar{I} \in T$ choose a representative $I \in \bar{I}$ and fix a $k[I]$-monomorphism

$$0 \to M(I) \to Q_I,$$

where Q_I is injective. If M is of type T then we get a monomorphism of $k\mathbb{I}$-modules

$$0 \to M \to \prod_{\bar{I} \in T} E_I Q_I.$$

In particular, it follows that any injective $k\mathbb{I}$-module of type T is a direct summand of a $k\mathbb{I}$-module $\prod_{\bar{I} \in T} E_I Q_I$, where Q_I are injective $k[I]$-modules for $\bar{I} \in T$.

The next result follows easily from the above definitions.

Lemma 1.1. (1) *The functors* E_I *and* Res_I *and the functors* S_I *and* In_I *are adjoint*, *i.e. there are natural isomorphisms of* k*-modules*

$$\mathrm{Hom}_{k\mathbb{I}}(M, E_I N) \to \mathrm{Hom}_{k[I]}(\mathrm{Res}_I M, N)$$

and

$$\mathrm{Hom}_{k[I]}(N, S_I M) \to \mathrm{Hom}_{k\mathbb{I}}(\mathrm{In}_I N, M).$$

(2) $S_I \circ E_I : k[I]$-Mod $\to k[I]$-Mod *is naturally equivalent to the identity functor. The composition* $S_J \circ E_I$ *is zero for* $\bar{I} \neq \bar{J}$.

(3) S_I *and* E_I *preserve products, monomorphisms and injective modules.*

The dual category k-Mod$^{\mathrm{op}}$ is isomorphic to the category k-Mod$^{\mathrm{c}}$ of linearly compact k-modules considered in [5]. For our purpose we briefly present some results on the category k-Mod$^{\mathrm{c}}$. A topological k-module M is said to be *linearly topological* if it is Hausdorff and there is a fundamental system $\mathcal{N}(M)$ of neighborhoods of zero consisting of k-submodules. A linearly topological k-module M is called *linearly compact* if for every collection $\{F_i\}_{i\in I}$ of closed affine subsets of M (i.e. $F_i = m_i + M_i$ for some closed k-submodule $M_i \subseteq M$) with the finite intersection property we have $\bigcap_{i\in I} F_i \neq \emptyset$. For linearly topological k-modules M and N let $\mathrm{Hom}_k^{\mathrm{t}}(M,N)$ be the set of all continuous k-linear maps. We topologize this k-module by requiring that for any linearly compact k-submodule $K \subseteq M$ and an open k-submodule $V \subseteq N$ the k-submodules $\{f \in \mathrm{Hom}_k^{\mathrm{t}}(M,N) : f(K) \subseteq V\}$ form a subbasis of a linear topology on $\mathrm{Hom}_k^{\mathrm{t}}(M,N)$. For a k-module M let $M^* = \mathrm{Hom}_k^{\mathrm{t}}(M,k)$ be its topological dual.

THEOREM 1.2 [5]. (1) *A linearly topological k-module M is linearly compact if and only if M^* is discrete.*

(2) *If M is linearly compact or discrete then the canonical map $M \to M^{**}$ is a topological isomorphism.*

(3) *If M and N are linearly compact or discrete k-modules then the canonical map* $\mathrm{Hom}_k^{\mathrm{t}}(M,N) \to \mathrm{Hom}_k^{\mathrm{t}}(N^*,M^*)$ *is a topological isomorphism.*

For a linearly topological k-module M and its closed k-submodule M' the quotient topology on M/M' is linear. In particular, if M' is an open submodule then this topology on M/M' is discrete. Let $\omega_{M'} : M \to M/M'$ be the canonical map. For $V_1, V_2 \in \mathcal{N}(M)$ such that $V_1 \subseteq V_2$, let $\omega_{V_2}^{V_1} : M/V_1 \to M/V_2$ be the canonical map and $M^\wedge = \varprojlim_{V\in\mathcal{N}(M)} M/V$. Write $\pi_V : M^\wedge \to M/V$ for the canonical projection. Then the collection $\{\ker \pi_V : V \in \mathcal{N}(M)\}$ of k-submodules forms a subbasis of a linear topology on $M^\wedge$. The k-module $M^\wedge$ with this topology is called the *completion* of M. The collection of maps $\omega_V : M \to M/V$ for $V \in \mathcal{N}(M)$ determines a continuous monomorphism $\omega : M \to M^\wedge$ such that $\omega(M)$ is dense in $M^\wedge$. A topological k-module M is said to be *complete* if the map ω is a topological isomorphism. Of course, if M is linearly compact or discrete then $\omega(M)$ is closed in $M^\wedge$ and thus M is complete as well.

For two linearly topological k-modules M and N let $M \otimes N$ be their tensor product over k. If $V \subseteq M$ and $W \subseteq N$ are two open k-submodules, we write $[V,W] = V \otimes N + M \otimes W$. Then the following lemma holds.

LEMMA 1.3. *If M and N are linearly topological k-modules then the collection of k-submodules $[V, W]$ of $M \otimes N$ with open k-submodules $V \subseteq M$ and $W \subseteq N$ forms a linear topology on $M \otimes N$ such that the canonical bilinear map $M \times N \to M \otimes N$ is universal with respect to uniformly continuous k-bilinear maps to linearly topological k-modules.*

Write $M \widehat{\otimes} N$ for the completion $(M \otimes N)^\wedge$ and call it the *complete tensor product* of M and N. Then the canonical map $M \times N \to M \widehat{\otimes} N$ is universal with respect to uniformly continuous k-bilinear maps to complete k-modules.

Now let $\mathbb{I}$ be an EI-category. A covariant functor from $\mathbb{I}$ to $k\text{-Mod}^{\mathrm{c}}$ is said to be a *linearly compact left* $k\mathbb{I}$-module. For two linearly compact left $k\mathbb{I}$-modules M, N we define their *complete tensor product* $M \widehat{\otimes} N$ as a linearly compact left $k\mathbb{I}$-module such that $(M \widehat{\otimes} N)(I) = M(I) \widehat{\otimes} N(I)$ for all $I \in \mathrm{Ob}(\mathbb{I})$.

Let DGA_k be the category of homologically connected commutative differential graded k-algebras (or simply k-algebras). We briefly recall some constructions presented in [4]. For a map $\gamma : B \to E$ in DGA_k, where B is augmented, Halperin [4] considers its "minimal factorization". Namely, he generalizes the notion of a minimal k-algebra [7] to a *minimal KS-extension* given by a special sequence of augmented k-algebras

$$\mathbb{E} : B \xrightarrow{i} C \xrightarrow{\pi} A,$$

where A is free as a graded commutative k-algebra generated by some graded k-module $M = \{M_i\}_{i \geq 0}$. If $M_0 = 0$ then the extension $\mathbb{E}$ is called *positive*. In [4] it is shown that for any map $\gamma : B \to E$ [4] of connected k-algebras, where B is augmented, there is a unique (up to isomorphism) minimal KS-extension

$$\mathbb{E} : B \xrightarrow{i} C \xrightarrow{\pi} A$$

and a homology isomorphism $\varrho : C \to E$ such that $\varrho \circ i = \gamma$.

The extension $\mathbb{E}$ together with the map $\varrho : C \to E$ is called a *KS-minimal model* for γ. In particular, for a k-algebra A and the canonical map $k \to A$ one gets a *minimal algebra* M_A together with a homology isomorphism $\varrho_A : M_A \to A$ called the *minimal model* for A.

An object $A = \{A^n\}_{n \geq 0}$ in DGA_k is called *complete* if

(1) A^n is a complete linearly topological k-module and the differential $d : A^n \to A^{n+1}$ is continuous for all $n \geq 0$,

(2) multiplication $A^n \times A^m \to A^{n+m}$ is uniformly continuous for all $n, m \geq 0$ (with respect to the linear product topology on $A^n \times A^m$).

The key example of a complete algebra is produced as follows. For the field of rationals $\mathbb{Q}$ and a simplicial set X one can form a $\mathbb{Q}$-algebra A_X^* by taking collections of $\mathbb{Q}$-polynomial forms on each simplex (sums of terms of type $\omega(t_0,\dots,t_n)\,dt_{i_1}\wedge\dots\wedge dt_{i_l}$, where ω is a $\mathbb{Q}$-polynomial) that agree when restricted to common faces (see [1] for more details). We define a *natural topology* on the $\mathbb{Q}$-module A^n_X of n-forms on X as follows: for any map $\widetilde{x}:\Delta(l)\to X$, the k-submodules $\ker(A^n_X(\widetilde{x}):A^n_X\to A^n_{\Delta(l)})$, where $\Delta(l)$ is the l-simplex, form a fundamental system of neighborhoods of zero in A^n_X. The following proposition holds.

PROPOSITION 1.4. *Let X be a simplicial set. Then*:

(1) *the natural topology on A^n_X is complete for all $n\ge 0$,*

(2) *the multiplication $A^n_X\times A^m_X\to A^{n+m}_X$ of differential forms is uniformly continuous* (*with respect to the product topology on $A^n_X\times A^m_X$*),

(3) *the differential $d^n_X:A^n_X\to A^{n+1}_X$ is continuous.*

Proof. (1) First, observe that for a simplicial map $\widetilde{x}:\Delta(l)\to X$ there is an isomorphism $A^n_X/\ker A^n_{\tilde{x}}\approx A^n_{\Delta(l)}$ of discrete $\mathbb{Q}$-modules. Then the map

$$\phi: A^n_X\to \varprojlim_{\tilde{x}:\Delta(l)\to X} A^n_X/\ker A^n_{\tilde{x}}\approx \varprojlim_{\tilde{x}:\Delta(l)\to X} A^n_{\Delta(l)}$$

such that $\phi(\omega)=(A^n_{\tilde{x}}(\omega))_{\tilde{x}:\Delta(l)\to X}$, for $\omega\in A^n_X$, is the required topological isomorphism.

(2) For a simplicial map $\widetilde{x}:\Delta(l)\to X$ and the corresponding open k-submodule $V=\ker(A^{n+m}(\widetilde{x}):A^{n+m}_X\to A^{n+m}_{\Delta(l)})$ consider the subspaces $U_1=\ker(A^n(\widetilde{x}):A^n_X\to A^n_{\Delta(l)})$ and $U_2=\ker(A^m(\widetilde{x}):A^m_X\to A^m_{\Delta(l)})$ of A^n_X and A^m_X, respectively. Then the image of $U_1\times A^m_X$ and $A^n_X\times U_2$ under the multiplication map of differential forms is contained in V, so the multiplication is uniformly continuous.

(3) The differential d^n_X is natural with respect to X, hence it is continuous as well. ∎

Write $\mathrm{DGA}^\wedge_k$ for the subcategory of DGA_k determined by complete differential graded k-algebras.

For a minimal k-algebra M let $M(n)$ be its subalgebra generated by elements of degree at most n. Then M is said to be *nilpotent* if each $M(n)$ is constructed from $M(n-1)$ by a finite number of elementary extensions (see [4] for details). A homologically connected k-algebra A is said to be *nilpotent* if its minimal model M_A is nilpotent. If X is a (connected) nilpotent simplicial set then the de Rham $\mathbb{Q}$-algebra A^*_X of differential forms is nilpotent as shown in [1]. If a k-algebra A is augmented let $\widetilde{A}=\ker(A\to k)$ be its augmentation ideal. Recall that *decomposability* of the differential d of A means that $d(A)\subseteq\widetilde{A}\cdot\widetilde{A}$.

Let $\mathbb{I}$ be an EI-category and $k\mathbb{I}$-DGA_k the category of all covariant functors from $\mathbb{I}$ to DGA_k called $k\mathbb{I}$-*algebras* (or *systems of k-algebras*). We say that a $k\mathbb{I}$-algebra $\mathcal{A}$ is complete if the algebras $\mathcal{A}(I)$ are complete for all $I \in \mathrm{Ob}(\mathbb{I})$ and $\mathcal{A}$ is *injective* if the left $k\mathbb{I}$-modules $\mathcal{A}^n$ are injective for $n \geq 0$, where $\mathcal{A}^n(I) = (\mathcal{A}(I))^n$ for all $I \in \mathrm{Ob}(\mathbb{I})$.

For any complete injective (as a $k\mathbb{I}$-module) $k\mathbb{I}$-algebra $\mathcal{A}$ and a complete left $k\mathbb{I}$-module M we consider two types of cohomology of $\mathcal{A}$.

(1) The $k\mathbb{I}$-module $\boldsymbol{H}^n(\mathcal{A})$ such that $\boldsymbol{H}^n(\mathcal{A})(I) = H^n(\mathcal{A}(I))$ for $I \in \mathrm{Ob}(\mathbb{I})$ and $n \geq 0$.

(2) The cohomology $H^n(\mathcal{A}, M) = H^n(\mathrm{Hom}(M, \mathcal{A}))$ with coefficients in M for $n \geq 0$, where $\{\mathrm{Hom}(M, \mathcal{A}^n)\}_{n\geq 0}$ is a cochain complex in the category of complete left $k\mathbb{I}$-modules. For a projective resolution $M^{(\star)}$ of M in the category of complete $k\mathbb{I}$-modules we form the double complex $\mathrm{Hom}(M^{(\star)}, \mathcal{A})$. The standard homological algebra arguments yield a spectral sequence

$$E_2^{pq} = \mathrm{Ext}^p(M, \boldsymbol{H}^q(\mathcal{A})) \Rightarrow H^{p+q}(\mathcal{A}, M).$$

Notice that the injectivity of $\mathcal{A}$ (as a $k\mathbb{I}$-module) implies the convergence of this sequence and $H^n(\mathcal{A}, M) = \mathrm{Hom}(M, \boldsymbol{H}^n(\mathcal{A}))$ if M is projective.

2. Injective extension of systems of algebras. The spectral sequence considered in the previous section plays a key role in a construction of an injective minimal model for a complete injective $k\mathbb{I}$-algebra $\mathcal{A}$, for an EI-category $\mathbb{I}$. This is the reason why the injectivity of $\mathcal{A}$ (as a $k\mathbb{I}$-module) is necessary. Theorem 2.1 in this section shows that for any complete $k\mathbb{I}$-algebra $\mathcal{A}$ there exists a complete injective $k\mathbb{I}$-algebra $\mathfrak{Q}(\mathcal{A})$ and a natural cohomology isomorphism $\mathcal{A} \to \mathfrak{Q}(\mathcal{A})$.

Hereafter, we assume that $\mathbb{I}$ is an EI-category with the filtration $\emptyset = T_0 \subset T_1 \ldots \subset T_m = \mathrm{Is}(\mathbb{I})$ such that $\bar{I} \in T_k$, $\bar{J} \in T_l$, $\bar{I} < \bar{J}$ implies $k > l$ and all $k\mathbb{I}$-algebras $\mathcal{A}$ are homologically connected, i.e. satisfy $\boldsymbol{H}^0(\mathcal{A}) = \underline{k}$, where $\underline{k}$ is the constant $k\mathbb{I}$-module determined by a field k. To show the main result we need some constructions. An augmented k-algebra A is called *acyclic* if $H^n(\widetilde{A}) = 0$ for all $n \geq 0$, where $\widetilde{A}$ is the augmentation ideal of A. If M is a graded k-module then the k-algebra $\mathfrak{F}(M)$ freely generated by $M \oplus sM$, where sM is a copy of M with a shift of degree $+1$ and $d(m) = sm$ for $m \in M$, is an augmented acyclic k-algebra. In particular, for a $k\mathbb{I}$-algebra $\mathcal{A}$ and $I \in \mathrm{Ob}(\mathbb{I})$ we get an associated system $\mathfrak{F}(E_I S_I \mathcal{A})$ of acyclic $k\mathbb{I}$-algebras such that $\mathfrak{F}(E_I S_I \mathcal{A})(J) = \mathfrak{F}(E_I S_I \mathcal{A}(J))$ for $J \in \mathrm{Ob}(\mathbb{I})$, where E_I and S_I are functors defined in the previous section. Now we are in a position to present a generalization of Theorem 1 in [2].

THEOREM 2.1. *If $\mathbb{I}$ is an EI-category such that $k[I]$ is a semisimple ring for all $I \in \mathrm{ob}(\mathbb{I})$ and there is a filtration*

$$\emptyset = T_0 \subset T_1 \subset \ldots \subset T_m = \mathrm{Is}(\mathbb{I})$$

satisfying the above condition then for any complete $k\mathbb{I}$-algebra $\mathcal{A}$ there is a complete and injective (*as a $k\mathbb{I}$-module*) *$k\mathbb{I}$-algebra $\mathfrak{Q}(\mathcal{A})$ and a natural inclusion $i_{\mathcal{A}} : \mathcal{A} \to \mathfrak{Q}(\mathcal{A})$ which is a cohomology isomorphism.*

P r o o f. We proceed by induction over the filtration of $\mathrm{Is}(\mathbb{I})$ to construct a sequence of $k\mathbb{I}$-algebras and natural inclusions

$$\mathcal{A} = \mathfrak{Q}_0(\mathcal{A}) \xrightarrow{i_0} \mathfrak{Q}_1(\mathcal{A}) \xrightarrow{i_1} \ldots \xrightarrow{i_{m-1}} \mathfrak{Q}_m(\mathcal{A}) = \mathfrak{Q}(\mathcal{A})$$

which are cohomology isomorphisms.

Let $\mathfrak{Q}_0(\mathcal{A}) = \mathcal{A}$ and $\mathfrak{Q}_1(\mathcal{A})$ be a $k\mathbb{I}$-algebra such that

$$\mathfrak{Q}_1(\mathcal{A})(J) = \begin{cases} \mathcal{A}(J) \mathbin{\widehat{\otimes}} \mathfrak{F}(\prod_{\bar{I} \in T_1} E_I S_I \mathcal{A})(J) & \text{if } \bar{J} \notin T_1, \\ \mathcal{A}(J) & \text{otherwise.} \end{cases}$$

The value of $\mathfrak{Q}_1(\mathcal{A})$ on a morphism $\phi : J \to K$ in the category $\mathbb{I}$ is defined as follows. If $\overline{K} \notin T_1$ then the map $\mathfrak{Q}_1(\mathcal{A})(\phi) : \mathfrak{Q}_1(\mathcal{A})(J) \to \mathfrak{Q}_1(\mathcal{A})(K)$ is induced by the maps $\mathcal{A}(\phi) : \mathcal{A}(J) \to \mathcal{A}(K)$ and $E_I S_I(\mathcal{A})(\phi) : E_I S_I(\mathcal{A})(J) \to E_I S_I(\mathcal{A})(K)$. For $\overline{K} \in T_1$ the map $\mathfrak{Q}_1(\mathcal{A})(\phi)$ is determined by the maps $\mathcal{A}(\phi) : \mathcal{A}(J) \to \mathcal{A}(K)$ and $\prod_{\bar{I} \in T_1} (E_I S_I \mathcal{A})(J) \xrightarrow{\pi_K} (E_K S_K \mathcal{A})(J) \xrightarrow{(E_K S_K \mathcal{A})(\phi)} (E_K S_K \mathcal{A})(K) = S_K \mathcal{A} \xrightarrow{\eta_K} \mathcal{A}(K)$, where π_K is the projection map and η_K the inclusion $S_K \mathcal{A} \to \mathcal{A}(K)$. Write $i_0 : \mathfrak{Q}_0(\mathcal{A}) \to \mathfrak{Q}_1(\mathcal{A})$ for the canonical inclusion; it is a cohomology isomorphism since $\mathfrak{F}(\prod_{\bar{I} \in T_1} E_I S_I \mathcal{A})(J)$ are acyclic k-algebras for all $J \in \mathrm{Ob}(\mathbb{I})$.

Given $\mathfrak{Q}_l(\mathcal{A})$ let $\mathfrak{Q}_{l+1}\mathcal{A}$ be a $k\mathbb{I}$-algebra such that

$$\mathfrak{Q}_{l+1}(\mathcal{A})(J) = \begin{cases} \mathfrak{Q}_l(\mathcal{A})(J) \mathbin{\widehat{\otimes}} \mathfrak{F}(\prod_{\bar{I} \in T_{l+1}} E_I S_I \mathfrak{Q}_l \mathcal{A})(J) & \text{if } \bar{J} \notin T_{l+1}, \\ \mathfrak{Q}_l(\mathcal{A})(J) & \text{otherwise.} \end{cases}$$

The values of $\mathfrak{Q}_{l+1}(\mathcal{A})$ on morphisms are defined in the same way as for $\mathfrak{Q}_1(\mathcal{A})$. Write $i_l : \mathfrak{Q}_l(\mathcal{A}) \to \mathfrak{Q}_{l+1}(\mathcal{A})$ for the canonical inclusion which is a cohomology isomorphism since $\mathfrak{F}(\prod_{\bar{I} \in T_{l+1}} E_I S_I \mathfrak{Q}_l \mathcal{A})(J)$ are acyclic k-algebras for all $J \in \mathrm{Ob}(\mathbb{I})$. Define $\mathfrak{Q}(\mathcal{A}) = \mathfrak{Q}_m(\mathcal{A})$ and $i_{\mathcal{A}} = i_{m-1} \circ \ldots \circ i_0 : \mathcal{A} \to \mathfrak{Q}(\mathcal{A})$. Then $i_{\mathcal{A}}$ is a cohomology isomorphism and from the construction it follows that $\mathfrak{Q}$ is a functor and $i : \mathrm{id}_{\mathbb{I}\text{-}\mathrm{DGA}_k} \to \mathfrak{Q}$ is a natural transformation, where $\mathrm{id}_{\mathbb{I}\text{-}\mathrm{DGA}_k}$ is the identity functor.

It remains to show that $\mathfrak{Q}(\mathcal{A})$ is injective, i.e. by [3] it can be written as a product of $k\mathbb{I}$-modules $E_I M$ for some $I \in \mathrm{Ob}(\mathbb{I})$ and $k[I]$-modules M. Again the argument goes inductively over the filtration of $\mathrm{Is}(\mathbb{I})$. First observe that $\mathfrak{Q}_1(\mathcal{A})$ as a graded $k\mathbb{I}$-module contains the injective graded $k\mathbb{I}$-module $\prod_{\bar{I} \in T_1} E_I \mathcal{A}(I)$. Therefore, there is a split short exact sequence of graded $k\mathbb{I}$-modules

$$0 \to \prod_{\bar{I} \in T_1} E_I \mathcal{A}(I) \to \mathfrak{Q}_1(\mathcal{A}) \to R_1 \to 0,$$

where $R_1(I) = 0$ for $\bar{I} \in T_1$ and $S_I \mathfrak{Q}_1(\mathcal{A}) = S_I R_1$ for $\bar{I} \notin T_1$. In particular,

$S_I\mathfrak{Q}_1\mathcal{A} = R_1(I)$ for $\bar{I} \in T_2 \setminus T_1$. Then from the construction of $\mathfrak{Q}_2(\mathcal{A})$ it follows that the injective $k\mathbb{I}$-module $\prod_{\bar{I}\in T_1} E_I\mathcal{A}(I) \oplus \prod_{\bar{I}\in T_2\setminus T_1} E_I R_1(I)$ is contained in $\mathfrak{Q}_2(\mathcal{A})$. Hence there is a split short exact sequence

$$0 \to \prod_{\bar{I}\in T_1} E_I\mathcal{A}(I) \oplus \prod_{\bar{I}\in T_2\setminus T_1} E_I R_1(I) \to \mathfrak{Q}_2(\mathcal{A}) \to R_2 \to 0,$$

where $R_2(I) = 0$ for $\bar{I} \in T_2$ and $S_I\mathfrak{Q}_2(\mathcal{A}) = S_I R_2$ for $\bar{I} \notin T_2$. In particular, $S_I\mathfrak{Q}_2(\mathcal{A}) = R_2(I)$ for $\bar{I} \in T_3 \setminus T_2$.

Assume that $\mathfrak{Q}_l(\mathcal{A}) \approx \prod_{\bar{I}\in T_1} E_I\mathcal{A}(I) \oplus \prod_{\bar{I}\in T_2\setminus T_1} E_I R_1(I) \oplus \ldots \oplus \prod_{\bar{I}\in T_l\setminus T_{l-1}} E_I R_{l-1}(I) \oplus R_l$ as $k\mathbb{I}$-modules, $R_l(I) = 0$ for $\bar{I} \in T_l$ and $S_I\mathfrak{Q}_l(\mathcal{A}) = S_I R_l$ for $\bar{I} \notin T_l$. Then $S_I\mathfrak{Q}_l\mathcal{A}(I) = R_l(I)$ for $\bar{I} \in T_{l+1} \setminus T_l$ and $\mathfrak{Q}_{l+1}\mathcal{A}$ contains an injective $k\mathbb{I}$-module $\prod_{\bar{I}\in T_1} E_I\mathcal{A}(I) \oplus \prod_{\bar{I}\in T_2\setminus T_1} E_I R_1(I) \oplus \ldots \oplus \prod_{\bar{I}\in T_{l+1}\setminus T_l} E_I R_l(I)$ and there is a split short exact sequence

$$0 \to \prod_{\bar{I}\in T_1} E_I\mathcal{A}(I) \oplus \prod_{\bar{I}\in T_2\setminus T_1} E_I R_1(I) \oplus \ldots \oplus \prod_{\bar{I}\in T_{l+1}\setminus T_l} E_I R_l(I) \to \mathfrak{Q}_{l+1}(\mathcal{A}) \to R_{l+1} \to 0,$$

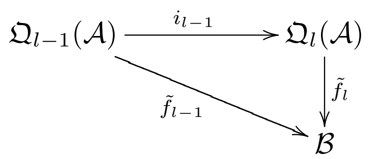
where $R_{l+1}(I) = 0$ for $\bar{I} \in T_{l+1}$ and $S_I\mathfrak{Q}_{l+1}\mathcal{A} = R_{l+1}$ for $\bar{I} \in T_{l+2} \setminus T_{l+1}$. Finally, we obtain $\mathfrak{Q}(\mathcal{A}) = \mathfrak{Q}_m(\mathcal{A}) \approx \prod_{\bar{I}\in T_1} E_I\mathcal{A}(I) \oplus \prod_{\bar{I}\in T_2\setminus T_1} E_I R_1(I) \oplus \ldots \oplus \prod_{\bar{I}\in T_m\setminus T_{m-1}} E_I R_{m-1}(I)$ as a graded $k\mathbb{I}$-module, since $R_m(I) = 0$ for $\bar{I} \in T_m$, so $\mathfrak{Q}(\mathcal{A})$ is injective as a graded $k\mathbb{I}$-module. ∎

If $\underline{k}$ is the constant $k\mathbb{I}$-algebra determined by a field k then $\underline{k}$ is not in general injective as $k\mathbb{I}$-module. But for any $k\mathbb{I}$-algebra $\mathcal{A}$ (injective as a $k\mathbb{I}$-module) there is a map $\mathfrak{Q}(\underline{k}) \to \mathcal{A}$ of $\mathbb{I}$-algebras extending the canonical inclusion $\underline{k} \to \mathcal{A}$ as follows from a more general fact.

PROPOSITION 2.2. *Let $\mathbb{I}$ be an EI-category satisfying the above conditions. If $f : \mathcal{A} \to \mathcal{B}$ is a map of $k\mathbb{I}$-algebras and $\mathcal{B}$ is injective as a $k\mathbb{I}$-module then there is an extension map $\widetilde{f} : \mathfrak{Q}(\mathcal{A}) \to \mathcal{B}$ of $k\mathbb{I}$-algebras.*

Proof. We construct by induction over the filtration of $\mathrm{Is}(\mathbb{I})$ a sequence of maps $\widetilde{f}_l : \mathfrak{Q}_l(\mathcal{A}) \to \mathcal{B}$ for $l = 0, 1, \ldots, n$.

Let $\widetilde{f}_0 = f$. Given $\widetilde{f}_l : \mathfrak{Q}_l(\mathcal{A}) \to \mathcal{B}$ such that the diagram



commutes we construct a map $\widetilde{f}_{l+1} : \mathfrak{Q}_{l+1}(\mathcal{A}) \to \mathcal{B}$ as follows. The $k\mathbb{I}$-algebra $\mathcal{B}$ is injective, so by [3] there is an isomorphism of $k\mathbb{I}$-modules $\mathcal{B} \approx$

$\prod_{\bar{I}\in \mathrm{Is}}(\mathbb{I}) E_I S_I \mathcal{B}$ and $\widetilde{f}_l$ induces maps $E_I S_I \widetilde{f}_l : E_I S_I \mathfrak{Q}_l(\mathcal{A}) \to E_I S_I \mathcal{B}$ for $I \in \mathrm{Ob}(\mathbb{I})$. Then $\widetilde{f}_l$ together with these maps determines a map $\widetilde{f}_{l+1} : \mathfrak{Q}_{l+1}(\mathcal{A}) \to \mathcal{B}$. The map $\widetilde{f} = \widetilde{f}_m$ has the required property. ∎

A $k\mathbb{I}$-algebra $\mathcal{A}$ with a map $\mathfrak{Q}(\underline{k}) \to \mathcal{A}$ is called a $k\mathbb{I}$-*algebra under* $\mathfrak{Q}(\underline{k})$ or a *based* $k\mathbb{I}$-*algebra*. A based injective, nilpotent and complete $k\mathbb{I}$-algebra $\mathcal{M}$ is said to be *minimal* if it satisfies the following:

(1) there is an inclusion $\mathfrak{Q}(\underline{k}) \hookrightarrow \mathcal{M}$;

(2) $\mathcal{M}(I)$ is a positive KS-extension of $\mathfrak{Q}(\underline{k})(I)$ for all $I \in \mathrm{Ob}(\mathbb{I})$;

(3) $\mathcal{M}(I)$ is a minimal KS-extension of $\mathfrak{Q}(\underline{k})(I)$ for all terminal $I \in \mathrm{Ob}(\mathbb{I})$;

(4) if d is the differential of $\mathcal{M}$ then $d_{|S_I\mathcal{M}}$ is decomposable for all $I \in \mathrm{Ob}(\mathbb{I})$.

A $k\mathbb{I}$-algebra $\mathcal{A}$ is called *nilpotent* if $\mathcal{A}(I)$ is nilpotent for all $I \in \mathrm{Ob}(\mathbb{I})$. We shall show in the forthcoming paper that injective minimal $k\mathbb{I}$-algebras play the same role in the category of nilpotent complete $k\mathbb{I}$-algebras as minimal algebras in the category of nilpotent k-algebras.

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Received 27 June 1996;
revised 11 July 1996