EXCEPTIONAL LIE ALGEBRAS AND RELATED ALGEBRAIC AND GEOMETRIC STRUCTURES

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1. *Introduction*

Certain algebraic structures, most notably associative, alternative, and Jordan algebras are strongly linked via construction and classification to simple Lie algebras and to interesting geometries. These geometries are in turn linked to simple Lie algebras via their groups of collineations. These linkages serve to illustrate how various notions of exceptionality in algebra and geometry (e.g., non-classical Lie algebras, non-associative alternative algebras, non-special Jordan algebras, and non-Desarguian projective planes) are just different manifestations of the same phenomenon. It is the intent of this survey to discuss briefly the general classes of structures in which the exceptional objects occur, to describe the linkage between the exceptional objects, and to illustrate the utility of these linkages in understanding the nature of these diverse exceptional structures.

In §2, we briefly survey the relevant areas in the theory of Lie algebras (§2.1) and the related Chevalley groups (§2.2), for these provide the principal motivation for our study. §§3, 4, 5 are devoted to some well known (alternative algebras in §3, Jordan algebras in §4) and not so well known ($\mathfrak{J}$-ternary algebras in §5) classes of algebraic structures, defined by identities, which provide linear realizations and hence a deeper understanding of the exceptional Lie algebras (other than E_8). In §6 we indicate how two algebras from the classes discussed in §§3 and 4 can be "pasted together" via a construction of Tits to give new versions of the exceptional Lie algebras (including E_8 this time). In §7 we show how information gleaned from previous sections enables one to show that in many cases the Tits' constructions yield (up to isomorphism) all exceptional simple Lie algebras of a particular type over a field Φ of characteristic zero. Finally, in §8 we indicate a method of constructing geometries (isomorphic to some geometries of Tits) from simple Lie algebras and investigate, via coordinatization of the geometries, their links with the exceptional algebraic structures introduced in earlier sections.

Throughout the article we have exercised our own prejudices relative to selection and presentation of material. Therefore it is our intention that this be considered an introduction to a broad area rather than an encyclopedic survey of it. Each facet of mathematics touched upon here, with the possible exception of the geometry, is blessed with one (or several) excellent related reference works. While not attempting to be complete, we mention the books of Bourbaki [8], Humphreys [33], Jacobson [39] and Seligman [66] on Lie algebras; of Carter [13] and Steinberg [77] on Chevalley groups; of Jacobson [42] on Jordan algebras and again [43] on exceptional Lie algebras; and of Schafer [63] on general concepts in non-associative algebra. In an attempt to keep our reference list short, we have restricted ourselves as much as possible to listing the most recent and/or most general result related to our statements.

Received 30 January, 1976

† NSF grant MPS-71-03061-A04 was held during the preparation of this paper.
‡ NSF grant GP38919 was held during the preparation of this paper.

Early, formative papers in the area can be found in the extensive reference lists of several of the above-mentioned texts.

Finally, we note that unless otherwise stated, we have restricted our attention to commutative ground fields Φ of *characteristic zero.* Often this is done in the interest of ease of exposition and comprehension, rather than because the results fail in characteristic $\neq 0$. We restrict ourselves as well to consideration of *finite dimensional* algebras.

2. *Lie algebras and related groups*

This section on preliminaries is divided into two subsections, one devoted to Lie algebras, one to Chevalley groups.

2.1. *Lie algebra preliminaries.* While the study of Lie algebras has its origin in the study of analytic groups, our purposes are better served by introducing the subject via a purely algebraic example.

2.1.1. *Example.* Let Φ_n be the algebra of $n \times n$ matrices with coefficients in Φ, $\mathfrak{S}(\Phi_n) = \{A \in \Phi | A^t = -A\}$ (the space of skew symmetric matrices). $\mathfrak{S}(\Phi_n)$ is not closed under the usual matrix product but is closed under $[A, B] = AB - BA$, where juxtaposition is the usual matrix product.

$\mathfrak{S}(\Phi_n)$ with the prescribed product is a *Lie algebra*; i.e., the identities

$$[A, A] = 0, \quad \forall A \in \mathfrak{S}(\Phi_n),$$

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0, \quad \forall A, B, C \in \mathfrak{S}(\Phi_n) \text{ (Jacobi identity)}$$

are satisfied. One sees in fact that Φ_n (indeed any associative algebra $\mathfrak{A}$) is a Lie algebra relative to this product. We denote this latter Lie algebra by Φ_n^- (resp. $\mathfrak{A}^-$). $\mathfrak{S}(\Phi_n)$ is thus a Lie subalgebra of Φ_n^-. A rather startling observation related to this example is that every finite dimensional Lie algebra over Φ is isomorphic to some subalgebra of Φ_n^- for suitable n, [34].

Example 2.1.1 has two generalizations which will be of interest to us.

2.1.2. *Example.* Let $\mathfrak{A}$ be a finite dimensional associative algebra over Φ with involution τ. Let $\mathfrak{S}(\mathfrak{A}, \tau) = \{a \in \mathfrak{A} | a^\tau = -a\}$. Then $\mathfrak{S}(\mathfrak{A}, \tau)$ is a Lie algebra relative to the product in $\mathfrak{A}^-$.

2.1.3. *Example.* Let $\mathfrak{B}$ be a finite dimensional vector space over Φ, f a non-degenerate bilinear form on $\mathfrak{B}$. Let

$$s(\mathfrak{B}, f) = \{A \in \operatorname{End} \mathfrak{B} | f(xA, y) = -f(x, yA) \quad \forall x, y \in \mathfrak{B}\}$$

where End $\mathfrak{B}$ is the algebra of Φ-linear transformations of $\mathfrak{B}$. Then $s(\mathfrak{B}, f)$ is a Lie subalgebra of $(\operatorname{End} \mathfrak{B})^-$.

If f in 2.1.3 is symmetric, $s(\mathfrak{B}, f)$ is called an *orthogonal* Lie algebra and denoted by $o(\mathfrak{B}, f)$. If f is alternating we call $s(\mathfrak{B}, f)$ a *symplectic* Lie algebra and denote it by $sp(\mathfrak{B}, f)$.

A further useful example is

2.1.4. *Example.* Let $\mathfrak{A}$ be a (not necessarily associative) algebra over Φ. Let $\operatorname{Der} \mathfrak{A} = \{A \in \operatorname{End} \mathfrak{A} | (xy)A = (xA)y + x(yA), \forall x, y \in \mathfrak{A}\}$ (the algebra of *derivations* of $\mathfrak{A}$). Der $\mathfrak{A}$ is a Lie subalgebra of $(\operatorname{End} \mathfrak{A})^-$.

Example 2.1.4, for suitably selected algebras, will be used in §§3 and 4 to describe certain "exceptional" simple Lie algebras. For the moment, we look at this example only in the event that $\mathfrak{A}$ itself is a Lie algebra. Then the Jacobi identity shows that $\operatorname{ad} x : y \to [y, x]$ is in $\operatorname{Der} \mathfrak{A}$, hence a Lie algebra acts on itself as derivations.

The mapping $x \to \operatorname{ad} x$ of $\mathfrak{L}$ to $\operatorname{End} \mathfrak{L}$ for a Lie algebra $\mathfrak{L}$ is a special case of a *representation* of $\mathfrak{L}$ on a vector space $\mathfrak{V}$; i.e., a Lie algebra homomorphism ϕ ($[x, y]\phi = [x\phi, y\phi]$) from $\mathfrak{L}$ to $(\operatorname{End} \mathfrak{V})^-$. This particular representation of $\mathfrak{L}$ on $\mathfrak{L}$ is called the *adjoint representation*. Given any representation $\phi : \mathfrak{L} \to (\operatorname{End} \mathfrak{V})^-$ there is a naturally associated *contragredient* representation $\phi^* : \mathfrak{L} \to (\operatorname{End} \mathfrak{V}^*)^-$, where $\mathfrak{V}^*$ is the dual space of $\mathfrak{V}$, given by $\langle \eta(x\phi^*), v \rangle = -\langle \eta, v(x\phi) \rangle$ for all $x \in \mathfrak{L}$, $v \in \mathfrak{V}$, $\eta \in \mathfrak{V}^*$. If $\mathfrak{V}$ admits a non-degenerate bilinear form f, the contragredient representation is actually realized (up to isomorphism) in $\mathfrak{V}$ via $f(v_1(x\phi^*), v_2) = -f(v_1, v_2(x\phi))$. In the special case of the adjoint representation of a semisimple $\mathfrak{L}$, using the *Killing form* $K(x, y) = \operatorname{trace}(ad x\, ad y)$ one sees that the contragredient representation on $\mathfrak{L}$ is again the adjoint representation.

The fundamental problem we address here is: describe explicitly all simple Lie algebras over a given field Φ. In the event Φ is algebraically closed of characteristic zero, this has been done in an elegant manner by Killing [47] and Cartan [12]. We describe their solution in some detail as it provides a setting for our entire discussion. For the remainder of this section we consider P an algebraically closed field, $\operatorname{char} P = 0$, $\mathfrak{L}$ a simple Lie algebra over P.

2.1.5. *Cartan decomposition.* $\mathfrak{L} = \mathfrak{H} \oplus \sum_{\alpha \in \Sigma} \oplus \mathfrak{L}_\alpha$ where $\mathfrak{H}$ is an abelian subalgebra ($[\mathfrak{H}, \mathfrak{H}] = 0$) of $\mathfrak{L}$ (a *Cartan subalgebra*), Σ (the set of *roots* of $\mathfrak{H}$) is a finite subset of $\mathfrak{H}^* \setminus \{0\}$ and $\mathfrak{L}_\alpha = \{\ell \in \mathfrak{L} \mid [\ell, h] = \alpha(h)\ell, \ \forall h \in \mathfrak{H}\}$ is one dimensional.

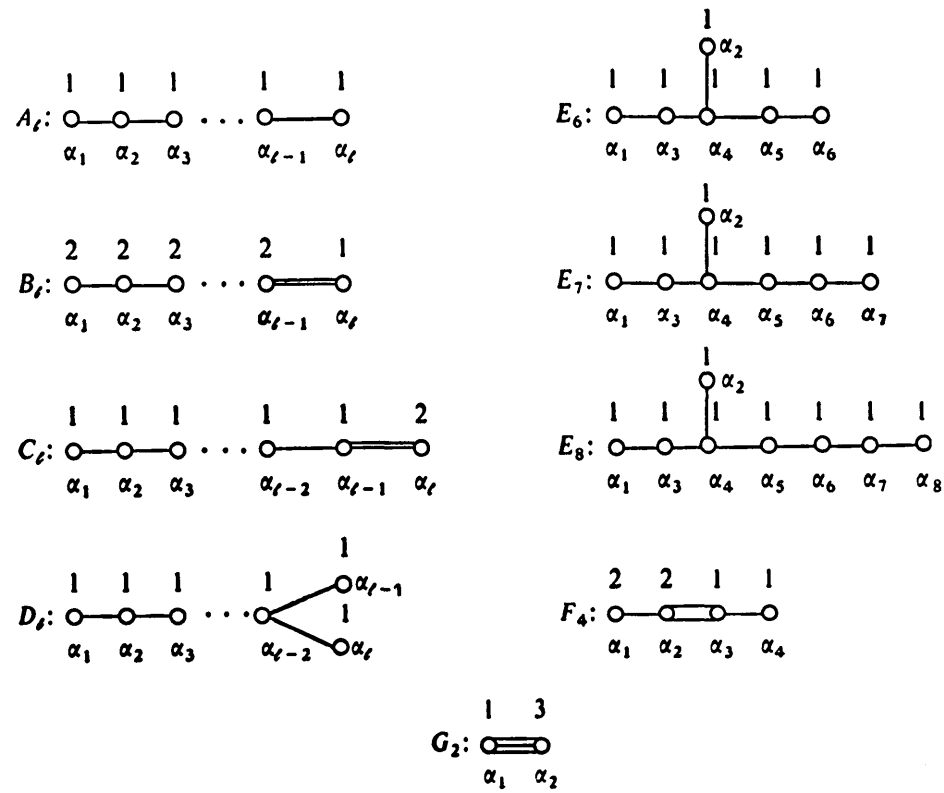
We write $\mathfrak{L}_\alpha = \mathrm{P}x_\alpha$ and note that $[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] = \mathfrak{L}_{\alpha+\beta}$ if $\alpha \neq -\beta$ (with the convention $\mathfrak{L}_\gamma = 0$ if $0 \neq \gamma \notin \Sigma$), and $[\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}] \subseteq \mathfrak{H}$.

Given the existence of a Cartan decomposition of $\mathfrak{L}$, one has a complete description of all simple Lie algebras over P, if one can determine all possible sets Σ of roots, for each such Σ the possible values for $N_{\alpha,\beta} \in \mathrm{P}$ such that $[x_\alpha, x_\beta] = N_{\alpha,\beta}\, x_{\alpha+\beta}$, and for each $\alpha \in \Sigma$, the possible $h_\alpha \in \mathfrak{H}$ with $[x_\alpha, x_{-\alpha}] = h_\alpha$. The task of determining the algebras is much simplified by the fact [13] that once Σ is determined, so (essentially) are the $N_{\alpha,\beta}$ and h_α, hence the algebras.

The analysis of Σ is an exercise in the geometry of Euclidean ℓ-space where $\ell = \dim \mathfrak{H}$ is the *rank* of $\mathfrak{L}$. Of particular interest is the rational span $\mathfrak{H}_0^*$ of Σ in $\mathfrak{H}^*$ where the inner product is given by $(\alpha, \beta) = \beta(h_\alpha)$, $h_\alpha \in \mathfrak{H}$ chosen such that $K(h_\alpha, h) = \alpha(h)$ for all $h \in \mathfrak{H}$. For each $\alpha \in \Sigma$, the reflection $\mathrm{w}_\alpha : \beta \to \beta - (2(\beta, \alpha)/(\alpha, \alpha))\alpha$ in $\mathfrak{H}_0^*$ permutes the elements in Σ. Thus $\{\mathrm{w}_\alpha \mid \alpha \in \Sigma\}$ generates a finite group W, the *Weyl group* associated with $\mathfrak{L}$. The constants $2(\beta, \alpha)/(\alpha, \alpha)$ are in $\mathbb{Z}$ and are called the *Cartan integers* (denoted $\langle \beta, \alpha \rangle$).

One can select a subset $\Pi \subseteq \Sigma$ (a *simple system of roots*) with the property that every $\beta \in \Sigma$ has form $\beta = \sum_{\alpha_i \in \Pi} k_i \alpha_i$ where $k_i \in \mathbb{Z}$ are all of the same sign. Those roots for which the sign is positive form the set Σ^+ of *positive roots*. Since $\Sigma = \Sigma^+ \cup (-\Sigma^+)$ it suffices to describe Σ^+, which can be done once the Cartan integers $\langle \alpha_i, \alpha_j \rangle$ are determined for $\alpha_i, \alpha_j \in \Pi$. All relevant information about these integers is displayed in the *Dynkin diagram* for $\mathfrak{L}$. This is a connected graph of ℓ nodes indexed by Π where the α_i and α_j nodes are connected by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges and the α_i node is weighted by $(\alpha_i, \alpha_i)^2/(\alpha_k, \alpha_k)^2$ where k is chosen such that (α_k, α_k) is minimal. Further analysis of these diagrams leads to

2.1.6. *Theorem. For* P *algebraically closed, characteristic zero, the isomorphism classes of simple Lie algebras over* P *are indexed by the diagrams*



The infinite classes A_ℓ, $\ell \geqslant 1$, B_ℓ, $\ell \geqslant 2$, C_ℓ, $\ell \geqslant 3$ and D_ℓ, $\ell \geqslant 4$ are referred to as *classical Lie algebras* (not to be confused with algebras of classical type [66]) since they correspond in a natural way (see §2.2) to the well-known classical groups. The remaining classes, each consisting of a single isomorphism class, are called *exceptional* (D_4 is often included here due to the exceptional nature of its automorphism group though it does in fact correspond to a classical group).

The classical algebras can be easily described as Lie algebras of linear transformation: Φ_n' (the algebra of matrices of trace zero) is of type A_{n-1}; $o(\mathfrak{V},f)$ is of type B_ℓ if $\dim \mathfrak{V} = 2\ell+1$; $o(\mathfrak{V},f)$ is of type D_ℓ if $\dim \mathfrak{V} = 2\ell$; and $\mathrm{sp}(\mathfrak{V},f)$ is of type C_ℓ for $\dim \mathfrak{V} = 2\ell$. There are no such simple realizations of the exceptional Lie algebras (other than D_4). It is the aim of §§3, 4 and 5 to introduce algebraic structures which admit realization of the exceptional algebras (except E_8) as linear Lie algebras in a natural way. One should note that though we introduce them in the context of a study of Lie algebras, the octonion algebra and the exceptional Jordan algebra arose originally in contexts far removed from Lie algebras. The relationship between the exceptional Lie algebras and other algebraic structures has been developed in some detail in [43] and [63].

Our description of simple Lie algebras in this section has been restricted to algebras defined over algebraically closed fields. An analogous description is valid for *split* Lie algebras (algebras for which there is a decomposition as in 2.1.5) for arbitrary fields of characteristic zero. In general, however, not every simple Lie algebra over such a field is the split. One of the principal applications of the realization of the split simple algebras as linear Lie algebras is to the study of non-split simple algebras which are *Φ-forms* of simple Lie algebras over P (i.e., simple Lie algebras $\mathfrak{L}$ over Φ such that $\mathfrak{L}_P$ ($\equiv \mathfrak{L} \otimes_\Phi P$) is simple for P the algebraic closure of Φ). We shall call an algebra $\mathfrak{L}$ over Φ of type X_ℓ ($X = A, B, C, D, E, F$ or G) if $\mathfrak{L}_P$ falls in the class X_ℓ of 2.1.6.

2.2. *Chevalley Group preliminaries.* The classification of Φ-forms of a simple Lie algebra $\mathfrak{L}$ over P is closely tied via Galois cohomology to the automorphism group Aut $\mathfrak{L}$ **[68]**. This latter group is in turn closely related to the adjoint Chevalley group $G_P(\mathfrak{L}, \mathfrak{L})$ **[15]**. We shall look in this section at the construction of Chevalley groups (a process closely related to that which yields a linear Lie group from its Lie algebra) and the process by which these groups lead to a determination of Aut $\mathfrak{L}$. In §7, these Chevalley groups arise again in a different context, as groups of collineations of certain geometric structures.

To begin, we sketch the module theory of simple Lie algebras over ℂ. If ϕ is a representation of $\mathfrak{L}$ in $\mathfrak{V}$, we write $v\ell$ for $v(\ell\phi)$ and call $\mathfrak{V}$ with this action of $\mathfrak{L}$ an *$\mathfrak{L}$-module.* $\mathfrak{V}$ is *irreducible* if there is no proper subspace invariant under $\mathfrak{L}$. Taking the Cartan decomposition 2.1.5 for $\mathfrak{L}$ one sees that $\mathfrak{H}$ is *toral* in the sense that $\mathfrak{H}\phi$ is diagonalizable for any finite dimensional representation ϕ. Thus, there is a finite subset $\Gamma_{\mathfrak{V}} \subseteq \mathfrak{H}^*$ (the set of *weights* of the representation ϕ) such that

$$\mathfrak{V} = \sum_{\rho \in \Gamma_{\mathfrak{V}}} \oplus \mathfrak{V}_\rho$$

where $\mathfrak{V}_\rho$ (the *weight space* for ρ in $\mathfrak{V}$) $= \{v \in \mathfrak{V} | vh = \rho(h)v \ \forall\, h \in \mathfrak{H}\} \neq 0$. Among the weights in $\Gamma_{\mathfrak{V}}$ is a unique "highest" weight Λ such that $\mathfrak{V}_\Lambda \mathfrak{L}_\alpha = 0 \ \forall\, \alpha \in \Sigma^+$, and $\dim \mathfrak{V}_\Lambda = 1$. $\mathfrak{V}_\rho$ is then spanned by weight vectors v_ρ where

$$\rho = \Lambda - \sum_{\substack{\alpha_i \in \Pi \\ k_i \geqslant 0}} k_i \alpha_i.$$

Picking judiciously, one can simultaneously find a basis of root vectors $\{e_\alpha, h_i\}$ for $\mathfrak{L}$ and a basis $\{m_\lambda\}$ of weight vectors for $\mathfrak{V}$ such that the ℤ-span of the basis for $\mathfrak{L}$ is closed under multiplication and $e_\alpha{}^m/m!$ has an integral matrix relative to the basis $\{m_\rho\}$. Moreover, for sufficiently large m, $(e_\alpha \phi)^m = 0$ so for any field Φ one can define the exponential

$$\exp tE_\alpha = \sum_0^\infty t^n \otimes \frac{E_\alpha^n}{n!} = x_\alpha(t)$$

for $t \in \Phi$, $E_\alpha = e_\alpha \phi$, in $\mathrm{End}(\Phi \otimes_{\mathbb{Z}} \mathfrak{M})$, $\mathfrak{M}$ the ℤ-span of $\{m_\lambda\}$. The Chevalley group $G_\Phi(\mathfrak{L}, \mathfrak{V})$ is the group generated by $\{x_\alpha(t) | t \in \Phi, \alpha \in \Sigma\}$ and is independent (up to isomorphism) of the particular selection of m_λ.

This construction is analogous to the construction of linear Lie groups from linear Lie algebras over ℂ via exponentiation, the essential difference being that only nilpotent linear transformations are exponentiated so that set of generators is considerably smaller. Even with the restricted set of generators, this process assigns to each classical Lie algebra $\mathfrak{L}$ a classical group G if $\Phi = \mathbb{C}$. In particular, if $\mathfrak{L} = \mathrm{sl}(n, \mathbb{C})$, $\mathfrak{V}$ the usual representation in n-dimensions, $G_{\mathbb{C}}(\mathfrak{L}, \mathfrak{V}) = \mathrm{SL}(n, \mathbb{C})$ the special linear group. If $\mathfrak{L} = \mathrm{o}(\mathfrak{V}, f)$, $G_{\mathbb{C}}(\mathfrak{L}, \mathfrak{V}) = \mathrm{SO}(\mathfrak{V}, f)$ the special orthogonal group, while if $\mathfrak{L} = \mathrm{sp}(\mathfrak{V}, f)$, $G_{\mathbb{C}}(\mathfrak{L}, \mathfrak{V}) = \mathrm{Sp}(\mathfrak{V}, f)$, the symplectic group. In the cases of the exceptional Lie algebras, the algebraic structures of §§3, 4 and 5 provide modules for

$\mathfrak{L}$ (except E_8) which give rise to corresponding Chevalley groups as automorphism groups of the structures.

In general, for characteristic zero, $G_P(\mathfrak{L}, \mathfrak{V})$ has finite centre Z and the quotient $G_P(\mathfrak{L}, \mathfrak{V})/Z$ is simple. If $\mathfrak{V} = \mathfrak{L}$, $G_P(\mathfrak{L}, \mathfrak{V})$ (the adjoint group) has trivial centre, hence is simple. Moreover, $G_P(\mathfrak{L}, \mathfrak{L})$ is a homomorphic image of every group $G_P(\mathfrak{L}, \mathfrak{V})$. The determination of the group Aut $\mathfrak{L}$ in specific cases depends on this fact and the general fact for P algebraically closed, that Aut $\mathfrak{L} = G_P(\mathfrak{L}, \mathfrak{L})A$, where the product is semidirect and A is isomorphic to the group of symmetries of the weighted Dynkin diagram. Determination of A is immediate using 2.1.6. The analysis of $G_P(\mathfrak{L}, \mathfrak{L})$ is often made simpler if one knows a suitable linear realization of $\mathfrak{L}$ over P and can determine the covering map, $G_P(\mathfrak{L}, \mathfrak{V}) \to G_P(\mathfrak{L}, \mathfrak{L})$. This question is thoroughly discussed in **[66]**. We shall here note only the general outline of the argument. In all cases we consider, we have a vector space $\mathfrak{V}$ over P, a "structure" s on $\mathfrak{V}$ (bilinear form f, trilinear form N, quartic form q, bilinear product m or trilinear product μ) such that $\mathfrak{L} \subseteq \operatorname{Inv}(\mathfrak{V}, s)$ (i.e., $\mathfrak{L}$ preserves s "infinitesimally"). Then $G_P(\mathfrak{L}, \mathfrak{V})$ is a subgroup of Aut $(\mathfrak{V}, s)$ (the group of non-singular transformations preserving s) and the mapping $x_\alpha(t) \to \operatorname{Inn} x_\alpha(t)$ (where $\operatorname{Inn} x_\alpha(t) : g \to x_\alpha(t)^{-1} g x_\alpha(t)$) induces a surjective homomorphism of $G_P(\mathfrak{L}, \mathfrak{V})$ onto $G_P(\mathfrak{L}, \mathfrak{L}) \subseteq$ Aut $\mathfrak{L}$. Indeed, in all such cases under consideration, $\mathfrak{L}$ is invariant under $\operatorname{Inn} u : \ell \to u^{-1}\ell u$ for all $u \in$ Aut $(\mathfrak{V}, s)$, so

2.2.1. $G_P(\mathfrak{L}, \mathfrak{L}) \subseteq \operatorname{Inn}(\operatorname{Aut}(\mathfrak{V}, s)) \subseteq$ Aut $\mathfrak{L}$.

For the classical Lie algebras in their usual realizations over algebraically closed Φ, this procedure yields:

$$\text{for } A_n(\mathfrak{L} = \mathrm{sl}_{n+1}(\mathrm{P})), \quad \operatorname{Aut}\mathfrak{L} = \langle \operatorname{Inn}(\mathrm{SL}_{n+1}(\mathrm{P})), \tau\rangle \text{ where}$$

$\ell^\tau = -\ell^t$, t denoting transpose (τ here accounts for the unique diagram automorphism); for B_n, $D_n(n > 4)$ $(\mathfrak{L} = \mathrm{so}(\mathfrak{V}, f))$, Aut $\mathfrak{L} = \operatorname{Inn} O(\mathfrak{V}, f)$ $(G_P(\mathfrak{L}, \mathfrak{L}) = \operatorname{Inn} \mathrm{SO}(\mathfrak{V}, f))$ and for D_n the diagram automorphism is effected by Inn u for suitable $(u \in O(\mathfrak{V}, f) \setminus \mathrm{SO}(\mathfrak{V}, f))$; for $C_n(\mathfrak{L} = \mathrm{sp}(\mathfrak{V}, f))$, Aut $\mathfrak{L} = \operatorname{Inn}(\mathrm{Sp}(\mathfrak{V}, f))$. We shall see analogous results for the exceptional algebras in §§3, 4 and 5. By changing the representation of A_n, one may make all automorphisms appear to be inner, a convenience in some applications. Indeed, if A_n acts on $\mathfrak{V} = \mathfrak{N} \oplus \mathfrak{N}^*$, $\mathfrak{N}$ an $(\ell+1)$-dimensional vector space over Φ, $\mathfrak{N}^*$ the dual space of $\mathfrak{N}$, via $(n, m^*)\ell = (n\ell, m^*\ell)$ where $m^*\ell$ is the contragredient action, then Aut $\mathfrak{L} = \operatorname{Inn} \mathfrak{U}$ where

$$\mathfrak{U} = \{T \in \operatorname{End}\mathfrak{V} \mid (n, m^*)T = (nt, m^*t) \text{ or } (n, m^*)T = (mt, n^*t) \text{ with } t \in \mathrm{SL}_{n+1}(\mathrm{P})\},$$

$n \to n^*$ denoting a linear mapping of $\mathfrak{N}$ to $\mathfrak{N}^*$ carrying a basis of $\mathfrak{N}$ to a dual basis for $\mathfrak{N}^*$.

The internal structure of the group $G_\Phi(\mathfrak{L}, \mathfrak{V})$ closely reflects that of $\mathfrak{L}$. The analogue of the Cartan subalgebra $\mathfrak{H}$ is the subgroup H generated by all

$$h_\alpha(t) = w_\alpha(t)w_\alpha(1)^{-1}$$

for $t \in \Phi$, $\alpha \in \Sigma$ where $w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$. The analogue of the nilpotent subalgebra $\sum_{\alpha \in \Sigma^+} \mathfrak{L}_\alpha = \mathfrak{N}$ is a unipotent subgroup

$$U = \prod_{\alpha \in \Sigma^+} X_\alpha, \quad X_\alpha = \{x_\alpha(t) \mid t \in \Phi\}.$$

Since H normalizes U, the complex product $B = UH$ is a subgroup of G (corresponding to the maximal solvable subalgebra $\mathfrak{H} \oplus \mathfrak{N}$). A feature unique to the group is that the Weyl group of Σ can be recovered directly from the structure of $G_\Phi(\mathfrak{L}, \mathfrak{B})$ since the mapping $w_\alpha \to Hw_\alpha(t)$ extends to an isomorphism from W onto N/H where $N = \langle w_\alpha(t) | \alpha \in \Sigma, t \in \Phi \rangle$. Abusing notation to identify $w \in W$ with a coset representative of its image in N, one has the Bruhat decomposition

2.2.2. $$G_\Phi(\mathfrak{L}, \mathfrak{B}) = \bigcup_{w \in W} BwB.$$

The following relationships between the component parts of the groups B and N which are important in establishing the Bruhat decomposition, play an important role in the study of geometries associated with the related Lie algebras

2.2.3.

(i) $U \cap U' = \{1\}$ where $U' = \langle x_\alpha(t), \alpha \in -\Sigma^+ \rangle$,

(ii) $(X_\alpha, X_\beta) \subseteq \prod_{\gamma = i\alpha + j\beta} X_\gamma$ for $\alpha \neq -\beta, i, j > 0$,

(iii) $w_\alpha(t) X_\beta w_\alpha(t)^{-1} = X_{\beta w_\alpha}$

where $(Y, Z) = \{yzy^{-1}z^{-1} | y \in Y, z \in Z\}$.

The geometric objects studied are related to the *parabolic* subgroups

$$G_\pi = \langle H, X_\alpha | \alpha \in \Sigma^+ \cup \Sigma_\pi \rangle$$

where π is a subset of the set Π of simple roots and Σ_π is the set of roots in $-\Sigma^+$ which are linear combinations of the roots in π. These subgroups can be characterized as all subgroups intermediate between B and G since one has

2.2.4.

(i) $P_\phi = B$.

(ii) If $B < P < G$ there is $\pi \subseteq \Pi$ such that $P = G_\pi$.

(iii) G_π is conjugate to $G_{\pi'}$ if and only if $\pi = \pi'$.

3. *The octonion algebra and* G_2

The exceptional Lie algebra G_2 was known by Cartan **[12]** to have a seven-dimensional representation, a representation describable in terms of the octonion algebra $\mathfrak{O}$—the eight-dimensional member of the class of *composition algebras* (algebras $\mathfrak{A}$ with a quadratic form n such that $n(ab) = n(a)n(b)$ for all $a, b \in \mathfrak{A}$).

To understand fully the structure of the octonion algebras, it is necessary to understand the simple iterative process which yields all composition algebras. Let $\mathfrak{B}$ be an algebra with involution—(i.e., $\bar{\bar{x}} = x$, $\overline{xy} = \bar{y}\bar{x}$, $\forall x, y \in \mathfrak{B}$) over Φ and let $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{B}$ with product $(x, y)(z, w) = (xz + \mu\bar{w}y, wx + y\bar{z})$ for some $\mu \in \Phi^*$. $\mathfrak{A}$ thus becomes an algebra with involution—defined by $\overline{(x, y)} = (\bar{x}, -y)$. If one begins with $\mathfrak{B} = \Phi$, $-$ = identity, and continues this process four times one obtains at each step an algebra on which is defined a quadratic form $n(x)1 = x\bar{x}$ and a linear form $t(x)1 = x + \bar{x}$. Moreover the form n permits compositions $(n(ab) = n(a)n(b))$, hence these constructions yield composition algebras. Indeed, these algebras exhaust the class of composition algebras **[37]**. Attempting to carry out the process beyond four steps yields new algebras which, while no longer composition algebras, are still of some intrinsic interest and have been studied in **[61]** and **[9]**.

It is of historical interest to observe that if we begin constructing algebras from $\mathbb{R}$, always selecting $\mu = -1$, we obtain algebras which were known in other contexts well before the general theory developed, namely $\mathbb{R}$, $\mathbb{C}$, Hamilton's quaternions, and the Cayley numbers [**14**]. For general fields Φ and arbitrary μ one obtains by the construction: Φ; a degree 2 separable, commutative, associative algebra over Φ; a central simple degree 2 associative (but not commutative) algebra over Φ (a *generalized quaternion algebra*); and an 8-dimensional, simple algebra which is alternative $\left(a^2 b = a(ab),\ ab^2 = (ab)b\ \forall\, a, b\right)$ but not associative. These latter algebras are the *octonion* (generalized Cayley) algebras.

The composition algebras are quadratic in the sense that every $x \in \mathfrak{A}$ satisfies $x^2 - t(x)x + n(x)1 = 0$. Thus, the forms t and n play a most important role in the structure of such algebras. One sees, for instance, that $\mathfrak{A}$ is a division algebra ($\forall\, a \in \mathfrak{A}\ \exists\, b \in \mathfrak{A}$ with $ab = ba = 1$) if and only if n is anisotropic $\left(n(a) \neq 0\ \forall\, a \neq 0\right)$. Somewhat more surprising is

3.1. *There is, up to isomorphism, exactly one composition of each dimension* 2, 4 *or* 8 *over* Φ *which is not a division algebra.*

Indeed, one can show that if n is not anisotropic, it must be of maximal Witt index (there is $\mathfrak{U} \subseteq \mathfrak{A}$ of dimension $\frac{1}{2}\dim \mathfrak{A}$ such that $n(a) = 0\ \forall\, a \in \mathfrak{U}$). 3.1 then is immediate from

3.2. *If* $(\mathfrak{A}_1, n_1)$, $(\mathfrak{A}_2, n_2)$ *are composition algebras over* Φ, *then* $\mathfrak{A}_1 \cong \mathfrak{A}_2$ *if and only if* n_1 *is equivalent to* n_2.

The unique non-division algebra is called a *split* composition algebra. Clearly, the split quaternion algebra, being central simple of degree 2 over Φ must be isomorphic to the matrix algebra Φ_2. Zorn [**89**] discovered that the split octonion algebra can also be viewed as an algebra of matrices as follows: Let

$$\mathfrak{O} = \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \middle| \alpha, \beta \in \Phi,\ a, b \in \Phi^{(3)} \right\} \quad \text{where} \quad \Phi^{(3)} = \{(\gamma_1, \gamma_2, \gamma_3) | \gamma_i \in \Phi\}.$$

Define addition in $\mathfrak{O}$ componentwise, and multiplication by

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma + a \cdot d & \alpha c + \delta a - b \times d \\ \gamma b + \beta d + a \times c & \beta\delta + b \cdot c \end{pmatrix}$$

where $\cdot$ and $\times$ are the usual vector dot and cross products. $\mathfrak{O}$ is then a split octonion algebra with involution

$$\overline{\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}} = \begin{pmatrix} \beta & -a \\ -b & \alpha \end{pmatrix}.$$

We shall see in subsequent sections that similar "matrix like" constructions play a role in representing other exceptional Lie algebras. Sagle, [**59**], has investigated similar structures in another context.

When looked at in the context of the structure theory of algebras, the octonion algebras are seen to play an exceptional role among the simple algebras (simple = no subspace $\mathfrak{I}$ with $\mathfrak{A}\mathfrak{I} \subseteq \mathfrak{I}$, $\mathfrak{I}\mathfrak{A} \subseteq \mathfrak{I}$) since, [**48**]:

3.3. *If $\mathfrak{A}$ is a simple alternative algebra with centre Φ which is not associative, then $\mathfrak{A}$ is isomorphic to an octonion algebra.*

It is an immediate consequence of 3.3 that if P is algebraically closed and $\Phi \subseteq P$ and if $\mathfrak{O}$ is an octonion algebra over P,

3.4. *Every Φ-form of $\mathfrak{O}$ is an octonion algebra over Φ,*

where by a *Φ-form* of an arbitrary algebra $\mathfrak{A}$ over P we mean a Φ-algebra $\mathfrak{B}$ such that $\mathfrak{B}_P (\equiv \mathfrak{B} \otimes_\Phi P)$ is P-isomorphic to $\mathfrak{A}$. For certain special fields, one can determine precisely the isomorphism classes of octonion algebras defined over Φ [3]. For algebraically closed Φ, there is only one, the split form. The same is true for any p-adic field. For $\Phi = \mathbb{R}$ there are two non-isomorphic forms while for Φ an algebraic number field there are 2^t non-isomorphic forms where t = number of real completions of Φ. We shall return to questions of Φ-forms of algebras, particularly exceptional Lie algebras, in subsequent sections.

The importance of the octonion algebras in Lie theory rests on

3.5. Der $\mathfrak{O}$ *is a simple Lie algebra of type G_2 if $\mathfrak{O}$ is an octonion algebra over an algebraically closed field* [36].

In this setting, the algebra G_2 appears little different from the classical Lie algebras which we observed in the form of linear Lie algebras leaving some structure invariant in §2. As is the case for the classical algebras, the linear realization of G_2 allows specific identification of the elements of the algebra as linear transformations, since we have [43]:

3.6. *Every derivation of an octonion algebra is inner,*

where by *inner derivations* in alternative algebras we mean linear combinations of transformations

3.7. $D_{a,b} = [L_a, L_b] + [L_a, R_b] + [R_a, R_b]$ *with R_a* (resp. L_a) *denoting right* (resp. *left*) *multiplication by a.*

The automorphism group of G_2 in this specific realization is easily obtained using 2.2.1 for $\mathfrak{B} = \mathfrak{O}$, s = multiplication in $\mathfrak{O}$, the fact that G_2 has no diagram automorphisms and the fact that $\mathfrak{L}_P(G_2, \mathfrak{O})$ has a trivial centre, to obtain

3.8. Aut (Der $\mathfrak{O}$) = Inn (Aut $\mathfrak{O}$) *or, simply put, every automorphism of* Der $\mathfrak{O}$ *is conjugation by an automorphism of $\mathfrak{O}$.*

While the octonions are of primary interest owing to the connection with the algebra G_2, they also provide a module for the algebra D_4 in the obvious way since $\dim \mathfrak{O} = 8$ and n is non-degenerate so

3.9. *$o(\mathfrak{O}, n)$ is an algebra of type D_4.*

In this realization the algebra structure of $\mathfrak{O}$ does not enter. However, it does enter in the determination of the automorphism group of D_4 via the *Principle of Local Triality.*

3.10. *Let* $A \in o(\mathfrak{O}, n)$. *Then there are uniquely defined* $A^{\sigma_1}, A^{\sigma_2} \in o(\mathfrak{O}, n)$ *such that* $(xy)A = (xA^{\sigma_1})y + x(yA^{\sigma_2})$ *for all* $x, y \in \mathfrak{O}$.

The mappings σ_1, σ_2 are automorphisms of $o(\mathfrak{O}, n)$ generating a subgroup J of $\mathrm{Aut}\, o(\mathfrak{O}, n)$ (the group of triality automorphisms) isomorphic to S_3 and complementary to the group of inner automorphisms $G_\Phi(\mathfrak{L}, \mathfrak{L})$. Since the group of diagram automorphisms is S_3 we obtain for algebraically closed Φ, using again 2.2.1.

3.11. $\mathrm{Aut}\, o(\mathfrak{O}, n) = \mathrm{Inn}\, O(\mathfrak{O}, n)\, J$ (*semidirect*).

We shall see in §4 that if we utilize a representation of D_4 in an exceptional Jordan algebra, the form of the automorphism group can be simplified to make all automorphisms appear inner.

4. *The exceptional Jordan algebra and* F_4

In §3 we saw that the exceptional Lie algebra G_2 was closely related to an exceptional alternative algebra $\mathfrak{O}$. In this section, we find an analogous connection between an exceptional member of the class of Jordan algebras and the exceptional Lie algebra of type F_4. We shall also observe that this Jordan algebra provides convenient representations of the algebras D_4 and E_6.

In its simplest algebraic setting, the class of Jordan algebras is encountered as follows. In Φ_n, we have $\Phi_n = \mathfrak{H}(\Phi_n) \oplus \mathfrak{S}(\Phi_n)$ where $\mathfrak{H}(\Phi_n)$ is the set of symmetric matrices and $\mathfrak{S}(\Phi_n)$ the set of skew symmetric matrices. We have seen in §2.1 that $\mathfrak{S}(\Phi_n)$ becomes a Lie algebra with a suitably defined product which we can identify here as twice the projection of the usual matrix product onto $\mathfrak{S}(\Phi_n)$. Defining a product $x \cdot y$ on $\mathfrak{H}(\Phi_n)$ similarly, we see $x \cdot y = \frac{1}{2}(xy + yx)$. A simple calculation shows

4.1. $x \cdot y = y \cdot x$

$$(x^2 \cdot y) \cdot x = x^2 \cdot (y \cdot x) \text{ where } x^2 = x \cdot x \text{ for all } x, y \in \mathfrak{H}(\Phi_n).$$

These identities were first observed in a very different context (quantum mechanics [**44**]) and are taken as defining identities of the class of *Jordan algebras*.

While our considerations involve only fields of characteristic zero, this definition of Jordan algebra is sufficient for all fields except those of characteristic 2, where pathologies arise. To circumvent these pathologies, as well as to make accessible the study of the structure of Jordan rings which are not necessarily finite dimensional algebras, an alternate definition of Jordan algebras based on axioms for a linear operator $U_x : \mathfrak{J} \to \mathfrak{J}$ ($yU_x = xyx$ for the above example) depending quadratically on x, has been developed by McCrimmon [**52**]. The relationship between the two theories is analogous to that between the theories of symmetric bilinear and quadratic forms, diverging only in characteristic 2.

Even in the context of Jordan algebras defined by 4.1, the operator

$$U_x : y \to 2(y \cdot x) \cdot x - y \cdot x^2$$

plays an important role. This is perhaps best illustrated in the analysis of the structure theory of Jordan algebras [**42**], which follows closely the Artin approach to structure theory of associative algebras with the one variation being that the role of left ideals in associative algebras is played by *inner* (quadratic) *ideals* for the Jordan algebra $\mathfrak{J}$ (i.e., subspaces $\mathfrak{N} \subseteq \mathfrak{J}$ such that $\mathfrak{J}U_{\mathfrak{N}} \subseteq \mathfrak{N}$ where $U_{\mathfrak{N}} = \{U_x | x \in \mathfrak{N}\}$). Note that if $\mathfrak{J} = \mathfrak{H}(\Phi_n)$, $\mathfrak{N}$ is inner if and only if $n\mathfrak{J}n \subseteq \mathfrak{N}\ \forall\, n \in \mathfrak{N}$. With this deviation in method of proof, the result obtained is familiar.

4.2. *If $\mathfrak{J}$ is a semisimple Jordan algebra, $\mathfrak{J}$ is a direct sum of simple ideals.*

Several interesting Jordan algebra constructions are easily obtained as analogues of the Lie algebra constructions of §2.

4.3. *Example.* Let $\mathfrak{A}$ be an associative algebra. $\mathfrak{A}^+$ ($= \mathfrak{A}$ with product $x \cdot y = \frac{1}{2}(xy+yx)$) is a Jordan algebra.

Since any subspace of $\mathfrak{A}^+$ closed under the Jordan product is again a Jordan algebra, we have

4.4. *Example.* Let $\mathfrak{A}$ be an associative algebra with involution τ, $\mathfrak{H} = \mathfrak{H}(\mathfrak{A}, \tau)$ the space of τ-symmetric elements ($x\tau = x$). Then $\mathfrak{H}$ is a Jordan subalgebra of $\mathfrak{A}^+$.

4.5. *Example.* Let $\mathfrak{J} = \Phi \oplus \mathfrak{B}$, $\mathfrak{B}$ a vector space over Φ with bilinear form f. Then $\mathfrak{J} = \mathfrak{J}(\mathfrak{B}, f)$ with product $x \cdot y = (\alpha\beta + f(v, w), \alpha w + \beta v)$ for $x = (\alpha, v), y = (\beta, w)$ is a Jordan algebra (the Jordan algebra of the form f). $\mathfrak{J}(\mathfrak{B}, f)$ can be identified naturally with a subalgebra of $C(\mathfrak{B}, f)^+$, $C(\mathfrak{B}, f)$ the Clifford algebra.

As a generalization of 4.4 we consider

4.6. *Example.* Let $\mathfrak{D}$ be a (non-associative) algebra over Φ with involution $d \to \bar{d}$. Let $\gamma = \{\gamma_1, \ldots, \gamma_n\}$, $\gamma_i \in \Phi^*$, and define $\tau_\gamma : X \to \gamma^{-1}\bar{X}^t\gamma$ on $\mathfrak{D}_n$, the algebra of $n \times n$ matrices with coefficients in $\mathfrak{D}$. τ_γ is an involution and the space of symmetric elements $\mathfrak{H}(\mathfrak{D}_h, \gamma)$ is an algebra relative to the product $x \cdot y = \frac{1}{2}(xy + yx)$. $\mathfrak{H}(\mathfrak{D}_n, \gamma)$ need not be Jordan. Indeed, if $n \geqslant 4$, $\mathfrak{H}(\mathfrak{D}_n, \gamma)$ is Jordan if and only if $\mathfrak{D}$ is associative (in which case we have here a special case of 4.4) while if $n = 3$, $\mathfrak{H}(\mathfrak{D}_n, \gamma)$ is Jordan if and only if $\mathfrak{D}$ is alternative.

In the event that $\mathfrak{D}$ is octonion, 4.6 yields a Jordan algebra which, at least superficially, appears not to be a subalgebra of $\mathfrak{A}^+$ for any associative $\mathfrak{A}$. It is the culmination of a detailed analysis of identities satisfied by Jordan algebras [**42**] that $\mathfrak{H}(\mathfrak{O}_3, \gamma)$ is not even a homomorphic image of any such subalgebra.

Defining a Jordan algebra to be *special* if it is isomorphic to a subalgebra of some $\mathfrak{A}^+$, $\mathfrak{A}$ associative, *exceptional* otherwise, the above result implies

4.7. $\mathfrak{H}(\mathfrak{O}_3, \gamma)$, $\mathfrak{O}$ *octonion, is exceptional.*

We shall throughout use $\mathfrak{H}(\mathfrak{O}_3)$ to denote $\mathfrak{H}(\mathfrak{O}_3, \gamma)$, $\gamma = \operatorname{diag}\{1, 1, 1\}$. We then have

4.8. *Let Φ be algebraically closed, $\mathfrak{J}$ a simple Jordan algebra over Φ, then $\mathfrak{J}$ is isomorphic to one of*

(i) $\Phi_n^{\ +}$,

(ii) $\mathfrak{H}(\Phi_n)$,

(iii) $\mathfrak{H}(\mathfrak{Q}_n, \tau)$, $\mathfrak{Q} = \Phi_2$, $\tau : A \to JA^t J^{-1}$, $J = \operatorname{diag}\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$,

(iv) $\mathfrak{J}(\mathfrak{B}, f)$, *f non-degenerate, or*

(v) $\mathfrak{H}(\mathfrak{O}_3)$, $\mathfrak{O}$ *octonion.*

This provides a second rationale for considering $\mathfrak{H}(\mathfrak{O}_3)$ as exceptional since, as with the exceptional Lie algebras, it is the only simple algebra which does not belong to an infinite class.

In our study of Lie algebras, $\mathfrak{H}(\mathfrak{O}_3, \gamma)$ is important since

4.9. Der $(\mathfrak{H}(\mathfrak{O}_3, \gamma))$ *is a simple Lie algebra of type* F_4 **[83]**.

This displays once again the exceptional nature of $\mathfrak{H}(\mathfrak{O}_3, \gamma)$ among simple Jordan algebras, as it is the only such algebra with an exceptional Lie algebra as derivation algebra.

The internal structure of $\mathfrak{H}(\mathfrak{O}_3, \gamma)$ is very similar to that of the matrix algebra $\mathfrak{H}(\Phi_3)$. In particular, if we write $a[i, j] = \gamma_j a e_{ij} + \gamma_i \bar{a} e_{ji}$, $a \in \mathfrak{O}$, $i \neq j$, where e_{ij} denotes the usual matrix unit, we can write any $x \in \mathfrak{H}(\mathfrak{O}_3, \gamma)$ as

4.10. $x = \sum_1^3 \alpha_i e_{ii} + \sum_{i=1}^3 a_i[j, k]$, $\alpha_i \in \Phi$, $a_i \in \mathfrak{O}$, (i, j, k) *a cyclic permutation of* $(1, 2, 3)$
and define

4.11.

(i) $N(x) = \alpha_1 \alpha_2 \alpha_3 + \alpha_1 \gamma_2 \gamma_3 n(a_1) + \gamma_1 \alpha_2 \gamma_3 n(a_2) + \gamma_1 \gamma_2 \alpha_3 n(a_3)$.

(ii) $T(x) = \alpha_1 + \alpha_2 + \alpha_3$.

(iii) $x^\# = \sum(\alpha_j \alpha_k - \gamma_j \gamma_k n(a_i)) e_{ii} + \sum(\gamma_i \overline{(a_j a_k)} - \alpha_i a_i)[j, k]$.

The cubic form N is analogous to the determinant in $\mathfrak{H}(\Phi_3)$, the linear form T is analogous to the trace, and $x^\#$ is analogous to the adjoint of x since $x \cdot x^\# = N(x)1$. Defining $x \in \mathfrak{H}(\mathfrak{O}_3, \gamma)$ to be *invertible* if there is $y \in \mathfrak{H}(\mathfrak{O}_3, \gamma)$ such that $x \cdot y = 1$, $x^2 \cdot y = x$, we see that in analogy with the situation in composition algebras and matrix algebras, x is invertible if and only if $N(x) \neq 0$, in which case $x^{-1} = N(x)^{-1} x^\#$. From this follows the standard matrix fact $(x^\#)^\# = N(x)x$ for x invertible, hence by a density argument for all x.

In terms of # one can define the *rank one* elements of $\mathfrak{H}(\mathfrak{O}_3, \gamma)$ $(x^\# = 0, x \neq 0)$ which play an important role in the structure theory (being either idempotent or nilpotent of order 2) and in the analysis of isomorphisms leading **[70]** to

4.12. $\mathfrak{H}(\mathfrak{O}_3{}^1, \gamma^1) \cong \mathfrak{H}(\mathfrak{O}_3{}^2, \gamma^2)$ *if and only if the norms* N_i *and traces* T_i *are equivalent.*

It can easily be shown by example that this is the best possible result in the direction of 3.2 since one can find algebras with equivalent norm forms which are non-isomorphic. Equivalence of norms in exceptional simple Jordan algebras is equivalent rather to isotopy of algebras where an *isotope* of a Jordan algebra is defined as follows: let $u \in \mathfrak{J}$ be invertible and define on $\mathfrak{J}$ a new product

$$x \cdot_u y = (x \cdot u) \cdot y + (y \cdot u) \cdot x - (x \cdot y) \cdot u.$$

This product satisfies 4.1, hence yields a new Jordan algebra we denote by $\mathfrak{J}^{(u)}$, the *u-isotope* of $\mathfrak{J}$. By a slight abuse of language, we call $\mathfrak{J}_1$ an isotope of $\mathfrak{J}_2$ if there is $u \in \mathfrak{J}_2$ such that $\mathfrak{J}_1 \cong \mathfrak{J}_2^{(u)}$. We then have [40]:

4.13. *$\mathfrak{H}(\mathfrak{O}_3{}^1, \gamma^1)$ is an isotope of $\mathfrak{H}(\mathfrak{O}_3{}^2, \gamma^2)$ if and only if the norm forms N_i are equivalent.*

A useful and interesting consequence of 4.13 is

4.14. *$\mathfrak{H}(\mathfrak{O}_3{}^1, \gamma^1)$ is an isotope of $\mathfrak{H}(\mathfrak{O}_3{}^2, \gamma^2)$ if and only if $\mathfrak{O}^1 \cong \mathfrak{O}^2$.*

In a particularly simple situation, where $u = \gamma 1$, it is easy to see that multiplication in the u-isotope of a Jordan algebra $\mathfrak{J}$ is given simply by $a \cdot_u b = \gamma(a \cdot b)$.

Since the cubic norm form plays such an important role in the study of the exceptional Jordan algebra $\mathfrak{H}(\mathfrak{O}_3, \gamma)$, it is not surprising that one can, beginning with a suitable cubic form, reconstruct a Jordan algebra for which it is the norm form. The construction, due to McCrimmon [53], proceeds as follows. Let $\mathfrak{J}$ be a vector space over Φ, N a cubic form on $\mathfrak{J}$, $c \in \mathfrak{J}$ with $N(c) = 1$. Suppose

$$T(x, y) = (\partial_x N|_c)(\partial_y N|_c) - \partial_x \partial_y N|_c$$

is a non-degenerate symmetric bilinear form where $\partial_u f|_a$ denotes the "directional derivative" in direction u, evaluated at a, of the polynomial function f. Define $x^\#$ by $T(x^\#, y) = \partial_y N|_x$. Then $x^\#$ depends quadratically on x and one can define a bilinear product $x \times y = (x+y)^\# - x^\# - y^\#$. Setting $T(x) = T(x, c)$ one has

4.15. *$\mathfrak{J}$, with product $x \cdot y = T(x)y + T(y)x - T(x \times y)c + x \times y$ is a Jordan algebra (in the setting of quadratic Jordan algebras, the structure is given by*

$$yU_x = T(x, y)x - x^\# \times y)$$

with identity c.

An algebra constructed in this way is called the *Jordan algebra of the (admissible) cubic form N* (with basepoint c). As special cases of these algebras we have

4.16. *Example.* Let $\mathfrak{J} = \mathfrak{H}(\mathfrak{C}_3, \gamma)$ for a composition algebra $\mathfrak{C}$. Define N by 4.11(i) and take $c = e_{11} + e_{22} + e_{33}$. The product of 4.15 gives $\mathfrak{H}(\mathfrak{C}_3, \gamma)$ the usual Jordan structure of 4.6.

4.17. *Example.* Let $\mathfrak{J} = \mathfrak{H}(\mathfrak{C}_3, \gamma)$ be as in 4.16, N as in 4.11, $c \in \mathfrak{J}$ with $N(c) \neq 0$. Define $N'(x) = N(x)N(c)$ so $N'(c^{-1}) = 1$. Then the algebra constructed from N' and c^{-1} is the c-isotope of $\mathfrak{H}(\mathfrak{C}_3, \gamma)$.

4.18. *Example.* Let $\mathfrak{J}$ be (i) $\Phi \oplus \Phi \oplus \Phi$ with $N(x) = \alpha_1 \alpha_2 \alpha_3$ for $x = (\alpha_1, \alpha_2, \alpha_3)$, $c = (1, 1, 1)$; (ii) $\Phi \oplus \Gamma$, Γ a quadratic field extension of Φ, $N(x) = \alpha_1 N_{\Gamma/\Phi}(\alpha_2)$ for $x = (\alpha_1, \alpha_2)$, $c = (1, 1)$ or (iii) P a separable cubic field extension of Φ, $N(x) = N_{P/\Phi}(x)$, $c = 1$. Then $\mathfrak{J}$ with the product of 4.15 has Jordan structure identical with the usual associative structure on $\mathfrak{J}$.

4.19. *Example.* Let $\mathfrak{J} = \Phi \oplus \mathfrak{V}$, $\mathfrak{V}$ a vector space with non-degenerate bilinear form f, $c_0 \in \mathfrak{V}$ with $f(c_0, c_0) = 1$. For $N(x) = \alpha f(v, v)$ when $x = (\alpha, v)$ and $c = (1, c_0)$, we obtain the direct sum of a 1-dimensional ideal and an ideal with the Jordan structure of 4.5 (f replaced by $f|_{c_0{}^\perp}$).

4.20. *Example.* Let $\mathfrak{A}$ be a separable, degree three associative algebra with 1 with reduced norm form n and trace form t. Let $\mathfrak{J} = \mathfrak{A} \oplus \mathfrak{A} \oplus \mathfrak{A}$ as vector space, $N((a_0, a_1, a_2)) = n(a_0) + \mu n(a_1) + \mu^{-1} n(a_2) - t(a_0 a_1 a_2)$ for some $\mu \in \Phi^*$, $c = (1, 0, 0)$. This yields a Jordan algebra which, in case $\mathfrak{A} = \Phi_3$, is isomorphic to $\mathfrak{H}(\mathfrak{O}_3)$, $\mathfrak{O}$ a split octonion algebra.

4.21. *Example.* Let $\mathfrak{A}$ be separable, degree three, associative over a quadratic extension Γ of Φ with involution $a \to \bar{a}$ of the second kind. Let $u \in \mathfrak{H}(\mathfrak{A})$ satisfy $n(u) = \mu\bar{\mu}$ for some $\mu \in \Gamma^*$. Set $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}) \oplus \mathfrak{A}$ as vector space over Φ and define $N((h, a)) = n(h) + \mu n(a) + \bar{\mu} n(\bar{a}) - t(hau\bar{a})$. For $c = (1, 0)$ this again yields a Jordan algebra.

Examples 4.20 and 4.21 are due to Tits and are of particular interest in view of **[42]**:

4.22. *Let $\mathfrak{J}$ be a Φ-form of $\mathfrak{H}(\mathfrak{O}_3)$, $\mathfrak{O}$ the octonion algebra over the algebraic closure* P *of Φ. Then $\mathfrak{J}$ is isomorphic to an algebra constructed as in* 4.20 *or* 4.21 *with $\mathfrak{A}$ central simple over Φ* (resp. Γ).

Moreover, for suitably selected Φ, μ, $\mathfrak{A}$ one can arrange that N be *anisotropic* $(N(x) \neq 0 \ \forall\, x \neq 0)$, hence that the above construction yields an *exceptional division algebra* (x invertible $\forall\, x \neq 0$) since in general one has **[53]**

4.23. *The Jordan algebra of the cubic form N is a division algebra if and only if N is anisotropic.*

For the exceptional simple Jordan algebras (forms of $\mathfrak{H}(\mathfrak{O}_3)$), $N(x)$ isotropic is equivalent to the existence of a rank one element in $\mathfrak{J}$ (in fact, it is equivalent to $\mathfrak{J}$ being *reduced* in the sense of containing three supplementary orthogonal idempotents). For reduced algebras we have **[60]**:

4.24. *Let $\mathfrak{J}$ be an exceptional simple Jordan algebra with centre Φ. Then $\mathfrak{J}$ is reduced if and only if $\mathfrak{J} \cong \mathfrak{H}(\mathfrak{O}_3, \gamma)$ for some octonion algebra $\mathfrak{O}$. Thus, every exceptional simple Jordan algebra is either a division algebra or of form $\mathfrak{H}(\mathfrak{O}_3, \gamma)$.*

Investigating the reduced exceptional simple Jordan algebras further one finds

4.25. $\mathfrak{H}(\mathfrak{O}_3, \gamma) \cong \mathfrak{H}(\mathfrak{O}_3, \gamma')$ *for any γ, γ' if $\mathfrak{O}$ is split octonion.*

Thus there is a unique reduced algebra with split coefficient algebra. We call this the *split* exceptional simple Jordan algebra.

For special fields Φ one can often make use of the arithmetic of the field to show that the class of exceptional simple Jordan algebras over Φ is quite restricted. In particular one has **[1]**

4.26. *Let $\mathfrak{J}$ be exceptional, simple Jordan over Φ. Then*

(i) *$\mathfrak{J}$ is split if $|\Phi| < \infty$, Φ is p-adic, or Φ is algebraically closed.*

(ii) *$\mathfrak{J}$ is reduced if $\Phi = \mathbb{R}$ or Φ is an algebraic number field.*

As a consequence of 4.14, 4.26, the remarks following 3.4 and further investigation of the relationship between γ and γ' when $\mathfrak{H}(\mathfrak{O}_3, \gamma) \simeq \mathfrak{H}(\mathfrak{O}_3, \gamma')$ it has been shown [3] that there are exactly 3^t isomorphism classes of exceptional simple Jordan algebras defined over an algebraic number field Φ with t real completions.

We note that the algebras constructed via 4.15 in the examples 4.16, 4.18 and 4.19 have been characterized by Schafer [62] as commutative algebras with cubic form permitting composition in the sense that $N(xU_y) = N(x)N(y^2)$, making them in a sense an analogue of composition algebras.

Returning again to our basic concern with Lie algebras, we recall that Der $\mathfrak{J}$ is of type F_4 if $\mathfrak{J}$ is a split exceptional simple Jordan algebra with centre Φ. In exact analogy with 3.8 we have

4.27. $\operatorname{Aut}(\operatorname{Der}\mathfrak{J}) = \operatorname{Inn}(\operatorname{Aut}\mathfrak{J})$.

Moreover, we have an exact description of the elements of Der $\mathfrak{J}$ as *inner derivations*, (sums of form $\sum_{a, b \in \mathfrak{J}}[R_a, R_b]$) since [43]:

4.28. *Every derivation of the split exceptional simple Jordan algebra is inner.*

If as in §3 we turn our attention from the algebra of derivations of $\mathfrak{J}$ to the algebra of transformations leaving the norm form invariant we find [16]:

4.29. *Let* $\mathfrak{J}$ *be the split exceptional simple Jordan algebra over algebraically closed* P, N *the norm form* (4.11) *of* $\mathfrak{J}$. Inv $(\mathfrak{J}, N)$ *is a Lie algebra of type* E_6.

Thus we see that E_6 can be considered in the same context as the classical groups as a linear Lie algebra on $\mathfrak{J}$. An investigation into the form of the particular transformations making up Inv $(\mathfrak{J}, N)$ show [40]

4.30. $\operatorname{Inv}(\mathfrak{J}, N) = R_{\mathfrak{J}_0} \oplus \operatorname{Der}\mathfrak{J} = \{R_a + D \mid T(a) = 0, D \in \operatorname{Der}\mathfrak{J}\}$.

In this form we see easily displayed the natural embedding of F_4 in E_6. Indeed, one sees [38]:

4.31. $\operatorname{Der}(\mathfrak{J}) = \{\ell \in \operatorname{Inv}(\mathfrak{J}, N) \mid 1\ell = 0\}$

which is the Lie algebra analogue of

4.32. $\operatorname{Aut}\mathfrak{J} = \{g \in \operatorname{Aut}(\mathfrak{J}, N) \mid 1g = 1\}$,

which characterizes the automorphisms of the Jordan algebra of a cubic form N with basepoint 1 in terms of the constituents N and 1 only

$$(\operatorname{Aut}(\mathfrak{J}, N) = \{g \in \operatorname{End}\mathfrak{J} \mid N(xg) = N(x)\ \forall\, x \in \mathfrak{J}\}).$$

As for the realization of D_4 in $\mathfrak{O}$ (3.9), the realization of E_6 as Inv $(\mathfrak{J}, N)$ allows explicit description of the automorphism groups as a semidirect product

4.33. $\operatorname{Aut}(\operatorname{Inv}(\mathfrak{J}, N)) = \operatorname{Inn}(\operatorname{Aut}(\mathfrak{J}, N))A$, when Φ is algebraically closed, where A is the group of order 2 generated by the outer automorphism $\tau : \ell \to -\ell^t$, t denoting transpose relative to $T(a, b)$. It is interesting to note that Der $\mathfrak{J}$ is precisely the set of fixed points of τ acting on Inv $(\mathfrak{J}, N)$. In the context of 2.2.1, Aut $(\mathfrak{J}, N) = G_\Phi(\mathfrak{L}, \mathfrak{J})$, and A is isomorphic to the group of diagram automorphisms.

Utilizing the split exceptional simple Jordan algebra $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3)$ we can also identify D_4 as a subalgebra of F_4 via

4.34. $\text{Der}\left(\mathfrak{J}\Big/\sum_1^3 \Phi e_i\right) = \left\{\ell \in \text{Der}\,\mathfrak{J}\,\Big|\left(\sum_1^3 \Phi e_i\right)\ell = 0\right\}$ *is a Lie algebra of type* D_4.

Indeed,

$$\text{Der}\left(\mathfrak{J}\Big/\sum_1^3 \Phi e_i\right)$$

is naturally isomorphic to $o(\mathfrak{O}, n)$ via the map $A \to \bar{A}$ where

$$x\bar{A} = a_1 A^{\sigma_2}[2, 3] + \overline{a_2 A}[3, 1] + a_3 A^{\sigma_1}[1, 2]$$

for x as in (4.10) and σ_1, σ_2 the triality automorphisms of (3.10) [43].

In this realization, the group of automorphisms of D_4 takes a much nicer form than (3.11) since [4] for Φ algebraically closed

4.35. $$\text{Aut}\left(\text{Der}\left(\mathfrak{J}\Big/\sum_1^3 \Phi e_i\right)\right) = \text{Inn}\left(\text{Aut}\left(\mathfrak{J}\Big/\sum_1^3 \Phi e_i\right)\right) \textit{ for}$$

$$\text{Aut}\left(\mathfrak{J}\Big/\sum_1^3 \Phi e_i\right) = \left\{g \in \text{Aut}\,\mathfrak{J}\,\Big|\left(\sum_1^3 \Phi e_i\right)g \subseteq \sum_1^3 \Phi e_i\right\}.$$

5. $\mathfrak{J}$-*ternary algebras*

In the preceding sections we have observed that all exceptional Lie algebras except E_7 and E_8 admit linear representation as algebras leaving invariant certain algebraic structures on vector spaces. Indeed, these algebraic structures are interesting in their own right and have been studied quite extensively prior to the discovery of their value in the context of Lie algebras. In this section we introduce another algebraic structure, an exceptional member of a class of ternary algebras (vector spaces with trilinear composition $(x, y, z) \to xyz$), which serves in an analogous way as module for E_7. (Algebras of type E_8 we shall ignore in this context, since no useful linear realization is known at this time.)

A $\mathfrak{J}$-ternary algebra [6], [32] is a vector space $\mathfrak{M}$ with trilinear composition satisfying

5.1.

$$\left.\begin{array}{l}\text{(i) } L_{u,v} - L_{v,u} = R_{u,v} - R_{v,u} \\ \text{(ii) } [R_{v,w}, R_{x,y}] = R_{vR_{x,y},w} + R_{v,wR_{y,x}}\end{array}\right\} \forall\, x, y, z, u, v, w \in \mathfrak{M}.$$

where $L_{u,v} : x \to uvx$ *and* $R_{u,v} : x \to xuv$.

It should be noted that the identity 5.1(ii) differs from the usual identity for Jordan [56] and Lie [51] triple systems only in the sign of one summand. Kantor [45] has axiomatically characterized a closely related class of ternary algebras using the latter identity.

The name $\mathfrak{J}$-ternary arises since by (i) and (ii) combined, the transformations $\langle u, v\rangle$ ($\equiv R_{u,v}-R_{v,u}$) span a Jordan subalgebra $\mathfrak{J}$ of $(\mathrm{End}\,\mathfrak{M})^+$ so that $\mathfrak{M}$ is a special $\mathfrak{J}$-module (in the sense of [42]).

Using the algebras we have previously encountered in this survey, as well as associative algebras, one can construct several interesting $\mathfrak{J}$-ternary algebras.

5.2. *Example* [6]. Let $\mathfrak{A}$ be an associative algebra with involution J, $\mathfrak{B}$ a left $\mathfrak{A}$-module, h a non-degenerate J-skew hermitian form on $\mathfrak{B}$. Define

$$uvw = \tfrac{1}{2}\big(h(v, w)u+h(u, w)v+h(u, v)w\big).$$

Then $\mathfrak{B}$ is a $\mathfrak{J}$-ternary algebra with $\mathfrak{J} \subseteq \mathfrak{H}(\mathfrak{A}, J)$ (acting in the natural way on $\mathfrak{B}$).

5.3. *Example* [7]. Let $\mathfrak{C}_1, \mathfrak{C}_2$ be composition algebras (see §2) over Φ with involutions $-_1$, $-_2$. Set $\mathfrak{A} = \mathfrak{C}_1 \otimes_\Phi \mathfrak{C}_2$ with the usual multiplication and let $J = -_1 \otimes -_2$. Then $(\mathfrak{A}, J)$ is an algebra with involution over Φ. Picking $t \in \mathfrak{A}$ such that $t^J = -t$ we define $xyz = \tfrac{1}{2}\big((xy^J)(tz)+(y(z^J t))x+(x(z^J t))y\big)$. $\mathfrak{A}$ again is a $\mathfrak{J}$-ternary algebra where $\mathfrak{J} \subseteq \mathfrak{H}(\mathfrak{A}, -J)$ (acting naturally on $\mathfrak{A}$). $\mathfrak{H}(\mathfrak{A}, -J)$ is a Jordan algebra relative to $a\cdot b = \tfrac{1}{2}(a(tb)+b(ta))$.

5.4. *Example* [19]. Let $\mathfrak{J}$ be either the Jordan algebra of an admissible non-degenerate cubic form N (§4) with basepoint 1, trace bilinear form T and cross-product $\times$, or the Jordan algebra of a non-degenerate quadratic form Q with base-point c [53] with $T(a, b) = Q(a, b^*)$ and N, $\times$ identically zero. Set

$$\mathfrak{M}(\mathfrak{J}) = \left\{\begin{pmatrix}\alpha & a\\ b & \beta\end{pmatrix} \Big| \alpha, \beta\in\Phi, a, b\in\mathfrak{J}\right\} \quad \text{and for} \quad x_i = \begin{pmatrix}\alpha_i & a_i\\ b_i & \beta_i\end{pmatrix}, \quad i = 1, 2, 3$$

define:

$$x_1 x_2 x_3 = \begin{pmatrix}\gamma & c\\ d & \delta\end{pmatrix}$$

where

$$\gamma = \alpha_1\beta_2\alpha_3+2\alpha_1\alpha_2\beta_3-\alpha_3 T(a_1, b_2)-\alpha_2 T(a_1, b_3)-\alpha_1 T(a_2, b_3)+T(a_1, a_2\times a_3)$$

$$c = (\alpha_2\beta_3-T(b_3, a_2))a_1+(\alpha_1\beta_3-T(b_3, a_1))a_2+(\alpha_1\beta_2-T(a_1, b_2))a_3-\alpha_1(b_2\times b_3)$$
$$-\alpha_2(b_1\times b_3)-\alpha_3(b_1\times b_2)+(a_1\times a_3)\times b_2+(a_2\times a_3)\times b_1+(a_1\times a_2)\times b_3$$

$$\delta = -\gamma^\sigma$$

$$d = -c^\sigma, \quad \text{where} \quad \sigma = (\alpha\beta)(ab).$$

$\mathfrak{M}(\mathfrak{J})$ is a Φ-ternary algebra.

The class of $\mathfrak{J}$-ternary algebras has only recently been introduced (primarily to facilitate the handling of algebras of type BC_1, in the sense of [67]) and the structure of such algebras has not been thoroughly investigated independent of the Lie algebra setting. In the special case $\mathfrak{J} = \Phi$ it has been shown [22] that for Φ algebraically closed of characteristic zero, the algebras of 5.2 with $\mathfrak{A} = \Phi$ and of 5.4 exhaust all isomorphism classes of $\mathfrak{J}$-ternary algebras (called *symplectic algebras* in **[22]**). Indeed, every such algebra becomes, upon symmetrization of the product, a *Freudenthal triple system* in the sense of **[55]** and the classification reduces to the classification of such triple systems.

Restricting ourselves to the triple systems of 5.4 which are of most interest for our purposes, one finds that $\langle x_1, x_2 \rangle = \alpha_1 \beta_2 - \alpha_2 \beta_1 - T(a_1, b_2) + T(a_2, b_1)$ is a skew symmetric bilinear form on $\mathfrak{M}(\mathfrak{J})$ and that $q(x) = \langle x, xxx \rangle$ is a quartic form on $\mathfrak{M}(\mathfrak{J})$. In the particular case that $\mathfrak{J}$ is the exceptional simple Jordan algebra over Φ, $\mathfrak{M}(\mathfrak{J})$ is 56-dimensional and we have **[55]** for $\operatorname{Der}\mathfrak{M}(\mathfrak{J}) = \{D \in \operatorname{End}\mathfrak{M}(\mathfrak{J}) | (xyz)D = (xD)yz + x(yD)z + xy(zD)\}$,

5.5. $\operatorname{Der}(\mathfrak{M}(\mathfrak{J}))$ *is a Lie algebra of type* E_7.

Indeed, as $\mathfrak{J}$ runs through the algebras $\mathfrak{H}(\mathfrak{C}_3)$ $\mathfrak{C}$ a composition algebra of dimension 1, 2, 4 or 8 over Φ, $\operatorname{Der}\mathfrak{M}(\mathfrak{J})$ runs through the algebras C_3, A_5, D_6 and E_7, a sequence we shall see again in §6. The algebra $\mathfrak{M}(\mathfrak{J})$, $\mathfrak{J}$ split exceptional simple, will be called the *split exceptional* Φ*-ternary algebra* because of its connection with the exceptional algebra E_7.

That $\operatorname{Der}\mathfrak{M}(\mathfrak{J})$ is of type E_7 is a consequence of the fact that

5.6. $\operatorname{Der}\mathfrak{M}(\mathfrak{J}) = \operatorname{Inv}(\mathfrak{M}(\mathfrak{J}), q) \cap \operatorname{Inv}(\mathfrak{M}(\mathfrak{J}), \langle,\rangle) = \operatorname{Inv}(\mathfrak{M}(\mathfrak{J}), q)$

and that the algebra $\operatorname{Inv}(\mathfrak{M}(\mathfrak{J}), q)$ can be shown by explicit calculation **[65]** to be of type E_7, a fact known to Cartan **[12]** (without reference to $\mathfrak{J}$) and elaborated on by Freudenthal **[27]** (in the context of Jordan algebras). Freudenthal, in fact, first introduced a ternary product on $\mathfrak{M}(\mathfrak{J})$ in this context, a product which can be derived from that of 5.4 by symmetrizing relative to two arguments.

Turning briefly to a discussion of the Φ-forms of $\mathfrak{M}(\mathfrak{J})$, $\mathfrak{J}$ exceptional simple Jordan over Φ, (the *exceptional* Φ*-ternary algebras*), one sees that the algebras $\mathfrak{M}(\mathfrak{J})$ are characterized in this class in a manner analogous to the characterization of the $\mathfrak{H}(\mathfrak{C}_3, \gamma)$ among the exceptional Jordan algebras, namely **[24]**:

5.7. *An exceptional* Φ*-ternary algebra is isomorphic to an algebra* $\mathfrak{M}(\mathfrak{J})$ *if and only if there is* $x \in \mathfrak{M}$ *such that* $U_x : y \to xyx$ *is a rank one linear transformation.*

An algebra satisfying the condition of 5.7 is called *reduced*, an element x with U_x of rank one is called a *rank one* element. As is the case of exceptional Jordan algebras, an exceptional Φ-ternary algebra which is reduced has a "norm form" q which is isotropic. It is not known whether a non-reduced algebra can have a norm form which is isotropic. Rather, one knows

5.8. *An exceptional* Φ*-ternary algebra* $\mathfrak{M}$ *is reduced if and only if there is* $x \in \mathfrak{M}$ *with* $q(x) = -6$, *where* $q(x)$ *is given in general by* $x(xxx)x = \frac{1}{2}q(x)x$ *for every* $x \in M$.

For special fields, we have **[24]**, **[26]**:

5.9. *Let* $\mathfrak{M}$ *be an exceptional* Φ*-ternary algebra,* Φ *real, p-adic, algebraically closed, or an algebraic number field. Then* $\mathfrak{M}$ *is reduced.*

For reduced algebras (in contrast to the Jordan case), isomorphism is completely determined by the Jordan coordinate algebra $\mathfrak{J}$ since

5.10. $\mathfrak{M}(\mathfrak{J}) \cong \mathfrak{M}(\mathfrak{J}')$ *if and only if* $\mathfrak{J}$ *is an isotope of* $\mathfrak{J}'$.

As consequences, one sees that there are, up to isomorphism, exactly two exceptional Φ-ternary algebras if $\Phi = \mathbb{R}$, one exceptional Φ-ternary if Φ is algebraically closed or p-adic, and 2^t exceptional Φ-ternary algebras if Φ is an algebraic number field with t real completions.

In the setting provided by 5.5 (or 5.6), one sees E_7 in the context of the classical linear groups and can apply 2.2.1 and related arguments to see for algebraically closed Φ

5.11. Aut (Der $\mathfrak{M}(\mathfrak{J})$) = Inn (Aut $\mathfrak{M}(\mathfrak{J})$), *where* $A \in \operatorname{End} \mathfrak{M}(\mathfrak{J})$ *is an automorphism of* $\mathfrak{M}(\mathfrak{J})$ *if* $(xyz)A = xAyAzA$ *for all* $x, y, z \in \mathfrak{M}(\mathfrak{J})$.

In analogy with the realization of D_4 as a subalgebra of E_6 (4.34), one has for

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

in $\mathfrak{M}(\mathfrak{J})$ and Φ-algebraically closed,

5.12. Der $\left(\mathfrak{M}(\mathfrak{J}) \Big/ \sum_1^2 \Phi e_i\right)$ *is a Lie algebra of type* E_6.

Moreover, the knowledge of Aut (Inv ($\mathfrak{J}, N$)) (4.33) allows us to show in this case

5.13. $\operatorname{Aut}\left(\operatorname{Der}\left(\mathfrak{M}(\mathfrak{J}) \Big/ \sum_1^2 \Phi e_i\right)\right) = \operatorname{Inn}\left(\operatorname{Aut}\left(\mathfrak{M}(\mathfrak{J}) \Big/ \sum_1^2 \Phi e_i\right)\right)$

so, as for D_4, even the outer automorphisms of E_6 appear inner in this context.

6. *Some constructions*

As well as providing convenient linear representations for the exceptional Lie algebras, the algebraic structures introduced in §§3, 4 and 5 provide the constituents of a very useful explicit construction due to Tits **[81]** which yields all exceptional Lie algebras over algebraically closed fields of characteristic zero.

Let $\mathfrak{A}$, $\mathfrak{B}$ be composition algebras over Φ, $\mathfrak{J} = \mathfrak{H}(\mathfrak{B}_3, \gamma)$ where $\gamma = \operatorname{diag}\{\gamma_1, \gamma_2, \gamma_3\}$ (Example 4.6). Denote by T the trace bilinear form on $\mathfrak{J}$, by t the analogous form on $\mathfrak{A}$, by $\mathfrak{A}_0$ (resp. $\mathfrak{J}_0$) the space of elements of trace zero in $\mathfrak{A}$ (resp. $\mathfrak{J}$). Define a product $*$ on $\mathfrak{A}_0$ (resp. $\mathfrak{J}_0$) by projecting the usual product relative to the decomposition $\mathfrak{A} = \Phi 1 \oplus \mathfrak{A}_0$ ($\mathfrak{J} = \Phi 1 \oplus \mathfrak{J}_0$). Define $D_{a,b} \in \operatorname{Der} \mathfrak{A}$ by 3.7 and recall that $[R_x, R_y] \in \operatorname{Der} \mathfrak{J}$. Let $\mathfrak{L}(\mathfrak{A}, \mathfrak{J})$ be the Φ-vector space

$$\operatorname{Der} \mathfrak{A} \oplus \mathfrak{A}_0 \otimes \mathfrak{J}_0 \oplus \operatorname{Der} \mathfrak{J}$$

with product

6.1.

(i) $[X, Y]$ *the usual Lie product in* $\operatorname{Der} \mathfrak{A} \oplus \operatorname{Der} \mathfrak{J}$.

(ii) $[a \otimes x, D+E] = aD \otimes x + a \otimes xE$ *for* $a \in \mathfrak{A}_0$, $x \in \mathfrak{J}_0$, $D \in \operatorname{Der} \mathfrak{A}$, $E \in \operatorname{Der} \mathfrak{J}$.

(iii) $[a \otimes x, b \otimes y] = \frac{1}{12} T(x, y) D_{a,b} + a*b \otimes x*y + \frac{1}{2} t(a, b)[R_x, R_y]$ *for* $a, b \in \mathfrak{A}_0$, $x, y \in \mathfrak{J}_0$.

With this product $\mathfrak{L}(\mathfrak{A}, \mathfrak{J})$ is a Lie algebra (see **[43]** for details). As $\mathfrak{A}$ and $\mathfrak{B}$ vary over the possible composition algebras, the algebras arising are those displayed in the " magic square " of Freudenthal

6.2.

		dim $\mathfrak{B}$					
		Φ	$\Phi^{(3)}$	1	2	4	8
dim $\mathfrak{A}$	1	0	0	A_1	A_2	C_3	F_4
	2	0	$\mathfrak{U}$	A_2	$A_2 \oplus A_2$	A_5	E_6
	4	A_1	$A_1 \oplus A_1 \oplus A_1$	C_3	A_5	D_6	E_7
	8	G_2	D_4	F_4	E_6	E_7	E_8

We see thus that using the octonion algebra and the exceptional simple Jordan algebra we can obtain a concrete realization of E_8 in this way. Carrying out the same construction for $\mathfrak{J} = \Phi$ and $\mathfrak{J} = \Phi^{(3)}$ we get the augmented table 6.2 where $\mathfrak{U}$ is two-dimensional abelian. The last row now displays all true exceptional Lie algebras along with D_4, giving further weight to the decision to consider D_4 as exceptional.

A number of other well-known constructions of the exceptional Lie algebras can be seen to be special cases of the Tits' construction.

6.3. *The* E_6 *construction* $\mathfrak{L}(\mathfrak{J}) = R_{\mathfrak{J}_0} \oplus \operatorname{Der} \mathfrak{J}$ *of* 4.30, $\mathfrak{J}$ *exceptional simple Jordan appears in row* 2, *column* 6 *of* 6.2 *when* $\mathfrak{A} = \Phi \oplus \Phi$. *If* $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3)$ *is reduced, this is also found in row* 4, *column* 4 *with* $\mathfrak{B} = \Phi \oplus \Phi$ (*i.e.*, $\mathfrak{H}(\mathfrak{B}_3) \cong \Phi_3{}^+$).

6.4. *The earlier construction of* E_7 *due to Tits* **[80]**, $\mathfrak{L} = \mathfrak{L}_1 \otimes \mathfrak{J} \oplus \operatorname{Der} \mathfrak{J}$, $\mathfrak{L}_1$ *a simple, three-dimensional Lie algebra, occurs in row* 3, *column* 6 (Der $\mathfrak{A}$ *identifies with* $\mathfrak{L}_1 \otimes 1$).

6.5. *The Koecher construction for* E_7 **[50]**, $\mathfrak{L} = \mathfrak{J} \oplus \bar{\mathfrak{J}} \oplus \widehat{\mathfrak{L}(\mathfrak{J})}$ *where* $\mathfrak{J}$ *is exceptional simple Jordan and* $\widehat{\mathfrak{L}(\mathfrak{J})} = \{R_x + D \mid x \in \mathfrak{J}, D \in \operatorname{Der} \mathfrak{J}\}$ *also occurs in row* 3, *column* 6 *if* $\mathfrak{A}$ *is split.*

6.6. *The derivation algebras of the exceptional algebras*; Der $\mathfrak{O}$ *occurs in row* 4, *column* 1, Der $\mathfrak{J}$ *occurs in row* 1, *column* 6, Der $\mathfrak{M}(\mathfrak{J})$ *occurs in row* 3, *column* 6 *for* $\mathfrak{A}$ *split.* (*Note the connection between the entries of row* 3 *and the algebras obtained as* Der $\mathfrak{M}(\mathfrak{J})$, $\mathfrak{J} = \mathfrak{H}(\mathfrak{C}_3)$, $\mathfrak{C}$ *a composition algebra in the remark following* 5.5).

6.7. *The* E_8 *construction* $\mathfrak{L} = \mathfrak{L}(\mathfrak{J}) \oplus \mathfrak{J} \otimes \mathfrak{V} \oplus \bar{\mathfrak{J}} \otimes \bar{\mathfrak{V}} \oplus \Phi_3'$ **[21]** *for* $\mathfrak{J}$ *exceptional simple Jordan,* $\mathfrak{V}$ *a* 3*-dimensional vector space over* Φ, Φ_3' *the algebra of transformations of trace zero in* $\mathfrak{V}$, $\bar{\mathfrak{V}}$ *the contragredient module to* $\mathfrak{V}$. *This appears in line* 4, *column* 6 *with* $\mathfrak{D}$ *split.*

The construction of Tits has been put in a slightly different form which utilizes directly the algebra $\mathfrak{B}$ rather than the Jordan algebra $\mathfrak{H}(\mathfrak{B}_3, \gamma)$ and thus displays the essential symmetry of the Tits' construction in $\mathfrak{A}$ and $\mathfrak{B}$ (note the symmetry of 6.2) **[87]**. In this setting the construction also allows for the use of " twisted composition algebras " [73] as constituent algebras in place of the algebras $\mathfrak{A}$ and $\mathfrak{B}$, thus giving a larger class of algebras over some fields which are not algebraically closed.

The algebras of §5 have been used **[6]**, [32], **[45]** in another construction of exceptional Lie algebras similar to that of Koecher (6.5) and Meyberg [56]. If $\mathfrak{M}$ is a $\mathfrak{J}$-ternary algebra, $\mathfrak{J}$ spanned by $\{\langle u, v\rangle | u, v \in \mathfrak{M}\}$ a simple Jordan algebra, $R_a : (u, b) \to (\frac{1}{2}ua, a \cdot b)$, $A_{v,w} : (u, b) \to (uvw, \langle v, bw\rangle)$ for $a, b \in \mathfrak{J}$, $v, w, u \in \mathfrak{M}$ one forms $\mathfrak{L}(\mathfrak{J}, \mathfrak{M}) = \bar{\mathfrak{J}} \oplus \bar{\mathfrak{M}} \oplus \mathfrak{L}_0 \oplus \mathfrak{M} \oplus \mathfrak{J}$ ($\mathfrak{L}_0$ = span of $\{A_{v,w} \in \mathfrak{M}\}$, $\bar{\mathfrak{J}}$, $\bar{\mathfrak{M}}$ copies of $\mathfrak{J}$, $\mathfrak{M}$) with product defined by:

$$[A, B] = \text{usual Lie product in } \mathfrak{L}_0$$

$$\left.\begin{array}{ll} [a+v, b+w] = \langle v, w\rangle, & [\overline{a+v}, \overline{b+w}] = \overline{\langle v, w\rangle} \\ [a+v, A] = (a+v)A, & [\overline{a+v}, A] = \overline{(a+v)A^{\varepsilon}} \end{array}\right\} a, b \in \mathfrak{J},\ v, w \in \mathfrak{M},\ A, B \in \mathfrak{L}_0$$

$$[a+v, \overline{b+w}] = 2R_{a \cdot b} - 2[R_a, R_b] + A_{v,w} + aw + \overline{bv}$$

where $\varepsilon : A \to A - 2R_{eA}$, e the identity of $\mathfrak{J}$.

$\mathfrak{L}(\mathfrak{J}, \mathfrak{M})$ is a Lie algebra which, in most cases, is simple **[6]**. For $\mathfrak{M}$ as in 5.4 with $\mathfrak{J}$ exceptional simple one obtains the algebra E_8 [18] in a form first introduced by Freudenthal [27]. In fact, as $\mathfrak{M}$ runs through the algebras $\mathfrak{M}(\mathfrak{J})$, $\mathfrak{J} = \mathfrak{H}(\mathfrak{C}_3, \gamma)$, $\mathfrak{C}$ a composition algebra, $\mathfrak{L}(\mathfrak{J}, \mathfrak{M})$ runs through the algebras listed in line 4 of 6.2. A similar result has been shown [7] for $\mathfrak{L}(\mathfrak{J}, \mathfrak{M})$ as $\mathfrak{M}$ runs through the algebras $\mathfrak{C} \otimes \mathfrak{D}$ of 5.3. It is likely that in fact the algebras $\mathfrak{L}(\mathfrak{J}, \mathfrak{M})$, $\mathfrak{M} = \mathfrak{C}_1 \otimes \mathfrak{C}_2$, yield another construction of all algebras in the magic square 6.2 for suitable selection of $\mathfrak{C}_1$ and $\mathfrak{C}_2$.

7. *Related results in the classification of simple Lie algebras over arbitrary fields*

In §2, we observed that if Φ is algebraically closed, characteristic zero, one can separate the simple algebras into isomorphism classes A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 and G_2. In subsequent sections we saw that explicit descriptions could be given, class by class, for these simple algebras, hence we can give a concrete realization for every simple algebra over Φ. If Φ is not algebraically closed, the situation is considerably different. In particular there are non-split algebras over Φ so the general classification results of §2 do not hold. In this section, we indicate how the linear realizations of the split Lie algebras give rise to associative algebra invariants for all simple Lie algebras (except those related to E_8) and survey the known classification results which have been obtained via an analysis of these invariants.

Throughout, we shall work in the setting Φ an arbitrary field of characteristic zero, P the algebraic closure of Φ, $\mathfrak{L}$ a *central simple* Lie algebra over Φ (i.e., $\mathfrak{L}_P$ is simple).

In general, if $\mathfrak{L}_P$ is isomorphic to a subalgebra of $\mathrm{End}_P(\mathfrak{V})$ for some vector space $\mathfrak{V}$ (i.e., $\mathfrak{L}_P$ has a faithful representation $\mathfrak{V}$), we denote the enveloping P-algebra of $\mathfrak{L}_P$ in $\mathrm{End}_P(\mathfrak{V})$ (the smallest associative P-subalgebra of $\mathrm{End}_P(\mathfrak{V})$ containing $(\mathfrak{L}_P)\phi$) by $\mathfrak{E}(\mathfrak{L}_P)$. Since $\mathfrak{L}\phi$ is a Φ-subspace of $\mathrm{End}_P(\mathfrak{V})$ we define $\mathfrak{E}(\mathfrak{L})$ to be the smallest associative Φ-subalgebra of $\mathrm{End}_P(\mathfrak{V})$ containing $\mathfrak{L}$. It will suffice, for our purposes, always to identify $\mathfrak{L}$ as a Φ-subalgebra of $\mathfrak{L}_P$ and $\mathfrak{L}_P$ as a P-subspace of End $\mathfrak{V}$ and we shall do so, avoiding henceforth the use of ϕ.

Using the realizations for the algebras $\mathfrak{L}_P$ we have discussed in earlier sections one can see that $\mathfrak{E}(\mathfrak{L}_P)$ is given by

7.1.

Type	*Realization*	$\mathfrak{E}(\mathfrak{L}_P)$
A_ℓ	$\{T \in \mathrm{End}(\mathfrak{V} \oplus \mathfrak{V}^*) \mid (x, y^*)T = (xt, y^*t),\ t \in \mathrm{End}\,\mathfrak{V}$ of trace $0\}$ $\dim \mathfrak{V} = \ell+1$	$P_\ell \oplus P_\ell$
B_ℓ	$o(\mathfrak{V}, f)$, $\dim \mathfrak{V} = 2\ell+1$	$P_{2\ell+1}$
C_ℓ	$sp(\mathfrak{V}, f)$, $\dim \mathfrak{V} = 2\ell$	$P_{2\ell}$
$D_\ell, \ell > 4$	$o(\mathfrak{V}, f)$, $\dim \mathfrak{V} = 2\ell$	$P_{2\ell}$
D_4	$\mathrm{Der}\left(\mathfrak{J} \Big/ \sum_1^3 \Phi e_i\right)$, $\mathfrak{J}$ exceptional simple Jordan	$P_8 \oplus P_8 \oplus P_8$
E_6	$\mathrm{Der}\left(\mathfrak{M}(\mathfrak{J}) \Big/ \sum_1^2 \Phi e_i\right)$	$P_{27} \oplus P_{27}$
E_7	$\mathrm{Der}\,\mathfrak{M}(\mathfrak{J})$	P_{56}
F_4	$\mathrm{Der}\,\mathfrak{J}$	P_{26}
G_2	$\mathrm{Der}\,\mathfrak{D}$, $\mathfrak{D}$ octonion	P_7

(Note that we have taken a non-standard realization for $A_\ell = \Phi'_{\ell+1}$.)

It is well known [41] that $\mathfrak{L}$ can be identified as the set of fixed points $\mathfrak{L}_P{}^G$, $G = \mathrm{Gal}\,(P/\Phi)$, where each element of G acts via an s-semilinear automorphism A_s ($[\ell_1, \ell_2]A_s = [\ell_1 A_s, \ell_2 A_s]$, $(\alpha\ell_1)A_s = \alpha^s(\ell_1 A_s)\ \forall\, \ell_i \in \mathfrak{L}_P, \alpha \in P$) and $s \to A_s$ is a homomorphism. A look at the form of the automorphism groups for each $\mathfrak{L}_P$ with the given realization shows $A \in \mathrm{Aut}\,\mathfrak{L}_P$ satisfies $A = \mathrm{Inn}\,a$ where a is an " automorphism " of the representing algebraic structure (3.8, 4.27, 4.35, 5.11, 5.13, and remarks following 2.2.1). It is easy to see that this implies each s-semilinear automorphism is of the form $\mathrm{Inn}\,a_s$ where a_s is an s-semilinear automorphism of the structure of the representing space $\mathfrak{V}$. It follows easily that A_s has a natural extension to a semilinear automorphism $\bar{A}_s = \mathrm{Inn}\,a_s$ of End $\mathfrak{V}$ which leaves $\mathfrak{E}(\mathfrak{L}_P)$ invariant. One then has

7.2. *In all cases in* 7.1, $\mathfrak{E}(\mathfrak{L}) = \mathfrak{E}(\mathfrak{L}_P)^G$, $s \in G$ *acting as* $\bar{A}_s$.

Thus with each Φ algebra $\mathfrak{L}$ we associate a Φ-form of $\mathfrak{E}(\mathfrak{L}_P)$. For the classical algebras, the algebra $\mathfrak{L}$ is easily described in terms of the structure of $\mathfrak{E}(\mathfrak{L})$ since one easily sees that $\mathfrak{E}(\mathfrak{L}_P)$ is an algebra with involution $\bar{\tau}$ leaving $\mathfrak{L}$, hence $\mathfrak{E}(\mathfrak{L})$, invariant so $\mathfrak{E}(\mathfrak{L})$ is simple as algebra with involution $\tau = \bar{\tau}|_{\mathfrak{E}(\mathfrak{L})}$ and one has [88]:

7.3. *If $\mathfrak{L}$ is a classical Lie algebra over Φ (type A_ℓ, B_ℓ, C_ℓ, D_ℓ ($\ell > 4$)), there is an associative algebra $\mathfrak{A}$ with involution τ (possibly of the second kind for A_ℓ) such that* $\mathfrak{L} = \mathfrak{S}(\mathfrak{A}, \tau)'$ (Example 2.1.2), where $\mathfrak{L}'$ *denotes the derived algebra* $[\mathfrak{L}, \mathfrak{L}]$.

Moreover, one can completely describe the isomorphisms of two classical Lie algebras over Φ in this setting since we have

7.4. $\mathfrak{S}(\mathfrak{A}, \tau)' \cong \mathfrak{S}(\mathfrak{A}^1, \tau^1)'$ *if and only if* $(\mathfrak{A}, \tau) \cong (\mathfrak{A}^1, \tau^1)$ *as algebras with involution.*

For the exceptional algebras the connection between the structure of $\mathfrak{L}$ and that of $\mathfrak{E}(\mathfrak{L})$ is less definitive and, in most cases, the description of all Φ-algebras of type X_ℓ remains unknown. The two cases which are completely known, types G_2 and F_4, are handled as a consequence of the general principle (which has been checked in case by case manner in the literature but which is amenable to a general treatment)

7.5. *Let $\mathfrak{B}$ be one of the algebraic structures listed in* 7.1 *for the exceptional algebras $\mathfrak{L}_P$, $\mathfrak{B}_0$ one of the specified subspaces such that* $\mathfrak{L}_P = \operatorname{Der}(\mathfrak{B}/\mathfrak{B}_0)$. *If $\mathfrak{E}(\mathfrak{L})$ is a sum of matrix algebras, then there is a Φ-form $\mathfrak{B}_\Phi$ of the algebra $\mathfrak{B}$, and a form $(\mathfrak{B}_0)_\Phi \subseteq \mathfrak{B}_\Phi$ such that* $\mathfrak{L} \cong \operatorname{Der}(\mathfrak{B}_\Phi/(\mathfrak{B}_0)_\Phi)$.

In the case G_2 (resp. F_4) each Φ-form is defined by $\{A_s = \operatorname{Inn} a_s | s \in G\}$ where a_s is a semiautomorphism of $\mathfrak{O}$ (resp. $\mathfrak{J}$), hence fixes the identity of that algebra. It is easy to see since $A_s A_t = A_{st}$ $\forall\, s, t \in G$, that $a_s a_t = a_{st}$ $\forall\, s, t \in G$, hence $\mathfrak{E}(\mathfrak{L})$ is isomorphic to Φ_7 (Φ_{26}). We thus have from 7.5 [36]

7.6. *Let $\mathfrak{L}$ be an algebra of type G_2 over Φ, then* $\mathfrak{L} \cong \operatorname{Der} \mathfrak{O}$, *$\mathfrak{O}$ an octonion algebra over Φ.*

Moreover, one can easily show using the characterization of automorphisms of Φ-forms in terms of the defining homomorphisms $s \to A_s$ [41]

7.7. $\operatorname{Der} \mathfrak{O} \cong \operatorname{Der} \mathfrak{O}'$ *if and only if* $\mathfrak{O} \cong \mathfrak{O}'$.

Similar results for F_4 yield [83]:

7.8. *Let $\mathfrak{L}$ be an algebra of type F_4 over Φ. Then* $\mathfrak{L} \cong \operatorname{Der} \mathfrak{J}$, *$\mathfrak{J}$ an exceptional simple Jordan algebra with centre Φ.*

7.9. $\operatorname{Der} \mathfrak{J} \cong \operatorname{Der} \mathfrak{J}'$ *if and only if* $\mathfrak{J} \cong \mathfrak{J}'$.

For algebras $\mathfrak{L}$ of type E_6, there are two possible forms $\mathfrak{E}(\mathfrak{L})$ can take, depending on whether one of the defining semiautomorphisms $\bar{A}_s$ permutes the summands of $\mathfrak{E}(\mathfrak{L}_P)$ or not. In the former case, $\mathfrak{E}(\mathfrak{L}) = \mathfrak{A}$ is central simple of degree 27 over a quadratic extension Γ of Φ and admits an involution of the second kind. $\mathfrak{L}$ is said to be of *type* $E_{6\text{II}}$ in this case. Otherwise, $\mathfrak{E}(\mathfrak{L}) = \mathfrak{A}_1 \oplus \mathfrak{A}_2$, $\mathfrak{A}_i$ antiisomorphic central simple algebras of degree 27 over Φ and $\mathfrak{L}$ is of *type* $E_{6\text{I}}$. In either case, since $A_s = \operatorname{Inn} a_s$ where a_s can be shown to be constructed from semilinear transformations of $\mathfrak{H}(\mathfrak{A}_3)$ which preserve the cubic norm N, one can show that $\mathfrak{A}(\mathfrak{A}_i)$ is of exponent 3 or 1, [**43**] hence in particular if $\Phi = \mathbb{R}$, $\mathfrak{A}(\mathfrak{A}_i)$ is a matrix algebra. If Φ is p-adic or an algebraic number field $\mathfrak{A}(\mathfrak{A}_i) = \mathfrak{B}_3$, $\mathfrak{B}$ cyclic of index 3 [**2**] or a matrix algebra, and since there are no degree three division algebras with involution of the second kind over p-adic Φ, $\mathfrak{A}$ is Γ_{27} in the case of p-adic $E_{6\text{II}}$. Combining these results with rather complicated normalization procedures for the transformations a_s one obtains [**23**]:

7.10. *Let $\mathfrak{L}$ be of type $E_{6\text{I}}$ over Φ, Φ real, p-adic, or an algebraic number field and let $\mathfrak{E}(\mathfrak{L}) = (\mathfrak{B}_1)_3 \oplus (\mathfrak{B}_2)_3$. Then there is an octonion algebra $\mathfrak{O}$ over Φ such that $\mathfrak{L} \cong \mathfrak{L}(\mathfrak{O}, \mathfrak{B}_1{}^+)$ (the Tits construction of* 6.1 *with $\mathfrak{J} = \mathfrak{B}_1{}^+$).*

Let $\mathfrak{L}$ be of type $E_{6\text{II}}$ over Φ, Φ real or p-adic so $\mathfrak{E}(\mathfrak{L}) = \Gamma_{27}$, $[\Gamma : \Phi] = 2$. Then there is an exceptional Jordan algebra $\mathfrak{J}$ over Φ such that $\mathfrak{L} \cong \mathfrak{L}(\Gamma, \mathfrak{J})$.

The classification for algebras of type $E_{6\text{II}}$ over an algebraic number field is not complete since the case $\mathfrak{E}(\mathfrak{L})$ of exponent exactly 3 has not been handled. If $\mathfrak{E}(\mathfrak{L}) = \Gamma_{27}$ the result is as in 7.10.

For arbitrary fields, one knows only the special result of 7.5

7.11. *Let $\mathfrak{L}$ be of type E_6 over Φ with $\mathfrak{E}(\mathfrak{L})$ either $\Phi_{27} \oplus \Phi_{27}$ or Γ_{27}, $[\Gamma : \Phi] = 2$. Then there exists an exceptional Φ-ternary algebra $\mathfrak{M}$ and subalgebra $\mathfrak{M}_0$ such that* $\mathfrak{L} \cong \operatorname{Der}(\mathfrak{M}/\mathfrak{M}_0)$.

In analogy with 7.7 and 7.9 one has also

7.12. $\operatorname{Der}(\mathfrak{M}/\mathfrak{M}_0) = \operatorname{Der}(\mathfrak{M}'/\mathfrak{M}_0')$ *if and only if there is an isomorphism $I : \mathfrak{M} \to \mathfrak{M}'$ such that $\mathfrak{M}_0 I = \mathfrak{M}_0'$.*

It is of interest to note that in the setting of 7.11, the E_6 type (I or II) is determined by the structure of $\mathfrak{M}_0$, since $\operatorname{Der}(\mathfrak{M}/\mathfrak{M}_0)$ is of type $E_{6\text{I}}$ if and only if $\mathfrak{M}_0$ is reduced.

When $\mathfrak{M}_0$ is reduced, one is thus in the case $\mathfrak{M} = \mathfrak{M}(\mathfrak{J})$ and

$$\mathfrak{L} = \operatorname{Der}\left(\mathfrak{M}(\mathfrak{J}) \Big/ \sum_1^2 \Phi e_i\right)$$

which one sees to be isomorphic to $\operatorname{Inv}(\mathfrak{J}, N)$. Hence an algebra of type $E_{6\text{I}}$ with split envelope is of particularly simple construction. Moreover, one can show in this case that the condition of 7.12 becomes (if $\mathfrak{M}' = \mathfrak{M}(\mathfrak{J}')$), $\operatorname{Inv}(\mathfrak{J}, N) \cong \operatorname{Inv}(\mathfrak{J}', N')$ if and only if $\mathfrak{J}$ is an isotope of $\mathfrak{J}'$ (5.10). If, further, $\mathfrak{J}$ is reduced, this isomorphism condition can be further specified in terms of isomorphism of coordinate octonion algebras (4.14).

For algebras of type E_7, $\mathfrak{E}(\mathfrak{L}) = \mathfrak{A}$ is clearly central simple of degree 56 over Φ and since the defining A_s are of form $\operatorname{Inn} a_s$ where a_s preserves both $\langle\, , \rangle$ and q in $\mathfrak{M}(\mathfrak{J})$, one can show [**43**], that exponent $\mathfrak{A} = 2$ or 1. As for E_6 one can work in $\mathfrak{M}(\mathfrak{J})$ to normalize the A_s to obtain, in special field cases [**24**], [**25**]:

7.13. *Let $\mathfrak{L}$ be of type E_7 over Φ, Φ real, p-adic, or an algebraic number field, so $\mathfrak{E}(\mathfrak{L}) = \mathfrak{B}_{28}$. Then there is an exceptional simple Jordan algebra $\mathfrak{J}$ such that $\mathfrak{L} \cong \mathfrak{L}(\mathfrak{B}, \mathfrak{J})$.*

Again for general fields of characteristic zero one can handle only those cases for which $\mathfrak{E}(\mathfrak{L})$ is split, obtaining from 7.5

7.14. *Let* $\mathfrak{L}$ *be of type* E_7 *over* Φ *with* $\mathfrak{C}(\mathfrak{L}) = \Phi_{56}$. *Then there is an exceptional* Φ*-ternary algebra* $\mathfrak{M}$ *such that* $\mathfrak{L} \cong \operatorname{Der}\mathfrak{M}$.

Also we have again in this setting

7.15. $\operatorname{Der}\mathfrak{M} \cong \operatorname{Der}\mathfrak{M}'$ *if and only if* $\mathfrak{M} \cong \mathfrak{M}'$. *(Note that for* $\mathfrak{M}$ *reduced,* $\mathfrak{M} \cong \mathfrak{M}(\mathfrak{J})$ *and* $\mathfrak{M}' \cong \mathfrak{M}(\mathfrak{J}')$ *by* 5.7, *and using* 5.10 *we have the more precise statement* $\operatorname{Der}\mathfrak{M} \cong \operatorname{Der}\mathfrak{M}'$ *if and only if* $\mathfrak{J}$ *is an isotope of* $\mathfrak{J}'$.)

Finally turning to the case of algebras of type D_4, one has four distinguished cases according to the action of G on the summands of $\mathfrak{C}(\mathfrak{L}_p)$. $\mathfrak{L}$ is of *type* $D_{4\text{I}}$ if G stabilizes all summands, $D_{4\text{II}}$ if G stabilizes one summand and permutes the remaining two, $D_{4\text{III}}$ if each element of G either stabilizes every summand or permutes the three summands cyclically or $D_{4\text{VI}}$ if every possible permutation of the summands is induced by some element of G.

Since arguments analogous to those yielding 7.3 are easily seen to apply to D_4 also in the cases of type $D_{4\text{I}}$ and $D_{4\text{II}}$, algebras of this type are considered *special* D_4*'s* and for those 7.3 is valid. Algebras of type $D_{4\text{III}}$ and $D_{4\text{VI}}$ require other arguments and are considered *exceptional* D_4*'s*.

The only known, constructible algebras of type D_4, aside from the special algebras of 7.3, are the *Jordan* D_4*'s*, i.e., algebras $\operatorname{Der}(\mathfrak{J}/\mathfrak{K})$ where $\mathfrak{J}$ is an exceptional Jordan algebra over Φ, $\mathfrak{K} \subseteq \mathfrak{J}$ a separable, degree three algebra over Φ. The pairs $(\mathfrak{J}, \mathfrak{K})$ are thus precisely the Φ-forms of the pair $(\mathfrak{J}_p, \sum\Phi_p e_i)$, so 7.5 yields

7.16. *Let* $\mathfrak{L}$ *be of type* D_4 *over* Φ. *Then* $\mathfrak{L}$ *is a Jordan* D_4 *if and only if* $\mathfrak{C}(\mathfrak{L})$ *is a sum of matrix algebras.*

For each algebra $\mathfrak{L}$ of type D_4 over Φ, there is a unique minimal extension field $\Gamma \supseteq \Phi$ such that $\mathfrak{L}_\Gamma$ is of type $D_{4\text{I}}$. The determination of whether $\mathfrak{L}$ is a Jordan D_4 is simplified by

7.17. *Let* $\mathfrak{L}$ *be of type* D_4 *over* Φ. *Then* $\mathfrak{L}$ *is a Jordan* D_4 *if and only if* $\mathfrak{L}_\Gamma$ *is a Jordan* D_4.

It is clear, even for $\Phi = \mathbb{R}$, that the Jordan D_4's do not exhaust the class of algebras of type D_4, since one has in general, the algebras $\mathfrak{S}(\mathfrak{A}, \tau)$ where $\mathfrak{A} = \mathfrak{Q}_4$, $\mathfrak{Q}$ a quaternion algebra over Φ, τ extending to the involution $x \to x^t$ in $\mathfrak{A}_P = P_8$. This is clearly of type D_4 (see 7.3) and is not Jordan since $\mathfrak{A}$ is a summand of $\mathfrak{C}(\mathfrak{L})$.

If we restrict ourselves to $\Phi = \mathbb{R}$ it is easy to see that every algebra of type D_4 is of type $D_{4\text{I}}$ or $D_{4\text{II}}$ and is described by 7.3. If we consider p-adic Φ, there exist algebras of type $D_{4\text{III}}$ and $D_{4\text{VI}}$. If $\mathfrak{L}$ is of one of these algebras,

$$\mathfrak{C}(\mathfrak{L}_\Gamma) = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_3,$$

one can show the $\mathfrak{A}_i$ are isomorphic as rings and $\mathfrak{A}_1 \otimes \mathfrak{A}_2 \otimes \mathfrak{A}_3$ is split. Moreover, one can show each $\mathfrak{A}_i$ is of exponent one or two. Since there is a unique exponent two algebra over Φ, it follows that $\mathfrak{A}_i \cong \Phi_8$, $i = 1, 2, 3$. Thus $\mathfrak{L}_\Gamma$ is a Jordan D_4 by 7.16, hence $\mathfrak{L}$ is a Jordan D_4 by 7.17. We thus have [4]

7.18. *Let* $\mathfrak{L}$ *be of type* D_4 *over p-adic* Φ. *Then either* $\mathfrak{L} \cong \mathfrak{S}(\mathfrak{A}, \tau)$ *for* $\mathfrak{A}$ *of exponent* $\leqslant 2$ *or* $\mathfrak{L}$ *is a Jordan* D_4.

If Φ is an algebraic number field, 7.18 is false [5] and the question of describing all algebras of type D_4 is open.

As in previous cases, one has simple isomorphism conditions for those algebras with $\mathfrak{C}(\mathfrak{L})$ split, namely

7.19. $\mathrm{Der}(\mathfrak{J}/\mathfrak{K}) = \mathrm{Der}(\mathfrak{J}'/\mathfrak{K}')$ *if and only if there is an isomorphism* $\phi : \mathfrak{J} \to \mathfrak{J}'$ *such that* $\mathfrak{K}\phi = \mathfrak{K}'$.

Also, in analogy with the situation for E_6, one can easily distinguish the D_4 type of an algebra $\mathrm{Der}(\mathfrak{J}/\mathfrak{K})$ by looking only at the structure of the algebra $\mathfrak{K}$.

We have sketched here but one of several approaches to the classification of central simple Lie algebras. The interested reader is referred to [49], [31], [29] and [67] for other fruitful approaches.

8. *Geometries related to exceptional Lie algebras*

The algebraic structures (vector spaces, vector spaces with (skew) hermitian forms) which yield linear realizations of the classical Lie algebras also serve as vehicles for the definition of geometric structures with collineation groups closely related to the corresponding adjoint Chevalley group. In this section we indicate how analogous constructions based on some "exceptional" algebraic structures give rise to similarly interesting geometries which turn out to be exceptional in a geometric sense. Finally, we briefly indicate a general procedure for defining geometries related to arbitrary finite dimensional Lie algebra modules and note this construction (yielding geometries closely related to those of Tits [78], [79]) yields the previously defined geometries as special cases.

The prototype for the geometries we consider is *projective geometry* $\mathscr{P}(\mathfrak{V})$ (all subspaces of a finite dimensional left vector space $\mathfrak{V}$ over an associative division ring Δ with incidence given by containment). Here subspaces of dimension $1, 2, 3, \ldots, k+1, \ldots \dim \mathfrak{V} - 1$ are called *points, lines, planes, k-planes* and *hyperplanes* respectively. Of interest also is the subgeometry $\mathscr{I}(\mathfrak{V}, h)$ of $\mathscr{P}(\mathfrak{V})$ consisting of all totally isotropic subspaces of $\mathfrak{V}$ relative to a nondegenerate (skew) hermitian form h.

If $T \in \Gamma\mathrm{L}(\mathfrak{V})$ (the group of semilinear isomorphisms of $\mathfrak{V}$) then T induces a collineation of $\mathscr{P}(\mathfrak{V})$ (i.e., an incidence and dimension preserving permutation of the subspaces of $\mathfrak{V}$). The kernel of this action is the set of scalar multiplications, so $\mathrm{P}\Gamma\mathrm{L}(\mathfrak{V}) = \Gamma\mathrm{L}(\mathfrak{V})/\Delta^*$ can be viewed as a subgroup of the collineation group of $\mathscr{P}(\mathfrak{V})$. Indeed [17] one has the algebraic characterization of collineations

8.1. *Fundamental Theorem of Projective Geometry. The collineation group of* $\mathscr{P}(\mathfrak{V})$, $\dim \mathfrak{V} \geqslant 3$, *is* $\mathrm{P}\Gamma\mathrm{L}(\mathfrak{V})$.

Similarly, if $\Gamma\mathrm{U}(\mathfrak{V}, h)$ is the group of semisimilarities of h on $\mathfrak{V}$ (i.e., $T \in \Gamma\mathrm{U}(\mathfrak{V}, h)$ is τ-semilinear and satisfies $h(xT, yT) = \rho h(x, y)^\tau$ for all $x, y \in \mathfrak{V}$, some $\rho \in \Delta^*$), $\mathrm{P}\Gamma\mathrm{U}(\mathfrak{V}, h)$ can be identified with a subgroup of the collineation group of $\mathscr{I}(\mathfrak{V}, h)$ and [17]:

8.2. *The collineation group of* $\mathscr{I}(\mathfrak{V}, h)$ *is* $\mathrm{P}\Gamma\mathrm{U}(\mathfrak{V}, h)$ *if* $\mathfrak{V}$ *has a totally isotropic subspace of dimension* $\geqslant 3$.

The group $\mathrm{P\Gamma L}(\mathfrak{V})$ (resp. $\mathrm{P\Gamma U}(\mathfrak{V}, h)$) clearly contains the adjoint Chevalley group $\mathrm{PSL}(\mathfrak{V})$ (resp. $\mathrm{PSO}(\mathfrak{V}, h)$ or $\mathrm{PSp}(\mathfrak{V}, h)$ if h is symmetric or skew symmetric bilinear of maximal Witt index). Moreover, these subgroups can be geometrically characterized in the collineation group. In particular, recalling that a *transvection* of $\mathscr{P}(\mathfrak{V})$ is a collineation fixing all points in a hyperplane $\mathfrak{W}$ and no point not in $\mathfrak{W}$, one can see that every transvection is induced by some $S \in \Gamma\mathrm{L}(\mathfrak{V})$ with $yS - y \in \mathfrak{W}$ for all $y \in \mathfrak{V}$ which satisfies $S|_{\mathfrak{W}} = id_{\mathfrak{W}}$.

8.3. *If* $\dim \mathfrak{V} \geqslant 3$, *the subgroup of the collineation group of* $\mathscr{P}(\mathfrak{V})$ *generated by all transvections is the group* $\mathrm{PSL}(\mathfrak{V})$.

An analogous result characterizes $\mathrm{PSO}(\mathfrak{V}, h)$ and $\mathrm{PSp}(\mathfrak{V}, h)$ acting on $\mathscr{I}(\mathfrak{V}, h)$ **[17]**.

The interplay between algebraic and geometric ideas evident in 8.1, 8.2 and 8.3 is made striking in the case of projective geometry via coordinatization results which, among other things, allow one to begin with a set of specific geometric properties of the geometry $\mathscr{P}(\mathfrak{V})$ and from them recover Δ and $\mathfrak{V}$ up to isomorphism. To isolate the necessary properties, it clearly suffices to consider only incidences among points and lines, for one may inductively define a k-plane as the set of all points lying on lines joining a fixed point P with some point in a fixed $(k-1)$-plane Π where P is not incident to Π.

A *projective geometry* $\mathscr{P}$ consists of points, lines, and an incidence relation satisfying

8.4.

(*a*) *Two points lie on a unique line.*

(*b*) *Coplanar lines intersect.*

(*c*) *Every line is incident to at least three points.*

A projective geometry $\mathscr{P}$ in which all points lie in a single plane (but not on a single line) is a *projective plane.*

To insure that a projective geometry is indeed of form $\mathscr{P}(\mathfrak{V})$, one needs also additional information about certain geometric configurations in $\mathscr{P}$. In particular, one needs the validity of

8.5. *Desargues Theorem. If* T *and* T' *are triangles with vertices* A, B, C *and* A', B', C' *respectively and sides* a, b, c *and* a', b', c' *respectively, then* AA', BB', CC' *are concurrent lines if and only if* aa', bb', cc' *are collinear points.*

If a projective geometry $\mathscr{P}$ contains more than one plane, Desargues Theorem is valid in $\mathscr{P}$ **[11]** and one can find a Δ and $\mathfrak{V}$ such that $\mathscr{P}$ is isomorphic with $\mathscr{P}(\mathfrak{V})$. In a projective plane, the theorem need not hold. For any plane $\mathscr{P}$ for which 8.5 holds, there is an associative division algebra Δ such that points of $\mathscr{P}$ can be coordinatized by the symbols (∞), (m), (x, y), the lines of $\mathscr{P}$ can be coordinatized by symbols $[\infty]$, $[x]$, $[m, b]$ where $x, y, m, b \in \Delta$ and incidence is given by **[58]**:

8.6. (∞), (m) *lie on* $[\infty]$

(∞), (x, y) *lie on* $[x]$

(m), (x, y) *lie on* $[m, b]$ *if* $y = mx+b$.

The plane defined by 8.6 is easily seen to be isomorphic with $\mathscr{P}(\mathfrak{V})$ for $\mathfrak{V}$ a 3-dimensional vector space over Δ. One thus has

8.7. *A projective geometry is of form $\mathscr{P}(\mathfrak{V})$ unless $\mathscr{P}$ is a non-desarguian projective plane.*

If one pursues further the possibility of coordinatizing projective planes $\mathscr{P}$ which are non-desarguian (8.5 fails to hold) one encounters new configurations, weaker than Desargues yet still strong enough to place interesting algebraic constraints on the coordinates. Returning to the considerations leading to 8.3, we consider the transvections of the plane $\mathscr{P}$, which we call *elations.* An elation can be characterized as a collineation σ fixing all points on a line ℓ (the *axis*) and all lines through a point P (the *centre*) where P lies on ℓ. It is easy to see that such a σ is uniquely determined by P, ℓ, Q and $Q'(=Q^\sigma)$ where Q and Q' are points not on ℓ collinear with P. Given any P, ℓ, Q, Q' satisfying the given conditions in a Desarguian plane, there is always an elation σ with $Q' = Q^\sigma$ [58]. We say thus that a Desarguian plane has *all possible elations.* The converse fails, since a projective plane having all possible elations need satisfy only the Little Desargues Theorem or, in characteristic not two, the Harmonic Point Theorem [58]. Coordinatizing this larger class of projective planes yields [57], [30], [58]

8.8. *A projective plane having all possible elations can be coordinatized as in* 8.6 *with $x, y, m, b \in \mathfrak{A}$ an alternative division algebra.*

In the light of 3.3, 8.8 shows that only one new "exceptional" geometry occurs when we relax our conditions on a projective plane to require only that it admit all elations, namely, the plane coordinatized by an octonion division algebra $\mathfrak{O}$. These planes are well known as *Moufang planes.*

Having begun with the geometric definition of elations as collineations in a Moufang plane, one would like to identify "algebraically" the collineations of this plane and give an algebraic characterization of the group generated by elations analogous to 8.3 for the projective Desarguian planes. To do this it is convenient to look anew at the geometry $\mathscr{P}(\mathfrak{V})$, $\mathfrak{V}$ the space of n-tuples with entries in a field Φ. Since the map $\mathfrak{W} \to \mathfrak{J}_{\mathfrak{W}}(=\{T \in \Phi_n | \mathfrak{V}T \subseteq \mathfrak{W}\})$ is an inclusion preserving bijection between the subspaces $\mathfrak{W}$ of $\mathfrak{V}$ and the left ideals of the associative algebra Φ_n, one has

8.9. *$\mathscr{P}(\mathfrak{V})$ is isomorphic with the geometry of left ideals in Φ_n with incidence given by containment.*

A further realization of $\mathscr{P}(\mathfrak{V})$, this time in the context of Jordan algebras, is suggested by the similarity between the roles of left ideals in associative theory and inner ideals in Jordan theory, namely [46]

8.10. *$\mathscr{P}(\mathfrak{V})$ is isomorphic with the geometry of inner ideals of $\mathfrak{H}(\Phi_n)$ with incidence given by containment, the isomorphism being given by $\mathfrak{W} \to \mathfrak{N}_{\mathfrak{W}} = \mathfrak{J}_{\mathfrak{W}} \cap \mathfrak{H}(\Phi_n)$.*

A look at the geometries of inner ideals of other simple Jordan algebras reveals that, if $\mathfrak{J} = \mathfrak{J}(\mathfrak{V}, f)$ is as in 4.5 and $h(x, y) = \alpha\beta - f(u, v)$, the inner ideal geometry in $\mathfrak{J}$ is precisely the geometry of totally isotropic subspaces $\mathscr{I}(\mathfrak{J}, h)$ defined above, since one sees directly that $yU_x = 2h(x, \bar{y})x - h(x, x)\bar{y}$ for x, y as in 4.5, $\bar{y} = (\beta, -w)$.

Of more interest in our context is the geometry of inner ideals in the exceptional algebra $\mathfrak{H}(\mathfrak{D}_3)$, $\mathfrak{D}$ an octonion division algebra. There are precisely two classes of inner ideals namely $\{a_* = \Phi a | a \in \mathfrak{J}, a \text{ rank } 1\}$ or $\{a^* = a \times \mathfrak{J} | a \in \mathfrak{J}, a \text{ rank } 1\}$. Moreover $a_* \subseteq b^*$ if and only if $T(a, b) = 0$ and $a_*, b_* \subseteq c^*$ or $a^*, b^* \supseteq c_*$ if and only if $c \in \Phi(a \times b)$. It follows that this geometry is a projective plane and, in fact **[71]**

8.11. *The geometry of inner ideals of* $\mathfrak{H}(\mathfrak{D}_3)$, $\mathfrak{D}$ *a division algebra, is isomorphic with the Moufang plane with coordinates from* $\mathfrak{D}$.

In this setting we have the analogues of 8.1 **[71]**

8.12. *Every collineation of the Moufang plane is induced by a semisimilarity* T *of the norm form* N *of* $\mathfrak{H}(\mathfrak{D}_3)$ *(i.e.,* T *is* τ*-semilinear in* $\Gamma L(\mathfrak{J})$ *and*

$$N(xT) = \rho N(x)^\tau \, \forall \, x \in \mathfrak{H}(\mathfrak{D}_3),$$

some $\rho \in \Phi^*$*).*

8.13. *The group generated by all elations of the Moufang plane is the simple group* $PS(\mathfrak{J})$ *of norm preserving transformations of* $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3)$ *modulo its centre (this is an algebraic group of type* E_6*).*

In the spirit of our previous sections, 8.13 suggests considering the octonion plane as an " exceptional " geometry.

In broadening one's investigations to include a study of the geometry of inner ideals in $\mathfrak{H}(\mathfrak{D}_3)$, $\mathfrak{D}$ split octonion, one finds in addition to the spaces $a \times \mathfrak{J}$ encountered before, subspaces consisting entirely of rank 1 elements (*point spaces*) which may have dimension greater than one **[54]**. Thus one obtains a new geometry which is no longer a projective plane. Nevertheless, the analogues of 8.12 and 8.13 are again valid and in this setting, $PS(\mathfrak{J})$ is a Chevalley group of type E_6 **[75]**, **[84]**, **[86]**. In much the same spirit as that which led to characterization (in §4) of F_4 and D_4 as subalgebras of E_6 in terms of interplay with algebraic properties of $\mathfrak{H}(\mathfrak{D}_3)$, one can distinguish certain interesting subgroups of the Chevalley group $PS(\mathfrak{J})$ of type E_6 in terms of interaction with the geometry of inner ideals. In particular, if π is a polarity in the geometry (an order two, incidence preserving map interchanging points and lines), the group generated by all elations which commute with π, for suitable selection of π, is a group of type F_4 (resp. a twisted group of type E_6) **[76]**, these latter being natural group analogues to the Lie algebras of type E_6 found in the second row of 6.2 where $\mathfrak{A}$ is a quadratic field extension of Φ and $\mathfrak{B} = \mathfrak{D}$ **[86]**, **[85]**.

The concept of inner ideal in a Jordan algebra $\mathfrak{J}$ is clearly connected less directly with the binary product on $\mathfrak{J}$ than with the ternary composition yU_x. If one uses the natural idea of inner ideal in an arbitrary ternary algebra $\mathfrak{M}$ (inner ideal = subspace $\mathfrak{N}$ such that $\mathfrak{N}\mathfrak{M}\mathfrak{N} \subseteq \mathfrak{N}$) and considers the geometry of inner ideals of $\mathfrak{M}$, one obtains from the $\mathfrak{J}$-ternary algebra of 5.2 another realization of the "classical" geometry $\mathfrak{I}(\mathfrak{B}, h)$, h skew hermitian on $\mathfrak{B}$. Of more interest is the inner ideal geometry $\mathscr{P}$ defined by the exceptional Φ-ternary algebra $\mathfrak{M}(\mathfrak{J})$ of 5.4 with $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3)$ since in this case one has **[19]**, **[20]**

8.14. *Every collineation of* $\mathscr{P}$ *is induced by a semisimilarity* T *of the quartic norm form* q *of* $\mathfrak{M}(\mathfrak{J})$.

When $\mathfrak{J}$ is split, 8.14 and [66] show that the Chevalley group E_7 is a subgroup of the collineation group of $\mathscr{P}$. Moreover, one knows in terms of $\mathfrak{M}(\mathfrak{J})$ a reasonable set of generators for this group [10]. Still open, however, is the question of existence of a geometric characterization of this subgroup analogous to 8.3 and 8.13.

Yet another interesting geometry of inner ideals, this one connected with the Lie algebra G_2, is that constructed from $\mathfrak{O}_0$, the space of elements of trace zero in an octonion algebra, with ternary product $aD_{b,c}$ (see 3.7) [64], [20]. In this case the collineation group is induced by the semiautomorphisms of $\mathfrak{O}$ and the Chevalley group of type G_2 can be geometrically characterized in the collineation group.

The previously described geometries, related to certain representations of Lie algebras, are special cases of a general geometric construction which we shall see later is closely connected to the geometries defined by Tits [78], [79] in terms of algebraic groups. One begins with a semisimple Lie algebra $\mathfrak{L}$ and a finite dimensional $\mathfrak{L}$-module $\mathfrak{B}$. For $y^* \in \mathfrak{B}^*$ (the contragredient $\mathfrak{L}$-module) and $z \in \mathfrak{B}$, the map $\ell \to \langle y^*, z\ell \rangle$ is a linear functional on $\mathfrak{L}$ so one may define an element $R(y^*, z) \in \mathfrak{L}$ by

8.15. $\langle y^*, z\ell \rangle = K(\ell, R(y^*, z))$,

$K(\ ,\ )$ the Killing form (non-degenerate) on $\mathfrak{L}$. Setting $xy^*z = xR(y^*, z)$ gives a trilinear map $\mathfrak{B} \times \mathfrak{B}^* \times \mathfrak{B} \to \mathfrak{B}$. An *inner ideal* $\mathfrak{N}$ is then a subspace of $\mathfrak{B}$ with $\mathfrak{N}\mathfrak{B}^*\mathfrak{N} \subseteq \mathfrak{N}$. We denote by $\mathscr{G}(\mathfrak{L}, \mathfrak{B})$ the geometry of all inner ideals of V with incidence given by containment. The geometries of inner ideals in Jordan algebras and $\mathfrak{J}$-ternary algebras which were discussed above can be shown to arise in this manner. In the event $\mathfrak{B} = \mathfrak{L}$, one may identify $\mathfrak{L}^*$ with $\mathfrak{L}$ via $K(\ ,\ )$, in which case $\mathscr{G}(\mathfrak{L}, \mathfrak{L})$ becomes the geometry of inner ideals in the Lie triple product $[x,[y,z]]$. Similar identifications of $\mathfrak{B}$ with $\mathfrak{B}^*$ via bilinear forms account for the fact that in the specific examples considered above, the contragredient representation never entered.

The fact that the Chevalley group $G_\Phi(\mathfrak{L}, \mathfrak{B})$ occurs in the special cases as a subgroup of the collineation group is also a consequence of a general phenomena. Since there is a natural isomorphism $g \to g^*$ of the group $G_\Phi(\mathfrak{L}, \mathfrak{B})$ with $G_\Phi(\mathfrak{L}, \mathfrak{B}^*)$ such that $(x_\alpha(t))^*$ is the element called $x_\alpha(t)$ in $G_\Phi(\mathfrak{L}, \mathfrak{B}^*)$ and since 8.15 implies

8.16. $(xy^*z)\ell = (x\ell)y^*z + x(y^*\ell^*)z + xy^*(z\ell)$ *for* $\ell \in \mathfrak{L}$, $x, z \in \mathfrak{B}$, $y^* \in \mathfrak{B}^*$ *one sees*

8.17. $(xy^*z)g = (xg)(y^*g^*)(zg)$ *for* $g \in G_\Phi(\mathfrak{L}, \mathfrak{B})$, $x, z \in \mathfrak{B}$, $y^* \in \mathfrak{B}$.

Thus if $\mathfrak{N}$ is an inner ideal $(\mathfrak{N})g$ is also, so

8.18. $G(\mathfrak{L}, \mathfrak{B})$ *acts as collineations on* $\mathscr{G}(\mathfrak{L}, \mathfrak{B})$.

Results analogous to 8.1, 8.3 and 8.8 namely, an algebraic characterization of the collineation group of $\mathscr{G}(\mathfrak{L}, \mathfrak{B})$, a geometric characterization of the Chevalley group $G_\Phi(\mathfrak{L}, \mathfrak{B})$ as a subgroup of the collineation group, and a set of geometric axioms characterizing the geometries are as yet unknown in general. One solution to the latter problem would appear to involve the use of an axiom scheme [28], [82] to reduce the question to consideration of subgeometries which are *planar* (geometries with exactly two types of objects—points and lines) and, using a suitable notion of elation, to characterize the planar geometries admitting all elations in terms of certain "coordinate" algebraic structures. The planar geometries known to occur among the $\mathscr{G}(\mathfrak{L}, \mathfrak{B})$ are n-gon geometries, for $n = 3$, 4 or 6, where a planar geometry is called a (generalized) *n-gon geometry* if every object is incident to at least three other objects, every pair of objects can be imbedded in an n-gon, and there are no k-gons for $2 \leqslant k < n$ (using the obvious notion to define a polygon).

8.19. *Example.* $\mathscr{P}$ is triangular (3-gon) if and only if $\mathscr{P}$ is a projective plane.

8.20. *Example.* If $\mathfrak{L}$ is a split Lie algebra of type B_2 (resp. G_2) over Φ, $\mathscr{G}(\mathfrak{L}, \mathfrak{L})$ is a quadrilateral (4-gon) (resp. hexagonal (6-gon)) geometry.

8.21. *Example.* $\mathscr{G}(\operatorname{Der}\mathfrak{O}, \mathfrak{O}_0)$, $\mathfrak{O}$ split octonion is hexagonal.

Example 8.19, together with earlier discussion, indicates what collineations are elations for $\mathscr{P}$ triangular. For $\mathscr{P}$ quadrilateral, an elation is a collineation fixing all points on a given line and all lines through either of two points on the line (or the dual concept) while for $\mathscr{P}$ hexagonal an elation fixes all lines intersecting a given line and all points on either of two lines which intersect the given line but not each other (or the dual concept).

If one assumes that $\mathscr{P}$ admits all elations and, moreover, has no elations of order two (resp. three) if $\mathscr{P}$ is quadrilateral (resp. hexagonal) the group $G(\mathscr{P})$ generated by elations is generated by non-trivial subgroups X_α, $x \in \Sigma$ (with $X_{2\alpha} \subseteq X_\alpha$) and

8.22.

(i) (X_α, X_β) *is contained in the subgroup generated by all* X_γ *with* $\gamma = i\alpha + j\beta \in \Sigma$, $i, j > 0$, $\alpha \neq -\beta$.

(ii) *If* $1 \neq x \in X_\alpha$, *there is* $x' \in X_{-\alpha}$, $x'' \in X_\alpha$ *with* $w = xx'x''$ *satisfying* $w^{-1} X_\beta w = X_{\beta w_\alpha}$.

(iii) $X_{\Sigma^+} \cap X_{\Sigma^-} = \{1\}$ *where* X_S *is the subgroup generated by* $\bigcup_{\alpha \in S} X_\alpha$ *where* Σ *is a root system of type* A_2 *(triangular),* B_2 *or* BC_2 *(quadrilateral) or* G_2 *(hexagonal). (Compare with* 2.2.3.)

The subgroups X_α can be parametrized in algebraic structures in a manner consistent with the group product in $G(\mathscr{P})$ (e.g., X_α and X_β may be parametrized by a Jordan algebra $\mathfrak{A}$ in such a manner that $(x_\alpha(a), x_\beta(b)) = x_{\alpha+\beta}(a \cdot b)$ and

$$x_\alpha(a)^{w_\alpha(b)} = x_{-\alpha}(aU_{b^{-1}})$$

where $w_\alpha(b)$ is the w corresponding to $x = x_\alpha(b)$ as in 8.22(ii)). These structures in turn coordinatize the geometry $\mathscr{P}$ in a natural way analogous to 8.8. Without going into further details of the coordinatization, we note that this process gives rise, in a geometric setting, to the algebraic structures introduced in §§3 and 4, since one obtains as coordinatizing structures

8.23.

(i) *For* Σ *of type* A_2: *an alternative division algebra* $\mathfrak{A}$.

(ii) *For* Σ *of type* B_2: *a pair* $(\mathfrak{A}, \mathfrak{H})$ *where either* $\mathfrak{A}$ *is an associative division algebra with involution* τ *and* $\mathfrak{H}$ *is the Jordan algebra of* τ *skew elements, or* $\mathfrak{A}$ *is the Jordan division algebra of a bilinear form over a field* $\mathfrak{H}$.

(iii) *For* Σ *of type* BC_2: *a triple* $(\mathfrak{A}, \mathfrak{H}, \mathfrak{B})$ *with* $\mathfrak{A}, \mathfrak{H}$ *as in* (ii) *and* $\mathfrak{B}$ *an* $\mathfrak{A}$*-module with a skew hermitian form h with* $h(x, x) \neq 0 \ \forall x \in \mathfrak{B}$ (*or a suitable analogue in the second case of* (ii)).

(iv) *For* Σ *of type* G_2: *a pair* $(\mathfrak{J}, \Phi)$ *with* $\mathfrak{J}$ *the Jordan algebra of a cubic form over the field* Φ.

At present, an algebraic characterization of the entire collineation group of the planar geometry $\mathscr{P}$ in terms of its coordinate algebras is not known. One does, however, have a good description of the group generated by elations via 8.22 and sees in these cases that this is the associated rank two Chevalley group precisely in those cases when $\mathfrak{A}$ (or $\mathfrak{J}$) is the base field Φ. Applying this, for example, to the case $\mathscr{P} = \mathscr{G}(\mathrm{Der}\,\mathfrak{O}, \mathfrak{O}_0)$ of 8.21 one obtains as a corollary of the general result that the group generated by elations in $\mathscr{P}$ is the Chevalley group of type G_2 as noted before.

In those cases where the coordinate algebras are not Φ, the planar geometries correspond in a natural way to non-split simple Lie algebras $\mathfrak{L}$ having restricted root system of rank 2 (See [67] for a related discussion of Lie algebra structure). For instance

8.24. *Example.* Let $\mathfrak{L}$ be the Lie algebra of type E_8 constructed as in 6.2 with $\mathfrak{A} = \mathfrak{O}$, the split octonions, $\mathfrak{J}$ an exceptional Jordan division algebra. The subgeometry of the inner ideal geometry of $\mathscr{G}(\mathfrak{L}, \mathfrak{L})$ consisting of all points (one dimensional inner ideals) and all lines with the property that every one dimensional subspace is an inner ideal is a hexagonal geometry with coordinates $(\mathfrak{J}, \Phi)$ (root system of type G_2).

The analysis of non-planar geometries of inner ideals has not yet been undertaken. However, it seems reasonable to assume that such an analysis could be carried out simply by investigating the planar subgeometries and their interrelation, reducing the problem basically to the planar case.

Turning finally to the connection between inner ideal geometries and the Tits' geometries [78], [79], we restrict ourselves to the case $\mathfrak{L}$ split simple, $\mathfrak{B}$ irreducible with highest weight Λ. For Π a simple system of roots for $\mathfrak{L}$ and $T \subseteq \Pi$, we denote by $\mathfrak{N}(T)$ the subspace of $\mathfrak{B}$ spanned by all weight vectors belonging to weights $\lambda = \Lambda - \sum_{\alpha_i \in \Pi} k_i \alpha_i$ (see §2.2) with $k_i = 0$ for $\alpha_i \in T$. Then [20]:

8.25. $\mathfrak{N}(T)$ *is an inner ideal in* $\mathfrak{B}$. *Moreover, if* $\mathfrak{N}_1 \subseteq \mathfrak{N}_2$ *are inner ideals, there is* $g \in G_\Phi(\mathfrak{L}, \mathfrak{B})$ *and* $T_1, T_2 \subseteq \Pi$ *such that* $\mathfrak{N}_i g = \mathfrak{N}(T_i)$.

It follows that all incidences among inner ideals are consequences of incidences among the $\mathfrak{N}(T)$'s. These latter incidences can be described solely in the context of roots. To do this, we identify Π with the set of vertices of the Dynkin diagram and say for subsets $S_1, S_2, S_3 \subseteq \Pi$ that S_2 separates S_1 and S_3 (written $S_1/S_2/S_3$) if every connected subdiagram containing an element of S_1 and an element of S_3 also contains an element of S_2. For $R = \{\alpha \in \Pi | (\Lambda, \alpha) \neq 0\}$ we have

8.26. $\mathfrak{N}(T_1) \subseteq \mathfrak{N}(T_2)$ *if and only if* $R/T_1/T_2$.

In particular, $\mathfrak{N}(T_1) = \mathfrak{N}(T_2)$ implies $R/T_1/T_2$ and $R/T_2/T_1$ so $\mathfrak{N}(T_1) = \mathfrak{N}(T_3)$ where $T_3 = T_1 \cap T_2$. Thus for any $T \subseteq \Pi$ there is a subset $T' \subseteq \Pi$ *reduced mod R* (no proper subset of T' separates R and T') such that $\mathfrak{N}(T) = \mathfrak{N}(T')$.

One is thus led to

8.27. *There is a bijection between the orbits of inner ideals under* $G_\Phi(\mathfrak{L}, \mathfrak{V})$ *and the reduced subsets of* Π *mod* R *since the stabilizer of* $\mathfrak{N}(T)$ *for* T *reduced is the parabolic subgroup* $G(T) = G_{\Pi-T}$ *and hence by* 2.2.4(iii) *distinct* $\mathfrak{N}(T)$*'s lie in distinct orbits.*

8.28. *Example.* If $\mathfrak{L} = o(\mathfrak{V}, f)$ for a symmetric bilinear form f of maximal Witt index n (hence $\mathfrak{L}$ has Dynkin diagram D_n as in 2.1.6) then $R = \{\alpha_1\}$ and the reduced subsets mod R are $\{\alpha_k\}$, $k = 1, \ldots, n$, and $\{\alpha_{n-1}, \alpha_n\}$. Under the action of

$$G_\Phi(\mathfrak{L}, \mathfrak{V}) = \mathrm{SO}(\mathfrak{V}, f),$$

the k-dimensional totally isotropic subspaces form an orbit of inner ideals corresponding to $\{\alpha_k\}$ for $1 \leqslant k \leqslant n-2$ and to $\{\alpha_{n-1}, \alpha_n\}$ for $k = n-1$. For $k = n$ there are two orbits, corresponding to $\{\alpha_{n-1}\}$ and $\{\alpha_n\}$.

8.29. *Example.* If $\mathfrak{L}$ is the split Lie algebra of type E_6 and $\mathfrak{V} = \mathfrak{H}(\mathfrak{O}_3)$ is the split exceptional simple Jordan algebra, we have (using notation as in 2.1.6) that $R = \{\alpha_1\}$ and the reduced subsets mod R are $\{\alpha_1\}$, $\{a_3\}$, $\{\alpha_4\}$ and $\{a_2, \alpha_5\}$ with corresponding point spaces (=subspace consisting of rank 1 elements) of dimensions 1, 2, 3 and 4 respectively, $\{\alpha_5\}$ and $\{\alpha_2, \alpha_6\}$ corresponding to the two types of point spaces of dimension 5 with the latter type contained in a point space of dimension 6, $\{\alpha_2\}$ corresponding to a point space of dimension 6, and $\{\alpha_6\}$ corresponding to inner ideals $a \times \mathfrak{V}$, a of rank one. This is the geometry of the split octonion plane.

For any semisimple algebraic group G (e.g., $G_\Phi(\mathfrak{L}, \mathfrak{L})$, $\mathfrak{L}$ split, simple) and any $R \subseteq \Pi$ (the system of simple roots), one defines a geometry $\mathcal{G}$, the *Tits' geometry*, as follows. The objects of $\mathcal{G}$ are all subsets Y of $X = G/G(R)$ ($G(R)$ the parabolic subgroup $G_{\Pi-R}$) of form $Y = \{G(R)h | h \in G(T)g\}$ for some $T \subseteq \Pi$, $g \in G$. Incidence is given by containment. Note that such a Y is in fact $N(T)g$ where

$$N(T) = \{G(R)h | h \in G(T)\}.$$

Since one can show that $N(T_1) \subseteq N(T_2)$ if and only if $R/T_1/T_2$ ($X/Y/Z$ defined as above), $(\mathfrak{N}(T))g \to N(T)\bar{g}$, where $g \to \bar{g}$ is the usual homomorphism of $G_\Phi(\mathfrak{L}, \mathfrak{V})$ onto $G_\Phi(\mathfrak{L}, \mathfrak{L})$ is a geometry isomorphism of $\mathcal{G}(\mathfrak{L}, \mathfrak{V})$ onto the Tits' geometry constructed from $G_\Phi(\mathfrak{L}, \mathfrak{L})$ and the set R defined by the highest weight Λ of $\mathfrak{V}$ for $\mathfrak{V}$ irreducible.

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