BRAID STATISTICS IN THREE-DIMENSIONAL LOCAL QUANTUM THEORY *

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Abstract. The general theory of superselection sectors and their statistics in three-dimensional local quantum theory is outlined. It is shown that abelian and non-abelian braid statistics can arise, provided that reflections at lines in two-dimensional space (parity) are not symmetries of the theory. Braid statistics is completely described by a family of braid- and fusion matrices satisfying polynomial equations. These braid- and fusion matrices have properties very similar to those of the corresponding matrices in two-dimensional conformal field theory and determine invariants for coloured links in S^3. The role of quantum group theory in three-dimensional local quantum theory is elucidated. Excitations with braid statistics are believed to play an important role in fractional quantum Hall systems and two-dimensional high-T_c superconductors. Three-dimensional gauge theories with Chern-Simons term exhibiting such excitations are briefly described.

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1. Introduction

1.1 Some remarks on the history of fractional statistics

Since the beginnings of quantum theory the notion of identical particles and their statistics played a key role in the study of quantum systems. Implicitly, Bose-Einstein statistics already appeared in Planck's law of black body radiation and more explicitly in the study of monatomic gases carried out by Bose and Einstein. Pauli discovered his famous exclusion principle in the context of the old quantum theory. After the discovery of quantum mechanics, Heisenberg showed that the statistics of identical particles is described by the symmetry properties of n-particle wave functions under permutations of their arguments. For bosons, the complete wave functions are totally symmetric, for fermions they are totally anti-symmetric.

In local, relativistic quantum field theory, statistics was cast in the form of commutation relations between pairs of local fields at space-like separated arguments: Two Bose fields at space-like separated arguments commute, while Fermi fields anti-commute. The principles of local, relativistic quantum field theory led to the discovery of another basic fact: <u>Bosons</u> have <u>integral spin</u>, while <u>fermions</u> have <u>half-integral spin</u>. This connection between spin and statistics, though previously known in examples, was first shown to be a general feature of local, relativistic quantum field theory by Fierz, in 1939, in the context of free-field theory [1]. It was later shown to be a general consequence of the basic principles of local quantum theory [2].

After the advent of the quark model in strong interaction physics [3], the statistics problem became important again: For hadrons to have the observed spin, quarks had to have half-integral spin. Assuming that the spin-statistics connection is correct for unobservable particles (quarks), one concluded that quarks had to be fermions. This, however, led to difficulties in constructing phenomenologically viable quark wave functions for hadrons, in particular for the proton. A possible way out of these difficulties appeared to be to view quarks as particles with <u>parastatistics</u>: The symmetry properties of n-particle wave functions would then be described by higher-dimensional representations of the permutation group, S_n, of n elements. However, ultimately the statistics problem in the quark model of hadrons turned out to be one motivation (among several ones) for the introduction of colour and was neatly resolved by it: By introducing additional internal degrees of freedom the apparent parastatistics of quarks could be reinterpreted as ordinary Fermi statistics, and the standard connection between spin and statistics was saved. [The possibility of converting parastatistics into ordinary Bose- or Fermi statistics through the introduction of additional internal degrees of freedom had apparently been suggested by Fierz, Glaser and others.]

A deep analysis of statistics based on fundamental postulates of local quantum theory was carried out by Doplicher, Haag and Roberts [4], at the beginning of the seventies. They classified all possible statistics (para-Bose and para-Fermi statistics of order $d = 1, 2, 3, ...$) compatible with locality and certain general assumptions on the nature of physical states, for theories in four or more dimensions. The starting point of their analysis was reconsidered and given a better and more general foundation that includes gauge theories by Buchholz and Fredenhagen [5]. In an awsome effort, Doplicher and Roberts finally succeeded in proving that the parastatistics of "charged" particles in local, relativistic quantum theory could always be reinterpreted as ordinary Bose- or Fermi statistics by introducing additonal, internal degrees of freedom on which a global, compact internal symmetry group acts [6].

It has been known for some time that in quantum theory in two and three space-time dimensions the statistics of particles and fields is not in general described by representations of the permutation groups. In two-dimensional space-time, the concept of particle statistics becomes meaningless. Particle positions form an ordered set in a one-dimensional space, and hence the symmetry properties of wave functions under exchanging particles are not physically relevant. [For example, the Pauli principle can be understood as a consequence of local hard-core interactions in a system of bosons.] However, the statistics of fields in local quantum theory in two-dimensional Minkowski space is an interesting concept, at least for a certain class of theories. In the early seventies, the quantum theory of solitons in simple models of two-dimensional, relativistic quantum field theory became topical among theorists. The commutation relations of what people nowadays call vertex operators of the massless free field in two space-time dimensions were studied by Streater and Wilde [7] and found to be different, in general, from local commutators or anti-commutators.

More subtle examples of "exotic" commutation relations between soliton fields and meson fields in two-dimensional, interacting scalar theories with soliton sectors ($\lambda\varphi_2^4$ - and, more generally, $P(\varphi)_2$-models exhibiting quantum kinks) were studied by one of us in [8]. It was recognized there that the key facts behind the appearance of soliton sectors and exotic field statistics in two space-time dimensional theories are vacuum degeneracy and the property of two-dimensional Minkowski space that the causal complement of a bounded double cone consists of two disconnected wedges. It was recognized only fairly recently by several people that the new field statistics encountered in two-dimensional models with quantum kinks [8] may lead to representations of the braid groups. Earlier, one of us (J.F.) noted that, in the Euclidean description of two-dimensional quantum field theory, exotic statistics manifests itself as non-trivial monodromy of Euclidean Green functions. This was a model-independent interpretation of the monodromy properties found in the study of order-disorder correlation functions of the two-dimensional Ising model by Kadanoff and Ceva [9] and exploited in the work of Jimbo, Miwa and Sato [10]. For a brief sketch of such results see [11], and [12] for interesting related results.

While, during the late seventies, there was little interest in two-dimensional models, the situation changed after a resurge of interest in string theory and the appearance of the fundamental paper of Belavin, Polyakov and Zamolodchikov on two-dimensional conformal field theory [13], in 1984. Soon, it was recognized that the chiral fields of two-dimensional conformal field theories provide interesting examples of exotic statistics, or "exchange algebras" [14]. This was conceptualized in [15]. In this paper, a general theory of exotic statistics in two-dimensional theories, not limited to conformal theories, was sketched, and it was suggested that the proper framework for a more rigorous analysis was algebraic field theory [16,4], in combination with the theory of Yang-Baxter representations of the braid groups [17]. Subsequently, this point of view was developed in [18]. It should be emphasized, however, that there are examples of field statistics in two-dimensional theories <u>not</u> covered by the framework of [18]. Similar themes in the context of conformal field theory were studied in [19].

In these notes, we focus on the study of field- and particle statistics in <u>three-dimensional local quantum theory</u> which is of considerable interest in condensed matter theory. It appears to be a rather old observation that, for quantum-mechanical systems in two-dimensional space, particle statistics is described by representations of the braid groups, [20]. In fact, the right story to tell about statistics in quantum mechanics is that it is described by unitary representations of the fundamental group of the classical configuration space of identical particles on the Hilbert space of quantum mechanical wave functions. In two space dimensions the classical n-particle configuration space is

$$M_n = [(\mathsf{E}^2)^{\times n} \backslash D_n]/S_n, \tag{1.1}$$

where

$$D_n = \{(\underline{x}_1, \cdots, \underline{x}_n) \in (\mathsf{E}^2)^{\times n} : \underline{x}_i = \underline{x}_j \quad ; \quad \text{for some } i \neq j\},$$

and S_n is the permutation group of n elements. It is easily seen that the fundamental group of M_n is given by

$$\pi_1(M_n) = B_n,$$

where B_n is the braid group on n strands [21,17].

Quantum mechanical systems in two-dimensional space with abelian braid statistics were first proposed and studied by Leinaas and Myrheim [22]. Consider an array of identical point particles carrying an electric charge q and a magnetic flux (vorticity) ϕ. Such particles have been termed "anyons" by Wilczek [23]. Naturally, such a system exhibits an Aharonov-Bohm effect: If one particle moves around a positively oriented loop enclosing k particles the total wave function picks up a phase factor $exp(2ikq\phi)$. If two particles are exchanged along paths whose composition forms a positively oriented loop enclosing k particles the wave function gets multiplied by a factor $exp(iq\phi + 2ikq\phi)$.

Elements of the braid group B_n label homotopy classes of loops in M_n of which the two loops just described are examples. The braid group B_n has generators $\tau_i, i = 1, \cdots, n-1$; the generator $\tau_i^{\pm 1}$ describes the exchange of particle i with particle $i+1$ (in some order chosen on the points of E^2) along paths whose composition forms a $\binom{positively}{negatively}$ oriented loop in the plane not enclosing any other particle. An element $b \in B_n$ can be written as a word in the generators $\{\tau_1, \ldots, \tau_{n-1}\}$ modulo the relations

$$\begin{aligned} \tau_i\,\tau_{i+1}\,\tau_i &= \tau_{i+1}\,\tau_i\,\tau_{i+1}, \\ \tau_i\,\tau_j &= \tau_j\,\tau_i, \quad \text{for } \mid i-j \mid \geq 2\,. \end{aligned} \tag{1.2}$$

The unitary representation of B_n carried by the space of n-anyon wave functions is defined by assigning a phase factor $exp(\pm i\, q\, \phi)$ to $\tau_j^{\pm 1}$, for all $j = 1, \ldots, n-1$. This is a one-dimensional (abelian) representation of B_n. We shall see that there are, in principle, other non-abelian representations of B_n that could describe the statistics of <u>non-abelian anyons</u>. Let ρ be a representation of B_n. If

$$\rho(\tau_i^2) \;=\; \rho(\tau_i)^2 \;= \mathbb{1}$$

then ρ factors through a representation of S_n, and the corresponding particles have ordinary permutation group statistics. Thus if

$$q\phi/2\pi \;\in\; \frac{1}{2}\mathbb{Z} \tag{1.3}$$

anyons are bosons or fermions. More precisely, if $q\phi/2\pi$ is an integer anyons are bosons, while if $q\phi/2\pi$ is a half-integer anyons are fermions. For other values of $\theta \equiv q\phi/2\pi$, anyons have what has been called fractional-, or intermediate-, or θ-statistics. We shall speak of <u>braid (group) statistics</u>, as opposed to <u>permutation (group) statistics</u>.

It should be emphasized that it is not necesary to think of anyons as particles carrying electric charge and vorticity. By a gauge transformation the vector potential can be gauged away. But then if $\theta \notin \frac{1}{2}\mathbb{Z}$ the Hilbert space of anyon wave functions must be chosen to be a space of multi-valued functions with half-monodromies given by the phase factors $exp(\pm 2\pi i\theta)$. Such wave functions can be viewed as single-valued functions on the universal cover, $\widetilde{M}_n$, of M_n. This description of n-anyon systems is more natural if there are no electrostatic interactions between anyons, i.e. if the Coulomb interactions are absent or nearly completely screened.

In this picture, an n-anyon wave function will have the form

$$\psi(\underline{x}_1, \ldots, \underline{x}_n) \;=\; \prod_{i<j}(z_i - z_j)^{2\theta}\; g(\underline{x}_1, \ldots, \underline{x}_n), \tag{1.4}$$

where $g(\underline{x}_1, \ldots, \underline{x}_n)$ is a single-valued, symmetric function on $(\mathrm{E}^2)^{\times n}$, and z_j is the complex number $(x_j^1 + i\, x_j^2)$ corresponding to $\underline{x}_j$. An example is a Laughlin-type wave function [25]

$$\psi(\underline{x}_1, \ldots, \underline{x}_n) \;=\; const.\; \prod_{i<j}(z_i - z_j)^{2\theta}\; \prod_{k=1}^{n}\; exp(-\mid z_k \mid^2 /4)\,. \tag{1.5}$$

To conclude this section, it might be mentioned that a somewhat systematic analysis of gauge theories with Chern-Simons term (and related 0(3) non-linear σ-models with a Hopf term) in three space-time dimensions describing particles with braid statistics was initiated in [23,26-28]. It was noted in [15,28] that the braid statistics of anyons is closely related to the 't Hooft commutation relations between Wilson loops and vortex creation operators. Some general comments on three-dimensional theories are also contained in [18].

1.2 Physical realizations of braid statistics

One should ask why, as physicists, we should care about anyons and braid statistics? Is there experimental evidence for excitations with braid statistics in two-dimensional systems of condensed matter physics? By now, the standard answer is that excitations with braid statistics appear to play an important role in systems exhibiting a fractional quantum Hall effect. Moreover, pure anyon gases, with $\theta \in \mathbb{Q}\backslash\frac{1}{2}\mathbb{Z}$, appear to be superconductors of a new type [29]. It has been speculated that anyon superconductivity may describe (essentially two-dimensional) high-T_c superconductors. This would be fairly plausible if one could exhibit doped anti-ferromagnets that admit a flux phase breaking parity and time reversal invariance.

Let us briefly recall Laughlin's argument explaining the role of anyons in the fractional quantum Hall effect: An idealized experimental set-up is sketched in Fig.1

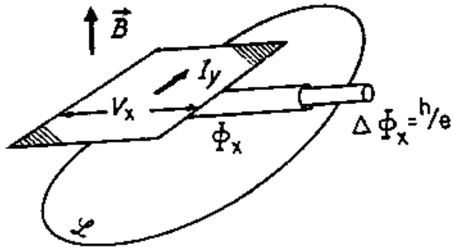


Fig. 1

The transverse conductivity is

$$\sigma_{xy} \;=\; {}^{I_y}/\,V_x \tag{1.6}$$

Experimentally, one finds that in films of $Ga\,Al_x\,As_{1-x}$ at low temperatures and in strong magnetic fields, σ_{xy}, plotted as a function of the electron density ρ, has plateaux at the values

$$\nu\, e^2/h\,, \text{ with } \nu \;=\; \begin{cases} 1,2,3,\ldots \\ \frac{1}{5},\frac{2}{7},\frac{1}{3},\frac{2}{5},\frac{2}{3},\ldots\,, \end{cases} \tag{1.7}$$

more generally, for $\nu = p/r$, where p and r do not have a common divisor, and r is odd. [More recently, plateaux appear to have been observed for even values of r, too.] Fractional values of ν could be understood as an effect of fractionally charged excitations in such a system: Suppose there are charge carriers of charge $q = f\,e$, $f \in \mathbb{Q}$, in the conducting rectangle depicted in Fig. 1. Imagine that the total magnetic flux through the loop $\mathcal{L}$ is increased adiabatically by one quantum of flux from ϕ_x to $\phi_x + \triangle\phi_x$, $(\triangle\phi_x = h/e)$. The microscopic quantum-mechanical Hamiltonians $H(\phi_x)$ and $H(\phi_x + \triangle\phi_x)$ of the system are then gauge-equivalent, hence have the same spectrum. If the Fermi energy of the system is in a mobility gap, the adiabatic process described above will therefore not change the total energy of the system. Suppose that during that process n charge carriers of charge fe move from one to another edge in the x-direction. This changes the total energy by an amount

$$\triangle\, U \;=\; n\, f\, e\, V_x. \tag{1.8}$$

By Faraday's induction law the work of the current in the y-direction, during the same adiabatic process, is given by

$$\triangle\, W \;=\; -\int_{-\infty}^{\infty} dt\; I_y \frac{d\phi_x}{dt} \;=\; -I_y\,\frac{h}{e}. \tag{1.9}$$

Since the total energy remains unchanged,

$$\triangle\, U \,+\, \triangle\, W \;=\; 0.$$

By (1.6), (1.8) and (1.9),

$$\sigma_{xy} \;=\; n\, f\;\, e^2/h. \tag{1.10}$$

Comparison with experimental data, (1.7), shows that if the naïve argument just sketched is correct then f must be fractional, for fractional values of ν. For strongly correlated many-electron systems, one might imagine mechanisms giving rise to "solitons" which correspond to a fraction, $1/r$, of an electron. [Solitons of this kind are well known in one-dimensional systems such as polyacethylene.] Solitons corresponding to a fraction of an electron carry fractional charge and fractional spin. Excitations with fractional spin in two dimensional systems necessarily obey fractional statistics, as we shall see in Sect. 3. Phenomenological wave functions for an assembly of n such excitations have the form (1.4), with $\theta = \frac{1}{2r}$, [25].

We believe that a detailed understanding of how anyons emerge in two-dimensional, highly correlated many-body systems is still missing.

1.3 Three-dimensional gauge theories with braid statistics

Next, we give a mini-review of three-dimensional gauge theories with Chern-Simons term describing particles with braid statistics. The simplest examples are abelian gauge theories. They have an action given by

$$\begin{aligned} S[A] \; = \; \frac{1}{2} \int_{M^3} [e^{-2} F^2 + \frac{\theta}{2\pi} \, \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - 2 j^\mu \, A_\mu \\ + \; (\text{pure matter terms})] \; . \qquad (1.11) \end{aligned}$$

The term proportional to θ is the Chern-Simons term, breaking parity, j^μ is the matter current. By varying the action with respect to A we find the modified Maxwell equations. In particular, one has that

$$\frac{1}{e^2} \, div \; \vec{E} \; = \; j^0 \; - \; \frac{\theta}{2\pi} \, B. \qquad (1.12)$$

This equation shows that if the electric field is screened vortices carry an electric charge $\frac{\theta}{2\pi} \, \phi$, ($\phi \equiv \int d^2x \; B(\vec{x}, t)$ is their vorticity), and particles with an electric charge q carry vorticity q/θ . The abelian braid statistics of charged particles and vortices in such theories can be understood as a consequence of the Aharonov-Bohm effect. This can be substantiated in models by means of non-perturbative calculations with functional integrals [26], (or in an operator formalism, using 't Hooft commutation relations [28]). Abelian gauge theories with Chern-Simons term have been studied in detail in [30], and their particle spectrum and statistics are analyzed in [23, 26-28, 31].

We now ask whether non-abelian gauge theories with Chern-Simons term in three dimensions might describe particles with non-abelian braid statistics? Here we only sketch a preliminary answer to this question.

Consider a simply connected, compact gauge group $G(\overset{e.g.}{=} SU(N))$. Let A be a vector potential (connection) with values in $Lie(G)$. Following Witten [32], we consider the pure Chern-Simons theory with action

$$S_{C.S.}[A] = \frac{k}{4\pi} \int_{M^3} tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{1.13}$$

This theory can be quantized using geometrical quantization [32] or functional integrals [33]. Gauge-invariance and unitarity constrain the coupling constant of the theory to be quantized,

$$k_{eff.} = n + h_G, \quad n = 1, 2, 3, \ldots, \tag{1.14}$$

where h_G is the dual Coxeter number of G; (e.g. $h_G = N$, for $G = SU(N)$).

Static colour sources in this theory have non-abelian braid statistics described by Yang-Baxter matrices that are identical to the braid matrices of the Wess-Zumino-Witten models corresponding to the group G, at level n. These braid matrices can be understood as holonomy matrices of the Knizhnik-Zamolodchikov connection. These results follow from [32-34]. Although pure Chern-Simons theory has interesting applications to pure mathematics [32], it is uninteresting for physics. It is a purely topological theory, and hence its Hamilton opertor vanishes on physical states. The physical state spaces are finite-dimensional.

An idea for constructing non-topological gauge theories with non-abelian braid statistics is to add non-topological terms to $S_{C.S.}$. Consider a theory with Euclidean action

$$\begin{aligned} S[A] \overset{e.g.}{=} g^{-2} \int tr(F^2) - \frac{ik}{4\pi} \int tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \\ + \lambda \int \bar{\psi}(\not{D}_A + m)\psi + \ldots, \end{aligned} \tag{1.15}$$

where g, λ and m are positive constants, and ψ is a two-component spinor field in the fundamental representation of G; (there may be further matter fields, e.g. Higgs fields). The conjectured properties of this theory are as follows.

1) The first two terms in $S[A]$ on the r.h.s. of (1.15) make the gluon massive [30]. Hence all interactions mediated by gluons are expected to be of short range. In particular, one expects that there is <u>no confinement of colour</u> in this theory.

2) Long-range interactions in this theory are purely topological, just like in pure Chern-Simons theory. The effective action at very large distance scales is essentially a pure Chern-Simons action with a renormalized value of k, the renormalization depending only on the number and nature of matter fields [30].

3) Since particle statistics can be determined at arbitrarily large distance scales, one expects, on the basis of 1) and 2), that the statistics of coloured particles in this theory is the same as the statistics of static colour sources in pure Chern-Simons theory, for a renormalized value of k, which we have described above.

4) Coloured one-particle states in this theory are expected to be created by applying Mandelstam string operators to the physical vacuum, Ω. The Mandelstam string operators, denoted by $\psi(\mathcal{C}_x)$, are smeared out versions of

$$\psi(y)\ P(exp \int_{\gamma_y} A_\mu(\xi)\ d\xi^\mu), \tag{1.16}$$

where γ_y is a space-like path starting at $y \in \mathsf{M}^3$ and reaching out to space-like ∞. By averaging the operators (1.16) over Poincaré transformations in the vicinity of $(\mathbb{1}, 0)$, we obtain operators, $\psi(\mathcal{C}_x)$, localized in some space-like cone $\mathcal{C}_x$ with apex at some point $x \in \mathsf{M}^3$ (and with arbitrarily small opening angle) which are densely defined, closed operators on the physical Hilbert space. [By polar decomposition, these operators could then be replaced by colour-carrying, bounded operators; see also Sect. 2.]

A more careful definition of the operators $\psi(\mathcal{C}_x)$ reveals that they are multi-valued. They are elements of fibres of a certain vector bundle of operators over the circle of asymptotic directions in two-dimensional space. In order to construct a section of operators in this bundle, we choose a "reference cone", $\mathring{\mathcal{C}}$, (corresponding to a boundary condition at ∞), as sketched in Fig. 2.

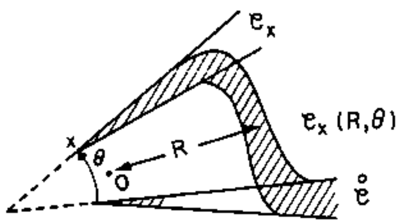


Fig. 2

The angle θ through which $\overset{\circ}{C}$ is rotated to have the direction of C_x is called the asymptotic direction, as (C_x), of C_x. We define the operator $\psi(C_x, \theta)$ as a limit

$$\psi(C_x, \theta) \;=\; \lim_{R \to \infty} \psi(C_x(R, \theta)). \tag{1.17}$$

The space-like cone C_x can also be reached by rotating $\overset{\circ}{C}$ through an angle $\theta - 2\pi$; see Fig. 3.

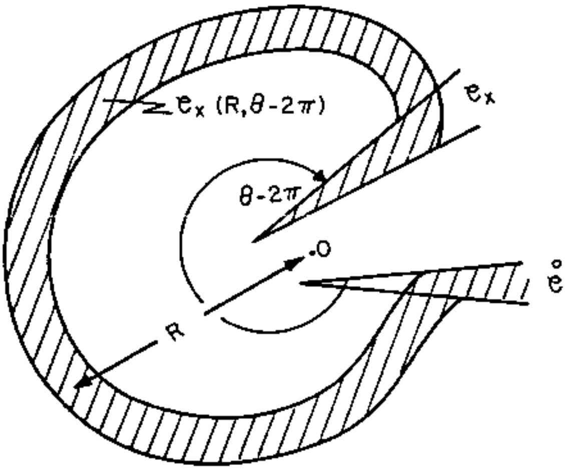


Fig. 3

We define

$$\psi(C_x, \theta - 2\pi) \;=\; \lim_{R \to \infty} \psi(C_x(R, \theta - 2\pi)). \tag{1.18}$$

The point is that $\psi(C_x, \theta)$ and $\psi(C_x, \theta - 2\pi)$ are distinct:

$$\psi(C_x, \theta - 2\pi) \;=\; V_{-2\pi}\, \psi(C_x, \theta), \tag{1.19}$$

where the operator $V_{-2\pi}$ commutes with all local observables of the theory and can be expressed in terms of the fractional spins of coloured particles.

The operators $\{\psi(C_x, \theta)\}$ are expected to obey (non-abelian) <u>braid statistics</u>: If C_{x_1} and C_{x_2} are space-like separated then

$$\psi(C_{x_1}, \theta)\, \psi(C_{x_2}, \varphi) \;=\; R^{\pm}\, \psi(C_{x_2}, \varphi)\, \psi(C_{x_1}, \theta), \tag{1.20}$$

for $\theta \gtrless \varphi$. The operators R^+ and R^- are unitary operators commuting with all local observables of the theory. Braid statistics (as opposed to permutation statistics) arises if $R^+ \neq R^-$ - which is expected for the gauge theories discussed here, unless the Chern-Simons term in the effective (large-scale) gauge field action vanishes.

One may wonder whether the gauge theories discussed here are of purely academic interest? It is conceivable, though far from established, that such theories arise as large-scale effective theories in highly correlated two-dimensional quantum many-body systems. It is a challenge to find physically plausible model systems of this type.

From now on, we outline a general analysis of braid statistics in three-dimensional local quantum theory which is mathematically rigorous [28]. It lends support to the idea that systems with infinitely many degrees of freedom in two space dimensions, with broken parity (and time reversal invariance), will generically exhibit excitations with braid statistics.

2. The algebraic formulation of local quantum theory

In the algebraic formulation of local, relativistic quantum theory [16,4], the basic object is an algebra of local observables. Physical properties of a system are extracted from the representation theory of that algebra.

The construction of algebras of local observables might proceed as follows: We imagine that we are given a local, relativistic quantum field theory, in the sense of Wightman [2], in the vacuum representation. The vacuum Hilbert space is denoted by $\mathcal{H}_1$. It contains a Poincaré-invariant vector Ω, the physical vacuum. Of particular interest for our purposes are gauge theories. The gauge-invariant, local observables of such a theory are Wilson loop operators $W(\mathcal{L})$, where $\mathcal{L}$ is a space-like loop in $\mathbb{M}^3$, Mandelstam string operators, denoted $\psi(\gamma_{xy})$, where γ_{xy} is a space-like curve starting at x and ending at y, and local, gauge-invariant currents, $J^{\alpha}_{\mu_1,\ldots,\mu_k}(\vec{x},t)$, where $\mu_1,\ldots,\mu_k$ are Lorentz indices, and α labels different currents with the same tensorial properties under Lorentz transformations. In order to obtain densely defined, closed operators on $\mathcal{H}_1$, the distributional fields introduced above must be smeared out with test functions. Let $\mathcal{O}$ be a bounded open region in $\mathbb{M}^3$, e.g. a double cone, and let f be a test function with $supp\, f \subset \mathcal{O}$. If $J^{\alpha}_{\mu_1,\ldots,\mu_k}$ is a real current, and f is a real test function one expects that the operator

$$J^{\alpha}_{\mu_1,\dots,\mu_k}(f) \;=\; \int d^3x \; J_{\mu_1,\dots,\mu_k}(\vec{x},t)\, f(\vec{x},t) \tag{2.1}$$

is selfadjoint on the vacuum sector $\mathcal{H}_1$. Similarly, by averaging Wilson loops and Mandelstam operators over (finite-dimensional) families of loops or curves, respectively, contained in $\mathcal{O}$, we can hope to construct further selfadjoint operators on $\mathcal{H}_1$. All these operators have the common feature that they are localized in the space-time region $\mathcal{O}$.

We now define the local algebra $\mathcal{A}(\mathcal{O})$ to be the von Neumann algebra generated by all bounded functions of all the gauge-invariant, selfadjoint operators localized in $\mathcal{O}$ introduced above. The algebra $\mathcal{A}(\mathcal{O})$ is closed in the weak operator topology determined by the scalar product on $\mathcal{H}_1$.

If $\mathcal{S}$ is an unbounded space-time region in M^3 we define the algebra $\mathcal{A}(\mathcal{S})$ by setting

$$\mathcal{A}(\mathcal{S}) \;=\; \overline{\bigcup_{\substack{\mathcal{O}\subset\mathcal{S} \\ \mathcal{O}\ \text{bounded}}} \mathcal{A}(\mathcal{O})}^{\,n}\,, \tag{2.2}$$

where the closure is taken in the operator norm. In particular, the "algebra of all (quasi-) local observables", $\mathcal{A}$, is defined to be

$$\mathcal{A} \;=\; \mathcal{A}(\mathcal{S} \;=\; \mathsf{M}^3)\,. \tag{2.3}$$

The algebras $\mathcal{A}(\mathcal{S}), \mathcal{A}$ are C^*-algebras. We define the "relative commutant", $\mathcal{A}^c(\mathcal{S})$, of $\mathcal{A}(\mathcal{S})$ in $\mathcal{A}$ by

$$\mathcal{A}^c(\mathcal{S}) \;=\; \{A \in \mathcal{A} : \, [A,B] \;=\; 0,\ \forall B \in \mathcal{A}(\mathcal{S})\}\,. \tag{2.4}$$

Let $\mathcal{C}_0$ be a wedge in two-dimensional space. The causal completion, $\mathcal{C}$, of $\mathcal{C}_0$ is defined as follows. Let $\mathcal{C}_0'$ (the causal complement of $\mathcal{C}_0$) be given by

$$\mathcal{C}_0' \;=\; \{x \in \mathsf{M}^3 : \, (x-y)^2 \;<\; 0,\ \forall y \in \mathcal{C}_0\}\,. \tag{2.5}$$

Then one sets

$$\mathcal{C} \;=\; (\mathcal{C}_0')'\,. \tag{2.6}$$

The causal completion of a wedge $\mathcal{C}_0$ is called a <u>simple domain</u>. If the opening angle of $\mathcal{C}_0$ is smaller than π $\mathcal{C}$ is called a <u>space-like cone</u>.

Let U_1 be the unitary representation of the quantum mechanical Poincaré group $\tilde{\mathcal{P}}_+^{\uparrow}$ on the vacuum sector $\mathcal{H}_1$. We define a representation, α, of $\mathcal{P}_+^{\uparrow}$ on $\mathcal{A}$ by setting

$$\alpha_{(\Lambda,a)}(A) \;=\; U_1[\Lambda,a]\; A\; U_1[\Lambda,a]^{-1}\,. \tag{2.7}$$

This is a representation of $\mathcal{P}_+^\uparrow$ as a group of *automorphisms on $\mathcal{A}$.

Next, we recall some basic properties of the net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\,\subset\,\mathsf{M}^3}$ of local observable algebras and the representation α of $\mathcal{P}_+^\uparrow$ on $\mathcal{A}$ which are believed to be true in every "reasonable" local, relativistic QFT.

(1) Locality: For all $A \in \mathcal{A}(\mathcal{S})$ and all $B \in \mathcal{A}(\mathcal{S}')$,

$$[A,B] \;=\; 0\,, \tag{2.8}$$

i.e. $\mathcal{A}(\mathcal{S}') \subseteq \mathcal{A}^c(\mathcal{S})$; ($\mathcal{S}'$ is the causal complement of $\mathcal{S}$, see (2.5)).

(2) For $\mathcal{S}_1 \subseteq \mathcal{S}_2$, $\mathcal{A}(\mathcal{S}_1) \subseteq \mathcal{A}(\mathcal{S}_2)$, and, for arbitrary $\mathcal{S}_1$ and $\mathcal{S}_2$,

$$\mathcal{A}(\mathcal{S}_1\; \cup\; \mathcal{S}_2) \;\supseteq\; \mathcal{A}(\mathcal{S}_1)\; \vee\; \mathcal{A}(\mathcal{S}_2)\,. \tag{2.9}$$

(3) Duality [4,5]: Let $\mathcal{B}$ be an algebra of bounded operators on $\mathcal{H}_1$. By $\mathcal{B}'$ we denote the algebra of all bounded operators on $\mathcal{H}_1$ commuting with all operators in $\mathcal{B}$, (the "commutant of $\mathcal{B}$"). It is reasonable to expect that

$$\mathcal{A}^c(\mathcal{S})' \;=\; \overline{\mathcal{A}(\mathcal{S})}^{\,w}\,, \tag{2.10}$$

where $\overline{(\cdot)}^{\,w}$ denotes closure in the weak operator topology. See [5] for more discussion of (2.10).

(4) Poincaré covariance: Let

$$\mathcal{O}_{(\Lambda,a)} \;=\; \{x \in \mathsf{M}^3:\; \Lambda^{-1}(x-a) \in \mathcal{O}\},\;\; (\Lambda,a) \in \mathcal{P}_+^\uparrow\}\,.$$

Then

$$\alpha_{(\Lambda,a)}\left(\mathcal{A}(\mathcal{O})\right) \;=\; \mathcal{A}(\mathcal{O}_{(\Lambda,a)})\,. \tag{2.11}$$

The basic objects in the algebraic approach to local quantum theory are

$$\left\{\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\,\subset\,\mathsf{M}^3},\;\alpha\right\} \tag{2.12}$$

satisfying properties (1)-(4), above. For a more precise description of this structure see [4-6, 28].

The physics of a system described by a given pair $\{\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\subset \mathbf{M}^3},\ \alpha\}$ can be inferred from the representation theory of $\{\mathcal{A}, \alpha\}$.

Definition 2.1 A *representation, j, of $\mathcal{A}$ on a separable Hilbert space $\mathcal{H}_j$ is called a covariant positive-energy representation iff there exists a unitary representation, U_j, of $\tilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}_j$ such that

- $j(\alpha_{(\Lambda,a)}(A)) \;=\; U_j[\Lambda,a]\, j(A)\, U_j[\Lambda,a]^{-1}$, for all $A \in \mathcal{A}$ and all $(\Lambda,a) \in \mathcal{P}_+^\uparrow$; ($[\Lambda,a]$ denotes an element in $\tilde{\mathcal{P}}_+^\uparrow$ projecting onto (Λ,a)).
- $U_j(\mathbb{1},a) \;=\; e^{i a\cdot P_j}$, with $spec(P_j) \subseteq \bar{V}_+$, where $\bar{V}_+$ denotes the closure of the forward light cone. This is the relativistic spectrum condition.

The superselection sectors of a system described by $\{\mathcal{A},\alpha\}$ are the representation spaces, $\mathcal{H}_j$, corresponding to irreducible covariant positive-energy representations, j, of $\mathcal{A}$.

The physical state space of the theory, $\mathcal{H}$, is defined to be

$$\mathcal{H} \;=\; \bigoplus_j \mathcal{H}_j \tag{2.13}$$

where the direct sum extends over all inequivalent, irreducible covariant positive-energy representations of $\mathcal{A}$. It carries a unitary representation

$$U \;=\; \bigoplus_j U_j \tag{2.14}$$

of $\tilde{\mathcal{P}}_+^\uparrow$ satisfying the relativistic spectrum condition.

Buchholz and Fredenhagen have analyzed the covariant positive-energy representations of $\mathcal{A}$ for systems which admit a complete particle interpretation and without zero-mass particles, [5]. For simplicity, we may suppose that there is only one vacuum sector, $\mathcal{H}_1$, containing a unique vacuum Ω. Then the results in [5] show that (under suitably precise hypotheses on the particle structure of the theory) a covariant positive-energy representation, j, of $\mathcal{A}$ has the property that, for an arbitrary space-like cone $C \subset \mathbf{M}^3$,

$$j\,|_{\mathcal{A}^c(C)} \;\simeq\; 1\,|_{\mathcal{A}^c(C)}\ , \tag{2.15}$$

where $\simeq$ denotes "unitarily equivalent". This implies that there exists a unitary operator, V_C, from $\mathcal{H}_j$ to the vacuum sector $\mathcal{H}_1$ such that

$$j(A) \;=\; V_C^*\, A\, V_C\ , \tag{2.16}$$

for all $A \in \mathcal{A}^c(C)$; (we are identifying the abstract element $A \in \mathcal{A}$ with the operator $1(A)$ on $\mathcal{H}_1$). The proof of (2.16) involves using the Reeh-Schlieder theorem [2].

We now define a representation, $\rho_C^j \equiv \rho_C$, of $\mathcal{A}$ on $\mathcal{H}_1$ by setting

$$\rho_C(A) \;=\; V_C\, j(A)\, V_C^* \,. \tag{2.17}$$

For $A \in \mathcal{A}$, $\rho_C(A)$ is a bounded operator on $\mathcal{H}_1$, and by (2.16),

$$\rho_C(A) \;=\; A, \;\text{ for all } A \in \mathcal{A}^c(C)\,. \tag{2.18}$$

Let $\mathcal{C}_a$ be some auxiliary space-like cone of arbitrarily small opening angle, and let

$$\mathcal{C}_a + x \;=\; \left\{ y \in \mathsf{M}^3 \;:\; y - x \in \mathcal{C}_a \right\}.$$

We define an enlarged C^*algebra, $\mathcal{B}^{\mathcal{C}_a}$, containing $\mathcal{A}$, by setting

$$\mathcal{B}^{\mathcal{C}_a} \;=\; \overline{\bigcup_{x \in \mathsf{M}^3} \overline{\mathcal{A}^c(\mathcal{C}_a + x)}^{w}}^{n}\,. \tag{2.19}$$

It has been shown in [5] that ρ_C has a continuous extension to $\mathcal{B}^{\mathcal{C}_a}$, and if C is space-like separated from $\mathcal{C}_a + x$, for some x, then ρ_C is a *morphism on $\mathcal{B}^{\mathcal{C}_a}$, i.e., ρ_C is a linear map from $\mathcal{B}^{\mathcal{C}_a}$ into $\mathcal{B}^{\mathcal{C}_a}$ such that

$$\rho_C(A \cdot B) \;=\; \rho_C(A) \cdot \rho_C(B) \;\text{ and }\; \rho_C(A^*) \;=\; \rho_C(A)^*\,. \tag{2.20}$$

We may often keep the choice of $\mathcal{C}_a$ fixed and then write $\mathcal{B}$ for $\mathcal{B}^{\mathcal{C}_a}$.

Next, we introduce the notion of "charge-transport operators" [4,5] which plays quite a basic role: Consider two *morphisms, ρ_{C_1} and ρ_{C_2}, of $\mathcal{B}$ equivalent to a given covariant positive-energy representation, j, of $\mathcal{B}$. Then there exists a unitary operator $\Gamma_{\rho_{C_1},\rho_{C_2}}$, a "charge-transport operator", such that

$$\rho_{C_1}(A) \;=\; \Gamma_{\rho_{C_1},\rho_{C_2}}\, \rho_{C_2}(A)\, \Gamma^*_{\rho_{C_1},\rho_{C_2}} \tag{2.21}$$

on $\mathcal{H}_1$. Let S be any simple domain containing $C_1 \cup C_2$. Then, since

$$\rho_{C_1}(A) \;=\; \rho_{C_2}(A) \;=\; A, \;\text{ for } A \in \mathcal{A}^c(C_1) \cap \mathcal{A}^c(C_2)\,, \tag{2.22}$$

it follows that

$$\begin{aligned}\Gamma_{\rho_{C_1},\rho_{C_2}} \;&\in\; (\mathcal{A}^c(C_1) \,\cap\, \mathcal{A}^c(C_2))' \\ &\subseteq\; (\mathcal{A}^c(S))' \;=\; \overline{\mathcal{A}(S)}^{w}\,. \end{aligned}\tag{2.23}$$

The last equality in (2.23) follows from duality, (2.10). Thus if $\mathcal{S}$ is in the causal complement of $\mathcal{C}_a + x$, for some x, $\Gamma_{\rho_{C_1},\rho_{C_2}} \in \mathcal{B}$, and hence $i(\Gamma_{\rho_{C_1},\rho_{C_2}})$ is defined, for an arbitrary covariant positive-energy representation, i, of $\mathcal{B}$.

Note that it does not follow from (2.23) that $\Gamma_{\rho_{C_1},\rho_{C_2}} \in \overline{\mathcal{A}(C_1)}^{w} \vee \overline{\mathcal{A}(C_2)}^{w}$. In fact, theories in which

$$\Gamma_{\rho_{C_1},\rho_{C_2}} \in \overline{\mathcal{A}(C_1)}^{w} \vee \overline{\mathcal{A}(C_2)}^{w} \tag{2.24}$$

have ordinary permutation statistics. In other words, braid statistics is tied to a failure of (2.24), [28]. The operators $\psi(\mathcal{C}_x, \theta)$ introduced in (1.17) are likely to determine charge transport operators for which (2.24) does not hold, as discussed in [28] for abelian gauge theories.

Summary According to [5], the class of representations of $\{\mathcal{A}, \alpha\}$ describing the physics of a system (at zero temperature) consists of all covariant positive-energy representations, j, localizable in space-like cones, in the sense of Eqs. (2.15), (2.16). These representations are unitarily equivalent to representations of $\mathcal{A}$ on $\mathcal{H}_1$ determined by *morphisms, ρ^j_C, of the extended algebra $\mathcal{B}$ localized in space-like cones C. If $\rho^j_{C_1}$ and $\rho^j_{C_2}$ are both unitarily equivalent to j, and C_1 and C_2 are space-like separated from $\mathcal{C}_a + x$, for some x, then there is a unitary intertwiner, $\Gamma_{\rho_{C_1},\rho_{C_2}}$, (a charge-transport operator), such that

$$\rho_{C_1}(A)\,\Gamma_{\rho_{C_1},\rho_{C_2}} = \Gamma_{\rho_{C_1},\rho_{C_2}}\,\rho_{C_2}(A), \quad \text{with } \Gamma_{\rho_{C_1},\rho_{C_2}} \in \mathcal{B}\,. \tag{2.25}$$

We denote by L the complete list of all inequivalent, irreducible, covariant positive-energy representations of $\{\mathcal{A}, \alpha\}$ localizable in space-like cones.

The fact that ρ_C is a *morphism of $\mathcal{B}$ (if $C \subset (\mathcal{C}_a + x)'$, for some x) and (2.23) permit us to define a composition of representations in L: For $C_1 \subset (\mathcal{C}_a + x)'$ and $C_2 \subset (\mathcal{C}_a + x)'$, for some x, $\rho^j_{C_1}(A) \in \mathcal{B}$, for $A \in \mathcal{A} \subseteq \mathcal{B}$, $j \in L$; hence, for $i \in L$,

$$\rho^i_{C_2} \circ \rho^j_{C_1}(A) \equiv \rho^i_{C_2}(\rho^j_{C_1}(A)) \tag{2.26}$$

is well defined. We define $i \times j$ to be the representation of $\mathcal{A}$, unique up to unitary equivalence, unitarily equivalent to the vacuum representation, 1, of $\rho^i_{C_2} \circ \rho^j_{C_1}(\mathcal{A})$.

If C_1 and C_2 are chosen to be space-like separated then $\rho^i_{C_2} \circ \rho^j_{C_1}(A) = \rho^j_{C_1} \circ \rho^i_{C_2}(A)$, hence

$$i \times j = j \times i . \tag{2.27}$$

Clearly, $i \times j$ is localizable in cones. One easily deduces from (2.23) that, for i and j in L, $i \times j$ is again a covariant, positive-energy representation of $\{\mathcal{A}, \alpha\}$; see [28] for details.

In [4,5,35] natural hypotheses on $\{\mathcal{A}, \alpha\}$ have been isolated which imply the following

Property 2.2

(P1) Every covariant positive-energy representation of $\{\mathcal{A}, \alpha\}$ is completely reducible into a direct sum of irreducible, covariant positive-energy representations.

(P2) There is a unique involution $^-$ ("charge conjugation") on L : $j \in L \mapsto \bar{j} \in L$, such that $j \times \bar{j}$ contains the vacuum representation, 1, of $\mathcal{A}$ precisely once as a subrepresentation.

These properties are deep properties, and it is a non-trivial task to derive them from the structure of $\{\mathcal{A}, \alpha\}$; see [5,35]. Henceforth, they will be assumed to hold.

As a corollary of the fact that, for i and j in L, $i \times j$ is a covariant positive-energy representation of $\{\mathcal{A}, \alpha\}$, and of (P1), we have that $i \times j$ can be decomposed into a direct sum of irreducible representations belonging to L:

$$i \times j = \bigoplus_{k \in L} \bigoplus_{\mu = 1}^{N_{kij}} k^{(\mu)} , \tag{2.28}$$

where $k^{(\mu)}$ is unitarily equivalent to $k \in L$, and $N_{kij} \in \{0, 1, 2, \ldots\}$ is the multiplicity of k in $i \times j$. By property (P2), N_{kij} can also be interpreted as the multiplicity of 1 in $\bar{k} \times i \times j$. This and (2.27) show that

$$N_{kij} = N_{kji} = N_{\bar{j}\bar{k}i} . \tag{2.29}$$

We define $| L | \times | L |$ matrices, N_j, $j \in L$, by setting

$$(\mathsf{N}_j)_{ki} = N_{kji} \in \{0, 1, 2, \ldots\} , \tag{2.30}$$

($| L |$ is the cardinality of L). Clearly

$$1 \times j = j \times 1 = j,$$

so that

$$\mathsf{N}_1 = \mathbb{1}\,. \tag{2.31}$$

As shown in [28], (2.27) implies that

$$\mathsf{N}_i \cdot \mathsf{N}_j = \mathsf{N}_j \cdot \mathsf{N}_i, \text{ for all } i, j \in L\,. \tag{2.32}$$

Properties (2.29)-(2.32) identify the matrices $\{\mathsf{N}_j\}$ as matrices of <u>fusion rules</u>. It is an outstanding open problem to classify all possible fusion rules. Standard examples of fusion rules are those derived from the representation theory of a compact group $G, (N_{k\,j\,i} =$ multiplicity of irrep. k in the tensor product, $j \otimes i$, of two irreps. j and i of G; in this connection see also [6]), or from the representation theory of a quantum group, $U_q(\mathcal{G})$, $q = exp(2\pi\, i/m)$. For example, if $\min_{j \neq 1} \|\mathsf{N}_j\| < 2$ then $\min_{j\neq 1} \|\mathsf{N}_j\| = 2\cos(\frac{\pi}{n+1})$, where $n \equiv |\, L\,|$, and

$$\begin{aligned} N_{k\,j\,i} &= 1, \text{ for } \ |\, i - j \,| < k < min\{i + j,\ 2(n+1) - i - j\} \\ &= 0, \text{ otherwise}. \end{aligned} \tag{2.33}$$

These fusion rules can be derived from the representation theory of $U_q(sl(2))$, $q = exp(2\pi\, i/n + 2)$.

Suppose now that $N_{k\,j\,i} \neq 0$. Then k appears as a subrepresentation of $j \times i$. By the definition of composition, $\times$, this implies that the representation $\rho^j_{\mathcal{C}_1} \circ \rho^i_{\mathcal{C}_2}$ contains $N_{k\,j\,i}$ subrepresentations $\rho^k_{\bar{\mathcal{C}},\mu}$, $\mu = 1, \ldots, N_{k\,j\,i}$; (here $\mathcal{C}_1, \mathcal{C}_2$ and $\bar{\mathcal{C}}$ are space-like cones space-like separated from $\mathcal{C}_a + x$, for some x). Equivalently, the representation i of $\rho^j_{\mathcal{C}}(\mathcal{A})$ contains $N_{k\,j\,i}$ subrepresentations $k^{(\mu)}, \mu = 1, \ldots, N_{k\,j\,i}$, unitarily equivalent to the representation k of $\mathcal{A}$. Hence the superselection sector $\mathcal{H}_i$ can be decomposed into a direct sum of spaces

$$\mathcal{H}_i = \bigoplus_{k \in L} \bigoplus_{\mu=1}^{N_{k\,j\,i}} \mathcal{H}_i(k, j; \mu) \tag{2.34}$$

with the property that the representation i of $\rho^j_{\mathcal{C}}(\mathcal{A})$ on $\mathcal{H}_i(k, j; \mu)$ is unitarily equivalent to the representation k of $\mathcal{A}$.

We now wish to construct a <u>complex vector bundle</u>, $\mathcal{J} \equiv \mathcal{J}_{k\,j\,i}$, <u>of intertwining operators</u> from $\mathcal{H}_k$ to $\mathcal{H}_i$. The <u>base space</u> of this bundle is the space, $\mathcal{M}_j$, of all *morphisms, ρ_C^j, of an extended algebra $\mathcal{B}^{C_a}$ localizable in some space-like cone $\mathcal{C}$ space-like separated from $\mathcal{C}_a$, for some choice of an auxiliary cone $\mathcal{C}_a$, and with the property that the representation $1(\rho_C^j(\cdot))$ of $\mathcal{A}$ is unitarily equivalent to the representation j of $\mathcal{A}$. The fibre space, $\mathcal{V}_k(\rho_C^j)_i$, above a point $\rho_C^j \in \mathcal{M}_j$ is a complex vector space of operators

$$V \;:\; \mathcal{H}_k \;\rightarrow\; \mathcal{H}_i\,, \tag{2.35}$$

satisfying the intertwining relations

$$i(\rho_C^j(A))V \;=\; V\,k(A), \text{ for all } A \in \mathcal{A}\,. \tag{2.36}$$

By (2.34), the range of V is contained in the subspace $\bigoplus_{\mu=1}^{N_{kji}} \mathcal{H}_i(k,j;\mu)$ of $\mathcal{H}_i$.

The space $\mathcal{V}_k(\rho_C^j)_i$ is equipped with a scalar product: For V and W in $\mathcal{V}_k(\rho_C^j)_i$, V^*W is an operator from $\mathcal{H}_k$ to $\mathcal{H}_k$ which, by (2.36), satisfies

$$k(A)\,V^*W \;=\; V^*W\,k(A)\,. \tag{2.37}$$

Since k is an irreducible representation of $\mathcal{A}$, it follows from Schur's lemma that

$$V^*W \;=\; c\cdot\mathbb{1},\; c\in\mathbb{C}\,. \tag{2.38}$$

The complex number c depends anti-linearly on V and linearly on W. Moreover, for $V = W \neq 0$, c is strictly positive. Hence c defines a scalar product,

$$\langle\, V,\, W\,\rangle\,, \tag{2.39}$$

on $\mathcal{V}_k(\rho_C^j)_i$.

Clearly, the multiplicity, N_{kji}, of k in $j \times i \simeq i(\rho_C^j(\cdot))$ does not depend on the choice of the *morphism $\rho_C^j \in \mathcal{M}_j$. Hence all fibres $\mathcal{V}_k(\rho_C^j)_i$ are isomorphic, as complex Hilbert spaces, to $\mathbb{C}^{N_{kji}}$, equipped with the usual scalar product; in particular,

$$dim\ \mathcal{V}_k(\rho)_i \;=\; N_{kji},\; \text{for all } \rho\in\mathcal{M}_j\,. \tag{2.40}$$

Physically speaking, local sections of $\mathcal{J}_{kji}$, (operators $V(\rho): \mathcal{H}_k \rightarrow \mathcal{H}_i, \rho\in\mathcal{M}_j$, satisfying the intertwining relations (2.36)), are interpreted as the unobservable "charged" fields of the theory which play an important rôle in the construction of scattering theory [4,5,28].

In order to describe the bundle $\mathcal{J}_{k\,j\,i}$ more explicitly, we propose to construct an atlas of local coordinate charts, along with its transition functions, for $\mathcal{J}_{k\,j\,i}$. The details of this construction are given in [28]. Here we just outline some basic ideas.

First, we define the manifold $\mathcal{M}_j$, of *morphisms of type j more precisely. We choose two space-like separated, space-like auxiliary cones, $\mathcal{C}_a^I$ and $\mathcal{C}_a^{II}$, where $\mathcal{C}_a^{II}$ is obtained from $\mathcal{C}_a^I$ by a Euclidean motion. The corresponding enlarged C^* algebras,

$$\mathcal{B}^I \;\equiv\; \mathcal{B}^{\mathcal{C}_a^I}, \quad \text{and} \quad \mathcal{B}^{II} \;\equiv\; \mathcal{B}^{\mathcal{C}_a^{II}}, \tag{2.41}$$

are defined as in (2.19). We pick two reference morphisms, ρ^I and ρ^{II}, with the properties that $1(\rho^{\#}(\cdot)) \simeq j$, and $\rho^{\#}$ is localized in a space-like cone $\mathcal{C}_{\#}$, for $\# = I, II$, such that $\mathcal{C}_I$ and $\mathcal{C}_{II}$ are space-like separated from $\mathcal{C}_a^I \cup \mathcal{C}_a^{II}$. We could choose $\rho^I = \rho^{II} \equiv \rho_0$, where ρ_0 is localized in a space-like cone $\mathcal{C}_0$ space-like separated from $\mathcal{C}_a^I \cup \mathcal{C}_a^{II}$.

Next, we define two groups, $\mathcal{U}^I$ and $\mathcal{U}^{II}$, of unitary operators, as follows:

$$\mathcal{U}^{\#} \;:=\; \{\Gamma \,:\, \Gamma \in \mathcal{B}^{\#},\ \Gamma^* \;=\; \Gamma^{-1}\}\,.$$

If ρ is a *morphism in $\mathcal{M}_j$ localized in a space-like cone $\mathcal{C}$ such that $\mathcal{C}$ is space-like separated from $\mathcal{C}_a^{\#} + x$, for some $x \in \mathsf{M}^3$ — this is written, for short, as $\rho \diagup\!\!\!\!\diagdown\, \mathcal{C}_a^{\#}$ — then there exists an operator $\Gamma^{\#} \in \mathcal{U}^{\#}$ such that

$$\rho(A)\,\Gamma^{\#}_{\rho,\rho^{\#}} \;=\; \Gamma^{\#}_{\rho,\rho^{\#}}\,\rho^{\#}(A),\ \text{for all } A \in \mathcal{A}\,; \tag{2.42}$$

see (2.21) and (2.23). If $\Gamma'_{\rho,\rho^{\#}}$ is another element of $\mathcal{U}^{\#}$ for which (2.42) holds then $\left(\Gamma^{\#}_{\rho,\rho^{\#}}\right)^*\Gamma'_{\rho,\rho^{\#}}$ commutes with $\rho^{\#}(\mathcal{A})$, and, since $\mathcal{M}_j$ consists of irreducible morphisms,

$$\left(\Gamma^{\#}_{\rho,\rho^{\#}}\right)^*\,\Gamma'_{\rho,\rho^{\#}} \;=\; e^{i\theta},\ \text{for some } \theta \in [0,2\pi)\,. \tag{2.43}$$

Thus $\Gamma^{\#}_{\rho,\rho^{\#}}$ is unique up to a phase factor. We define

$$\left[\Gamma^{\#}_{\rho,\rho^{\#}}\right] \;=\; \left\{\Gamma'_{\rho,\rho^{\#}} \in \mathcal{U}^{\#} \,:\, \Gamma'_{\rho,\rho^{\#}} = e^{i\theta}\,\Gamma^{\#}_{\rho,\rho^{\#}}\right\} \in P\mathcal{U}^{\#}\,. \tag{2.44}$$

It is straightforward to verify the following properties: Let ρ_1, ρ_2, and ρ_3 be three *morphisms in $\mathcal{M}_j$, with $\rho_i \bigtimes \mathcal{C}_a^{\#}$, for $i = 1, 2, 3$. Then

(a) $\quad \left[\Gamma^{\#}_{\rho_i, \rho_i}\right] \;=\; \text{identity}\,;$

(b) $\quad \left[\left(\Gamma^{\#}_{\rho_i, \rho_k}\right)^*\right] \;=\; \left[\Gamma^{\#}_{\rho_k, \rho_i}\right];$

(c) $\quad \left[\Gamma^{\#}_{\rho_1, \rho_2}\right]\left[\Gamma^{\#}_{\rho_2, \rho_3}\right] \;=\; \left[\Gamma^{\#}_{\rho_1, \rho_3}\right];$

(d) $\quad \left[\Gamma^{I}_{\rho_i, \rho_k}\right] \;=\; \left[\Gamma^{II}_{\rho_i, \rho_k}\right]$

if ρ_i and ρ_k are localized in a simple domain $\mathcal{S}$ space-like separated from $\mathcal{C}_a^I \cup \mathcal{C}_a^{II}$.

Local coordinates on $\mathcal{M}_j$ in the vicinity of a reference morphism $\rho^{\#}$ are given by the coordinate map

$$\phi^{\#}_{\rho^{\#}} \;:\; \rho \in \mathcal{M}_j \;\mapsto\; \left[\Gamma^{\#}_{\rho, \rho^{\#}}\right] \;\in\; P\mathcal{U}^{\#}\,. \tag{2.45}$$

By (b) and (c), the transition function $\phi^{I}_{\rho^I} \circ \left(\phi^{II}_{\rho^{II}}\right)^{-1}$ is given by right multiplication by

$$\left[\Gamma^{I}_{\rho^{II}, \rho^{I}}\right].$$

This defines $\mathcal{M}_j$ as an infinite-dimensional topological manifold modelled on the projective unitary group $P\mathcal{U}$, where $\mathcal{U}$ is the group of unitary operators in a C^*algebra $\mathcal{B}$ isomorphic to $\mathcal{B}^{\#}$; (note that, as abstract C^*algebras, $\mathcal{B}^I$ and $\mathcal{B}^{II}$ are isomorphic). It is not hard to see that the fundamental group of $\mathcal{M}_j$ is given by

$$\pi_1(\mathcal{M}_j) \;=\; \mathbb{Z}\,. \tag{2.46}$$

The geometrical fact underlying (2.46) is that the manifold of asymptotic directions of the space-like cones in which the *morphisms $\rho \in \mathcal{M}_j$ are localized is a circle, the circle of points at infinity in two-dimensional space. For additional details see [28].

Next, we shall construct coordinate charts for the bundle $\mathcal{J}_{kji}$. For any *morphism $\rho \in \mathcal{M}_j$, with $\rho \bigtimes \mathcal{C}_a^{\#}$, we shall choose a representative $\Gamma^{\#}_{\rho, \rho^{\#}} \in \left[\Gamma^{\#}_{\rho, \rho^{\#}}\right]$, in a definite way explained in [28]. By (2.23), $\Gamma^{\#}_{\rho, \rho^{\#}} \in \mathcal{B}^{\#}$. Let $V(\rho^{\#})$ be an intertwiner satisfying (2.36). Then, by (2.42) and (2.36), the operator

$$V(\rho) \;=\; i\left(\Gamma^{\#}_{\rho, \rho^{\#}}\right) V(\rho^{\#}) \tag{2.47}$$

is an element of the fibre $\mathcal{V}_k(\rho)_i$ above ρ. [Note that $i(\Gamma^{\#}_{\rho,\rho^{\#}})$ is a well-defined, unitary operator on $\mathcal{H}_i$, because $\Gamma^{\#}_{\rho,\rho^{\#}} \in \mathcal{B}^{\#}$, and i is a representation of $\mathcal{B}^{\#}$.] Let us choose an orthonormal basis

$$\left\{V^{ik}_{\mu}(\rho^{\#})\right\}_{\mu=1}^{N_{kji}} \subset \mathcal{V}_k(\rho^{\#})_i . \tag{2.48}$$

Then (2.47) shows that

$$\left\{i\left(\Gamma^{\#}_{\rho,\rho^{\#}}\right) V^{ik}_{\mu}\left(\rho^{\#}\right)\right\}_{\mu=1}^{N_{kji}} \tag{2.49}$$

is an orthonormal basis in $\mathcal{V}_k(\rho)_i$.

Two coordinate charts on $\mathcal{J}_{kji}$ are now constructed as follows: A pair $\{\rho,\, V(\rho)\}$, $\rho \in \mathcal{M}_j$, belongs to the chart $\mathcal{N}^{\#}$ of $\mathcal{J}_{kji}$ iff $\rho \times \mathcal{C}^{\#}_a$, (i.e. ρ is localized in a space-like cone C with the property that C is space-like separated from $C^{\#}_a + x$, for some $x \in \mathbb{M}^3$); $\# = I$, or II. The $\mathcal{N}^{\#}$-coordinates of $\{\rho, V(\rho)\}$ are given by

$$\left\{[\Gamma^{\#}_{\rho,\rho^{\#}}] \in P\mathcal{U}^{\#},\ \langle V(\rho),\, i(\Gamma^{\#}_{\rho,\rho^{\#}})\, V^{ik}_{\mu}(\rho^{\#}),\rangle\ \mu = 1,\ldots,N_{kji}\right\}, \tag{2.50}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product on $\mathcal{V}_k(\rho)_i$ constructed in (2.39).

An atlas for $\mathcal{J}_{kji}$ consists of the two coordinate charts $\mathcal{N}^{I}$ and $\mathcal{N}^{II}$, together with transition functions on $\mathcal{N}^{I} \cap \mathcal{N}^{II}$. In order to calculate the transition functions, we consider the geometrical situation sketched in Fig. 4:

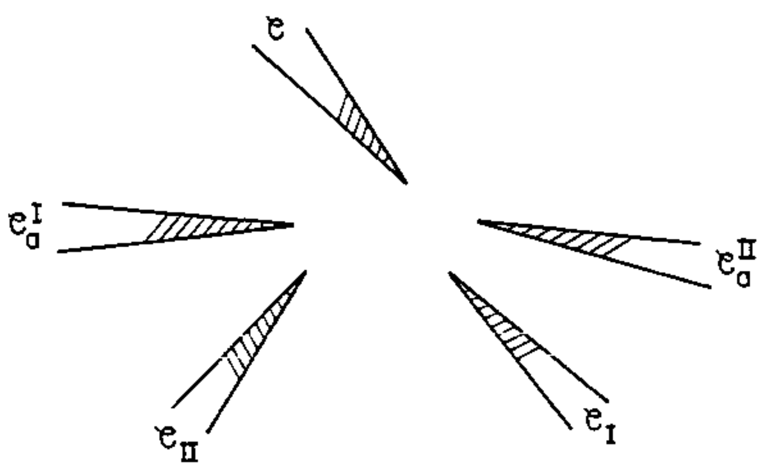


Fig. 4

The reference morphisms ρ^I and ρ^{II} are localized in the space-like cones C_I and C_{II}; the morphism ρ is localized in C. The cones C, C_I and C_{II} are space-like separated from the two auxiliary cones C_a^I and C_a^{II}. The $\mathcal{N}^{\#}$-coordinates of $\{\rho,\, V(\rho)\}$ are given by

$$\left\{[\Gamma^{\#}_{\rho,\rho^{\#}}],\ \langle V(\rho), i(\Gamma^{\#}_{\rho,\rho^{\#}})\ V_\mu^{ik}(\rho^{\#})\rangle,\ \mu = 1,\ldots,N_{kji}\right\}, \tag{2.51}$$

with $\Gamma^{\#}_{\rho,\rho^{\#}} \in \mathcal{U}^{\#} \subset \mathcal{B}^{\#}$. The transition functions are calculated by comparing (2.51), for $\# = I$, with (2.51), for $\# = II$. This requires some work [28] which we now sketch.

(1) Since $\rho^{II} \in \mathcal{N}^I$, (see Fig. 4), we can express $\{\rho^{II},\, V_\nu^{ik}(\rho^{II})\}$ in $\mathcal{N}^I$-coordinates. They are given by

$$\left\{[\Gamma^I_{\rho^{II},\rho^I}],\ a_{\nu\mu}(\rho^{II},\rho^I),\ \mu = 1,\ldots,N_{kji}\right\}, \tag{2.52}$$

where

$$a_{\nu\mu}(\rho^{II},\rho^I) = \langle V_\nu^{ik}(\rho^{II}),\ i(\Gamma^I_{\rho^{II},\rho^I})\ V_\mu^{ik}(\rho^I)\rangle. \tag{2.53}$$

(2) The calculation of transition functions on $\mathcal{N}^I \cap \mathcal{N}^{II}$ is complicated by the circumstance that there is no abstract C^*algebra containing Γ^I_{ρ,ρ^I}, $\Gamma^I_{\rho^{II},\rho^I}$ <u>and</u> $\Gamma^{II}_{\rho,\rho^{II}}$, for arbitrary $\rho \in \mathcal{N}^I \cap \mathcal{N}^{II}$. The operators Γ^I_{ρ,ρ^I} and $\Gamma^I_{\rho^{II},\rho^I}$ are elements of $\mathcal{B}^I$, while $\Gamma^{II}_{\rho,\rho^{II}}$ is an element of $\mathcal{B}^{II}$. Though not disjoint, the algebras $\mathcal{B}^I$ and $\mathcal{B}^{II}$ are distinct, so that, a priori, multiplication of $(\Gamma^I_{\rho,\rho^I})^*$ with $\Gamma^{II}_{\rho,\rho^{II}}$ is not defined. However, $i(\mathcal{B}^I)$ and $i(\mathcal{B}^{II})$ are both naturally imbedded in $B(\mathcal{H}_i)$, the algebra of all bounded operators on the representation space $\mathcal{H}_i$. Hence $i(\Gamma^I_{\rho,\rho^I})^*\, i(\Gamma^{II}_{\rho,\rho^{II}})$ is defined as multiplication of operators in $B(\mathcal{H}_i)$. We define

$$V^i_{-2\pi} = i(\Gamma^I_{\rho^{II},\rho^I})\ i(\Gamma^I_{\rho,\rho^I})^*\ i(\Gamma^{II}_{\rho,\rho^{II}}), \tag{2.54}$$

where ρ is localized in a space-like cone C located as sketched in Fig. 4.* From Fig. 4 we see that $V^i_{-2\pi}$ has the interpretation of rotating the *morphism ρ^{II} through an angle -2π. One might therefore expect that $V^i_{-2\pi}$ can be expressed in terms of <u>spins</u>.

---

* *This definition is unambiguous only if the phases of the charge transport operators are chosen in a definite way; see [28].*

(3) In order to compute $V^i_{-2\pi}$, we compare an arbitrary intertwiner

$$V(\rho^{II}) \;=\; \sum_{\mu=1}^{N_{kji}} b_\mu \, V^{ik}_\mu(\rho^{II}) \;\in\; \mathcal{V}_k(\rho^{II})_i$$

with the operator $V^i_{-2\pi}\,V(\rho^{II})$. We note that $V(\rho^{II})$ and $V^i_{-2\pi}\,V(\rho^{II})$ both satisfy the intertwining relations

$$\begin{aligned} i(\rho^{II}(A))\,V(\rho^{II}) \;&=\; V(\rho^{II})\,k(A), \text{ and} \\ i(\rho^{II}(A))\,V^i_{-2\pi}\,V(\rho^{II}) \;&=\; V^i_{-2\pi}\,V(\rho^{II})\,k(A), \end{aligned} \tag{2.55}$$

for all $A \in \mathcal{A}$; (the second equation in (2.55) follows from the first one and from (2.54)). Hence $V^i_{-2\pi}\,V(\rho^{II})$ belongs to $\mathcal{V}_k(\rho^{II})_i$, too, and by Schur's lemma, there exists an $N_{kji} \times N_{kji}$-matrix, $\left(V^-_{\nu\mu}(i,k)\right)^{N_{kji}}_{\nu,\mu=1}$, such that

$$V^i_{-2\pi}\,V(\rho^{II}) \;=\; \sum_\nu c_\nu \, V^{ik}_\nu(\rho^{II}),$$

with

$$c_\nu \;=\; \sum_\mu V^-_{\nu\mu}(i,k)\,b_\mu\,. \tag{2.56}$$

We propose to calculate the matrices $V^-(i,k)$.

(4) We recall that all representations in L are irreducible, covariant positive-energy representations of $\{\mathcal{A},\alpha\}$. Thus, for $k \in L$, there is a representation U_k of $\tilde{\mathcal{P}}^\uparrow_+$ on $\mathcal{H}_k$. Let $U_k(2\pi)$ be the unitary operator representing the space rotation through an angle 2π. Clearly $U_k(2\pi)$ commutes with $k(\mathcal{A})$, and, since k is irreducible, it follows that

$$U_k(2\pi) \;=\; e^{2\pi i s_k}, \tag{2.57}$$

where s_k is called the spin of the representation k. Since the little group in $\tilde{\mathcal{P}}^\uparrow_+$ of a time-like vector is isomorphic to the covering group, $\widetilde{SO}(2) \;=\; \mathbf{R}$, of the subgroup of space rotations in $\mathbf{M}^3$, the spin s_k can be an arbitrary real number in the interval [0,1).

Consider the following loop of intertwiners:

$$\{V(\rho^{II},\theta) \,:\, -2\pi \,\le\, \theta \,\le\, 0\},$$

with

$$V(\rho^{II},\theta) := U_i(\theta)\, V(\rho^{II})\, U_k(-\theta), \tag{2.58}$$

where $V(\rho^{II}) \in \mathcal{V}_k(\rho^{II})_i$, and $U_k(\theta)$ represents a space rotation through an angle θ on $\mathcal{H}_k$. One easily checks that

$$V(\rho^{II},\theta) \;\in\; \mathcal{V}_k\big(\alpha_\theta \circ \rho^{II} \circ \alpha_{-\theta}\big)_i\,,$$

where α_θ is the *automorphism of $\mathcal{A}$ representing the space rotation through an angle θ. Hence $V(\rho^{II},-2\pi) \in \mathcal{V}_k(\rho^{II})_i$. Given the geometrical situation sketched in Fig. 4, it is not surprizing that one can choose the charge-transport operators $\{\Gamma^I_{\rho,\rho^I},\, \rho \in \mathcal{N}^I\}$ and $\{\Gamma^{II}_{\rho,\rho^{II}},\, \rho \in \mathcal{N}^{II}\}$ such that

$$V^i_{-2\pi}\, V(\rho^{II}) \;=\; V(\rho^{II},-2\pi)\,. \tag{2.59}$$

A complete analysis of this point is non-trivial, and we refer the reader to [28] for details.

From (2.57), (2.58) and (2.59) we conclude the following theorem proven in [28].

<u>Theorem</u>

<u>The matrix</u> $V^-(i,k)$ <u>introduced in</u> (2.56) <u>is given by</u>

$$V^-(i,k) \;=\; e^{2\pi i(s_k - s_i)}\, \mathbb{I}\,; \tag{2.60}$$

<u>equivalently</u>,

$$V^i_{-2\pi}\, V(\rho^{II}) \;=\; e^{2\pi i(s_k - s_i)}\, V(\rho^{II}), \tag{2.61}$$

<u>for all</u> $V(\rho^{II}) \in \mathcal{V}_k(\rho^{II})_i$.

□

(5) We are now ready to calculate the <u>transition functions</u> on $\mathcal{N}^I \cap \mathcal{N}^{II}$. By (2.52), left multiplication by $i\big(\Gamma^I_{\rho^{II},\rho^I}\big)^*$ maps $\mathcal{V}_k(\rho^{II})_i$ onto $\mathcal{V}_k(\rho^I)_i$. By (2.54) and (2.61), we have that, for an arbitrary intertwiner $V(\rho^{II}) \in \mathcal{V}_k(\rho^{II})_i$,

$$\begin{aligned} i\big(\Gamma^{II}_{\rho,\rho^{II}}\big)\, V(\rho^{II}) \;&=\; i\big(\Gamma^I_{\rho,\rho^I}\big)\, i\big(\Gamma^I_{\rho^{II},\rho^I}\big)^*\, e^{2\pi i(s_k - s_i)}\, V(\rho^{II}) \\ &=\; i\big(\Gamma^I_{\rho,\rho^{II}}\big)\, e^{2\pi i(s_k - s_i)}\, V(\rho^{II}), \end{aligned} \tag{2.62}$$

where $\Gamma^I_{\rho,\rho^{II}} \equiv \Gamma^I_{\rho,\rho^I}\left(\Gamma^I_{\rho^{II},\rho^I}\right)^*$. Using (2.62) and (2.52), we find that

$$
\begin{aligned}
&\left\langle V(\rho), i\left(\Gamma^{II}_{\rho,\rho^{II}}\right) V^{ik}_\nu(\rho^{II})\right\rangle \\
&\quad = \left\langle V(\rho),\ i\left(\Gamma^I_{\rho,\rho^{II}}\right) V^{ik}_\nu(\rho^{II})\right\rangle e^{2\pi i(s_k - s_i)} \\
&\quad = \sum_\mu \bar{a}_{\nu\mu}\, e^{2\pi i(s_k - s_i)} \left\langle V(\rho), i\left(\Gamma^I_{\rho,\rho^I}\right) V^{ik}_\mu(\rho^I)\right\rangle,
\end{aligned} \tag{2.63}
$$

where $a_{\nu\mu} = a_{\nu\mu}(\rho^{II}, \rho^I)$ is given by (2.53). Eq. (2.45) and (2.63) show that the transition function on the component of $\mathcal{N}^I \cap \mathcal{N}^{II}$ described in Fig. 4 is given by

$$
\left\{ R\left[\Gamma^I_{\rho^I,\rho^{II}}\right],\ \left(\bar{a}_{\nu\mu}\, e^{2\pi i(s_k - s_i)}\right)^{N_{kji}}_{\nu,\mu=1} \right\}, \tag{2.64}
$$

where $R[\Gamma]$ denotes right-multiplication by $[\Gamma]$ and corresponds to the transformation

$$
\left[\Gamma^I_{\rho,\rho^I}\right] \mapsto \left[\Gamma^{II}_{\rho,\rho^{II}}\right] = \left[\Gamma^I_{\rho,\rho^I}\right]\left[\Gamma^I_{\rho^I,\rho^{II}}\right], \tag{2.65}
$$

and $\left(\bar{a}_{\nu\mu}\, e^{2\pi i(s_k - s_i)}\right)$ describes the transformation (2.63). The transition function on the component of $\mathcal{N}^I \cap \mathcal{N}^{II}$ shown in Fig. 4', below, is given by

$$
\left\{ R\left[\Gamma^I_{\rho^I,\rho^{II}}\right],\ \left(\bar{a}_{\nu\mu}\right)^{N_{kji}}_{\nu\mu=1} \right\}. \tag{2.66}
$$

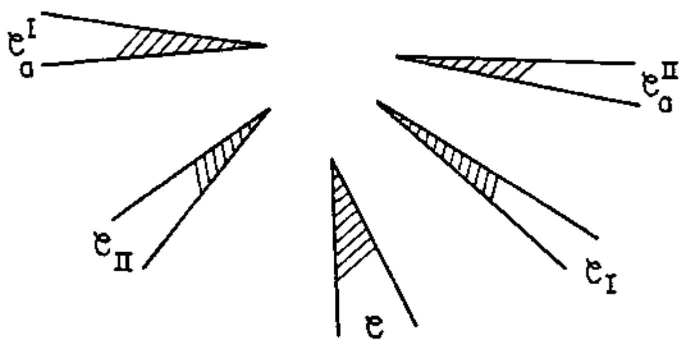


Fig. 4'

If we choose $\rho^I = \rho^{II}$ then

$$
\begin{aligned}
a_{\nu\mu} &= \delta_{\nu\mu}, \text{ and} \\
V^I_{-2\pi} &= i\left(\Gamma^I_{\rho,\rho^I}\right)^* i\left(\Gamma^{II}_{\rho,\rho^I}\right).
\end{aligned} \tag{2.67}
$$

This completes our sketch of the construction of the vector bundles $\mathcal{J}_{kji}$ of intertwiners ('charged fields') from $\mathcal{H}_k$ to $\mathcal{H}_i$.

Formulas (2.64) and (2.66) reflect the non-trivial topology of $\mathcal{M}_j$ which, in turn, reflects the topology of the manifold of space-like asymptotic directions of space-time.

This concludes our brief review of the algebraic approach to local, relativistic quantum theory [4,5,6,28]. In the next section, we shall discuss the structure of the algebra of intertwiners.

3. Statistics and fusion of intertwiners

Let $\mathcal{C}_a$ be some space-like cone in three-dimensional Minkowski space M^3, and let $\mathcal{S}$ be a simple domain (region) contained in the space-like complement of $\mathcal{C}_a$. [Space-like cones and simple domains were defined in (2.5),(2.6).] Let $\mathcal{C} \subset \mathcal{S}$ be some space-like cone. With $\mathcal{C}$ we associate an angle $\theta(\mathcal{C})$ as follows: We choose polar coordinates (r, θ) in two-dimensional space, $\{(\vec{x}, t) \in \mathsf{M}^3 : t = 0\}$. Let $\sigma(\mathcal{C})$ be the half-line in space bisecting the wedge $\mathcal{C}_0$ whose causal completion is the cone $\mathcal{C}$; see (2.5),(2.6). Let $\theta(\mathcal{C})$ be the asymptotic angle of $\sigma(\mathcal{C})$; $\theta(\mathcal{C})$ is called the asymptotic direction of $\mathcal{C}$. If ρ is some *morphism of $\mathcal{B}^{\mathcal{C}_a}$ localized in $\mathcal{C}$ then $\theta(\mathcal{C})$ is called the <u>asymptotic direction</u> of ρ and is denoted by $as(\rho)$. We may choose our coordinates such that $\theta(\mathcal{C}_a) = \pi$, and require that

$$| \, as(\rho) \, | < \pi \,, \tag{3.1}$$

for all *morphisms ρ localized in space-like cones contained in $\mathcal{S}$.

Let ρ_1 and ρ_2 be two *morphisms of $\mathcal{B}^{\mathcal{C}_a}$ localized in space-like cones $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. We say that ρ_1 and ρ_2 are <u>causally independent</u>, denoted by $\rho_1 \times \rho_2$, iff $\mathcal{C}_1$ and $\mathcal{C}_2$ are space-like separated.

For every irreducible, covariant, positive-energy representation $j \in L$, we choose a <u>reference morphism</u> $\rho_0^j \in \mathcal{M}_j$, localized in a space-like cone $\mathcal{C}_0^j \subset \mathcal{S}$, and a basis of intertwiners

$$V_\mu^{ik}(\rho_0^j) \, : \, \mathcal{H}_k \to \mathcal{H}_i \,, \tag{3.2}$$

satisfying the intertwining relations

$$i(\rho_0^j(A))\, V_\mu^{ik}(\rho_0^j) \;=\; V_\mu^{ik}(\rho_0^j)\, k(A)\,, \tag{3.3}$$

for $\mu = 1, \ldots, N_{kji}$. [We recall that $\mathcal{M}_j$ is the space of all *morphisms, ρ^j, localized in space-like cones with the property that the representation j of $\mathcal{A}$ is unitarily equivalent to the vacuum representation, 1, of $\rho^j(\mathcal{A})$.]

Let ρ^j be some other *morphism of $\mathcal{B}^{\mathcal{C}_*}$ contained in $\mathcal{M}_j$ and localized in a space-like cone contained in $\mathcal{S}$. As explained in Sect. 2, (2.21), (2.23) there is then a unitary operator $\Gamma_{\rho^j,\rho_0^j} \equiv \Gamma^{\mathcal{S}}_{\rho^j,\rho_0^j} \in \overline{\mathcal{A}(\mathcal{S})}^w$ such that

$$\rho^j(A)\, \Gamma^{\mathcal{S}}_{\rho^j,\rho_0^j} \;=\; \Gamma^{\mathcal{S}}_{\rho^j,\rho_0^j}\, \rho_0^j(A)\,, \tag{3.4}$$

for all $A \in \mathcal{A}$. Moreover, if ρ_1, ρ_2 and ρ_3 are *morphisms in $\mathcal{M}_j$ localized in space-like cones $\subset \mathcal{S}$

$$\left[\Gamma^{\mathcal{S}}_{\rho_i,\rho_i}\right] \;=\; \mathbb{1}, \;\; \left[\left(\Gamma^{\mathcal{S}}_{\rho_i,\rho_j}\right)^*\right] \;=\; \left[\Gamma^{\mathcal{S}}_{\rho_j,\rho_i}\right], \tag{3.5}$$

for $i, j = 1, 2, 3$, and

$$\left[\Gamma^{\mathcal{S}}_{\rho_1,\rho_2}\right] \left[\Gamma^{\mathcal{S}}_{\rho_2,\rho_3}\right] \;=\; \left[\Gamma^{\mathcal{S}}_{\rho_1,\rho_3}\right]. \tag{3.6}$$

See Sect. 2, (a)-(c), after (2.44).

A basis of intertwiners, $\left\{V_\mu^{ik}(\rho^j)\right\}_{\mu=1}^{N_{kji}}$, associated with ρ^j is obtained by setting

$$V_\mu^{ik}(\rho^j) \;=\; i\left(\Gamma^{\mathcal{S}}_{\rho^j,\rho_0^j}\right) V_\mu^{ik}(\rho_0^j)\,; \tag{3.7}$$

see (2.49). They satisfy the intertwining relations

$$i(\rho^j(A))\; V_\mu^{ik}(\rho^j) \;=\; V_\mu^{ik}(\rho^j)\; k(A)\,. \tag{3.8}$$

3.1. The statistics of intertwiners

The structure of the algebra of intertwiners is described in the following basic result.

<u>Theorem 1</u> [28]

<u>For</u> p <u>and</u> q <u>in</u> L, <u>let</u> $\rho^p \in \mathcal{M}_p$ <u>and</u> $\rho^q \in \mathcal{M}_q$ <u>be two *morphisms of</u> $\mathcal{B}^{\mathcal{C}_*}$ <u>localized in space-like cones contained in</u> $\mathcal{S}$. <u>Let the intertwiners</u> $\left\{V_\mu^{ik}(\rho^{p,q})\right\}$ <u>be defined as in</u> (3.7). <u>Then there are matrices, called statistics matrices,</u>

$$\left(R^{\pm}(j,p,q,k)^{l\kappa\lambda}_{i\mu\nu}\right)$$

only depending on the classes $\mathcal{M}_p$ and $\mathcal{M}_q$, such that

$$\begin{aligned} &V^{ji}_{\mu}(\rho^p)\, V^{ik}_{\nu}(\rho^q) \\ &\quad = \sum_{l,\alpha,\beta} R^{\pm}(j,p,q,k)^{l\alpha\beta}_{i\mu\nu}\, V^{jl}_{\alpha}(\rho^q)\, V^{lk}_{\beta}(\rho^p)\,, \end{aligned} \tag{3.9}$$

provided $\rho^p \bigtimes \rho^q$ and $as(\rho^p) \gtrless as(\rho^q)$.

□

Remarks The proof of Theorem 1 is given in [28]. That there is a relation of the form of (3.9) is not difficult to see. It is a straightforward consequence of Schur's lemma: Consider the operator

$$V \;\equiv\; V^{lk}_{\beta}(\rho^p)^*\, V^{jl}_{\alpha}(\rho^q)^*\, V^{ji}_{\mu}(\rho^p)\, V^{ik}_{\nu}(\rho^q)\,. \tag{3.10}$$

From the intertwining relations (3.8) and their adjoint it follows that

$$k(A)V \;=\; V\,k(A), \quad \text{for all } A \in \mathcal{A}\,. \tag{3.11}$$

Since k is an irreducible representation of $\mathcal{A}$, it follows from Schur's lemma that

$$V \;=\; \lambda\,\mathbb{1},\;\; \lambda \in \mathbb{C}\,. \tag{3.12}$$

We denote λ by $R(j,\rho^p,\rho^q,k)^{l\alpha\beta}_{i\mu\nu}$. Next, we note that

$$\left\{V^{jl}_{\alpha}(\rho^q)\, V^{lk}_{\beta}(\rho^p) \,:\, l \in L,\; \alpha = 1,\ldots,N_{lqj},\; \beta = 1,\ldots,N_{kpl}\right\} \tag{3.13}$$

is a basis of intertwiners from $\mathcal{H}_k$ to $\mathcal{H}_j$ intertwining the representations $j(\rho^q \circ \rho^p(\,\cdot\,))$ and $k(\cdot)$ of the algebra $\mathcal{A}$; see e.g. [28]. From (3.10),(3.12) and (3.13) we conclude that

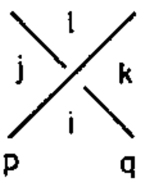
$$V^{ji}_{\mu}(\rho^p)\, V^{ik}_{\nu}(\rho^q) \;=\; \sum_{l,\alpha,\beta} R(j,\rho^p,\rho^q,k)^{l\alpha\beta}_{i\mu\nu}\, V^{jl}_{\alpha}(\rho^q)\, V^{lk}_{\beta}(\rho^p)\,. \tag{3.14}$$

Next, one shows, using an argument invented in the proof of Lemma 2.6 of [4], that if $\rho^p \bigtimes \rho^q$ then

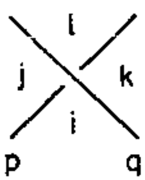
$$R(j,\rho^p,\rho^q,k)^{l\alpha\beta}_{i\mu\nu} \;=\; R^{\pm}(j,p,q,k)^{l\alpha\beta}_{i\mu\nu}\,, \tag{3.15}$$

for $as(\rho^p) \gtrless as(\rho^q)$, where the matrices $\left(R^{\pm}(j,p,q,k)^{l\alpha\beta}_{i\mu\nu}\right)$ <u>only</u> depend on the <u>classes</u> $\mathcal{M}_p$ and $\mathcal{M}_q$ of *morphisms to which ρ^p and ρ^q belong, but are <u>independent</u> of the specific choice of ρ^p and ρ^q; they are also <u>independent</u> of the choice of the auxiliary cone C_a, (as long as C_a is space-like separated from the localization cones of ρ^p and ρ^q). Although these facts are not difficult to prove, technically, they are somwhat more subtle than (3.14). For proofs see [28].

Next, we investigate the properties of the statistics matrices $\left(R^{\#}(j,p,q,k)^{l\alpha\beta}_{i\mu\nu}\right)$ somewhat systematically. For that purpose we introduce a <u>graphical notation</u>:

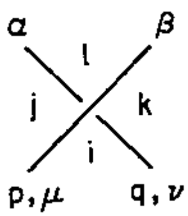


$$\longleftrightarrow \qquad R^{+}(j,p,q,k)^{l}_{i} \tag{3.16}$$



$$\longleftrightarrow \qquad R^{-}(j,p,q,k)^{l}_{i}\,. \tag{3.17}$$

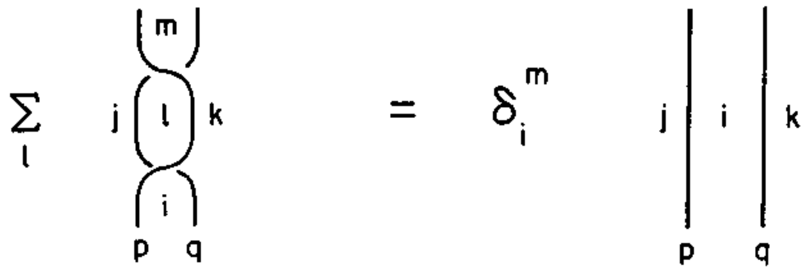
We have dropped the Greek multiplicity indices μ,ν,α and β; (a more complete notation would be



, etc.; repeated indices are to be summed over).

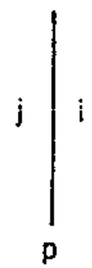
Iterating equ. (3.9), we find that

(1)



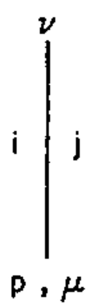
(3.18)

where



$$\longleftrightarrow \quad \mathbb{1}\big|_{\mathbb{C}^{N_{ipj}}}\,, \tag{3.19}$$

or



$$\longleftrightarrow \quad \delta^{\nu}_{\mu}\,,\ \ \mu,\nu = 1,\ldots,N_{ipj}\,. \tag{3.19'}$$

By considering

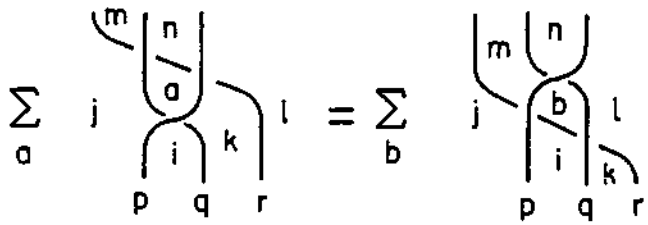
$$V_\mu^{ji}(\rho^p)\, V_\nu^{ik}(\rho^q)\, V_\kappa^{kl}(\rho^r)\,, \tag{3.20}$$

and assuming that $as(\rho^p)$, $as(\rho^q)$ and $as(\rho^r)$ are ordered in some way and ρ^p, ρ^q and ρ^r are pairwise causally independent, (i.e. $\rho^p \times \rho^q$, $\rho^p \times \rho^r$ and $\rho^q \times \rho^r$), we find by permuting the order of the factors in (3.20) to

$$V_\alpha^{jm}(\rho^r)\, V_\beta^{mn}(\rho^q)\, V_\gamma^{nl}(\rho^p) \tag{3.21}$$

in two distinct ways that

(2)

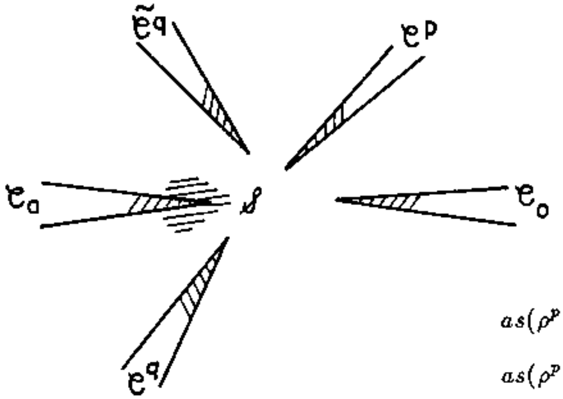


(3.22)

here it is assumed that $as(\rho^p) > as(\rho^q) > as(\rho^r)$. Other related identities are found for other orderings of $as(\rho^p)$, $as(\rho^q)$ and $as(\rho^r)$. The equations (3.22) are homogeneous, cubic equations in the matrices $R^\pm$. They represent the sos-form of the Yang-Baxter equations (YBE) without spectral parameter. The derivation of (3.22) from (3.20) and (3.21) was first given in [15].

From (1) and (2) we conclude that the matrices $(R^\pm(j,p,q,k)_i^l)$ generate representations of the groupoid, B_n^c, of coloured braids on n strands.

Next, we derive a basic relation between R^+ and R^-: We consider two *morphisms ρ^p and ρ^q, localized in space-like cones C^p and C^q whose projection onto two-dimensional space is shown in Fig. 5. We suppose that the reference morphisms, ρ_0^p and ρ_0^q are localized in a space-like cone C_0. The cones C^p, C^q and C_0 are assumed to be contained in a simple region S whose space-like complement, S', is the auxiliary cone C_a.



$$as(\rho^p) > as(\rho^q),$$
$$as(\rho^p) < as(\tilde{\rho}^q).$$

Fig. 5

We also consider a *morphism, $\tilde{\rho}^q$, localized in a space-like cone $\tilde{\mathcal{C}}^q$, as shown in Fig. 5. Then we have from Theorem 1 that

$$V^{ij}(\rho^p)\, V^{jk}(\rho^q) \;=\; \sum R^{+}(i,p,q,k)^l_j\; V^{il}(\rho^q)\, V^{lk}(\rho^p)\,, \tag{3.23}$$

and

$$V^{ij}(\rho^p)\, V^{jk}(\tilde{\rho}^q) \;=\; \sum R^{-}(i,p,q,k)^l_j\; V^{il}(\tilde{\rho}^q)\, V^{lk}(\rho^p)\,. \tag{3.24}$$

[We omit the multiplicity indices $\mu, \nu, \ldots$ everywhere.] Since $R^{-}(i,p,q,k)^l_j$ is independent of the choice of the auxiliary cone, $\mathcal{C}_a$, (3.24) does not change if $\mathcal{C}_a$ is replaced by a new auxiliary cone, $\hat{\mathcal{C}}_a$, chosen as indicated in Fig. 6.

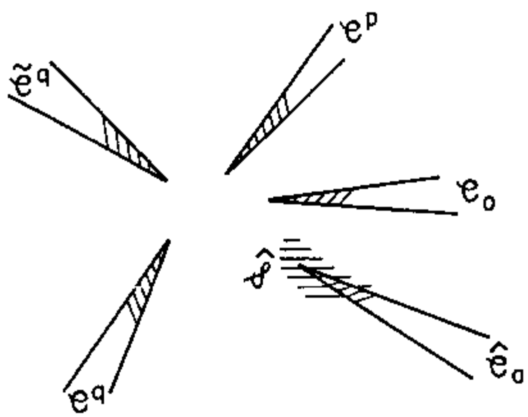


Fig. 6

In the situation shown in Fig. 6,

$$as(\rho^q) \geq as(\tilde{\rho}^q) > as(\rho^p). \tag{3.25}$$

We define

$$V^{ij}_{\hat{S}}(\rho^\#) = i(\Gamma^{\hat{S}}_{\rho^\#,\rho^\#_0})\, V^{ij}(\rho^\#_0), \tag{3.26}$$

for $\# = p, q$. We also recall that

$$V^{ij}(\rho^\#) = i(\Gamma^{S}_{\rho^\#,\rho^\#_0})\, V^{ij}(\rho^\#_0); \tag{3.27}$$

see (3.7). Thus we conclude from (3.24) and the remark that R^- does not change if C_a is replaced by $\hat{C}_a$ that

$$V^{ij}_{\hat{S}}(\rho^p)\, V^{jk}_{\hat{S}}(\tilde{\rho}^q) = \sum R^-(i,p,q,k)^l_j\, V^{il}_{\hat{S}}(\tilde{\rho}^q)\, V^{lk}_{\hat{S}}(\rho^p). \tag{3.28}$$

But in the situation shown in Fig. 6, $\tilde{C}^q$ can be rotated to C^q in the positive direction inside $\hat{S}$, and, since $R^-(i,p,q,k)^l_j$ only depends on $\mathcal{M}_p$ and $\mathcal{M}_q$, but not on $\tilde{\rho}^q$, we can replace $\tilde{\rho}^q$ by ρ^q and conclude that

$$V^{ij}_{\hat{S}}(\rho^p)\, V^{jk}_{\hat{S}}(\rho^q) = \sum R^-(i,p,q,k)^l_j\, V^{il}_{\hat{S}}(\rho^q)\, V^{lk}_{\hat{S}}(\rho^p). \tag{3.29}$$

From (3.26) and (3.27) we obtain that

$$\begin{aligned} V^{ij}(\rho^q) &= i(\Gamma^{S}_{\rho^q,\rho^q_0})\, i(\Gamma^{\hat{S}}_{\rho^q,\rho^q_0})^*\, V^{ij}_{\hat{S}}(\rho^q) \\ &= i(\Gamma^{S}_{\rho^q,\rho^q_0})\, i(\Gamma^{\hat{S}}_{\rho^q_0,\rho^q})\, V^{ij}_{\hat{S}}(\rho^q). \end{aligned} \tag{3.30}$$

But from the calculations in Sect. 2, (2.54),(2.56),(2.60), (2.61) and (2.67), we infer that

$$\begin{aligned} & i(\Gamma^{S}_{\rho^q,\rho^q_0})\, i(\Gamma^{\hat{S}}_{\rho^q_0,\rho^q})\, V^{ij}_{\hat{S}}(\rho^q) \\ &\quad = V^i_{-2\pi}\, V^{ij}_{\hat{S}}(\rho^q) \\ &\quad = e^{2\pi i(s_j - s_i)}\, V^{ij}_{\hat{S}}(\rho^q), \end{aligned}$$

where s_j is the spin of representation j. Hence

$$V^{ij}(\rho^q) = e^{2\pi i(s_j - s_i)}\, V^{ij}_{\hat{S}}(\rho^q). \tag{3.31}$$

From Fig. 5 and Fig. 6 we also learn that

$$V^{ij}(\rho^p) = V^{ij}_{\hat{S}}(\rho^p). \tag{3.32}$$

Inserting (3.31) and (3.32) into (3.29) we have that

$$\begin{aligned} V^{ij}(\rho^p)\, V^{jk}(\rho^q) &= e^{2\pi i(s_j - s_k)}\, V^{ij}_{\hat{S}}(\rho^p)\, V^{jk}_{\hat{S}}(\rho^q) \\ &= e^{2\pi i(s_j - s_k)} \sum_l R^-(i,p,q,k)^l_j\, V^{il}_{\hat{S}}(\rho^q)\, V^{lk}_{\hat{S}}(\rho^p) \\ &= \sum_l e^{2\pi i(s_j - s_k + s_l - s_i)}\, R^-(i,p,q,k)^l_j\, V^{il}(\rho^q)\, V^{lk}(\rho^p)\,. \end{aligned} \tag{3.33}$$

Comparing (3.33) with (3.23) we arrive at the following fundamental identity:

$$(3) \qquad \boxed{R^+(i,p,q,k)^{l\alpha\beta}_{j\mu\nu} \;=\; e^{2\pi i(s_j + s_l - s_i - s_k)}\, R^-(i,p,q,k)^{l\alpha\beta}_{j\mu\nu}} \tag{3.34}$$

where s_j is the spin of representation j.

<u>Remark</u> If s_j is reinterpreted as the conformal dimension of a representation j of some chiral algebra then identities (1),(2) and (3), (see (3.18),(3.22) and (3.34)), become well known identities for the braid matrices of conformal field theory [19].

Equ. (3.34) has the following obvious, but important corollary: If all representations $j \in L$ have integer spins, i.e.

$$s_j \;=\; 0 \;mod. \mathbb{Z}, \;\text{ for all }\; j \in L\,,$$

then

$$R^+(i,p,q,k)^l_j \;=\; R^-(i,p,q,k)^l_j \;\equiv\; R(i,p,q,k)^l_j\,, \tag{3.35}$$

for arbitrary i,p,q,k,j and l in L. In this case, (3.18) and (3.22) imply that the matrices $(R(i,p,q,k)^l_j)$ define representations of the <u>permutation groups</u>, S_n, of n elements. Hence, <u>in a theory in which all representations have integer spin, the statistics of the intertwiners</u> $\{V^{ij}(\rho^p)\}$ <u>is ordinary permutation group statistics</u>, as analyzed by Doplicher, Haag and Roberts in [4].

Next, we prove a <u>connection between spin and statistics</u>. It is based on the following simple, but basic result: Given a representation $j \in L$, $\bar{j}$ denotes its conjugate representation; $\bar{j}$ is the unique representation of $\rho^j(\mathcal{A})$, $\rho^j \in \mathcal{M}_j$, containing precisely one subrepresentation unitarily equivalent to the vacuum representation, 1, of $\mathcal{A}$.

<u>Lemma 2</u> [28]

$$(1) \quad R^-(l,\bar{q},\bar{p},j)^{m\alpha\beta}_{k\mu\nu} \;=\; \overline{R^+(j,p,q,l)^{m\beta\alpha}_{k\nu\mu}}$$

(2) $R^{\pm}(j,p,q,1)^{l\alpha\beta}_{k\mu\nu} \;=\; \delta^{\bar q}_{k}\,\delta^{l}_{\bar p}\,\delta^{1}_{\nu}\,\delta^{\beta}_{1}\; R^{\pm}(j,p,q,1)^{\bar p\alpha 1}_{\bar q\mu 1}$

(3) $R^{\pm}(1,p,q,j)^{l\alpha\beta}_{k\mu\nu} \;=\; \delta^{p}_{k}\,\delta^{l}_{q}\,\delta^{1}_{\mu}\,\delta^{\alpha}_{1}\; R^{\pm}(1,p,q,j)^{q1\beta}_{p1\nu}\,.$

□

We omit the proof of this lemma.

We now note that, by Lemma 2, parts (2) and (3), the only non-zero matrix elements of the matrices $R_{\pm}(1,p,\bar p,1)^{m\alpha\beta}_{k\mu\nu})$ are $R^{\pm}(1,p,\bar p,1)^{\bar p 1 1}_{p 1 1}$. By Lemma 2, part (1), and since $\bar{\bar p} = p$,

$$R^{-}(1,p,\bar p,1)^{\bar p 1 1}_{p 1 1} \;=\; \overline{R^{+}(1,p,\bar p,1)^{\bar p 1 1}_{p 1 1}}\,. \tag{3.36}$$

If one chooses the intertwiners to be partial isometries then one sees that

$$|\, R^{\pm}(1,p,\bar p,1)^{\bar p 1 1}_{p 1 1}\,| \;=\; 1.$$

We may therefore introduce the notation

$$R^{+}(1,p,\bar p,1)^{\bar p 1 1}_{p 1 1} \;=\; e^{2\pi i\,\theta_p}\,. \tag{3.37}$$

By (3.36)

$$R^{-}(1,p,\bar p,1)^{\bar p 1 1}_{p 1 1} \;=\; e^{-2\pi i\,\theta_p}\,. \tag{3.38}$$

Next, we apply (3.34) to conclude that

$$e^{2\pi i\,\theta_p} \;=\; e^{2\pi i(s_p+s_{\bar p})}\; e^{-2\pi i\,\theta_p}\,, \tag{3.39}$$

and we have used that $s_1 = 0 \; mod.\, \mathbb{Z}$. Finally, we note that

$$s_p \;=\; s_{\bar p}. \tag{3.40}$$

Thus, combining (3.39) and (3.40) we have that

$$\boxed{s_p \;=\; \theta_p \;\, mod.\, \frac{1}{2}\mathbb{Z}\,.} \tag{3.41}$$

This is the simplest connection between spin and statistics. More precise results of a similar nature will be proven below.

3.2 Fusion of intertwiners

For p and q in L, we consider *morphisms $\rho^p \in \mathcal{M}_p$ and $\rho^q \in \mathcal{M}_q$ localized in space-like cones C^p and C^q, respectively, which are contained in the interior of a simple region $\mathcal{S} \subset \mathrm{M}^3$. The space-like complement of $\mathcal{S}$ is assumed to contain a non-empty, space-like auxiliary cone, $\mathcal{C}_a$. Then ρ^p and ρ^q are *morphisms of the extended algebra $\mathcal{B}^{\mathcal{C}_a}$, defined in (2.19). In particular, the composition, $\rho^p \circ \rho^q$, of ρ^p with ρ^q is well defined on the algebra $\mathcal{A}$ of quasi-local observables. Property 2.2, (P1), (Sect. 2, after (2.27)) guarantees that the product representation $p \times q$ can be decomposed into a direct sum of irreducible, localizable, covariant positive-energy representations, i.e.,

$$p \times q \;=\; \bigoplus_{r \in L} \bigoplus_{\mu=1}^{N_{rpq}} r^{(\mu)}\,, \tag{3.42}$$

see (2.28). Let C^r be a space-like cone contained in the interior of $\mathcal{S}$, and let $\rho^r \in \mathcal{M}_r$ be a *morphism of $\mathcal{B}^{\mathcal{C}_a}$ localized in C^r with the property that the representation r of $\mathcal{A}$ is unitarily equivalent to the vacuum representation, 1, of $\rho^r(\mathcal{A})$. Then there exist N_{rpq} partial isometries,

$$\Gamma^{\mathcal{S}}_{\rho^p \circ \rho^q, \rho^r}(\mu) \in \overline{\mathcal{A}(\mathcal{S})}^{w} \subset \mathcal{B}^{\mathcal{C}_a} \tag{3.43}$$

such that

$$\rho^p \circ \rho^q(A)\, \Gamma^{\mathcal{S}}_{\rho^p \circ \rho^q, \rho^r}(\mu) \;=\; \Gamma^{\mathcal{S}}_{\rho^p \circ \rho^q, \rho^r}(\mu)\, \rho^r(A)\,, \tag{3.44}$$

for all $A \in \mathcal{A}$, $\mu = 1, \dots, N_{rpq}$; see [4,5].

If $\mathcal{S}_0 \subseteq \mathcal{S}$ is a simple domain containing the cones C^p, C^q and C^r in the interior then, actually,

$$\Gamma^{\mathcal{S}}_{\rho^p \circ \rho^q, \rho^r}(\mu) \;\in\; \overline{\mathcal{A}(\mathcal{S}_0)}^{w}\,. \tag{3.45}$$

Next, we consider a product of intertwiners $V^{ij}_{\alpha}(\rho^p)\, V^{jk}_{\beta}(\rho^q)$. They satisfy the intertwining relations

$$\begin{aligned} i(\rho^p \circ \rho^q(A))\, V^{ij}_{\alpha}(\rho^p)\, V^{jk}_{\beta}(\rho^q) \\ = V^{ij}_{\alpha}(\rho^p)\, V^{jk}_{\beta}(\rho^q)\, k(A). \end{aligned} \tag{3.46}$$

We wish to compare the properties of $V^{ij}_{\alpha}(\rho^p)\, V^{jk}_{\beta}(\rho^q)$ to those of the operators

$$i\big(\Gamma^{\mathcal{S}}_{\rho^p \circ \rho^q, \rho^r}(\mu)\big)\, V^{ik}_{\nu}(\rho^r) \tag{3.47}$$

which, by (3.44), satisfy the same intertwining relations

$$
\begin{aligned}
i(\rho^p \circ \rho^q(A))\, i\big(\Gamma^{S}_{\rho^p \circ \rho^q, \rho^r}(\mu)\big)\; V^{ik}_{\nu}(\rho^r) \\
= i\big(\Gamma^{S}_{\rho^p \circ \rho^q, \rho^r}(\mu)\big)\; V^{ik}_{\nu}(\rho^r)\, k(A),
\end{aligned}
\tag{3.48}
$$

for all $A \in \mathcal{A}$, $\mu = 1, \ldots, N_{rpq}$. Equs. (3.46),(3.48) and Schur's lemma suggest that $V^{ij}_{\alpha}(\rho^p)\, V^{jk}_{\beta}(\rho^q)$ can be expanded in a sum over the operators $i\big(\Gamma^{S}_{\rho^p \circ \rho^q, \rho^r}(\mu)\big)\, V^{ik}_{\nu}(\rho^r)$. This expansion will be called fusion.

In order to make these ideas precise, we start with a special case of fusion: Consider the operators $V^{\bar{r}q}_{\alpha}(\rho^p)\, V^{\bar{q}1}(\rho^q)$ and $V^{\bar{r}1}(\rho^r)$. [We recall that to every representation $r \in L$ there exists a unique conjugate representation $\bar{r} \in L$ such that $\bar{r} \times r$ contains the vacuum representation, 1, precisely once; see Property 2.2, (P2), Sect. 2, after (2.27). From this one can conclude that

$$
V^{j1}(\rho^r) \;=\; 0, \text{ unless } j \;=\; \bar{r}, \tag{3.49}
$$

and that there exists precisely one partial isometry $V^{\bar{r}1}(\rho^r) : \mathcal{H}_1 \;\rightarrow\; \mathcal{H}_{\bar{r}}$ which is unique up to a phase.] By (3.42) and (3.44), there exist complex numbers, $\sigma_{\alpha}(r; p, q)$, $\alpha = 1, \ldots, N_{rpq}$, such that

$$
V^{\bar{r}q}_{\alpha}(\rho^p)\; V^{\bar{q}1}(\rho^q) \;=\; \sigma_{\alpha}(r; p, q)\, \bar{r}\big(\Gamma^{S}_{\rho^p \circ \rho^q, \rho^r}(\alpha)\big)\; V^{\bar{r}1}(\rho^r). \tag{3.50}
$$

Since we choose the operators $\{V^{ij}_{\alpha}(\rho^p)\}$ to be partial isometries, it follows that

$$
\|\bar{r}\big(\Gamma^{S}_{\rho^p \circ \rho^q, \rho^r}(\alpha)\big)\; V^{\bar{r}1}(\rho^r)\,\Omega\| \;=\; 1,
$$

where $\Omega \in \mathcal{H}_1$ is the vacuum vector. Hence

$$
\sigma_{\alpha}(r; p, q) \;=\; \big\langle \bar{r}\,\big(\Gamma^{S}_{\rho^p \circ \rho^q, \rho^r}(\alpha)\big)\; V^{\bar{r}1}(\rho^r)\,\Omega,\; V^{\bar{r}\bar{q}}_{\alpha}(\rho^p)\; V^{\bar{q}1}(\rho^q)\,\Omega \big\rangle. \tag{3.51}
$$

This formula is quite useful: Suppose that ρ^p and ρ^q are causally independent, i.e. the cones C^p and C^q are space-like separated. Then $\rho^p \circ \rho^q = \rho^q \circ \rho^p$, and hence we may choose the intertwining opertors $\Gamma^{S}_{\rho^p \circ \rho^q, \rho^r}(\alpha)$ and $\Gamma^{S}_{\rho^q \circ \rho^p, \rho^r}(\alpha)$ to be equal. Moreover, by Theorem 1, (3.9), and Lemma 2, part (2),

$$
\begin{aligned}
V^{\bar{r}\bar{q}}_{\alpha}(\rho^p)\; V^{\bar{q}1}(\rho^q)\,\Omega \\
= \sum_{\mu} R^{\pm}(\bar{r}, p, q, 1)^{\bar{p}\mu 1}_{\bar{q}\alpha 1}\; V^{\bar{r}\bar{p}}_{\mu}(\rho^q)\; V^{\bar{p}1}(\rho^p)\,\Omega
\end{aligned}
\tag{3.52}
$$

if $as(\rho^p) \gtrless as(\rho^q)$. Hence

$$\sigma_\alpha(r;p,q) = \sum_\mu R^\pm(\bar{r},p,q,1)^{\bar{p}\mu 1}_{\bar{q}\alpha 1}\, \sigma_\mu(r;q,p), \tag{3.53}$$

provided $\rho^p \bigtimes \rho^q$ and $as(\rho^p) \gtrless as(\rho^q)$.

We now state our basic result on the fusion of intertwiners.

Theorem 3 [28]

There exist matrices $\left(F(i,p,q,k)^{r\mu\nu}_{j\alpha\beta}\right)$ only depending on the representations i,p,q,k,j and r, but not on the specific choice of ρ^p, ρ^q and ρ^r, such that

$$\begin{aligned} &V^{ij}_\alpha(\rho^p)\, V^{jk}_\beta(\rho^q) \\ &= \sum_{r,\mu,\nu} F(i,p,q,k)^{r\mu\nu}_{j\alpha\beta}\, \sigma_\mu(r;p,q)\, i\big(\Gamma^{S}_{\rho^p\circ\rho^q,\rho^r}(\mu)\big)\, V^{ik}_\nu(\rho^r). \end{aligned} \tag{3.54}$$

The matrices $\left(F(i,p,q,k)^{r\mu\nu}_{j\alpha\beta}\right)$ can be expressed in terms of the matrices $\left(R^\pm(l,a,b,m)^{k\gamma\delta}_{n\lambda\kappa}\right)$.

□

We shall outline the main ideas going into the proof of Theorem 3: Let $\rho^{\bar{k}}$ be a *morphism of B^{C_a} localized in a space-like cone $C^{\bar{k}} \subset S$ and suppose that C^p, C^q and $C^{\bar{k}}$ are pairwise space-like separated. Let us suppose, for example, that $as(\rho^p),\ as(\rho^q) > as(\rho^{\bar{k}})$. We consider the operator $V^{ij}_\alpha(\rho^p)\, V^{jk}_\beta(\rho^q)\, k(A)\, V^{k1}(\rho^{\bar{k}})$, where A is an arbitrary element of $\mathcal{A}$, and apply the intertwining relations (3.8) and the commutation relations (3.9) between intertwiners. Then

$$\begin{aligned} &V^{ij}_\alpha(\rho^p)\; V^{jk}_\beta(\rho^q)\, k(A)\; V^{k1}(\rho^{\bar{k}}) \\ &\quad = i\big(\rho^p \circ \rho^q(A)\big)\; V^{ij}_\alpha(\rho^p)\; V^{jk}_\beta(\rho^q)\; V^{k1}(\rho^{\bar{k}}) \\ &\quad = \sum R^+(j,q,\bar{k},1)^{\bar{q}\mu 1}_{k\beta 1}\; i\big(\rho^p \circ \rho^q(A)\big)\; V^{ij}_\alpha(\rho^p)\; V^{j\bar{q}}_\mu(\rho^{\bar{k}})\; V^{\bar{q}1}(\rho^q) \\ &\quad = \sum R^+(j,q,\bar{k},1)^{\bar{q}\mu 1}_{k\beta 1}\; R^+(i,p,\bar{k},\bar{q})^{\bar{r}\gamma\nu}_{j\alpha\mu}\, i\big(\rho^p \circ \rho^q(A)\big) \\ &\qquad\qquad \times\; V^{i\bar{r}}_\gamma(\rho^{\bar{k}})\; V^{\bar{r}\bar{q}}_\nu(\rho^p)\; V^{\bar{q}1}(\rho^q) \\ &\quad = \sum R^+(j,q,\bar{k},1)^{\bar{q}\mu 1}_{k\beta 1}\; R^+(i,p,\bar{k},\bar{q})^{\bar{r}\gamma\nu}_{j\alpha\mu}\, \sigma_\nu(r;p,q) \\ &\qquad\qquad \times\; i\big(\rho^p \circ \rho^q(A)\big)\; V^{i\bar{r}}_\gamma(\rho^{\bar{k}})\, \bar{r}\big(\Gamma^{S}_{\rho^p\circ\rho^q,\rho^r}(\nu)\big)\; V^{\bar{r}1}(\rho^r), \end{aligned} \tag{3.55}$$

and we have used (3.50). Let $S_{p,q} \subset S$ be a simple domain containing the cones C^p and C^q and space-like separated from $C^{\bar{k}}$. We may choose ρ^r to be localized in $S_{p,q}$.

Then $\Gamma^{S}_{\rho^p \circ \rho^q, \rho^r}(\nu) \in \overline{\mathcal{A}(S_{p,q})}^{w}$, and hence

$$V_{\gamma}^{i\bar{r}}(\rho^{\bar{k}})\,\bar{r}\big(\Gamma^{S}_{\rho^p \circ \rho^q, \rho^r}(\nu)\big) \;=\; i\big(\Gamma^{S}_{\rho^p \circ \rho^q, \rho^r}(\nu)\big)\; V_{\gamma}^{i\bar{r}}(\rho^{\bar{k}}),$$

by (3.8). We therefore derive from (3.55) that

$$\begin{aligned}
&V_{\alpha}^{ij}(\rho^p)\; V_{\beta}^{jk}(\rho^q)\, k(A)\; V^{k1}(\rho^{\bar{k}}) \\
&\quad = \sum R^{+}(j,q,\bar{k},1)_{k\beta 1}^{\bar{q}\mu 1}\; R^{+}(i,p,\bar{k},\bar{q})_{j\alpha\mu}^{\bar{r}\gamma\nu}\; R^{-}(i,\bar{k},r,1)_{\bar{r}\gamma 1}^{k\delta 1} \\
&\quad \times\; \sigma_{\nu}(r;p,q)\, i\big(\rho^p \circ \rho^q(A)\big)\, i\big(\Gamma^{S}_{\rho^p \circ \rho^q, \rho^r}(\nu)\big)\; V_{\delta}^{ik}(\rho^r)\; V^{k1}(\rho^{\bar{k}}).
\end{aligned} \tag{3.56}$$

From the intertwining relations (3.44) and (3.8) we derive that

$$\begin{aligned}
&i\big(\rho^p \circ \rho^q(A)\big)\; i\big(\Gamma^{S}_{\rho^p \circ \rho^q, \rho^r}(\nu)\big)\; V_{\delta}^{ik}(\rho^r) \\
&\qquad = i\big(\Gamma^{S}_{\rho^p \circ \rho^q, \rho^r}(\nu)\big)\; i\big(\rho^r(A)\big)\; V_{\delta}^{ik}(\rho^r) \\
&\qquad = i\big(\Gamma^{S}_{\rho^p \circ \rho^q, \rho^r}(\nu)\big)\; V_{\delta}^{ik}(\rho^r)\, k(A).
\end{aligned} \tag{3.57}$$

Combining (3.56) and (3.57) we find that

$$\begin{aligned}
&V_{\alpha}^{ij}(\rho^p)\; V_{\beta}^{jk}(\rho^q)\, k(A)\; V^{k1}(\rho^{\bar{k}}) \\
&\quad = \sum R^{+}(j,q,\bar{k},1)_{k\beta 1}^{\bar{q}\mu 1}\; R^{+}(i,p,\bar{k},\bar{q})_{j\alpha\mu}^{\bar{r}\gamma\nu}\; R^{-}(i,\bar{k},r,1)_{\bar{r}\gamma 1}^{k\delta 1} \\
&\quad \times\; \sigma_{\nu}(r;p,q)\; i\big(\Gamma^{S}_{\rho^p \circ \rho^q, \rho^r}(\nu)\big)\; V_{\delta}^{ik}(\rho^r)\, k(A)\; V^{k1}(\rho^{\bar{k}}).
\end{aligned} \tag{3.58}$$

Next, we note that

$$\big\{k(A)\; V^{k1}(\rho^{\bar{k}})\xi \;:\; A \in \mathcal{A},\; \xi \in \mathcal{H}_1\big\}$$

is dense in $\mathcal{H}_k$. Thus (3.58) proves (3.54), with

$$\begin{aligned}
F(i,p,q,k)_{j\alpha\beta}^{r\nu\delta} \;=\; &\sum_{\mu,\gamma} R^{+}(j,q,\bar{k},1)_{k\beta 1}^{\bar{q}\mu 1}\; R^{+}(i,p,\bar{k},\bar{q})_{j\alpha\mu}^{\bar{r}\gamma\nu} \\
&\times\; R^{-}(i,\bar{k},r,1)_{\bar{r}\gamma 1}^{k\delta 1}.
\end{aligned} \tag{3.59}$$

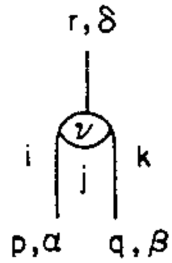
By choosing $as(\rho^{\bar{k}}) > as(\rho^p)$, $as(\rho^q)$, we find that

$$\begin{aligned}
F(i,p,q,k)_{j\alpha\beta}^{r\nu\delta} \;=\; &\sum_{\mu,\gamma} R^{-}(j,q,\bar{k},1)_{k\beta 1}^{\bar{q}\mu 1}\; R^{-}(i,p,\bar{k},\bar{q})_{j\alpha\mu}^{\bar{r}\gamma\nu} \\
&\times\; R^{+}(i,\bar{k},r,1)_{\bar{r}\gamma 1}^{k\delta 1}.
\end{aligned} \tag{3.60}$$

We must ask whether (3.59) and (3.60) are consistent? The reader verifies without the slightest difficulty that the consistency of (3.59) and (3.60) follows from the fundamental identity (3.34).

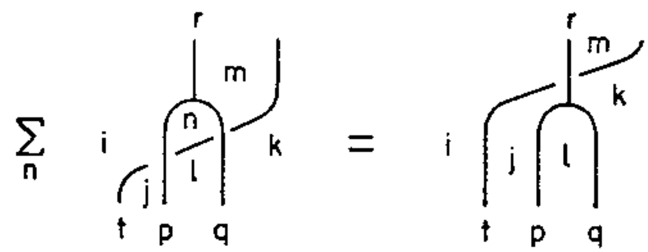
In order to discuss further properties of the fusion matrices $\left(F(i,p,q,k)^{r\nu\delta}_{j\alpha\beta}\right)$ it is helpful to introduce a graphical notation:

We denote $F(i,p,q,k)^{r\nu\delta}_{j\alpha\beta}$ by



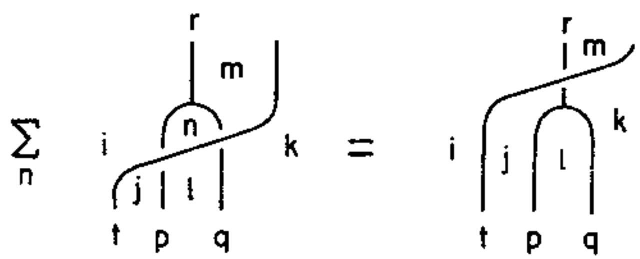
(3.61)

As in conformal field theory [19,36] it is easy to derive the following "polynomial equations":



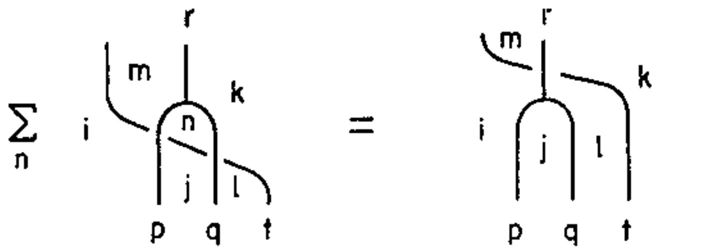
(3.62)

where we have used notation (3.17). Similarly,



(3.63)

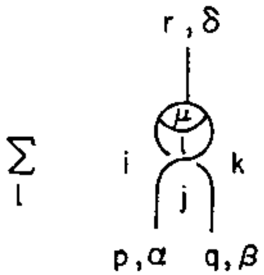
where we have used (3.16), and



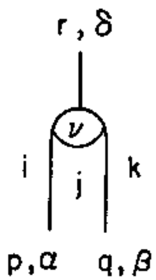
(3.64)

etc.. In (3.62)-(3.64) and henceforth we omit the Greek indices $\alpha, \beta, \nu, \delta, \ldots$.

Furthermore, from (3.9),(3.53) and (3.54) one easily derives [28,37] that



$$= \sum_{\nu} R^{+}(\bar{r}, p, q, 1)^{\bar{p}\nu 1}_{\bar{q}\mu 1}$$



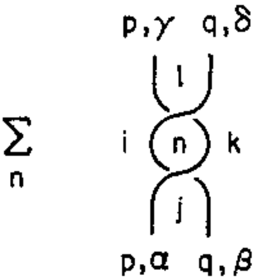
(3.65)

A similar equation holds with R^- replacing R^+.

Next, we introduce the monodromy matrices

$$M(i,p,q,k)^{l\gamma\delta}_{j\alpha\beta} = \sum_{n\mu\nu} R^{+}(i,p,q,k)^{n\mu\nu}_{j\alpha\beta}\, R^{+}(i,q,p,k)^{l\gamma\delta}_{n\mu\nu} \,. \qquad (3.66)$$

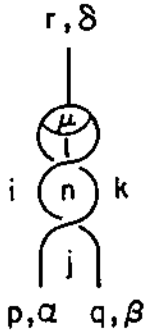
Graphically,



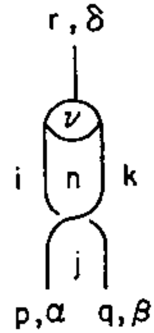
$$\longleftrightarrow \quad M(i,p,q,k)^{l\gamma\delta}_{j\alpha\beta} \,. \qquad (3.67)$$

Iterating (3.65) we find that

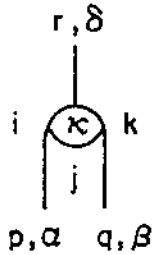
$$\sum_{n,l}$$



$$= \sum_{n,\nu} R^{+}(\bar{r},q,p,1)^{\bar{q}\nu 1}_{\bar{p}\mu 1}$$



$$= \sum_{n,\nu,\kappa} R^{+}(\bar{r},q,p,1)^{\bar{q}\nu 1}_{\bar{p}\mu 1}\, R^{+}(\bar{r},p,q,1)^{\bar{p}\kappa 1}_{\bar{q}\nu 1} \tag{3.68}$$



Our fundamental identity (3.34) says that

$$R^{+}(\bar{r},p,q,1)^{\bar{p}\kappa 1}_{\bar{q}\nu 1} \;=\; e^{2\pi i(s_q+s_p-s_r)}\, R^{-}(\bar{r},p,q,1)^{\bar{p}\kappa 1}_{\bar{q}\nu 1}\,, \tag{3.69}$$

where we have used that $s_1 = 0\ mod.\, Z$, and (3.18) says that

$$\sum_{\nu} R^{+}(\bar{r},q,p,1)^{\bar{q}\nu 1}_{\bar{p}\mu 1}\, R^{-}(\bar{r},p,q,1)^{\bar{p}\kappa 1}_{\bar{q}\nu 1} \;=\; \delta^{\kappa}_{\mu}\,. \tag{3.70}$$

By combining (3.68), (3.69) and (3.70) we find that

$$\boxed{\sum_{l\gamma\delta} M(i,p,q,k)^{l\gamma\delta}_{j\alpha\beta}\, F(i,p,q,k)^{r\mu\nu}_{l\gamma\delta} \;=\; e^{2\pi i(s_p+s_q-s_r)}\, F(i,p,q,k)^{r\mu\nu}_{j\alpha\beta}\,,} \tag{3.71}$$

where we have also used that $s_j = s_{\bar{j}}$, for all $j \in L$.

3.3 Spin spectrum, spin addition rules, spin-statistics

As shown in [37] and refs. given there, equ. (3.71) has rather interesting consequences:

(1) The fusion matrices $F(i,p,q,k)^{r\mu\nu}_{l\gamma\delta}$ diagonalize the monodromy matrices $\left(M(i,p,q,k)^{l\gamma\delta}_{j\alpha\beta}\right)$.

(2) The spectrum of $M(i,p,q,k)$ is given by $\left\{e^{2\pi i(s_p+s_q-s_r)} : r \in L,\ N_{rpq} \neq 0\right\}$.

(3) Let us assume that we are dealing with a theory which has only finitely many distinct superselection sectors, i.e., $|L| < \infty$. Then we have the following result.

<u>Theorem 4</u> [37]

<u>If the number,</u> $|L|$, <u>of superselection sectors is finite all the spins</u> $s_j, j \in L$, <u>are rational numbers</u>.

In analogy with conventional jargon in conformal field theory [38], field theories in three space-time dimensions with only finitely many distinct superselection sectors, $|L| < \infty$, are called <u>rational</u> theories.

(4) Next, we consider a three-dimensional theory with <u>permutation group statistics</u>, i.e.,

$$R^+(j,p,q,k) = R^-(j,p,q,k) . \tag{3.72}$$

Then, using (3.18) one concludes that all monodromy matrices are trivial, i.e.,

$$M(i,p,q,k)^{l\gamma\delta}_{j\alpha\beta} = \delta^l_j\, \delta^\gamma_\alpha\, \delta^\delta_\beta , \tag{3.73}$$

and hence all their eigenvalues are equal to 1,

$$e^{2\pi i(s_p+s_q-s_r)} = 1 , \tag{3.74}$$

for all p, q and r for which $N_{rpq} \neq 0$.

If $q = \bar{p}$ then $s_q = s_p$ and $N_{1pq} \neq 0$. In this case (3.74) implies that

$$e^{4\pi i s_p} = 1, \text{ for all } p , \tag{3.75}$$

or, equivalently,

$$\boxed{s_p \in \frac{1}{2}\mathbb{Z}, \text{ for all } p.} \tag{3.76}$$

We define "spin parity", σ_p, by setting

$$\sigma_p = e^{2\pi i s_p},\ p \in L . \tag{3.77}$$

Since, by (3.74), (3.72) implies that

$$s_p + s_q - s_r \in \mathbb{Z} ,$$

for all p, q and r satisfying the fusion rules, i.e., $N_{rpq} \neq 0$, we conclude, using (3.76), that

$$\sigma_p \sigma_q = \sigma_r , \tag{3.78}$$

for p, q and r such that $N_{rpq} \neq 0$; in words: spin parity is conserved under fusion.

Conversely, let us assume that we consider a theory with the property that

$$s_p \in \frac{1}{2}\mathbb{Z}, \text{ for all } p \in L, \text{ and} \tag{3.79}$$

$\sigma_p \sigma_q = \sigma_r$, for all p, q and r for which $N_{rpq} \neq 0$.

Then, by (3.74)

$$M(i,p,q,k)^{l\gamma\delta}_{j\alpha\beta} = \delta^l_j \, \delta^\gamma_\alpha \, \delta^\delta_\beta \tag{3.80}$$

for all i, p, q, k, j and l satisfying the fusion rules. Hence

$$R^+(i,p,q,k) = R^-(i,p,q,k), \tag{3.81}$$

i.e. the theory has standard permutation group statistics.

We summarize these findings in a theorem.

Theorem 5 [28]

The conditions

(i) $R^+(i,p,q,k) = R^-(i,p,q,k)$, for all i,p,q,k, and

(ii) $s_p \in \frac{1}{2}\mathbb{Z}$, for all $p \in L$, and $\sigma_p \cdot \sigma_q = \sigma_r$, for all p,q,r for which $N_{rpq} \neq 0$, are equivalent.

(5) The analysis developed in Sects. 2 and 3 can also be applied to theories in $d \geq 4$ space-time dimensions or to theories in three space-time dimensions with superselection sectors generated by *morphisms of an algebra $\mathcal{A}$ of quasi-local observables localized in bounded space-time regions, as described in [4]. In both cases, one shows easily (see also [4,5]) that

$$R^+(i,p,q,k) = R^-(i,p,q,k),$$

for all i, p, q, k. Hence all sectors of such theories have integral or half-integral spin and spin parity is conserved in such theories. [The fact that, in such theories, $s_j \in \frac{1}{2}\mathbb{Z}$, for all $j \in L$, can also be derived from the structure of the Poincaré group ($d \geq 4$)

and from locality and the relativistic spectrum condition $(d = 3)$.]

A fundamental theorem due to Doplicher and Roberts [6] says that, in a theory with standard permutation group statistics, the fusion rules (N_{rpq}) can be derived from the representation theory of some compact group, G , and, by introducing internal degrees of freedom on which the representations of G act, the permutation group statistics can be reduced to ordinary Bose- and Fermi statistics.

(6) It follows from (3.66) and (3.71) that the 1×1 matrix $M(1,p,\bar{p},1)_p^p = (M(1,p,\bar{p},1)_{p11}^{p11})$ is given by

$$\begin{aligned} M(1,p,\bar{p},1)_p^p &= R^+(1,p,\bar{p},1)_p^{\bar{p}} \ R^+(1,\bar{p},p,1)_{\bar{p}}^p \\ &= e^{4\pi i s_p} \ , \end{aligned} \tag{3.82}$$

and we have used that $s_{\bar{p}} = s_p$. In a relativistically covariant theory, a field-theoretic argument suggests that *

$$R^+(1,p,\bar{p},1)_p^{\bar{p}} = R^+(1,\bar{p},p,1)_{\bar{p}}^p = e^{2\pi i s_p} \ . \tag{3.83}$$

By definition of the angle θ_p , see (3.37),

$$R^+(1,p,\bar{p},1)_p^{\bar{p}} = e^{2\pi i\theta_p} \ .$$

Thus

$$s_p = \theta_p\,, \ mod.\ \mathbb{Z}\,, \tag{3.84}$$

for all p . This connection between spin and statistics strengthens the one found in (3.41). Equ. (3.84) is well known for theories with standard permutation group statistics [2]. For more details see [28].

In all examples known to us, the matrices $R^{\pm}(i,p,q,k)_j^l$ and $F(i,p,q,k)_j^r$ can be obtained from the representation theory of some group $(R^+ = R^-$, [6]) or some quantum group $(R^+ \neq R^-)$ with the help of the socalled *vertex* $-$ *sos* transformation: see Sect. 4. By analogy with the results of Doplicher and Roberts [6], one might conjecture that this is always true. In these cases, the spectrum $\left\{e^{2\pi i(s_p+s_q-s_r)} : N_{rpq} \neq 0\right\}$ of the monodromy matrices, $M(i,p,q,k)_j^l$, is com-

* *Under somewhat stronger hypotheses on the structure of the theory, (3.83) can be proven.*

pletely determined by the representation theory of the quantum group. This ob-servation and equ. (3.84) permit us to calculate all spins s_p *mod.* $\mathbb{Z}$, for all $p \in L$.

Let us briefly consider the example of a quantum deformation, $U_q(\mathcal{G})$, of the universal enveloping algebra of a classical Lie algebra $\mathcal{G}$, with $q^N = 1$, for some integer $N \geq 2$.

Then one has the formula

$$s_p = \pm\frac{1}{2N}\left(\langle\nu_p,\nu_p\rangle + \langle\rho,\nu_p\rangle\right) + s_p^{(a)} \text{ mod. } \mathbb{Z}\,, \tag{3.85}$$

where ν_p is the highest weight of some finite-dimensional, unitarizable, irreducible highest-weight representation of $U_q(\mathcal{G})$, ρ is the sum of positive roots of $\mathcal{G}$, and $s_p^{(a)}$ is a contribution to s_p that comes from an abelian factor in the braid group statistics of the theory. (For example, $s_p^{(a)} = \frac{1}{2}n_p, n_p = 0,1$, would describe a contribution to s_p due to an ordinary Fermi field.) Below, the structure of $s_p^{(a)}$ is described completely. The proof of (3.85) is obtained by comparing equ. (3.71) with a formula in [45]. See Sect. 4 for more details concerning connections between our theory and quantum group theory.

The example of abelian braid group statistics is elementary and can be analysed completely. It has interesting applications in quantum field theory and condensed matter physics. The results reviewed below have been conjectured in [26] on the basis of an analysis of concrete models describing anyons. As shown in [28], abelian braid group statistics implies that all representations $p \in L$ are unitarily equivalent to representations $1(\rho^p(\,\cdot\,))$ of $\mathcal{A}$, where ρ^p is a *automorphism of $\mathcal{B}^{\mathcal{C}_a}$. Then $\bar{p}$ is equivalent to $1((\rho^p)^{-1}(\,\cdot\,))$, hence $\bar{p} \times p = 1$, and every power $p^{\times n} = p \times \cdots \times p$ of the representation p belongs to L. The set $\{p^{\times n} : n \in \mathbb{Z}\}$ is a subset of L invariant under composition, whose fusion rules are described by $\mathbb{Z}$. The braid matrices of such theories have the form [28].

$$R^{\pm}(j,p,q,k)_l^m = \begin{cases} e^{\pm 2\pi i \theta_{p,q}}, & \text{if all fusion rules are satisfied;} \\ 0\,, & \text{otherwise.} \end{cases} \tag{3.86}$$

It then follows from equ. (3.64), (with $q = \bar{p}$, $t = p$, $r = 1$, $k = 1$, $l = \bar{p}$, $j = 1$, $i = \bar{p}$ and $m = 1$) that

$$R^+(1,\bar{p},p,1)_{\bar{p}}^p\, R^+(\bar{p},p,p,p)_1^1 = 1\,,$$

i.e.

$$e^{2\pi i \theta_{\bar{p},p}}\; e^{2\pi i \theta_{p,p}} \;=\; 1\,, \tag{3.87}$$

for all $p \in L$. Previously, $\theta_{\bar{p},p}$ has been denoted by θ_p which was shown to be given by s_p *mod.* $\mathbb{Z}$; see (3.84). Hence (3.87) and (3.84) yield

$$\boxed{\theta_{p,p} \;=\; -\,\theta_{\bar{p},p} \;=\; -\,s_p \;\; mod.\, \mathbb{Z}\,,} \tag{3.88}$$

for all $p \in L$. By (3.86),

$$R^{+}(\bar{p}\,\times\,\bar{p},p,p,1)^{\bar{p}}_{\bar{p}} \;=\; e^{2\pi i \theta_{p,p}}\,. \tag{3.89}$$

Furthermore (3.66) and (3.71) show that

$$\left(R^{+}(\bar{p}\,\times\,\bar{p},p,p,1)^{\bar{p}}_{\bar{p}}\right)^2 \;=\; e^{2\pi i (2\,s_p - s_{p\times p})}\,. \tag{3.90}$$

Combining (3.88)-(3.90), we conclude that

$$s_{p\times p} \;=\; 4\,s_p \;\; mod.\, \mathbb{Z}\,, \tag{3.91}$$

for all $p \in L$. Iterating these arguments, one finds that

$$\boxed{s_{p^{\times n}} \;=\; n^2\, s_p \;\; mod.\, \mathbb{Z}\,,} \tag{3.92}$$

for all $p \in L$.

Equs. (3.86), (3.88) and (3.92) represent a completely general, model-independent proof of relations conjectured in [26] on the basis of an analysis of specific models describing anyons.

In the non-abelian case, results analogous to (3.92) can be proven by using (3.39) and the polynomial equations (3.64), provided the fusion matrices $\{F(i,p,q,k)^{r}_{j}\}$ can be calculated without using equs. (3.59) or (3.60). This is the case if, for example, the matrices $R^{\pm}$ and F can be derived from the representation theory of some quantum group via the vertex-sos transformation; see [28,37].

Finally, we wish to point out that the norms, $\|\mathsf{N}_p\|$, of the multiplicity matrices, N_p, defined by $(\mathsf{N}_p)_{rq} \;=\; N_{rpq}$ (fusion rules) define "statistical dimensions" which can be interpreted in terms of indices of subfactors in the sense of Jones, [18,39].

Further results are discussed in [28].

4. Connections to knot theory and to quantum group theory

Let us first draw attention to a deep connection between the theory developed in Sect. 3 and the theory of knots and links in $S^3 = \dot{\mathbb{R}}^3$. This connection is the same as one described in detail in [40] in the context of conformal field theory; (see also [37]). The point is that from a family of matrices $\{R^{\pm}(i,p,q,k)^l_j,\ F(i,p,q,k)^r_j\}$ satisfying

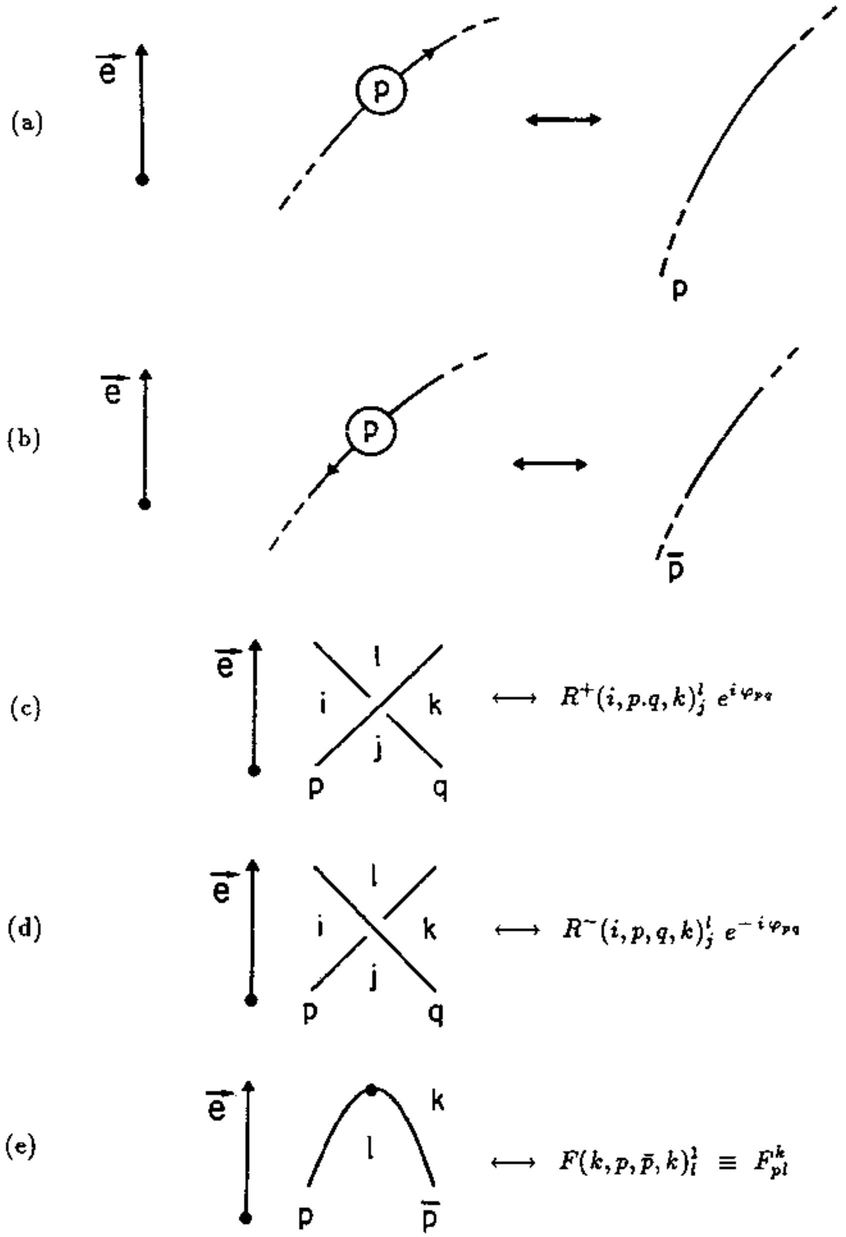
(a) equs. (3.18) and (3.22) (YBE);

(b) equs. (3.62) - (3.65); equs. (3.83);

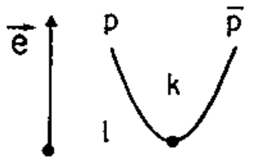
one can derive a family of invariants for oriented knots and links in S^3, by an explicit construction described in detail in [40]. Furthermore, as was sketched in [40], one can derive a family of invariants for tri-valent ribbon graphs.

We briefly sketch the construction of invariants for knots and links from $\{R^{\pm}, F\}$: In $\dot{\mathbb{R}}^3$ we choose a two-dimensional plane, π, a unit vector $\vec{n}$ normal to π and a unit vector $\vec{e} \in \pi$. Given an oriented link, $\mathcal{L}$, in $\mathbb{R}^3$, we choose a representative of $\mathcal{L}$, (i.e. a system of loops in $\dot{\mathbb{R}}^3$ representing $\mathcal{L}$), which has a non-degenerate projection along $\vec{n}$ onto π. Such a projection is called a shadow of $\mathcal{L}$. If under- and overcrossings in the ordering fixed by $\vec{n}$, are recorded on the crossings of lines in the shadow of $\mathcal{L}$ we speak of a diagram, $D(\mathcal{L})$, of $\mathcal{L}$. The link $\mathcal{L}$ can obviously be reconstructed from $\vec{n}$ and $D(\mathcal{L})$. The diagram $D(\mathcal{L})$ is marked as follows: If the coordinate function in the direction of $\vec{e}$ on π has a local maximum or a local minimum at a point $p \in D(\mathcal{L})$ then p is marked by a dot:

Next, a marked diagram, $D(\mathcal{L})$, is decorated as follows: The shadow of $\mathcal{L}$ on π decomposes π into disjoint regions $\Omega_1, \ldots, \Omega_N$, where Ω_1 is defined to be the region containing the point at infinity, and $N \geq 2$ is determined by $D(\mathcal{L})$. Then we assign to every region Ω_l a representation k_l of $\mathcal{A}$ belonging to the list L introduced in Sect. 2. Without loss of generality, we may assign the vacuum representation, 1, to Ω_1. A <u>component</u> of $D(\mathcal{L})$ is the projection of an <u>oriented</u> loop (connected component) of the link in $\dot{R}^3$ onto π. Every component, C_i, of $D(\mathcal{L})$ is assigned a representation $j_i \in L$. The resulting marked, decorated diagram of $\mathcal{L}$ is denoted by $\mathcal{D}(\mathcal{L}; j_1, \ldots, j_n; k_2, \ldots, k_N)$; ($n$ = # of components of $\mathcal{L}$). The <u>elements</u> of a decorated diagram of $\mathcal{L}$ are defined in Fig. 7 and are assigned matrices R^+, R^- or F, as shown in Fig. 7, (a)-(f):



(f)



$\longleftrightarrow \quad (F^k_{pl})^{-1}$

Fig. 7

The angles φ_{pq} are given by

$$\varphi_{pq} \;=\; 2\pi\sqrt{s_p \cdot s_q}\,. \tag{4.1}$$

By $(F^k_{pl})^{-1}$ is meant the inverse of the matrix F^k_{pl} with matrix elements

$$\left(F^k_{pl}\right)_{\alpha\beta} \;=\; F\left(k,p,\bar{p},k\right)^{1\,1\,1}_{l\alpha\beta}\,. \tag{4.2}$$

With $\mathcal{D}(\mathcal{L};j_1,\ldots j_n;k_2,\ldots k_N)$ we associate a complex number $i(\mathcal{L};\ j_1,\ldots,j_n;\ k_2,\ldots,k_N)$ by calculating the sum of products of matrix elements of the matrices $(R^{\pm},F,F^{-1})$ associated to all the elements of $\mathcal{D}(\mathcal{L};j_1,\ldots,j_n;\,k_2,\ldots,k_N)$ by the rules specified in Fig. 7, (a) - (f).

We may now quote one of the main results of [40] (Theorem 6.1): Let $\mathcal{L}$ be an oriented link in S^3 with diagram $D(\mathcal{L})$. The component, i, of $\mathcal{L}$ projected onto C_i is assigned the "colour" $j_i \in L$. With an oriented, coloured link $(\mathcal{L};j_1,\ldots,j_n)$ we associate a complex number

$$I(\mathcal{L};j_1,\ldots,j_n) \;=\; \sum_{k_2,\ldots,k_N\in L} i(\mathcal{L};j_1,\ldots,j_n;k_2,\ldots,k_N)\,. \tag{4.3}$$

Then we have the following result

Theorem 6 The numbers $I(\mathcal{L};j_1,\ldots,j_n)$ define invariants for oriented, coloured links in S^3.

Remark. The proof is identical to the proof of Theorem 6.1 in [40]. Since it is somewhat lengthy, we shall not repeat it here.

It has been outlined in [40] how, from the same data $\{R^{\pm},F\}$ one can, in principle, construct invariants for oriented, coloured links embedded in general three-dimensional manifolds. See also [42] for related results.

Next, we wish to briefly address the question of connections between the theory developed in Sects. 2 and 3 and quantum group theory. This question has been discussed in the context of two-dimensional conformal field theory in [40,37,43]. The results, or better: speculations, found there carry over to the present framework essentially without change.

Let $\mathcal{K}$ be a Hopf algebra with co-multiplication $\triangle : \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ and universal R-matrix $\mathcal{R} \in \mathcal{K} \otimes \mathcal{K}$. Let I be a list of finite-dimensional, unitarizable, irreducible highest-weight representations of $\mathcal{K}$. Comultiplication, $\triangle$, is what is needed to define tensor product representations of $\mathcal{K}$: For i and j in I we define the representation $i \otimes j$ of $\mathcal{K}$ by

$$i \otimes j : \ A \in \mathcal{K} \longrightarrow i \otimes j\big(\triangle(A)\big) \in \ End(V_i \otimes V_j), \tag{4.4}$$

where V_i is the representation space of $i \in I$. Let $P_{ij} : V_i \otimes V_j \to V_j \otimes V_i$ be the transposition operator. We define

$$R_{ij} \ = \ P_{ij}\, i \otimes j\,(\mathcal{R}). \tag{4.5}$$

The matrices $\{R_{ij}\}$ are Yang-Baxter matrices intertwining $i \otimes j$ with $j \otimes i$, i.e.

$$j \otimes i\big(\triangle(A)\big)\, R_{ij} \ = \ R_{ij}\, i \otimes j\,\big(\triangle(A)\big). \tag{4.6}$$

Since $\mathcal{K}$ will in general not be a semi-simple algebra, it need not be possible to decompose a tensor product representation $i \otimes j$ of $\mathcal{K}$ into a direct sum of representations in I. But $i \otimes j$ may contain some $k \in I$ as a subrepresentation. We let N_{kij} denote the multiplicity of k in $i \otimes j$. There is then a basis of Clebsch-Gordan matrices

$$P_{k,ij}(\mu) : \ V_i \otimes V_j \ \longrightarrow \ V_k,\ \mu = 1, \ldots, N_{kij}\,, \tag{4.7}$$

intertwining k with $i \otimes j$, i.e.,

$$k(A)\ P_{k,ij}\,(\mu) \ = \ P_{k,ij}\,(\mu)\, i \otimes j\,\big(\triangle(A)\big). \tag{4.8}$$

Experience with the representation theory of some simple quantum groups suggests to introduce the following assumption on the structure of $\{\mathcal{K}, I\}$: Define a representation $i \otimes j \otimes l, i, j, l \in I$, by

$$i \otimes j \otimes l : \ A \mapsto i \otimes j \otimes l\big((\triangle \otimes \mathbb{1})\, \triangle\,(A)\big). \tag{4.9}$$

Let N_{kijl} be the multiplicity of $k \in I$ in $i \otimes j \otimes l$. Then

$$\text{(A1)} \quad N_{kijl} \ = \ \sum_{r \in I} N_{kir}\, N_{rjl}\,. \tag{4.10}$$

Let $P_{k,ijl}(\lambda)$, $\lambda = 1, \dots, N_{kijl}$, be a basis of Clebsch-Gordan matrices intertwining k with $i \otimes j \otimes l$.

(A2) We assume that every $P_{k,ijl}(\lambda)$ can be represented as

$$P_{k,ijl}(\lambda) = \sum_{r \in I, \mu, \nu} c(\lambda, \mu, \nu; r)\, P_{k,ir}(\mu)\, P_{r,jl}(\nu)\,, \tag{4.11}$$

for some coefficients $c(\lambda, \mu, \nu; r) \in \mathbf{C}$.

Now we can describe the vertex-sos transformation mentioned at the end of Sect. 3: We consider two further families of N_{kijl} Clebsch-Gordan-matrices intertwining k with $i \otimes j \otimes l$:

$$\begin{aligned} &\text{(i)} \quad P_{k,jr}(\alpha)\, P_{r,il}(\beta)\, (R_{ij}^{\pm 1} \otimes \mathbb{1}_l)\,, \\ &\text{(ii)} \quad P_{k,rl}(\gamma)\, P_{r,ij}(\delta)\,, \end{aligned} \tag{4.12}$$

with $r \in I$. By assumption (A2), these matrices can be expanded in the basis (4.11):

$$\begin{aligned} &P_{k,jr}(\alpha)\, P_{r,il}(\beta)\, (R_{ij}^{\pm 1} \otimes \mathbb{1}_l) \\ &\qquad = \sum_{p\mu\nu} \rho^{\pm}(k,i,j,l)_{p\mu\nu}^{r\alpha\beta}\, P_{k,ip}(\mu)\, P_{p,jl}(\nu)\,, \end{aligned} \tag{4.13}$$

and

$$P_{k,rl}(\gamma)\, P_{r,ij}(\delta) = \sum_{p\mu\nu} \varphi(k,i,j,l)_{p\mu\nu}^{r\gamma\delta}\, P_{k,ip}(\mu)\, P_{p,jl}(\nu)\,, \tag{4.14}$$

for complex numbers $\rho^{\pm}(k,i,j,l)_{p\mu\nu}^{r\alpha\beta}$ and $\varphi(k,i,j,l)_{p\mu\nu}^{r\gamma\delta}$. By arguments very similar to those used in Sect. 3, one can show that the matrices $\rho^{\pm}(k,i,j,l)$ and $\varphi(k,i,j,l)$ satisfy polynomial equations analogous to (3.18),(3.22),(3.62)-(3.65); see [45,40]. Thus a Hopf algebra $\mathcal{K}$ and the set I of finite-dimensional, unitarizable, irreducible highest-weight representations, subject to assumptions (A1) and (A2), determine "six-index symbols" $\rho^{\pm}(k,i,j,l)_p^r$ and $\varphi(k,i,j,l)_p^r$ which have the same properties as $R^{\pm}(k,i,j,l)_p^r$ and $F(k,i,j,l)_p^r$.

The general problem one would like to solve is to find conditions on families of matrices $\{\rho^{\pm}(k,i,j,l)_p^r,\ \varphi(k,i,j,l)_p^r\}$, satisfying equs. (3.18),(3.22) and (3.62)-(3.65), which guarantee that these matrices are derivable from a Hopf algebra via the vertex-sos transformations (4.13),(4.14). This problem appears to be open.

Let us now suppose that there is a bijection from L to I taking the vacuum representation, 1, of $\mathcal{A}$ to the trivial representation of $\mathcal{K}$, also denoted by **1**, and preserving the fusion rules $\{N_{kij}\}$. A point in L and its image in I will then be denoted by the same lower-case Latin letter. Let us further suppose that

$$R^{\pm}(k,i,j,l)^{r}_{p} \;=\; \rho^{\pm}(k,i,j,l)^{r}_{p}\,, \tag{4.15}$$

and

$$F(k,i,j,l)^{r}_{p} \;=\; \varphi(k,i,j,l)^{r}_{p}\,. \tag{4.16}$$

Then the algebra $\mathcal{K}$ plays the rôle of a "<u>global symmetry algebra</u>" of the quantum theory described by $\{\mathcal{A}, L\}$, which acts trivially on all observables in $\mathcal{A}$ but can be made to act <u>non-trivially</u> on the <u>unobservable, charged fields of the theory</u>. This is analogous to what has been found in [6]. In order to make these remarks more concrete, we define "$\mathcal{K}-vertices$" $v^{ki}_{\mu}(x^j) : V_i \to V_k$, for $x^j \in V_j$, by setting

$$v^{ki}_{\mu}(x^j)\,x \;\equiv\; P_{k,ji}(\mu)\,(x^j \otimes x)\,, \tag{4.17}$$

for all $x \in V_i$, $\mu = 1,\ldots,N_{kji}$.

Let D_{ipj} be the "structure constants" discussed in Sect. 4 of [40], and refs. given there. Then we may introduce new charged fields, $\psi(\rho^p, x^p)$, $\rho^p \in \mathcal{M}_p$, $x^p \in V_p$, by setting

$$\psi(\rho^p, x^p) \;=\; \sum_{ij\mu\nu} (D_{ipj})^{\mu\nu}\; V^{ij}_{\mu}(\rho^p)\; v^{ij}_{\nu}(x^p)\,. \tag{4.18}$$

Let ρ^p and ρ^q be two *morphisms localized in space-like separated, space-like cones, C^p and C^q, respectively, contained in a simple domain $S \subset \mathbb{M}^3$. Let $\{e^p_\alpha\}$ and $\{e^q_\beta\}$ be bases for V_p, V_q, respectively, and set

$$\psi_\alpha(\rho^p) \;\equiv\; \psi(\rho^p, e^p_\alpha),\;\; \psi_\beta(\rho^q) \;\equiv\; \psi(\rho^q, e^q_\beta).$$

Then the <u>commutation relations</u> between $\psi_\alpha(\rho^p)$ ans $\psi_\beta(\rho^q)$ are given by

$$\psi_\alpha(\rho^p)\,\psi_\beta(\rho^q) \;=\; \sum_{\gamma\delta} \left(R^{\pm 1}_{pq}\right)_{\alpha\beta,\gamma\delta}\, \psi_\gamma(\rho^q)\,\psi_\delta(\rho^p)\,, \tag{4.19}$$

if $as(\rho^p) \gtrless as(\rho^q)$. Here $(R^{\pm 1}_{pq})_{\alpha\beta,\gamma\delta}$ are the matrix elements of $R^{\pm 1}_{pq}$, defined in (4.5), in the bases $\{e^p_\alpha\}$ and $\{e^q_\beta\}$. See [37,43] for details. It will be shown elsewhere how to construct a *algebra out of the fields $\{\psi_\alpha(\rho^p)\}$.

Remarks

(1) It is clear how to define an action of $\mathcal{K}$ on the fields $\psi(\rho^p, x^p)$.

(2) If we work with the fields $\psi(\rho^p, x^p)$ it is natural to define the physical Hilbert space of the quantum theory described by $\{\mathcal{A}, L\}$ as follows:

$$\mathcal{H}_{phys.} = \bigoplus_{p \in L} \mathcal{H}_p \otimes V_p .$$

Let π be the representation of $\mathcal{A}$ on $\mathcal{H}_{phys.}$. Then we have that

$$\pi\big(\rho^p(A)\big)\ \psi(\rho^p, x^p) = \psi(\rho^p, x^p)\ \pi(A), \tag{4.20}$$

for all $A \in \mathcal{A}$.

See [6] for a complete theory in the case where $R^+ = R^-$, where $\mathcal{K}$ can be replaced by a compact group. An ansatz of the form (4.19) for commutation relations between unobservable, charged fields was first discussed in [15], (for theories in two space-time dimensions).

5. Back to physics

It is now time to ask how the braid statistics of charged fields (intertwiners), found in Sect. 3 and further discussed in the second half of Sect. 4, will manifest itself physically? One answer to this question is found by studying collision (scattering) theory in quantum theories with charged fields obeying braid statistics. A form of collision theory in the algebraic formulation of local, relativistic quantum theory, inspired by Haag-Ruelle theory [2], has been developed in [4,5] and can be adapted to theories with fields obeying braid statistics, as sketched in [26,28]. The result of the analysis presented in these papers is that the momentum-space wave functions describing incoming or outgoing states of charged particles have symmetry properties under exchanging the momenta (and spins, internal quantum numbers) of charged particles along oriented paths in momentum space that can be described in terms of the braid matrices $\{R^+, R^-\}$. Thus, the statistics of charged fields (intertwiners) determines the statistics of asymptotic charged particles which manifests itself in symmetry properties of scattering amplitudes and cross sections. For all details we must refer the reader to [4,5,26,28].

Next, we wish to reconsider the question of what kinds of Lagrangian models of local, relativistic quantum theories in three space-time dimensions have charged fields obeying braid statistics? Heuristically, a rather clear-cut answer can be given.

It is known from the work of Doplicher, Haag and Roberts [4] that if the charged fields (intertwiners) can be localized in bounded space-time regions and the dimension of space-time is $d \geq 3$ then $R^+(i,p,q,k)^l_j = R^-(i,p,q,k)^l_j$, and, by Theorem 5, Sect. 3,

$$s_p \in \frac{1}{2}\mathbb{Z}, \text{ for all } p \in L,$$

and "spin parity" is conserved under fusion, (i.e, $\sigma_p \cdot \sigma_q = \sigma_r$ if $N_{rpq} \neq 0$) .

Thus, in order to find examples of theories with charged fields satisfying braid group statistics we must look for theories whose charged fields cannot be localized in bounded space-time regions, but only in space-like cones. Charged fields only localizable in space-like cones are a typical feature of gauge theories.

Consider a three-dimensional theory with charged fields only localizable in space-like cones. Let P denote space-reflection at a line $l \subset \{(\vec{x},t) \in \mathsf{M}^3 \,:\, t = 0\}$. Let $\mathcal{O}$ be a region in M^3. We define

$$\mathcal{O}^P = \{(\vec{x},t) \in \mathsf{M}^3 \,:\, (P\vec{x},t) \in \mathcal{O}\}. \tag{5.1}$$

We suppose that P is represented on $\mathcal{A}$ by a *automorphism α_P with the property that

$$\alpha_P(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\mathcal{O}^P). \tag{5.2}$$

Let ρ^p be a *morphism of an extended algebra $\mathcal{B}^{C_a}$, with $p \in L$. We define ρ^p_P by setting

$$\rho^p_P(A) = \alpha_P \circ \rho^p \circ \alpha_P(A). \tag{5.3}$$

Let ρ^p and ρ^q be *morphisms of $\mathcal{B}^{C_a}$ localized in space-like separated, space-like cones, C^p and C^q, (space-like separated from the auxiliary cone C_a). It follows easily from (5.2), (5.3) and (2.18) that

$$as(\rho^p) \gtrless as(\rho^q) \iff as(\rho^p_P) \lessgtr as(\rho^q_P). \tag{5.4}$$

Next, suppose that, for all $p \in L$, the *morphisms ρ^p and ρ^p_P are unitarily equivalent and that α_P can be implemented unitarily on the total physical Hilbert space of states, $\mathcal{H}_{phys.}$, of the theory, i.e.,

$$\rho^p \sim \rho^p_P, \text{ for all } p \in L, \ \pi(\alpha_P(A)) = U(P)\,\pi(A)\,U(P)^*, \tag{5.5}$$

for all $A \in \mathcal{A}$, where $U(P)$ is a unitary involution on $\mathcal{H}_{phys.}$. Let us recall the commutation relations (3.9) between charged fields established in Theorem 1 of Sect. 3:

$$\begin{aligned} &V^{ji}_{\mu}(\rho^p)\, V^{ik}_{\nu}(\rho^q) \\ &\quad = \sum_{l\alpha\beta} R^{\pm}(j,p,q,k)^{l\alpha\beta}_{i\mu\nu}\, V^{jl}_{\alpha}(\rho^q)\, V^{lk}_{\beta}(\rho^p), \end{aligned} \tag{5.6}$$

if $\rho^p \bigtimes \rho^q$ and $as(\rho^p) \gtrless as(\rho^q)$.

By applying (5.5) and (5.4) to (5.6) we obtain the following

<u>Theorem 7</u> <u>If space reflection at a line</u>, P, <u>is a symmetry of the theory, in the sense of equ.</u> (5.5), <u>then</u>

$$R^+(j,p,q,k)^l_i = R^-(j,p,q,k)^l_i,$$

<u>i.e., the theory has ordinary permutation group statistics</u>, $s_p \in \frac{1}{2}\mathbb{Z}$, <u>and "spin parity" is conserved under fusion.</u>

Thus, in order for a three-dimensional, local quantum theory to exhibit <u>non-trivial braid group statistics</u> it must have charged fields which <u>cannot</u> be localized in bounded space-time regions and it must <u>break</u> the symmetry of space-reflection at a line.

Let us suppose that we look for a relativistic theory with these features. Then it must likely be a gauge theory without confinement which breaks space-reflection at a line. Thinking in terms of Lagrangian theories, we conclude that it must be a gauge theory with a Chern-Simons term in the effective gauge field Langrangian, as discussed in Sect. 1, or a theory which can be reformulated as a Chern-Simons theory, such as an $O(3)$ non-linear σ-model with a Hopf term in the effective Lagrangian.

In view of applications of our theory to condensed matter physics, it is useful to recall the essential assumptions on which the general theory developed in Sects. 2, 3 and 5 rests. They are as follows:

(1) The algebra of observables, $\mathcal{A}$, of the theory has a local structure

$$\mathcal{A} = \overline{\cup \mathcal{A}(C)}^{n}, \tag{5.7}$$

where $\mathcal{A}(C)$ is an algebra of observables corresponding to measurements in a localized region, C, of space-time; (C might be a wedge contained in a time slice). One requires some suitable form of locality, duality and the Reeh-Schlieder theorem; (see Sect. 2).

(2) Existence of a space-time translation- and rotation covariant "ground state representation", with the property that the generator of time translations, (the Hamiltonian), H satisfies the spectrum condition

$$H \geq 0. \tag{5.8}$$

(3) One considers then the class of all space-time translation- and rotation-covariant representations, p, of $\mathcal{A}$ satisfying the spectrum condition (5.8) and requires that p be localizable in an arbitrary "wedge", with respect to the local structure (5.7), in the sense of (2.18).

We emphasize that full relativistic covariance is not needed; covariance under the projective group of Euclidean motions in space, and time translations is already a little more than what we need for our analysis.

From these remarks we conclude that braid statistics can, in principle, be encountered in non-relativistic systems of condensed matter theory with broken space-reflection symmetry! Archetypal examples are correlated electronic systems in a strong external magnetic field, such as quantum Hall systems. Other systems are two-dimensional systems with flux phases, (perhaps of relevance in high T_c superconductivity).

These matters will be discussed in more detail in a future publication. Our approach to these problems is somewhat comparable in its spirit to the topological approach to classifying defects in ordered media [46]: Very general arguments based on symmetry and topological considerations yield a considerable amount of insight.

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