Lemma 3.98.1. Let R be a Noetherian ring and M an R-module. Then $\mathbf{Supp}\,(M) = \bigcup V(\mathfrak{p})$ where $\mathfrak{p}$ runs through $\mathbf{Ass}\,(M)$.

Proof. Indeed, one sees easily that $M_{\mathfrak{p}} \neq 0$ if and only if $\mathbf{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq \emptyset$. But this holds if and only if there is a $\mathfrak{q} \in \mathbf{Ass}\,(M)$ such that

$$\mathfrak{q} \cap (R - \mathfrak{p}) = \emptyset,$$

that is, if and only if $\mathfrak{p} \supset \mathfrak{q}$ for some $\mathfrak{q} \in \mathbf{Ass}\,(M)$. □

3.99 An Interlude with Hilbert's Nullstellensatz

We have already proved various weak forms of Hilbert's *Nullstellensatz*, but here we state another weak version and use the "celebrated trick of Rabinowitsch" to get the strong form.

Theorem 3.99.1 (Another Weak Nullstellensatz)**.** If k is an algebraically closed field and I is a proper ideal of $k[X_1, \ldots, X_n]$ then $V(I) \neq \emptyset$.

Proof. It suffices to prove this in the case of a maximal ideal, since if $I \subset J$ then $V(J) \subset V(I)$. Thus we may assume that $L = k[X_1, \ldots, X_n]/I$ is a field, and, of course, $L \supset k$. If one knew that $L = k$ then the result would be established since for each X_i the I-residue a_i of X_i would belong to k, and then the point $(a_1, \ldots, a_n) \in V(I)$, so $V(I) \neq \emptyset$. Thus it suffices to establish

> If an algebraically closed field k is a subfield of a field L, and there is a k-algebra homomorphism of $k[X_1, \ldots, X_n]$ onto L (that is a ring homomorphism that is the identity on k), then $k = L$.

□

We postpone the proof of this last fact, and now show how this Weak Nullstellensatz gives the classical

Theorem 3.99.2 (Hilbert's Nullstellensatz)**.** Let k be an algebraically closed field and let I be an ideal of $k[X_1, \ldots, X_n]$. Then $I(V(I)) = \sqrt{I}$, the nilradical of I.

Proof. The inclusion $\sqrt{I} \subset I\,(V(I))$ is easy ($f^n \in I$ and $x \in V(I)$ imply $f^n(x) = 0$ whence $f(x) = 0$ and so $f \in I\,(V(I))$. To see the reverse inclusion, let $I = (F_1, \ldots, F_r)$ and suppose that $G = I(V(I))$. Let $J = (F_1, \ldots, F_r, X_{n+1}G - 1) \subset k[X_1, \ldots, X_n, X_{n+1}]$. Then $V(J) \subset \mathbb{A}^{n+1} = \emptyset$ by construction, since G vanishes where all the F_i are zero. We apply the Weak Nullstellensatz to J to conclude that $1 \in J$. Thus we have an equation of the form

$$1 = \sum_i A_i(X_1, \ldots, X_n, X_{n+1})F_i(X_1, \ldots, X_n) + B(X_1, \ldots, X_n, X_{n+1})(X_{n+1}G(X_1, \ldots, X_n) - 1).$$

Let $Y = 1/X_{n+1}$, and multiply the equation by a sufficiently high power of Y to obtain

$$Y^N = \sum_i C_i(X_1, \ldots, X_n, Y)F_i + D(X_1, \ldots, X_n, Y)(G - Y)$$

in $k[X_1, \ldots, X_n, Y]$. Now put $Y = G(X_1, \ldots, X_n)$ to obtain the desired result. □

To complete the proof of the Weak Nullstellensatz (and thus justify its use in the proof of Hilbert's Nullstellensatz) we must prove the result whose proof was postponed.

First some remarks on field extensions. Suppose K is a subfield of a field L and suppose $L = K(v)$ for some $v \in L$. Let $\phi : K[X] \longrightarrow L$ be the K-homomorphism taking X to v, and let $\ker\phi = (F)$ with $F \in K[X]$. Since $K[X]/(F) \cong K[v]$ which is an integral domain, the ideal (F) is prime. There are two possible cases

1. $F = 0$. In this case $K[v]$ is isomorphic to $K[X]$ and so $L = k(v)$ is isomorphic to $K(X)$, and one sees easily that L is not finitely generated as a ring over K. Indeed $L = K[X]$ contains infinitely many monic irreducible polynomials, and any purported finite set of ring generators for L over K could involve only finitely many such irreducibles as denominators, while $L = K(X)$ clearly contains the reciprocals of all its irreducible monics.

2. $F \neq 0$. In this case we may assume F to be monic, and as above it must clearly be irreducible since the quotient ring is a domain. But in $K[X]$ prime ideals are maximal, so $K[v] = K(v)$. Moreover $F(v) = 0$, so v is algebraic over K, and $L = K[v]$ is finitely generated as a K-module.

Exercise 1. Show that if k is algebraically closed and L is a field containing k, then any element of L algebraic over k must belong to k. Show also that an algebraically closed field k has no field extensions L which are finitely generated over k as vector spaces, except k itself. (These are almost tautologies.)

Exercise 2. Show that if k is a field, the $k[X]$ is integrally closed in its field of quotients, that is, show that if $f \in k(X)$ satisfies a monic equation with coefficients in k, then $f \in k[X]$.

Solution: If $f = n/d$ with $(n, d) = 1$ then from

$$\left(\frac{n}{d}\right)^e + a_{n-1}\left(\frac{n}{d}\right)^{e-1} \cdots a_1 \frac{n}{d} + a_0 = 0$$

with $a_i \in k[x]$ one finds, on clearing denominators that

$$n^e = d(\text{element of } k[x])$$

which contradicts $(n, d) = 1$, unless $d \in k$, that is, unless the quotient really belongs to $k[x]$.

In view of the preceding exercise to prove the postponed part of the Weak Nullstellensatz it suffices to prove

Proposition 1 (Zariski)**.** *If a field L is finitely generated as a ring over a subfield K, then L is finitely generated as a module over K, and hence it algebraic over K.*

Proof. Suppose $L = K[v_1, \ldots, v_n]$. We proceeda by induction on n, the discussion preceding Exercise 1 settling the case $n = 1$. So assume the result for all extensions generated by $n - 1$ elements, and put $K_1 = K(v_1)$. By the induction hypothesis $L = K_1[v_2, \ldots, v_n]$ is a finitely generated module over K_1. If v_1 is algebraic over K, the result is easy (take generators $v^j u_i$ where $0 \leq j <$ degree of v over K and the u_i are generators of L as a K_1-module). So we may assume v_1 not algebraic over K. Then for $i = 2, \ldots, n$ each v_i satisfies an equation

$$v_i^{n_i} * a_{i1} v_i^{n_i - 1} + \cdots + a_{i,n_i} = 0 \qquad \text{with all } a_{ij} \in K_1.$$

If we choose $a \in K[v_1]$ to be a multiple of all the denominators of the a_{ij} we get equations

$$(av_i)^{n_i} + \cdots + a^{n_i} a_{i,n_i} = 0.$$

Since the set of elements of L that satisfy monic equations (i.e. the integral elements of L over K_1) is a subring of L containing K_1, we find that in fact there is an integer N such that for all $z \in L$ one has $a^N z$ is integral over $K[v_1]$. In particular this holds for $z \in K(v_1)$. But, as one easily checks using the techniques of Exercises 1 and 2, this is impossible. Indeed, it suffices to choose an element z of $K[v_1]$ which is prime to a, and consider $1/z$. No power of a can push $1/z$ into $k[v]$, and only such elements are integral over $k[v]$. $\square$

We mention in passing (*i.e.* , without proof, a modern version of the Nullstellensatz which holds over an arbitrary field.

Theorem 3.99.3. If a ring R is finitely generated over a field, then the Jacobson radical of R coincides with the nilradical of R.

One recovers a more classical interpretation by taking R to be the ring $k[X_1, \ldots, X_n]/J$ and noting that $f \in R$ belongs to the Jacobson radical iff and only f one has $f(\mathfrak{m}) = 0$ for all maximal ideals $\mathfrak{m}$ of R (where, of course, $f(\mathfrak{m})$ means the reduction of f modulo $\mathfrak{m}$, which may be naturally identified with $f(x)$ when $\mathfrak{m}$ corresponds to the point x). So f lies in the Jacobson radical if and only if $f \in I(V(J))$, this latter set being identified with the closed points of $\mathbf{Spec}(R)$. On the other hand, f belongs to the nilradical of R iff and only if its preimages in $k[X_1, \ldots, X_n]$ belong to the nilradical of J.

Exercise 3. Re-examine your counterexamples to the classical versions of the Nullstellensatz (weak and Hilbert versions) in the light of the "modern version".

3.100 Artinian Rings

Let R be a ring. An R-module M is said to be *Artinian* if every nonempty set of R-submodules of M has a minimal element, (or equivalently, if every descending chain of submodules stops). A ring R is Artinian if it is Artinian as an R-module, that is, if every descending chain of ideals stops. This definition can also be given when R is not commutative, in which case one is lead to the study of such objects as matrix rings over a field, division rings, and Brauer groups. Many familiar and important rings are NOT Artinian, $\mathbf{Z}$ and $k[X_1, \ldots, X_n]$ being important examples, whose localizations provide many other examples. However Artinian rings also play an important role in classical algebraic geometry where they appear, for example, as the rings associated with 0-dimensional intersections of algebraic varieties, in particular with the intersection of two plane algebraic curves without common components. Indeed, Artinian rings turn out to be precisely the 0-dimensional Noetherian rings.

Proposition 2. *Let R be a ring. An R-module M has finite length if and only if it is both artinian and noetherian.*

Proof. If M has finite length, then by the Jordan-Hölder Theorem, every chain of submodules has finite length. Hence M is both Artinian and Noetherian. Conversely, one may construct a filtration $\{M_i\}$ of M as follows: let $M_0 = M$ and construct the filtration recursively by taking M_{i+1} to be a maximal proper submodule of M_i for each i. This descending chain must stop (by the Artinian hypothesis), and by construction it is a composition series for M, so M has finite length. $\square$

Lemma 3.100.1. Let R be a ring in which (0) is the product of a finite number of maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$. Then any prime ideal $\mathfrak{p}$ of R is one of the $\mathfrak{m}_i$, and R is both Noetherian and Artinian. Moreover, if the $R/\mathfrak{m}_i$ are all finitely generated algebras over a field k, then A has finite k-dimension.

Proof. Since any prime $\mathfrak{p}$ contains $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$, it follows that $\mathfrak{p} = \mathfrak{m}_i$ for some index i. Let $I_j = \mathfrak{m}_1 \cdots \mathfrak{m}_j$. The I_j provide R with a finite filtration $I_0 \supset I_1 \supset \cdots \supset I_n = (0)$ with quotients I_{j-1}/I_j which are finite dimensional vector spaces over $R/\mathfrak{m}_j$, indeed, with quotients which are isomorphic to the $R/\mathfrak{m}_j$. Hence by Proposition 2 we see that R is both Artinian and Noetherian. Moreover, if each $A/\mathfrak{m}_i$ is of finitely generated as an algebra over k,then by Proposition 1 of the preceding section $A/\mathfrak{m}_i$ has finite k-dimension, and from this the last assertion follows.

$\square$

Theorem 3.100.2. A ring R is Artinian if and only if the following two conditions hold:

1. R is Noetherian.

2. R has dimension zero, that is, EVERY prime ideal of R is maximal.

Moreover, if R is Artinian, then R has only a finite number of primes, and the Jacobson radical $J(R)$ is nilpotent. If, moreover, R is finitely generated as a ring over a field k, then R has finite dimension over k.

Proof. Suppose that R is Noetherian. Then every ideal of R contains a finite product of prime ideals. This may be seen via "Noetherian induction": indeed, if the thesis is false, let I be an ideal maximal with respect to NOT containing a finite product of primes. The I is surely not prime and also not equal to all of R. Since I is not prime there exist $a, b \in R \backslash I$ with $ab \in I$. By maximality of I both $I + Ra$ and $I + Rb$ contain a finite product of prime ideals, and the product of those products is again a finite product of prime ideals contained in the product $(I + aR)(I + bR) \subset I$. This contradicts the choice of I. If, furthermore, every prime ideal of R is maximal then (0) may be written as a product of maximal ideals, and so by Lemma 3.100.1 R is Artinian.

Conversely, suppose that R is Artinian, and let m be the smallest product of maximal ideals of R. Let S be the set of ideals I contained in m such that $Im \neq (0)$. If I is minimal in S, then

$$m^2 I = mI \neq (0).$$

Hence by minimality we must have $mI = I$. Since $m \subset J(R)$, the Jacobson radical of R, if $I = xR$, then $I = 0$ by Nakayama's Lemma. Hence, if $x \in I$, then $xm = (0)$ so $Im = 0$, which contradicts the minimality of I with respect to the property $Im \neq (0)$. Hence the set S must be empty, and so $m = m^2 = 0$. Thus by Lemma 3.100.1 R is Noetherian, and every prime ideal of R is maximal. □

Exercise 4. Give a counterexample (very easy) to show that the situation described by the phrase beginning "If, moreover, R is finitely generated ..." does not always hold.

Corollary 3.100.3. Let R be an Artinian ring and M a finite R-module. Then M has finite length, and $\mathbf{Ass}\,(M) = \mathbf{Supp}\,(M)$.

Proof. This follows from the previous result and Lemma 3.98.1. □

Proposition 3. *Let R be an Artinian ring and $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ the maximal ideals of R. Then*

1. *The natural map $u : A \longrightarrow \prod A_{\mathfrak{m}_i}$ is an isomorphism.*
2. *For sufficiently large n, the natural maps*

$$v_i : R_{\mathfrak{m}_i} \longrightarrow A/\mathfrak{m}_i^n$$

are isomorphisms.

Proof. Since $X = \mathbf{Spec}(R)$ is discrete, the map u appearing in 1) is just the natural isomorphism of R with the ring of global sections on $\mathbf{Spec}(R)$. As to the second statement, any $s \notin \mathfrak{m}_i$ becomes a unit in the local ring $R/\mathfrak{m}_i^n$. Hence, by the universal property of localization the maps v_i exist. For fixed i consider $u_i : R \longrightarrow R_{\mathfrak{m}_i}$. For each positive integer n one can certainly find

$$s \in \left(\cap_{j \neq i} \mathfrak{m}_j^n\right) \backslash \mathfrak{m} i$$

Then by the preceding Theorem, for n sufficiently large one has $sa = 0$ for all $a \in \mathfrak{m}_i^n$, so u_i induces $u_i' : R/\mathfrak{m}_i^n \longrightarrow R_{\mathfrak{m}_i}$, which is an inverse for v_i. □