Lecture 1. Basic Systems

1.1. What is an exterior differential system?

An *exterior differential system* (EDS) is a pair $(M, \mathcal{I})$ where M is a smooth manifold and $\mathcal{I} \subset \Omega^*(M)$ is a graded ideal in the ring $\Omega^*(M)$ of differential forms on M that is closed under exterior differentiation, i.e., for any ϕ in $\mathcal{I}$, its exterior derivative $d\phi$ also lies in $\mathcal{I}$.

The main interest in an EDS $(M, \mathcal{I})$ centers around the problem of describing the submanifolds $f : N \to M$ for which all the elements of $\mathcal{I}$ vanish when pulled back to N, i.e., for which $f^*\phi = 0$ for all $\phi \in \mathcal{I}$. Such submanifolds are said to be *integral manifolds* of $\mathcal{I}$. (The choice of the adjective 'integral' will be explained shortly.)

In practice, most EDS are constructed so that their integral manifolds will be the solutions of some geometric problem one wants to study. Then the techniques to be described in these lectures can be brought to bear.

The most common way of specifying an EDS $(M, \mathcal{I})$ is to give a list of generators of $\mathcal{I}$. For $\phi_1, \ldots, \phi_s \in \Omega^*(M)$, the 'algebraic' ideal consisting of elements of the form

$$\phi = \gamma^1 \wedge \phi_1 + \cdots \gamma^s \wedge \phi_s$$

will be denoted $\langle \phi_1, \ldots, \phi_s \rangle_{\mathrm{alg}}$ while the differential ideal $\mathcal{I}$ consisting of elements of the form

$$\phi = \gamma^1 \wedge \phi_1 + \cdots \gamma^s \wedge \phi_s + \beta^1 \wedge d\phi_1 + \cdots \beta^s \wedge d\phi_s$$

will be denoted $\langle \phi_1, \ldots, \phi_s \rangle$.

Exercise 1.1: Show that $\mathcal{I} = \langle \phi_1, \ldots, \phi_s \rangle$ really is a differentially closed ideal in $\Omega^*(M)$. Show also that a submanifold $f : N \to M$ is an integral manifold of $\mathcal{I}$ if and only if $f^*\phi_\sigma = 0$ for $\sigma = 1, \ldots, s$.

The p-th graded piece of $\mathcal{I}$, i.e., $\mathcal{I} \cap \Omega^p(M)$, will be denoted $\mathcal{I}^p$. For any $x \in M$, the evaluation of $\phi \in \Omega^p(M)$ at x will be denoted ϕ_x and is an element of $\Omega^p_x(M) = \Lambda^p(T^*_x M)$. The symbols $\mathcal{I}_x$ and $\mathcal{I}^p_x$ will be used for the corresponding concepts.

Exercise 1.2: Make a list of the possible ideals in $\Lambda^*(V)$ up to isomorphism, where V is a vector space over $\mathbb{R}$ of dimension at most 4. (Keep this list handy. We'll come back to it.)

1.2. Differential equations reformulated as EDSs

Élie Cartan developed the theory of exterior differential systems as a coordinate-free way to describe and study partial differential equations. Before I describe the general relationship, let's consider some examples:

Example 1.1: *An Ordinary Differential Equation.* Consider the system of ordinary differential equations

$$\begin{aligned} y' &= F(x, y, z) \\ z' &= G(x, y, z) \end{aligned}$$

where F and G are smooth functions on some domain $M \subset \mathbb{R}^3$. This can be modeled by the EDS $(M, \mathcal{I})$ where

$$\mathcal{I} = \langle\, dy - F(x, y, z)\, dx,\ dz - G(x, y, z)\, dx \,\rangle.$$

It's clear that the 1-dimensional integral manifolds of $\mathcal{I}$ are just the integral curves of the vector field

$$X = \frac{\partial}{\partial x} + F(x, y, z)\, \frac{\partial}{\partial y} + G(x, y, z)\, \frac{\partial}{\partial z}.$$

Example 1.2: *A Pair of Partial Differential Equations.* Consider the system of partial differential equations

$$z_x = F(x, y, z)$$
$$z_y = G(x, y, z)$$

where F and G are smooth functions on some domain $M \subset \mathbb{R}^3$. This can be modeled by the EDS $(M, \mathcal{I})$ where

$$\mathcal{I} = \langle\, dz - F(x, y, z)\, dx - G(x, y, z)\, dy \,\rangle.$$

On any 2-dimensional integral manifold $N^2 \subset M$ of $\mathcal{I}$, the differentials dx and dy must be linearly independent (Why?). Thus, N can be locally represented as a graph $\bigl(x, y, u(x, y)\bigr)$ The 1-form

$$dz - F(x, y, z)\, dx - G(x, y, z)\, dy$$

vanishes when pulled back to such a graph if and only if the function u satisfies the differential equations

$$u_x(x, y) = F\bigl(x, y, u(x, y)\bigr)$$
$$u_y(x, y) = G\bigl(x, y, u(x, y)\bigr)$$

for all (x, y) in the domain of u.

Exercise 1.3: Check that a surface $N \subset M$ is an integral manifold of $\mathcal{I}$ if and only if each of the vector fields

$$X = \frac{\partial}{\partial x} + F(x, y, z)\, \frac{\partial}{\partial z} \qquad \text{and} \qquad Y = \frac{\partial}{\partial y} + G(x, y, z)\, \frac{\partial}{\partial z}$$

is tangent to N at every point of N. In other words, N must be a union of integral curves of X and also a union of integral curves of Y. By considering the special case $F = y$ and $G = -x$, show that there need not be any 2-dimensional integral manifolds of $\mathcal{I}$ at all.

Example 1.3: *Complex Curves in* $\mathbb{C}^2$. Consider $M = \mathbb{C}^2$, with coordinates $z = x + i\, y$ and $w = u + i\, v$. Let $\mathcal{I} = \langle \phi_1, \phi_2 \rangle$ where ϕ_1 and ϕ_2 are the real and imaginary parts, respectively, of

$$dz \wedge dw = dx \wedge du - dy \wedge dv + i\, (dx \wedge dv + dy \wedge du).$$

Since $\mathcal{I}^1 = (0)$, any (real) curve in $\mathbb{C}^2$ is an integral curve of $\mathcal{I}$. A (real) surface $N \subset \mathbb{C}^2$ is an integral manifold of $\mathcal{I}$ if and only if it is a complex curve. If dx and dy are linearly independent on N, then locally N can be written as a graph $\bigl(x, y, u(x, y), v(x, y)\bigr)$ where u and v satisfy the Cauchy-Riemann equations: $u_x - v_y = u_y + v_x = 0$. Thus, $(M, \mathcal{I})$ provides a model for the Cauchy-Riemann equations.

In fact, any 'reasonable' system of partial differential equations can be described by an exterior differential system. For concreteness, let's just stick with the first order case. Suppose, for example, that you have a system of equations of the form

$$F^\rho\bigl(\mathbf{x}, \mathbf{z}, \frac{\partial \mathbf{z}}{\partial \mathbf{x}}\bigr) = 0, \qquad \rho = 1, \ldots, r,$$

where $\mathbf{x} = (x^1, \ldots, x^n)$ are the independent variables, $\mathbf{z} = (z^1, \ldots, z^s)$ are the dependent variables, and $\frac{\partial \mathbf{z}}{\partial \mathbf{x}}$ is the Jacobian matrix of $\mathbf{z}$ with respect to $\mathbf{x}$. The hypotheses that I want to place on the functions F^ρ is that they are smooth on some domain $D \subset \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^{ns}$ and that, at every point $(\mathbf{x}, \mathbf{z}, \mathbf{p}) \in D$ at which all of the F^ρ vanish, one can smoothly solve the above equations for r of the $\mathbf{p}$-coordinates in terms of $\mathbf{x}$, $\mathbf{z}$, and the $ns{-}r$ remaining $\mathbf{p}$-coordinates. If we then let $M^{n+s+ns-r} \subset D$ be the common zero locus of the F^ρ, set

$$\theta^\alpha = dz^\alpha - p^\alpha_i\, dx^i$$

and let $\mathcal{I} = \langle\, \theta^1, \ldots, \theta^s \rangle$. Then any n-dimensional integral manifold $N \subset M$ of $\mathcal{I}$ on which the $\{dx^i\}_{1 \le i \le n}$ are linearly independent is locally the graph of a solution to the original system of first order PDE.

Obviously, one can 'encode' higher order PDE as well, by simply regarding the intermediate partial derivatives as dependent variables in their own right, constrained by the obvious PDE needed to make them be the partials of the lower order partials. For example, in the classical literature, one frequently sees a second order scalar PDE

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

written in the standard classical notation

$$\begin{aligned}
0 &= F(x, y, u, p, q, r, s, t)\\
0 &= du - p\,dx - q\,dy\\
0 &= dp - r\,dx - s\,dy\\
0 &= dq - s\,dx - t\,dy
\end{aligned}$$

We would interpret this to mean that the equation $F = 0$ defines a smooth hypersurface M^7 in $xyupqrst$-space and the differential equation is then modeled by the differential ideal $\mathcal{I} \subset \Omega^*(M)$ given by

$$\mathcal{I} = \langle\, du{-}p\,dx{-}q\,dy,\ dp{-}r\,dx{-}s\,dy,\ dq{-}s\,dx{-}t\,dy\,\rangle.$$

The assumption that the PDE be 'reasonable' is then that not all of the partials (F_r, F_s, F_t) vanish along the locus $F = 0$, so that x, y, u, p, q, and two of r, s, and t can be taken as local coordinates on M.

Exercise 1.4: Show that a second order scalar equation of the form

$$\begin{aligned}
A(x, y, u, p, q)\, r + 2B(x, y, u, p, q)\, s + C(x, y, u, p, q)\, t&\\
+D(x, y, u, p, q)\,(rt - s^2) + E(x, y, u, p, q) &= 0
\end{aligned}$$

(in the classical notation described above) can be modeled on $xyupq$-space (i.e., $M = \mathbb{R}^5$) via the ideal $\mathcal{I}$ generated by $\theta = du{-}p\,dx{-}q\,dy$ together with the 2-form

$$\Upsilon = A\,dp\wedge dy + B\,(dq\wedge dy{-}dp\wedge dx) - C\,dq\wedge dx + D\,dp\wedge dq + E\,dx\wedge dy.$$

(Equations of this kind are known as *Monge-Ampere equations*. They come up very frequently in differential geometry.)

Example 1.4: *Linear Weingarten Surfaces.* This example assumes that you know some differential geometry. Let $M^5 = \mathbb{R}^3 \times S^2$ and let $\mathbf{x} : M \to \mathbb{R}^3$ and $\mathbf{u} : M \to \mathbb{S}^2 \subset \mathbb{R}^3$ be the projections on the two factors. Notice that the isometry group G of Euclidean 3-space acts on M in a natural way, with translations acting only on the first factor and rotations acting 'diagonally' on the two factors together.

Consider the 1-form $\theta = \mathbf{u}\cdot d\mathbf{x}$, which is G-invariant. If $\iota : N \hookrightarrow \mathbb{R}^3$ is an oriented surface, then the lifting $f : N \to M$ given by $f(p) = \big(\iota(p), \nu(p)\big)$ where $\nu(p) \in S^2$ is the oriented unit normal to the immersion ι at p, is an integral manifold of θ. (Why?) Conversely, any integral 2-manifold $f : N \to M$ of θ for which the projection $\mathbf{x}{\circ}f : N \to \mathbb{R}^3$ is an immersion is such a lift of a canonically oriented surface $\iota : N \hookrightarrow \mathbb{R}^3$.

Exercise 1.5: Prove this last statement.

In the classical literature, the elements of M are called the (first order) contact elements of (oriented) surfaces in $\mathbb{R}^3$. (The adjective 'contact' refers to the image from mechanics of two oriented surfaces making contact to first order at a point if and only if they pass through the point in question and have the same unit normal there.)

It is not hard to show that any G-invariant 1-form on M is a constant multiple of θ. However, there are several G-invariant 2-forms (in addition to $d\theta$). For example, the 2-forms

$$\Upsilon_0 = \tfrac{1}{2}\mathbf{u}\cdot(d\mathbf{x}\times d\mathbf{x}),\quad \Upsilon_1 = \tfrac{1}{2}\mathbf{u}\cdot(d\mathbf{u}\times d\mathbf{x}),\quad \Upsilon_2 = \tfrac{1}{2}\mathbf{u}\cdot(d\mathbf{u}\times d\mathbf{u}).$$

are all manifestly G-invariant.

Exercise 1.6: For any oriented surface $\iota : N \hookrightarrow \mathbb{R}^3$ with corresponding contact lifting $f : N \to M$, show that

$$f^*(\Upsilon_0) = dA, \qquad f^*(\Upsilon_1) = -H\,dA, \qquad f^*(\Upsilon_2) = K\,dA.$$

where dA is the induced area form of the immersion ι and H and K are its mean and Gauss curvatures, respectively. Moreover, an integral 2-manifold of θ is a contact lifting if and only if Υ_0 is nonvanishing on it.

From this exercise, it follows, for example, that the contact liftings of minimal surfaces in $\mathbb{R}^3$ are integral manifolds of $\mathcal{I} = \langle \theta, \Upsilon_1 \rangle$. As another example, it follows that the surfaces with Gauss curvature $K = -1$ are integral manifolds of the ideal $\mathcal{I} = \langle \theta, \Upsilon_2 + \Upsilon_0 \rangle$. In fact, any constant coefficient linear equation of the form $a\,K + b\,H + c = 0$ is modeled by $\mathcal{I} = \langle \theta, a\,\Upsilon_2 - b\,\Upsilon_1 + c\,\Upsilon_0 \rangle$. Such equations are called *linear Weingarten equations* in the literature.

Exercise 1.7: Fix a constant r and consider the mapping $\Phi_r : M \to M$ satisfying $\Phi_r(x, u) = (x + ru, u)$. Show that $\Phi^*\theta = \theta$ and interpret what this means with regard to the integral surfaces of θ. Compute $\Phi^*\Upsilon_i$ for $i = 0, 1, 2$ and interpret this in terms of surface theory. In particular, what does this say about the relation between surfaces with $K = +1$ and surfaces with $H = \pm\frac{1}{2}$?

Exercise 1.8: Show that the cone $z^2 = x^2 + y^2$ is the projection to $\mathbb{R}^3$ of an embedded smooth cylinder in M that is an integral manifold of $\langle \theta, \Upsilon_2 \rangle$. Show that the double tractrix (or pseudosphere), a rotationally invariant singular 'surface' with Gauss curvature $K = -1$ at its smooth points, is the projection to $\mathbb{R}^3$ of an embedded cylinder in M that is an integral manifold of $\langle \theta, \Upsilon_2 + \Upsilon_0 \rangle$.

1.3. The Frobenius Theorem

Of course, reformulating a system of PDE as an EDS might not necessarily be a useful thing to do. It will be useful if there are techniques available to study the integral manifolds of an EDS that can shed light on the set of integral manifolds and that are not easily applicable to the original PDE system. The main techniques of this type will be discussed in lectures later in the week, but there are a few techniques that are available now.

The first of these is when the ideal $\mathcal{I}$ is algebraically as simple as possible.

Theorem 1: (The Frobenius Theorem) Let $(M, \mathcal{I})$ be an EDS with the property that $\mathcal{I} = \langle \mathcal{I}^1 \rangle_{\text{alg}}$ and so that $\dim \mathcal{I}^1_p$ is a constant r indepdendent of $p \in M$. Then for each point $p \in M$ there is a coordinate system $\mathbf{x} = (x^1, \ldots, x^{n+r})$ on a p-neighborhood $U \subset M$ so that

$$\mathcal{I}_U = \langle dx^{n+1}, \ldots, dx^{n+r} \rangle.$$

In other words, if $\mathcal{I}$ is algebraically generated by 1-forms and has constant 'rank', then $\mathcal{I}$ is locally equivalent to the obvious 'flat' model. In such a case, the n-dimensional integral manifolds of $\mathcal{I}$ are described locally in the coordinate system $\mathbf{x}$ as 'slices' of the form

$$x^{n+1} = c^1, \qquad x^{n+2} = c^2, \qquad \ldots, \qquad x^{n+r} = c^r.$$

In particular, each connected integral manifold of $\mathcal{I}$ lies in a unique maximal integral manifold, which has dimension n. Moreover, these maximal integral manifolds foliate the ambient manifold M.

If you look back at Example 1.2, you'll notice that $\mathcal{I}$ is generated algebraically by $\mathcal{I}^1$ if and only if it is generated algebraically by

$$\zeta = dz - F(x, y, z)\,dx - G(x, y, z)\,dy,$$

and this, in turn, is true if and only if $\zeta \wedge d\zeta = 0$. (Why?) Now

$$\zeta \wedge d\zeta = \big(F_y - G_x + G\,F_z - F\,G_z\big)\,dx \wedge dy \wedge dz.$$

Thus, by the Frobenius Theorem, if the two functions F and G satisfy the PDE $F_y - G_x + G\,F_z - F\,G_z = 0$, then for every point $(x_0, y_0, z_0) \in M$, there is a function u defined on an open neighborhood of $(x_0, y_0) \in \mathbb{R}^2$ so that $u(x_0, y_0) = z_0$ and so that u satisfies the equations $u_x = F(x, y, u)$ and $u_y = G(x, y, u)$.

Exercise 1.9: State and prove a converse to this last statement.

Note, by the way, that it may not be easy to actually *find* the 'flat' coordinates $\mathbf{x}$ for a given $\mathcal{I}$ that satisfies the Frobenius condition.

Exercise 1.10: Suppose that u and v are functions of x and y that satisfy the equations

$$u_x - v_y = e^u \sin v, \qquad\qquad u_y + v_x = e^u \cos v.$$

Show that $u_{xx} + u_{yy} = e^{2u}$ and that $v_{xx} + v_{yy} = 0$. Conversely, show that if $u(x,y)$ satisfies $u_{xx} + u_{yy} = e^{2u}$, then there exists a one parameter family of functions v so that the pair (u,v) satisfies the displayed equations. Prove a similar existence theorem for a given arbitrary solution of $v_{xx} + v_{yy} = 0$. (This peculiar system is an elementary example of what is known as a *Bäcklund transformation*. More on this later.)

1.4. The Pfaff Theorem

There is another case (or rather, sequence of cases) in which there is a simple local normal form.

Theorem 2: (The Pfaff Theorem) Let $(M,\mathcal{I})$ be an EDS with the property that $\mathcal{I} = \langle\omega\rangle$ for some nonvanishing 1-form ω. Let $r \geq 0$ be the smallest integer for which $\omega{\wedge}(d\omega)^{r+1} \equiv 0$. Then for each point $p \in M$ at which $\omega{\wedge}(d\omega)^r$ is nonzero, there is a coordinate system $\mathbf{x} = (x^1,\ldots,x^{n+2r+1})$ on a p-neighborhood $U \subset M$ so that $\mathcal{I}_U = \langle dx^{n+1}\rangle$ if $r = 0$ and, if $r > 0$, then

$$\mathcal{I}_U = \langle\, dx^{n+1} - x^{n+2}\, dx^{n+3} - x^{n+4}\, dx^{n+5} - \cdots x^{n+2r}\, dx^{n+2r+1}\,\rangle.$$

Note that the case where $r = 0$ is really a special case of the Frobenius Theorem. Points $p \in M$ for which $\omega{\wedge}(d\omega)^r$ is nonzero are known as the *regular* points of the ideal $\mathcal{I}$. The regular points are an open set in M.

Exercise 1.11: Explain why the integer r is well-defined, i.e, if $\mathcal{I} = \langle\omega\rangle = \langle\eta\rangle$, then you will get the same integer r if you use η as the generator and you will get the same notion of regular points.

In fact, the Pfaff Theorem has a slightly stronger form. It turns out that the maximum dimension of an integral manifold of $\mathcal{I}$ that lies in the regular set is $n{+}r$. Moreover, if $N^{n+r} \subset M$ is such a maximal dimensional integral manifold and N is embedded, then for every $p \in N$, one can choose the coordinates $\mathbf{x}$ so that $N \cap U$ is described by the equations

$$x^{n+1} = x^{n+2} = x^{n+4} = \cdots = x^{n+2r} = 0.$$

Any integral manifold in U near this one on which the $n{+}r$ functions $x^1,\ldots,x^n,x^{n+3},x^{n+5},\ldots,x^{n+2r+1}$ form a coordinate system can be described by equations of the form

$$\begin{aligned} x^{n+1} &= f(x^{n+3},x^{n+5},\ldots,x^{n+2r+1}),\\ x^{n+2k} &= \frac{\partial f}{\partial y^k}(x^{n+3},x^{n+5},\ldots,x^{n+2r+1}), \quad 1 \leq k \leq r \end{aligned}$$

for some suitable function $f(y^1,\ldots,y^r)$. Thus, one can informally say that the integral manifolds of maximal dimension depend on one arbitrary function of r variables.

Exercise 1.12: Consider the contact ideal $\bigl(\mathbb{R}^3{\times}S^2, \langle \mathbf{u}{\cdot}d\mathbf{x}\rangle\bigr)$ introduced in Example 1.4. Show that one can introduce local coordinates (x, y, z, p, q) in a neighborhood of any point of $M^5 = \mathbb{R}^3{\times}S^2$ so that

$$\langle \mathbf{u}{\cdot}d\mathbf{x}\rangle = \langle\, dz{-}p\,dx{-}q\,dy \,\rangle$$

and conclude that $\theta = \mathbf{u}{\cdot}d\mathbf{x}$ satisfies $\theta{\wedge}(d\theta)^2 \neq 0$. Explain how this shows that each of the ideals $\mathcal{I} = \langle \theta, a\,\Upsilon_2 - b\,\Upsilon_1 + c\,\Upsilon_0\rangle$ is locally equivalent to the ideal associated to a Monge-Ampere equation, as defined in Exercise 1.4.

1.5. Jørgen's Theorem

I want to conclude this lecture by giving one example of the advantage one gets by looking at even a very classical problem from the point of view of an exterior differential system.

Consider the Monge-Ampere equation

$$z_{xx}\,z_{yy} - {z_{xy}}^2 = 1.$$

It is easy to see that this has solutions of the form

$$z = u(x, y) = a\,x^2 + 2b\,xy + c\,y^2 + d\,x + e\,y + f$$

for any constants $a, \ldots, f$ satisfying $4(ac - b^2) = 1$. According to a theorem of Jørgen, these are the only solutions whose domain is the entire xy-plane. I now want to give a proof of this theorem.

As in Exercise 1.4, every (local) solution $z = u(x, y)$ of this equation gives rise to an integral manifold of an ideal $\mathcal{I}$ on $xyupq$-space where

$$\begin{aligned}\mathcal{I} &= \langle\, du - p\,dx - q\,dy,\ dp\wedge dq - dx\wedge dy \,\rangle\\ &= \langle\, du - p\,dx - q\,dy,\ dp\wedge dx + dq\wedge dy,\ dp\wedge dq - dx\wedge dy \,\rangle_{\text{alg}}.\end{aligned}$$

Now, consider the mapping $\Phi : \mathbb{R}^5 \to \mathbb{R}^5$ defined by

$$\Phi(x, y, u, p, q) = (x, q, u{-}qy, p, -y).$$

Then Φ is a smooth diffeomorphism of $\mathbb{R}^5$ with itself and it is easy to check that

$$\Phi^*(\mathcal{I}) = \langle\, du - p\,dx - q\,dy,\ dp\wedge dy + dx\wedge dq \,\rangle$$

However, this latter ideal is the ideal associated to $u_{xx} + u_{yy} = 0$! In other words, 'solutions' to the Monge-Ampere equation are transformed into 'solutions' of Laplace's equation by this mapping.

The reason for the scare quotes around the word 'solution' is that, while we know that the integral surfaces of the two ideals correspond under Φ, not all of the integral surfaces actually represent solutions, since, for example, some of the integral surfaces of $\mathcal{I}$ won't even have dx and dy be linearly independent, and these must somehow be taken into account.

Still, the close contact with the harmonic equation and thence the Cauchy-Riemann equations suggests an argument: Namely, the integral surface $N \subset \mathbb{R}^5$ of a solution to the Monge-Ampere equation must satisfy

$$0 = dp\wedge dx + dq\wedge dy + i(dp\wedge dq - dx\wedge dy) = (dp + i\,dy)\wedge(dx + i\,dq).$$

Thus the projection of N into $xypq$-space is a complex curve when $p{+}i\,y$ and $x{+}i\,q$ are regarded as complex coordinates on this $\mathbb{R}^4$. In particular, N can be regarded as a complex curve for which each of $p{+}i\,y$ and $x{+}i\,q$ are holomorphic functions.

Since dx and dy are linearly independent on N, it follows that neither of the 1-forms $dp{+}i\,dy$ nor $dq{-}i\,dx$ can vanish on N. Thus, there exists a holomorphic function λ on N so that

$$dp + i\,dy = \lambda\,(dx + i\,dq).$$

Because $dx{\wedge}dy$ is nonvanishing on N, the real part of λ can never vanish.

Suppose that the real part of λ is always positive. (I'll leave the other case to you.) Then $|\lambda+1|^2 > |\lambda-1|^2$, which implies that

$$|\lambda+1|^2(dx^2+dq^2) > |\lambda-1|^2(dx^2+dq^2) > 0$$

and, by the above relation, this is

$$|(dp+i\,dy)+(dx+i\,dq)|^2 > |(dp+i\,dy)-(dx+i\,dq)|^2$$

or, more simply,

$$d(p{+}x)^2 + d(q{+}y)^2 > d(p{-}x)^2 + d(q{-}y)^2.$$

In particular, the left hand quadratic form is greater than the average of the left and right hand quadratic forms, i.e.,

$$d(p{+}x)^2 + d(q{+}y)^2 > dp^2 + dx^2 + dq^2 + dy^2 > dx^2 + dy^2.$$

If the solution is defined on the whole plane, then the right hand quadratic form is complete on N, so the left hand quadratic form must be complete on N also. It follows from this that the holomorphic map

$$(p{+}x) + i(y{+}q) : N \to \mathbb{C}$$

is a covering map and hence must be a biholomorphism, so that N is equivalent to $\mathbb{C}$ as a Riemann surface. By Liouville's Theorem, it now follows that λ (which takes values in the right half plane) must be constant. The constancy of λ implies that dp and dq are constant linear combinations of dx and dy, which forces u to be a quadratic function of x and y. QED.

Exercise 1.13: Is it necessarily true that any entire solution of

$$u_{xx}\,u_{yy} - {u_{xy}}^2 = -1$$

must be a quadratic function of x and y? Prove or give a counterexample.

Lecture 2. Applications 1: Scalar first order PDE, Lie Groups

2.1. The contact system

For any vector space V of dimension N over $\mathbb{R}$, let $G_n(V)$ denote the set of n-dimensional subspaces of V. When $0 < n < N$ (which I will assume from now on), the set $G_n(V)$ can naturally be regarded as a smooth manifold of dimension $n(N-n)$. To see this, set $s = N - n$ and, for any $E \in G_n(V)$ choose linear coordinates $(\mathbf{x}, \mathbf{u}) = (x^1, \ldots, x^n; u^1, \ldots, u^s)$ so that the x^i restrict to E to be linearly independent. Let $G_n(V, \mathbf{x}) \subset G_n(V)$ denote the set of $\tilde{E} \in G_n(V)$ to which the x^i restrict to be linearly independent. Then there are unique numbers $p^a_i(\tilde{E})$ so that the defining equations of $\tilde{E}$ are

$$u^a - p^a_i(\tilde{E})\, x^i = 0, \qquad 1 \le a \le s.$$

Give $G_n(V)$ the manifold structure so that the maps $(p^a_i) : G_n(V, \mathbf{x}) \to \mathbb{R}^{ns}$ are smooth coordinate charts.

Exercise 2.1: Check that this does work, i.e., that these charts are smooth on overlaps.

Now let X be a manifold of dimension N. The set of n-dimensional subspaces of the tangent spaces T_xX as x varies over X will be denoted by $G_n(TX)$. Any $E \in G_n(TX)$ is an n-dimensional subspace $E \subset T_{\pi(E)}X$ for a unique $\pi(E) \in X$. Obviously, the fiber of the map $\pi : G_n(TX) \to X$ over the point $x \in X$ is $G_n(T_xX)$. It should not be surprising, then, that there is a natural manifold structure on $G_n(TX)$ for which π is a submersion and for which $G_n(TX)$ has dimension $n{+}s{+}ns$.

In fact, consider a coordinate chart $(\mathbf{x}, \mathbf{u}) : U \to \mathbb{R}^n \times \mathbb{R}^s$ defined on some open set $U \subset X$, where $\mathbf{x} = (x^1, \ldots, x^n)$ and $\mathbf{u} = (u^1, \ldots, u^s)$. Let $G_n(TU, \mathbf{x}) \subset G_n(TU)$ denote the set of n-planes to which the differentials dx^i restrict to be independent. Then each $E \in G_n(TU, \mathbf{x})$ satisfies a set of linear relations of the form

$$du^a - p^a_i(E)\, dx^i = 0, \qquad 1 \le a \le s.$$

for some unique real numbers $p^a_i(E)$. Set $\mathbf{p} = (p^a_i) : G_n(TU, \mathbf{x}) \to \mathbb{R}^{ns}$. Then the map

$$(\mathbf{x}, \mathbf{u}, \mathbf{p}) : G_n(TU, \mathbf{x}) \to \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^{ns}$$

embedds $G_n(TU, \mathbf{x})$ as an open subset of $\mathbb{R}^{n+s+ns}$. Give $G_n(TX)$ the manifold structure for which these maps are smooth coordinate charts.

Exercise 2.2: Check that this does work, i.e., that these charts are smooth on overlaps.

The coordinate chart $\big((\mathbf{x}, \mathbf{u}, \mathbf{p}), G_n(TU, \mathbf{x})\big)$ will be called the *canonical extension* of the coordinate chart $\big((\mathbf{x}, \mathbf{u}), U\big)$.

Any diffeomorphism $\phi : X \to Y$ lifts to a diffeomorphism $\phi^{(1)} : G_n(TX) \to G_n(TY)$ defined by the rule

$$\phi^{(1)}(E) = d\phi(E) \subset T_{\phi\left(\pi(E)\right)}Y.$$

Now $G_n(TX)$ comes endowed with a canonical exterior system $\mathcal{C}$ called the *contact system*. Abstractly, it can be defined as follows: There is a canonical $(n{+}ns)$-plane field $C \subset TG_n(TX)$ defined by

$$C_E = d\pi^{-1}(E) \subset T_E G_n(TX).$$

Then $\mathcal{C}$ is the ideal generated by the set of 1-forms on $G_n(TX)$ that vanish on C. From the canonical nature of the deifinition, it's clear that for any diffeomorphism $\phi : X \to Y$, the corresponding lift $\phi^{(1)} : G_n(TX) \to G_n(TY)$ will identify the two contact systems.

Now, why is $\mathcal{C}$ called a 'contact' system? Consider an immersion $f : N \to X$ where N has dimension n. This has a canonical 'tangential' lift $f^{(1)} : N \to G_n(TX)$ defined by

$$f^{(1)}(p) = df(T_pN) \subset T_{f(p)}X.$$

Almost by construction, $df^{(1)}(T_pN) \subset C_{f^{(1)}(p)}$, so that $f^{(1)} : N \to G_n(TX)$ is an integral manifold of $\mathcal{C}$. Conversely, if $F : N^n \to G_n(TX)$ is an integral manifold of $\mathcal{C}$ that is transverse to the fibration $\pi : G_n(TX) \to X$, i.e., $f = \pi \circ F : N^n \to M$ is an immersion, then $F = f^{(1)}$.

Exercise 2.3: Prove this last statement.

Thus, the contact system $\mathcal{C}$ essentially distinguishes the tangential lifts of immersions of n-manifolds into X from arbitrary immersions of n-manifolds into X. As for the adjective 'contact', it comes from the interpretation that two different immersions $f, g : N \to X$ will satisfy $f^{(1)}(p) = g^{(1)}(p)$ if and only if $f(p) = g(p)$ and the two image submanifolds share the same tangent n-plane at p. Intuitively, the two image submanifolds $f(N)$ and $g(N)$ have 'first order contact' at p.

Exercise 2.4: (IMPORTANT!) Show that, in canonically extended coordinates $(\mathbf{x}, \mathbf{u}, \mathbf{p})$ on $G_n(TX, \mathbf{x})$,

$$\mathcal{C}_{G_n(TX,\mathbf{x})} = \langle\, du^1 - p^1_i\, dx^i,\, \ldots,\, du^s - p^s_i\, dx^i \,\rangle,$$

I.e., $\mathcal{C}$ is locally generated by the 1-forms $\theta^a = du^a - p^a_i\, dx^i$ for $1 \le a \le s$ in any canonically extended coordinate system.

As a consequence of the previous exercise, we see that the integral manifolds of $\mathcal{C}$ in $G_n(TX, \mathbf{x})$ to which $\mathbf{x}$ restricts to be a coordinate system are described by equations of the form

$$u^a = f^a(x^1, \ldots, x^n), \qquad p^a_i = \frac{\partial f^a}{\partial x^i}(x^1, \ldots, x^n)$$

for some differentiable functions f^a on an appropriate domain in $\mathbb{R}^n$.

Once the construction of the contact system $\big(G_n(TX), \mathcal{C}\big)$ is in place, it can be used to construct other canonical systems and manifolds. For example, Let X have dimension n and U have dimension s. Let $J^1(X, U) \subset G_n\big(T(X{\times}U)\big)$ denote the open (dense) set consisting of the n-planes $E \subset T_{(x,u)}X{\times}U$ that are transverse to the subspace $0 \oplus T_uU \subset T_{(x,u)}X{\times}U$. The graph $(\mathrm{id}, f) : X \to X{\times}U$ of any smooth map $f : X \to U$ then has the property that $j^1 f = (\mathrm{id}, f)^{(1)}$ lifts X into $J^1(X, U)$. In fact, two maps $f, g : X \to U$ satisfy $j^1 f(p) = j^1 g(p)$ if and only if f and g have the same 1-jet at p. Thus, $J^1(X, U)$ is canonically identified with the space of 1-jets of mappings of X into U. The contact system then restricts to $J^1(X, U)$ to be the usual contact system defined in the theory of jets.

If one chooses a submanifold $M \subset G_n(TX)$ and lets $\mathcal{I}$ be the differential ideal on M generated by the pullbacks to M of elements of $\mathcal{C}$, then the integral manifolds of $(M, \mathcal{I})$ can be thought of as representing the n-dimensional submanifolds of X whose tangent planes lie in M. In other words, M can be thought of as a system of first order partial differential equations for submanifolds of X. As we will see, this is a very useful point of view.

Exercise 2.5: Let X^4 be an almost complex 4-manifold and let $M \subset G_2(TX)$ be the set of 2-planes that are invariant under complex multiplication. Show that M has (real) dimension 6 and describe the fibers of the projection $M \to X$. What can you say about the surfaces in X whose tangential lifts lie in M?

2.2. THE METHOD OF CHARACTERISTICS

I now want to apply some of these ideas to the classical problem of solving a single, scalar first order PDE

$$F\big(x^1, \ldots, x^n, u, \tfrac{\partial u}{\partial x^1}, \ldots, \tfrac{\partial u}{\partial x^1}\big) = 0.$$

As explained before, I am going to regard this as an exterior differential system as follows: Using the standard coordinates $\mathbf{x} = (x^1, \ldots, x^n)$ on $\mathbb{R}^n$ and $\mathbf{u} = (u)$ on $\mathbb{R}$, the canonical extended coordinates on $J^1(\mathbb{R}^n, \mathbb{R}) = G_n\big(T(\mathbb{R}^n{\times}\mathbb{R}), \mathbf{x}\big)$ become $(\mathbf{x}, \mathbf{u}, \mathbf{p})$ where $\mathbf{p} = (p_1, \ldots, p_n)$. The equation

$$F(x^1, \ldots, x^n, u, p_1, \ldots, p_n) = 0$$

then defines a subset $M \subset J^1(\mathbb{R}^n, \mathbb{R})$. I am going to suppose that F is smooth and that not all of the partials $\partial F/\partial p_i$ vanish at any single point of M. By the implicit function theorem, it follows then that M is a smooth manifold of dimension $2n$ and that the projection $(\mathbf{x}, \mathbf{u}) : M \to \mathbb{R}^n{\times}\mathbb{R}$ is a smooth submersion. Let $\mathcal{I}$ be the exterior differential system on M generated by the contact 1-form

$$\theta = du - p_i\, dx^i.$$

Note that, on M, the 1-forms dx^i, du, dp_i are not linearly independent (there are too many of them), but satisfy a single linear relation

$$0 = dF = \tfrac{\partial F}{\partial x^i}\, dx^i + \tfrac{\partial F}{\partial u}\, du + \tfrac{\partial F}{\partial p_i}\, dp_i\,.$$

Of course, $\theta{\wedge}(d\theta)^n = 0$, but $\theta{\wedge}(d\theta)^{n-1}$ is nowhere vanishing.

Exercise 2.6: Prove this last statement.

By the Pfaff theorem, it follows that every point in M has a neighborhood U on which there exist coordinates $(z, y^1, \ldots, y^{n-1}, v, q_1, \ldots, q_{n-1})$ so that

$$\langle\theta\rangle = \langle\, dv - q_1\, dy^1 - q_2\, dy^2 - \cdots - q_{n-1}\, dy^{n-1}\,\rangle.$$

I.e., there is a nonvanishing function μ on U so that

$$\theta = \mu\bigl(dv - q_1\, dy^1 - q_2\, dy^2 - \cdots - q_{n-1}\, dy^{n-1}\bigr).$$

Notice what this says about the vector field $Z = \frac{\partial}{\partial z}$. Not only does it satisfy $\theta(Z) = 0$, but it also satisfies

$$Z \lrcorner\, d\theta = d\mu(Z)\, \theta.$$

Moreover, up to a multiple, Z is the only vector field that satisfies $\theta(Z) = 0$ and $Z \lrcorner\, d\theta \equiv 0 \mod \theta$.

Exercise 2.7: Prove this last statement. Moreover, show that the vector field

$$Z = \frac{\partial F}{\partial p_i}\frac{\partial}{\partial x^i} + p_i\frac{\partial F}{\partial p_i}\frac{\partial}{\partial u} - \left(\frac{\partial F}{\partial x^i} + p_i\frac{\partial F}{\partial u}\right)\frac{\partial}{\partial p_i}$$

defined on $J^1(\mathbb{R}^n, \mathbb{R})$ is tangent to the level sets of F (and $M = F^{-1}(0)$ in particular), satisfies $\theta(Z) = 0$, and satisfies $Z \lrcorner\, d\theta \equiv 0 \mod \{\theta, dF\}$. Conclude that this Z is, up to a multiple, equal to the Z described above in Pfaff coordinates on M. This vector field is known as the *Cauchy characteristic vector field* of the function F.

A solution to the above equation is then represented by a function $f : \mathbb{R}^n \to \mathbb{R}$ so that $N = j^1 f(\mathbb{R}^n)$ lies in M. In other words, $j^1 f : \mathbb{R}^n \to M$ is an integral manifold of $\mathcal{I}$. Now, an n-dimensional integral manifolds of $\mathcal{I}$ is locally described in some Pfaff normal coordinates as above in the form

$$v = g(y^1, \ldots, g^{n-1}), \quad q_i = \frac{\partial g}{\partial y_i}(y^1, \ldots, g^{n-1})$$

for a suitable function g on a domain in $\mathbb{R}^{n-1}$. In particular, such an integral manifold is always tangent to the Cauchy characteristic vector field.

This gives a prescription for solving a given initial value problem for the above partial differential equation: Use initial data for the equation to find an $(n-1)$-dimensional integral manifold $P^{n-1} \subset M$ of $\mathcal{I}$ that is transverse to the Cauchy characteristic vector field Z. Then construct an n-dimensional integral manifold of $\mathcal{I}$ by taking the union of the integral curves of Z that pass through P.

This method of solving a single scalar PDE via ordinary differential equations (i.e., integrating the flow of a vector field) is known as the *method of characteristics*. For some explicit examples, consult pp. 25–27 of the EDS notes.

2.3. Maps into Lie groups – existence and uniqueness

Let G be a Lie group with Lie algebra $\mathfrak{g} = T_eG$, and let η be its canonical left-invariant 1-form. Thus, η is a 1-form on G with values in $\mathfrak{g}$ that satisfies the conditions that, first $\eta_e : T_eG = \mathfrak{g} \to \mathfrak{g}$ is the identity, and, second, that η is left invariant, i.e., $L_a^*(\eta) = \eta$ for all $a \in G$, where $L_a : G \to G$ is left multiplication by a.

Exercise 2.8: Show that if G is a matrix group, with $g : G \to M_n(\mathbb{R})$ the inclusion into the n-by-n matrices, then

$$\eta = g^{-1}\,dg.$$

It is well-known (and easy to prove) that η satisfies the Maurer-Cartan equation

$$d\eta = -\tfrac{1}{2}\,[\eta, \eta].$$

(In the matrix case, this is equivalent to the perhaps-more-familiar equation $d\eta = -\eta{\wedge}\eta$.)

There are many cases in differential geometry where a geometric problem can be reduced to the following problem: Suppose given a manifold N and a $\mathfrak{g}$-valued 1-form γ on N that satisfies the Maurer-Cartan equation $d\gamma = -\frac{1}{2}\,[\gamma, \gamma]$. Prove that there exists a smooth map $g : N \to G$ so that $\gamma = g^*(\eta)$.

The fundamental result concerning this problem is due to Elie Cartan and is the foundation of the method of the moving frame:

Theorem 3: (Maurer-Cartan) If N is connected and simply connected and γ is a smooth $\mathfrak{g}$-valued 1-form on N that satisfies $d\gamma = -\frac{1}{2}\,[\gamma, \gamma]$, then there exists a smooth map $g : N \to G$, unique up to composition with a constant left translation, so that $g^*\eta = \gamma$.

I want to sketch the proof as an application of the Frobenius theorem. Here are the ideas: Let $M = N{\times}G$ and consider the $\mathfrak{g}$-valued 1-form

$$\theta = \eta - \gamma.$$

It's easy to compute that

$$d\theta = -\tfrac{1}{2}\,[\theta, \theta] - [\theta, \gamma].$$

In particular, writing $\theta = \theta^1\,x_1 + \cdots + \theta^s\,x_s$ where $x_1, \ldots, x_s$ is a basis of $\mathfrak{g}$, the differential ideal

$$\mathcal{I} = \langle\, \theta^1, \ldots, \theta^s \,\rangle$$

satisfies $\mathcal{I} = \langle\, \theta^1, \ldots, \theta^s \,\rangle_{\text{alg}}$. Moreover, the θ^a are manifestly linearly independent since they restrict to each fiber $\{n\}{\times}G$ to be linearly independent. Thus, the hypotheses of the Frobenius theorem are satisfied, and M is foliated by maximal connected integral manifolds of $\mathcal{I}$, each of which can be shown to project onto the first factor N to be a covering map.

Exercise 2.9: Prove this. (You will need to use the fact that the foliation is invariant under the maps $\mathrm{id}{\times}L_a : N{\times}G \to N{\times}G$.)

Since N is connected and simply connected, each integral leaf projects diffeomorphically onto N and hence is the graph of a map $g : N \to G$. This g has the desired property. QED

Exercise 2.10: Use Cartan's Theorem to prove that for every Lie algebra $\mathfrak{g}$, there is, up to isomorphism, at most one connected and simply connected Lie group G with Lie algebra $\mathfrak{g}$. (Such a Lie group does exist for every Lie algebra, but this is proved by other techniques.) Hint: If G_1 and G_2 satisfy these hypotheses, consider the map $g : G_1 \to G_2$ that satisfies $g^*\eta_2 = \eta_1$ and $g(e_1) = e_2$.

2.4. The Gauss and Codazzi equations

As another typical application of the Frobenius Theorem, I want to consider one of the fundamental theorems of surface theory in Euclidean space.

Let $x : \Sigma \to \mathbb{R}^3$ be an immersion of an oriented surface Σ and let $u : \Sigma \to S^2$ be its Gauss map. In particular $u \cdot dx = 0$. The two quadratic forms

$$\mathrm{I} = dx \cdot dx\,, \qquad\qquad \mathrm{I\!I} = -\,du \cdot dx$$

are known as the first and second fundamental forms of the oriented immersion x.

It is evident that if $y = Ax{+}b$ where A lies in $\mathrm{O}(3)$ and b lies in $\mathbb{R}^3$, then y will be an immersion with the same first and second fundamental forms. (NB. The Gauss map of y will be $v = \det(A)\,Au = \pm Au$.) One of the fundamental results of surface theory is a sort of converse to this statement, namely that if $x, y : \Sigma \to \mathbb{R}^3$ have the same first and second fundamental forms, then they differ by an ambient isometry. (Note that the first or second fundamental form alone is not enough to determine the immersion up to rigid motion.) This is known as Bonnet's Theorem, although it appears to have been accepted as true long before Bonnet's proof appeared.

The standard argument for Bonnet's Theorem goes as follows: Let $\pi : F \to \Sigma$ be the oriented orthonormal frame bundle of Σ endowed with the metric I. Elements of F consist of triples (p, v_1, v_2) where (v_1, v_2) is an oriented, I-orthonormal basis of $T_p\Sigma$ and $\pi(p, v_1, v_2) = p$. There are unique 1-forms on F, say $\omega_1, \omega_2, \omega_{12}$ so that

$$d\pi(w) = v_1\,\omega_1(w) + v_2\,\omega_2(w)$$

for all $w \in T_{(p,v_1,v_2)}F$ and so that

$$d\omega_1 = -\omega_{12} \wedge \omega_2\,, \qquad\qquad d\omega_2 = \omega_{12} \wedge \omega_1\,.$$

Then

$$\pi^*\mathrm{I} = {\omega_1}^2 + {\omega_2}^2, \qquad\qquad \pi^*\mathrm{I\!I} = h_{11}\,{\omega_1}^2 + 2h_{12}\,\omega_1\omega_2 + h_{22}\,{\omega_2}^2,$$

for some functions h_{11}, h_{12}, and h_{22}. Defining $\omega_{31} = h_{11}\,\omega_1 + h_{12}\,\omega_2$ and $\omega_{32} = h_{12}\,\omega_1 + h_{22}\,\omega_2$, it is not difficult to see that the $\mathbb{R}^3$-valued functions x, $e_1 = x'(v_1)$, $e_2 = x'(v_2)$, and $e_3 = e_1 \times e_2$ must satisfy the matrix equation

$$d\begin{bmatrix} 1 & 0 & 0 & 0 \\ x & e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & e_1 & e_2 & e_3 \end{bmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega_1 & 0 & \omega_{12} & -\omega_{31} \\ \omega_2 & -\omega_{12} & 0 & -\omega_{32} \\ 0 & \omega_{31} & \omega_{32} & 0 \end{pmatrix}.$$

Now, the matrix

$$\gamma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega_1 & 0 & \omega_{12} & -\omega_{31} \\ \omega_2 & -\omega_{12} & 0 & -\omega_{32} \\ 0 & \omega_{31} & \omega_{32} & 0 \end{pmatrix}$$

takes values in the Lie algebra of the group $G \subset \mathrm{SL}(4, \mathbb{R})$ of matrices of the form

$$\begin{bmatrix} 1 & 0 \\ b & A \end{bmatrix}, \qquad b \in \mathbb{R}^3,\ A \in \mathrm{SO}(3),$$

while the mapping $g : F \to G$ defined by

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & e_1 & e_2 & e_3 \end{bmatrix}$$

clearly satisfies $g^{-1}\,dg = \gamma$. Thus, by the uniqueness in Cartan's Theorem, the map g is uniquely determined up to left multiplication by a constant in G.

Exercise 2.11: Explain why this implies Bonnet's Theorem as it was stated.

Perhaps more interesting is the application of the existence part of Cartan's Theorem. Given any pair of quadratic forms $(\mathrm{I}, \mathrm{I\!I})$ on a surface Σ with I being positive definite, the construction of F and the accompanying forms $\omega_1, \omega_2, \omega_{12}, \omega_{31}, \omega_{32}$ and thence γ can obviously be carried out. However, it won't necessarily be true that $d\gamma = -\gamma\wedge\gamma$. In fact,

$$d\gamma + \gamma \wedge \gamma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega_{12} & -\Omega_{31} \\ 0 & -\Omega_{12} & 0 & -\Omega_{32} \\ 0 & \Omega_{31} & \Omega_{32} & 0 \end{pmatrix}$$

where, for example,

$$\Omega_{12} = \left(K - h_{11}h_{22} + {h_{12}}^2\right)\omega_1 \wedge \omega_2$$

where K is the Gauss curvature of the metric I. Thus, a necessary condition for the pair $(\mathrm{I}, \mathrm{I\!I})$ to come from an immersion is that the *Gauss equation* hold, i.e.,

$$\det\nolimits_{\mathrm{I}} \mathrm{I\!I} = K.$$

The other two expressions $\Omega_{31} = h_1\,\omega_1{\wedge}\omega_2$ and $\Omega_{32} = h_2\,\omega_1{\wedge}\omega_2$ are such that there is a well-defined 1-form η on Σ so that $\pi^*\eta = h_1\,\omega_1 + h_2\,\omega_2$. The mapping δ_{I} from quadratic forms to 1-forms that $\mathrm{I\!I} \mapsto \eta$ defines is a first order linear differential operator. Thus, another necessary condition that the pair $(\mathrm{I}, \mathrm{I\!I})$ come from an immersion is that the *Codazzi equation* hold, i.e.,

$$\delta_{\mathrm{I}}(\mathrm{I\!I}) = 0.$$

By Cartan's Theorem, if a pair $(\mathrm{I}, \mathrm{I\!I})$ on a surface Σ satisfy the Gauss and Codazzi equations, then, at least locally, there will exist an immersion $x : \Sigma \to \mathbb{R}^3$ with $(\mathrm{I}, \mathrm{I\!I})$ as its first and second fundamental forms.

Exercise 2.12: Show that this immersion can be defined on all of Σ if Σ is simply connected. (Be careful: Just because Σ is simply connected, it does not follow that F is simply connected. How do you deal with this?) Is this necessarily true if Σ is not simply connected?

Exercise 2.13: Show that the quadratic forms on $\Sigma = \mathbb{R}^2$ defined by

$$\mathrm{I} = \cos^2 u\,dx^2 + \sin^2 u\,dy^2$$
$$\mathrm{I\!I} = \cos u \sin u\left(dx^2 - dy^2\right)$$

satisfy the Gauss and Codazzi equations if and only if the function $u(x, y)$ satisfies $u_{xx} - u_{yy} = \sin u\, \cos u \neq 0$. What sorts of surfaces in $\mathbb{R}^3$ correspond to these solutions? What happens if u satisfies the differential equation but either $\sin u$ or $\cos u$ vanishes? Does Cartan's Theorem give anything?

Lecture 3. Integral Elements and the Cartan-Kähler Theorem

The lecture notes for this section will mostly be definitions, some basic examples, and exercises. In particular, I will not attempt to give the proofs of the various theorems that I state. The full details can be found in Chapter III of *Exterior Differential Systems*.

Before beginning the lecture proper, let me just say that our method for constructing integral manifolds of a given exterior differential system will be to do it by a process of successively 'thickening' p-dimensional integral manifolds to $(p{+}1)$-dimensional integral manifolds by solving successive initial value problems. This will require some tools from partial differential equations, phrased in a geometric language, but it will also require us to understand the geometry of certain 'infinitesimal' integral manifolds known as 'integral elements'. It is to this study that I will first turn.

3.1. Integral elements and their extensions

Let $(M, \mathcal{I})$ be an EDS. An n-dimensional subspace $E \subset T_xM$ is said to be an *integral element* of $\mathcal{I}$ if

$$\phi(v_1, \ldots, v_n) = 0$$

for all $\phi \in \mathcal{I}^n$ and all $v_1, \ldots, v_n \in E$. The set of all n-dimensional integral elements of $\mathcal{I}$ will be denoted $V_n(\mathcal{I}) \subset G_n(TM)$.

Our main interest in integral elements is that the tangent spaces to any n-dimensional integral manifold $N^n \subset M$ are integral elements. Our ultimate goal is to answer the 'converse' questions: When is an integral element tangent to an integral manifold? If so, in 'how many' ways?

It is certainly not always true that every integral element is tangent to an integral manifold.

Example 3.1: *Non-existence.* Consider

$$(M, \mathcal{I}) = \big(\mathbb{R}, \langle\, x\, dx \,\rangle\big).$$

The whole tangent space $T_o\mathbb{R}$ is clearly a 1-dimensional integral element of $\mathcal{I}$, but there can't be any 1-dimensional integral manifolds of $\mathcal{I}$.

For a less trivial example, do the following exercise.

Exercise 3.1: Show that the ideal $\mathcal{I}_1 = \langle\, dx{\wedge}dz, dy{\wedge}(dz - y\, dx) \,\rangle$ has exactly one 2-dimensional integral element at each point, but that it has no 2-dimensional integral manifolds. Compare this with the ideal $\mathcal{I}_2 = \langle\, dx{\wedge}dz, dy{\wedge}dz \,\rangle$.

Now, $V_n(\mathcal{I})$ is a closed subset of $G_n(TM)$. To see why this is so, let's see how the elements of $\mathcal{I}$ can be used to get defining equations for $V_n(\mathcal{I})$ in local coordinates. Let $(\mathbf{x}, \mathbf{u}) : U \to \mathbb{R}^{n+s}$ be any local coordinate chart and let $(\mathbf{x}, \mathbf{u}, \mathbf{p}) : G_n(TX, \mathbf{x}) \to \mathbb{R}^{n+s+ns}$ be the canonical extension described in Lecture 2. Every $E \in G_n(TX, \mathbf{x})$ has a well-defined basis $\big(X_1(E), \ldots, X_n(E)\big)$, where

$$X_i(E) = \frac{\partial}{\partial x^i} + p^a_i(E)\,\frac{\partial}{\partial u^a}.$$

(This is the basis of E that is dual to the basis $dx^1, \ldots, dx^n$) of E^*.) Using this basis, we can define a function $\phi_{\mathbf{x}}$ on $G_n(TX, \mathbf{x})$ associated to any n-form ϕ by the rule

$$\phi_{\mathbf{x}}(E) = \phi\big(X_1(E), \ldots, X_n(E)\big).$$

It's not hard to see that $\phi_{\mathbf{x}}$ will be smooth as long as ϕ is smooth.

Exercise 3.2: Prove this last statement.

With this notation, $V_n(\mathcal{I}) \cap G_n(TX, \mathbf{x})$ is seen to be the simultaneous zero locus of the set of functions $\{\, \phi_{\mathbf{x}} | \phi \in \mathcal{I}^n \}$. Thus $V_n(\mathcal{I}) \cap G_n(TX, \mathbf{x})$ is closed. It follows that $V_n(\mathcal{I})$ is a closed subset of $G_n(TX)$, as desired.

Exercise 3.3: Describe $V_1(\mathcal{I})$ and $V_2(\mathcal{I})$ for

i. $(M,\mathcal{I}) = \big(\mathbb{R}^4, \langle\, dx^1{\wedge}dx^2 + dx^3{\wedge}dx^4 \,\rangle\big)$.

ii. $(M,\mathcal{I}) = \big(\mathbb{R}^4, \langle\, dx^1{\wedge}dx^2,\ dx^3{\wedge}dx^4 \,\rangle\big)$.

iii. $(M,\mathcal{I}) = \big(\mathbb{R}^4, \langle\, dx^1{\wedge}dx^2 + dx^3{\wedge}dx^4,\ dx^1{\wedge}dx^4 - dx^3{\wedge}dx^2 \,\rangle\big)$.

Now, there are some relations among the various $V_k(\mathcal{I})$. An easy one is that if E belongs to $V_n(\mathcal{I})$, then every p-dimensional subspace of E is also an integral element, i.e, $G_p(E) \subset V_p(\mathcal{I})$. This follows because $\mathcal{I}$ is an ideal. The point is that if $E' \subset E$ were a p-dimensional subspace and $\phi \in \mathcal{I}^p$ did not vanish when pulled back to E', then there would exist an $(n{-}p)$-form α so that $\alpha{\wedge}\phi$ (which belongs to $\mathcal{I}$) did not vanish when pulled back to E.

Exercise 3.4: Prove this last statement.

On the other hand, obviously not every extension of an integral element is an integral element. In fact, from the previous exercise, you can see that the topology of the space of integral elements of a given degree can be surprisingly complicated. However, describing the integral extensions one dimension at a time turns out to be reasonably simple:

Let $E \in V_k(\mathcal{I})$ be an integral element and let $(e_1,\ldots,e_k)$ be a basis for $E \subset T_xM$. The set

$$H(E) = \{\, v \in T_xM \;\big|\; \kappa(v, e_1, \ldots, e_k) = 0, \forall \kappa \in \mathcal{I}^{k+1} \} \subseteq T_xM$$

is known as the *polar space* of E, though it probably ought to be called the *extension space* of E, since a vector $v \in T_xM$ lies in $H(E)$ if and only if either it lies in E (the trivial case) or else $E^+ = E + \mathbb{R}v$ lies in $V_{k+1}(\mathcal{I})$. In other words, a $(k{+}1)$-plane E^+ containing E is an integral element of $\mathcal{I}$ if and only if it lies in $H(E)$.

Now, from the very definition of $H(E)$, it is a vector space and contains E. It is traditional to define the function $r : V_k(\mathcal{I}) \to \{-1, 0, 1, 2, \ldots\}$ by the formula

$$r(E) = \dim H(E) - k - 1.$$

The reason for subtracting 1 is that then $r(E)$ is the dimension of the set of $(k{+}1)$-dimensional integral elements of $\mathcal{I}$ that contain E, with $r(E) = -1$ meaning that there are no such extensions. When $r(E) \geq 0$, we have

$$\{\, E^+ \in V_{k+1}(\mathcal{I}) \;\big|\; E \subset E^+ \,\} \simeq \mathbb{P}\big(H(E)/E\big) \simeq \mathbb{RP}^{r(E)}.$$

Exercise 3.5: Compute the function $r : V_1(\mathcal{I}) \to \{-1, 0, 1, 2, \ldots\}$ for each of the examples in Exercise 3.3. Show that $V_3(\mathcal{I})$ is empty in each of these cases. What does this say about r on $V_2(\mathcal{I})$?

3.2. Ordinary and Regular Elements

Right now, we only have that $V_k(\mathcal{I})$ is a closed subset of $G_n(TX)$ and closed subsets can be fairly nasty objects in the eyes of a geometer. We want to see if we can put a nicer structure on $V_k(\mathcal{I})$.

First, some terminology. If $S \subset C^\infty(M)$ is some set of smooth functions on M, we can look at the common zero set of S, i.e.,

$$Z_S = \{\ x \in M \mid f(x) = 0,\ \forall f \in S\ \}.$$

Of course, this is a closed set, but we'd like to find conditions that will make it be a smooth manifold. One such case is provided by the implicit function theorem: Say that $z \in Z_S$ is an *ordinary zero* of S if there is an open neighborhood U of z in M and a set of functions $f_1, \ldots, f_c \in S$ so that

(1). $df_1{\wedge}df_2{\wedge}\cdots{\wedge}df_c \neq 0$ on U, and

(2). $Z_S \cap U = \{\, y \in U \mid f_1(y) = \cdots = f_c(y) = 0 \,\}$.

By the implicit function theorem, $Z_S \cap U$ is an embedded submanifold of U of codimension c. Let $Z_S^o \subset Z_S$ denote the set of ordinary zeros of S.

Exercise 3.6: Show that Z_S and Z_S^o depend only on the ideal generated by S in $C^\infty(M)$. Also, show that for $z \in Z_S^o$, the integer c described above is well-defined, so that one can speak without ambiguity of the codimension of Z_S^o at z.

This idea can now be applied to the $V_n(\mathcal{I})$. Say that $E \in V_n(\mathcal{I})$ is an *ordinary* integral element if it is an ordinary zero of the set

$$S_{\mathbf{x}} = \{\, \phi_{\mathbf{x}} \mid \phi \in \mathcal{I}^n \}$$

for some local coordinate chart $(\mathbf{x}, \mathbf{u}) : U \to \mathbb{R}^{n+s}$ with E in $G_n(TM, \mathbf{x})$.

Exercise 3.7: Show that on the intersection $G_n(TM, \mathbf{x}) \cap G_n(TM, \mathbf{y})$, the two sets of functions $S_{\mathbf{x}}$ and $S_{\mathbf{y}}$ generate the same ideal. Conclude that this notion of ordinary does not depend on the choice of a coordinate chart, only on the ideal $\mathcal{I}$.

Let $V_n^o(\mathcal{I}) \subset V_n(\mathcal{I})$ be the set of ordinary integral elements of dimension n. By the implicit function theorem, the connected components of $V_n^o(\mathcal{I})$ are smooth embedded submanifolds of $G_n(TM)$. They may not be closed or even all have the same dimension, but at least they are smooth manifolds and are cut out 'cleanly' by the condition that the n-forms vanish on them.

Exercise 3.8: Find an example of an integral element that is not ordinary. Now find a non-trivial example.

Exercise 3.9: Check to see whether or not all the integral elements you found in Exercise 3.3 are ordinary.

Even the ordinary integral elements aren't quite as nice as you could want. For example, the function $r : V_n(\mathcal{I}) \to \{-1, 0, 1, \ldots\}$ might not be locally constant on $V_n^o(\mathcal{I})$.

Example 3.2: *Polar Jumping.* Look back to the first ideal given in Exercise 3.1. There, $V_1(\mathcal{I}_1) = G_1(T\mathbb{R}^3)$ because ${\mathcal{I}_1}^1 = (0)$. Now a 1-dimensional integral element E based at (x, y, z) will be spanned by a vector

$$e_1 = a\,\frac{\partial}{\partial x} + b\,\frac{\partial}{\partial y} + c\,\frac{\partial}{\partial z}$$

where not all of a, b, and c vanish. Using the definition of the polar space, we see that

$$v = f\,\frac{\partial}{\partial x} + g\,\frac{\partial}{\partial y} + h\,\frac{\partial}{\partial z}$$

lies in $H(E)$ if and only if $dx{\wedge}dz(v, e_1) = \big(dy{\wedge}(dz - y\,dx)\big)(v, e_1) = 0$, i.e.,

$$c\,f - a\,h = -yb\,f - (c - y\,a)\,g + b\,h = 0.$$

These two linear equations for (f, g, h) will be linearly independent, forcing $H(E) = E$ and $r(E) = -1$, unless $c - y\,a = 0$, in which case the two equations are linearly dependent and $\dim H(E) = 2$, so that $r(E) = 0$.

We say that an ordinary integral element $E \in V_n^o(\mathcal{I})$ is *regular* if r is locally constant in a neighborhood of E in $V_n^o(\mathcal{I})$. Denote the set of regular integral elements by $V_n^r(\mathcal{I}) \subset V_n^o(\mathcal{I})$.

The regular integral elements are extremely nice. Not only do they 'vary smoothly', but their possible extensions 'vary smoothly' as well.

Exercise 3.10: Show that $V_n^r(\mathcal{I})$ is a dense open subset of $V_n^o(\mathcal{I})$. Hint: Show that if $E \subset T_xM$ is regular, then one can choose a fixed set of $(n{+}1)$-forms, say $\kappa^1, \ldots, \kappa^m \in \mathcal{I}^{n+1}$, where m is the codimension of $H(E)$ in T_xM, so that

$$H(E^*) = \{\, v \in T_xM \mid \kappa^\mu(v, e_1, \ldots, e_k) = 0,\ 1 \le \mu \le m \,\}$$

for all E^* in a neighborhood of E in $V_n^o(\mathcal{I})$. This shows that it is open. To get denseness, explain why r is upper semicontinuous and use that.

One more bit of terminology: An integral manifold $N^k \subset M$ of $\mathcal{I}$ will be said to be *ordinary* if all of its tangent planes are ordinary integral elements and *regular* if all of its tangent planes are regular integral elements. Note that if $N \subset M$ is a connected regular integral manifold of $\mathcal{I}$ then the numbers $r(T_xN)$ are all the same, so it makes sense to define $r(N) = r(T_xN)$ for any $x \in N$.

3.3. The Cartan-Kähler Theorem

I can now state one of the fundamental theorems in the subject. A discussion of the proof will be deferred to the next lecture. Here, I am just going to state the theorem, discuss the need for the hypotheses, and do a few examples. In the next lecture, I'll try to give you a feeling for why it works.

Theorem 4: (Cartan-Kähler) Let $(M, \mathcal{I})$ be a real analytic EDS and suppose that

(1) $P \subset M$ is a connected, k-dimensional, real analytic, regular integral manifold of $\mathcal{I}$ with $r(P) \geq 0$ and

(2) $R \subset M$ is a real analytic submanifold of codimension $r(P)$ containing P and having the property that $T_pR \cap H(T_pP)$ has dimension $k{+}1$ for all $p \in P$.

There exists a unique, connected, $(k{+}1)$-dimensional, real analytic integral manifold X of $\mathcal{I}$ that satisfies $P \subset X \subset R$.

The sudden appearance of the hypothesis of real analyticity is somewhat unexpected. However the PDE results that the enter in the proof of the Cartan-Kähler theorem require this assumption and, as will be seen, the theorem is not even true without this hypothesis in the generality stated.

Example 3.3: *The importance of regularity for existence.* Consider the case of Exercise 3.1. For either of the ideals, the line L defined by $x = z = 0$ is an integral curve of the ideal with the property that $r(T_pL) = 0$ for all $p \in L$. However, $\mathcal{I}_1$ has no integral surfaces while $\mathcal{I}_2$ has the integral surface $z = 0$ that contains L. In both cases, however, L is an ordinary integral manifold but not a regular one, so the Cartan-Kähler Theorem does not apply.

Example 3.4: *The importance of regularity for existence.* Consider the case of Exercise 3.3,*ii*. The line L defined by $x^2 = x^3 = x^4 = 0$ is a non-regular integral curve of this ideal, and has $r(T_pL) = 1$ for all $p \in L$, with the polar space $H(T_pL)$ being spanned by the vectors

$$\frac{\partial}{\partial x^1}, \; \frac{\partial}{\partial x^3}, \; \frac{\partial}{\partial x^4}.$$

for all $p \in L$. If you take R to be the 3-plane defined by $x^3 = 0$, then $T_pR \cap H(T_pL)$ has dimension 2 for all $p \in L$, but there is no integral surface X of $\mathcal{I}$ satisfying $L \subset X \subset R$, even though there are integral surfaces of $\mathcal{I}$ that contain L.

Example 3.5: *The meaning of R.* The manifold R that appears in the Cartan-Kähler Theorem is sometimes known as the 'restraining manifold'. You need it when $r(P) > 0$ because then the extension problem is actually underdetermined in a certain sense. (I'll try to make that precise in the next lecture.) However, you can see a little bit of why you need it by looking at the case of Exercise 3.3, (*i*). There, you should have computed that all of the integral elements $E \in V_1(\mathcal{I}) = G_1(T\mathbb{R}^4)$ are regular, with $r(E) = 1$. This means that every integral element has a 1-dimensional family of possible extensions to a 2-dimensional integral element. Suppose, for example, that you start with the curve $P \subset \mathbb{R}^4$ defined by the equations $x^2 = x^3 = x^4 = 0$. Then it is easy to compute that $H(T_pP)$ is spanned by the vectors

$$\frac{\partial}{\partial x^1}, \; \frac{\partial}{\partial x^3}, \; \frac{\partial}{\partial x^4}.$$

for all $p \in P$. In particular, any (real analytic) hypersurface R given by an equation $x^4 = F(x^1, x^2, x^3)$ where F satisfies $F(x^1, 0, 0) = 0$ will satisfy the conditions of the Theorem. If we pull the ideal $\mathcal{I}$ back to this hypersurface and use x^1, x^2, x^3 as coordinates on R, then the ideal on R is generated by the 2-form

$$dx^1 \wedge dx^2 + dx^3 \wedge (F_1\, dx^1 + F_2\, dx^2) = \left(dx^1 + F_2\, dx^3\right) \wedge \left(dx^2 - F_1\, dx^3\right)$$

Of course, this is a closed 2-form on R and its integral surfaces are swept out by integral curves of the vector field

$$X = -F_2 \frac{\partial}{\partial x^1} + F_1 \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}.$$

(Why?). Thus, to get the integral surface X, we take the union of these integral curves that pass through the initial curve P. Clearly, x^1 and x^3 are independent coordinates on a neighborhood of P in X, so X can also be written locally as a graph

$$x^2 = f(x^1, x^3), \qquad x^4 = g(x^1, x^3).$$

where f and g are functions that satisfy $f(x^1, 0) = g(x^1, 0) = 0$. The condition that these define an integral surface then turns out to be that there is another function h so that

$$x^2 = \frac{\partial h}{\partial x^1}(x^1, x^3), \qquad x^4 = \frac{\partial h}{\partial x^3}(x^1, x^3).$$

On the other hand, any such function works as long as its first partials vanish along the line $x^3 = 0$. This shows why you don't usually get uniqueness without a restraining manifold.

Example 3.6: *The importance of real analyticity.* Consider the case of Exercise 3.3, (*iii*). You'll probably recognize this as the ideal generated by the real and imaginary parts of the complex 2-form

$$\big(dx^1 - i\, dx^3\big) \wedge \big(dx^2 + i\, dx^4\big),$$

so the 2-dimensional integral manifolds are complex curves in $\mathbb{R}^4 \simeq \mathbb{C}^2$. Now, if you have done the exercises up to this point, you know that all of the 1-dimensional elements $E \in V_1(\mathcal{I}) = G_1(TM)$ are regular and satisfy $r(E) = 0$, so that each one can be extended uniquely to a 2-dimensional integral element. The Cartan-Kähler theorem then says that any real analytic curve in M lies in a unique connected, real analytic integral surface of $\mathcal{I}$ (i.e., a complex curve). As you know, a complex curve is necessarily real analytic when considered as a surface in $\mathbb{R}^4$. Now suppose that you had a curve described by

$$x^2 = f(x^1), \qquad x^3 = 0, \qquad x^4 = g(x^1),$$

where f and g are smooth, but not real analytic. Then I claim that there is no complex curve that can contain this curve, because if there were, it could be described locally in the form $x^2 + i\, x^4 = F(x^1 - i\, x^3)$ where F is a holomorphic function of one variable. However, setting $x^3 = 0$ in this equation shows that the original curve would be described by $x^2 + i\, x^4 = F(x^1)$, which is absurd because the real and imaginary parts of a holomorphic function are themselves real analytic.

Example 3.7: *Linear Weingarten Surfaces, again.* I now want to return to Example 1.4 and compute the integral elements, determine the notions of ordinary and regular, etc., and see what the Cartan-Kähler Theorem tells us about the integral manifolds.

For example, I claim that, for the EDS $\big(M, \langle\theta, \Upsilon_1\rangle\big)$, the space $V_1(\mathcal{I})$ is a smooth bundle over M, whose fiber at every point is diffeomorphic to $\mathbb{RP}^3$, that $V_1(\mathcal{I})$ consists entirely of regular integral elements, and that $r(E) = 0$ for all $E \in V_1(\mathcal{I})$. By the Cartan-Kähler Theorem, it will then follow that every real analytic integral curve of $\mathcal{I}$ lies in a unique real analytic integral surface.

Now, the integral curves of $\mathcal{I}$ are easy to describe: They are just of the form $\big(x(t), u(t)\big)$, where $x : (a, b) \to \mathbb{R}^3$ is a space curve and $u : (a, b) \to S^2$ is a unit length curve with $u(t) \cdot x'(t) = 0$. The condition that this describe an immersed curve in M is, of course, that x' and u' do not simultaneously vanish.

We have already said that the integral surfaces of $\mathcal{I}$ are 'generalized' minimal surfaces, so what the Cartan-Kähler Theorem says in this case is the geometric theorem that every real analytic 'framed curve', $\big(x(t), u(t)\big)$ in space lies on a unique, oriented minimal surface S for which $u(t)$ is the unit normal.

Exercise 3.11: Use this result to show that every nondegenerate real analytic space curve is a geodesic on a unique connected minimal surface. Also, use this result to prove the existence of a minimal Möbius band. (You'll have to think of a trick to get around the non-orientability of the Möbius band.)

Now, here is how this computation can be done. The principal difficulty in working with $M = \mathbb{R}^3 \times S^2$ is that, unlike $\mathbb{R}^4$ and other simple manifolds that we have been mostly dealing with, there is no obvious basis of 1-forms in which to compute. However, we can remedy this situation by regarding M as a homogeneous space of the group G of rigid motions of $\mathbb{R}^3$. Recall that

$$G = \left\{ \begin{bmatrix} 1 & 0 \\ b & A \end{bmatrix} \;\middle|\; b \in \mathbb{R}^3, A \in \mathrm{SO}(3) \right\}.$$

and that G acts on $\mathbb{R}^3$ by

$$\begin{bmatrix} 1 & 0 \\ b & A \end{bmatrix} \cdot y = Ay + b.$$

Writing out the columns of the inclusion map $g : G \to \mathrm{GL}(4, \mathbb{R})$ as

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathbf{x} & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix},$$

we have the structure equations

$$d \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathbf{x} & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathbf{x} & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega_1 & 0 & \omega_{12} & -\omega_{31} \\ \omega_2 & -\omega_{12} & 0 & -\omega_{32} \\ \omega_3 & \omega_{31} & \omega_{32} & 0 \end{pmatrix},$$

i.e., the classical structure equations

$$d\mathbf{x} = \mathbf{e}_j \, \omega_j, \qquad d\mathbf{e}_i = \mathbf{e}_j \, \omega_{ji}$$

where ω_i and $\omega_{ij} = -\omega_{ji}$ satisfy

$$d\omega_i = -\omega_{ij} \wedge \omega_j, \qquad d\omega_{ij} = -\omega_{ik} \wedge \omega_{kj}.$$

Now, consider the map $\pi : G \to M = \mathbb{R}^3 \times S^2$ given by

$$\pi(g) = (\mathbf{x}, \mathbf{e}_3).$$

This map is a smooth submersion and its fibers are the circles that are the left cosets of the circle subgroup H consisting of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, the 1-forms $\omega_1, \omega_2, \omega_3, \omega_{12}, \omega_{31}, \omega_{32}$ are a convenient basis for the left-invariant 1-forms on G, so we should be able to express the pullbacks of the various forms we have constructed on M in terms of these.

Exercise 3.12: Prove the formulae:

$$\begin{aligned} \pi^*\theta &= \omega_3\,, \\ \pi^*\Upsilon_0 &= \omega_1 \wedge \omega_2\,, \\ \pi^*\Upsilon_1 &= -\tfrac{1}{2}\big(\omega_{31} \wedge \omega_2 + \omega_1 \wedge \omega_{32}\big)\,, \\ \pi^*\Upsilon_2 &= \omega_{31} \wedge \omega_{32}\,. \end{aligned}$$

In particular, it follows from this exercise that

$$\begin{aligned} \pi^*\big(\langle \theta, \Upsilon_1 \rangle\big) &= \langle \omega_3, \omega_{31} \wedge \omega_2 + \omega_1 \wedge \omega_{32} \rangle \\ &= \langle\ \omega_3,\ \omega_{31} \wedge \omega_1 + \omega_{32} \wedge \omega_2,\ \omega_{31} \wedge \omega_2 + \omega_1 \wedge \omega_{32}\ \rangle_{\mathrm{alg}}. \end{aligned}$$

Now, let $E \subset T_{(x,u)}M$ be a 1-dimensional integral element of $\langle \theta, \Upsilon_1 \rangle = \langle \theta, d\theta, \Upsilon_1 \rangle_{\text{alg}}$. I want to compute the polar space $H(E)$. If $e_1 \in E$ is a basis element, then

$$H(E) = \{ \ v \in T_{(x,u)} \bigm| \ \theta(v) = d\theta(v, e_1) = \Upsilon_1(v, e_1) = 0 \ \},$$

so, *a priori*, the dimension of $H(E)$ could be anywhere from 2 (if the three equations on v are all linearly independent) to 4 (if the three equations on v are all multiples of $\theta(v) = 0$, which we know to be nontrivial). To see what actually happens, fix a $g \in G$ so that $\pi(g) = (x, u)$ and choose vectors $\tilde{e}_1$ and $\tilde{v}$ in T_gG so that $\pi_*(\tilde{e}_1) = e_1$ and $\pi_*(\tilde{v}) = v$. Define $a_i = \omega_i(\tilde{e}_1)$ and $a_{ij} = \omega_{ij}(\tilde{e}_1)$ and define $v_i = \omega_i(\tilde{v})$ and $v_{ij} = \omega_{ij}(\tilde{v})$. Then by the formulae from the exercise, we have

$$\begin{aligned}
\theta(v) &= \omega_3(\tilde{v}) \\
&= v_3 \\
d\theta(v, e_1) &= -(\omega_{31} \wedge \omega_1 + \omega_{32} \wedge \omega_2)(\tilde{v}, \tilde{e}_1) \\
&= a_{31}\, v_1 \ + a_{32}\, v_2 \ - a_1 \ v_{31} - a_2 \ v_{32} \\
-2\Upsilon_1(v, e_1) &= (\omega_{31} \wedge \omega_2 + \omega_1 \wedge \omega_{32})(\tilde{v}, \tilde{e}_1) \\
&= a_{32}\, v_1 \ - a_{31}\, v_2 \ + a_2 \ v_{31} - a_1 \ v_{32}
\end{aligned}$$

Now, unless $a_1 = a_2 = a_{31} = a_{32} = 0$, these are three linearly independent relations for $(v_1, v_2, v_3, v_{31}, v_{32})$. However, since e_1 is nonzero, we cannot have $a_1 = a_2 = a_{31} = a_{32} = 0$ (Why?). Thus, the three relations are linearly independent and it follows that $H(E)$ has dimension 2 for all $E \in V_1(\mathcal{I})$, as I wanted to show.

Exercise 3.13: Show that the same conclusion holds for all of the ideals of the form $\mathcal{I} = \langle\, \theta,\ \Upsilon_1 + c\,\Upsilon_0 \,\rangle$. Thus, every real analytic framed curve $\bigl(x(t), u(t)\bigr)$ lies in a unique (generalized) surface S with mean curvature $H = c$. Do the same for the ideal $\mathcal{I} = \langle\, \theta,\ \Upsilon_2 - c^2\,\Upsilon_0 \,\rangle$, and give a geometric interpretation of this result.

However, it is not always true that every integral element is regular, even for the linear Weingarten ideals.

Example 3.8: *Surfaces with* $K = -1$. Consider $\mathcal{I} = \langle\, \theta,\ \Upsilon_2 + \Upsilon_0 \,\rangle$, whose integrals correspond to surfaces with $K \equiv -1$. If you go through the same calculation as above for this ideal, everything runs pretty much the same until you get to

$$\begin{aligned}
\theta(v) &= \omega_3(\tilde{v}) \\
&= v_3 \\
d\theta(v, e_1) &= -(\omega_{31} \wedge \omega_1 + \omega_{32} \wedge \omega_2)(\tilde{v}, \tilde{e}_1) \\
&= a_{31}\, v_1 \ + a_{32}\, v_2 \ - a_1 \ v_{31} - a_2 \ v_{32} \\
(\Upsilon_2 + \Upsilon_1)(v, e_1) &= (\omega_{31} \wedge \omega_{32} + \omega_1 \wedge \omega_2)(\tilde{v}, \tilde{e}_1) \\
&= a_2 \ v_1 \ - a_1 \ v_2 \ + a_{32}\, v_{31} - a_{31}\, v_{32}
\end{aligned}$$

These three relations on $(v_1, v_2, v_3, v_{31}, v_{32})$ will be independent except when $(a_{31}, a_{32}) = \pm(a_2, -a_1)$, when the last two relations become dependent. For such integral elements $E \in V_1(\mathcal{I})$, we have $r(E) = 1$ but for all the other integral elements, we have $r(E) = 0$.

Exercise 3.14: Show that if $\bigl(x(t), u(t)\bigr)$ is an integral curve of θ, then its tangent vectors are all irregular if and only if $x : (a, b) \to \mathbb{R}^3$ is an immersed space curve of torsion $\tau = \pm 1$ and $u(t)$ is its binormal (up to a sign). Thus, these are the framed curves for which we cannot say that there exists a surface with $K = -1$ containing the curve with u as the surface normal along the curve. Even if there exists one, we cannot claim that it is unique.

Exercise 3.15: Determine which of the ideals $\mathcal{I} = \langle\, \theta,\ a\,\Upsilon_2 + b\,\Upsilon_1 + c\,\Upsilon_0 \rangle$ (where a, b, and c are constants, not all zero) have irregular integral elements in $V_1(\mathcal{I})$.

Lecture 4. The Cartan-Kähler Theorem: Ideas in the Proof

4.1. The Cauchy-Kowalewski Theorem

The basic PDE result that we will need is an existence and uniqueness theorem for initial value problems of a very special kind. You are probably familiar with the ODE existence and uniqueness theorem: If $D \subset \mathbb{R}\times\mathbb{R}^n$ is an open set and $F : D \to \mathbb{R}^n$ is a smooth map, then for any $(t_0, \mathbf{u}_0) \in D$, the initial value problem

$$\mathbf{u}'(t) = F\big(t, \mathbf{u}(t)\big), \qquad \mathbf{u}(t_0) = \mathbf{u}_0$$

has a solution $\mathbf{u} : I \to \mathbb{R}^n$ on some open interval $I \subset \mathbb{R}$ containing t_0, this solution is smooth, and this solution is unique in the sense that, if $\tilde{\mathbf{u}} : \tilde{I} \to \mathbb{R}^n$ is another solution for some interval $\tilde{I}$ containing t_0, then $\tilde{\mathbf{u}} = \mathbf{u}$ on the intersection $\tilde{I} \cap I$. Of course, smoothness of F is a much more restrictive assumption than one actually needs; one can get away with locally Lipschitz, but the idea of the theorem is clear.

When one comes to initial value problems for PDE, the theorem we will need is the oldest known such result.

Theorem 5: (Cauchy-Kowalewski) Suppose that $D \subset \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^s\times\mathbb{R}^{ns}$ is an open set and suppose that $F : D \to \mathbb{R}^s$ is *real analytic*. Suppose that $U \subset \mathbb{R}^n$ is an open set and that $\phi : U \to \mathbb{R}^s$ is a *real analytic* function with the property that its '1-graph'

$$\left\{ \left(t_0, \mathbf{x}, \phi(\mathbf{x}), \frac{\partial\phi}{\partial\mathbf{x}}(\mathbf{x})\right) \,\middle|\, \mathbf{x} \in U \right\}$$

lies in D for some t_0. Then there exists a domain $V \subset \mathbb{R}\times\mathbb{R}^n$ for which $\{t_0\}\times U \subset V$ and a *real analytic* function $\mathbf{u} : V \to \mathbb{R}^s$ satisfying

$$\begin{aligned} \frac{\partial\mathbf{u}}{\partial t}(t,\mathbf{x}) &= F\Big(t, \mathbf{x}, \mathbf{u}(t,\mathbf{x}), \frac{\partial\mathbf{u}}{\partial\mathbf{x}}(t,\mathbf{x})\Big), \qquad \text{for } (t,\mathbf{x}) \in V \\ \mathbf{u}(t_0,\mathbf{x}) &= \phi(\mathbf{x}), \qquad \text{for } \mathbf{x} \in U. \end{aligned}$$

Moreover, $\mathbf{u}$ is unique as a *real analytic* solution in the sense that any other such $(\tilde{V}, \tilde{\mathbf{u}})$ with $\tilde{\mathbf{u}}$ real analytic satisfies $\tilde{\mathbf{u}} = \mathbf{u}$ on any component of $\tilde{V} \cap V$ that meets $\{t_0\}\times U$.

This may seem to be a complicated theorem, but it basically says that if the equation and initial data are real analytic and they have domains so that the initial data make sense, then you can find a solution $\mathbf{u}$ by expanding it out in a power series

$$\mathbf{u}(t,\mathbf{x}) = \phi(\mathbf{x}) + \phi_1(\mathbf{x})(t-t_0) + \tfrac{1}{2}\phi_2(\mathbf{x})(t-t_0)^2 + \cdots.$$

The equation will allow you to recursively solve for the sequence of analytic functions ϕ_k and the domains of convergence of the functions F and ϕ give you estimates that allow you to show that the above series converges on some domain V containing $\{t_0\}\times U$. (In fact, proving convergence of the series is the only really subtle point.)

Without the hypothesis of real analyticity, this theorem would not be true. The problem can fail to have a solution or can have more than one solution. There are even examples with F smooth for which there are no solutions to the equation at all, whatever the initial conditions.

In any case, it is traditional to refer to a system of PDE written in the form

$$\frac{\partial\mathbf{u}}{\partial t} = F\big(t, \mathbf{x}, \mathbf{u}, \mathbf{v}, \frac{\partial\mathbf{u}}{\partial\mathbf{x}}, \frac{\partial\mathbf{v}}{\partial\mathbf{x}}\big)$$

as a *system in Cauchy form*, 'underdetermined' if there are 'unconstrained' functions $\mathbf{v}$ present. In this case, we can always reduce to the determined case by simply specifying the functions $\mathbf{v}$ 'arbitrarily' (subject to the condition that the equations still make sense after the specification).

4.2. Equations not in Cauchy form.

Many interesting equations cannot be put in Cauchy form by any choice of coordinates. For example, consider the equation familiar from vector calculus $\operatorname{curl}\mathbf{u} = \mathbf{f}$ where $\mathbf{f}$ is a known vector field in $\mathbb{R}^3$ and $\mathbf{u}$ is an unknown vector field. Certainly, by inspection of the equations

$$\frac{\partial u^2}{\partial x^3} - \frac{\partial u^3}{\partial x^2} = f^1, \quad \frac{\partial u^3}{\partial x^1} - \frac{\partial u^1}{\partial x^3} = f^2, \quad \frac{\partial u^1}{\partial x^2} - \frac{\partial u^2}{\partial x^1} = f^3$$

it is hard to imagine how one might solve for all of the u-partials in some direction. This appears even more doubtful when you realize that there is no hope of uniqueness in this problem: If $\mathbf{u}$ is a solution, then so is $\mathbf{u} + \operatorname{grad} g$ for any function g. Even worse, assuming that $\mathbf{f}$ is real analytic doesn't help either since it is also clear that there can't be any solution at all unless $\operatorname{div}\mathbf{f} = 0$.

Of course, this is a very special equation, and we know how to treat it by ordinary differential equations means (e.g., the proof of Poincaré's Lemma).

Example 4.1: *Self-Dual Equations.* A more interesting problem is to consider the so-called 'self-dual equations' in dimension 4. Remember that there is the Hodge star operator $* : \Omega^p(\mathbb{R}^n) \to \Omega^{n-p}(\mathbb{R}^n)$, which is invariant under rigid motions in $\mathbb{R}^n$ and satisfies $**\alpha = (-1)^{p(n-p)}\,\alpha$. In particular, when $n = 4$ and $p = 2$, the 2-forms can be split into the forms that satisfy $*\alpha = \alpha$, the self-dual 2-forms $\Omega^2_+(\mathbb{R}^4)$, and the forms that satisfy $*\alpha = -\alpha$, the anti-self-dual 2-forms $\Omega^2_-(\mathbb{R}^4)$. For example, every $\phi \in \Omega^2_+(\mathbb{R}^4)$ is of the form

$$\begin{aligned} \phi = u^1\,(dx^2 \wedge dx^3 &+ dx^1 \wedge dx^4) \\ &+ u^2\,(dx^3 \wedge dx^1 + dx^2 \wedge dx^4) + u^3\,(dx^1 \wedge dx^2 + dx^3 \wedge dx^4). \end{aligned}$$

The equation $d\phi = 0$ then represents four equations for the three unknown coefficients u^1, u^2, u^3. Obviously, this overdetermined system cannot be put in Cauchy form. This raises the interesting question: How can one describe the space of local solutions of these equations? Well, let's look at the equations. They can be written in the form

$$\begin{aligned} 0 &= \frac{\partial u^1}{\partial x^1} + \frac{\partial u^2}{\partial x^2} + \frac{\partial u^3}{\partial x^3}\,, \\ \frac{\partial u^1}{\partial x^4} &= \frac{\partial u^2}{\partial x^3} - \frac{\partial u^3}{\partial x^2}\,, \\ \frac{\partial u^2}{\partial x^4} &= \frac{\partial u^3}{\partial x^1} - \frac{\partial u^1}{\partial x^3}\,, \\ \frac{\partial u^3}{\partial x^4} &= \frac{\partial u^1}{\partial x^2} - \frac{\partial u^2}{\partial x^1}\,. \end{aligned}$$

Setting aside the first one, the remaining equations are certainly in Cauchy form and we could solve them (at least near $x^4 = 0$) for any real analytic initial conditions

$$u^i(x^1, x^2, x^3, 0) = f^i(x^1, x^2, x^3), \qquad \text{for } i = 1, 2, 3.$$

Unfortunately, there's no reason to believe that the resulting functions will satisfy the first equation. Indeed, unless the functions f^i satisfy

$$0 = \frac{\partial f^1}{\partial x^1} + \frac{\partial f^2}{\partial x^2} + \frac{\partial f^3}{\partial x^3},$$

the resulting u^i *can't* satisfy the first equation.

However, suppose that we choose the f^i on $\mathbb{R}^3$ to satisfy the above equation on $\mathbb{R}^3$ (and to be real analytic, of course). Then do we have a hope that the resulting u^i will satisfy the remaining equation? In fact, we do, for they will *always* satisfy it! Here is how you can see this: Define the 'error' to be

$$E = \frac{\partial u^1}{\partial x^1} + \frac{\partial u^2}{\partial x^2} + \frac{\partial u^3}{\partial x^3}.$$

By the choice of f, we know that $E(x^1, x^2, x^3, 0) = 0$. Moreover, by the above equations and commuting partials, we have

$$\begin{aligned}
\frac{\partial E}{\partial x^4} &= \frac{\partial}{\partial x^1}\left(\frac{\partial u^1}{\partial x^4}\right) + \frac{\partial}{\partial x^2}\left(\frac{\partial u^2}{\partial x^4}\right) + \frac{\partial}{\partial x^3}\left(\frac{\partial u^3}{\partial x^4}\right) \\
&= \frac{\partial}{\partial x^1}\left(\frac{\partial u^2}{\partial x^3} - \frac{\partial u^3}{\partial x^2}\right) + \frac{\partial}{\partial x^2}\left(\frac{\partial u^3}{\partial x^1} - \frac{\partial u^1}{\partial x^3}\right) + \frac{\partial}{\partial x^3}\left(\frac{\partial u^1}{\partial x^2} - \frac{\partial u^2}{\partial x^1}\right) \\
&= 0.
\end{aligned}$$

Of course, this implies that $E(x^1, x^2, x^3, x^4) = 0$, which is what we wanted to be true.

Thus, the solutions to the full system are found by choosing initial conditions f^i to satisfy the single equation on $\mathbb{R}^3$

$$0 = \frac{\partial f^1}{\partial x^1} + \frac{\partial f^2}{\partial x^2} + \frac{\partial f^3}{\partial x^3}.$$

Of course, this can be regarded as an equation in Cauchy form, now underdetermined, by writing it in the form

$$\frac{\partial f^3}{\partial x^3} = -\frac{\partial f^1}{\partial x^1} - \frac{\partial f^2}{\partial x^2}.$$

By Cauchy-Kowalewski, we can solve this equation uniquely by choosing f^1 and f^2 as arbitrary real analytic functions and then choosing the initial value $f^3(x^1, x^2, 0)$ as a real analytic function on $\mathbb{R}^2$.

Exercise 4.1: Show that you don't need to invoke the Cauchy-Kowalewski Theorem for this problem on $\mathbb{R}^3$ and you also don't need real analyticity to solve the initial value problem. However, show that any solutions u^i on $\mathbb{R}^4$ to the self-dual equations are harmonic and so must be real analytic. What does this tell you about the need for Cauchy-Kowalewski in the system for the u^i?

The upshot of all this discussion is that, although the system can't be put in Cauchy form, it can be regarded as a sequence of Cauchy problems. Moreover, this sequence has the unexpectedly nice property that, when you solve one of the Cauchy problems then use the solution as initial data for the next Cauchy problem, the satisfaction of the first set of equations is 'propagated' by the equations at the next level.

Exercise 4.2: Consider the overdetermined system

$$\begin{aligned}
z_x &= F(x, y, z) \\
z_y &= G(x, y, z)
\end{aligned} \qquad\qquad z(0,0) = z_0$$

for z as a function of x and y. Show that if you set it up as a sequence of Cauchy problems, first

$$w_x(x) = F\big(x, 0, w(x)\big), \qquad w(0) = z_0$$

and then use the resulting function w to consider the equation

$$z_y(x, y) = G\big(x, y, z(x, y)\big), \qquad z(x, 0) = w(x),$$

then the resulting solutions will satisfy the equation $z_x = F(x, y, z)$ for all choices of z_0 only if F and G satisfy the condition needed for the system $\langle dz - F(x, y, z)\, dx - G(x, y, z)\, dy\rangle$ to be Frobenius.

Exercise 4.3: Go back to the equation $\operatorname{curl}\mathbf{u} = \mathbf{f}$ and show that you can write that as a sequence of Cauchy problems. Show also that they won't have this 'propagation' property unless $\operatorname{div}\mathbf{f} = 0$.

Exercise 4.4: Now consider the equation $\operatorname{curl}\mathbf{u} = \mathbf{u}+\mathbf{f}$. Of course, this equation can't be put in Cauchy form either, since it differs from the previous one only by terms that don't involve any derivatives. However, show now that when you apply the divergence operator to both sides, you get, not a condition on $\mathbf{f}$, but *another* first order equation on $\mathbf{u}$. Show that you can write this system of four equations for the three unknowns as a sequence of Cauchy problems and that this system *does* have the good 'propagation' property. How much freedom do you get in specifying the initial data to determine a solution?

Exercise 4.5: Back to the self-dual equations: Now consider the u^i as free coordinates and set $M = \mathbb{R}^4 \times \mathbb{R}^3$ with coordinates $x^1, x^2, x^3, x^4, u^1, u^2, u^3$. Define the 3-form

$$\Phi = du^1 \wedge (dx^2 \wedge dx^3 + dx^1 \wedge dx^4) \\ + du^2 \wedge (dx^3 \wedge dx^1 + dx^2 \wedge dx^4) + du^3 \wedge (dx^1 \wedge dx^2 + dx^3 \wedge dx^4).$$

Explain why the 4-dimensional integral manifolds in M of $\mathcal{I} = \langle \Phi \rangle$ on which $dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \neq 0$ can be thought of locally as representing closed self-dual 2-forms. Describe $V_4(\mathcal{I}) \cap G_4(TM, \mathbf{x})$. Are these ordinary or regular integral elements? What about $V_3(\mathcal{I}) \cap G_3\bigl(TM, (x^1, x^2, x^3)\bigr)$?

With all these examples in mind, I can now describe how the proof of the Cartan-Kähler Theorem goes: Remember that we start with a real analytic EDS $(M, \mathcal{I})$ and $P \subset M$ a connected, k-dimensional, real analytic, regular integral manifold of $\mathcal{I}$ with $r = r(P) \geq 0$. For each $p \in P$, the dimension of $H(T_pP) \subset T_pM$ is $r{+}k{+}1$ and the generic subspace $S \subset T_pM$ of codimension r will intersect $H(T_pP)$ is a subspace $S \cap H(T_pP)$ of dimension $k + 1$. Thus, choosing the 'generic' codimension r submanifold $R \subset M$ that contains P will have the property that $T_pR \cap H(T_pP)$ has dimension $k{+}1$ and so will be an integral element. So now suppose that we have a real analytic R containing P and satisfying this genericity condition. We now want to find a $(k{+}1)$-dimensional integral manifold X satisfying $P \subset X \subset R$.

Because of the real analyticity assumption, it's enough to prove the existence and uniqueness of X in a neighborhood of any point $p \in P$, so fix such a p and let $e_1, \ldots, e_k$ be a basis of T_pP. Choose $\kappa^1, \ldots, \kappa^m \in \mathcal{I}^{k+1}$ so that

$$H(T_pP) = \{\ v \in T_pM \mid \kappa^\mu(v, e_1, \ldots, e_k) = 0,\ 1 \leq \mu \leq m\ \}$$

where $m = \dim T_pM - (r{+}k{+}1)$. Because of the regularity assumption, the forms $\kappa^1, \ldots, \kappa^m$ can be used to compute the polar space of any integral element $E \in V_k(\mathcal{I})$ that is sufficiently near T_pP.

Now R has dimension $m{+}k{+}1$ and, when you pull back the forms κ^μ to R, they are 'independent' near p because we assumed $T_pR \cap H(T_pP)$ to have dimension $k{+}1$. When you write them out in local coordinates, they become a system of m PDE in Cauchy form for extending P to a $(k{+}1)$-dimensional integral manifold of the system $\mathcal{J} = \langle \kappa^1, \ldots, \kappa^m \rangle$, and P itself provides the initial condition. Thus, the Cauchy-Kowalewski Theorem applies: there is a unique, connected, real analytic X of dimension $k{+}1$ satisfying $P \subset X \subset R$ that is an integral manifold of $\mathcal{J}$.

Now, all of the k-forms in $\mathcal{I}$ vanish when pulled back to P, but we need them to vanish when pulled back to X. Here, finally, is where the assumption that $\mathcal{I}$ be differentially closed comes in, as well as the need for the integral elements to be ordinary in the first place. What we do is show that the differential closure condition plus the ordinary assumption allows us to write down a system in Cauchy form for the coefficients of the k-forms in $\mathcal{I}$ pulled back to X. This system has 'zero' initial conditions since P is an integral manifold of $\mathcal{I}$ and to have all of the coefficients be zero is a solution of the system. By the uniqueness part of the Cauchy-Kowalewski Theorem, it follows that 'zero' is the only solution, i.e., that all of the k-forms of $\mathcal{I}$ must vanish on X. However, this, coupled with the vanishing of the κ^μ and the fact that they determine the integral extensions (at least near p) forces all of the tangent spaces to X to be integral elements of $\mathcal{I}$, i.e., forces X to be an integral manifold of the whole ideal $\mathcal{I}$.

Well, that, in outline, is the proof of the Cartan-Kähler Theorem. The full details are in Chapter III of the EDS book. I encourage you to look at them at some point, probably after you have been convinced, by seeing its applications, that the Cartan-Kähler Theorem is worth knowing.

Exercise 4.6: How would you describe the 2-forms on $\mathbb{R}^5$ that are both closed and coclosed? What I'm asking for is an analysis of the local solutions to the equations $d\alpha = d(*\alpha) = 0$ for $\alpha \in \Omega^2(\mathbb{R}^5)$. If you think you have a handle on this, you might want to go ahead and try the general case: $d\alpha = d(*\alpha) = 0$ for $\alpha \in \Omega^p(\mathbb{R}^n)$.

4.3. Integral Flags and Cartan's Test

In light of the Cartan-Kähler Theorem, there is a simple sufficient condition for the existence of an integral manifold tangent to $E \in V_n(\mathcal{I})$.

Theorem 6: Let $(M, \mathcal{I})$ be a real analytic EDS. If $E \in V_n(\mathcal{I})$ contains a flag of subspaces

$$(0) = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E \subset T_pM$$

where $E_i \in V_i^r(\mathcal{I})$ for $0 \leq i < n$, then there is a real analytic n-dimensional integral manifold $P \subset M$ passing through p and satisfying $T_pP = E$.

The proof is the obvious one: Just apply the Cartan-Kähler Theorem one step at a time, noting that, because $V_k^r(\mathcal{I})$ is an open subset of $V_k(\mathcal{I})$, any k-dimensional integral manifold of $\mathcal{I}$ that is tangent to $E_k \in V_k^r(\mathcal{I})$ will perforce be a regular integral manifold in some neighborhood of p.

Now this is a nice result but it leaves a few things to be desired. First of all, this sufficient condition is not necessary. As we will see, there are quite a few cases in which the integral manifolds we are interested in cannot be constructed by the above process, simply because the integral elements to which they would be tangent are not the terminus of a flag of regular integral elements. Second, as things stand, it is a lot of work to check whether or not a given integral element *is* the terminus of a flag of regular integral elements.

Exercise 4.7: Look back at the two ideals of Exercise 3.1. Show that in neither case does any $E \in V_2(\mathcal{I}_i)$ contain a $E_1 \in V_1^r(\mathcal{I}_i)$. Now, $\mathcal{I}_1$ has no 2-dimensional integral manifolds anyway. For $\mathcal{I}_2$, however, ...

Exercise 4.8: For Exercise 4.5, determine which integral elements of $\mathcal{I}$ are the terminus of a flag of regular integral elements.

As you can see, computing with flags of subspaces can be a bit of work. I am now going to describe a simplification of this process that will make these computations almost routine. First, though, some simplifications and terminology.

A flag of integral elements

$$(0) = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E \subset T_pM$$

where $E_i \in V_i^r(\mathcal{I})$ for $0 \leq i < n$ and $E_n \in V_n(\mathcal{I})$ will be known as a *regular flag* for short. (Note that the terminus E_n of a regular flag is not required to be regular and, in fact, it can fail to be. However, it does turn out that E_n is ordinary.)

Note that the assumption that $E_0 = 0_p \subset T_pM$ be regular implies, in particular, that is it ordinary, i.e., E_0 is an ordinary zero of the set of functions $\mathcal{I}^0 \subset \Omega^0(M)$. Now, the set $V_0^o(\mathcal{I})$ is a smooth submanifold of $G_0(TM) = M$.

Exercise 4.9: Explain why any n-dimensional integral element $E \subset T_pM$ with $p \in V_0^o(\mathcal{I})$ must be tangent to $V_0^o(\mathcal{I})$. Is this necessarily true if p does not lie in $V_0^o(\mathcal{I})$?

Obviously, every integral manifold of $\mathcal{I}$ that is constructed by the 'regular flag' approach will lie in $V_0^o(\mathcal{I})$ anyway. Thus, at least on theoretical grounds, nothing will be lost if we simply replace M by $V_0^o(\mathcal{I})$, i.e., restrict to the ordinary part of the zero locus of the functions in $\mathcal{I}$. I am going to do this for the rest of this section. This amounts to the blanket assumption that $\mathcal{I}^0 = (0)$, i.e., that $\mathcal{I}$ is generated in positive degree.

Now, corresponding to any integral flag

$$(0) = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E \subset T_pM$$

(regular or not), there is the *descending* flag of corresponding polar spaces

$$T_pM \supseteq H(E_0) \supseteq H(E_1) \supseteq \cdots \supseteq H(E_{n-1}) \supseteq H(E_n) \supseteq E_n \,.$$

It will be convenient to keep track track of the dimensions of these spaces in terms of their codimension in T_pM. For $k < n$, set

$$c(E_k) = \dim(T_pM) - \dim H(E_k) = n + s - k - 1 - r(E_k)$$

where $\dim M = n{+}s$. It works out best to make the special convention that $c(E_n) = s$. (In practice, it is usually the case that $H(E_n) = E_n$, in which case, the above formula for $c(E_k)$ works even when you set $k = n$.) Since $\dim H(E_k) \geq \dim E_n = n$, we have $c(E_k) \leq s$. Because of the nesting of these spaces, we have

$$0 \leq c(E_0) \leq c(E_1) \leq \cdots \leq c(E_n) \leq s.$$

For notational convenience, set $c(E_{-1}) = 0$. The *Cartan characters* of the flag $F = (E_0, E_1, \ldots, E_n)$ are the numbers

$$s_k(F) = c(E_k) - c(E_{k-1}) \geq 0.$$

They will play an important role in what follows.

I'm now ready to describe *Cartan's Test*, a necessary and sufficient condition for a given flag to be regular. First, let me introduce some terminology: A subset $X \subset M$ will be said to have *codimension at least q at $x \in X$* if there is an open x-neighborhood $U \subset M$ and a codimension q submanifold $Q \subset U$ so that $X \cap U$ is a subset of Q. In the other direction, X will be said to have *codimension at most q at $x \in X$* if there is an open x-neighborhood $U \subset M$ and a codimension q submanifold $Q \subset U$ containing x so that $Q \subset X \cap U$.

Theorem 7: (Cartan's Test) Let $(M, \mathcal{I})$ be an EDS and let $F = (E_0, E_1, \ldots, E_n)$ be an integral flag of $\mathcal{I}$. Then $V_n(\mathcal{I})$ has codmension at least

$$c(F) = c(E_0) + c(E_1) + \cdots + c(E_{n-1})$$

in $G_n(TM)$ at E_n. Moreover, $V_n(\mathcal{I})$ is a smooth submanifold of $G_n(TM)$ of codimension $c(F)$ in a neighborhood of E_n if and only if the flag F is regular.

This is a very powerful result, because it allows one to test for regularity of a flag by simple linear algebra, computing the polar spaces $H(E_k)$ and then checking that $V_n(\mathcal{I})$ is smooth near E_n and of the smallest possible codimension, $c(F)$. In many cases, these two things can done by inspection.

Example 4.2: *Self-Dual 2-Forms.* Look back at Exercise 4.5. Any integral element $E \in V_4(\mathcal{I}) \cap G_4(T\mathbb{R}^7, d\mathbf{x})$ is defined by linear equations of the form

$$\pi^a = du^a - p^a_i(E)\, dx^i = 0.$$

In order that Φ vanish on such a 4-plane, it suffices that the $p^a_i(E)$ satisfy four equations:

$$p^1_1{+}p^2_2{+}p^3_3 = p^1_4{-}p^2_3{+}p^3_2 = p^2_4{-}p^3_1{+}p^1_3 = p^3_4{-}p^1_2{+}p^2_1 = 0$$

It's clear from this that $V_4(\mathcal{I}) \cap G_4(T\mathbb{R}^7, d\mathbf{x})$ is a smooth manifold of codimension 4 in $G_4(T\mathbb{R}^7)$. On the other hand, if we let $E_k \subset E$ be defined by the equation $dx^{k+1} = dx^{k+2} = \cdots = dx^4 = 0$ for $0 \leq k < 4$, then it is easy to see that

$$\begin{aligned} H(E_0) = H(E_1) &= T_p(M) \\ H(E_2) &= \{v \in T_p(M) \,|\, \pi_3(v) = 0\,\} \\ H(E_3) &= \{v \in T_p(M) \,|\, \pi_1(v) = \pi_2(v) = \pi_3(v) = 0\,\} \\ H(E_4) &= \{v \in T_p(M) \,|\, \pi_1(v) = \pi_2(v) = \pi_3(v) = 0\,\} \end{aligned}$$

so $c(E_0) = c(E_1) = 0$, $c(E_2) = 1$, $c(E_3) = 3$, and $c(E_4) = 3$. Since $c(F) = 0{+}0{+}1{+}3 = 4$, which is the codimension of $V_4(\mathcal{I})$ in $G_4(T\mathbb{R}^7)$, Cartan's Test is verified and the flag is regular.

Exercise 4.10: Show that, for $E \in V_4(\mathcal{I}) \cap G_4(T\mathbb{R}^7, d\mathbf{x})$, every flag is regular. (Hint: Rotations in $\mathbb{R}^4$ preserve the self-dual equations.)

Exercise 4.11: Write the exterior derivative $d : \Omega^1(\mathbb{R}^4) \to \Omega^2(\mathbb{R}^4)$ as a sum $d_+ + d_-$ where $d_\pm : \Omega^1(\mathbb{R}^4) \to \Omega^2_\pm(\mathbb{R}^4)$. Show that if a 1-form λ satisfies $d_+\lambda = 0$, then locally it can be written in the form $\lambda = df + \psi$, where ψ is real analytic. Use this result to show that if $d_+\lambda = 0$, then there exist non-vanishing self-dual 2-forms Υ satisfying $d\Upsilon = \lambda{\wedge}\Upsilon$. (Hint: You will want to recall that any closed self-dual or anti-self-dual 2-form is real analytic and also that if Υ is self-dual while Λ is anti-self dual, then $\Upsilon{\wedge}\Lambda$ vanishes identically. What can you say about the local solvability of the equation $d\Upsilon = \lambda{\wedge}\Upsilon$ for $\Upsilon \in \Omega^2_+(\mathbb{R}^4)$ if you don't have $d_+\lambda = 0$? (I don't expect a complete answer to this yet. I just want you to think about the issue. We'll come back to this later.

Example 4.3: *Special Lagrangian Manifolds in* $\mathbb{C}^n$. Let $M = \mathbb{C}^n$ with standard complex coordinates $z^1, \ldots, z^n$. Write $z^k = x^k + i\,y^k$, as usual. Let $\mathcal{I}$ be the ideal generated by the Kähler 2-form

$$\omega = dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n$$

and the n-form

$$\begin{aligned}\Phi &= \mathrm{Im}(dz^1 \wedge \cdots \wedge dz^n)\\ &= dy^1 \wedge dx^2 \wedge \cdots \wedge dx^n + dx^1 \wedge dy^2 \wedge \cdots \wedge dx^n + \cdots\\ &\quad + dx^1 \wedge dx^2 \wedge \cdots \wedge dy^n + (\text{higher order terms in } \{dy^k\})\end{aligned}$$

The n-dimensional integral manifolds of $\mathcal{I}$ are known as *special Lagrangian*. They and their generalizations to the special Lagrangian submanifolds of Kähler-Einstein manifolds are the subject of much interest now in mathematical physics.

Consider the integral element $E \in V_n(\mathcal{I})$ based at $0 \in \mathbb{C}^n$ defined by the relations

$$dy^1 = dy^2 = \cdots = dy^n = 0.$$

Let $E_k \subset E$ be defined by the additional relations $dx^j = 0$ for $j > k$. Then, for $k < n{-}1$, the polar space for E_k is easily seen to be defined by the relations $dy^j = 0$ for $j \le k$. In particular, $c(E_k) = k$ for $k < n{-}1$. However, for $k = n{-}1$, the form Φ enters into the computation of the polar equations, showing that $H(E_{n-1}) = E_n$. Consequently, $c(E_{n-1}) = n$. It follows that $V_n(\mathcal{I})$ must have codimension at least

$$0 + 1 + \cdots + (n{-}2) + n = \tfrac{1}{2}(n^2{-}n{+}2).$$

On the other hand, on any nearby integral element E^*, the 1-forms dx^i are linearly independent, so it can be described by relations of the form

$$dy^a - p^a_i\, dx^i = 0.$$

The condition that ω vanish on E^* is just that $p^a_i = p^i_a$, while the condition that Φ vanish on E^* is a polynomial equation in the p^a_i of the form

$$0 = p^1_1{+}p^2_2{+}\cdots{+}p^n_n + (\text{higher order terms in } \{p^a_i\}).$$

This equation has independent differential from the equations $p^a_i = p^i_a$ at the integral element E (defined by $p^a_i = 0$). Consequently, $V_n(\mathcal{I})$ is smooth near E and of codimension $\tfrac{1}{2}(n^2{-}n{+}2)$ in $G_n(T\mathbb{C}^n)$. Thus, by Cartan's Test, the flag is regular.

4.4. The notion of generality of integral manifolds

It is very useful to know not only that integral manifolds exist, but 'how many' integral manifolds exist. I now want to make this into a precise notion and give the answer.

Suppose that $F = (E_0, E_1, \ldots, E_n)$ is a regular flag of a real analytic EDS $(M, \mathcal{I})$. By the Cartan-Kähler Theorem, there exists at least one real analytic integral manifold $N^n \subset M$ containing the basepoint p of E_n and satisfying $T_pN = E_n$. Set

$$c_k = \begin{cases} 0 & \text{for } k = -1;\\ c(E_k) & \text{for } 0 \le k < n; \text{ and}\\ s & \text{for } k = n.\end{cases}$$

and define $s_k = c_k - c_{k-1}$ for $0 \le k \le n$.

Choose a real analytic coordinate system

$$(\mathbf{x}, \mathbf{u}) = (x^1, \ldots, x^n, u^1, \ldots u^s) : U \to \mathbb{R}^{n+s}$$

centered on $p \in U$ with the following properties:

(i) $N \cap U \subset U$ is defined by $u^a = 0$.

(ii) $E_k \subset E_n$ is defined by $dx^j = 0$ for $j > k$.

(iii) $H(E_k)$ is defined by $du^a = 0$ for $a \le c_k$ when $0 \le k < n$.

Exercise 4.12: Explain why such a coordinate system must exist.

Define the *level* $\lambda(a)$ of an integer a between 1 and s to be the smallest integer $k \ge 0$ for which $a \le c_k$. Note that $0 \le \lambda(a) \le n$. Note that there are exactly s_k indices of level k.

Now, let $\mathcal{C}$ denote the collection of real analytic integral manifolds of $(U, \mathcal{I})$ that are 'near' N in the following sense: An integral manifold N^* belongs to $\mathcal{C}$ if it can be represented by equations of the form

$$u^a = F^a(x^1, \ldots, x^n)$$

where the F^a are real analytic functions defined on a neighborhood of $\mathbf{x} = 0$ and, moreover, these functions and their first partial derivatives are 'sufficiently small' near $\mathbf{x} = 0$. ('Sufficiently small' can be made precise in terms of a connected neighborhood of the flag $F = (E_0, \ldots, E_n)$ in the space of regular flags.)

If the index a has level k, define the function f^a on a neighborhood of 0 in $\mathbb{R}^k$ by

$$f^a(x^1, \ldots, x^k) = F^a(x^1, \ldots, x^k, 0, \ldots, 0).$$

Then f^a is a function of k variables. (By convention, we will sometimes refer to a constant as a function of 0 variables.) We then have a mapping

$$N^* \longmapsto \{f^a\}_{1 \le a \le s}.$$

A close analysis of the proof of the Cartan-Kähler Theorem then shows that this correspondance between the elements of $\mathcal{C}$ and collections of 'small' functions $\{f^a\}_{1 \le a \le s}$ consisting of

$$\begin{array}{ll} s_0 & \text{constants,} \\ s_1 & \text{functions of one variable,} \\ s_2 & \text{functions of two variables,} \\ \vdots & \\ s_n & \text{functions of } n \text{ variables.} \end{array}$$

is one-to-one and onto.

Example 4.4: *Self-Dual* 2*-Forms again.* Looking at the self-dual 2-forms example, one sees the real analytic functions

$$\begin{aligned} f^1(x^1, x^2) &= u^3(x^1, x^2, 0, 0) \\ f^2(x^1, x^2, x^3) &= u^1(x^1, x^2, x^3, 0) \\ f^3(x^1, x^2, x^3) &= u^2(x^1, x^2, x^3, 0) \end{aligned}$$

can be specified arbitrarily and that there is only one solution with any such triple of functions f^i as it's 'initial data'.

Exercise 4.13: Consider the EDS $(\mathbb{R}^{2n}, \langle dx^1 \wedge dy_1 + \cdots + dx^n \wedge dy_n \rangle)$ whose n-dimensional integral manifolds are the Lagrangian submanifolds of $\mathbb{R}^{2n}$. Compare and contrast the Cartan-Kähler description of these integral manifolds near the n-plane N defined by $y_1 = y_2 = \cdots = y_n = 0$ with the more common description as the solutions of the equations

$$y_i = \frac{\partial f}{\partial x^i}$$

where f is an arbitrary differentiable function of n variables. Is there a contradiction here?

Lecture 6. Applications 2: Weingarten Surfaces, etc.

This lecture will consist entirely of examples drawn from geometry, so that you can get some feel for the variety of applications of the Cartan-Kähler Theorem.

6.1. Weingarten surfaces

Let $x : N \to \mathbb{R}^3$ be an immersion of an oriented surface and let $u : N \to S^2$ be the associated oriented normal, sometimes known as the Gauss map. Recall that we have the two fundamental forms

$$\mathrm{I} = dx \cdot dx, \qquad \mathrm{I\!I} = -du \cdot dx.$$

The eigenvalues of $\mathrm{I\!I}$ with respect to I are known as the *principal curvatures* of the immersion. On the open set $N^* \subset N$ where the two eigenvalues are distinct, they are smooth functions on N. The complement $N \setminus N^*$ is known as the *umbilic locus*. For simplicity, I am going to suppose that $N^* = N$, though many of the constructions that I will do can, with some work, be made to go through even in the presence of umbilics.

Possibly after passing to a double cover, we can define vector-valued functions $e_1, e_2 : N \to \mathbb{S}^2$ so that $e_1 \times e_2 = u$ and so that, setting $\eta^i = e_i \cdot dx$, we can write

$$\begin{aligned} dx &= e_1 \quad\ \ \eta_1 + e_2 \quad\ \ \eta_2\,, \\ -du &= e_1\ \kappa_1\,\eta_1 + e_2\ \kappa_2\,\eta_2\,, \end{aligned}$$

where $\kappa_1 > \kappa_2$ are the principal curvatures. The immersion x defines a *Weingarten surface* if the principal curvatures satisfy a (non-trivial) relation of the form $F(\kappa_1, \kappa_2) = 0$. (For a generic immersion, the functions κ_i satisfy $d\kappa_1 \wedge d\kappa_2 \neq 0$, at least on a dense open set.) For example, the equations $\kappa_1 + \kappa_2 = 0$ and $\kappa_1\kappa_2 = 1$ define Weingarten relations, perhaps better known as the relations $H = 0$ (minimal surfaces) and $K = 1$, respectively.

I want to describe a differential system whose integral surfaces are the Weingarten surfaces. For underlying manifold M, I will take $G \times \mathbb{R}^2$ where G is the group of rigid motions of 3-space as described in Example 3.7 (I will mantain the notation established there) and the coordinates on the $\mathbb{R}^2$ factor will be κ_1 and κ_2. Consider the ideal $\mathcal{I} = \langle\, \theta_0, \theta_1, \theta_2, \Upsilon\,\rangle$, where

$$\theta_0 = \omega_3\,, \quad \theta_1 = \omega_{31} - \kappa_1\,\omega_1\,, \quad \theta_2 = \omega_{32} - \kappa_2\,\omega_2\,, \quad \Upsilon = d\kappa_1 \wedge d\kappa_2\,.$$

Exercise 6.1: Explain how every Weingarten surface without umbilic points gives rise to an integral 2-manifold of $(M, \mathcal{I})$ and, conversely why every integral 2-manifold of $(M, \mathcal{I})$ on which $\omega_1 \wedge \omega_2$ is nonvanishing comes from a Weingarten surface in $\mathbb{R}^3$ by the process you have described.

Now let's look a little closer at the algebraic structure of $\mathcal{I}$. First of all, by the structure equations

$$\begin{aligned} d\theta_0 = d\omega_3 &= -\omega_{31} \wedge \omega_1 - \omega_{32} \wedge \omega_2 \\ &= -(\theta_1 + \kappa_1\,\omega_1) \wedge \omega_1 - (\theta_2 + \kappa_2\,\omega_2) \wedge \omega_2 \\ &= -\theta_1 \wedge \omega_1 - \theta_2 \wedge \omega_2\,. \end{aligned}$$

Then, again, by the structure equations

$$\begin{aligned} d\theta_1 &= d\omega_{31} - d\kappa_1 \wedge \omega_1 - \kappa_1\, d\omega_1 \\ &= -\omega_{32} \wedge \omega_{21} - d\kappa_1 \wedge \omega_1 + \kappa_1\,(\omega_{12} \wedge \omega_2 + \omega_{13} \wedge \omega_3) \\ &= -(\theta_2 + \kappa_2\,\omega_2) \wedge \omega_{21} - d\kappa_1 \wedge \omega_1 + \kappa_1\,(-\omega_{21} \wedge \omega_2 + \omega_{13} \wedge \theta_0) \\ &\equiv -d\kappa_1 \wedge \omega_1 - (\kappa_1 - \kappa_2)\omega_{12} \wedge \omega_2 \ \mathrm{mod}\ \{\theta_0, \theta_1, \theta_2\}. \end{aligned}$$

A similar computation gives

$$d\theta_2 \equiv -(\kappa_1 - \kappa_2)\omega_{21} \wedge \omega_1 - d\kappa_2 \wedge \omega_2 \ \mathrm{mod}\ \{\theta_0, \theta_1, \theta_2\}.$$

Thus, setting $\pi_1 = d\kappa_1$, $\pi_2 = (\kappa_1{-}\kappa_2)\omega_{21}$, and $\pi_3 = d\kappa_2$, we have

$$\mathcal{I} = \langle\, \theta_0, \theta_1, \theta_2,\ \pi_1\wedge\omega_1{+}\pi_2\wedge\omega_2,\ \pi_2\wedge\omega_1{+}\pi_3\wedge\omega_2,\ \pi_1\wedge\pi_3 \,\rangle_{\mathrm{alg}}$$

Now, on the open set $M^+ \subset M$ where $\kappa_1 > \kappa_2$, the 1-forms

$$\omega_1,\ \omega_2,\ \theta_0,\ \theta_1,\ \theta_2,\ \pi_1,\ \pi_2,\ \pi_3$$

are linearly independent and are a basis for the 1-forms. For any $e \in TM^+$ we can write its components in this basis as

$$\begin{aligned} \omega_i(e) &= a_i\,,\ \ (i = 1,2),\\ \theta_j(e) &= t_j\,,\ \ (j = 0,1,2),\\ \pi_k(e) &= p_k\,,\ \ (k = 1,2,3). \end{aligned}$$

The vector e spans a 1-dimensional integral element E if and only if it is nonzero and satisfies $t_0 = t_1 = t_2 = 0$.

Exercise 6.2: Explain why this shows that all of the elements in $V_1(\mathcal{I})$ are ordinary.

Now, assuming e spans $E \in V_1(\mathcal{I})$, the polar space $H(E)$ is then defined as the set of vectors v that annihilate the 1-forms θ_i and the three 1-forms

$$\begin{aligned} e \lrcorner (\pi_1\wedge\omega_1{+}\pi_2\wedge\omega_2) &= p_1\,\omega_1 + p_2\,\omega_2 - a_1\,\pi_1 - a_2\,\pi_2\\ e \lrcorner (\pi_2\wedge\omega_1{+}\pi_3\wedge\omega_2) &= p_2\,\omega_1 + p_3\,\omega_2 \qquad\quad - a_1\,\pi_2 - a_2\,\pi_3\\ e \lrcorner (\pi_1\wedge\pi_3) &= \qquad\qquad\quad - p_3\,\pi_1 \qquad\quad + p_1\,\pi_3\,. \end{aligned}$$

Clearly, for any 'generic' choice of the quantities $(a_1, a_2, p_1, p_2, p_3)$, these three 1-forms will be linearly independent, so that $H(E)$ will have dimension 2. (Remember that M^+ has dimension 8.) In this case, the flag $\bigl(0, E, H(E)\bigr)$ will be regular with characters $(s_0, s_1, s_2) = (3, 3, 0)$. From the description of the generality of solutions given in the last Lecture, it follows that the 'general' Weingarten surface depends on 3 constants and 3 functions of one variable.

Exercise 6.3: Describe the set of $E_2 \subset V_2(\mathcal{I})$ on which $\omega_1{\wedge}\omega_2$ is nonzero. Show that this is not a smooth submanifold of $G_2(TM)$ and describe the singular locus. Show, however, that every $E_2 \in V_2^r(\mathcal{I})$ on which $\omega_1{\wedge}\omega_2$ is nonzero does contain a regular flag.

Exercise 6.4: Describe which curves $\bigl(x(t), e_1(t), e_2(t), e_3(t), \kappa_1(t), \kappa_2(t)\bigr)$ in M^+ represent regular 1-dimensional integral manifolds of $\mathcal{I}$.

Exercise 6.5: Suppose that you want to prescribe the relation $F(\kappa_1, \kappa_2) = 0$ beforehand and then describe all of the (umbilic-free) surfaces in $\mathbb{R}^3$ that satisfy $F(\kappa_1, \kappa_2) = 0$. How would you set this up as an exterior differential system? What are its characters?

6.2. Orthogonal Coordinates on 3-manifolds

Suppose now that N^3 is a 3-manifold and that $g : TN \to \mathbb{R}$ is a Riemmanian metric, i.e., a smooth function on TN with the property that, on each T_xN, g is a positive definite quadratic form. A coordinate chart $(x^1, x^2, x^3) : U \to \mathbb{R}^3$ is said to be *g-orthogonal* if, on U,

$$g = g_{11}\,(dx^1)^2 + g_{22}\,(dx^2)^2 + g_{33}\,(dx^3)^2,$$

i.e., if the coordinate expression $g = g_{ij}\,dx^i\,dx^j$ satisfies $g_{ij} = 0$ for i different from j. This is three equations for the three coordinate functions x^i. I now want to describe an exterior differential system whose 3-dimensional integral manifolds describe the solutions to this problem.

First, note that, if you have a solution, then the 1-forms $\eta_i = \sqrt{g_{ii}}\,dx^i$ form a g-orthonormal coframing, i.e.,

$$g = {\eta_1}^2 + {\eta_2}^2 + {\eta_3}^2.$$

This coframing is not the most general coframing, though, because it satisfies

$$\eta_1 \wedge d\eta_1 = \eta_2 \wedge d\eta_2 = \eta_3 \wedge d\eta_3 = 0,$$

since each η_i is a multiple of an exact 1-form. Conversely, any g-orthonormal coframing (η_1, η_2, η_3) that satisfies $\eta_i{\wedge}d\eta_i = 0$ for $i = 1, 2, 3$ is locally of the form $\eta_i = A_i\, dx_i$ for some functions $A_i > 0$ and x^i, by the Frobenius Theorem. (Why?)

Thus, up to an application of the Frobenius Theorem, the problem of finding g-orthogonal coordinates is equivalent to finding g-orthonormal coframings (η_1, η_2, η_3) satisfying $\eta_i{\wedge}d\eta_i = 0$. I now want to set up an exterior differential system whose integral manifolds reprsent these coframings.

To do this, let $\pi : F \to N$ be the g-orthonormal coframe bundle of N, i.e, a point of F is a quadruple $f = (x, u_1, u_2, u_3)$ where $x = \pi(f)$ belongs to N and $u_i \in T_xN$ are g-orthonormal. This is an O(3)-bundle over N and hence is a manifold of dimension 6. There are canonical 1-forms $\omega_1, \omega_2, \omega_3$ on F that satisfy

$$\omega_i(v) = u_i\bigl(\pi'(v)\bigr), \qquad \text{for all } v \in T_fM \text{ with } f = (x, u_1, u_2, u_3).$$

These 1-forms have the 'reproducing property' that, if $\eta = (\eta_1, \eta_2, \eta_3)$ is a g-orthonormal coframing on $U \subset M$, then regarding η as a section of F over U via the map

$$\sigma_\eta(x) = \bigl(\, x,\, (\eta_1)_x,\, (\eta_2)_x,\, (\eta_3)_x\, \bigr),$$

we have $\sigma_\eta^*(\omega_1, \omega_2, \omega_3) = (\eta_1, \eta_2, \eta_3)$.

Exercise 6.6: Prove this statement. Prove also that $\pi^*(*1) = \omega_1{\wedge}\omega_2{\wedge}\omega_3$, and that a 3-dimensional submanifold $P \subset F$ can be locally represented as the graph of a local section $\sigma : U \to F$ if and only if $\omega_1{\wedge}\omega_2{\wedge}\omega_3$ is nonvanishing on P.

Consider the ideal $\mathcal{I} = \langle\, \omega_1{\wedge}d\omega_1,\ \omega_2{\wedge}d\omega_2,\ \omega_3{\wedge}d\omega_3\,\rangle$. The 3-dimensional integral manifolds of $\mathcal{I}$ on which $\omega_1{\wedge}\omega_2{\wedge}\omega_3$ is nonvanishing are then the desired local sections. We now want to describe these integral manifolds.

First, it is useful to note that, just as for the orthonormal (co-)frame bundle of Euclidean space, there are unique 1-forms $\omega_{ij} = -\omega_{ji}$ that satisfy the structure equations

$$d\omega_i = -\sum_{j=1}^{3} \omega_{ij} \wedge \omega_j\,.$$

The 1-forms $\omega_1, \omega_2, \omega_3, \omega_{23}, \omega_{31}, \omega_{12}$ are then a basis for the 1-forms on F.

By the structure equations, an alternative description of $\mathcal{I}$ is

$$\mathcal{I} = \langle\, \omega_2 \wedge \omega_3 \wedge \omega_{23},\ \omega_3 \wedge \omega_1 \wedge \omega_{31},\ \omega_1 \wedge \omega_2 \wedge \omega_{12}\,\rangle.$$

Let $G_3(TF, \omega)$ denote the set of tangent 3-planes on which $\omega_1{\wedge}\omega_2{\wedge}\omega_3$ is nonvanishing. Any $E \in G_3(TF, \omega)$ is defined by equations of the form

$$\begin{aligned}
\omega_{23} - p_{11}\,\omega_1 - p_{12}\,\omega_2 - p_{13}\,\omega_3 &= 0\\
\omega_{31} - p_{21}\,\omega_1 - p_{22}\,\omega_2 - p_{23}\,\omega_3 &= 0\\
\omega_{12} - p_{31}\,\omega_1 - p_{32}\,\omega_2 - p_{33}\,\omega_3 &= 0
\end{aligned}$$

Such a plane E is an integral element of $\mathcal{I}$ if and only the coefficients p_{ij} satisfy $p_{11} = p_{22} = p_{33} = 0$, which shows that $V_3(\mathcal{I}) \cap G_3(TF, \omega)$ consists entirely of ordinary integral elements. (Why?) Since $\mathcal{I}$ is generated in degree 3, each 1-plane or 2-plane is an ordinary integral element of $\mathcal{I}$. Moreover, since $\mathcal{I}$ is generated by three 3-forms, it follows that for any $E_2 \in V_2(\mathcal{I})$, the codimension of $H(E_2)$ in T_pF is at most 3. In particular, every such E_2 has at least one extension to a 3-dimensional integral element, so that $r(E_2) \geq 0$ for every $E_2 \in V_2(\mathcal{I})$.

If the metric g is real analytic, then the Cartan-Kähler Theorem applies and it follows that there will be 3-dimensional integral manifolds of $\mathcal{I}$ and that, in fact, the generic real analytic surface in F lies in such an integral manifold.

This would be enough to solve our problem, but it is useful to determine the explicit condition that makes a surface in F be a regular integral manifold. To do this, we need to determine $V_2^r(\mathcal{I})$. Now, suppose that E_2 is spanned by two vectors a and b and set $a_i = \omega_i(a)$ and $b_i = \omega_i(b)$. A vector v will lie in the polar space of E_2 if and only if it is annihilated by the three 1-forms

$$\left.\begin{aligned} (a\wedge b) \lrcorner (\omega_2\wedge\omega_3\wedge\omega_{23}) &\equiv (a_2b_3 - a_3b_2)\,\omega_{23} \\ (a\wedge b) \lrcorner (\omega_3\wedge\omega_1\wedge\omega_{31}) &\equiv (a_3b_1 - a_1b_3)\,\omega_{31} \\ (a\wedge b) \lrcorner (\omega_1\wedge\omega_2\wedge\omega_{12}) &\equiv (a_1b_2 - a_2b_1)\,\omega_{12} \end{aligned}\right\} \mod \{\omega_1,\omega_2,\omega_3\}.$$

In particular, $r(E_2) = 0$ and $H(E_2) = E_3$ lies in $G_3(TF,\omega)$ when all of the numbers

$$\{(a_2b_3 - a_3b_2), (a_3b_1 - a_1b_3), (a_1b_2 - a_2b_1)\}$$

are nonzero.

Exercise 6.7: Show that this computation leads to the following geometric description of the regular integral surfaces of $\mathcal{I}$. A regular integral surface can be seen as a surface $S \subset M$ and a choice of a g-orthonormal coframing $\eta = (\eta_1, \eta_2, \eta_3)$ along S such that none of the $\eta_i\wedge\eta_j$ $(i \neq j)$ vanish on the tangent planes to the surface S. By the Cartan-Kähler Theorem, a real analytic coframing satisfying this nondegeneracy condition defined along a real analytic surface S can be 'thickened' uniquely to a real analytic coframing in a neighborhood of S in such a way that each of the η_i become integrable (i.e., locally exact up to multiples).

6.3. The existence of local Lie groups

As you know, every Lie group G has an associated Lie algebra structure on the tangent space $\mathfrak{g} = T_eG$. This Lie algebra structure is a skewsymmetric bilinear pairing $[,] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ that satisfies the *Jacobi identity*

$$\big[u,[v,w]\big] + \big[v,[w,u]\big] + \big[w,[u,v]\big] = 0$$

for all $u, v, w \in \mathfrak{g}$. One way this shows up in the geometry of G (there are many ways) is that, as discussed in Lecture 2, the canonical left invariant 1-form η on G satisfies the Mauer-Cartan equation $d\eta = -\frac{1}{2}[\eta,\eta]$.

We have already seen Cartan's Theorem, which says that any $\mathfrak{g}$-valued 1-form ω on a connected and simply connected manifold M that satisfies $d\omega = -\frac{1}{2}[\omega,\omega]$ is of the form $\omega = g^*\eta$ for some $g : M \to G$, unique up to composition with left translation. This implies, in particular, that there is at most one connected and simply connected Lie group associated to each Lie algebra.

I now want to consider the existence question: Suppose that we are given a Lie algebra, i.e., a vector space $\mathfrak{g}$ over $\mathbb{R}$ with a skewsymmetric bilinear pairing $[,] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ that satisfies the Jacobi identity. Does there exist a Lie group G with Lie algebra $\mathfrak{g}$? Now, the answer is known to be 'yes', but it's rather delicate because of certain global topological issues that I don't want to get into here. What I want to do instead is use the Cartan-Kähler Theorem to give a quick, simple proof that there exists a *local* Lie group with Lie algebra $\mathfrak{g}$.

What this amounts to is showing that there exists a $\mathfrak{g}$-valued 1-form η on a neighborhood U of $0 \in \mathfrak{g}$ with the property that $\eta_0 : T_0\mathfrak{g} \to \mathfrak{g}$ is the identity and that it satisfies the Maurer-Cartan equation $d\eta = -\frac{1}{2}[\eta,\eta]$.

Exercise 6.8: Assuming such an η exists, prove that there exists some 0-neighborhood $V \subset U$ and a smooth (in fact, real analytic) map $\mu : V\times V \to U$ satisfying

1. (*Identity*) $\mu(0,v) = \mu(v,0) = v$ for all $v \in V$,
2. (*Inverses*) For each $v \in V$, there is a $v^* \in V$ so that $\mu(v,v^*) = \mu(v^*,v) = 0$.
3. (*Associativity*) For $u, v, w \in V$, $\mu\big(\mu(u,v),w\big) = \mu\big(u,\mu(v,w)\big)$ when both sides make sense,

and so that, if L_v is defined by $L_v(u) = \mu(v,u)$, then $(L_v)'\big(\eta_v(w)\big) = w$ for all $w \in T_v\mathfrak{g} = \mathfrak{g}$.

To prove the existence of η, we proceed as follows: First identify $\mathfrak{g}$ with $\mathbb{R}^n$ by choosing linear coordinates $\mathbf{x} = (x^i)$. Now, let $M = \mathrm{GL}(n,\mathbb{R})\times\mathbb{R}^n$, with $\mathbf{u} : M \to \mathrm{GL}(n,\mathbb{R})$ and $\mathbf{x} : M \to \mathbb{R}^n$ being the projections onto the first and second factors. Now set

$$\Theta = d(\mathbf{u}\,d\mathbf{x}) + \tfrac{1}{2}\big[\mathbf{u}\,d\mathbf{x}, \mathbf{u}\,d\mathbf{x}\big] = \big(\Theta^i\big).$$

Exercise 6.9: Show that the Jacobi identity implies that (in fact, is equivalent to the fact that) $\big[\psi,[\psi,\psi]\big] = 0$ for any $\mathfrak{g}$-valued 1-form ψ. Conclude that Θ satisfies $d\Theta = \frac{1}{2}\big[\Theta, \mathbf{u}\,d\mathbf{x}\big] - \frac{1}{2}\big[\mathbf{u}\,d\mathbf{x}, \Theta\big]$.

From this exercise, it follows that the ideal $\mathcal{I}$ generated by the n component 2-forms Θ^i is generated algebraically by these 2-forms.

Exercise 6.10: If $\mathbf{u} = (u^i_j)$, then show that there exist (linearly independent) 1-forms π^i_j satisfying

$$\pi^i_j \equiv du^i_j \mod \{dx^1, \dots, dx^n\}$$

for which $\Theta^i = \pi^i_j{\wedge}dx^j$.

The existence of η will be established if we can show that there exists an n-dimensional integral manifold $N \subset M$ of $\mathcal{I}$ passing through $p = (I_n, 0)$ on which the n-form $dx^1{\wedge}\cdots{\wedge}dx^n$ is nonvanishing.

To do this, consider the integral element $E_n \subset T_pM$ defined by the equations $\pi^i_j = 0$, and let $E_k \subset E_n$ be defined by the additional equations $dx^j = 0$ for $j > k$ for $0 \le k \le n$. Since the π^i_j are linearly independent, it follows that

$$H(E_k) = \{\, v \in T_pM \,\big|\, \pi^i_j(v) = 0 \text{ for } 1 \le j \le k \,\},$$

so $c(E_k) = nk$ for $0 \le k \le n$. Thus, for the flag $F = (E_0, \dots, E_n)$, we have

$$c(F) = 0 + n + 2n + \cdots + (n{-}1)n = \tfrac{1}{2}n^2(n{-}1).$$

On the other hand an n-plane $E \in G_n(TM, \mathbf{x})$ is defined by equations of the form

$$\pi^i_j - p^i_{jk}\,dx^k = 0$$

and it will be an integral element if and only if the $\frac{1}{2}n^2(n{-}1)$ linear equations $p^i_{jk} = p^i_{kj}$ hold. Consequently, Cartan's Test is satisfied and the flag is regular. The Cartan-Kähler Theorem now implies that there is an integral manifold of $\mathcal{I}$ tangent to E_n. QED

Exercise 6.11: If you are familiar with the proof of this theorem that uses only ODE techniques (see, for example, [Helgason]), compare that proof with this one. Can you see how the two are related?

6.4. Hyper-Kähler metrics

This example is somewhat more advanced that the previous ones. I'm including it for the sake of those who might be interested in seeing how the Cartan-Kähler theorem can be used to study more advanced problems in differential geometry.

A hyper-Kähler structure on a manifold M^{4n} is a quadruple (g, I, J, K) where g is a Riemannian metric and $I, J, K : TM \to TM$ are g-parallel and orthogonal skewcommuting linear transformations of TM that satisfy

$$I^2 = J^2 = K^2 = -1, \qquad IJ = -K, \quad JK = -I, \quad KI = -J.$$

In other words (I, J, K) define a *right* quaternionic structure on the tangent bundle of M that is orthogonal and parallel with respect to g.

Suppose we have such a structure on M. Set

$$\omega_1(v,w) = g(Iv,w), \qquad \omega_2(v,w) = g(Jv,w), \qquad \omega_3(v,w) = g(Kv,w).$$

Then these three 2-forms are g-parallel and hence closed. Moreover, these three 2-forms are enough data to recover I, J, K and even g. For example, I is the unique map that satisfies $\omega_3(Iv,w) = -\omega_2(v,w)$ and then $g(v,w) = \omega_1(v, Iw)$.

There remains the question of 'how many' such hyper-Kähler metrics there are locally. One obvious example is to take $M = \mathbb{H}^n$ with its standard metric and let I, J, and K be the usual multiplication (on the *right*) by the obvious unit quaternions. However, this is not a very interesting example.

Two of these 2-forms at a time can indeed be made flat in certain coordinates: If we set $\Omega = \omega_2 - i\,\omega_3$, it is easy to compute that

$$\Omega(Ix, y) = \Omega(x, Iy) = i\,\Omega(x, y)$$

for all tangent vectors $x, y \in T_pM$. Thus, Ω is a closed 2-form of type $(2, 0)$ with respect to the complex structure I. Moreover, it is easy to compute that Ω^n is nowhere vanishing but that $\Omega^{n+1} = 0$. It follows from the complex version of the Darboux Theorem that every $p \in M$ has a neighborhood U on which there exist complex coordinates $z^1, \ldots, z^{2n}$ that are holomorphic for the complex structure I and for which

$$\Omega = dz^1 \wedge dz^{n+1} + dz^2 \wedge dz^{n+2} + \cdots + dz^n \wedge dz^{2n}.$$

These coordinates are unique up to a holomorphic symplectic transformation. Meanwhile, the 2-form ω_1 in these coordinates takes the form

$$\omega_1 = \tfrac{\sqrt{-1}}{2}\, u_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$$

where $U = (u_{i\bar{j}})$ is a positive definite Hermitian matrix of functions that satisfies the equation ${}^tU\,Q\,U = Q$ where

$$Q = \begin{pmatrix} \mathrm{O}_n & \mathrm{I}_n \\ -\mathrm{I}_n & \mathrm{O}_n \end{pmatrix}.$$

One cannot generally choose the coordinates to make U be the identity matrix. Indeed, this is the necessary and sufficient condition that the hyper-Kähler structure be locally equivalent to the flat structure mentioned above.

Conversely, if one can find a smooth function U on a domain $D \subset \mathbb{C}^{2n}$ with values in positive definite Hermitian $2n$-by $2n$ matrices satisfying the algebraic condition ${}^tU\,Q\,U = Q$ as well as the differential condition that the 2-form

$$\omega_1 = \tfrac{\sqrt{-1}}{2}\, u_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$$

be closed, then setting

$$\omega_2 - i\,\omega_3 = dz^1 \wedge dz^{n+1} + dz^2 \wedge dz^{n+2} + \cdots + dz^n \wedge dz^{2n}$$

defines a triple $(\omega_1, \omega_2, \omega_3)$ on D that determines a hyper-Kähler structure on D.

This suggests the construction of a differential ideal whose integral manifolds will represent the desired functions U. First, define

$$Z = \{ H \in \mathrm{GL}(2n, \mathbb{C}) \mid H = {}^t\bar{H} > 0,\ {}^tH\,Q\,H = Q \}.$$

Exercise 6.12: Show that Z can also be described as the space of matrices $H = {}^t\bar{A}\,A$ with $A \in \mathrm{Sp}(n, \mathbb{C}) = \{ A \in \mathrm{GL}(2n, \mathbb{C}) \mid {}^tA\,Q\,A = Q \}$ and hence that Z is just the Riemannian symmetric space $\mathrm{Sp}(n, \mathbb{C})/\mathrm{Sp}(n)$, whose dimension is $2n^2{+}n$. In particular, Z is a smooth submanifold of $\mathrm{GL}(2n, \mathbb{C})$.

Now define $M = Z \times \mathbb{C}^{2n}$ and let $H = (h_{i\bar{j}}) : M \to Z$ be the projection onto the first factor and $z : M \to \mathbb{C}^{2n}$ be the projection onto the second factor. Let $\mathcal{I}$ be the ideal generated by the (real) 3-form

$$\Theta = \tfrac{\sqrt{-1}}{2}\, dh_{i\bar{j}} \wedge dz^i \wedge d\bar{z}^j = d\left(\tfrac{\sqrt{-1}}{2}\, h_{i\bar{j}}\, dz^i \wedge d\bar{z}^j \right).$$

Obviously $\mathcal{I}$ is generated algebraically by Θ, since Θ is closed. One integral manifold of Θ is given by the equations $H = \mathrm{I}_{2n}$, which corresponds to the flat solution. We want to determine how general the space of solutions is near this solution.

First, let me note that the group $\mathrm{Sp}(n, \mathbb{C})$ acts on M preserving Θ via the action

$$A \cdot (H, z) = \left({}^t\bar{A}\,H\,A,\ \bar{A}^{-1}z \right).$$

The additive group $\mathbb{C}^{2n}$ also acts on M via translations in the $\mathbb{C}^{2n}$-factor, and this action also preserves Θ. These two actions combined generate a transitive action on M preserving Θ, so the ideal $\mathcal{I}$ is homogeneous. Thus, we can do our computations at any point, say $p = (\mathrm{I}_{2n}, 0)$, which I fix from now on.

Let $E_{4n} \subset T_pM$ be the tangent space at p to the flat solution $H = \mathrm{I}_{2n}$. Let $F = (E_0, \ldots, E_{4n})$ be any flag. Because $\mathcal{I}$ is generated by a single 3-form, it follows that

$$c(E_k) \le \binom{k}{2}$$

for all k. (Why?) On the other hand, since the codimension of E_{4n} in T_pM is $2n^2{+}n = \dim Z$, equality cannot hold for $k > 2n{+}1$.

Now, I claim that there exists a flag F for which $c(E_k) = \binom{k}{2}$ for $k \le 2n{+}1$ while $c(E_k) = 2n^2{+}n$ when $2n{+}1 < k \le 4n$. Moreover, I claim that Cartan's Test is satisfied for such a flag, i.e., $V_{4n}(\mathcal{I}) \cap G_{4n}(TM, \mathbf{z})$ is a smooth submanifold of $G_{4n}(TM, \mathbf{z})$ of codimension

$$c(F) = c(E_0) + \cdots + c(E_{4n-1}) = \frac{4}{3}n(2n-1)(2n+1).$$

Consequently, such a flag is regular.

Since $s_k(F) = k{-}1$ for $0 < k \le 2n{+}1$ and $s_k(F) = 0$ for $k > 2n{+}1$, the description of the generality of solutions near the flat solution now shows that the solutions depend on $2n$ 'arbitrary' functions of $2n{+}1$ variables and that a solution is determined by its restriction to a generic (real analytic) submanifold of dimension $2n{+}1$. Since the symplectic biholomorphisms depend only on arbitrary functions of $2n$ variables (why?), it follows that the generic hyper-Kähler structure is not flat. In fact, as we shall see in the next lecture, this calculation will yield much more detailed information about the local solutions.

I'm only going to sketch out the proof of these claims and leave much of the linear algebra to you.

The first thing to do is to get a description of the relations among the components of dH_p. Computing the exterior derivatives of the defining relations $H = {}^t\bar{H}$ and ${}^tH\,Q\,H = Q$ gives

$$dH = {}^t\overline{dH}, \qquad {}^t(dH)\,Q\,H + {}^tH\,Q\,dH = 0.$$

Evaluating this at $p = (\mathrm{I}_{4n}, 0)$, gives

$$dH_p = {}^t\overline{dH_p}, \qquad {}^t(dH_p)\,Q + Q\,dH_p = 0,$$

so it follows that

$$dH_p = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & -\bar{\alpha} \end{pmatrix}$$

where $\alpha = {}^t\bar{\alpha}$ and $\beta = {}^t\beta$ are n-by-n matrices of complex-valued 1-forms. Writing $z^i = u^i$ and $z^{i+n} = v^i$ for $1 \le i \le n$, it follows that

$$\Theta_p = \tfrac{\sqrt{-1}}{2}\,\alpha_{i\bar{j}} \wedge \left(du^i \wedge d\bar{u}^j - dv^j \wedge d\bar{v}^i\right) + \tfrac{\sqrt{-1}}{2}\left(\beta_{ij} \wedge du^i \wedge d\bar{v}^j + \bar{\beta}_{ij} \wedge dv^i \wedge d\bar{u}^j\right)$$

Now the only relations among the α-components and the β-components are $\alpha_{i\bar{j}} - \overline{\alpha_{j\bar{\imath}}} = \beta_{ij} - \beta_{ji} = 0$. Using this information, you can verify the computation of the $c(E_k)$ simply by finding an $E_{2n+1} \subset E_{4n}$ for which $c(E_{2n+1}) = 2n^2{+}n$, since this forces all the rest of the formulae for $c(E_k)$. (Why?) (Such an E_{2n+1} shouldn't be hard to find, since the generic element of $G_{2n+1}(E_{4n})$ works.)

Now, to verify the codimension of $V_{4n}(\mathcal{I})$, note that any $E^*_{4n} \subset T_pM$ that is transverse to the Z factor can be defined by equations of the form

$$\begin{aligned}
\alpha_{i\bar{j}} &= A_{i\bar{j}k}\,du^k - \overline{A_{j\bar{\imath}k}}\,d\bar{u}^k + B_{i\bar{j}k}\,dv^k - \overline{B_{j\bar{\imath}k}}\,d\bar{v}^k \\
\beta_{ij} &= P_{ijk}\,du^k + Q_{ij\bar{k}}\,d\bar{u}^k + R_{ijk}\,dv^k + S_{ij\bar{k}}\,d\bar{v}^k
\end{aligned}$$

where the coefficients are arbitrary subject to the relations $P_{ijk} = P_{jik}$, $Q_{ij\bar{k}} = Q_{ji\bar{k}}$, $R_{ijk} = R_{jik}$, $S_{ij\bar{k}} = S_{ji\bar{k}}$. Now you just need to check that the condition that Θ_p vanish on E^*_{4n} is exactly $\frac{4}{3}n(2n{+}1)(2n{-}1)$ linear relations on the coefficients A, B, P, Q, R, and S.

Exercise 6.13: Fill in the details in this proof.

Lecture 7. Prolongation

Almost all of the previous examples have been carefully chosen so that there will exist regular flags, so that the Cartan-Kähler theorem can be applied. Unfortunately, this is not always the case, in which case other methods must be applied. In this lecture, I'm going to describe those other methods.

7.1. When Cartan's Test fails.

We have already seen one case where the Cartan-Kähler approach fails, in the sense that it fails to find the integral manifolds that are actually there. That was in Exercise 3.1, where $M = \mathbb{R}^3$ and the ideal was

$$\mathcal{I} = \langle\, dx \wedge dz,\ dy \wedge dz \,\rangle.$$

Although there *are* 2-dimensional integral manifolds, namely the planes $z = c$, no 2-dimensional integral element is the terminus of a regular flag. It's not too surprising that this should happen, though, because the ideal

$$\mathcal{I}' = \langle\, dx \wedge dz,\ dy \wedge (dz{-}y, dx) \,\rangle$$

which is virtually indistinguisable from $\mathcal{I}$ in terms of the algebraic properties of the spaces of integral elements does not have any 2-dimensional integral manifolds. Some finer invariant of the ideals must be brought to light in order to distinguish the two cases.

Now, the above examples are admittedly a little artificial, so you might be surprised to see that they and their 'cousins' come up quite a bit.

Example 7.1: *Surfaces in $\mathbb{R}^3$ with constant principal curvatures.* You may already know how to solve this problem, but let's see what the naïve approach via differential systems will give. Looking back at the discussion of surface theory in Lecture 5, you can see that if we want to find the surfaces in $\mathbb{R}^3$ with principal curvatures equal to some fixed constants κ_1 and κ_2 (distinct), then we should look for integral manifolds of the ideal $\mathcal{I}$ on G that is generated by the three 1-forms

$$\theta_0 = \omega_3\,, \quad \theta_1 = \omega_{31} - \kappa_1\,\omega_1\,, \quad \theta_2 = \omega_{32} - \kappa_2\,\omega_2\,.$$

Now, if you compute the exterior derivatives of these forms, you'll get

$$\left.\begin{aligned} d\theta_0 &\equiv 0 \\ d\theta_1 &\equiv -(\kappa_1{-}\kappa_2)\omega_{12} \wedge \omega_2 \\ d\theta_2 &\equiv -(\kappa_1{-}\kappa_2)\omega_{21} \wedge \omega_1 \end{aligned}\right\} \bmod \{\theta_0, \theta_1, \theta_2\}.$$

So

$$\mathcal{I} = \langle\, \theta_0, \theta_1, \theta_2, (\kappa_1{-}\kappa_2)\omega_{12} \wedge \omega_2, (\kappa_1{-}\kappa_2)\omega_{21} \wedge \omega_1 \,\rangle)_{\mathrm{alg}}.$$

Now there are two cases: One is that $\kappa_1 = \kappa_2$, in which case $\mathcal{I}$ is Frobenius.

Exercise 7.1: Explain why the integral manifolds in the case $\kappa_1 = \kappa_2$ correspond to the planes in $\mathbb{R}^3$ when $\kappa_1 = \kappa_2 = 0$ and to spheres in $\mathbb{R}^3$ when $\kappa_1 = \kappa_2 \neq 0$.

In the second case, where $\kappa_1 \neq \kappa_2$, you'll see that the 2-dimensional integral elements on which $\omega_1{\wedge}\omega_2$ is non-zero (and there are some) never contain any regular 1-dimensional integral elements, just as in our 'toy' example.

Exercise 7.2: Describe $V_2(\mathcal{I})$ when $\kappa_1 \neq \kappa_2$. Show that there is a unique 2-dimensional integral element of $\mathcal{I}$ at each point of G on which $\omega_1{\wedge}\omega_2$ is nonvanishing. Explain why this shows, via Cartan's Test, that such integral elements cannot be the terminus of a regular flag.

Exercise 7.3: Redo this problem assuming that the ambient 3-manifold is of constant sectional curvature c, not necessarily 0. You may want to recall that the structure equations in this case are of the form

$$d\omega_i = -\omega_{ij} \wedge \omega_j\,, \qquad d\omega_{ij} = -\omega_{ik} \wedge \omega_{kj} + c\,\omega_i \wedge \omega_j\,.$$

Does anything significant change?

Exercise 7.4: Set up an exterior differential system to model the solutions of the system $u_{xx} = u_{yy} = 0$ (where u is a function of x and y). Compare this to the analogous model of the system $u_{xx} = u_{xy} = 0$. In particular, compare the regular integral curves of the two systems and their 'thickenings' via the Cartan-Kähler Theorem.

Exercise 7.5: What can you say about the surfaces in $\mathbb{R}^3$ with the property that each principal curvature κ_i is constant on each of its corresponding principal curves?

7.2. Prolongation, systems in good form

In each of the examples in the previous subsection, we found an exterior differential system for which the interesting integral manifolds (if there are any) cannot be constructed by thickening along a regular flag. Cartan proposed a process of 'regularizing' these ideals which he called 'prolongation'. Intuitively, prolongation is just differentiating the equations you have and then adjoining those equations as new equations in the system. You can see why such a thing might work by looking at the following situation:

We know how to check whether the system

$$z_x = f(x,y), \qquad z_y = g(x,y)$$

is compatible. You just need to see whether or not $f_y = g_x$, a first order condition on the equations that is by looking at the exterior ideal generated by the 1-form $\zeta = dz - f(x,y)\,dx - g(x,y)\,dy$. On the other hand, if you consider the system

$$z_{xx} = f(x,y), \qquad z_{yy} = g(x,y),$$

the compatibility condition is not revealed until you differentiate twice, i.e, $f_{yy} = g_{xx}$. Now, it's not clear how to get to this condition by looking at the ideal on $xyzpqs$-space generated by

$$\begin{aligned}
\theta_0 &= dz - p\,dx - q\,dy\\
\theta_1 &= dp - f(x,y)\,dx - s\,dy\\
\theta_2 &= dq - s\,dx - g(x,y)\,dy
\end{aligned}$$

because the exterior derivatives of these forms will only contain first derivatives of the functions f and g. And, sure enough, Cartan's Test fails for this system.

However, if you differentiate the given equations once, you can see that they imply

$$z_{xxy} = f_y(x,y), \qquad z_{xyy} = g_x(x,y)$$

which suggests looking at the ideal on $xyzpqs$-space generated by

$$\begin{aligned}
\theta_0 &= dz - p\,dx - q\,dy\\
\theta_1 &= dp - f(x,y)\,dx - s\,dy\\
\theta_2 &= dq - s\,dx - g(x,y)\,dy\\
\theta_3 &= ds - f_y(x,y)\,dx - g_x(x,y)\,dy.
\end{aligned}$$

Now, this ideal is Frobenius if and only if $f_{yy} = g_{xx}$, so the obvious compatibility condition is the necessary and sufficient condition for there to exist solutions to the original problem.

Exercise 7.6: What can you say about the solutions of the system

$$z_{xx} = z \qquad\qquad z_{yy} = z?$$

A systematic way to 'adjoin derivatives as new variables' for the general exterior differential system $(M,\mathcal{I})$ is this: Suppose that you are interested in studying the n-dimensional integral manifolds of $(M,\mathcal{I})$ whose tangent planes lie in some smooth submanifold (usually a component)

$$Z \subset V_n(\mathcal{I}) \subset G_n(TM).$$

As explained in Lecture 2, every such integral manifold $f : N \hookrightarrow M$ has a canonical lift to a submanifold $f^{(1)} : N \hookrightarrow Z$ defined simply by

$$f^{(1)}(p) = f'(T_pN) \subset T_{f(p)}M.$$

Now, $f^{(1)} : N \hookrightarrow Z \subset G_n(TM)$ is an integral manifold of the contact system $\mathcal{C}$ and is transverse to the projection $\pi : Z \to M$. Conversely, if $F : N \to Z \subset G_n(TM)$ is an integral manifold of the contact system $\mathcal{C}$ that is transverse to the projection π, then $F = f^{(1)}$ where $f = \pi \circ F$, and so, *a fortiori*, the tangent spaces of the immersion $f : N \to M$ all lie in $Z \subset V_n(\mathcal{I})$. (In particular, $f : N \to M$ is an integral manifold of $\mathcal{I}$.)

Let $\mathcal{I}^{(1)} \subset \Omega^*(Z)$ denote the exterior ideal on Z induced by pulling back $\mathcal{C}$ on $G_n(TM)$ via the inclusion $Z \subset G_n(TM)$. The pair $\bigl(Z, \mathcal{I}^{(1)}\bigr)$ is known as the *Z-prolongation* of $\mathcal{I}$. Our argument in the above paragraph has established that the integral manifolds of $\mathcal{I}$ whose tangent planes lie in Z are in one-to-one correspondance with the integral manifolds of $\bigl(Z, \mathcal{I}^{(1)}\bigr)$ that are transverse to the projection $\pi : Z \to M$.

Usually, there is only one component of $V_n^o(\mathcal{I})$ of interest anyway. In this case, it is common to refer to it as $M^{(1)} \subset V_n^o(\mathcal{I})$ and then simply say that $\bigl(M^{(1)}, \mathcal{I}^{(1)}\bigr)$ is the prolongation of $\mathcal{I}$, imprecise though this is.

Example 7.2: *The toy model again.* Look at the EDS

$$\bigl(\mathbb{R}^3, \langle\, dx\wedge dz,\ dy\wedge dz\rangle\,\bigr).$$

There is exactly one 2-dimensional integral element at each point, namely, the 2-plane defined by $dz = 0$. Since these 2-dimensional integral elements define a Frobenius system on $\mathbb{R}^3$, there is a unique integral surface passing through each point of $\mathbb{R}^3$.

It's instructive to go through the above prolongation process explicitly here: Using coordinates (x, y, z), consider the open set $G_2\bigl(T\mathbb{R}^3, (x, y)\bigr)$ consisting of the 2-planes on which $dx\wedge dy$ is nonzero. This 5-manifold has coordinates (x, y, z, p, q) so that $E \in G_2\bigl(T\mathbb{R}^3, (x, y)\bigr)$ is spanned by

$$\left\{ \frac{\partial}{\partial x} + p(E)\,\frac{\partial}{\partial z}\ ,\ \frac{\partial}{\partial y} + q(E)\,\frac{\partial}{\partial z} \right\}.$$

In these coordinates, the contact system $\mathcal{C}$ is generated by the 1-form

$$\theta = dz - p\,dx - q\,dy.$$

Now, $Z = V_2(\mathcal{I}) \subset G_2\bigl(T\mathbb{R}^3, (x, y)\bigr)$ is defined by the equations $p = q = 0$, so pulling back the form θ to this locus yields that (x, y, z) are coordinates on Z and that $\mathcal{I}^{(1)} = \langle\, dz\,\rangle$, an ideal to which the Frobenius Theorem applies.

Exercise 7.7: Repeat this analysis for the EDS

$$\bigl(\mathbb{R}^3, \langle dx\wedge dz,\ dy\wedge(dz - y\,dx)\,\rangle\bigr).$$

Frequently, M has a coframing $(\omega^1, \dots, \omega^n, \pi^1, \dots, \pi^s)$ (i.e., a basis for the 1-forms on M) and one is interested in the n-dimensional integral manifolds of some $\mathcal{I}$ on which the 1-forms $(\omega^1, \dots, \omega^n)$ are linearly independent. Let $V_n(\mathcal{I}, \omega)$ denote the integral elements on which $\omega = \omega^1\wedge \dots \wedge\omega^n$ is nonvanishing. The usual procedure is then to describe $V_n(\mathcal{I}, \omega)$ as the set of n-planes defined by equations of the form

$$\pi^a - p^a_i\,\omega^i = 0$$

where the p^a_i are subject to the constraints that make such an n-plane be an integral element. In this way, the p^a_i become functions on $V_n(\mathcal{I}, \omega)$. Moreover, the contact ideal $\mathcal{C}$ pulls back to $V_n(\mathcal{I}, \omega)$ to be generated by the 1-forms

$$\theta^a = \pi^a - p^a_i\,\omega^i\,,$$

thus giving us an explicit expression for the ideal $\mathcal{I}^{(1)}$ as

$$\mathcal{I}^{(1)} = \langle\ \theta^1, \dots, \theta^s\ \rangle.$$

Example 7.3: *Constant principal curvatures.* Look back at Example 7.1, with $\kappa_1 \neq \kappa_2$, where we found an ideal

$$\mathcal{I} = \langle\, \omega_3, \omega_{31}{-}\kappa_1\,\omega_1, \omega_{32}{-}\kappa_2\,\omega_2, (\kappa_1{-}\kappa_2)\omega_{12}\wedge\omega_2, (\kappa_1{-}\kappa_2)\omega_{21}\wedge\omega_1\,\rangle_{\text{alg}}.$$

There is a unique 2-dimensional integral element at each point of G, defined by the equations

$$\omega_3 = \omega_{31}{-}\kappa_1\,\omega_1 = \omega_{32}{-}\kappa_2\,\omega_2, = \omega_{12} = 0.$$

Thus $V_2(\mathcal{I})$ is diffeomorphic to G. By the same reasoning employed above, we have that

$$\mathcal{I}^{(1)} = \langle\, \omega_3,\, \omega_{31}{-}\kappa_1\,\omega_1,\, \omega_{32}{-}\kappa_2\,\omega_2,\, \omega_{12}\,\rangle$$

Computing exterior derivatives and using the structure equations, we find that

$$\mathcal{I}^{(1)} = \langle\, \omega_3,\, \omega_{31}{-}\kappa_1\,\omega_1,\, \omega_{32}{-}\kappa_2\,\omega_2,\, \omega_{12},\, \kappa_1\kappa_2\,\omega_1\wedge\omega_2\,\rangle.$$

Now we can see a distinction: If $\kappa_1\kappa_2 \neq 0$, then this ideal has no 2-dimensional integral elements at all, and hence no integral surfaces. On the other hand, if $\kappa_1\kappa_2 = 0$ (i.e., one of the κ_i is zero), then $\mathcal{I}^{(1)}$ is a Frobenius system and is foliated by 2-dimensional integral manifolds.

Exercise 7.8: Repeat this analysis of the surfaces with constant principal curvatures for the other 3-dimensional spaces of constant sectional curvature. What changes? (You may want to look back at Exercise 7.3, for the structure equations.)

I want to do one more example of this kind of problem so that you can get some sense of what the process can be like. (I warn you that this is a rather involved example.)

Example 7.4: *Restricted Principal Curvatures.* Consider the surfaces described in Exercise 7.5, i.e., the surfaces with the property that each principal curvature κ_i is constant along each of its corresponding principal curves. A little thought, together with reference to the discussion of Weingarten surfaces in Lecture 6 should convince you that these surfaces are the integral manifolds in $M = G{\times}\mathbb{R}^2$ of the ideal

$$\mathcal{I} = \langle\, \theta_0,\, \theta_1,\, \theta_2,\, \pi_1\wedge\omega_1{+}\pi_2\wedge\omega_2,\, \pi_2\wedge\omega_1{+}\pi_3\wedge\omega_2,\, \pi_1\wedge\omega_2,\, \pi_3\wedge\omega_1\,\rangle_{\text{alg}}\,.$$

(I am maintaining the notation established in Lecture 6.1.) Now, each 2-dimensional integral element on which $\omega_1{\wedge}\omega_2$ is non-vanishing is defined by equations of the form

$$\theta_0 = \theta_1 = \theta_2 = \pi_1 - p_1\,\omega_2 = \pi_2 - p_1\,\omega_1 - p_2\,\omega_2 = \pi_3 - p_2\,\omega_1 = 0.$$

where p_1 and p_2 are arbitrary parameters. I'll leave it to you to check that these integral elements are *not* the terminus of any regular flag. Consequently, we cannot apply the Cartan-Kähler Theorem to construct examples of such surfaces.

It is computationally advantageous to parametrize the integral elements by $q_1 = p_1/(\kappa_1{-}\kappa_2)$ and $q_2 = p_2/(\kappa_1{-}\kappa_2)$ rather than by p_1 and p_2 as defined above, so that is what we will do. (This change of scale avoids having to divide by $(\kappa_1{-}\kappa_2)$ several times later.)

Now, following the prescription already given, construct $\big(M^{(1)}, \mathcal{I}^{(1)}\big)$ as follows: We let $M^{(1)} = M\times\mathbb{R}^2$, with p_1 and p_2 being the coordinates on the $\mathbb{R}^2$-factor and set

$$\mathcal{I}^{(1)} = \langle\, \theta_0,\, \theta_1,\, \theta_2,\, \theta_3,\, \theta_4,\, \theta_5\,\rangle$$

where

$$\begin{aligned}
\theta_3 &= \pi_1 \qquad\qquad\;\; - p_1\,\omega_2 &&= \qquad\; d\kappa_1 \qquad\qquad - p_1\,\omega_2\\
(\kappa_1{-}\kappa_2)\,\theta_4 &= \pi_2 - p_1\,\omega_1 - p_2\,\omega_2 &&= (\kappa_1{-}\kappa_2)\,\omega_{21} - p_1\,\omega_1 - p_2\,\omega_2\\
\theta_5 &= \pi_3 - p_2\,\omega_1 &&= \qquad\; d\kappa_2 - p_2\,\omega_1
\end{aligned}$$

or, in terms of the q_i, we have

$$\begin{aligned}\theta_3 &= d\kappa_1 - (\kappa_1{-}\kappa_2)q_1\,\omega_2\,,\\ \theta_4 &= \omega_{21} - q_1\,\omega_1 - q_2\,\omega_2\,,\\ \theta_5 &= d\kappa_2 - (\kappa_1{-}\kappa_2)q_2\,\omega_1\,.\end{aligned}$$

Now, it should not be a surprise that

$$d\theta_0 \equiv d\theta_1 \equiv d\theta_2 \equiv 0 \bmod \theta_0,\,\theta_1,\,\theta_2,\,\theta_3,\,\theta_4,\,\theta_5$$

(you should check this if you are surprised). Moreover, using the structure equations and the definitions of the forms given so far, we can compute that

$$\begin{aligned}d\theta_3 &\equiv -(\kappa_1{-}\kappa_2)\,dq_1\wedge\omega_2\\ d\theta_4 &\equiv\ -dq_1\wedge\omega_1 - dq_2\wedge\omega_2 - ({q_1}^2{+}{q_2}^2{+}\kappa_1\kappa_2)\,\omega_1\wedge\omega_2\\ d\theta_5 &\equiv -(\kappa_1{-}\kappa_2)\,dq_2\wedge\omega_1\end{aligned}$$

where the congruences are taken modulo θ_0, θ_1, θ_2, θ_3, θ_4, θ_5. It follows from this computation that the 2-dimensional integral elements on which the 2-form $\omega_1{\wedge}\omega_2$ is nonzero are all of the form

$$\begin{aligned}\theta_0 = \cdots = \theta_5 &= 0\\ dq_1 - (q_3 + {q_1}^2 + \tfrac12\kappa_1\kappa_2)\,\omega_2 &= 0\\ dq_2 - (q_3 - {q_2}^2 - \tfrac12\kappa_1\kappa_2)\,\omega_1 &= 0\end{aligned}$$

for some q_3. Thus, there is a 1-parameter family of such integral elements at each point. Unfortunately, none of these integral elements are the terminus of a regular flag, so the Cartan-Kähler Theorem *still* cannot be applied.

There's nothing to do now, but do it again: We now parametrize the space of these integral elements of $\bigl(M^{(1)},\mathcal{I}^{(1)}\bigr)$ as $M^{(2)} = M^{(1)}\times\mathbb{R}$ with q_3 being the coordinate on the $\mathbb{R}$-factor and we consider the ideal

$$\mathcal{I}^{(2)} = \langle\,\theta_0,\,\theta_1,\,\theta_2,\,\theta_3,\,\theta_4,\,\theta_5,\,\theta_6,\,\theta_7\,\rangle$$

where

$$\begin{aligned}\theta_6 &= dq_1 - (q_3 + {q_1}^2 + \tfrac12\kappa_1\kappa_2)\,\omega_2,\\ \theta_7 &= dq_2 - (q_3 - {q_2}^2 - \tfrac12\kappa_1\kappa_2)\,\omega_1.\end{aligned}$$

Now we get

$$d\theta_0 \equiv \cdots \equiv d\theta_5 \equiv 0 \bmod \theta_0,\,\theta_1,\,\theta_2,\,\theta_3,\,\theta_4,\,\theta_5,\,\theta_6,\,\theta_7$$

(again, this should be no surprise), but we must still compute $d\theta_6$ and $d\theta_7$. Well, using the structure equations, we can do this and we get

$$\left.\begin{aligned}d\theta_6 &\equiv -\theta_8\wedge\omega_2,\\ d\theta_7 &\equiv -\theta_8\wedge\omega_1,\end{aligned}\right\}\ \bmod \theta_0,\,\ldots\,\theta_7$$

where

$$\theta_8 = dq_3 + (q_3{+}{q_1}^2{+}\tfrac12{\kappa_1}^2)q_2\,\omega_1 - (q_3{-}{q_2}^2{-}\tfrac12{\kappa_2}^2)q_1\,\omega_2\,.$$

(Whew!) At this point, it is clear that there is only one 2-dimensional integral element of $\mathcal{I}^{(2)}$ on which $\omega_1{\wedge}\omega_2$ is nonzero at each point of $M^{(2)}$ and it is defined by

$$\theta_0 = \theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6 = \theta_7 = \theta_8 = 0.$$

Thus $M^{(3)} = M^{(2)}$ and we can take

$$\mathcal{I}^{(3)} = \langle\,\theta_0,\,\theta_1,\,\theta_2,\,\theta_3,\,\theta_4,\,\theta_5,\,\theta_6,\,\theta_7,\,\theta_8\,\rangle$$

As before, it is clear that

$$d\theta_0 \equiv \cdots \equiv d\theta_7 \equiv 0 \bmod \theta_0,\, \theta_1,\, \theta_2,\, \theta_3,\, \theta_4,\, \theta_5,\, \theta_6,\, \theta_7,\, \theta_8\,.$$

What is surprising (perhaps) is that

$$d\theta_8 \equiv 0 \bmod \theta_0,\, \theta_1,\, \theta_2,\, \theta_3,\, \theta_4,\, \theta_5,\, \theta_6,\, \theta_7,\, \theta_8\,!$$

In other words, $\mathcal{I}^{(3)}$ is a Frobenius system! Consequently, $M^{(3)}$, a manifold of dimension 11 (count it up) is foliated by 2-dimensional integral manifolds of $\mathcal{I}^{(3)}$.

Now this rather long example is meant to convince you that the process of prolongation can actually lead you to some answers. Unfortunately, although we now know that there is a 9-parameter family of such surfaces (i.e., the solutions depend on $s_0 = 9$ constants), we don't know what the surfaces are in any explicit way.

Exercise 7.9: Show that cylinders, circular cones and tori of revolution where the profile curve is a standard circle are examples of such surfaces. How do you know that this is not all of them? Does every such surface have at least a 1-parameter family of symmetries?

Exercise 7.10: Note that the forms $\theta_0, \ldots, \theta_8$ are well defined on the locus $\kappa_1 - \kappa_2 = 0$. Show that any leaf of $\mathcal{I}^{(3)}$ that intersects this locus stays entirely in this locus. What do these integral surfaces mean? (After all, an umbilic surface does not have well-defined principal curvatures.)

Exercise 7.11: What would have happened if, instead, we had looked for surfaces for which each principal curvature was constant on each principal curve belonging to the *other* principal curvature? Write down the appropriate exterior differential system and analyse it.

7.3. The Cartan-Kuranishi Theorem

Throughout this section, I am going to assume that all the ideals in question are generated in positive degrees, i.e., that they contain no nonzero functions. This is just to simplify the statements of the results. I'll let you worry about what to do when you have functions in the ideal.

Let $\bigl(M, \mathcal{I}\bigr)$ be an EDS and let $Z \subset V_n^o(\mathcal{I})$ be a connected open subset of $V_n^o(\mathcal{I})$. We say that Z is *involutive* if every $E \in Z$ is the terminus of a regular flag. Usually, in applications, there is only one such Z to worry about anyway, or else the 'interesting' component Z is clear from context, in which case we simply say that $(M, \mathcal{I})$ is involutive.

The first piece of good news about the prolongation process is that it doesn't destroy involutivity:

Theorem 8: (Persistence of Involutivity) Let $\bigl(M, \mathcal{I}\bigr)$ be an EDS with $\mathcal{I}^0 = (0)$ and let $M^{(1)} \subset V_n^o(\mathcal{I})$ be a connected open subset of $V_n^o(\mathcal{I})$ that is involutive. Then the character sequence $\bigl(s_0(F), \ldots, s_n(F)\bigr)$ is the same for all regular flags $F = (E_0, \ldots, E_n)$ with $E_n \in M^{(1)}$. Moreover, the EDS $\bigl(M^{(1)}, \mathcal{I}^{(1)}\bigr)$ is involutive on the set $M^{(2)} \subset V_n\bigl(\mathcal{I}^{(1)}\bigr)$ of elements that are transverse to the projection $\pi : M^{(1)} \to M$ and its character sequence $\bigl(s_0^{(1)}, \ldots, s_n^{(1)}\bigr)$ is given by

$$s_k^{(1)} = s_k + s_{k+1} + \cdots + s_n\,.$$

Exercise 7.12: Define $\bigl(M^{(k)}, \mathcal{I}^{(k)}\bigr)$ by the obvious induction, starting with $\bigl(M^{(0)}, \mathcal{I}^{(0)}\bigr) = (M, \mathcal{I})$ and show that

$$\dim M^{(k)} = n + s_0 + \binom{k{+}1}{1} s_1 + \binom{k{+}2}{2} s_2 + \cdots + \binom{k{+}n}{n} s_n\,.$$

Explain why $M^{(k)}$ can be interpreted as the space of k-jets of integral manifolds of $\mathcal{I}$ whose tangent planes lie in $M^{(1)}$.

Now Theorem 8 is quite useful, as we will see in the next lecture, but what we'd really like to know is whether prolongation will help with components $Z \subset V_n(\mathcal{I})$ that are not involutive. The answer is a sort of qualified 'yes':

Theorem 9: (Cartan-Kuranishi) Suppose that one has a sequence of manifolds M_k for $k \geq 0$ together with embeddings $\iota_k : M_k \hookrightarrow G_n(TM_{k-1})$ for $k > 0$ with the properties

(1) The composition $\pi_{k-1} \circ \iota_k : M_k \to M_{k-1}$ is a submersion,
(2) For all $k \geq 2$, $\iota_k(M_k)$ is a submanifold of $V_n(\mathcal{C}_{k-2}, \pi_{k-2})$, the integral elements of the contact system $\mathcal{C}_{k-2}$ on $G_n(TM_{k-2})$ transverse to the fibers of $\pi_{k-2} : G_n(TM_{k-2}) \to M_{k-2}$.

Then there exists a $k_0 \geq 0$ so that for $k \geq k_0$, the submanifold $\iota_{k+1}(M_{k+1})$ is an involutive open subset of $V_n\big(\iota_k^*\mathcal{C}_{k-1}\big)$, where $\iota_k^*\mathcal{C}_{k-1}$ is the EDS on M_k pulled back from $G_n(TM_{k-1})$.

A sequence of manifolds and immersions as described in the theorem is sometimes known as a *prolongation sequence*.

Now, you can imagine how this theorem might be useful. When you start with an EDS $(M, \mathcal{I})$ and some submanifold $\iota : Z \hookrightarrow V_n(\mathcal{I})$ that is not involutive, you can start building a prolongation sequence by setting $M_1 = Z$ and looking for a submanifold $M_2 \subset V_n(\iota^*\mathcal{C}_0)$ that is some component of $V_n(\iota^*\mathcal{C}_0)$. You keep repeating this process until either you get to a stage M_k where $V_n(\iota^*\mathcal{C}_{k-1})$ is empty, in which case there aren't any integral manifolds of this kind, or else, eventually, this will have to result in an involutive system, in which case you can apply the Cartan-Kähler Theorem (if the system that you started with is real analytic).

The main difficulty that you'll run into is that the spaces $V_n(\mathcal{I})$ can be quite wild and hard to describe. I don't want to dismiss this as a trivial problem, but it really is an algebra problem, in a sense. The other difficulty is that the the components $M_1 \subset V_n(\mathcal{I})$ might not submerse onto $M_0 = M$, but onto some proper submanifold, in which case, you'll have to restrict to that submanifold and start over.

In the case that the original EDS $(M, \mathcal{I})$ is real analytic, the set $V_n(\mathcal{I}) \subset G_n(TM)$ will also be real analytic and so has a canonical stratification into submanifolds

$$V_n(\mathcal{I}) = \bigcup_{\beta \in B} Z_\beta \,.$$

One can then consider the family of prolongations $(Z_\beta, \mathcal{I}_\beta^{(1)})$ and analyse each one separately. (Fortunately, in all the interesting cases I'm aware of, the number of strata is mercifully small.)

Now, there are precise, though somewhat technical, hypotheses that will ensure that this *prolongation Ansatz*, when iterated and followed down all of its various branches, terminates after a finite number of steps, with the result being a finite (possibly empty) set of EDSs $\big\{\,(M_\gamma, \mathcal{I}_\gamma)\,\big|\,\gamma \in \Gamma\,\big\}$ that are involutive. This result (with the explicit technical hypotheses) is due to Kuranishi and is known as the Cartan-Kuranishi Prolongation Theorem. (Cartan had conjectured/stated this result in his earlier writings, but never provided adequate justification for his claims.) In practice, though, Kuranishi's result is used more as a justification for carrying out the process of prolongation as part of the analysis of an EDS, when it is necessary.

Exercise 7.13: Analyse the system

$$\frac{\partial^n z}{\partial x^n} = f(x, y), \qquad\qquad \frac{\partial^n z}{\partial y^n} = g(x, y),$$

and explain why you'll have to prolong it $(n-1)$ times before you reach either a system with no 2-dimensional integral elements or one that has 2-dimensional integral elements that can be reached by a regular flag. In the latter case, do you actually need the full Cartan-Kähler Theorem to analyse the solutions?

Exercise 7.14: Analyse the system for $u(x, y, z)$ given by

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = u, \qquad\qquad \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = u.$$

Show that the natural system generated by four 1-forms you would write down on $\mathbb{R}^{11}$ to model the solutions is not involutive but that its first prolongation is. How much data do you get to specify in a solution?

This is the Maple file that I used to compute the prolongations of the exterior differential system whose integral manifolds model the surfaces in 3-space with the property that the principal curvatures are contstant along their principal curves. You need to read the treatment of this example in Lecture 6 to see what things mean.

The first thing we do is load in the difforms package.

```
> with(difforms);
```

$$[\&\wedge, d, \mathit{defform}, \mathit{formpart}, \mathit{parity}, \mathit{scalarpart}, \mathit{simpform}, \mathit{wdegree}]$$

Now I want to set up the structure equations and define the variables and forms that I want to use. Note that I have to tell it explicitly that w is skew-symmetric. (The extra single quotes around things are just so that I can repeat this assignment if I decide to change my notation. It's a Maple thing, so don't worry about it right now.)

```
> omega = table(): kappa := table():
  q := table():  theta := table():
  assign('omega[1,2]'=-omega[2,1],
         'omega[1,3]'=-omega[3,1],
         'omega[2,3]'=-omega[3,2],
         ''omega[i,i]'=0'$'i'=1..3);
  defform(
  omega=1,theta=1,kappa=0,q=0,
  ''d(omega[i])'
      =sum('-&^(omega[i,j],omega[j])','j'=1..3)'$'i'=1..3,
  'd(omega[2,1])'=-omega[2,3] &^ omega[3,1],
  'd(omega[3,1])'=-omega[3,2] &^ omega[2,1],
  'd(omega[3,2])'=-omega[3,1] &^ omega[1,2]);
```

Set up the ideal on the space of integral elements. We list the 1-forms that generate the ideal:

```
> Ideal[1] := [
  theta[0] = omega[3],
  theta[1] = omega[3,1]-kappa[1]*omega[1],
  theta[2] = omega[3,2]-kappa[2]*omega[2],
  theta[3] = d(kappa[1])-(kappa[1]-kappa[2])*q[1]*omega[2],
  theta[4] = omega[2,1] - q[1]*omega[1]-q[2]*omega[2],
  theta[5] = d(kappa[2])-(kappa[1]-kappa[2])*q[2]*omega[1]
  ];
```

$$\mathit{Ideal}_1 := [\theta_0 = \omega_3, \theta_1 = \omega_{3,1} - \kappa_1\,\omega_1, \theta_2 = \omega_{3,2} - \kappa_2\,\omega_2, \theta_3 = \mathrm{d}(\kappa_1) - (\kappa_1 - \kappa_2)\,q_1\,\omega_2,$$
$$\theta_4 = \omega_{2,1} - q_1\,\omega_1 - q_2\,\omega_2, \theta_5 = \mathrm{d}(\kappa_2) - (\kappa_1 - \kappa_2)\,q_2\,\omega_1]$$

Since I'm going to need to back substitute to do computations modulo the theta[i], I go ahead and figure out how to express everything in terms of the theta[i] and omega[1] and omega[2]:

```
> [op(solve({op(Ideal[1])},
  {omega[3],omega[3,1],omega[3,2],omega[2,1],
   d(kappa[1]),d(kappa[2])} ) ) ]:
  ReverseIdeal[1] := map(simpform,%);
```

$$\mathit{ReverseIdeal}_1 := [\omega_3 = \theta_0, \omega_{3,2} = \theta_2 + \kappa_2\,\omega_2, \omega_{3,1} = \theta_1 + \kappa_1\,\omega_1, \omega_{2,1} = \theta_4 + q_1\,\omega_1 + q_2\,\omega_2,$$
$$\mathrm{d}(\kappa_1) = (q_1\,\kappa_1 - q_1\,\kappa_2)\,\omega_2 + \theta_3, \mathrm{d}(\kappa_2) = (q_2\,\kappa_1 - q_2\,\kappa_2)\,\omega_1 + \theta_5]$$

Now we check on the exterior derivatives of these 1-forms, which have to be added to the ideal Ideal[1]. We know that d(theta[i]) is in

Ideal[1] for i=0,1,2 for theoretical reasons, so we start with:

```
> d(subs(Ideal[1],theta[3])):
  subs(ReverseIdeal[1],%): subs(['theta[i]=0'$'i'=0..5],%):
  Theta[3] := simpform(%);
```

$$\Theta_3 := (-\kappa_1 + \kappa_2)\,(\mathrm{d}(q_1) \mathbin{\&\wedge} \omega_2)$$

```
> d(subs(Ideal[1],theta[5])):
  subs(ReverseIdeal[1],%): subs(['theta[i]=0'$'i'=0..5],%):
  Theta[5] := simpform(%);
```

$$\Theta_5 := (-\kappa_1 + \kappa_2)\,(\mathrm{d}(q_2) \mathbin{\&\wedge} \omega_1)$$

```
> d(subs(Ideal[1],theta[4])):
  subs(ReverseIdeal[1],%): subs(['theta[i]=0'$'i'=0..5],%):
  Theta[4] := simpform(%);
```

$$\Theta_4 := (\kappa_2\,\kappa_1 + q_1^{\,2} + q_2^{\,2})\,(\omega_2 \mathbin{\&\wedge} \omega_1) - (\mathrm{d}(q_1) \mathbin{\&\wedge} \omega_1) - (\mathrm{d}(q_2) \mathbin{\&\wedge} \omega_2)$$

It follows that on an integral element d(q[1]) will be a multiple of omega[2] and d(q[2]) will be a multiple of omega[1]. These two multiples are related by a single equation (coming from annihlating Theta[4]), so we get a solution in the form below (I have chosen q[3] so that the formulas will be as symmetric as possible). Now we check that they work:

```
> subs(
  d(q[1])=(q[3]+q[1]^2+kappa[1]*kappa[2]/2)*omega[2],
  d(q[2])=(q[3]-q[2]^2-kappa[1]*kappa[2]/2)*omega[1],
  [Theta[3],Theta[4],Theta[5]] ):
  simpform(%);
```

$$[0, 0, 0]$$

Now we add the two 1-forms to the ideal with the extra parameter q[3] that parametrizes these integral elements:

```
> Ideal[2] := [op(Ideal[1]),
  theta[6]=d(q[1])-(q[3]+q[1]^2+kappa[1]*kappa[2]/2)*omega[2],
  theta[7]=d(q[2])-(q[3]-q[2]^2-kappa[1]*kappa[2]/2)*omega[1]
  ];
```

$$\mathit{Ideal}_2 := \Big[\theta_0 = \omega_3,\ \theta_1 = \omega_{3,1} - \kappa_1\,\omega_1,\ \theta_2 = \omega_{3,2} - \kappa_2\,\omega_2,\ \theta_3 = \mathrm{d}(\kappa_1) - (\kappa_1 - \kappa_2)\,q_1\,\omega_2,$$
$$\theta_4 = \omega_{2,1} - q_1\,\omega_1 - q_2\,\omega_2,\ \theta_5 = \mathrm{d}(\kappa_2) - (\kappa_1 - \kappa_2)\,q_2\,\omega_1,\ \theta_6 = \mathrm{d}(q_1) - \left(q_3 + q_1^{\,2} + \frac{1}{2}\,\kappa_2\,\kappa_1\right)\omega_2,$$
$$\theta_7 = \mathrm{d}(q_2) - \left(q_3 - q_2^{\,2} - \frac{1}{2}\,\kappa_2\,\kappa_1\right)\omega_1\Big]$$

```
> [op(solve({op(Ideal[2])},
  {omega[3],omega[3,1],omega[3,2],omega[2,1],
  d(kappa[1]),d(kappa[2]),d(q[1]),d(q[2])}))]:
  ReverseIdeal[2] := map(simpform,%);
```

$$\mathit{ReverseIdeal}_2 := \Big[\mathrm{d}(q_1) = \left(q_3 + q_1^{\,2} + \frac{1}{2}\,\kappa_2\,\kappa_1\right)\omega_2 + \theta_6,\ \omega_3 = \theta_0,\ \omega_{3,2} = \theta_2 + \kappa_2\,\omega_2,$$
$$\omega_{3,1} = \theta_1 + \kappa_1\,\omega_1,\ \omega_{2,1} = \theta_4 + q_1\,\omega_1 + q_2\,\omega_2,\ \mathrm{d}(\kappa_1) = (q_1\,\kappa_1 - q_1\,\kappa_2)\,\omega_2 + \theta_3,$$

$$d(\kappa_2) = (q_2\,\kappa_1 - q_2\,\kappa_2)\,\omega_1 + \theta_5,\ d(q_2) = \left(q_3 - q_2^{\ 2} - \frac{1}{2}\,\kappa_2\,\kappa_1\right)\omega_1 + \theta_7\Bigg]$$

Now we check on the exterior derivatives of these 1-forms,
which have to be added to the ideal Ideal[2]. We know that d(theta[i]) is in
Ideal[2] for i=0,1,2,3,4,5 for theoretical reasons, so we start with:

```
> d(subs(Ideal[2],theta[6])):
  subs(ReverseIdeal[2],%): subs(['theta[i]=0'$'i'=0..7],%):
  Theta[6] := simpform(%);
```

$$\Theta_6 := \left(q_2\,q_3 + q_2\,q_1^{\ 2} + \frac{1}{2}\,q_2\,\kappa_1^{\ 2}\right)(\omega_2\ \&\!\wedge\ \omega_1) - (d(q_3)\ \&\!\wedge\ \omega_2)$$

```
> d(subs(Ideal[2],theta[7])):
  subs(ReverseIdeal[2],%): subs(['theta[i]=0'$'i'=0..7],%):
  Theta[7] := simpform(%);
```

$$\Theta_7 := \left(-\frac{1}{2}\,q_1\,\kappa_2^{\ 2} + q_1\,q_3 - q_1\,q_2^{\ 2}\right)(\omega_2\ \&\!\wedge\ \omega_1) - (d(q_3)\ \&\!\wedge\ \omega_1)$$

It follows that on an integral element d(q[3]) is a linear combination of omega[1]
and omega[2] and these coefficients are determined by the requirement that it
annihilate Theta[6] and Theta[7]. Here, we check, that this works:

```
> subs(
  d(q[3])
  = -(q[2]*q[3]+q[2]*q[1]^2+1/2*kappa[1]^2*q[2])*omega[1]
    +(q[1]*q[3]-q[1]*q[2]^2-1/2*kappa[2]^2*q[1])*omega[2],
  [Theta[6],Theta[7]]):simpform(%);
```

$$[0,\ 0]$$

Now we add this expression that vanishes on all integral elements into the ideal:

```
> Ideal[3] := [op(Ideal[2]),
  theta[8]=d(q[3])
  +(q[2]*q[3]+q[2]*q[1]^2+1/2*kappa[1]^2*q[2])*omega[1]
  -(q[1]*q[3]-q[1]*q[2]^2-1/2*kappa[2]^2*q[1])*omega[2]
  ];
```

$$\mathit{Ideal}_3 := \Big[\theta_0 = \omega_3,\ \theta_1 = \omega_{3,\,1} - \kappa_1\,\omega_1,\ \theta_2 = \omega_{3,\,2} - \kappa_2\,\omega_2,\ \theta_3 = d(\kappa_1) - (\kappa_1 - \kappa_2)\,q_1\,\omega_2,$$

$$\theta_4 = \omega_{2,\,1} - q_1\,\omega_1 - q_2\,\omega_2,\ \theta_5 = d(\kappa_2) - (\kappa_1 - \kappa_2)\,q_2\,\omega_1,\ \theta_6 = d(q_1) - \left(q_3 + q_1^{\ 2} + \frac{1}{2}\,\kappa_2\,\kappa_1\right)\omega_2,$$

$$\theta_7 = d(q_2) - \left(q_3 - q_2^{\ 2} - \frac{1}{2}\,\kappa_2\,\kappa_1\right)\omega_1,$$

$$\theta_8 = d(q_3) + \left(q_2\,q_3 + q_2\,q_1^{\ 2} + \frac{1}{2}\,q_2\,\kappa_1^{\ 2}\right)\omega_1 - \left(-\frac{1}{2}\,q_1\,\kappa_2^{\ 2} + q_1\,q_3 - q_1\,q_2^{\ 2}\right)\omega_2\Big]$$

```
> [op(solve({op(Ideal[3])},
  {omega[3],omega[3,1],omega[3,2],omega[2,1],
  d(kappa[1]),d(kappa[2]),d(q[1]),d(q[2]),d(q[3])}))]:
  ReverseIdeal[3] := map(simpform,%);
```

$$ReverseIdeal_3 := \left[d(q_1) = \left(q_3 + q_1^2 + \frac{1}{2}\kappa_2\,\kappa_1\right)\omega_2 + \theta_6,\ \omega_3 = \theta_0,\ \omega_{3,2} = \theta_2 + \kappa_2\,\omega_2,\right.$$
$$\omega_{3,1} = \theta_1 + \kappa_1\,\omega_1,\ \omega_{2,1} = \theta_4 + q_1\,\omega_1 + q_2\,\omega_2,\ d(\kappa_1) = (q_1\,\kappa_1 - q_1\,\kappa_2)\,\omega_2 + \theta_3,$$
$$d(\kappa_2) = (q_2\,\kappa_1 - q_2\,\kappa_2)\,\omega_1 + \theta_5,\ d(q_2) = \left(q_3 - q_2^2 - \frac{1}{2}\kappa_2\,\kappa_1\right)\omega_1 + \theta_7,$$
$$\left. d(q_3) = \left(-\frac{1}{2}q_1\,\kappa_2^2 + q_1\,q_3 - q_1\,q_2^2\right)\omega_2 + \left(-q_2\,q_3 - q_2\,q_1^2 - \frac{1}{2}q_2\,\kappa_1^2\right)\omega_1 + \theta_8 \right]$$

Now we check on the exterior derivatives of this 1-form theta[8],
which has to be added to the ideal Ideal[3]. We know that d(theta[i]) is in
Ideal[3] for i=0,1,2,3,4,5,6,7 for theoretical reasons, so we just need to compute

```
> d(subs(Ideal[3],theta[8])):
  subs(ReverseIdeal[3],%): subs(['theta[i]=0'$'i'=0..8],%):
  Theta[8] := simpform(%);
```

$$\Theta_8 := 0$$

```
>
```

It follows that Ideal[3] is a Frobenius system, so that the underlying 11-manifold is
foliated by 2-dimensional integral manifolds of Ideal[3].

Lecture 9. Applications 3: Geometric Systems Needing Prolongation

9.1. Orthogonal coordinates in dimension n.

In this example, I take up the question of orthogonal coordinates in general dimensions, as opposed to dimension 3, as was discussed in Lecture 5.

Let N be a manifold of dimension n endowed with a Riemannian metric g. If $U \subset N$ is an open set, a coordinate chart $(x^1, \ldots, x^n) : U \to \mathbb{R}^n$ is said to be *orthogonal* if, on U,

$$g = g_{11}\,(dx^1)^2 + g_{22}\,(dx^2)^2 + \cdots + g_{nn}\,(dx^n)^2,$$

i.e., if the coordinate expression $g = g_{ij}\,dx^i\,dx^j$ satisfies $g_{ij} = 0$ for i different from j. This is $\binom{n}{2}$ equations for the n coordinate functions x^i. When $n > 3$, this is an overdetermined system and one should not expect there to be solutions. Indeed, very simple examples in dimension 4 show that there are metrics for which there are no orthogonal coordinates, even locally. (I'll say more about this below.)

I want to describe an EDS whose n-dimensional integral manifolds describe the solutions to this problem. Note that, if you have a solution, then the 1-forms $\eta_i = \sqrt{g_{ii}}\,dx^i$ form an orthonormal coframing, i.e.,

$$g = {\eta_1}^2 + {\eta_2}^2 + \cdots + {\eta_n}^2.$$

This coframing is not the most general orthonormal coframing, though, because it satisfies $\eta_i{\wedge}d\eta_i = 0$ since each η_i is a multiple of an exact 1-form. Conversely, any g-orthonormal coframing $(\eta_1, \ldots, \eta_n)$ that satisfies $\eta_i{\wedge}d\eta_i = 0$ for $i = 1, \ldots, n$ is locally of the form $\eta_i = A_i\,dx_i$ for some functions $A_i > 0$ and x^i, by the Frobenius Theorem. (Why?)

Thus, up to an application of the Frobenius Theorem, the problem of finding g-orthogonal coordinates is equivalent to finding g-orthonormal coframings $(\eta_1, \ldots, \eta_n)$ satisfying $\eta_i{\wedge}d\eta_i = 0$. I now want to set up an exterior differential system whose integral manifolds reprsent these coframings.

To do this, let $\pi : F \to N$ be the g-orthonormal coframe bundle of N, i.e, a point of F is of the form $f = (x, u_1, \ldots, u_n)$ where $x = \pi(f)$ belongs to N and $u_i \in T_xN$ are g-orthonormal. This is an $\mathrm{O}(n)$-bundle over N and hence is a manifold of dimension $n + \binom{n}{2}$. There are the canonical 1-forms $\omega_1, \ldots, \omega_n$ on F that satisfy

$$\omega_i(v) = u_i\bigl(\pi'(v)\bigr), \qquad \text{for all } v \in T_fM \text{ with } f = (x, u_1, \ldots, u_n).$$

These 1-forms have the 'reproducing property' that, if $\eta = (\eta_1, \ldots, \eta_n)$ is a g-orthonormal coframing on $U \subset N$, then regarding η as a section of F over U via the map

$$\sigma_\eta(x) = \bigl(\,x,\,(\eta_1)_x,\,\ldots\,(\eta_n)_x\,\bigr),$$

we have $\sigma_\eta^*(\omega_1, \ldots, \omega_n) = (\eta_1, \ldots, \eta_n)$.

Exercise 9.1: Prove this statement. Prove also that a n-dimensional submanifold $P \subset F$ can be locally represented as the graph of a local section $\sigma : U \to F$ if and only if $\omega_1{\wedge}\cdots{\wedge}\omega_n$ is nonvanishing on P.

Consider the ideal $\mathcal{I} = \langle\,\omega_1{\wedge}d\omega_1,\,\ldots,\,\omega_n{\wedge}d\omega_n\,\rangle$ defined on F. The n-dimensional integral manifolds of $\mathcal{I}$ on which $\omega_1{\wedge}\cdots{\wedge}\omega_n$ is nonvanishing are then the desired local sections. We now want to describe these integral manifolds, so we start by looking at the integral elements.

Now, by the classical Levi-Civita existence and uniqueness theorem, there are unique 1-forms $\omega_{ij} = -\omega_{ji}$ that satisfy the structure equations

$$d\omega_i = -\sum_{j=1}^{n} \omega_{ij} \wedge \omega_j\,.$$

The 1-forms ω_i, ω_{ij} $(i < j)$ are then a basis for the 1-forms on F.

By the structure equations, an alternative description of $\mathcal{I}$ is

$$\mathcal{I} = \left\langle\ \omega_1 \wedge \Bigl(\sum_{j=1}^{n} \omega_{1j} \wedge \omega_j\Bigr), \ldots, \omega_n \wedge \Bigl(\sum_{j=1}^{n} \omega_{nj} \wedge \omega_j\Bigr)\ \right\rangle.$$

Let $G_n(TF, \omega)$ denote the set of tangent n-planes on which $\omega_1 \wedge \cdots \wedge \omega_n$ is nonvanishing. Any $E \in G_n(TF, \omega)$ is defined by equations of the form

$$\omega_{ij} = \sum_{k=1}^{n} p_{ijk}\, \omega_k$$

Such an n-plane will be an integral element if and only if the $p_{ijk} = -p_{jik}$ (which are $n\binom{n}{2}$ in number) satisfy the equations

$$0 = \omega_i \wedge \big(\sum_{j=1}^{n} \omega_{ij} \wedge \omega_j\big) = \omega_i \wedge \big(\sum_{j,k=1}^{n} p_{ijk}\, \omega_k \wedge \omega_j\big) \qquad \text{for } i = 1, \ldots, n.$$

Exercise 9.2: Show that these conditions imply that $p_{ijk} = 0$ unless k is equal to i or j and then that every integral element is defined by equations of the form

$$\omega_{ij} = p_{ij}\, \omega_i - p_{ji}\, \omega_j$$

where the $n(n-1)$ numbers $\{\, p_{ij} \,|\, i \neq j \,\}$ are arbitrary. Explain why the p_{ii} don't matter, and conclude that the codimension of the space $V_n(\mathcal{I}, \omega)$ in $G_n(TF, \omega)$ is

$$n\binom{n}{2} - n(n-1) = \tfrac{1}{2}n(n-1)(n-2).$$

Now, to check Cartan's Test, we need to compute the polar spaces of some flag in $E = E_n$. We already know from Lecture 5 that there are regular flags when $n = 3$, so we might as well assume that $n > 3$ from now on. I am going to argue that, in this case, there cannot be a regular flag, so Cartan-Kähler cannot be applied and we must prolong.

Let $F = (E_0, E_1, \ldots, E_n)$ be any flag. Because $\mathcal{I}$ is generated by n 3-forms, it follows that $c(E_0) = c(E_1) = 0$ and that $c(E_2) \leq n$. Moreover, because E_n has codimension $\frac{1}{2}n(n-1)$, it follows that $c(E_k) \leq \frac{1}{2}n(n-1)$ for all k. Combining these, we see that

$$c(F) \leq c(E_0) + \cdots + c(E_{n-1}) \leq 0 + 0 + n + (n-3) \cdot \tfrac{1}{2}n(n-1).$$

When $n > 3$, this last number is strictly *less* than $\frac{1}{2}n(n-1)(n-2)$, the codimension of $V_n(\mathcal{I}, \omega)$ in $G_n(TF, \omega)$ that we computed above. Thus Cartan's Test shows that the flag F is not regular.

Thus, if we want to find solutions, we will have to prolong. We make a new manifold $F^{(1)} = F \times \mathbb{R}^{n(n-1)}$, with $\{\, p_{ij} \,|\, i \neq j \,\}$ as coordinates on the second factor, and define $\mathcal{I}^{(1)}$ to be the ideal generated by the $\binom{n}{2}$ 1-forms

$$\theta_{ij} = \omega_{ij} - p_{ij}\, \omega_i + p_{ji}\, \omega_j \,.$$

Of course, if we are going to study the algebraic properties of this ideal, we are going to have to know $d\theta_{ij}$ and this will require that we know $d\omega_{ij}$. Now, the second structure equations of Cartan are

$$d\omega_{ij} = -\sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} + \tfrac{1}{2} \sum_{k,l=1}^{n} R_{ijkl}\, \omega_k \wedge \omega_l \,,$$

where the functions R_{ijkl} are the Riemann curvature functions.

Now, using this, if you compute, you will get

$$d\theta_{ij} \equiv \tfrac{1}{2} \sum_{k,l=1}^{n} R_{ijkl}\, \omega_k \wedge \omega_l - \pi_{ij} \wedge \omega_i + \pi_{ji} \wedge \omega_j \mod \{\theta_{kl}\}_{k<l}$$

for some 1-forms π_{ij} $(i \neq j)$, with $\pi_{ij} \equiv dp_{ij} \mod \{\omega_1, \ldots, \omega_n\}$.

Right away, this says that there is trouble: If there is a point $f \in F$ for which there exist (i, j, k, l) distinct and $R_{ijkl}(f) \neq 0$, then the prolonged ideal will not have any integral elements passing through f on which $\omega_1 \wedge \ldots \wedge \omega_n$ is nonzero. (Why not?)

Now, it turns out that the functions R_{ijkl} with (i, j, k, l) distinct are all identically zero if and only if the Weyl curvature of the metric g vanishes, i.e., (since $n \geq 4$) if and only if g is conformally flat. Since orthogonal coordinates don't care about conformal factors (why not?), if we are going to restrict to the conformally flat case, then we might as well go whole hog and restrict to the flat case, i.e., the case where $R_{ijkl} = 0$ for all quadruples of indices. In this case, the structure equations of $\mathcal{I}^{(1)}$ become

$$d\theta_{ij} \equiv -\pi_{ij} \wedge \omega_i + \pi_{ji} \wedge \omega_j \mod \{\theta_{kl}\}_{k<l}$$

for some 1-forms π_{ij} $(i \neq j)$, with $\pi_{ij} \equiv dp_{ij} \mod \{\omega_1, \ldots, \omega_n\}$.

Exercise 9.3: Use these structure equations to show that $\left(F^{(1)}, \mathcal{I}^{(1)}\right)$ is involutive, with Cartan characters

$$(s_0, s_1, \cdots, s_n) = (\tfrac{1}{2}n(n-1), \tfrac{1}{2}n(n-1), \tfrac{1}{2}n(n-1), 0, 0, \ldots, 0).$$

In particular, the last nonzero Cartan character is $s_2 = \frac{1}{2}n(n{-}1)$. Explain the geometric meaning of this result: How much freedom do you get in constructing local orthogonal coordinates on $\mathbb{R}^n$?

Exercise 9.4: (*somewhat nontrivial*) Using the above analysis as starting point, show that the Fubini-Study metric g on $\mathbb{CP}^2$ does not allow any orthogonal coordinate systems, even locally.

9.2. Isometric Embedding of Surfaces with Prescribed Mean Curvature

Consider a given abstract oriented surface N^2 endowed with a Riemannian metric g and a choice of a smooth function H. The question we ask is this: When does there exist an isometric embedding $x : N^2 \to \mathbb{R}^3$ such that the mean curvature function of the immersion is H? If you think about it, this is four equations for the map x (which has three components), three of first order (the isometric embedding condition) and one of second order (the mean curvature restriction).

Since $H^2 - K = (\kappa_1 {-} \kappa_2)^2 \geq 0$ for any surface in 3-space, one obvious restriction coming from the Gauss equation is that $H^2 - K$ must be nonnegative, where K is the Gauss curvature of the metric g. I'm just going to treat the case where $H^2 - K$ is strictly positive, though there are methods for dealing with the 'umbilic locus' (I just don't want to bother with them here). In fact, set $r = \sqrt{H^2 - K} > 0$.

The simplest way to set up the problem is to begin by fixing an oriented, g-orthonormal coframing (η_1, η_2), with dual frame field (u_1, u_2). We know that there exists a unique 1-form η_{12} so that

$$d\eta_1 = -\eta_{12} \wedge \eta_2, \qquad d\eta_2 = \eta_{12} \wedge \eta_1, \qquad d\eta_{12} = K\, \eta_1 \wedge \eta_2\,.$$

Now, any solution $x : N \to \mathbb{R}^3$ of our problem will define a lifting $f : N \to F$ (the oriented orthonormal frame bundle of $\mathbb{R}^3$) via

$$f = \left[\; x \quad x'(u_1) \quad x'(u_2) \quad x'(u_1){\times}x'(u_2) \;\right]$$

Of course, this will mean that

$$\begin{aligned} f^*\omega_3 &= 0 \\ f^*\omega_1 &= \eta_1 \\ f^*\omega_1 &= \eta_2 \\ f^*\omega_{31} &= h_{11}\,\eta_1 + h_{12}\,\eta_2 \\ f^*\omega_{32} &= h_{12}\,\eta_1 + h_{22}\,\eta_2 \end{aligned}$$

where $h_{11} + h_{22} = 2H$. We also know, by the uniqueness of the Levi-Civita connection, that

$$f^*\omega_{12} = \eta_{12}$$

and the Gauss equation tells us that $h_{11}h_{22} - {h_{12}}^2 = K$. This is two algebraic equations for the three h_{ij}. Because $H^2 - K = r^2 > 0$, these can be solved in terms of an extra parameter in the form

$$\begin{aligned} h_{11} &= H + r\,\cos\phi \\ h_{12} &= \phantom{H + {}} r\,\sin\phi \\ h_{22} &= H - r\,\cos\phi\,. \end{aligned}$$

This suggests setting up the following exterior differential system for the 'graph' of f in $N \times F$. Let $M = N \times F \times S^1$, with ϕ being the 'coordinate' on the S^1 factor and consider the ideal $\mathcal{I}$ generated by the five 1-forms

$$
\begin{aligned}
\theta_0 &= \omega_3 \\
\theta_1 &= \omega_1 - \eta_1 \\
\theta_2 &= \omega_2 - \eta_2 \\
\theta_3 &= \omega_{12} - \eta_{12} \\
\theta_4 &= \omega_{31} - (H{+}r\cos\phi)\,\eta_1 - r\,\sin\phi\,\eta_2 \\
\theta_5 &= \omega_{32} - r\,\sin\phi\,\eta_1 - (H{-}r\cos\phi)\,\eta_2
\end{aligned}
$$

It's easy to see (and you should check) that

$$d\theta_0 \equiv d\theta_1 \equiv d\theta_2 \equiv d\theta_3 \equiv 0 \mod \{\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}.$$

The interesting case will come when we look at the other two 1-forms. In fact, the formula for these is simply

$$\left.\begin{aligned}
d\theta_4 &\equiv \quad r\tau \wedge (\sin\phi\,\eta_1 - \cos\phi\,\eta_2) \\
d\theta_5 &\equiv -r\tau \wedge (\cos\phi\,\eta_1 + \sin\phi\,\eta_2)
\end{aligned}\right\} \mod \{\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$$

where, setting $dr = r_1\,\eta_1 + r_2\,\eta_2$ and $dH = H_1\,\eta_1 + H_2\,\eta_2$,

$$
\begin{aligned}
\tau = d\phi - 2\,\eta_{12} - r^{-1}(r_2 + H_2\cos\phi - H_1\sin\phi)\,\eta_1 \\
+ r^{-1}(r_1 - H_1\cos\phi - H_2\sin\phi)\,\eta_2\,.
\end{aligned}
$$

It is clear that there is a unique integral element at each point of M and that it is described by $\theta_0 = \cdots = \theta_5 = \tau = 0$. Thus, $M^{(1)} = M$ and

$$\mathcal{I}^{(1)} = \langle\, \theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \tau \,\rangle.$$

To get the structure of $\mathcal{I}^{(1)}$ is is only necessary to compute $d\tau$ now and the result of that is

$$d\tau \equiv r^{-2}\bigl(C\,\cos\phi + S\,\sin\phi + T\bigr)\,\eta_1 \wedge \eta_2 \mod \{\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \tau\}$$

where the functions C, S, and T are defined on the surface by

$$
\begin{aligned}
C &= 2r_1H_1 - 2r_2H_2 - rH_{11} + rH_{22}\,, \\
S &= 2r_2H_1 + 2r_1H_2 - 2rH_{12}\,, \\
T &= 2r^4 - 2H^2r^2 + r(r_{11}{+}r_{22}) - {r_1}^2 - {r_2}^2 - {H_1}^2 - {H_2}^2\,.
\end{aligned}
$$

and I have defined H_{ij} and r_{ij} by the equations

$$
\begin{aligned}
dH_1 &= -H_2\,\eta_{12} + H_{11}\,\eta_1 + H_{12}\,\eta_2\,, \\
dH_2 &= \quad H_1\,\eta_{12} + H_{12}\,\eta_1 + H_{22}\,\eta_2\,, \\
dr_1 &= -r_2\,\eta_{12} + r_{11}\,\eta_1 + r_{12}\,\eta_2\,, \\
dr_2 &= \quad r_1\,\eta_{12} + r_{12}\,\eta_1 + r_{22}\,\eta_2\,.
\end{aligned}
$$

Exercise 9.5: Why do such functions H_{ij} and r_{ij} exist? (What you need to explain is why H_{12} and r_{12} can appear in two places in these formulae.)

Clearly, there are no integral elements of $\mathcal{I}^{(1)}$ except along the locus where $C\,\cos\phi + S\,\sin\phi + T = 0$, so it's a question of what this locus looks like.

First, off, note that if $T^2 > S^2 + C^2$, then this locus is empty. Now, this inequality is easily seen not to depend on the choice of coframing (η_1, η_2) that we made to begin with. It depends only on the metric g and the function H. One way to think of this is that the condition $T^2 \le S^2 + C^2$ is a differential inequality any g and H satisfy if they are the metric and mean curvature of a surface in $\mathbb{R}^3$.

Now, when $T^2 < C^2+S^2$, there will be exactly two values of ϕ (mod 2π) that satisfy $C\,\cos\phi+S\,\sin\phi+T=0$, say ϕ_+ and ϕ_-, thought of as functions on the surface N. If you restrict to this double cover $\phi=\phi_\pm$, we now have an ideal $\mathcal{I}^{(1)}$ on an 8-manifold that is generated by seven 1-forms. In fact, $\theta_0, \ldots, \theta_5$ are clearly independent, but now

$$\tau = E_1\,\eta_1 + E_2\,\eta_2$$

where E_1 and E_2 are functions on the surface $\tilde{N} \subset N\times S^1$ defined by the equation $C\,\cos\phi+S\,\sin\phi+T=0$. Wherever either of these functions is nonzero, there is clearly no solution. On the other hand, if $E_1 = E_2 = 0$ on $\tilde{N}$, then there are exactly two geometrically distinct ways for the surface to be isometrically embeded with mean curvature H. If you unravel this, you will see that it is a pair of fifth order equations on the pair (g, H). (The expressions T and $S^2{+}C^2$ are fourth order in g and second order in H. Why?)

Exercise 9.6: (*somewhat nontrivial*) See if you can reproduce Cartan's result that the set of surfaces that admit two geometrically distinct isometric embeddings with the same mean curvature depend on four functions of one variable. (In the literature, such pairs of surfaces are known as *Bonnet pairs* after O. Bonnet, who first studied them.)

Another possibility is that $T=C=S=0$, in which case $\mathcal{I}^{(1)}$ becomes Frobenius.

Exercise 9.7: Explain why $T=C=S=0$ implies that the surface admits a one-parameter family (in fact, a circle) of geometrically distinct isometric embeddings with mean curvature H.

Of course, this raises the question of whether there exist any pairs (g, H) satisfying these equations. One way to try to satisfy the equations is to look for special solutions. For example, if H were constant, then H_1, H_2, H_{11}, H_{12}, and H_{22} would all be zero, of course, so this would automatically make $C=S=0$ and then there is only one more equation to satisfy, which can now be reëxpressed, using $K = H^2 - r^2$, as

$$T = r^2\big(\Delta_g \ln(H^2 - K) - 4K\big) = 0$$

where Δ_g is the Laplacian associated to g.

It follows that any metric g on a simply connected surface N that satisfies the fourth order differential equation $\Delta_g \ln(H^2-K) - 4K = 0$ can be isommetrically embedded in $\mathbb{R}^3$ as a surface of constant mean curvature H in a 1-parameter family (in fact, an S^1) of ways. In particular, we have Bonnet's Theorem: *Any simply connected surface in $\mathbb{R}^3$ with constant mean curvature can be isometrically deformed in an circle of ways preserving the constant mean curvature.*

However, the cases where H is constant give only one special class of solutions of the three equations $C = S = T = 0$. Could there be others?

Well, let's restrict to the open set $U \subset N$ where $dH \neq 0$, i.e., where ${H_1}^2 + {H_2}^2 > 0$. Remember, the original coframing (η_1, η_2) we chose was arbitrary, so we might as well use the nonconstancy of H to tack this down. In fact, let's take our coframing so that the dual frame field (u_1, u_2) has the property that u_1 points in the direction of steepest increase for H, i.e., in the direction of the gradient of H. This means that, for this coframing $H_2 = 0$ and $H_1 > 0$.

The equations $C=S=0$ now simplify to

$$H_{12} = (r_2/r)\,H_1, \qquad\qquad H_{11} - H_{22} = (2r_1/r)H_1\,.$$

Moreover, looking back at the structure equations found so far, this implies that $dH = H_1\,\eta_1$ and that there is a function P so that

$$\begin{aligned} {H_1}^{-1} dH_1 &= (rP + r_1/r)\,\eta_1 + (r_2/r)\,\eta_2,\ , \\ -\eta_{12} &= (r_2/r)\,\eta_1 + (rP - r_1/r)\,\eta_2\,. \end{aligned}$$

The first equation can be written in the form

$$d\big(\ln(H_1/r)\big) = rP\,\eta_1.$$

Differentiating this and using the structure equations we have so far then yields that $dP\wedge\eta_1 = 0$, so that there is some λ so that $dP = \lambda\,\eta_1$. On the other hand, differentiating the second of the two equations above and using $T = 0$ to simplify the result, we see that the multiplier λ is determined. In fact, we must have

$$dP = (r^2H^2 + {H_1}^2 - r^4 - r^4P^2)\,\eta_1\,.$$

Differentiating this relation and using the equations we have found so far yields

$$0 = 2r^{-4}\big({H_1}^2 + r^2H^2\big)\,r_2\,\eta_1\wedge\eta_2\,.$$

In particular, we must have $r_2 = 0$. Of course, this simplifies the equations even further. Taking the components of $0 = dr_2 = r_1\eta_{12} + r_{11}\,\eta_1 + r_{22}\,\eta_2$ together with the equation $T = 0$ allows us to solve for r_{11}, r_{12}, and r_{22} in terms of $\{r, H, r_1, H_1, P\}$.

In fact, collecting all of this information, we get the following structure equations for any solution of our problem:

$$\begin{aligned}
d\eta_1 &= 0\\
d\eta_2 &= (rP - r_1/r)\,\eta_1\wedge\eta_2\\
dr &= r_1\,\eta_1\\
dH &= H_1\,\eta_1\\
dr_1 &= (2r^3 - 2H^2r + r_1rP - 2{r_1}^2/r - {H_1}^2/r)\,\eta_1\\
dH_1 &= H_1(rP + r_1/r)\,\eta_1\\
dP &= (r^2H^2 + {H_1}^2 - r^4 - r^4P^2)\,\eta_1
\end{aligned}$$

These may not look promising, but, in fact, they give a rather complete description of the pairs (g, H) that we are seeking. Suppose that N is simply connected. The first structure equation then says that $\eta_1 = dx$ for some function x, uniquely defined up to an additive constant. The last 5 structure equations then say that the functions (r, H, r_1, H_1, P) are solutions of the ordinary differential equation system

$$\begin{aligned}
r' &= r_1\\
H' &= H_1\\
r_1' &= (2r^3 - 2H^2r + r_1rP - 2{r_1}^2/r - {H_1}^2/r)\\
H_1' &= H_1(rP + r_1/r)\\
P' &= (r^2H^2 + {H_1}^2 - r^4 - r^4P^2)
\end{aligned}$$

Obviously, this defines a vector field on the open set in $\mathbb{R}^5$ defined by $r > 0$, and there is a four parameter family of integral curves of this vector field. Given a solution of this ODE system on some maximal x-interval, there will be a function F uniquely defined up to an additive constant so that

$$F' = (rP - r_1/r).$$

Now by the second structure equation, we have $d(e^{-F}\eta_2) = 0$, so that there must exist a function y on the surface N so that $\eta_2 = e^F\,dy$. Thus, in the (x, y)-coordinates, the metric is of the form

$$g = dx^2 + e^{2F(x)}\,dy^2$$

where (r, H, r_1, H_1, P, F) satisfy the above equations.

Exercise 9.8: Explain why this shows that the space of inequivalent solutions (g, H) with H nonconstant can be thought of as being of dimension 4. Also, note that the metric g has a symmetry, namely translation in y. Can you use this to understand the circle of isometric embeddings of (N, g) into $\mathbb{R}^3$ with mean curvature H? (Hint: Look back at the EDS analysis we did earlier and apply Bonnet's Theorem.

Exercise 9.9: Redo this analysis for isometric immersion with prescribed curvature in a 3-dimensional space form of constant sectional curvature c. Does anything significant change?

Exercise 9.10: (*somewhat nontrivial*) Regarding the equations $S = C = T = 0$ as a set of three partial differential equations for the pair (g, H), show that they are not involutive as they stand, carry out the prolongation process and show how the space of integral manifolds breaks into two distinct pieces because the space of integral elements has two distinct components at a certain level of prolongation. Show that one of these (the one corresponding to the case where H is constant) goes into involution right away, but that the other (corresponding to the Bonnet surfaces that we found above) takes considerably longer.

Exercise 9.11: (*also somewhat nontrivial*) Suppose that we want to isometrically embedd (N^2, g) into $\mathbb{R}^3$ in such a way that a given g-orthogonal coframing (η_1, η_2) defines the principal coframing. Set up the exterior differential system and carry out the prolongations to determine how many solutions to this problem there are in general and whether there are any special metrics and coframings for which there is a larger than expected space of solutions.