ON CHERN-SIMONS MATRIX MODELS

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Abstract. The contribution of reducible connections to the $U(N)$ Chern-Simons invariant of a Seifert manifold M can be expressed in some cases in terms of matrix integrals. We show that the $U(N)$ evaluation of the LMO invariant of any rational homology sphere admits a matrix model representation which agrees with the Chern-Simons matrix integral for Seifert spheres at the trivial connection.

Contents

1. Introduction

Chern-Simons invariants of links and three-manifolds have been a rich arena for the interactions of mathematics and physics in the last years. More recently, there has been a growing connection between Chern-Simons invariants and topological string theory/Gromov-Witten theory. This has motivated various developments and results. One of these developments has been the representation of Chern-Simons invariants of three-manifolds in terms of matrix integrals over a Lie algebra [Mar05, AKMV04]. This representation has its origin in the work of Rozansky on the trivial connection contribution to the Chern-Simons invariant

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[Roz96], and on the results of Lawrence and Rozansky on the $SU(2)$ Chern-Simons invariant of Seifert homology spheres [LR99].

In this short note we clarify these results in the light of the LMO invariant [LMO98] and its Aarhus integral representation [BNGRT02]. After reviewing in section 2 the connection between Chern–Simons theory and matrix integrals, we show in section 3 that the LMO invariant of a rational homology sphere M, evaluated for the $U(N)$ weight system, can be always represented as a matrix integral. If the manifold M is obtained through surgery on a link $\mathcal{L}$ in $\mathbf{S}^3$, the matrix model 'potential' involved in the matrix integral is related to the Kontsevich integral of $\mathcal{L}$. Since the LMO invariant is conjectured to capture the trivial connection contribution to the Chern-Simons invariant of M, this suggests that this contribution always has a matrix integral representation, as shown in [Mar05].

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2. The Witten-Reshetikhin-Turaev invariant as a matrix integral

2.1. **The Witten-Reshetikhin-Turaev invariant.** The Witten-Reshetikhin-Turaev (WRT) invariant of a three-manifold M was originally defined by Witten in [Wit89] as the partition function of a certain topological quantum field theory on M, the so-called Chern-Simons theory. Fortunately, the invariant can be defined in a purely combinatorial way, as we now describe.

The WRT invariant depends on a choice of gauge group G and of an integer k related to the level of the affine Lie algebra based on G. We will use the following notations: r denotes the rank of G, and d its dimension. y denotes the dual Coxeter number. The fundamental weights will be denoted by λ_i, and the simple roots by α_i, with $i = 1, \cdots, r$. ρ denotes as usual the Weyl vector, given by the sum of the fundamental weights. The weight and root lattices of G are denoted by $\Lambda_{\rm w}$ and $\Lambda_{\rm r}$, respectively. Finally, we put $l = k + y$.

The WRT invariant can be defined in terms of a surgery presentation of M. By theorem due to Lickorish, any three-manifold M can be obtained by surgery on a link $\mathcal{L}$ in $\mathbf{S}^3$. Let us denote by $\mathcal{K}_i$, $i = 1, \cdots, L$, the components of $\mathcal{L}$. The surgery operation means that around each of the knots $\mathcal{K}_i$ we take a tubular neighborhood $\mathrm{Tub}(\mathcal{K}_i)$ that we remove from $\mathbf{S}^3$. This tubular neighborhood is a solid torus with a contractible cycle α_i and a noncontractible cycle β_i. We then glue the solid torus back after performing an $\mathrm{SL}(2, \mathbf{Z})$ transformation given by the matrix

$$U^{(p_i,q_i)} = \begin{pmatrix} p_i & r_i \\ q_i & s_i \end{pmatrix}. \tag{1}$$

This means that the cycles $p_i\alpha_i + q_i\beta_i$ and $r_i\alpha_i + s_i\beta_i$ on the boundary of the complement of $\mathcal{K}_i$ are identified with the cycles α_i, β_i in $\mathrm{Tub}(\mathcal{K}_i)$.

To define the WRT invariant we use the representation of $\mathrm{SL}((2, \mathbf{Z})$ in the space of integrable representations of the affine Lie algebra associated to G. A representation given by a highest weight Λ is integrable if the weight $\rho + \Lambda$ is in the fundamental chamber $\mathcal{F}_l$ The fundamental chamber is given by $\Lambda_{\rm w}/l\Lambda_{\rm r}$ modded out by the action of the Weyl group. In the following, the basis of integrable representations will be labeled by the weights in $\mathcal{F}_l$. In the case of simply-laced gauge groups, the $\mathrm{Sl}(2,\mathbf{Z})$ transformation given by $U^{(p,q)}$ has the following matrix elements in the above basis [Roz96, HT04]:

$$\begin{aligned}\mathcal{U}^{(p,q)}_{\alpha\beta} &= \frac{[i\,\mathrm{sign}(q)]^{|\Delta_+|}}{(l|q|)^{r/2}}\exp\Big[-\frac{id\pi}{12}\Phi(U^{(p,q)})\Big]\bigg(\frac{\mathrm{Vol}\,\Lambda_{\mathrm{w}}}{\mathrm{Vol}\,\Lambda_{\mathrm{r}}}\bigg)^{\frac{1}{2}}\\ &\cdot \sum_{n\in\Lambda_{\mathrm{r}}/q\Lambda_{\mathrm{r}}}\sum_{w\in\mathcal{W}}\epsilon(w)\exp\Big\{\frac{i\pi}{lq}(p\alpha^2-2\alpha(ln+w(\beta))+s(ln+w(\beta))^2\Big\}.\end{aligned} \tag{2}$$

In this equation, $|\Delta_+|$ denotes the number of positive roots of G, and the second sum is over the Weyl group $\mathcal{W}$ of G. $\Phi(U^{(p,q)})$ is the Rademacher function:

$$\Phi\begin{bmatrix} p & r\\ q & s\end{bmatrix}=\frac{p+s}{q}-12s(p,q), \tag{3}$$

where $s(p,q)$ is the Dedekind sum

$$s(p,q)=\frac{1}{4q}\sum_{n=1}^{q-1}\cot\Big(\frac{\pi n}{q}\Big)\cot\Big(\frac{\pi np}{q}\Big). \tag{4}$$

In terms of the above data, the WRT invariant of M is given by:

$$Z(M,l)=\mathrm{e}^{i\phi_{\mathrm{fr}}}\sum_{\alpha_1,\cdots,\alpha_L\in\mathcal{F}_l}Z_{\alpha_1,\cdots,\alpha_L}(\mathcal{L})\,\mathcal{U}^{(p_1,q_1)}_{\alpha_1\rho}\cdots\mathcal{U}^{(p_L,q_L)}_{\alpha_L\rho}. \tag{5}$$

In this equation, $Z_{\alpha_1,\cdots,\alpha_L}(\mathcal{L})$ is the quantum group invariant of the link $\mathcal{L}$ with representation $\alpha_i-\rho$ attached to its i-th component (recall that the weights in $\mathcal{F}_l$ are of the form $\rho+\Lambda$). The phase factor $\mathrm{e}^{i\phi_{\mathrm{fr}}}$ is a framing correction that guarantees that the resulting invariant is in the canonical framing for the three-manifold M. Its explicit expression is

$$\phi_{\mathrm{fr}}=\frac{\pi kd}{12l}\bigg(\sum_{i=1}^{L}\Phi(U^{(p_i,q_i)})-3\sigma\left(\mathcal{L}\right)\bigg), \tag{6}$$

where $\sigma(\mathcal{L})$ is the signature of the linking matrix of $\mathcal{L}$. One can show that the above definition of WRT is invariant under Kirby moves, therefore it defines a topological invariant of the three-manifold M.

The WRT invariant was originally defined by Witten in quantum-field theoretic terms, as the partition function of Chern-Simons theory on the three-manifold M. The action of Chern-Simons theory is given by

$$S_{\mathrm{CS}}(A)=\frac{k}{4\pi}\int_M\mathrm{Tr}\Big(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\Big), \tag{7}$$

where A is a G-connection on M, and the WRT invariant is given by

$$Z_k(M)=\int\mathcal{D}A\mathrm{e}^{iS_{\mathrm{CS}}(A)}. \tag{8}$$

The description given above in terms of combinatorial data can be derived from Chern-Simons theory in the context of canonical quantization.

2.2. **Matrix integral representation of the WRT invariant.** The fact that the WRT invariant is the partition function of a quantum field theory suggests that it can be evaluated semiclassically as a sum over critical points of the action. In the case of the Chern-Simons functional (7), the critical points are flat connections on M. Each term in the sum is in turn an asymptotic, perturbative expansion around the flat connection in powers of the coupling constant of the model. Moreover, it can be argued that the terms in this perturbative expansion contain important topological information about the three-manifold M. For example, the one-loop contribution involves the analytic torsion of the flat connection [Wit89], while the two-loop contribution around the trivial flat connection turns out to be equal to the Casson-Walker invariant of M [Roz96].

From the point of view of combinatorial definition of the WRT invariant, the properties that emerge in the asymptotic expansion in powers of the coupling constant are far from being obvious. In the case of Seifert spaces, it was shown in [LR99] that the WRT invariant for gauge group $SU(2)$ can be written as a sum of contour integrals and residues which correspond to the contributions associated to the different flat connections. Some of the results of [LR99] were generalized in [Mar05] to general simply-laced groups, where it was shown that the contribution of reducible flat connections can be written as a matrix integral. In [BW05], Beasley and Witten have presented a very elegant derivation of this matrix integral representation in the case of the trivial flat connection, by using non-abelian localization. A similar result has been recently obtained in [BT06].

Seifert homology spheres can be constructed by performing surgery on a link $\mathcal{L}$ in $\mathbf{S}^3$ with $n+1$ components, consisting on n parallel and unlinked unknots together with a single unknot whose linking number with each of the other n unknots is one. The surgery data are p_j/q_j for the unlinked unknots, $j = 1, \cdots, n$, and 0 on the final component. p_j is coprime to q_j for all $j = 1, \cdots, n$, and the p_j's are pairwise coprime. After doing surgery, one obtains the Seifert space $M = X(\frac{p_1}{q_1}, \cdots, \frac{p_n}{q_n})$. This is rational homology sphere whose first homology group $H_1(M, \mathbf{Z})$ has order $|H|$, where

$$H = P \sum_{j=1}^{n} \frac{q_j}{p_j}, \quad \text{and} \quad P = \prod_{j=1}^{n} p_j. \tag{9}$$

We will denote $e = H/P$. For $n = 1, 2$, Seifert homology spheres reduce to lens spaces, and one has that $L(p, q) = X(q/p)$. For $n = 3$, we obtain the Brieskorn homology spheres $\Sigma(p_1, p_2, p_3)$ (in this case the manifold is independent of q_1, q_2, q_3). By using the formulae for the WRT invariant presented above, one can write the contribution of reducible flat connections to the Chern-Simons partition function of $X(\frac{p_1}{q_1}, \cdots, \frac{p_n}{q_n})$ as

$$\begin{aligned} &\frac{(-1)^{|\Delta_+|}}{|\mathcal{W}|\,(2\pi i)^r} \left(\frac{\mathrm{Vol}\,\Lambda_{\mathrm{w}}}{\mathrm{Vol}\,\Lambda_{\mathrm{r}}}\right) \frac{[\mathrm{sign}(P)]^{|\Delta_+|}}{|P|^{r/2}} \mathrm{e}^{\frac{\pi i d}{4}\mathrm{sign}(H/P) + \frac{\pi i d y}{12 l}\phi} \\ &\cdot \sum_{t \in \Lambda_{\mathrm{r}}/H\Lambda_{\mathrm{r}}} \int d\beta\, \mathrm{e}^{-\hbar e \beta^2/2 - l t\cdot\beta} \frac{\prod_{i=1}^{n}\prod_{\alpha>0} 2\sinh\frac{\beta\cdot\alpha}{2p_i}}{\prod_{\alpha>0}\left(2\sinh\frac{\beta\cdot\alpha}{2}\right)^{n-2}} \end{aligned} \tag{10}$$

In this equation, β is an element of $\Lambda_w \otimes \mathbf{R}$, ϕ is given by

$$\phi = e - 3\text{sign}\,(e) - 12\sum_{i=1}^{n} s(q_i, p_i). \tag{11}$$

and we have introduced

$$\hbar = \frac{2\pi i}{k+y}.$$

The lattice $\Lambda_r/H\Lambda_r$ decomposes in different Weyl orbits, and each of these orbits correspond to a different, reducible flat connection. The contribution of the trivial flat connection is obtained by setting $t = 0$ in (10).

3. Matrix integrals and the LMO invariant

In this section, we will show that the LMO invariant of a rational homology sphere, evaluated through the $U(N)$ weight system, can be always expressed as a matrix integral. This follows very simply by the definition of the LMO invariant in terms of formal Gaussian integration given in [BNGRT02], and the detailed structure of the $U(N)$ weight system. Since the LMO invariant is conjectured to capture the contribution of the trivial connection to the WRT invariant, our result indicates that this contribution is expected to have a representation in terms of matrix integrals, as it happens with the Seifert homology spheres. In particular, we will show that the result (10) for Seifert spheres agrees with the $U(N)$ evaluation of the LMO invariant calculated in [BNL04].

3.1. **A review of the Kontsevich integral.** The physics origin of the Kontsevich integral of a link in $\mathbf{S}^3$ is Chern-Simons perturbation theory along the trivial flat connection of the backround 3-space. The Feynmann diagrams of the theory are trivalent graphs with legs (so-called unitrivalent graphs). The legs are colored by the components of the link, and the edges along the trivalent vertices are equipped with a cyclic ordering. The graphs are considered modulo the AS and IHX relations. The graphs can be multiplied (using the disjoint union) and can be graded (where the degree of a graph is half the number of vertices). Using formal linear combinations (with coefficients in $\mathbb{Q}$) of these graphs, we can define a completed graded algebra $\mathcal{A}(\star_X)$ where X is a set in 1-1 correspondence with the components of the link.

It turns out that the Konstevich integral of a link is a group-like element of $\mathcal{A}(\star_X)$ (i.e., the exponential of a series of connected diagrams), thus we can define its logarithm

$$\mathcal{F}(S^3, L) = \log \mathcal{Z}(S^3, L)$$

which lies in the completed vector space $\mathcal{A}^c(\star_X)$ generated by *connected* unitrivalent graphs, modulo the AS and IHX relations. $\mathcal{A}^{gp}(\star_X)$ will denote the set of group-like elements.

There are two degrees of a diagram D in $\mathcal{A}(\star_X)$:

- The *Vassiliev degree* $\deg_1(D)$, which equals to half the number of vertices.
- The *Euler degree* $\deg_2(D)$ which equals to $-\chi(D)$.

Notice that the Euler degree of a connected diagram is $\text{rk}H_1(D) - 1 \geq -1$, and that $\deg_1(D) = \deg_2(D) + |\text{Legs(D)}|$.

3.2. **A review of the LMO invariant.** In this section we review the LMO integral. The physics origin of the LMO invariant (and its cousin, the Aarhus integral) is Chern-Simons perturbation theory along the trivial flat connection.

Consider a framed link L of r components in S^3, and let M denote the 3-manifold obtained by surgery on L. The Aarhus integration map:

$$\int dX : \mathcal{A}(\star_X) \longrightarrow \mathcal{A}(\emptyset)$$

takes values in the completed vector space $\mathcal{A}(\emptyset)$ of trivalent graphs with vertex orientations modulo the AS and IHX relations. If the integrand a is group-like (i.e., the exponential of a series of connected diagrams), so is the result of integration.

This allows us to define

$$\mathcal{Z}_0(L) = \int \mathcal{Z}(S^3, L)$$

as well as

$$\mathcal{Z}(M) = \frac{\mathcal{Z}_0(L)}{\mathcal{Z}_0(S^3, U^+)^{\sigma^+(L)} \mathcal{Z}_0(S^3, U^-)^{\sigma^-(L)}}$$

where $U^{\pm}$ is the unknot with framing ± 1 and $\sigma^{\pm}(L)$ is the number of positive (resp. negative) eigenvalues of the linking matrix of L. It turns out that $\mathcal{Z}(M)$ is depends only on M and not on the framed link L. Moreover, $\mathcal{Z}(M)$ is group-like, that is we can define its logarithm:

$$\mathcal{F}(M) = \log \mathcal{Z}(M)$$

which takes values in $\mathcal{A}^c(\emptyset)$.

Here is a rough description of the Aarhus integration $\int$.

- Consider an element $a \in \mathcal{A}(\star_X)$. Concentrate on a_{2m}, the piece of a that contains diagrams with exactly $2m$ legs of each color.
- Then, $\int a\, dX$ is the sum of all $((2m-1)!!)^{|X|}$ ways of pairing up the legs of each color X.
- We consider the result in $\mathcal{A}(\emptyset)$.
- If a is group-like, then we can assemble the pieces $\int a\, dX$ into a group-like element in $\mathcal{A}(\emptyset)$.

3.3. **Weight systems.** For every semisimple Lie algebra $\mathfrak{g}$, we have a *weight system* map:

$$W_{\mathfrak{g}} : \mathcal{A}^c(\star_X) \longrightarrow \hbar S(\mathfrak{g}^{\oplus |X|})^{\mathfrak{g}}[[\hbar]]$$

Specifically, if D is a unitrivalent graph with $2m$ vertices and l legs then $W_{\mathfrak{g}}(D) \in S^l(\mathfrak{g}^{\oplus |X|})^{\mathfrak{g}}\, \hbar^m$. Notice that $m = -\chi(D) + l$.

Let $W^U = W_{\mathfrak{gl}_N}$, for arbitrary N. We now describe in detail the W^U weight system. Following Bar-Natan, let us introduce the vector space spanned by marked surfaces.

Definition 3.1. An X-*marked surface* (Σ, γ) is an oriented compact topological surface Σ with nonempty boundary, together with a choice γ of points (colored by X) on $\partial\Sigma$. Let $\mathcal{M}_X$ denote the completed vector space of formal $\mathbb{Q}$-linear combinations of X-marked surfaces.

If $|X| = 1$ and Σ is connected, γ gives rise to a partition $(0^{\gamma_0}1^{\gamma_1}\dots)$, where γ_j is the number of boundary components of Σ with j points.

Like unitrivalent graphs, marked surfaces have two degrees:

- The *Vassiliev degree* $\deg_1(\Sigma, \gamma)$ of a marked surface is $-\chi(\Sigma)+|\gamma|$, where $|\gamma| = \sum_j j\gamma_j$.
- The *Euler degree* $\deg_2(\Sigma, \gamma)$ of a marked surface is $-\chi(\Sigma)$.

Thus, $|\gamma|$ in the case of a marked surface plays the role of the number of legs.

Marked surfaces can be multiplied (via the disjoint union) and graded (by the Vassiliev degree). Let $\mathcal{M}^c_X$ be defined analogously.

There is a map:

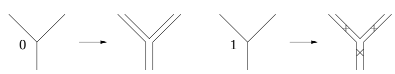
$$\Psi : \mathcal{A}(\star_X) \longrightarrow \mathcal{M}_X$$

defined by:

$$D \longrightarrow \sum_M (-1)^{s_M}(\Sigma_{D,M}, \gamma_{D,M})$$

where

- the sum is over all possible markings M of the trivalent vertices of D by 0 or 1,
- s_M is the sum, over the set of trivalent vertices, of the values of M
- $\Sigma_{D,M}$ denotes the X-marked surface obtained by thickening the trivalent vertices of D as follows:

(12) 

  and thickening the edges of D, and connected up to a surface. The legs of D become the choice $\gamma_{D,M}$ of points in $\Sigma_{D,M}$. It turns out that $\Sigma_{D,M}$ is well-defined and oriented.

The above map preserves the Vassiliev and Euler degrees $\deg_1$ and $\deg_2$.

Moreover, there is a map:

$$\Phi : \mathcal{M}^c_X \longrightarrow \Lambda^{\otimes r}[N, \hbar]$$

where Λ is the ring of symmetric polynomials. This map is defined by:

$$(\Sigma, \gamma) \longrightarrow N^{\gamma_0}(\prod_{n=1}^{\infty} p_n^{\gamma_n})\, \hbar^m \in \Lambda_l^{\otimes r}[N, \hbar]$$

where $m = \deg_1(\Sigma, \gamma)$, $l = \deg_1(\Sigma) - \deg_2(\Sigma)$, and p_n is the power sum $\sum_j x_j^n$. We remind that products of power sums provide a basis for the ring of symmetric polynomials in the variables x_j. We will also use in the following the basis of Λ given by Schur functions s_λ, which are labeled by a partition λ. An r-uple of partitions will be denoted by $\lambda = (\lambda_1, \cdots, \lambda_r)$. A basis of $\Lambda^{\otimes r}$ is therefore provided by the products $s_\lambda = s_{\lambda_1} \cdots s_{\lambda_r}$.

Remark 3.2. In fact, the map Φ is a vector space isomorphism, although we will not use this.

Proposition 3.3. [BN95, CD99] *We have a commutative diagram*

$$\begin{tikzcd}
\exp(\mathcal{A}^c(\star_X)) \arrow[rr, "\Psi"] \arrow[dr, "W^U"'] & & \exp(\mathcal{M}^c_X) \arrow[dl, "\Phi"] \\
& \exp(\mathrm{Str}^{\mathrm{p}}_r) &
\end{tikzcd}$$

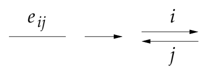
where we define

$$\mathrm{Str}^{\mathrm{p}}_r := \begin{cases} \left\{ \sum_{g=0}^{\infty} \sum_{\boldsymbol{\lambda} \neq \mathbf{0}} a^g_{\boldsymbol{\lambda}} \, s_{\boldsymbol{\lambda}} \;\middle|\; a^g_{\boldsymbol{\lambda}} \in \hbar^{2g-2+|\boldsymbol{\lambda}|} \mathbb{Q}[N, \hbar] \right\} \subset \Lambda^{\otimes r}[N, \hbar] & \text{if} \quad r > 0 \\ \frac{1}{\hbar^2} \mathbb{Q}[N, \hbar] & \text{if} \quad r = 0. \end{cases} \tag{13}$$

Proof. Let $e_{i,j}$ for $1 \leq i, j \leq N$ be a basis for $\mathfrak{gl}_N$. We have

$$\begin{aligned} \mathrm{tr}(e_{i,j} e_{k,l}) &= \delta_{i,l} \, \delta_{j,k} \\ [e_{i,j}, e_{k,l}] &= \delta_{j,k} \, e_{i,l} - \delta_{l,i} \, e_{k,j}. \end{aligned}$$

A diagram consists of a number of Y graphs some of whose half-edges are glued in pairs. The weight system colors each half-edge by an element of $\mathfrak{gl}_N$. Graphically, the above equations become:

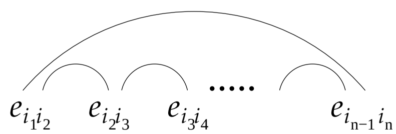


and Equation (12).

This computes the corresponding element in $S(\mathfrak{gl}_N)^{\mathfrak{gl}_N}$.

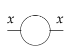
Let $E = (e_{ij})$ denote the N by N matrix with commutative entries e_{ij}. Consider the X-marked ribbon graph of genus 0:

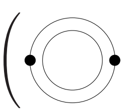
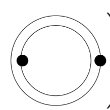
$$R_n := \sum_{i_1, \ldots, i_n = 1}^{N}$$



Then, it is easy to see that $W_{\mathfrak{gl}_N}(R_n) = \mathrm{tr}(E^n)$.

We have $S(\mathfrak{gl}_N)^{\mathfrak{gl}_N} \cong S(\mathfrak{h}_N)^{\mathrm{Sym}_N}$, where $\mathfrak{h}_N$ is the Cartan subalgebra spanned by $x_i := E_{ii}$ for $i = 1, \ldots, N$. Under this isomorphism, E maps to a diagonal matrix $\mathrm{diag}(x_1, \ldots, x_N)$, thus $\mathrm{tr}(E^n)$ gets mapped to $\sum_{i=1}^N x_i^n = p_n$. $\square$

Example 3.4. If $w_2 :=$  is the wheel with 2 legs colored by $X = \{x\}$, we have:

$w_2 \xrightarrow{\Psi} 2\Big($  $-$  $\Big) \xrightarrow{\Phi} 2(Np_2 - p_1^2)\hbar^2.$

More generally, for a wheel w_{2n} with $2n$ legs colored by $\{x\}$, we obtain

$$(\Phi \, \Psi)(w_{2n}) = \sum_{1 \leq i,j \leq N} (x_i - x_j)^{2n} = \hbar^{2n} \sum_{s=0}^{2n} (-1)^s \binom{2n}{s} p_s p_{2n-s},$$

where we set $p_0 = N$.

We will define

$$Z^{\mathrm{U}} := W^U \, \mathcal{Z} \qquad F^{\mathrm{U}} := W^U \, \mathcal{F}. \tag{14}$$

We may think of $F^{\mathrm{U}}(S^3, L)$ as the potential for a matrix model. As we will see, this potential the 3-manifold invariant, by integration on the full Lie algebra.

3.4. **Weight systems commute with LMO integration.**

Definition 3.5. Let us define

$$\int dX : \mathcal{M}_X \longrightarrow \mathcal{M}_\emptyset$$

as follows:

$$\int \prod_j p_j^{\gamma_j}\, dX = \begin{cases} \left\langle \sqcup_i(\sqcup^{\gamma_i} R_i), \prod_{x\in X} \frac{1}{m!}\left(\frac{\overset{x \quad x}{}}{2}\right)\right\rangle_X \hbar^{-m} & \text{if} \quad |\gamma| = 2m \\ 0 & \text{if} \quad |\gamma| \neq 2m \end{cases}$$

where for two ribbon surfaces Σ and Σ' with X-marked boundary, $\langle \Sigma, \Sigma' \rangle_X$ is the sum over all pairings of the X-marked ends of Σ with those of Σ' (if they match, otherwise zero).

Example 3.6. For $m = 1$, we have:

$$\begin{aligned} \int p_2\, dX &= \left\langle R_2, \frac{\overset{x \quad x}{}}{2} \right\rangle = \left\langle {}_{x}\ \ {}_{x}, \frac{\overset{x \quad x}{}}{2} \right\rangle = \quad = N^2 \\ \int p_1^2\, dX &= \left\langle R_1 \sqcup R_1, \frac{\overset{x \quad x}{}}{2} \right\rangle = \left\langle {}_{x}\ {}_{x}, \frac{\overset{x \quad x}{}}{2} \right\rangle = \quad = N \end{aligned}$$

and

$$\int w_2\, dX = 2(N^3 - N)\hbar.$$

Definition 3.5 leads naturally to a map

$$\int^U : \exp\left(\mathrm{Str}_r^{\mathrm{p}}\right) \to \left(\mathrm{Str}_0^{\mathrm{p}}\right) \tag{15}$$

such that the following diagram commutes:

$$\begin{tikzcd} \exp\left(\mathcal{A}^c(\star_X)\right) \arrow[r, "\Psi"] \arrow[d, "\int"'] & \exp\left(\mathcal{M}_X^c\right) \arrow[r, "\Phi"] \arrow[d, "\int"] & \exp\left(\mathrm{Str}_r^{\mathrm{p}}\right) \arrow[d, "\int^U"] \\ \exp\left(\mathcal{A}^c(\emptyset)\right) \arrow[r, "\Psi"] & \exp\left(\mathcal{M}_\emptyset^c\right) \arrow[r, "\Phi"] & \exp\left(\mathrm{Str}_0^{\mathrm{p}}\right) \end{tikzcd}$$

3.5. **The $U(N)$-version of the LMO invariant as a matrix model.**

The next theorem identifies the $Z^{\mathrm{U}}(M)$ invariant of a closed 3-manifold with a matrix model. Consider a framed link L in S^3, and let M denote the closed 3-manifold resulting from Dehn surgery on L. Then, the image $Z^{\mathrm{U}}(S^3, L)$ of the Kontsevich integral of L under the $\mathfrak{gl}_N$ weight system lies in $\exp(\mathrm{Str}_r^{\mathrm{p}})$, where $r = |L|$. We want to show that the integration (15) induced by Definition 3.5 agrees with Gaussian matrix model integration.

We first define Gaussian matrix model integration.

Definition 3.7. For each component L_i of L framed by ϵ_i, define the map

$$\int_{\mathcal{H}_N} : \exp\left(\mathrm{Str}_1^{\mathrm{p}}\right) \to \left(\mathrm{Str}_0^{\mathrm{p}}\right)$$

as follows:

$$\int_{\mathcal{H}_N} p_\lambda = \frac{1}{Z} \int dM \prod_j (\operatorname{tr} M^j)^{k_j} \exp\left(\epsilon_i \frac{1}{2} \operatorname{tr}(M^2)\right),$$

where $\lambda = (1^{k_1} 2^{k_2} \cdots)$, M is a Hermitian, $N \times N$ matrix, and

$$Z = \int dM \, \exp\left(\epsilon_i \frac{1}{2} \operatorname{tr}(M^2)\right).$$

The measure dM in the matrix integral is given by

$$dM = \prod_{i=1}^N dM_{ii} \prod_{1 \leq i < j \leq N} d(\operatorname{Re} M_{ij}) d(\operatorname{Im} M_{ij}).$$

The above definition can be extended to $\exp\left(\mathrm{Str}_r^{\mathrm{p}}\right)$ as follows

$$\int_{\mathcal{H}_N^r} p_{\boldsymbol{\mu}} = \int_{\mathcal{H}_N} p_{\mu_1} \times \cdots \times \int_{\mathcal{H}_N} p_{\mu_r}.$$

Theorem 3.1. *We have:*

$$\int^U = \int_{\mathcal{H}_N^r}$$

As a result, the $\mathfrak{gl}_N$ version of the LMO invariant is given by a matrix model:

$$\int_{\mathcal{H}_N^r} dX \, Z^{\mathrm{U}}(S^3, L) = Z^{\mathrm{U}}(M).$$

This theorem follows immediately from the fact that Definition 3.5 is simply the description of Gaussian integration in terms of Wick contractions, see for example [IZ90, Eqn.2.13], [DFI93, Lem.3] and [BIZ80].

Example 3.8. We have:

$$\begin{aligned}
\int_{\mathcal{H}_N} s_2 \, dX &= \frac{N(N+1)}{2} \\
\int_{\mathcal{H}_N} s_{1,1} \, dX &= -\frac{N(N-1)}{2}
\end{aligned}$$

and $p_2 = s_2 - s_{1,1}$ and $p_1^2 = s_2 + s_{1,1}$. Comparing with Example 3.6, we confirm the above claim for partitions with 2 boxes.

Example 3.9. The result of [BNL04] expresses the LMO invariant of a Seifert sphere as

$$Z = \exp\Big(\frac{\theta}{48}(e_0 - 3\mathrm{sign}(e_0) - \sum_i S(\frac{q_i}{p_i}))\Big) \int^{(m)} \Omega^{2-n}_{x/e_0^{1/2}} \prod_{\ell=1}^n \Omega_{x/(e_0^{1/2} p_\ell)} \tag{16}$$

where Ω_x is the element in $\mathcal{A}(\star_X)$ (with $X = \{x\}$) introduced in [BNGRT00] and given by

$$\Omega_x = \exp \sum_{m=1}^{\infty} b_{2m} w_{2m},$$

where

$$\sum_{m=0}^{\infty} b_{2m} x^{2m} = \frac{1}{2} \log \frac{\sinh x/2}{x/2}.$$

One easily calculates

$$(\Phi\,\Psi)(\Omega_x^{2-n} \prod_{\ell=1}^{n} \Omega_{x/p_\ell}) = P^{|\Delta_+|} \Delta^{-2}(x) \prod_{i<j} \Big(2\sinh(\frac{x_i - x_j}{2})\Big)^{2-n} \prod_{\ell=1}^{n} \prod_{i<j} \Big(2\sinh(\frac{x_i - x_j}{2p_\ell})\Big),$$

where $\Delta(x) = \prod_{i<j}(x_i - x_j)$. On the other hand, it is well known that Gaussian integration can be expressed in terms of eigenvalues as (see for example [BIZ80])

$$\int_{\mathcal{H}_N} p_\lambda = \frac{\int \prod_{i=1}^{N} dx_i e^{-x_i^2/2} \Delta^2(x) \prod_j p_j^{k_j}}{\int \prod_{i=1}^{N} dx_i e^{-x_i^2/2} \Delta^2(x)}.$$

After writing $\beta = \sum_i x_i e_i$, where e_i is an orthonormal basis in Λ_w, we find that Z^{U} is indeed given by the matrix integral (10), up to an overall factor.

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