Chiral Lagrangian from a functional integral over quarks

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A low-energy effective Lagrangian can be derived for a nonet of pseudoscalar mesons (incorporating the Wess-Zumino-Witten interaction) directly from the fundamental Lagrangian of quantum chromodynamics.

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Dynamic breaking of chiral symmetry in quantum chromodynamics has the consequence that the phase shift for γ_5 transformations of the quark fields describes Goldstone particles: pseudoscalar mesons π^a ($a = 1, ..., 9$). We can derive an effective low-energy action $W[\pi]$ for them, treating π^a as the elementary external field interacting with the quarks:

$$\exp - W[\pi] \equiv \frac{\int DG_\mu D\psi D\psi^+ \exp\{-S[G_\mu] + \int d^4x \psi^+ e^{i\Pi\gamma_5}\gamma_\mu(i\partial_\mu + G_\mu)e^{i\Pi\gamma_5}\psi\}}{\int DG_\mu D\psi D\psi^+ \exp\{-S[G_\mu] + \int d^4x\, \psi^+\gamma_\mu(i\partial_\mu + G_\mu)\psi\}}. \tag{1}$$

Here G_μ is the gluon field; $S[G_\mu] = \mathrm{Tr}\int d^4x G^2_{\mu\nu}/2g^2$ is the action for gluons; the ψ are the fields of the u, d, and s quarks; u, d, s, $\Pi = \pi^a t^a/F$ is the 3×3 Hermitian matrix of the pseudoscalar nonet; $\mathrm{Tr} t^a t^b = \delta^{ab}/2, t^9 = I/\sqrt{6}$; and F is a dimensional constant (see below). We are using the Euclidean formulation of quantum chromodynamics.

We immediately note that we have $W[\pi] \neq 0$ only by virtue of an axial anomaly. We could introduce the new variable $\psi' = \exp(i\Pi\gamma_5)\psi$ and apparently eliminate the field Π, but because of the anomaly the corresponding Jacobian of the transformation is nontrivial,[1] so that we have $W[\pi] \neq 0$. It is this circumstance that allows us to evaluate the integral over the quark fields. We rewrite the fermion part of the action, introducing left and right gauge fields:

$$\mathcal{L}_\psi = \psi^+\gamma_\mu\left[\frac{1+\gamma_5}{2}(i\partial_\mu + L_\mu) + \frac{1-\gamma_5}{2}(i\partial_\mu + R_\mu)\right]\psi \equiv \psi^+ i\hat{\nabla}\psi, \tag{2}$$

$$L_\mu = G_\mu \otimes I + I \otimes iU^+\partial_\mu U, \quad R_\mu = G_\mu \otimes I + I \otimes iU\partial_\mu U^+, \quad U = \exp i\Pi. \tag{3}$$

The first factor in the direct products refers to color space, while the second refers to flavor space.

Let us consider the fermion determinant

$$Y[\pi, G_\mu] \equiv \ln\det i\hat{\nabla} = -\frac{1}{2}\mathrm{Sp}\int_\epsilon^\infty \frac{dt}{t}\exp(t\hat{\nabla}^2), \tag{4}$$

where we have used a proper-time regularization, which conserves the vector current. We find the axial anomaly $\mathscr{A}^a(\pi, G_\mu)$ by acting on Y with the generator of the infinitesimal axial transformation: $T_A^a = T_L^a - T_R^a$, where the $T_L^a(x) = -D_\mu^{ab} gL\delta/\delta L_\mu^b(x)$ and $T_R^a(x) = D_\mu^{ab}(R)\delta/\delta R_\mu^b(x)$ are the generators of the gauge transformations of the left and right fields. We use the customary equation

$$T_A^a \,\mathrm{Sp}\, f(\hat{\nabla}) = -2i\,\mathrm{Sp}\, t^a \gamma_5 \hat{\nabla} \frac{\partial}{\partial \hat{\nabla}} f(\hat{\nabla}) \cdot$$

We find

$$\mathscr{A}^a(\pi, G_\mu) \equiv T_A^a Y[\pi, G_\mu] = 2i\,\mathrm{Sp}\, t^a \gamma_5 \int_\epsilon^\infty dt\, \hat{\nabla}^2 \exp t\hat{\nabla}^2$$

$$= -2i\,\mathrm{Sp}\, t^a \gamma_5 \exp \epsilon \hat{\nabla}^2 = -2i \int d^4x \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\, t^a \gamma_5 \exp[\epsilon(\hat{\nabla} + i\hat{p})^2]. \tag{5}$$

As expected, the anomaly is determined by the lower limit of the integration over the proper time. Using (3) to evaluate the last expression, we find

$$\mathscr{A}^a(\pi, G_\mu) = -\frac{1}{2\pi^2\epsilon} \mathrm{Tr}\, t^a D_\alpha(V) A_\alpha + \delta^{a9} \frac{i\sqrt{6}}{16\pi^2} \mathrm{Tr}\, G_{\mu\nu} \tilde{G}_{\mu\nu}$$

$$-\frac{2i}{3\pi^2} \epsilon_{\alpha\beta\gamma\delta}\, \mathrm{Tr}\, t^a A_\alpha A_\beta A_\gamma A_\delta + \mathscr{A}' + O(\epsilon). \tag{6}$$

Here $V_\alpha = (L_\alpha + R_\alpha)/2$, $A_\alpha = (L_\alpha - R_\alpha)/2 = (i/2)I \otimes (U^+ \partial_\alpha U - U\partial_\alpha U^+)$, $D_\alpha(V) = \partial_\alpha - i[V_\alpha, ...]$, $G_{\mu\nu}$ is the gluon stress, and $\tilde{G}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} G_{\alpha\beta}/2$. Here $\mathscr{A}'$ represents an expression which is finite in the ϵ cutoff and which does not contai the antisymmetric tensor $\epsilon_{\alpha\beta\gamma\delta}$. This expression leads to terms in the effective Lagrangian with higher-order derivatives, and we will not consider it further here; we will also ignore the following terms $\mathscr{O}(\epsilon)$.

Expression (6) could also be derived for other regularizations of the anomaly which conserve the vector current.[2] We wish to emphasize that the first (quadratically diverging) term in (6) is an unavoidable price paid for this regularization.

It should be kept in mind, however, that we have introduced the external field as an elementary field. Consequently, the ultraviolet cutoff of ϵ has the objective meaning of being that dimension at which the deviation of the pseudoscalar mesons from being point entities becomes manifested.

Substituting the first term in (6) into Eq. (5), we can reconstruct the corresponding term in $Y[\pi, G_\mu]$:

$$\frac{1}{4\pi^2\epsilon} \int d^4x\, \mathrm{Tr}\, A_\alpha^2 = \frac{N_c}{16\pi^2\epsilon} \int d^4x\, \mathrm{Tr}\, \partial_\mu U^{-2} \partial_\mu U^2. \tag{7}$$

This is the standard chiral Lagrangian of pseudoscalar mesons. Using the expansion $U \cong 1 + i\pi^a t^a / F$, and normalizing in terms of the kinetic energy, $(\partial_\mu \pi^a)^2/2$, we find $\sqrt{\epsilon} = \sqrt{N_c}/2\pi F$, where the constant $F = F_\pi \cong 95$ MeV is found by comparing the following terms in the expansion in π^a with the standard Lagrangian. We note that $F_\pi \propto \sqrt{N_c}$, so that the "size of the pion," $\sqrt{\epsilon}$, does not depend on the number of colors N_c. This is a reasonable result.

Integrating the second term in (6), we find a term in $Y[\pi, G_\mu]$ which breaks the U_1 symmetry[3]:

$$-\frac{1}{16\pi^2} \int d^4x \ln \det U^2 \cdot \mathrm{Tr}\, G_{\mu\nu} \tilde{G}_{\mu\nu}. \tag{8}$$

If there exists a nonzero topological-charge correlation function $\int d^4x \langle \mathrm{Tr} G\tilde{G}(x) \mathrm{Tr} G\tilde{G}(0) \rangle \equiv \lambda^4 (16\pi^2)^2$, in pure gluon dynamics (without light quarks), then in substituting (8) into (1) we find a term in $W[\pi]$ which gives us the mass of the ninth (singlet) meson,[3,4] $\pi^9 = \eta'$:

$$-\frac{\lambda^4}{2} \int d^4x (\ln \det U^2)^2 = \frac{1}{2} \int d^4x\, m_{\eta'}^2 \eta'^2, \quad m_{\eta'}^2 = \frac{6\lambda^4}{F_\pi^2}. \tag{9}$$

Finally, the third term in (6) gives us the Wess-Zumino-Witten interaction[5] (the detailed derivation will be published separately):

$$-\frac{iN_c}{240\pi^2} \int d^5x\, \epsilon_{\alpha\beta\gamma\delta\epsilon} \mathrm{Tr}\,(U^{-2} \partial_\alpha U^2)(_\beta)(_\gamma)(_\delta)(_\epsilon).$$

This is essentially the integral of the complete 5-divergence, which reduces to an ordinary 4-dimensional action. This term can be rewritten

$$\frac{N_c}{24\pi^2} \int d^4x \int_0^1 d\tau\, \epsilon_{\alpha\beta\gamma\delta} \cdot \mathrm{Tr}\, \Pi\, (e^{-2i\tau\Pi} \partial_\alpha e^{2i\tau\Pi})(_\beta)(_\gamma)(_\delta)$$

$$= \frac{2N_c}{15\pi^2} \int d^4x\, \epsilon_{\alpha\beta\gamma\delta} \mathrm{Tr}\, \Pi\, \partial_\alpha \Pi\, \partial_\beta \Pi\, \partial_\gamma \Pi\, \partial_\delta \Pi + O(\Pi^7). \tag{10'}$$

The effective action $W[\pi]$ which we are seeking is given by the sum of (7), (9), and (10).

We have thus shown that the low-energy Lagrangian of the nonet of pseudoscalar mesons can be derived directly for the functional integral over quarks. The constant F_π has the meaning of an ultraviolet cutoff associated with the circumstance that the mesons are not point entities.

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