Motivic stable homotopy groups of spheres

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Basic setup

Fix a field F.

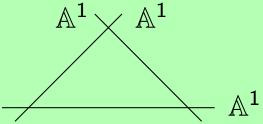
Morel-Voevodsky: It is possible to talk about the homotopy theory of "motivic spaces" over F, and also the associated stable homotopy theory of "motivic spectra".

$$
\begin{array}{ccccc}
 & & & & \color{blue}{X \mapsto X(\mathbb{C})} \\
 & & (\text{mot. spectra}/\mathbb{C}) & \longrightarrow & (\text{spectra}) \\
 & \nearrow & \uparrow & & \uparrow \\
 & & (\text{mot. spectra}/\mathbb{R}) & \longrightarrow & (\mathbb{Z}/2\text{-spectra}) \\
\color{red}{(\text{mot. spectra}/\mathbb{Z})} & \nearrow & \uparrow & & \\
\color{red}{?????} & \searrow & (\text{mot. spectra}/\mathbb{Q}) & & \\
 & \searrow & & & \\
 & & (\text{mot. spectra}/\mathbb{F}_q) & &
\end{array}
$$

Basic setup

In motivic homotopy theory we have a bigraded family of spheres:

$$S^{1,1} = \mathbb{A}^1 - 0 \qquad S^{1,0} =$$



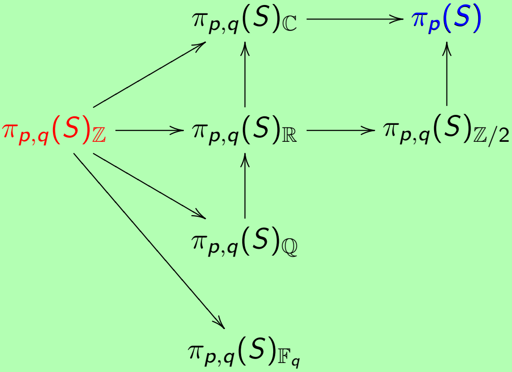
For $p \geq q$ define $\qquad S^{p,q} = (S^{1,1})^{\wedge(q)} \wedge (S^{1,0})^{\wedge(p-q)}$.

p is called the **topological dimension** of the sphere, and q is called the **weight**.

We have the sphere spectrum S, and we can talk about $\pi_{p,q}(S)$.

Should probably write $\pi_{p,q}(S)_F$ to keep track of the ground field.

Basic setup



Rough goal: Understand as much as we can about the different spots in this picture, and the maps between them.

Review of ordinary stable homotopy groups

- $\pi_i(S) = 0$ for $i < 0$ (connectivity)
- $\pi_0(S) \cong \mathbb{Z}$ (via the Hurewicz isomorphism)
- We have

i	0	1	2	3	4	5	6	7	8	9
π_i	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$
gen	1	η	η^2	ν			ν^2	σ	$\eta\sigma, ??$	$\nu^3, ??$

- The easiest elements to understand are the so-called *Hopf elements*: η, ν, and σ.

Classical Hopf elements

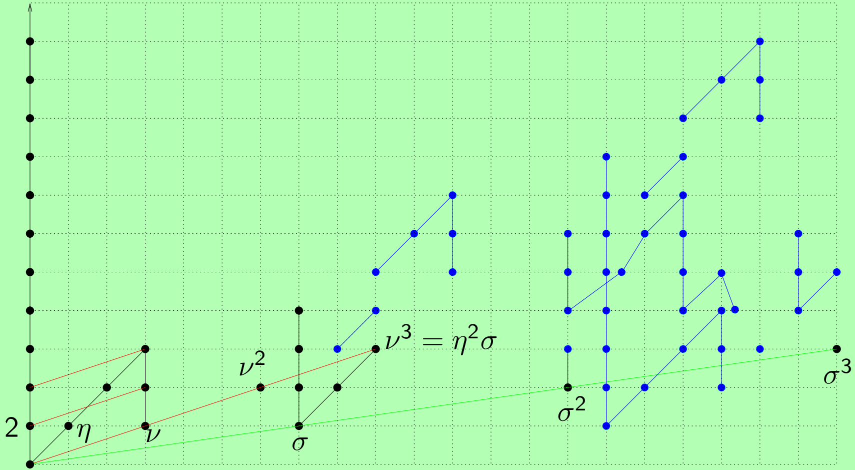
- These are the elements 2, η, ν, and σ:
  (i) $S^1 \simeq \mathbb{R}^2 - 0 \longrightarrow \mathbb{R}P^1 = S^1 \quad \rightsquigarrow \quad 2 \in \pi_0$
  (ii) $S^3 \simeq \mathbb{C}^2 - 0 \longrightarrow \mathbb{C}P^1 = S^2 \quad \rightsquigarrow \quad \eta \in \pi_1$
  (iii) $S^7 \simeq \mathbb{H}^2 - 0 \longrightarrow \mathbb{H}P^1 = S^4 \quad \rightsquigarrow \quad \nu \in \pi_3$
  (iv) $S^{15} \simeq \mathbb{O}^2 - 0 \longrightarrow \mathbb{O}P^1 = S^8 \quad \rightsquigarrow \quad \sigma \in \pi_7$
- The story stops here because there are no more division algebras continuing the sequence $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$.

Relations between the Hopf elements:

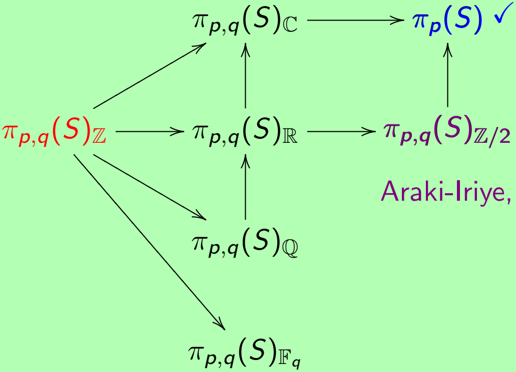
- $2\eta = 0$, $\eta\nu = 0$, $\nu\sigma = 0$
- $2\nu^2 = 0$, $\eta\sigma^2 = 0$
- $\eta^3 = 3 \cdot 2^2\nu$ (and hence $24\nu = 0$), $\nu^3 = 3 \cdot \eta^2\sigma = \eta^2\sigma$
- $240\sigma = 0$, $2\sigma^2 = 0$, $\sigma^4 = 0$

The 2-localization of the stable homotopy groups:

We get an approximation to $\pi_*(S) \otimes \mathbb{Z}_{(2)}$ via the Adams spectral sequence:



Basic setup



Araki-Iriye, 1980s ($p \leq 13$)

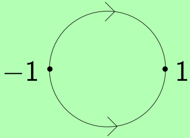
The $\mathbb{Z}/2$-equivariant homotopy groups

For $p \geq q$, $S^{p,q}$ is the compactification of $\mathbb{R}^{p-q} \oplus \mathbb{R}_-^q$.

Note that the fixed set of $S^{p,q}$ is S^{p-q}.

$S^{n,0}$ is the n-sphere with trivial $\mathbb{Z}/2$-action

$S^{1,1}$ is the compactification of $\mathbb{R}_-$:



$S^{2,1}$ is the compactification of $\mathbb{R} \oplus \mathbb{R}_- = \mathbb{C}$: so $S^{2,1} \simeq \mathbb{C}P^1$.

The $\mathbb{Z}/2$-equivariant homotopy groups

Two useful maps:

$\psi\colon \pi_{p,q}(S) \to \pi_p(S)$ "forgetful map"

$\phi\colon \pi_{p,q}(S) \to \pi_{p-q}(S)$ "restriction to the fixed set"

$f\colon S^{p,q} \to S^{0,0} \quad \rightsquigarrow \quad f^{\mathbb{Z}/2}\colon S^{p-q} \to S^0$

The $\mathbb{Z}/2$-equivariant homotopy groups

Let $\eta \colon \mathbb{C}^2 - 0 \to \mathbb{C}P^1$ be the Hopf map.

This is a map $S^{3,2} \to S^{2,1}$, so $\eta \in \pi_{1,1}(S)$.

Note that $\eta^{\mathbb{Z}/2}$ is the Hopf map $\mathbb{R}^2 - 0 \to \mathbb{R}P^1$, which is multiplication by 2. In other words, $\phi(\eta) = 2$.

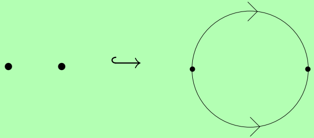
In particular, η is not a torsion class and is not nilpotent. This is different than what we're used to in classical algebraic topology.

NOTE: Actually, it will be better to set things up so that $\phi(\eta) = -2$. Don't ask why.

The $\mathbb{Z}/2$-equivariant homotopy groups

Another new feature is that we have nonzero groups in negative dimensions:

$\rho\colon S^{0,0} \hookrightarrow S^{1,1}$ is essential:

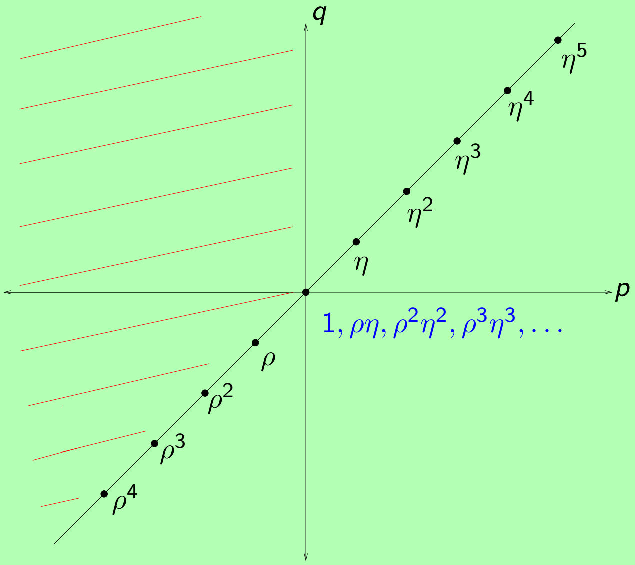


Note that $\phi(\rho) = 1$, so once again we deduce that ρ is not torsion and not nilpotent.

However, this phenomenon is limited. All maps $S^{a,b} \to S^{1,0} \wedge S^{a,b} \wedge S^{c,d}$ are null, by the usual argument.

It follows that $\pi_{p,q}(S) = 0$ if $p < 0$ and $q > p$.

Picture of the equivariant homotopy groups $\pi_{p,q}(S)_{\mathbb{Z}/2}$



The group $\pi_{0,0}$

We have the equivariant degree map:

$$\mathrm{Deg}\colon \pi_{0,0}(S) \to \mathbb{Z}^2, \qquad \mathrm{Deg}(f) = (deg(f), deg(f^{\mathbb{Z}/2})).$$

This map is an injection, and its image consists of all pairs (a, b) such that $a \equiv b \bmod 2$.

Notice that $\mathrm{Deg}(1) = (1, 1)$ and $\mathrm{Deg}(\rho\eta) = (0, -2)$.

So 1 and $\rho\eta$ generate $\pi_{0,0}(S) \cong \mathbb{Z}^2$.

Notice that $\mathrm{Deg}(\rho^2\eta^2) = (0, 4) = \mathrm{Deg}(-2\rho\eta)$, so $\rho^2\eta^2 = -2\rho\eta$.

In fact $\rho\eta^2 = -2\eta$, or $\rho\eta^2 + 2\eta = 0$.

The group $\pi_{0,0}$

Let $\epsilon\colon S^{1,1} \wedge S^{1,1} \to S^{1,1} \wedge S^{1,1}$ be the twist map.

Then $\epsilon \in \pi_{0,0}(S)$, so it is a linear combination of 1 and $\rho\eta$.

$\mathrm{Deg}(\epsilon) = (-1, 1) = -(1, 1) + (0, 2)$, so $\epsilon = -1 - \rho\eta$.

The relation $\rho\eta^2 + 2\eta = 0$ (previous slide) is equivalent to saying $\epsilon\eta = \eta$.

$\pi_{0,0}(S) = \mathbb{Z}\langle 1, \epsilon \rangle$

One more piece

Any non-equivariant map of spheres $S^a \to S^b$ can be regarded as an equivariant map $S^{a,0} \to S^{b,0}$.

This gives maps $\pi_k(S) \to \pi_{k,0}(S)$.

The forgetful map ψ gives a splitting, so that

$$\pi_{k,0}(S) \cong \pi_k(S) \oplus (????).$$

			$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}\oplus(\mathbb{Z}_2)^2$
			$\mathbb{Z}$	$\mathbb{Z}/4$	0	$\mathbb{Z}/12$	$\mathbb{Z}/2$	0	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}/240$
			$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	0	$\mathbb{Z}$	0	0
			$\mathbb{Z}$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/12$	$\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}_{480}\mathbb{Z}_{12}\mathbb{Z}_4$
			$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/8\oplus\pi_3$	$(\mathbb{Z}/2)^2\oplus\pi_4$
			$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}/24$	$0\oplus\pi_2$	$\mathbb{Z}/8\oplus\pi_3$	$\mathbb{Z}/4\oplus\pi_4$	$\mathbb{Z}_{240}\oplus\pi_5$
			0	$\mathbb{Z}$	$\mathbb{Z}/2\oplus\pi_1$	$\mathbb{Z}/2\oplus\pi_2$	$\mathbb{Z}/8\oplus\pi_3$	$\mathbb{Z}/2\oplus\pi_4$	$0\oplus\pi_5$	$\mathbb{Z}/2\oplus\pi_6$
			$\mathbb{Z}^2$	$(\mathbb{Z}/2)^2\oplus\pi_1$	$(\mathbb{Z}/2)^2\oplus\pi_2$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_8\oplus\pi_3$	$\mathbb{Z}/2\oplus\pi_4$	$0\oplus\pi_5$	$(\mathbb{Z}_2)^2\oplus\pi_6$	$\mathbb{Z}_{240}\mathbb{Z}_{16}\mathbb{Z}_2\pi_7$
		$\mathbb{Z}$	$\mathbb{Z}/2\oplus\pi_1$	$\mathbb{Z}/2\oplus\pi_2$	$\mathbb{Z}/8\oplus\pi_3$	$\mathbb{Z}/2\oplus\pi_4$	$0\oplus\pi_5$	$\mathbb{Z}/2\oplus\pi_6$	$\mathbb{Z}_{16}\oplus\mathbb{Z}_2\oplus\pi_7$	$(\mathbb{Z}_2)^3\oplus\pi_8$
	$\mathbb{Z}$	π_1	$\mathbb{Z}\oplus\pi_2$	$\mathbb{Z}/4\oplus\pi_3$	$0\oplus\pi_4$	$\mathbb{Z}/12\oplus\pi_5$	$\mathbb{Z}/2\oplus\pi_6$	$\mathbb{Z}/16\oplus\pi_7$	$(\mathbb{Z}_2)^2\oplus\pi_8$	$\mathbb{Z}_2\mathbb{Z}_{240}\pi_9$
$\mathbb{Z}$	π_1	π_2	$\mathbb{Z}/2\oplus\pi_3$	$0\oplus\pi_4$	$\mathbb{Z}/2\oplus\pi_5$	$(\mathbb{Z}/2)^2\oplus\pi_6$	$\mathbb{Z}/16\oplus\pi_7$	$\mathbb{Z}/2\oplus\pi_8$	$\mathbb{Z}/2\oplus\pi_9$	$\mathbb{Z}/2\oplus\pi_{10}$
π_1	π_2	π_3	$\mathbb{Z}\oplus\pi_4$	$(\mathbb{Z}/2)^2\oplus\pi_5$	$(\mathbb{Z}/2)^2\oplus\pi_6$	$\mathbb{Z}_{16}\,\mathbb{Z}_{12}\,\pi_7$	$\mathbb{Z}/2\oplus\pi_8$	$0\oplus\pi_9$	$0\oplus\pi_{10}$	$\mathbb{Z}_4\mathbb{Z}_{240}\pi_{11}$

Araki-Iriye computations

On to the motivic setting

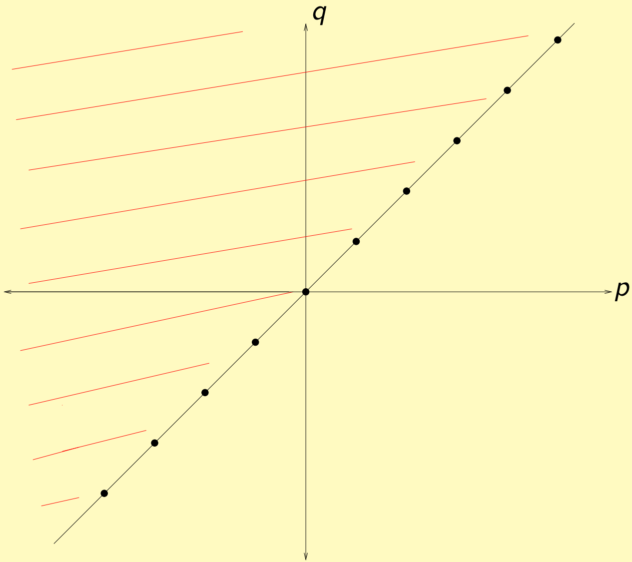
We now investigate the groups $\pi_{p,q}(S)_F$.

Morel's theorems:

(1) Connectivity: $\pi_{p,q}(S) = 0$ for $q > p$.

(2) $\oplus_n \pi_{n,n}(S)$ can be determined explicitly (more on this in a moment).

Motivic stable homotopy groups

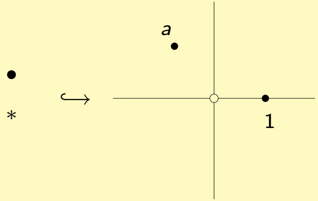


Motivic stable homotopy groups

We again have the Hopf map $\mathbb{A}^2 - 0 \to \mathbb{P}^1$, which is a map $S^{3,2} \to S^{1,1}$. Get the element $\eta \in \pi_{1,1}(S)$.

Recall that $S^{1,1} = \mathbb{A}^1 - 0$. For every $a \in F - \{0\}$ we have the corresponding rational point of $\mathbb{A}^1 - 0$, giving a map

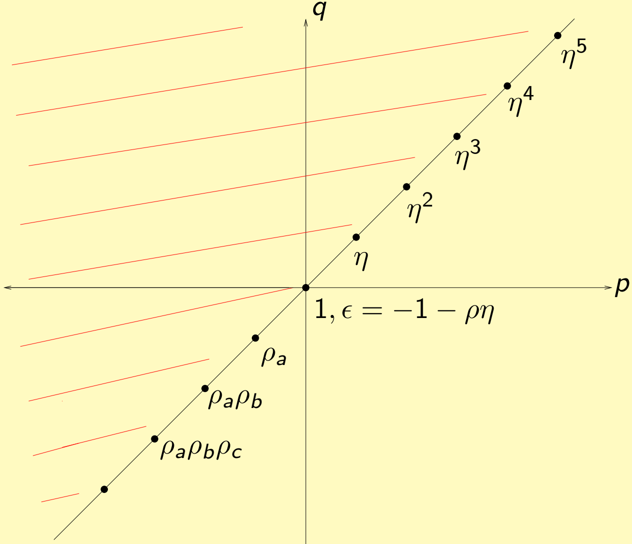
$$\boxed{\rho_a \colon S^{0,0} \hookrightarrow S^{1,1}}$$



The element ρ that we saw in the $\mathbb{Z}/2$-setting is ρ_{-1}.

We again have the twist map $\epsilon \colon S^{1,1} \wedge S^{1,1} \to S^{1,1} \wedge S^{1,1}$.

Motivic stable homotopy groups



Work over $\mathbb{Z}$ to get close parallel with $\pi_{*,*}(S)_{\mathbb{Z}/2}$

Motivic stable homotopy groups

Morel proved that $\bigoplus_n \pi_{n,n}$ is generated by the elements ρ_a, ϵ, and η, subject to the following relations:

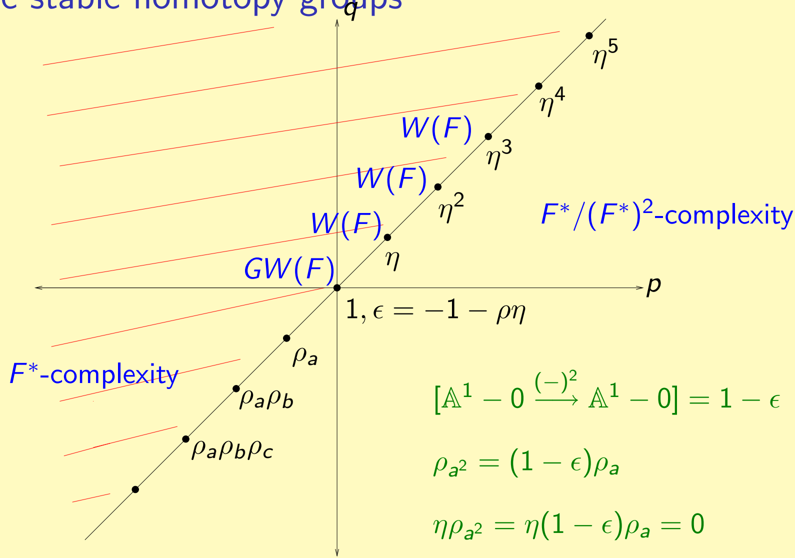
(i) $\epsilon\eta = \eta$, $\epsilon\rho_a = \rho_{a^{-1}}$, $\epsilon^2 = 1$
(ii) $\rho\eta = -(1+\epsilon)$
(iii) $\rho_a\eta = \eta\rho_a$, $\rho_a\rho_b = \rho_{b^{-1}}\rho_a = \rho_b\rho_{a^{-1}}$
(iv) $\rho_a\rho_{1-a} = 0$ (Steinberg relation)
(v) $\rho_{ab} = \rho_a + \rho_b + \eta\rho_a\rho_b$

$\pi_{0,0} \cong GW(F)$, the Grothendieck-Witt ring of quadratic forms $/F$

$$1 + \rho_a\eta \leftrightarrow (F, q_a; q_a(x) = ax^2).$$

The maps $\pi_{1,1} \xrightarrow{\eta} \pi_{2,2} \xrightarrow{\eta} \pi_{3,3} \longrightarrow \cdots$ are all isomorphisms, and these groups are all isomorphic to $W(F)$ (the Witt ring of F).

Motivic stable homotopy groups

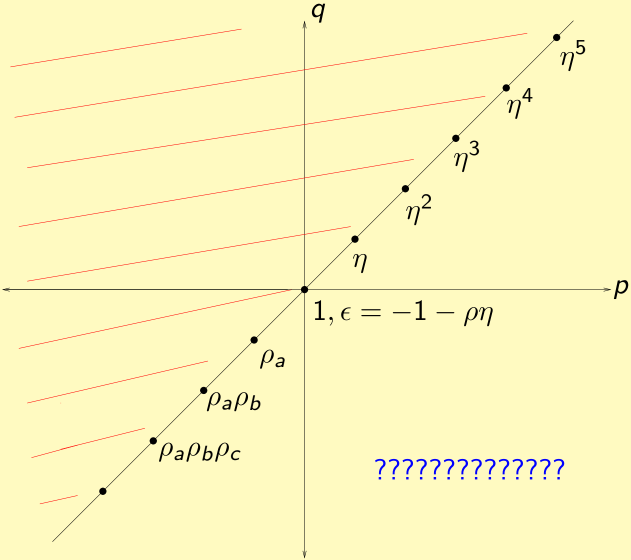


$$[\mathbb{A}^1 - 0 \xrightarrow{(-)^2} \mathbb{A}^1 - 0] = 1 - \epsilon$$

$$\rho_{a^2} = (1 - \epsilon)\rho_a$$

$$\eta\rho_{a^2} = \eta(1 - \epsilon)\rho_a = 0$$

Moving away from the 0-line



??????????????

Moving away from the 0-line

Two basic approaches:

(1) Write down explicit elements, and try to verify relations by direct geometric construction.

(2) Use the motivic version of the Adams spectral sequence. This only computes $\pi_{p,q}(S^{\wedge}_{H})$, but this is still interesting.

Warning: We don't know the groups $\pi_{p,q}(S)$ are finitely-generated, and in fact in negative degrees they are usually not. The relation between $\pi_{p,q}(S)$ and $\pi_{p,q}(S^{\wedge}_{H})$ is difficult to pin down.

Constructing the geometric ν and σ

- The classical Hopf elements come from the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. How can this work over other fields F?
- Cayley-Dickson algebras: If A is an F-algebra with an anti-involution $x \mapsto x^*$ (so $(ab)^* = b^* a^*$) and $\gamma \in F$, define

$$A_\gamma^{dbl} = A \oplus A, \qquad (a, b)(c, d) = (ac + \gamma d^* b, da + bc^*).$$

  This again has an anti-involution given by $(a, b) \mapsto (a^*, -b)$.
- Given a sequence of constants $\gamma_1, \gamma_2, \ldots \in F$, the doubling process can be repeated to give a sequence of algebras

$$A_1 = A_{\gamma_1}^{dbl}, \quad A_2 = (A_1)_{\gamma_2}^{dbl}, \quad \ldots$$

- Starting with $A = \mathbb{R}$ and doing this doubling process several times (with $\gamma = -1$ in each case) produces the sequence $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$.

Motivic Hopf elements (continued)

- An algebra with anti-involution has a *norm form* $N(x) = xx^*$.
- Fact: If A is associative and commutative and normed in the sense that $N(xy) = N(x)N(y)$, then the algebras A_1, A_2, and A_3 are also normed algebras.
  [Note: The A_i's are not necessarily division algebras.]
- Start with $A = F$ and $x^* = x$. Use the sequence where $\gamma_1 = 1$ and all other $\gamma_i = -1$.
- One can check that $A_1 = \mathbb{A}^2$ with $(a, b)(c, d) = (ac, db)$, $(a, b)^* = (b, a)$, and $N(a, b) = ab$.
- Then $A_2 = \mathbb{A}^4$ with ????? and so on.

Motivic Hopf elements (still continued)

- Write $\mathbb{S}_n$ for the affine variety in $A_n = \mathbb{A}^{2^n}$ defined by $N(x) = 1$. When $n \in \{1, 2, 3\}$, multiplication in A_n gives maps
$$\mathbb{S}_n \times \mathbb{S}_n \longrightarrow \mathbb{S}_n.$$
- The "Hopf construction" on this pairing is the composite
$$\Sigma(\mathbb{S}_n \wedge \mathbb{S}_n) \xrightarrow{\chi} \Sigma(\mathbb{S}_n \times \mathbb{S}_n) \longrightarrow \Sigma\mathbb{S}_n.$$
- Under our definitions the norm form on each A_n is split, so $\mathbb{S}_n \simeq S^{2^n-1,2^{n-1}}$. We have therefore produced maps
$$S^{2^{n+1}-1,2^n} \longrightarrow S^{2^n,2^{n-1}} \quad \rightsquigarrow \quad h_n \in \pi_{2^n-1,2^{n-1}}.$$
- $h_1 = \eta \in \pi_{1,1}, \quad h_2 = \nu \in \pi_{3,2}, \quad$ and $h_3 = \sigma \in \pi_{7,4}.$

Another table of the $\pi_{p,q}$ groups

q

									η^4			σ		
								η^3						
							η^2	ν						
						η								
					$1, \epsilon$	η_{top}	η^2_{top}	ν_{top}				σ_{top}		
				ρ_a										
			$\rho_a\rho_b$											

p

Some useful notation

- Write $A_{\mathbb{R}} = F$, $A_{\mathbb{C}} = A_1$, $A_{\mathbb{H}} = A_2$, and $A_{\mathbb{O}} = A_3$.

- These algebras have the "usual" properties: $A_{\mathbb{C}}$ is commutative, $A_{\mathbb{H}}$ is only associative, and $A_{\mathbb{O}}$ is neither.

- Exercise: $A_{\mathbb{H}} \cong M_{2x2}(F)$ with $X^* = \mathrm{adj}(X)$ and $N(X) = \det(X)$.

- Write $S_{\mathbb{R}}$, $S_{\mathbb{C}}$, $S_{\mathbb{H}}$, and $S_{\mathbb{O}}$ for the quadric $N(x) = 1$ inside of $A_{\mathbb{R}}$, $A_{\mathbb{C}}$, etc.

- Example: $A_{\mathbb{C}} = \mathbb{A}^2$ with the multiplication $(a, b)(c, d) = (ac, bd)$ and conjugation $(a, b)^* = (b, a)$. Then $N((a, b)) = (a, b)(b, a) = (ab, ab) = ab \cdot 1_{A_{\mathbb{R}}}$. So $S_{\mathbb{C}}$ is the subvariety of $\mathbb{A}^2$ consisting of points (a, b) with $ab = 1$. That is, $S_{\mathbb{C}} \cong \mathbb{A}^1 - 0 = S^{1,1}$.

The first Hopf relation

In the classical world we have $2\eta = 0$, but in the motivic world this is not true. Instead we have the relation

$$(1-\epsilon)\eta = 0, \qquad \text{or} \qquad \eta = \epsilon\eta.$$

The proof follows from the commutativity of $A_{\mathbb{C}}$:

$$\begin{array}{ccc}
\mathbb{S}_{\mathbb{C}} \times \mathbb{S}_{\mathbb{C}} & \xrightarrow{\mu} & \mathbb{S}_{\mathbb{C}} \\
\downarrow t & & \| \\
\mathbb{S}_{\mathbb{C}} \times \mathbb{S}_{\mathbb{C}} & \xrightarrow{\mu} & \mathbb{S}_{\mathbb{C}}
\end{array}$$

Applying Hopf constructions shows immediately that $\epsilon\eta = \eta$. This argument is due to Morel.

Moral: $1 - \epsilon$ plays the role of the 0th motivic Hopf element.

More on the first Hopf relation

A generalization of the previous argument shows that $\eta\nu = 0 = \nu\sigma$.

If A is associative and $\alpha \in A$ has norm 1, then $(a, b) \mapsto (a, \alpha b)$ is an endomorphism of A^{dbl}.
We then define maps $e_i \colon \mathbb{S}(A_i) \times \mathbb{S}(A_{i+1}) \longrightarrow \mathbb{S}(A_{i+1})$ by

$$\alpha, (a, b) \mapsto (a, \alpha b).$$

and this leads to a big diagram (given for $i = 1$)

$$\begin{array}{ccccc}
\mathbb{S}_{\mathbb{C}} \times \mathbb{S}_{\mathbb{H}} \times \mathbb{S}_{\mathbb{H}} & \xrightarrow{\Delta\times 1} & \mathbb{S}_{\mathbb{C}} \times \mathbb{S}_{\mathbb{C}} \times \mathbb{S}_{\mathbb{H}} \times \mathbb{S}_{\mathbb{H}} & \xrightarrow{1\times T\times 1} & \mathbb{S}_{\mathbb{C}} \times \mathbb{S}_{\mathbb{H}} \times \mathbb{S}_{\mathbb{C}} \times \mathbb{S}_{\mathbb{H}} \\
 & & & & \downarrow{\scriptstyle e\times e} \\
\downarrow{\scriptstyle 1\times\mu} & & & & \mathbb{S}_{\mathbb{H}} \times \mathbb{S}_{\mathbb{H}} \\
 & & & & \downarrow{\scriptstyle \mu} \\
\mathbb{S}_{\mathbb{C}} \times \mathbb{S}_{\mathbb{H}} & & \xrightarrow{\qquad\qquad e \qquad\qquad} & & \mathbb{S}_{\mathbb{H}}
\end{array}$$

A slightly painful analysis ends up showing $\eta\nu = 0$.

A new Hopf relation

One can use properties of $A_{\mathbb{H}}$ to show that $\epsilon\nu = -\nu$.

- Let $c\colon A_{\mathbb{H}} \to A_{\mathbb{H}}$ be the map $(a, b) \mapsto (a, -b)$. This is an automorphism of the algebra, so we have the diagram

$$\begin{array}{ccc} S_{\mathbb{H}} \times S_{\mathbb{H}} & \xrightarrow{\mu} & S_{\mathbb{H}} \\ \downarrow{\scriptstyle c\times c} & & \downarrow{\scriptstyle c} \\ S_{\mathbb{H}} \times S_{\mathbb{H}} & \xrightarrow{\mu} & S_{\mathbb{H}}. \end{array}$$

- Applying Hopf constructions gives that $\nu \cdot c^2 = c \cdot \nu$.

- The map $c\colon S_{\mathbb{H}} \to S_{\mathbb{H}}$ can be seen (with some trouble) to represent $-\epsilon$.

- So $\nu \cdot (-\epsilon)^2 = -\epsilon\nu$, hence $\nu = -\epsilon\nu$.

The next goal:

It is not possible that $\eta^3 = 12\nu$ in the motivic world, as $\eta^3 \in \pi_{3,3}$ whereas $\nu \in \pi_{3,2}$.

The best guess is the relation $\eta^2 \cdot \eta_{top} = 12\nu$.

You might suspect that instead of 12 you need $3(1-\epsilon)^2$, but the relation $\epsilon\nu = -\nu$ from the previous page tells us that this is unnecessary.

Our hope is to find some proof of this relation coming from properties of Cayley-Dickson algebras. So far it is still a mystery.

Note: The failure of the relation $\eta^3 = 12\nu$ in some sense explains why the motivic η is not nilpotent!

A consequence

If we know $12\nu = \eta^2\eta_{top}$ then we also know that $24\nu = 0$, because $2\eta_{top} = 0$.

The 24th power map $\mathbb{A}^1 - 0 \to \mathbb{A}^1 - 0$ is $12(1-\epsilon)$ in $\pi_{0,0}(S)$.

So $\rho_{a^{24}} \cdot \nu = \rho_a \cdot 12(1-\epsilon)\nu = \rho_a \cdot 24\nu = 0$.

We therefore get groups with the "complexity" of $F^*/(F^*)^{24}$, or equivalently of $F^*/(F^*)^2 \oplus F^*/(F^*)^3$.

Adams spectral sequence techniques

$\mathbb{M}_2 = H^{*,*}(\operatorname{Spec} F; \mathbb{Z}_2) = [K_*^M(F)/2][\tau]$

There is a tri-graded Adams spectral sequence

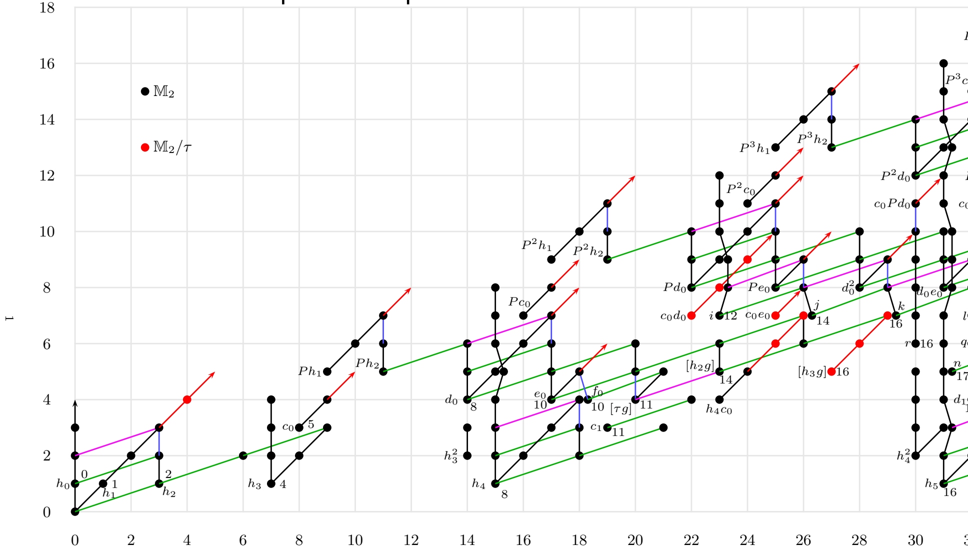
$$\mathrm{Ext}_{A_{mot}}^{s,(t,u)}(\mathbb{M}_2, \mathbb{M}_2) \Rightarrow \pi_{t-s,u}(S).$$

Let $\rho = [-1] \in F^*/(F^*)^2 = H^{1,1}(\operatorname{Spec} F; \mathbb{Z}/2)$. This is the image of $\rho \in \pi_{-1,-1}(S)$ under the Hurewicz map.

IMPORTANT FACT: The only part of $\mathbb{M}_2$ that is relevant to A_{mot} are the elements τ and ρ. So the crucial cases to understand are $F = \mathbb{C}$ and $F = \mathbb{R}$.

If $F = F^2$ then $\rho = 0$, $\mathbb{M}_2 = \mathbb{F}_2[\tau]$, and the Ext groups are easy to compute:

Motivic Adams spectral sequence when $F = \mathbb{C}$



Adams spectral sequence techniques

Next, there is a Bockstein spectral sequence that allows one to put the ρ's back into the picture for the case $F = \mathbb{R}$:

$$\mathrm{Ext}_{A_{\mathbb{C}}}(\mathbb{F}_2[\tau], \mathbb{F}_2[\tau])[\rho] \Rightarrow \mathrm{Ext}_{A_{\mathbb{R}}}(\mathbb{M}_2, \mathbb{M}_2).$$

The spectral sequence does not collapse at a finite page, and a lot of bookkeeping is required, but the differentials can be completely determined (Isaksen).

The patterns that show up are VERY close to what we saw in the chart for $\pi_{*,*}(S)_{\mathbb{Z}/2}$, for the portion of that chart below the $p = q$ line.

One finds the "Clifford periodicities" from the $\mathbb{Z}/2$-equivariant setting appearing in the pattern of differentials, but going off in only one direction (the direction of negative weight).

Adams spectral sequence techniques

Based on this data we make the following conjecture for the 1-line: as a module over the 0-line it is the quotient of

$$\left[\oplus_n \pi_{n,n}(S)\right]\langle \nu, \eta_{top}\rangle$$

by the relations

- $\epsilon\nu = -\nu, \quad \eta\nu = 0, \quad 2\eta_{top} = 0$
- $\eta^2\eta_{top} = 12\nu$
- $\rho^4\nu = \rho^2\eta_{top}$

This completely matches with the $\pi_{*,*}(S)_{\mathbb{Z}/2}$ groups, as well.

Adams spectral sequence techniques

We can also make conjectures for the first few "lines" beyond the 1-line, but they are more difficult to state and we are not yet sure that all the elements predicted by the ASS are really there.

The Adams spectral predicts an element $\theta \in \pi_{0,-2}(S)$ that is non-torsion. In the $\mathbb{Z}/2$-equivariant world it is easy to identify θ, but the model is non-algebraic. We do not know if there is an algebraic model for it.

			$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}\oplus(\mathbb{Z}_2)^2$
			$\mathbb{Z}$	$\mathbb{Z}/4$	0	$\mathbb{Z}/12$	$\mathbb{Z}/2$	0	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}/240$
			$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	0	$\mathbb{Z}$	0	0
			$\mathbb{Z}$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/12$	$\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}_{480}\,\mathbb{Z}_{12}\mathbb{Z}_4$
			$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/8\oplus\pi_3$	$(\mathbb{Z}/2)^2\oplus\pi_4$
			$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}/24$	$0\oplus\pi_2$	$\mathbb{Z}/8\oplus\pi_3$	$\mathbb{Z}/4\oplus\pi_4$	$\mathbb{Z}_{240}\oplus\pi_5$
			0	$\mathbb{Z}$	$\mathbb{Z}/2\oplus\pi_1$	$\mathbb{Z}/2\oplus\pi_2$	$\mathbb{Z}/8\oplus\pi_3$	$\mathbb{Z}/2\oplus\pi_4$	$0\oplus\pi_5$	$\mathbb{Z}/2\oplus\pi_6$
			$\mathbb{Z}^2$	$(\mathbb{Z}/2)^2\oplus\pi_1$	$(\mathbb{Z}/2)^2\oplus\pi_2$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_8\oplus\pi_3$	$\mathbb{Z}/2\oplus\pi_4$	$0\oplus\pi_5$	$(\mathbb{Z}_2)^2\oplus\pi_6$	$\mathbb{Z}_{240}\mathbb{Z}_{16}\mathbb{Z}_2\pi_7$
		$\mathbb{Z}$	$\mathbb{Z}/2\oplus\pi_1$	$\mathbb{Z}/2\oplus\pi_2$	$\mathbb{Z}/8\oplus\pi_3$	$\mathbb{Z}/2\oplus\pi_4$	$0\oplus\pi_5$	$\mathbb{Z}/2\oplus\pi_6$	$\mathbb{Z}_{16}\oplus\mathbb{Z}_2\oplus\pi_7$	$(\mathbb{Z}_2)^3\oplus\pi_8$
	$\mathbb{Z}$	π_1	$\mathbb{Z}\oplus\pi_2$	$\mathbb{Z}/4\oplus\pi_3$	$0\oplus\pi_4$	$\mathbb{Z}/12\oplus\pi_5$	$\mathbb{Z}/2\oplus\pi_6$	$\mathbb{Z}/16\oplus\pi_7$	$(\mathbb{Z}_2)^2\oplus\pi_8$	$\mathbb{Z}_2\mathbb{Z}_{240}\pi_9$
$\mathbb{Z}$	π_1	π_2	$\mathbb{Z}/2\oplus\pi_3$	$0\oplus\pi_4$	$\mathbb{Z}/2\oplus\pi_5$	$(\mathbb{Z}/2)^2\oplus\pi_6$	$\mathbb{Z}/16\oplus\pi_7$	$\mathbb{Z}/2\oplus\pi_8$	$\mathbb{Z}/2\oplus\pi_9$	$\mathbb{Z}/2\oplus\pi_{10}$
π_1	π_2	π_3	$\mathbb{Z}\oplus\pi_4$	$(\mathbb{Z}/2)^2\oplus\pi_5$	$(\mathbb{Z}/2)^2\oplus\pi_6$	$\mathbb{Z}_{16}\,\mathbb{Z}_{12}\,\pi_7$	$\mathbb{Z}/2\oplus\pi_8$	$0\oplus\pi_9$	$0\oplus\pi_{10}$	$\mathbb{Z}_4\mathbb{Z}_{240}\pi_{11}$

Araki-Iriye computations


Thank you!