Differential Geometry

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Preface

The present text evolved from differential geometry courses that I taught at the University of Bonn in 1983-1984 and at the Ohio State University between 1987 and 2005.

The reader is expected to be familiar with basic linear algebra and calculus of several real variables. Additional background in topology, differential equations and functional analysis, although obviously useful, is not necessary: self-contained expositions of all needed facts from those areas are included, partly in the main text, partly in appendices.

This book may serve either as the basis of a course sequence, or for self-study. It is with the latter use in mind that I included over 600 practice problems, along with hints for those problems that seem less than completely routine.

The exposition uses the coordinate-free language typical of modern differential geometry. However, whenever appropriate, traditional local-coordinate expressions are presented as well, even in cases where a coordinate-free description would suffice. Although seemingly redundant, this feature may teach the reader to recognize when and how to take advantage of shortcuts in arguments provided by local-coordinate notation.

I selected the topics so as to include what is needed for a reader who wishes to pursue further study in geometric analysis or applications of differential geometry to theoretical physics, including both general relativity and classical gauge theory of particle interactions.

The text begins with a rapid but thorough presentation of manifolds and differentiable mappings, followed by the definition of a Lie group, along with some examples. A list of all the topics covered can best be glimpsed from the table of contents.

One topic which I left out, despite its prominent status, is complex differential geometry (including Kähler manifolds). This choice seems necessary due to limitations of space.

Finally, I need to acknowledge several books from which I first learned differential geometry and which, consequently, influenced my view of the subject. These are *Riemannsche Geometrie im Großen* by Gromoll, Klingenberg and Meyer, Milnor's *Morse Theory*, Sulanke and Wintgen's *Differentialgeometrie und Faserbündel*, Kobayashi and Nomizu's *Foundations of Differential Geometry* (both volumes), *Le spectre d'une variété riemannienne* by Berger, Gauduchon and Mazet, Warner's *Foundations of Differentiable Manifolds and Lie Groups*, and Spivak's *A Comprehensive Introduction to Differential Geometry.*

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CHAPTER 1

Differentiable Manifolds

1. Manifolds

Topics: Coordinate systems; compatibility; atlases; topology; convergence; maximal atlases; the Hausdorff axiom; manifolds; vector spaces as manifolds.

Let r be a natural number ($r = 0, 1, 2, \dots$), or infinity ($r = \infty$), or an additional symbol ω ($r = \omega$). We order these values so that $0 < 1 < \dots < \infty < \omega$. A mapping F between open subsets of Euclidean spaces is said to be *of class* C^r if $r = 0$ and F is continuous, or $0 < r < \infty$ and F has continuous partial derivatives up to order r, or $r = \infty$ and F has continuous partial derivatives of all orders or, finally, $r = \omega$ and F is real-analytic. (Each of the regularity properties just named for F means the corresponding property for every real-valued component function of F.)

An *n-dimensional coordinate system* (or *chart*) in a set M is a pair (U, φ), where U (the chart's *domain*) is a nonempty subset of M and $\varphi : U \to \varphi(U)$ is a one-to-one mapping of U onto an open subset $\varphi(U)$ of $\mathbf{R}^n$.

Two n-dimensional coordinate systems (U, φ), $(\widetilde{U}, \widetilde{\varphi})$ in M are C^r *compatible*, $0 \le r \le \omega$, if

a. The images $\varphi(U \cap \widetilde{U})$, $\widetilde{\varphi}(U \cap \widetilde{U})$ are both open in $\mathbf{R}^n$,
b. The (bijective) composite mapping $\widetilde{\varphi} \circ \varphi^{-1} : \varphi(U \cap \widetilde{U}) \to \widetilde{\varphi}(U \cap \widetilde{U})$, and its inverse, are of class C^r.

We call $\widetilde{\varphi} \circ \varphi^{-1}$ the *transition mapping* between (U, φ) and $(\widetilde{U}, \widetilde{\varphi})$.

An *n-dimensional C^r atlas* $\mathcal{A}$ on a set M is a collection of n-dimensional coordinate systems in M which are mutually C^r compatible and whose domains cover M. When $\mathcal{A}$ is fixed, a set $Y \subset M$ is said to be *open* if $\varphi(U \cap Y)$ is open in $\mathbf{R}^n$ for each $(U, \varphi) \in \mathcal{A}$. The family of all open sets is called the *topology* in M determined by the C^r atlas $\mathcal{A}$. By a *neighborhood* of a point $x \in M$ we mean any open set containing x. A sequence x_k, $k = 1, 2, \dots$ of points in M then is said to *converge to a limit* $x \in M$ if every neighborhood of x contains the x_k for all but finitely many k.

An n-dimensional C^r atlas $\mathcal{A}$ on M is called *maximal* if it is contained in no other n-dimensional C^r atlas. Every n-dimensional C^r atlas $\mathcal{A}$ on M is contained in a unique maximal C^r atlas $\mathcal{A}_{\max}$. The topologies in M determined by $\mathcal{A}$ and $\mathcal{A}_{\max}$ coincide (Problem 2).

The topology determined by an atlas $\mathcal{A}$ on M is said to satisfy the *Hausdorff axiom* (or, to be a *Hausdorff topology*) if any two different points $x, y \in M$ have disjoint neighborhoods.

An *n-dimensional C^r manifold*, $0 \le r \le \omega$, consists of a nonempty set M along with a fixed n-dimensional maximal C^r atlas $\mathcal{A}$ that determines a Hausdorff topology. We will often suppress $\mathcal{A}$ from the notation and simply speak of "the n-dimensional manifold M". For instance, we will write $n = \dim M$. The coordinate systems $(U, \varphi) \in \mathcal{A}$ will be referred to as *local* coordinate systems (or local charts) in the manifold M, or simply coordinate systems (charts) in M; those among them having the domain $U = M$ will be called *global* (rather than local).

Rather than C^0 manifolds, one often uses the term *topological manifolds.*

Any real vector space V with $\dim V = n < \infty$ is an n-dimensional C^ω manifold, with the atlas

$$\mathcal{A} = \{(V, \varphi) : \varphi \quad \text{is a linear isomorphism of} \quad V \quad \text{onto} \quad \mathbf{R}^n\}\,. \tag{1.1}$$

The charts forming $\mathcal{A}$ are global, so just one of them would suffice to define the corresponding maximal atlas (Problem 9); however, our choice of $\mathcal{A}$ emphasizes that the manifold structure is determined by the vector space structure alone.

Problems

1. Call two n-dimensional C^r atlases $\mathcal{A}$, $\mathcal{A}'$ on a set M *equivalent* if their union $\mathcal{A} \cup \mathcal{A}'$ is a C^r atlas. Show that $\mathcal{A}$, $\mathcal{A}'$ are equivalent if and only if they are contained in the same maximal C^r atlas. (This establishes a natural bijective correspondence between the set of all maximal C^r atlases on M, and the set of equivalence classes of C^r atlases.)

2. Prove that equivalent C^r atlases (Problem 1) lead to the same topology.

3. For an n-dimensional C^r atlas $\mathcal{A}$ on a set M, show that the following two conditions are equivalent:
(i) the topology determined by $\mathcal{A}$ satisfies the Hausdorff axiom,
(ii) the limit of any convergent sequence of points in M is unique.

4. Given an n-dimensional C^r atlas $\mathcal{A}$ on a set M, a coordinate system $(U, \varphi) \in \mathcal{A}$, and a point $x \in U$, verify that a sequence x_k of points in M converges to x if and only if $x_k \in U$ for all sufficiently large k and $\varphi(x_k) \to \varphi(x)$ in $\mathbf{R}^n$ as $k \to \infty$.

5. A subset K of a manifold M is called *closed* if its complement $M \setminus K$ is open. Show that the class of all closed subsets of M contains M and the empty set, and is closed under finite unions and arbitrary intersections.

6. Verify that a subset K of a manifold M is closed if and only if K contains all limits of all sequences of points in K that converge in M.

7. Let $0 \le r \le s \le \omega$. Verify that every C^s manifold M is naturally a C^s manifold, that is, the maximal C^s atlas of M is contained in a unique maximal C^r atlas on the set M.

8. Show that any linear operator $F : \mathbf{R}^n \to \mathbf{R}^m$ is of class C^ω.

9. Given an n-dimensional coordinate system (U, φ) on a set M that is global (i.e., $U = M$), verify that $\mathcal{A} = \{(U, \varphi)\}$ is an n-dimensional C^ω atlas on M.

10. Let (U, φ) be a coordinate system in a C^r manifold M. Show that a subset Y of U is open if and only if $\varphi(Y)$ is open in $\mathbf{R}^n$.

2. Examples of manifolds

Topics: Affine spaces as manifolds; cosets and nonhomogeneous linear equations; open submanifolds; Cartesian products; the gluing construction; real and complex projective spaces; spheres; tori; local geometric properties; compactness.

Any real affine space $(M, V, +)$ of dimension $n < \infty$ (see the appendix below) can naturally be turned into an n-dimensional C^ω manifold, with the atlas formed by all *affine coordinate systems* in M, that is, those global charts (M, φ) for which $\varphi : M \to \mathbf{R}^n$ satisfies $\varphi(y) = \psi(y - x)$ for some "origin" $x \in M$, some linear isomorphism $\psi : V \to \mathbf{R}^n$, and all $y \in M$.

From now on, defining a specific atlas $\mathcal{A}$ on a set M, we will allow the coordinate mappings $\varphi : U \to \varphi(U)$ of the n-dimensional coordinate systems forming (U, φ) with $U \subset M$ to be one-to-one mappings of U onto open subsets $\varphi(U)$ of various n-dimensional affine spaces M' rather than just $\mathbf{R}^n$. These φ then have to be composed with all possible affine coordinate mappings $\varphi' : M' \to \mathbf{R}^n$ as defined above; since linear (and affine) mappings between $\mathbf{R}^n$'s are analytic (Problem 8 in §1, and Problem 15 below), compatibility for the resulting $\mathbf{R}^n$-valued charts does not depend on how the φ' were chosen.

Let V be a *Euclidean space*, i.e., a finite-dimensional real vector space endowed with a fixed inner product $\langle\,,\rangle$. The *unit sphere*

$$S(V) = \{v \in V : |v| = 1\}$$

of V (where $|\ |$ is the *Euclidean norm* with $|v| = \sqrt{\langle v, v\rangle}$) then carries a natural structure of a C^ω manifold with $\dim S(V) = \dim V - 1$, defined by the atlas $\mathcal{A} = \{(U_v, \varphi_v) : v \in S(V)\}$, $\varphi_v : U_v \to A_v$ being the *stereographic projection* with the "pole" v. More precisely, $U_v = S(V) \setminus \{v\}$, while $A_v = v^\perp - v$, that is, $A_v = \{y \in V : \langle y, v\rangle = -1\}$, is an affine space with the translation space $v^\perp$ (the coset of $v^\perp$ through $-v$; see the preceding paragraph), and, for $x \in U_v$, $\varphi_v(x)$ is the unique intersection point of the line through v and x with A_v. Compatibility is easily seen even without computing the transition mappings explicitly; their components are rational functions of the coordinates. When $V = \mathbf{R}^{n+1}$ with the standard Euclidean inner product, we write S^n instead of $S(V)$.

The 1-dimensional sphere S^1 is usually called the *circle*. By the *n-dimensional torus* we mean the manifold $T^n = S^1 \times \ldots \times S^1$ obtained as the Cartesian product (Problem 3 in §2) of n copies of the circle S^1. Thus, $\dim T^n = n$ and $T^1 = S^1$, while $T^2 = S^1 \times S^1$ can be visualized as a surface shaped like a donut (or an inner tube):

Fig. 1. The 2-dimensional torus

Consider now a real or complex vector space V of dimension $n < \infty$; thus, its *real* dimension is dn, where $d \in \{1, 2\}$ is the dimension of the scalar field over $\mathbf{R}$. We define the *projective space* of V to be the C^ω manifold of dimension $d(n-1)$ with the underlying set

$$P(V) = \{L : L \text{ is a 1-dimensional vector subspace of } V\}\,.$$

Its manifold structure is determined by the atlas $\mathcal{A} = \{(U_f, \varphi_f) : f \in V^* \setminus \{0\}\}$ indexed by all nonzero scalar-valued linear functions on V, and defined as follows. Set $U_f = \{L \in P(V) : L \text{ is not contained in } \operatorname{Ker} f\}$ and $A_f = f^{-1}(1)$ (this is an affine space with the translation space $\operatorname{Ker} f$, namely, a coset of $\operatorname{Ker} f$), and let $\varphi_f : U_f \to A_f$ send each $L \in U_f$ onto its unique intersection point with A_f; thus, if $L = \mathbf{R}v$ or $L = \mathbf{C}v$, then $\varphi_f(L) = v/f(v)$. Compatibility follows since, for $f, h \in V^* \setminus \{0\}$, $\varphi_f(U_f \cap U_h) = A_f \setminus \operatorname{Ker} h$ and $(\varphi_f \circ \varphi_h^{-1})(w) = w/f(w)$.

When $V = \mathbf{R}^{n+1}$ or $V = \mathbf{C}^{n+1}$, rather than $P(V)$ one speaks of the *real projective space* $\mathbf{R}\mathrm{P}^n$ and the *complex projective space* $\mathbf{C}\mathrm{P}^n$. The 1-dimensional subspace $L \in P(V)$ spanned by a nonzero vector $(x^0, \dots, x^n)$ in V then is denoted by $[x^0, \dots, x^n] \in P(V)$. (One then refers to $x^0, \dots, x^n$ as *homogeneous coordinates* of $L = [x^0, \dots, x^n]$.)

We say, informally, that a property pertaining to (subsets of) manifolds has a *local geometric character* if it can be defined/verified using just any particular collection of coordinate systems covering the set in question, without resorting to studying *all* charts in the maximal atlas of the manifold. Examples of such properties are: openness of sets (just cover the set U in question with any family of charts and see if each of them makes U appear open), and convergence of sequences: to see if $x_k \to x$ as $k \to \infty$, fix just one chart (U, φ) with $x \in U$ and ask if $\varphi(x_k) \to \varphi(x)$ (Problem 4 in §1).

Another important example of this kind is *compactness* of sets. (We say that a subset K of a manifold M is *compact* if every sequence x_k, $k = 1, 2, \dots$, of points in K has a subsequence that converges in M to a point $x \in K$.)

Problems

1. Let there be coordinate systems (U, φ) in a set M and (U', φ') in a set M', of dimensions n and n', respectively. Verify that $(U \times U', \varphi \times \varphi')$ with $(\varphi \times \varphi')(x, x') = (\varphi(x), \varphi'(x'))$ then is an $(n + n')$-dimensional coordinate system in the Cartesian product $M \times M' = \{(x.x') : x \in M\,,\ x' \in M'\}$.

2. Given coordinate systems (U_1, φ_1), (U_2, φ_2) in a set M and coordinate systems (U_1', φ_1'), (U_2', φ_2') in a set M', such that the first two and the last two are C^r compatible, show that the coordinate system $(U_1 \times U_1', \varphi_1 \times \varphi_1')$ in $M \times M'$ is C^r compatible with $(U_2 \times U_2', \varphi_2 \times \varphi_2')$.

3. Given manifolds M, M' of class C^r, prove that the set $M \times M'$ carries a unique manifold structure of dimension $\dim M + \dim M'$ and class C^r, whose maximal atlas contains all $(U \times U', \varphi \times \varphi')$ such that (U, φ) and (U', φ') belongs to the maximal atlas of M and M', respectively. (With this structure, $M \times M'$ is called the *product manifold* of M and M'.) Generalize this to the product $M_1 \times \ldots \times M_k$ of k factors.

4. Verify that the class of all open subsets of a manifold M (also called the *topology* of M) contains M and the empty set, and is closed under finite intersections and arbitrary unions.

5. Verify that every subset A of M is contained in a unique set $\operatorname{clos}(A)$ such that $\operatorname{clos}(A)$ is closed and $\operatorname{clos}(A) \subset K$ for any closed set $K \subset M$ containing A. (We call $\operatorname{clos}(A)$ the *closure* of A.)

6. Suppose we are given four open sets Y_1, Y_2, Z_1, Z_2 in $\mathbf{R}^n$ with $Z_\alpha \subset Y_\alpha$ for $\alpha \in \{1, 2\}$, and a mapping $h : Z_1 \to Z_2$ which is a C^r diffeomorphism, $0 \le l \le r \le \omega$ (that is, h is one-to-one and onto, and h and its inverse h^{-1} are both of class C^r). Define M to be the set obtained from the disjoint union of Y_1 and Y_2 by identifying Z_1 with Z_2 along h. Specifically, M is the quotient of $(Y_1 \times \{1\}) \cup (Y_2 \times \{2\})$ under the equivalence relation $\sim$, with $(y,\alpha) \sim (y',\alpha')$ if and only if $(y,\alpha) = (y',\alpha')$, or $\alpha \neq \alpha'$, $y \in Z_\alpha$, $y' \in Z'_\alpha$ and either $y' = h(y)$ with $\alpha = 1$, or $y = h(y')$ with $\alpha = 2$.

(i) Verify that the quotient projection $\pi : (Y_1 \times \{1\}) \cup (Y_2 \times \{2\}) \to M$, sending each (y,α) onto its $\sim$ equivalence class, is injective when restricted to either of $Y_\alpha \times \{\alpha\}$.

(ii) Set $U_\alpha = \pi(Y_\alpha \times \{\alpha\}) \subset M$, and let $\varphi_\alpha : U_\alpha \to Y_\alpha \subset \mathbf{R}^n$ be the inverse of $\pi : Y_\alpha \times \{\alpha\} \to U_\alpha$ followed by the identification $Y_\alpha \times \{\alpha\} \to Y_\alpha$. Show that $\mathcal{A} = \{(U_1,\varphi_1),(U_2,\varphi_2)\}$ is an n-dimensional C^r atlas on M.

7. In Problem 6, let $Z_1 = Z_2$ be an open subset of $Y_1 = Y_2 \subset \mathbf{R}^n$ such that the complement $Y_1 \setminus Z_1$ is not an open set, and let h be the identity mapping. Show that the resulting atlas $\mathcal{A}$ gives rise to a *non-Hausdorff topology* on M.

8. Construct M and the atlas $\mathcal{A}$ as in Problem 6, using $n = 2$, $Y_1 = Y_2 = \mathbf{C}$ (the set of complex numbers, identified with $\mathbf{R}^2$), $Z_1 = Z_2 = \mathbf{C} \setminus \{0\}$ and h given by $h(z) = 1/z$ (in the sense of the complex multiplication). Prove that $\mathcal{A}$ determines a Hausdorff topology on M. (The 2-dimensional C^ω manifold thus obtained is known as the *Riemann sphere*.)

9. Let M and $\mathcal{A}$ be again obtained as in Problem 6, with $n = 2$, $Y_1 = Y_2 = (-1,1) \times (-1,1)$, $Z_1 = Z_2 = \{(x,y) \in Y_1 : x \neq 0\}$, and with h given by $h(x,y) = (x+1,y)$ for $x < 0$ and $h(x,y) = (x-1,-y)$ if $x > 0$. Show that $\mathcal{A}$ determines a Hausdorff topology on M. (The resulting 2-dimensional C^ω manifold is called the *Möbius strip*. See Fig. 2 in §3.)

10. Suppose that (U,φ) be a coordinate system in a manifold M. For any open set $U' \subset M$ contained in U, let φ' stand for the restriction of φ to U'. Verify that the pair (U',φ') then also belongs to the maximal atlas of M.

11. Let U be an open subset of an n-dimensional C^r manifold M, and let $\mathcal{A}_U$ be the subset of the maximal C^r atlas $\mathcal{A}$ of M formed by all coordinate systems whose domains are contained in U. Show that $\mathcal{A}_U$ is an n-dimensional maximal C^r atlas on the set U. (The n-dimensional C^r manifold thus obtained is said to be an *open submanifold* of M.)

3. Differentiable mappings

Topics: Continuous mappings; homeomorphisms; differentiable mappings; functions; diffeomorphisms; compactness and continuity; curves; piecewise differentiability; connected sets; connectivity and continuity; connected components; disjoint sums of manifolds; gluing constructions; connected sums.

Given C^s manifolds M, N, a subset K of M and a mapping $f : K \to N$, we say that f is *continuous* if $f(x_k) \to f(x)$ in N as $k \to \infty$ whenever x_k, $k = 1,2,\dots$, is a sequence of points in K that converges in M to a point $x \in K$. A continuous mapping $f : K \to N$ is called a *homeomorphism* between K and the image $f(K) \subset N$ (or, briefly, a *homeomorphic mapping*) if f is one-to-one and the inverse mapping $f^{-1} : f(K) \to M$ is continuous.

A mapping $f : M \to N$ between C^s manifolds is said to be *of class* C^l, where $0 \leq l \leq s \leq \omega$, if it is continuous and, for any coordinate systems (U,φ), (U',φ') in M and N, respectively, the composite $\varphi' \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(U')) \to \mathbf{R}^n$, $n = \dim N$, is of class C^l. (Note that $\varphi(U \cap f^{-1}(U'))$ is open in $\mathbf{R}^m$, $m = \dim M$;

see Problems 2(i) below and 4 in §2.) When $0 \le l \le \infty$, we will then also say that f is C^l *differentiable*; in particular, C^0-differentiability is nothing else than continuity. On the other hand, C^ω mappings will often be referred to as *real-analytic*.

The C^l class of a given mapping will also be referred to as its *regularity*. Regularity of mappings between manifolds is another important example of a local geometric property (see §2). In fact, to verify if a mapping is of class C^l, we only need to use, in both manifolds involved, some simple atlases of our choice, rather than the full maximal atlases. (Regularity of the transition mappings between charts then guarantees the result for all the maximal atlases as well.)

We say that $f : M \to N$ is a C^l *diffeomorphism* if it is one-to-one and onto, while both f and $f^{-1} : N \to M$ are of class C^l. Two C^l manifolds are called *diffeomorphic* (or, C^l-*diffeomorphic*) if there is a C^l diffeomorphism between them.

Mappings $f : M \to \mathbf{R}$ are usually called (real-valued) *functions* on M.

A *curve* in a C^s manifold M is, by definition, a mapping $\gamma : I \to M$, where $I \subset \mathbf{R}$ is any interval containing more than one point. (Thus, I may be open, closed, or half-open, bounded or unbounded.) The curve γ is said to be of class C^l, $l = 0, 1, 2, \dots, r$, if it has a C^l extension to some open interval containing I (note that open intervals are 1-dimensional manifolds). More generally, we say that a curve $\gamma : I \to M$ is *piecewise* C^l, $l = 1, 2, \dots, r$, if it is continuous and there exist $t_1, \dots, t_k$ in the interior of I, for some integer $k \ge 0$, such that the restrictions of γ to $I \cap (-\infty, t_1]$, $I \cap [t_1, t_2]$,..., $I \cap [t_{k-1}, t_k]$, $I \cap [t_k, \infty)$ are all of class C^l.

Let K be a subset of a manifold M. Recall (§2) that K is said to be *compact* if every sequence x_k, $k = 1, 2, \dots$, of points in K has a subsequence that converges in M to a point $x \in K$. On the other hand, K is called (*pathwise*) *connected* if any two points $x, y \in K$ can be joined by a continuous curve in K, that is, if there exists a continuous curve $\gamma : [a, b] \to M$ with $-\infty < a < b < \infty$, $\gamma([a, b]) \subset K$, and $\gamma(a) = x$, $\gamma(b) = y$.

Every manifold M can be uniquely decomposed into a disjoint union of its *connected components*, that is, its maximal pathwise connected subsets. Every connected component is both open and closed as a subset of M (see Problem 10).

Conversely, given a family $\{M_\lambda\}_{\lambda\in\Lambda}$ of n-dimensional C^s manifolds, we can form their *disjoint sum* (or *disjoint union*), which is the n-dimensional C^s manifold with the underlying set

$$M = \bigcup_{\lambda\in\Lambda} (M_\lambda \times \{\lambda\}) \tag{3.1}$$

and with the manifold structure determined by the atlas which is the union of atlases describing the structures of the M_λ. (Note that each M_λ may be treated as a subset of M via the injective mapping $M_\lambda \ni x \mapsto (x, \lambda)$, and so a chart in M_λ is also a chart in M.)

The following assertions are immediate from the above definitions.

Lemma 3.1. *Let M be a C^s manifold with a family $\{U_\lambda\}_{\lambda\in\Lambda}$ of nonempty open sets $U_\lambda \subset M$ such that $\bigcup_\lambda U_\lambda = M$ and $U_\lambda \cap U_{\lambda'} = \emptyset$ whenever $\lambda \ne \lambda'$. Then there exists a unique C^s diffeomorphism between M and the disjoint sum of the U_λ (treated as open submanifolds of M), whose restriction to each U_λ is the identity inclusion mapping.*

Corollary 3.2. *Every manifold is naturally diffeomorphic to the disjoint sum of its connected components.*

As a consequence, in most situations problems concerning manifolds can be directly reduced to questions about *connected* manifolds.

The disjoint sum operation, applied to a family containing more than one manifold, always results in a manifold which is disconnected (that is, not connected; this is immediate from Problems 3.10 and 3.11). There is, however, a class of "sum" or "union" operations that does preserve connectedness. Specifically, given n-dimensional C^s manifolds M' and M'', open subsets $U' \subset M'$, $U'' \subset M''$, and a C^s diffeomorphism $h : U' \to U''$, let

$$M \,=\, M' \cup_h M'' \tag{3.2}$$

be the set obtained by "gluing" M' to M'' with the aid of h, i.e., forming first the disjoint union of M' and M'' and then identifying each $x \in U'$ with $h(x)$. (For details, see Problem 12 below, where Y_1, Y_2 stand for M', M''.) Both M' and M'' then may be regarded as (open) subsets of M, and the union of their maximal atlases is an n-dimensional C^s atlas $\mathcal{A}$ on M (Problem 12). Consequently, M acquires the structure of a manifold, *provided that $\mathcal{A}$ satisfies the Hausdorff axiom.* The latter condition need not hold in general (see Problem 7 in §2), and has to be verified on a case-to-case basis. For instance, it does hold for the gluing procedure used to obtain the Möbius strip (Problem 7 in §2):

Fig. 2. The the Möbius strip

An important class of examples in which the Hausdorff axiom does hold for $\mathcal{A}$ described as above arises in the so-called *connected-sum* constructions. Here M' and M'' are arbitrary n-dimensional C^s manifolds with open subsets $U' \subset B' \subset M'$ and $U'' \subset B'' \subset M''$, chosen so that B', B'' may be C^s-diffeomorphically identified with an open ball in $\mathbf{R}^n$, of some radius $a > 0$, centered at $\mathbf{0}$ and, under those identifications, either of U', U'' is a "spherical shell" obtained from a smaller concentric open "ball" B'_r, B''_r, of radius r, by removing an even smaller *closed* ball K'_ε, K''_ε of some radius $\varepsilon > 0$ (all balls centered at $\mathbf{0}$), with $0 < \varepsilon < r < a$. The C^s-diffeomorphism $h : U' \to U''$ is an *inversion mapping*, transforming the closure in $\mathbf{R}^n$ of the spherical shell that both U', U'' are identified with onto itself in such a way that the inner and outer boundary spheres become interchanged. Specifically,

$$h(\mathbf{x}) \,=\, \varepsilon\, r\, \frac{\varPhi\mathbf{x}}{|\mathbf{x}|^2}\,, \tag{3.3}$$

where $\varPhi : \mathbf{R}^n \to \mathbf{R}^n$ is any norm-preserving linear isomorphism (i.e., any orthogonal $n \times n$ matrix). We now use h to glue together, along U' and U'', *not* the original manifolds M' and M'', but their open submanifolds M_0' and M_0'' with $M_0' = M' \setminus K_\varepsilon'$, $M_0'' = M'' \setminus K_\varepsilon''$. The atlas $\mathcal{A}$ obtained as above on the set $M = M_0' \cup_h M_0''$ defined as in (3.2) then satisfies the Hausdorff axiom (see Problem 21), and so M becomes an n-dimensional C^s manifold, called a *connected sum* of M' and M''.

The connected-sum manifold just constructed depends not only on M' and M'', but also on the additional "parameters" that have to be fixed: The sets $B' \subset M'$ and $B'' \subset M''$, their diffeomorphic identifications with a ball of radius a in $\mathbf{R}^n$, the smaller radii r and ε, and the norm-preserving isomorphism F. Nevertheless, one often uses the informal notation

$$M = M' \# M'' \tag{3.4}$$

for the connected sum. (As a matter of fact, connected sums of the given manifolds M' and M'' may represent either one, or two diffeomorphic types, and if there are two of them, the difference between them results from the two different possible values of $\det F$, namely 1 and -1.)

A 2-dimensional C^s manifold is usually referred to as a C^s *surface*. Let $k \geq 0$ be an integer. By a *closed orientable* C^s *surface of genus* k we mean a C^s surface which is C^s diffeomorphic to

a. The sphere S^2, if $k = 0$.
b. A connected sum (3.4) of a closed orientable C^s surface M' of genus $k-1$ and a C^s surface M'' diffeomorphic to the torus T^2, if $k \geq 1$.

Thus, starting from the sphere S^2 and successively "adding" the torus T^2, we obtain examples of surfaces of all possible genera.

Fig. 3. The closed orientable surface of genus 2

Problems

1. Set $\mathbf{R}^0 = \{0\}$. By definition, let all mappings to/from any $\mathbf{R}^n$ from/to $\mathbf{R}^0$ be of class C^ω (and continuous), and let both subsets of $\mathbf{R}^0$ be open. Our definitions of charts, atlases and manifolds thus make sense in the 0-dimensional case as well. Verify that, for a 0-dimensional atlas on a set M, every subset of M is open (this is referred to as the *discrete topology*), and the Hausdorff axiom always holds. Show that every nonempty set M carries a unique structure of a 0-dimensional manifold, and that this manifold is compact if and only if M is finite.

2. Let $f : M \to N$ be a mapping between manifolds. Verify that the following three conditions are equivalent:

(i) f is continuous.
(ii) The f-preimage $f^{-1}(U)$ of any open set $U \subset N$ is open in M.

(iii) The f-preimage $f^{-1}(K)$ of any closed set $K \subset N$ is closed in M.

3. Let M be an n-dimensional C^s manifold. Show that, for a pair (U, φ), the following two conditions are equivalent:

(i) (U, φ) is a coordinate system in M (that is, an element of the maximal atlas forming the manifold structure of M).

(ii) U is an open subset of M and φ is a C^s diffeomorphism between U and an open submanifold of $\mathbf{R}^n$. (See also Problems 11 in §2 and 7 in §1.)

4. Let $f : K \to N$ be a continuous mapping from a compact subset K of a manifold M into a manifold N. Prove that

(a) The image $f(K)$ is also compact.

(b) If, in addition, f restricted to K is injective, then $f : K \to f(K)$ is a homeomorphism.

5. Verify that a subset K of $\mathbf{R}^n$ is compact if and only if it is closed and bounded (in the Euclidean norm).

6. Let $f : K \to \mathbf{R}$ be a continuous function on a compact subset K of a manifold M. Show that

(a) f is bounded, i.e., $|f| \le C$ for some constant $C \ge 0$.

(b) f assumes its largest and smallest values $\max f$, $\min f$ somewhere in K.

7. Let $f : K \to N$ be a continuous mapping from a subset K of a manifold M into a manifold N. Verify that, if K is (pathwise) connected, then so is the image $f(K)$.

8. Prove that compactness and connectedness of subsets of manifolds are both preserved by the Cartesian product operation.

9. Let K be a subset of a manifold M. By a *connected component* of K we mean any pathwise connected subset $K' \subset K$ which is maximal (i.e., not contained in any other pathwise connected subset of K). The pathwise connected components of M itself are simply called the *connected components* of M.

(a) Show that the connected components of K are pairwise disjoint, and their union equals K.

(b) Verify that two points $x, y \in K$ lie in the same connected component of K if and only if they can be joined by a continuous curve in K.

10. Prove that the connected components of any manifold M are both open and closed as subsets of M. Verify that the connected components of a 0-dimensional manifold M are the one-point subsets of M.

11. Show that a manifold M is connected if and only if the only subsets of M that are simultaneously open and closed are $\emptyset$ and M. (Hint below.)

12. Prove that the assertion of Problem 6 in §2 remains valid when modified as follows: Y_1 and Y_2, instead of being open sets in $\mathbf{R}^n$, are arbitrary n-dimensional C^s manifolds; (i) is the same, while (ii) is replaced by

(ii)' Let $\mathcal{A}$ be the union of atlases determining the manifold structures of Y_1 and Y_2. (Note that each Y_α may be treated as a subset of M via the injective mapping $Y_\alpha \ni x \mapsto [(x, \lambda)]_\sim$.) Show that $\mathcal{A}$ is an n-dimensional C^s atlas on M.

13. Given a real or complex vector space V of dimension $n < \infty$, let us denote by $\pi : V \setminus \{0\} \to P(V)$ the *standard projection* mapping, sending each nonzero vector onto the subspace it spans. Show that π is of class C^ω and onto. Verify that $V \setminus \{0\}$ is connected if $n \ge 2$. (Hint below.)

14. For a Euclidean space V, let the *normalization* mapping $\nu : V \setminus \{0\} \to S(V)$ be given by $\nu(w) = w/|w|$. Prove that ν is of class C^ω and onto.

15. Prove that the spheres $S(V)$ (with $\dim V > 1$), the tori T^n, and real/complex projective spaces $P(V)$ defined above are all connected and compact.

16. For a Euclidean space V, define the mapping $F : S(V) \to P(V)$ by $F(v) = \mathbf{R}v$. Verify that F is of class C^ω, onto, and the preimage under F of any point in $P(V)$ is a pair of opposite vectors ("antipodal points") in $S(V)$. (Thus, the underlying set of the real projective space $P(V)$ may be regarded as obtained by identifying antipodal points in the sphere $S(V)$.)

17. Show that the real projective line $\mathbf{R}\mathrm{P}^1$ is diffeomorphic to the circle (1-dimensional sphere) S^1. Cf. Problem 4 in §14. (Hint below.)

18. Prove that the complex projective line $\mathbf{C}\mathrm{P}^1$ is diffeomorphic to the Riemann sphere (Problem 8). Cf. Problem 4 in §14. (Hint below.)

19. Verify that the Riemann sphere (Problem 8 in §2) is diffeomorphic to the 2-dimensional sphere S^2. Cf. Problem 4 in §14. (Hint below.)

20. Prove that the Möbius strip (Problem 9 in §2) is diffeomorphic to the real projective plane $\mathbf{R}\mathrm{P}^2$ minus a point. (Hint below.)

21. Show that the connected-sum operations described in the text actually leads to manifolds, i.e., the atlases they produce always satisfy the Hausdorff axiom. (Hint below.)

22. Prove that every C^s manifold is C^s-diffeomorphic to a connected sum $M \# S^n$, with $n = \dim M$. (Hint below.)

23. Verify that the torus T^2 is a closed orientable C^ω surface of genus 1.

24. Given pathwise connected subsets K', K'' of a manifold M such that $K' \cap K''$ is nonempty, verify that $K' \cup K''$ is pathwise connected. Generalize this statement to the case of arbitrary (not just 2-element) families of pathwise connected sets.

25. Given a C^s manifold M, let B be an open subset of M that admits a C^s-diffeomorphic identification with a ball of radius a centered at $\mathbf{0}$ in $\mathbf{R}^n$, and let $K \subset B$ be the set corresponding under such an identification to an open or closed ball of a smaller radius $\varepsilon \geq 0$, also centered at $\mathbf{0}$. (Thus, it is possible that $K = \{\mathbf{0}\}$.) Prove that, if M is connected and $\dim M \geq 2$, then $M \setminus K$ is a pathwise connected subset of M. (Hint below.)

26. Prove that connectedness (in dimensions $n \geq 2$) and compactness (in all dimensions) are both preserved under the connected-sum constructions. What about dimension 1? (Hint below.)

Hint. In Problem 13, connectedness: Fix $v, w \in V \setminus \{0\}$. If w is not a negative multiple of v, the line segment connecting v to w lies entirely in $V \setminus \{0\}$. If $w = \lambda v$ with $\lambda < 0$, we may choose $u \in V \setminus \mathbf{R}v$ and join both v, w to u in $V \setminus \{0\}$ as before.

Hint. In Problem 17, define $F : \mathbf{R}\mathrm{P}^1 \to S^1$ by $F(\mathbf{R}z) = (z/|z|)^2 = z/\overline{z}$, treating $z \in \mathbf{R}^2 = \mathbf{C}$ as a complex number.

Hint. In Problem 18, define $F_\alpha : Y_\alpha \to \mathbf{C}\mathrm{P}^1$ by $F_1(z) = [z, 1]$, $F_2(z) = [1, z]$, with Y_α as in Problem 8 in §2 and $[\,,\,]$ referring to homogeneous coordinates (§2).

Hint. In Problem 19, define $F_\alpha : Y_\alpha \to S^2$ by $F_1(z) = (|z|^2 + 4)^{-1}(4z, |z|^2 - 4)$ (the inverse of the stereographic projection, mentioned in §2), and $F_2(z) = (4|z|^2 + 1)^{-1}(4z, 1 - 4|z|^2)$.

Hint. In Problem 20, define $F_\alpha : Y_\alpha \to \mathbf{R}\mathrm{P}^2 \setminus \{L\}$ (notation of Problem 9 in §2) with $L = [0, 0, 1]$ (homogeneous coordinates) by

$$F_1(x, y) = [\cos(\pi y/2)\cos(\pi x/2), \cos(\pi y/2)\sin(\pi x/2), \sin(\pi y/2)]\,,$$
$$F_2(x, y) = [-\cos(\pi y/2)\sin(\pi x/2), \cos(\pi y/2)\cos(\pi x/2), \sin(\pi y/2)]\,.$$

Hint. In Problem 21, consider two different points $x, y \in M = M' \# M''$. Since M_0' and M_0'' (notation as in the text) may be treated as open subsets of M, we may assume that $x \in M_0' \setminus M_0''$ and $y \in M_0'' \setminus M_0'$. (In fact, if x, y are both in M_0' or both in M_0'', they can be separated there.) Hence $x \in M'$ lies outside the closure $\mathrm{clos}(B_r')$ of B_r' and $y \in M''$ lies $\mathrm{clos}(B_r'')$, and so x, y have the disjoint neighborhoods $M' \setminus \mathrm{clos}(B_r')$ and $M'' \setminus \mathrm{clos}(B_r'')$ in M.

Hint. In Problem 22, set $M' = M$ and choose $B' \subset M'$ and a diffeomorphic identifications of B' with a ball of radius a centered at $\mathbf{0}$ in $\mathbf{R}^n$, as well as the smaller radii r and ε, as in the text. Using a scale-factor transformation of $\mathbf{R}^n$, we may always select $a > 2$, and then let us choose $r = 2$ and $\varepsilon = 1/2$. Writing elements of $\mathbf{R}^{n+1}$ as pairs $(\mathbf{x}, t)$ with $\mathbf{x} \in \mathbf{R}^n$ and $t \in \mathbf{R}$, let us denote by $\varphi = \varphi_{(\mathbf{0},1)} : U_{(\mathbf{0},1)} \to A_{(\mathbf{0},1)}$ the stereographic projection (see §2) with the pole $(\mathbf{0}, 1)$, where $U_{(\mathbf{0},1)} = S^n \setminus \{(\mathbf{0}, 1)\}$, S^n is the unit sphere centered at $\mathbf{0}$ in $\mathbf{R}^{n+1}$, and $A_{(\mathbf{0},1)} = \{(\mathbf{x}, -1) : \mathbf{x} \in \mathbf{R}^n\}$. Still treating B' as a subset of $\mathbf{R}^n$, let us set $M'' = S^n$, $K_{1/2}'' = \{(\mathbf{0}, 1)\} \cup \varphi^{-1}([\mathbf{R}^n \setminus B_1'] \times \{-1\})$ and $B_1'' = \{(\mathbf{0}, 1)\} \cup \varphi^{-1}([\mathbf{R}^n \setminus K_{1/2}'] \times \{-1\})$. The stereographic projection (see §2) with the pole $(\mathbf{0}, -1)$ followed by the projection $(\mathbf{x}, 1) \mapsto \mathbf{x}$ then makes $K_{1/2}''$ and B_1'' appear as a closed/open ball, centered at $\mathbf{0}$, and having the respective radius $1/2$ or 1. The C^s-diffeomorphism $h : U' \to U''$ (notation as in the text) comes from the inversion mapping (3.3) with $\Phi = \mathrm{Id}$.

To describe a C^s-diffeomorphism $F : M' \# M'' \to P$, for any manifold P, we just need to provide mappings $F' : M' \setminus K_\varepsilon' \to P$ and $F'' : M'' \setminus K_\varepsilon'' \to P$, which both are C^s-diffeomorphisms onto open submanifolds of P, while the intersection of their images is $F'(U') = F''(U'')$ and $F'' \circ h$ coincides with the restriction of F' to U'. In our case, $\varepsilon = 1/2$ and $P = M = M'$, and we may declare F' to be the identity mapping of $M' \setminus K_{1/2}' \subset M$, and set $F''(w) = \chi(w)$ with $\varphi(w) = (\chi(w), 1)$. Note that the inversion mapping (3.3) with $\varepsilon r = 1$ and $\Phi = \mathrm{Id}$ coincides with the transition between two stereographic projections with mutually antipodal (i.e., opposite) poles.

Hint. In Problem 25, let S be the boundary sphere of any closed ball Q centered at $\mathbf{0}$ in $\mathbf{R}^n$ with $K \subset Q \subset B$. Every $x \in \tilde{M} = (M \setminus Q) \cup S$ can be joined to some $y \in S$ by a continuous curve lying entirely in $\tilde{M}$; to see this, choose a continuous curve $\gamma : [a, b] \to M$ connecting x to a point in K and then replace it with γ restricted to $[a, c]$, where c is the supremum of those $t \in [a, b]$ with $\gamma([a, t]) \subset \tilde{M}$. Since S is connected (Problem 15), it follows that $\tilde{M}$ is connected. Now, if K is an open ball, we may choose Q to be the closure of K, and then $M \setminus K = \tilde{M}$. On the other hand, if K is an closed ball, $M \setminus K$ is a union of sets of the form $\tilde{M}$ (with the radii of the corresponding Q approaching ε from above), and we can use Problem 24.

Hint. In Problem 26, let $Q' \subset B_r'$, $Q'' \subset B_r''$ be open balls slightly larger than K_ε' or, respectively, K_ε'' (notation as in the text). Then the connected-sum manifold (3.4) is a union of the connected or, respectively, compact subsets $M' \setminus Q'$ and $M'' \setminus Q''$ (see Problem 25 in §3). The compact case now is obvious, and for connectedness we can use Problem 24 in §3.

4. Lie groups

Topics: Groups; Abelian groups; Lie groups; examples.

Recall that a *group* is a set G endowed with a fixed binary operation (called the *multiplication*), that is, a mapping

$$G \times G \ni (a, b) \mapsto ab \in G\,, \tag{4.1}$$

having the following properties:

a. $a(bc) = (ab)c$ for all $a, b, c \in G$ (*associativity*). Thus, we may skip the parentheses and simply write abc instead of $a(bc)$.
b. There exists a *neutral element* $1 \in G$ with $1a = a1 = a$ for all $a \in G$. (It follows that such an element is unique.)
c. Each $a \in G$ has an *inverse* $a^{-1} \in G$ with $aa^{-1} = a^{-1}a = 1$. (Again, a^{-1} is uniquely determined by a.)

The group G is said to be *Abelian* if its multiplication is *commutative*, that is, $ab = ba$ for all $a, b \in G$. For Abelian groups one sometimes uses the *additive* rather than multiplicative notation, writing $a + b$, 0, $-a$ and $a - b$ instead of ab, 1, a^{-1} and ab^{-1}, respectively.

By a *Lie group* of class C^s, $0 \le s \le \omega$, we mean a C^s manifold G with a fixed group structure such that both the multiplication and the inverse

$$G \times G \ni (a, b) \mapsto ab \in G\,, \qquad G \ni a \mapsto a^{-1} \in G \tag{4.2}$$

are C^s mappings. (Here $G \times G$ is the Cartesian product manifold; see Problem 3 in §2.)

Example 4.1. For any finite-dimensional real vector space V, the underlying additive group of V is a Lie group of class C^ω.

Example 4.2. Let $\mathcal{A}$ be a finite-dimensional associative algebra with unit over the field $\mathbf{R}$ of real numbers. The open submanifold G of $\mathcal{A}$ consisting of all invertible elements, with the algebra multiplication of $\mathcal{A}$ restricted to G, then is a Lie group of class C^ω, with $\dim G = \dim \mathcal{A}$. (See Problems 2, 4.)

Example 4.3. Let V be a real or complex vector space of real/complex dimension $\dim V = n < \infty$. The set $\mathrm{GL}(V)$ of all real/complex linear isomorphisms of V onto itself, with the composition operation, then is naturally a Lie group of dimension n^2 (when V is real) or $2n^2$ (when V is complex). This is a special case of Example 4.2, with $\mathrm{GL}(V) = G$ for the real algebra $\mathfrak{gl}(V) = \mathcal{A}$ of all real/complex linear operators $V \to V$.

Example 4.4. The sets $\mathrm{GL}\,(n, \mathbf{R})$ and $\mathrm{GL}\,(n, \mathbf{C})$ of all nonsingular real (or, respectively, complex) $n \times n$ matrices, with the matrix multiplication, carry natural Lie group structures of dimensions n^2 and $2n^2$, respectively. (This is a special case of Example 4.3, with $V = \mathbf{R}^n$ or $V = \mathbf{C}^n$, since linear operators between numerical spaces may be identified with matrices.)

Example 4.5. In particular, the multiplicative groups $\mathbf{R} \setminus \{0\} = \mathrm{GL}\,(1, \mathbf{R})$ and $\mathbf{C} \setminus \{0\} = \mathrm{GL}\,(1, \mathbf{C})$ of all nonzero real/complex numbers are Lie groups with their structures of open submanifolds of $\mathbf{R}$ and $\mathbf{C}$, respectively.

Example 4.6. Every group G may be viewed as a Lie group with $\dim G = 0$. (Such Lie groups are called *discrete*.) In fact, the maximal atlas of G then consists of charts with one-point (or empty) domains. Thus, the group structure of a Lie group does not, in general, determine its manifold structure.

Given groups G and H, by a *homomorphism* from G to H we mean a mapping $F : G \to H$ such that $F(ab) = F(a)F(b)$ for all $a, b \in G$. In the case where G and H are Lie groups of class C^s, we will say that F is a C^l homomorphism, $0 \le l \le r$, if it is both a group homomorphism and a C^l mapping between manifolds. (For examples, see Problems 6 – 7 in §4.) A C^l Lie-group homomorphism is called a C^l *isomorphism* if it also is a C^l diffeomorphism of the underlying manifolds. Two Lie groups G and H will be called C^l*-isomorphic* if there is a C^l isomorphism $G \to H$.

By the *algebra of quaternions*, denoted by $\mathbf{H}$, we mean the vector space $\mathbf{R}^4$ endowed with the bilinear operation of (quaternion) multiplication, written $\mathbf{H} \times \mathbf{H} \ni (p, q) \mapsto pq \in \mathbf{H}$, and described as follows. Let 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be the standard basis of $\mathbf{H} = \mathbf{R}^4$. Then

$$\begin{aligned} &\mathbf{i}^2 \,=\, \mathbf{j}^2 \,=\, \mathbf{k}^2 \,=\, -1\,, \\ &\mathbf{ij} \,=\, -\,\mathbf{ji} \,=\, \mathbf{k}\,, \quad \mathbf{jk} \,=\, -\,\mathbf{kj} \,=\, \mathbf{i}\,, \quad \mathbf{ki} \,=\, -\,\mathbf{ik} \,=\, \mathbf{j}\,, \end{aligned} \tag{4.3}$$

while 1 is the neutral element of the quaternion multiplication.

It follows easily from (4.3) that the quaternion multiplication is associative, and that the mapping $\mathbf{R} \ni t \mapsto t \cdot 1 \in \mathbf{H}$ is an algebra homomorphism. From now on we will identify $\mathbf{R}$ with its image and write

$$\mathbf{R} \,\subset\, \mathbf{H}\,. \tag{4.4}$$

The *conjugation* of quaternions is the real-linear operator $\mathbf{H} \ni x \mapsto \overline{x} \in \mathbf{H}$, defined by

$$\overline{1} \,=\, 1\,,\ \overline{\mathbf{i}} \,=\, -\,\mathbf{i}\,,\ \overline{\mathbf{j}} \,=\, -\,\mathbf{j}\,,\ \overline{\mathbf{k}} \,=\, -\,\mathbf{k}\,. \tag{4.5}$$

It follows easily that

$$\overline{xy} \,=\, \overline{y}\,\overline{x}\,, \qquad \overline{\overline{x}} \,=\, x \tag{4.6}$$

for all $x, y \in \mathbf{H}$. We call $\mathbf{i}, \mathbf{j}, \mathbf{k}$ for the *pure quaternion units*. More generally, by *pure quaternions* we mean elements of the 3-dimensional subspace

$$1^\perp \,\subset\, \mathbf{H} \tag{4.7}$$

spanned by $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Here $^\perp$ denotes the orthogonal complement relative to the standard Eulidean inner product $\langle\,,\rangle$ in $\mathbf{H} = \mathbf{R}^4$. We thus have the direct-sum decomposition (cf. also Problem 26 in §9):

$$\mathbf{H} \,=\, \mathbf{R} \oplus 1^\perp\,, \tag{4.8}$$

which, for any given quaternion x, will be written as

$$x \,=\, \operatorname{Re} x \,+\, \operatorname{Pu} x\,, \qquad \operatorname{Re} x \in \mathbf{R}\,, \qquad \operatorname{Pu} x \in 1^\perp\,. \tag{4.9}$$

Then, for all quaternions x, y,

$$\operatorname{Re} \overline{x} \,=\, \operatorname{Re} x\,, \qquad \operatorname{Pu} \overline{x} \,=\, -\operatorname{Pu} x\,, \tag{4.10}$$

$$x\overline{x} \,=\, |x|^2 \,\in\, \mathbf{R} \,\subset\, \mathbf{H}\,, \tag{4.11}$$

and, consequently (since a symmetric bilinear form is determined by its quadratic function),

$$\langle x, y\rangle \,=\, \operatorname{Re} x\overline{y} \,=\, \operatorname{Re} \overline{x}y\,. \tag{4.12}$$

(Here $|\ |$ is the Euclidean norm with $|x|^2 = \langle x, x\rangle$.) In view of (4.11), $\mathbf{H}$ is an associative, noncommutative *field*, that is, every $x \in \mathbf{H}\setminus\{0\}$ has a multiplicative inverse x^{-1} with $xx^{-1} = x^{-1}x = 1$. Namely,

$$x^{-1} \ = \ \frac{\overline{x}}{|x|^2}\,. \tag{4.13}$$

Also, the norm $|\ |$ is *multiplicative* in the sense that

$$|xy| \ = \ |x| \cdot |y| \tag{4.14}$$

for all $x, y \in \mathbf{H}$. See Problem 12 in §4.

Example 4.7. For a finite-dimensional real vector space V, the set $\mathrm{GL}^+(V)$ of all linear isomorphisms $A : V \to V$ with $\det A > 0$ is a Lie group of class C^ω. In fact, it is an open subgroup of $\mathrm{GL}(V)$ (cf. Problem 2 in §4). In particular, we have the C^ω Lie group $\mathrm{GL}^+(n,\mathbf{R}) = \mathrm{GL}^+(\mathbf{R}^n)$ of all real $n \times n$ matrices with positive determinants.

Example 4.8. The multiplicative group $\mathbf{H}\setminus\{0\}$ of all nonzero quaternions is a C^ω Lie group, namely, as the set of all invertible elements of the quaternion algebra $\mathbf{H}$. (See Example 4.2.)

It turns out that the spheres S^0, S^1 and S^3 are the underlying manifolds of Lie groups. Before discussing those examples, let us observe that, instead the atlas of stereographic projections on the unit sphere $S(V)$ in a Euclidean space V (see §2), it is sometimes more convenient to use the *projective atlas*, described below. (Both atlases are contained in the same maximal C^ω atlas, since transitions between them are given by simple algebraic formulas.) Specifically, the projective atlas $\tilde{\mathcal{A}}$ is the set of all projective charts $(\tilde U_v, \tilde\varphi_v)$, indexed by $v \in S(V)$ and given by $\tilde U_v = \{x \in S(V) : \langle v, x\rangle > 0\}$, with $\tilde\varphi_v : \tilde U_v \to A_{-v}$ given by

$$\tilde\varphi_v(x) \ = \ x/\langle v, x\rangle\,. \tag{4.15}$$

Here, as in §2, A_{-v} is the affine space $v^\perp + v$, i.e., $A_v = \{y \in V : \langle y, v\rangle = 1\}$. Then,

$$[\tilde\varphi_v]^{-1}(y) \ = \ y/|y|\,. \tag{4.16}$$

Example 4.9. Let

$$S^n \ = \ \{x \in \mathbf{K} : |x| = 1\} \tag{4.17}$$

be the multiplicative group (cf. (4.14)) of the unit elements of the algebra $\mathbf{K}$ of real numbers ($\mathbf{K} = \mathbf{R}$, $n = 0$), complex numbers ($\mathbf{K} = \mathbf{C}$, $n = 1$), or quaternions ($\mathbf{K} = \mathbf{H}$, $n = 3$). Each of the spheres S^0, S^1 and S^3 thus becomes a C^ω Lie group (which is obvious from (4.15) and (4.16) along with bilinearity of the algebra multiplication). The 2-element group $S^0 = \{1, -1\}$ is also denoted by $\mathbf{Z}_2$.

Problems

1. Show that, in the definition of a Lie group of class C^s, the requirement that the multiplication and inverse be both C^s, is equivalent to the condition

$$G \times G \ni (a, b) \mapsto ab^{-1} \in G \ \text{ is of class } C^s. \tag{4.18}$$

2. Show that any open subgroup of a C^s Lie group, with the open-submanifold structure, is a Lie group of class C^s.

3. Verify that the direct product $G \times H$ of two C^s Lie groups G and H, with the product-manifold structure, is a C^s Lie group.

4. Prove the assertion in Example 4.2 above. In particular, show that G is open in $\mathcal{A}$. (Hint below.)

5. Verify the claims made in Examples 4.3 – 4.6 above.

6. For V as in Example 4.3, show that the *determinant mapping*

$$\det : \mathrm{GL}(V) \to \mathbf{K} \setminus \{0\}\,, \tag{4.19}$$

where $\mathbf{K}$ is the scalar field, is a real-analytic homomorphism of Lie groups.

7. For $\mathcal{A}$ and G as in Example 4.2, verify that the formula $(\mathrm{ad}\, a)x = axa^{-1}$ defines a Lie-group homomorphism $\mathrm{ad} : G \to \mathrm{GL}(\mathcal{A})$ of class C^ω.

8. Let $T : V \to W$ be a linear operator between finite-dimensional vector spaces V, W. Verify that T is injective (surjective) if and only if there exists a linear mapping $T' : W \to V$ with $T'T = \mathrm{Id}_V$ (or, respectively, $TT' = \mathrm{Id}_W$).

9. Given a group G with the multiplication written as in (4.1), let G' be the same set with the new "reverse-order" multiplication given by $(x, y) \mapsto yx$. Show that G' is a group isomorphic to G. Verify the analogous statement for Lie groups. (See also Problem 1 in §12.)

10. Verify that, for any $a \in \mathbf{R}$ and $c \in \mathbf{C}$, the assignments

$$t \mapsto |t|\,, \qquad t \mapsto \operatorname{sign} t = t/|t|\,, \qquad t \mapsto e^{at}\,,$$
$$t \mapsto e^{iat}\,, \qquad z \mapsto |z|\,, \qquad z \mapsto z/|z|\,, \qquad z \mapsto e^{cz}$$

are C^ω Lie-group homomorphisms

$$\mathbf{R} \setminus \{0\} \to (0, \infty)\,, \qquad \mathbf{R} \setminus \{0\} \to \mathbf{Z}_2 = \{1, -1\}\,, \qquad \mathbf{R} \to (0, \infty)\,,$$
$$\mathbf{R} \to S^1\,, \qquad \mathbf{C} \setminus \{0\} \to (0, \infty)\,, \qquad \mathbf{C} \setminus \{0\} \to S^1\,, \qquad \mathbf{C} \to \mathbf{C} \setminus \{0\}\,,$$

$(0, \infty)$ being the multiplicative group of positive real numbers.

11. Find C^ω Lie-group isomorphisms $S^n \times (0, \infty) \to \mathbf{K} \setminus \{0\}$, where $\mathbf{K} = \mathbf{R}$ and $n = 0$, or $\mathbf{K} = \mathbf{C}$ and $n = 1$, or $\mathbf{K} = \mathbf{H}$ and $n = 3$.

12. Obtain (4.14) as a direct consequence of (4.11) and (4.6).

13. Show that quaternionic square roots of unity are precisely the pure quarternions of norm 1, that is,

$$S^3 \cap 1^\perp = \{x \in \mathbf{H} : x^2 = -1\}\,. \tag{4.20}$$

14. The *center* of a group G is the set of all $a \in G$ which commute with every element of G. Verify that the center of G is a normal subgroup of G.

15. Show that the centers of the Lie groups $\mathbf{H} \setminus \{0\}$ and S^3 are $\mathbf{R} \setminus \{0\}$ and, respectively, $\mathbf{Z}_2 = \{1, -1\}$.

16. Let p, q be quaternions such that, for each quaternion x, $px = xq$. Show that then $p = q \in \mathbf{R}$.

Hint. In Problem 4, first note that the matrix inverse $\mathfrak{M} \mapsto \mathfrak{M}^{-1}$ is a real-analytic mapping $\mathrm{GL}(n, \mathbf{K}) \to \mathrm{GL}(n, \mathbf{K})$ (with $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$), since the entries of $\mathfrak{M}^{-1}$ are rational functions of those of $\mathfrak{M}$. For V as in Example 4.3, this implies real-analyticity of $\mathrm{GL}(V) \ni F \mapsto F^{-1} \in \mathrm{GL}(V)$ (use an isomorphism between V and $\mathbf{K}^n$.)

Now let $\mathcal{A}$ be as in Example 4.2. The linear operator $P : \mathcal{A} \to \mathfrak{gl}(\mathcal{A})$ given by $(Pa)b = ab$ is injective as $(Pa)1 = a$ (where $\mathcal{A}$ is treated as a real vector space), and so (Problem 8) we may choose a linear operator $Q : \mathfrak{gl}(\mathcal{A}) \to \mathcal{A}$ with $QP = 1$ (the identity mapping of $\mathcal{A}$). Also, $P(ab) = (Pa)(Pb)$ and $P1 = 1$ (that is, P is an algebra homomorphism). Thus, $a \in G$ if and only if $Pa \in \mathrm{GL}(\mathcal{A})$ and then $P(a^{-1}) = (Pa)^{-1}$. (In fact, if $Pa \in \mathrm{GL}(\mathcal{A})$, the element $b = (Pa)^{-1}1$ is the inverse of a in $\mathcal{A}$. Namely, we have $ba = 1$, so $P(ab) = (Pa)(Pb) = (Pb)(Pa) = 1$ and hence $ab = 1$; in other words, a one-sided inverse of an element in $\mathcal{A}$ must be a two-sided inverse, since the same is true in $\mathrm{GL}(\mathcal{A})$.)

Thus, G is open in $\mathcal{A}$, as it is the P-preimage of $\mathrm{GL}(\mathcal{A})$ (which in turn is open, being the det-preimage of $\mathbf{K} \setminus \{0\}$).

Finally, the mapping $G \ni a \mapsto a^{-1} \in G$ is real-analytic since it is the composite in which the restriction $P : G \to \mathrm{GL}(\mathcal{A})$ is followed first by $\mathrm{GL}(\mathcal{A}) \ni F \mapsto F^{-1} \in \mathrm{GL}(\mathcal{A})$ and then by $Q : \mathrm{GL}(\mathcal{A}) \to \mathcal{A}$ (both P, Q being here linear operators restricted to open subsets.)

On the other hand, real-analyticity of the multiplication of G in Example 4.2 is a trivial consequence of its bilinearity.

CHAPTER 2

Tangent Vectors

5. Tangent and cotangent vectors

Topics: Index notation; partial derivatives; chain rule and group property; curves and tangentiality; tangent vectors; velocity; vector components; transformation rule; tangent vector space; tangent spaces in vector and affine spaces; directional derivative; germs; components of mappings; differentials of differentiable mappings; chain rule for differentials; tangent spaces for open submanifolds; cotangent spaces and vectors; differentials of C^1 functions; bases of tangent and cotangent spaces naturally distinguished by a given coordinate system; invariance of the dimension under diffeomorphisms.

All manifolds studied from now on are of class C^r with $r \geq 1$. Thus, we exclude C^0 manifolds, also known as *topological manifolds*, and restrict our discussion to manifolds that are C^r-differentiable, $1 \leq r \leq \infty$, or real-analytic.

Coordinate systems (U, φ) in a given manifold M will often be written as $x^1, \dots, x^n$ (or, briefly, x^j), where $n = \dim M$ and the $x^j : U \to \mathbf{R}$ are the components of $\varphi : U \to \mathbf{R}^n$, i.e., $\varphi(y) = (x^1(y), \dots, x^n(y))$ (and U is presumed to be known from the context, or irrelevant). Another coordinate system (U', φ') in M then may be written as $x^{1'}, \dots, x^{n'}$ (or, briefly, $x^{j'}$), with the same basis letter x and a different range of indices. The index sets $\{1, \dots, n\}$ and $\{1', \dots, n'\}$ are just two *disjoint sets* with $\dim M$ elements and each of them, formally, may be thought of as the Cartesian product of the set of the first $\dim M$ positive integers with the one-element set $\{(U, \varphi)\}$ or, respectively, $\{(U', \varphi')\}$. Different manifolds may be distinguished by using different alphabet ranges to label coordinates: j (and $k, l, \dots$) in M, α (and $\beta, \gamma, \dots$) in N, etc.

Another part of the *index notation* consists of the following conventions:

a. In each term ("monomial") forming a given expression, any index may appear at most twice.
b. If an index appears once in one term, then it must appear once in every other term of the given expression, always in the same position (up or down).
c. If an index appears twice in one term, then it must appear *once as a subscript and once as a superscript*, and the term is to be summed over that index.

Given coordinates x^j in a manifold M, corresponding to a coordinate system (U, φ), and a C^1 function $f : U \to \mathbf{R}$, we define the (continuous) partial derivatives $\partial_j f : U \to \mathbf{R}$ by

$$\partial_j f = \frac{\partial (f \circ \varphi^{-1})}{\partial x^j} \circ \varphi \,. \tag{5.1}$$

For two coordinate systems (U, φ) and (U', φ'), this leads to the functions

$$p_j^{j'} = \partial_j x^{j'} : U \cap U' \to \mathbf{R} \,. \tag{5.2}$$

Functions of this type appear in the *chain rule*

$$\partial_{j'} = p^j_{j'} \partial_j \,, \tag{5.3}$$

which means that $\partial_{j'} f = p^j_{j'} \partial_j f$ for any C^1 function $f : U \cap U' \to \mathbf{R}$. Also, when three coordinate systems are involved,

$$p^{j''}_j = p^{j'}_j p^{j''}_{j'} \tag{5.4}$$

in the intersection of all three coordinate domains (this is known as the *group property*). In particular,

$$p^k_j = \delta^k_j = \begin{cases} 1\,, & \text{if } j = k \\ 0\,, & \text{if } j \neq k \end{cases} \tag{5.5}$$

(the Kronecker delta), so that $p^{j'}_j p^k_{j'} = \delta^k_j$, (i.e., at each $y \in U \cap U'$, the matrix $[p^j_{j'}(y)]$ is the inverse of $[p^{j'}_j(y)]$ rather than its transpose).

Let y be a fixed point in a C^r manifold M $(r \geq 1)$. Denote by $\mathcal{G}_y$ the set of all pairs (γ, t) such that $\gamma : I_\gamma \to M$ is a C^1 curve in M defined on an interval I_γ, and $t \in I_\gamma$ with $\gamma(t) = y$. For (γ, t), $(\delta, s) \in \mathcal{G}_y$, write $(\gamma, t) \sim (\delta, s)$ if $\dot\gamma^j(t) = \dot\delta^j(s)$ for *some* coordinates x^j at y (that is, a coordinate system whose domain contains y) and each $j = 1, \ldots, n = \dim M$; the dot stands for the derivative. Since

$$\dot\gamma^{j'}(t) = p^{j'}_j(y)\, \dot\gamma^j(t) \tag{5.6}$$

(by the ordinary chain rule), the word *some* can be replaced with *any*, and $\sim$ is an equivalence relation. The set

$$T_y M = \mathcal{G}_y / \sim \tag{5.7}$$

of all equivalence classes of $\sim$ is called the *tangent space* of M at Y, and its elements are referred to as the *tangent vectors*. The $\sim$ equivalence class of any $(\gamma, t) \in \mathcal{G}_y$ is denoted by $\dot\gamma(t)$ or $\dot\gamma_t$, and called the *velocity* of the curve γ at the parameter value ("time") t.

Fig. 4. Tangent vectors as equivalence classes of curves

Any fixed coordinates x^j at y lead to the one-to-one surjective mapping $T_y M \to \mathbf{R}^n$ written as $v \mapsto (v^1, \ldots, v^n)$ and characterized by

$$v^j = \dot\gamma^j(t) \qquad \text{whenever} \qquad v = \dot\gamma(t)\,. \tag{5.8}$$

The numbers v^j are called the *components* of the vector $v \in T_y M$ relative to the coordinates x^j. In view of (5.6), they satisfy the *transformation rule*

$$v^{j'} = p^{j'}_j v^j \,, \tag{5.9}$$

where $p_j^{j'}$ stands for $p_j^{j'}(y)$. Consequently, T_yM carries a unique structure of a vector space such that for some (or any) coordinates x^j at y, the mapping

$$T_yM \ni v \mapsto (v^1, \dots, v^n) \in \mathbf{R}^n \tag{5.10}$$

is an isomorphism. In particular,

$$\dim T_yM = \dim M \tag{5.11}$$

EXAMPLE 5.1. For a real vector space V with $\dim V = n < \infty$, treated as a manifold (§1) and any $u \in V$, there is a canonical isomorphic identification $T_uV = V$ obtained by sending each $v = \dot\gamma(t) \in T_uV$ to the ordinary derivative (velocity) vector

$$\frac{d}{ds}\gamma(s)\bigg|_{t=0} = \lim_{s\to t}\frac{\gamma(s)-\gamma(t)}{s-t} \in V \tag{5.12}$$

(often denoted by $\dot\gamma(t)$ as well). Note that any basis e_j of V leads to the *linear* coordinate system x^j consisting of the linear homogeneous functions $x^j : V \to \mathbf{R}$ which form a basis of the dual space V^* dual to the basis e_j in the sense that $x^j(e_k) = \delta^j_k$. The element of V associated under the above identification with $v = \dot\gamma(t) \in T_uV$ then is v^je_j, which provides a description of our identification in terms of linear coordinate systems and components (rather than curves and velocities). Since both descriptions coincide, we conclude that the identification is really independent of the choice of γ and t, as well as the choice of the x^j.

EXAMPLE 5.2. Similarly, for a finite-dimensional real affine space $(M, V, +)$, regarded as a manifold (§1) and any $x \in M$, we have a natural isomorphic identification $T_xM = V$. See Problem 7.

Given a manifold M, a point $y \in M$, a tangent vector $v \in T_yM$, and a C^1 function $f : U \to \mathbf{R}$ on a neighborhood U of y, we define the *directional derivative of f in the direction of* v to be the real number, denoted by d_vf (or, sometimes, vf), and given by

$$d_{\dot\gamma(t)}f = (f\circ\gamma)^{\boldsymbol{\cdot}}(t) \tag{5.13}$$

for any $(\gamma, t) \in \mathcal{G}_y$ with $v = \dot\gamma(t)$, where the dot on the right-hand side represents the ordinary differentiation of real-valued functions of a real variable. From the ordinary chain rule,

$$d_vf = v^j\partial_jf\,, \tag{5.14}$$

and so d_vf is well-defined (independent of γ), and linear in v. To discuss its dependence on f, let us call two C^l functions defined on neighborhoods of y *equivalent* (or, C^l-*equivalent*) at y, if they coincide on some (possibly smaller) neighborhood of y. The equivalence classes of this relation are known as *germs* of C^l functions at y, and they form a vector space $\mathcal{F}^l_y$ (with the obvious operations applied to functions). The directional derivative d_vf now depends only on v and the C^1 germ of f, and constitutes a *bilinear mapping* $T_yM \times \mathcal{F}^1_y \to \mathbf{R}$.

Let $F : M \to N$ be a mapping between manifolds. Any local coordinates y^α in N then give rise to the *component functions* F^α of F, which are real-valued functions on the F-preimage of the domain of the y^α, given by

$$F^\alpha = y^\alpha \circ F\,. \tag{5.15}$$

Given manifolds $M,\ N$, a C^1 mapping $F : M \to N$, and a point $z \in M$, we define the *differential* of F at z to be the mapping $dF_z : T_zM \to T_{F(z)}N$ with

$$dF_z v \;=\; (F \circ \gamma)^{\dot{}}(t) \tag{5.16}$$

whenever $v = \dot\gamma(t) \in T_zM$. In coordinates $x^j,\ y^\alpha$ at z and $F(z)$, respectively, we thus have

$$(dF_z v)^\alpha \;=\; v^j(\partial_j F^\alpha)(z)\,, \tag{5.17}$$

and hence $dF_z v$ is well-defined (independent of γ) for the same reasons as in Example 5.1 above. Furthermore, (5.17) shows that dF_z is *linear.*

As an immediate consequence of (5.16) we have the *chain rule*

$$d(G \circ F)_x \;=\; dG_{F(x)} \circ dF_x\,. \tag{5.18}$$

for C^1 mappings $F : M \to N,\ G : N \to P$ between manifolds and any $x \in M$.

Example 5.3. The tangent space T_xU to an open submanifold U of a manifold M at any point $x \in M$ can be naturally identified with T_xM, as the differential dF_x of the inclusion mapping $F : U \to M$ is an isomorphism. (See Problem 1.) Thus, we will usually write $T_xU = T_xM$.

The *cotangent space* T_y^*M of a manifold M at a point $y \in M$ is, by definition, the dual of the tangent space T_yM, i.e., the vector space of all linear homogeneous functions $\xi : T_yM \to \mathbf{R}$. Elements ξ of T_y^*M are called *cotangent vectors*, or *dual vectors*, or *covariant vectors*, or 1*-forms* at y. Every real-valued C^1 function f defined in a neighborhood of y gives rise to the cotangent vector $df_y \in T_y^*M$, called the *differential* of f at y and characterized by

$$df_y v \;=\; d_v f\,. \tag{5.19}$$

(There is no conflict between this terminology and notation, and the case of differentials for mappings between arbitrary manifolds; see Problem 2 in §5.)

A fixed coordinate system x^j at a point y in manifold M gives rise to the tangent vectors $p_j = p_j(y) \in T_xM,\ j = 1, \ldots, n,\ n = \dim M$, characterized by their components relative to the coordinates x^j:

$$p_j^k \;=\; \delta_j^k \tag{5.20}$$

(the Kronecker delta). The p_j are called the *coordinate vectors* at y corresponing to the coordinates x^j. Obviously, they form a basis of T_xM, namely, the preimage under the isomorphism (5.10) of the standard basis of $\mathbf{R}^n$. For any tangent vector $v \in T_xM$ we then have (see Problem 4)

$$v \;=\; v^j p_j\,. \tag{5.21}$$

We define the *components of a cotangent vector* $\xi \in T_y^*M$ relative to any fixed coordinate system x^j at the point y in the manifold M to be the numbers

$$\xi_j \;=\; \xi(p_j) \tag{5.22}$$

with $p_j = p_j(y) \in T_yM$ defined as in (5.20). Thus, for instance, any C^1 function f defined in a neighborhood of y satisfies

$$(df)_j \;=\; \partial_j f \tag{5.23}$$

in the sense that $(df_y)_j = (\partial_j f)(y)$, while for any $\xi \in T_y^*M$ and any $v \in T_yM$ we have

$$\xi(v) \;=\; \xi_j v^j \tag{5.24}$$

Furthermore, the components ξ_j of a fixed cotangent vector $\xi \in T_y^*M$ obey the *transformation rule*

$$\xi_{j'} = p^j_{j'}\xi_j \tag{5.25}$$

under coordinate changes at y. Finally, denoting by dx^j the differentials of the coordinate functions at any point x of the coordinate domain, we easily see that the dx^j form a basis of T_x^*M; in fact, it is the *dual basis* for the basis $p_j = p_j(x)$ in T_xM, that is,

$$(dx^j)(p_k) \;=\; \delta^j_k\,. \tag{5.26}$$

Every $\xi \in T_x^*M$ then can be expanded as

$$\xi \;=\; \xi_j\, dx^j\,. \tag{5.27}$$

Problems

1. Show that $dF_x : T_xU \to T_xM$ in Example 5.3 is an isomorphism. (Hint below.)

2. For a C^1 function $f : M \to \mathbf{R}$ on a manifold M, treated as a mapping between the manifolds M and $\mathbf{R}$, a point $x \in M$ and a vector $v \in T_xM$, show that $df_x v = d_v f$ under the canonical identification $T_{f(x)}\mathbf{R} = \mathbf{R}$ (Example 5.1).

3. Given a manifold M, a point $y \in M$, a vector $v \in T_yM$, and coordinates x^j at y, verify that $v^j = d_v x^j$.

4. Given coordinates x^j at a point y in manifold M, define the vectors $p_j = p_j(y) \in T_xM$ by (5.20). Verify that
  (a) Relation (5.21) holds for each $v \in T_yM$.
  (b) The directional derivative $d_{p_j} f$ of any C^1 function f defined near y coincides with the partial derivative $(\partial_j f)(y)$.
  (c) There is no clash between the notations used in (5.2) and (5.8), i.e., the components of $v = p_j$ relative to any coordinate system $x^{j'}$ at y are $p_j^{j'} = \partial_j x^{j'}$.

5. Prove (5.23) – (5.27).

6. An *affine coordinate system* in an n-dimensional affine space $(M, V, +)$, with $n < \infty$, consists of an *origin* $o \in M$ and a basis e_j of V, $j = 1, \ldots, n$. The corresponding affine coordinates $x^j : M \to \mathbf{R}$ then are characterized by $M \ni x \mapsto x^j$ with $x = o + x^j e_j$. Verify that the coordinate functions x^j then are affine mappings (functions) $M \to \mathbf{R}$, and that their linear parts ψ^j form the basis of the dual space V^* dual to the basis e_j of V.

7. Proceeding as in Example 5.1 above (with affine coordinates introduced in Problem 6, rather than linear ones), describe the identification in Example 5.2 and show that it is well-defined.

8. Show that $dF_x = F$ for any linear operator $F : V \to W$ between finite-dimensional real vector spaces V, W and any $x \in V$, with the identifications $T_xV = V$, $T_yW = W$ described in Example 5.1.

9. Verify that $dF_x = \psi$ for any affine mapping $F : M \to N$ between finite-dimensional real affine spaces M, N (§69 in Appendix A) and any $x \in M$, where $\psi : V \to W$ is the linear part of F, while V, W denote the translation vector spaces of M, N, and $T_xM = V$, $T_yN = W$ as in Example 5.2.

10. Verify that the dimension is a diffeomorphic invariant, that is, $\dim N = \dim M$ whenever the C^1 manifolds M, N are C^1-diffeomorphic. (Hint below.)

11. For a C^∞ manifold M and a point $x \in M$, let $\mathcal{T}_x$ be the vector space of all *derivations of germs of* C^∞ *functions at* x, i.e., linear operators $A : \mathcal{F}_x^\infty \to \mathbf{R}$ (notation as in the paragraph following (5.14)) such that $A(fh) = (Af)\cdot h(x) + f(x)\cdot Ah$ for any (germs of) C^∞ functions f, h defined near x. Verify that

1. $Af = 0$ if $A \in \mathcal{T}_x$ and f is constant near x.
2. $d_v \in \mathcal{T}_x$ for each $v \in T_xM$.
3. The mapping $T_xM \ni v \mapsto d_v \in \mathcal{T}_x$ is linear and injective.

Hint. In Problem 1, use local coordinates x^j at x in U, treating them simultaneously as local coordinates x^j at x in M, and note that, for $v \in T_xU$, $(dF_xv)^j = v^j$.
Hint. In Problem 10, let $F : M \to N$ be a C^1 diffeomorphism. Applying the chain rule (5.18) to $P = M$ and $G = F^{-1}$, with any fixed $x \in M$, we see that dF_x then is an isomorphism $T_xM \to T_{F(x)}N$ (and $(dF_x)^{-1} = dG_{F(x)}$). Hence $\dim N = \dim M$ by (5.11).

6. Vector fields

Topics: Tangent vector fields on manifolds; directional derivatives along vector fields; the Lie bracket; Lie bracket as a commutator of differentiations; projectable vector fields; projectability of Lie brackets.

Let U be an open subset of a C^r manifold M, $1 \le r \le \omega$. By a *vector field* w in M defined on U we mean a mapping assigning to each $x \in U$ a vector $w(x)$ (sometimes written as w_x) tangent to M at x, i.e.,

$$U \ni x \mapsto w(x) = w_x \in T_xM\,. \tag{6.1}$$

Without specifying U (the *domain* of w) we will simply refer to w as a *local vector field* in M. When $U = M$, such w will be called a *global* vector field, or a vector field *on* M.

Given a local vector field w in M, any local coordinate system x^j in M gives rise to the *component functions* w^j of w relative to the x^j, which are the real-valued functions on the intersection of the domain of w and the coordinate domain, characterized by

$$w^j(x) = [w(x)]^j\,. \tag{6.2}$$

Under a change of coordinates, we then have, with $p_j^{j'} = \partial_j x^{j'}$ (§5)

$$w^{j'} = p_j^{j'} w^j\,. \tag{6.3}$$

Let V be any fixed finite-dimensional real or complex vector space. A local vector field w defined on an open subset U of a manifold M then leads to the corresponding *directional derivative operator* d_w assigning to each V-valued C^1 function $f : U' \to V$ (where U' is any open subset of U) the function $d_wf : U' \to V$ given by $(d_wf)(x) = d_{w(x)}f$ (see §5; the generalization of (5.13), (5.14) to V-valued C^1 functions f is straightforward). Thus,

$$d_wf = w^j\partial_j f\,. \tag{6.4}$$

for C^1 functions f. In particular, the component functions w^j of w relative to any local coordinates x^j in M can be expressed as

$$w^j = d_wx^j \tag{6.5}$$

By (6.5), a vector field w is uniquely determined by the operator d_w.

We say that a vector field w on an open subset U of a C^r manifold M is of class C^l, $l = 0, 1, 2, \ldots, r-1$, if so are its component functions w^j in all local coordinates x^j for M. (We use the conventions that $\infty - 1 = \infty$ and $\omega - 1 = \omega$.) This is a local geometric property, in view of (6.3) and the fact that $p_j^{j'} = \partial_j x^{j'}$ are of class C^{r-1}. Furthermore, vector fields of class C^l on a fixed open subset U of a manifold M admit the natural *pointwise operations* of addition and multiplication by real scalars c, with $(v+w)(x) = v(x) + w(x)$ and $(cw)(x) = cw(x)$; thus, they form a real vector space. More generally, such vector fields can be *multiplied by functions* $f : U \to \mathbf{R}$ so that $(fw)(x) = f(x)w(x)$ and then fw is of class C^l if so are f and w.

Let M now be a C^r manifold with $r \geq 2$. Given two vector fields v, w of class C^l, $1 \leq l \leq r-1$, on an open subset U of M, we define their *Lie bracket* to be the C^{l-1} vector field $[v, w]$ on U with

$$[v,w]^j \,=\, d_v w^j \,-\, d_w v^j\,, \tag{6.6}$$

i.e., having the component functions

$$[v,w]^j \,=\, v^k \partial_k w^j \,-\, w^k \partial_k v^j \tag{6.7}$$

relative to any local coordinates x^j in M. That this definition is correct, i.e., $[v, w]$ does not depend on the coordinates used, follows from the the transformation rule (6.3), which the $[v, w]^j$ satisfy whenever so do the v^j and w^j. In fact, by (6.3) we have $[v,w]^{j'} = d_v w^{j'} - d_w v^{j'} = d_v(p_j^{j'} w^j) - d_w(p_j^{j'} v^j) = p_j^{j'}(d_v w^j - d_w v^j) + w^j d_v p_j^{j'} - v^j d_w p_j^{j'} = p_j^{j'}[v,w]^j + v^k w^j(\partial_k p_j^{j'} - \partial_j p_k^{j'})$, while $\partial_k p_j^{j'} = \partial_k \partial_j x^{j'} = \partial_j p_k^{j'}$ due to symmetry properties of second-order partial derivatives. (See also Problem 3.)

A coordinate-free description of the Lie bracket operation $[\,,]$ can easily be obtained using directional derivative operators. Specifically, for C^1 vector fields v, w on $U \subset M$ as above and any vector-valued C^2 function f on an open subset of U, (6.4) yields $d_v d_w f = (v^k \partial_k w^j)\partial_j f + v^k w^j \partial_k \partial_j f$, and so

$$d_{[v,w]} \,=\, d_v d_w \,-\, d_w d_v \tag{6.8}$$

since, as before, $\partial_k \partial_j f = \partial_j \partial_k f$.

Let $F : M \to N$ be a C^1 mapping between manifolds. Vector fields w on M and v on N are called *F-related* if $dF_x(w(x)) = v(F(x))$ for each $x \in M$. We then also write

$$(dF)w \,=\, v \qquad \text{on} \quad F(M)\,. \tag{6.9}$$

Note that in general, when F is fixed, such a v (or w) need not exist or be unique for a given w or v (see Problem 9); if v does exist, and v, w are both of class C^1, one says that w is *F-projectable*. The local-coordinate expression of (6.9) is

$$w^j \partial_j F^\alpha = v^\alpha \circ F \tag{6.10}$$

in arbitrary local coordinates x^j in M and y^α in N.

Furthermore, if F happens to be a C^r diffeomorphism between the C^r manifolds M, N and w is any given vector field on M, then there obviously exists a unique vector field v on N with (6.9). That unique v, called the *push-forward* of w under the diffeomorphism F, is denoted by $(dF)w$ or simply Fw. Note that, by (6.10), $(dF)w$ is of class C^l $(0 \leq l \leq r-1)$ whenever so is w.

THEOREM 6.1. *Suppose that $F : M \to N$ is a C^2 mapping between manifolds and $w, \widetilde{w}$ are C^1 vector fields on M that are F-projectable, with $(dF)w = v$ on $F(M)$ and $(dF)\widetilde{w} = \widetilde{v}$ on $F(M)$ for some C^1 vector fields $v, \widetilde{v}$ on N. Then the Lie bracket $[w, \widetilde{w}]$ is F-projectable, with*

$$(dF)[w, \widetilde{w}] = [v, \widetilde{v}] \qquad \text{on} \quad F(M)\,.$$

In fact, $\partial_k\partial_j F^\alpha = \partial_j\partial_k F^\alpha$, and so differentiation by parts, the chain rule, (6.8) and (6.10) give $[w, \widetilde{w}]^j\partial_j F^\alpha = w^k(\partial_k\widetilde{w}^j)\partial_j F^\alpha - \widetilde{w}^k(\partial_k w^j)\partial_j F^\alpha = w^k\partial_k(\widetilde{w}^j\partial_j F^\alpha) - \widetilde{w}^k\partial_k(w^j\partial_j F^\alpha) + w^k\widetilde{w}^j(\partial_k\partial_j F^\alpha - \partial_j\partial_k F^\alpha) = w^k\partial_k(\widetilde{v}^\alpha\circ F) - \widetilde{w}^k\partial_k(v^\alpha\circ F) = (w^k\partial_k F^\beta)[(\partial_\beta\widetilde{v}^\alpha)\circ F] - (\widetilde{w}^k\partial_k F^\beta)[(\partial_\beta v^\alpha)\circ F] = (v^\beta\circ F)[(\partial_\beta\widetilde{v}^\alpha)\circ F] - (\widetilde{v}^\beta\circ F)[(\partial_\beta v^\alpha)\circ F] = [v, \widetilde{v}]^\alpha\circ F$, as required.

Problems

1. *First-order Taylor approximation.* Given coordinates x^j at a point y in a C^r manifold M, $r \ge 1$, show that there exists a neighborhood U of y contained in the coordinate domain such that every C^l function $f : U \to \mathbf{R}$, $l = 1, 2, \ldots, r$, can be written as

$$f = f(y) + (x^j - y^j)h_j$$

where $y^j = x^j(y)$ are the components of y and $h_j : U \to \mathbf{R}$ are C^{l-1} functions (depending on f) with $h_j(y) = (\partial_j f)(y)$. (Hint below.)

2. Let the x^j be coordinates at a point y in a C^∞ manifold M. Prove that

(a) $Af = 0$ whenever $A \in \mathcal{T}_x$ (notation as in Problem 11 of §5) and f is a (germ of) a C^∞ function defined near x, and constant in a neighborhood of x.

(b) $Af = (Ax^j)\partial_j f(y)$ if $A \in \mathcal{T}_x$ and f is a (germ of) a C^∞ function defined near x.

(c) the linear operator $T_xM \ni v \mapsto d_v \in \mathcal{T}_x$ is an isomorphism. (Thus, $\mathcal{T}_x$ may serve as an alternative description of T_xM.) (Hint below.)

3. Verify that

$$d_w(f + f') = d_w f + d_w f'\,, \qquad d_w(ff') = (d_w f)f' + f d_w f'$$

whenever U is an open subset of a C^1 manifold M, w is a vector field on U, and f, f' are C^1 functions $U \to \mathbf{R}$.

4. Denoting by $\mathcal{F}^\infty$ the ring of all C^∞ functions $F : M \to \mathbf{R}$ on a given C^∞ manifold M, prove that the assignment $w \mapsto d_w$ is a linear isomorphism between the vector space of all C^∞ vector fields on M and the space of all real-linear operators $B : \mathcal{F}^\infty \to \mathcal{F}^\infty$ satisfying the *Leibniz rule*

$$B(ff') = (Bf)f' + fBf'$$

for all $f, f' \in \mathcal{F}^\infty$. (Hint below.)

5. Show that the Lie bracket operation $[\,,]$ is skew-symmetric: $[u, v] = -[v, u]$, and satisfies the *Jacobi identity*

$$[[u, v], w] + [[v, w], u] + [[w, u], v] = 0 \tag{6.11}$$

for any C^2 vector fields u, v, w on a given C^3 manifold M.

6. Verify that $[fv, hw] = fh[v, w] + f(d_v h)w - h(d_w f)v$ for C^1 differentiable vector fields v, w and C^1 functions f, h on a C^2 manifold M.

7. Let $r \geq 2$. We say that two local C^1 vector fields w, u defined on the same open set U in a C^r manifold M *commute* if their Lie bracket $[w, u]$ is zero at every point of U. Verify that, with $n = \dim M$,

(a) For any local coordinates x^j in M and any $k, l \in \{1, \dots, n\}$, the *coordinate vector field* p_k commutes with p_l.

(b) For every $y \in M$ there exist C^{r-1} vector fields w, u defined near y that are not of the form $w = p_k$ and $u = p_l$ for any local coordinates x^j at y in M and any $k, l \in \{1, \dots, n\}$.

8. A subset K of a manifold M is said to be *dense* in M if every point of M is the limit of a sequence of points in K. Show that a subset K of a manifold M is dense in M if and only if it intersects every nonempty open set in M.

9. Let C^1-differentiable vector fields w and v on C^2 manifolds M and N be F-related for a C^2 mapping $F : M \to N$, so that $(dF)w = v$ on $F(M)$. Verify that

(a) v is unique for a given w, if the image $F(M)$ is dense in N (Problem 8);

(b) w is uniquely determined by v if, at each point $x \in M$, the differential $dF_x : T_xM \to T_{F(x)}N$ is injective.

10. Let U be an open subset of a finite-dimensional real vector space V. A C^l vector field w on U then may be identified with a C^l mapping $U \to V$, namely $U \ni x \mapsto v(x) \in V = T_xV = T_xU$. Show that, under this identification, the Lie bracket of C^1 vector fields v, w on U is given by $[v, w] = d_vw - d_wv$ (notation as in (6.4)), i.e., $[v, w](x) = (dw)_xv - (dv)_xw$ for all $x \in U$.

11. Let V be a finite-dimensional real or complex vector space. A vector field v on V is called *linear* if it has the form $v(x) = Fx \in V = T_xV$ for a linear operator $F : V \to V$. (We then write $v = F$.) Verify that v then is C^∞-differentiable. Show that for any two linear vector fields $v = F$, $v' = F'$ on V, the Lie bracket $[v, v']$ is also linear, with $[v, v'] = F'F - FF'$. (Hint below.)

12. Let $F : V \to W$ be a linear operator between finite-dimensional real or complex vector spaces. Find an algebraic condition necessary and sufficient for a given linear vector field v on V to be F-projectable.

Hint. In Problem 1, use the coordinates to identify a neighborhood of x with an open set in $\mathbf{R}^n$, $n = \dim M$, then choose U that is convex and note that $f(x) - f(y) = \int_0^1 \frac{d}{dt} f(y + t(x - y))\, dt$ for any $x \in U$.

Hint. In Problem 2, apply A to the equality in Problem 1. Then use Problem 4 in §5.

Hint. In Problem 4, let B be linear and satisfy the Leibniz rule. Then, for any $f \in \mathcal{F}^\infty$ with $f = 0$ in a neighborhood U of a given point $x \in M$, we have $(Bf)(x) = 0$, as $f = (1 - \phi)f$, where ϕ is chosen as in Problem 19 below for $K = M \setminus U$. Therefore, for $f \in \mathcal{F}^\infty$ and $x \in M$, the number $(Bf)(x)$ depends only on the *germ* of f at x. Since every germ at x is obtained from some $f \in \mathcal{F}^\infty$ (Problem 20 below), our B defines an assignment $M \ni x \mapsto B_x \in \mathcal{T}_x$ (notation of Problem 11 in §5), and we may use Problem 2(c).

Hint. In Problem 11, use Problem 10 of this section and Problem 8 in §5.

7. Lie algebras

Topics: Lie algebras; homomorphisms; examples.

A real/complex *Lie algebra* is a real/complex vector space $\mathfrak{g}$ with a fixed bilinear mapping

$$\mathfrak{g} \times \mathfrak{g} \ni (u, v) \mapsto [u, v] \in \mathfrak{g},$$

called the *multiplication* or *bracket*, which is skew-symmetric ($[u, v] = -[v, u]$ whenever $u, v \in \mathfrak{g}$) and satisfies the *Jacobi identity*: for all $u, v, w \in \mathfrak{g}$,

$$[[u, v], w] + [[v, w], u] + [[w, u], v] = 0\,. \tag{7.1}$$

By a *Lie subalgebra* of $\mathfrak{g}$ we then mean any vector subspace $\mathfrak{h}$ of $\mathfrak{g}$ which is closed under the multiplication $[\,,\,]$. The restriction of $[\,,\,]$ to $\mathfrak{h}$ then makes $\mathfrak{h}$ into a Lie algebra. Given Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, a *homomorphism* $\Phi : \mathfrak{g} \to \mathfrak{g}'$ is a linear mapping with $\Phi[u, v] = [\Phi u, \Phi v]$ for all $u, v \in \mathfrak{g}$. Composites of Lie-algebra homomorphisms are Lie-algebra homomorphisms, and so are the identity $\mathfrak{g} \to \mathfrak{g}$ and the zero mapping $\mathfrak{g} \to \mathfrak{g}'$. We call a Lie-algebra homomorphism Φ an *isomorphism* if, in addition, it is one-to-one and onto; the inverse Φ^{-1} then is an isomorphism as well. Two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ are said to be *isomorphic* if there exists an isomorphism between them.

A Lie algebra is called *Abelian* if its multiplication is identically zero. Any linear operator between Abelian Lie algebras is a Lie-algebra homomorphism.

EXAMPLE 7.1. Any real/complex vector space V, with the trivial (zero) multiplication, is an Abelian Lie algebra.

EXAMPLE 7.2. Let $\mathcal{A}$ be a real or complex associative algebra, with the multiplication operation denoted by $(a, b) \mapsto ab$. The bracket given by the *commutator*

$$[a, b] = ab - ba$$

then turns $\mathcal{A}$ into a Lie algebra, also denoted by $\mathcal{A}$.

EXAMPLE 7.3. As a special case of Example 7.2, every real or complex vector space V gives rise to the Lie algebra $\mathfrak{gl}(V)$ of all linear operators $A : V \to V$ with the commutator bracket

$$[A, B] \,=\, AB - BA\,,$$

AB being the composite of the mappings A and B. For $V = \mathbf{K}^n$ (with $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$) we may regard the Lie algebra $\mathfrak{gl}(V) = \mathfrak{gl}(n, \mathbf{K})$ as consisting of all real (complex) $n \times n$ matrices, with the commutator bracket induced by the matrix multiplication.

EXAMPLE 7.4. Given an open subset U of a C^r manifold M with $r = \infty$ or $r = \omega$, the set $\mathfrak{g}$ of all C^r vector fields in M defined on U is a real Lie algebra with the obvious (pointwise) vector space structure and the multiplication provided by the Lie bracket. (See Problem 5 in §6.)

EXAMPLE 7.5. Suppose that $\mathcal{B}$ is a real or complex algebra. (Its multiplication, denoted by $(u, v) \mapsto uv$, is assumed to be just bilinear, and does not have to be associative or satisfy the Jacobi identity.) The Lie algebra $\mathfrak{gl}(\mathcal{B})$ then contains a naturally distingushed Lie subalgebra $D(\mathcal{B})$ formed by all *derivations* of $\mathcal{B}$, that is, linear operators $A : \mathcal{B} \to \mathcal{B}$ suth that $A(uv) = (Au)v + u(Av)$ for all $u, v \in \mathcal{B}$.

EXAMPLE 7.6. When $\mathcal{B}$ in Example 7.5 is the algebra $\mathcal{F}^\infty$ of all C^∞ functions $F : M \to \mathbf{R}$ on a given C^∞ manifold M, the assignment $w \mapsto d_w$ is a Lie algebra isomorphism between the Lie algebra $\mathfrak{g}$ of all C^∞ vector fields on M and $D(\mathcal{F}^\infty)$.

Given a Lie algebra $\mathfrak{g}$ and a real/complex vector space V, by a real/complex *representation* of $\mathfrak{g}$ in V we mean any Lie-algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$. The representation is called finite-dimensional if so is V.

Example 7.7. The *adjoint representation* of any given Lie algebra $\mathfrak{g}$ is the Lie-algebra homomorphism $\mathrm{Ad} : \mathfrak{g} \to D(\mathfrak{g})$ given by $(\mathrm{Ad}\, u)v = [u, v]$.

Bu an *ideal* in a real/complex Lie algebra $\mathfrak{g}$ we mean any vector subspace $\mathfrak{h} \subset \mathfrak{g}$ with $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$ (that is, $[u, v] \in \mathfrak{h}$ whenever $u \in \mathfrak{g}$ and $v \in \mathfrak{h}$). It follows that $\mathfrak{h}$ then is a Lie subalgebra of $\mathfrak{g}$. Obvious examples of ideals are provided by the kernels $\mathfrak{h} = \mathrm{Ker}\, h$ of Lie-algebra homomorphisms $h : \mathfrak{g} \to \mathfrak{g}'$.

Problems

1. Verify the statements in Examples 7.2, 7.5, 7.6 and 7.7. (Hint below.)

2. Let V be a finite-dimensional real or complex vector space. Show that the trace function $\mathrm{Trace} : \mathfrak{gl}(V) \to \mathbf{K}$ is a Lie-algebra homomorphism of $\mathfrak{gl}(V)$ into the scalar field $\mathbf{K}$ treated as an Abelian Lie algebra.

3. Given a finite-dimensional real or complex vector space V, verify that the set $\mathfrak{sl}(V)$ of all linear operators $A : V \to V$ with $\mathrm{Trace}\, A = 0$ is an ideal, and hence also a Lie subalgebra, of $\mathfrak{gl}(V)$.

4. Let V be a finite-dimensional real vector space endowed with a fixed (positive-definite) inner product $\langle\, ,\rangle$. Prove that the space $\mathfrak{so}(V)$ of all linear operators $A : V \to V$ which are *skew-adjoint* in the sense that $\langle Au, v\rangle + \langle u, Av\rangle = 0$ for all $u, v \in V$, is a Lie subalgebra of $\mathfrak{sl}(V)$.

Hint. In Problem 1, use (6.8) and Problem 4 in §6 for Example 7.6, and (7.1) for Example 7.7.

8. The Lie algebra of a Lie group

Topics: The Lie algebra of left-invariant vector fields on a Lie group; projectability of left-invariant fields under Lie-group homomorphisms; the Lie-algebra homomorphism induced by a Lie-group homomorphism; a regularity theorem for Lie-group homomorphisms.

Given a group G and $a \in G$, the mappings $L_a, R_a : G \to G$ of the *left* and *right translations* by a, defined by

$$L_a(x) = ax\,, \qquad R_a(x) = xa \tag{8.1}$$

are bijections (with the inverses $L_{a^{-1}}$, $R_{a^{-1}}$). If, moreover, G is a Lie group of class C^s, both L_a and R_a are C^s diffeomorphisms. For any C^s Lie group G, $1 \le s \le \omega$, and any $x, a \in G$ and $v \in T_xG$, we will use the notation

$$av \in T_{ax}G\,, \qquad va \in T_{xa}G$$

for the following vectors (see also Problems 1 – 6):

$$av = (dL_a)_x v\,, \qquad va = (dR_a)_x v\,. \tag{8.2}$$

A vector field w on a Lie group G of class C^s ($s \ge 1$) is called *left-invariant* if $(dL_a)w = w$ for each $w \in G$ (notation of (6.7)), that is, w is pushed-forward onto itself by each left-translation diffeomorphism. We then denote by $\mathfrak{g}$ the real vector space of all left-invariant vector fields on G. Consequently, $w \in \mathfrak{g}$ if and only if

$$aw_x = w_{ax} \qquad \text{for all } a, x \in G\,, \tag{8.3}$$

(where $w_x = w(x)$), which, by (8.7) below, is equivalent to

$$w(x) = xv \qquad \text{for all } x \in G\,, \text{ with } v = w(1) \in T_1G\,. \tag{8.4}$$

On the other hand, for any fixed $v \in T_1G$, formula (8.4) defines a unique vector field $w \in \mathfrak{g}$ with $w(1) = v$. More generally, for any $a \in G$, the *evaluation mapping*

$$\mathfrak{g} \ni w \mapsto w(a) \in T_aG \tag{8.5}$$

is an isomorphism of real vector spaces. From now on we will often use (8.5) with $a = 1$ to identify $\mathfrak{g}$ with the tangent space T_1G at the identity (unit element) $1 \in G$. Thus, whenever convenient, we will just write

$$\mathfrak{g} = T_1G\,. \tag{8.6}$$

Every left-invariant vector field $w \in \mathfrak{g}$ is automatically of class C^{s-1} (if G is of class C^s). In fact, choosing any local coordinates x^j at a fixed $z \in G$ and y^α at $1 \in G$, we have, for x near z and y near 1,

$$(xy)^j = \Phi^j(x^1, \dots, x^n, y^1, \dots, y^n)\,,$$

where $n = \dim G$ and the Φ^j are some C^s functions of $2n$ real variables. Now (8.4) becomes

$$w^j(x) = v^\alpha \frac{\partial \Phi^j}{\partial y^\alpha}(x^1, \dots, x^n, u^1, \dots, u^n)$$

(where u stands for $1 \in G$, to replace the awkward sumbols 1^α by u^α); hence the w^j are functions of class C^{s-1}.

In the case where G is a Lie group of class C^s with $s \geq 3$, the space $\mathfrak{g}$ is closed under the Lie bracket operation $[\,,]$ (in view of Theorem 6.1), and $[\,,]$ restricted to $\mathfrak{g}$ satisfies the Jacobi identity (Problem 5 in §6). Therefore, $\mathfrak{g}$ with the multiplication $[\,,]$ forms a Lie algebra, called the *Lie algebra of the Lie group* G. Under the identification (8.6), the Lie algebra of G may be thought of as the tangent space T_1G of G at the identity, with the bracket multiplication in T_1G (also denoted by $[\,,]$) which assigns to $u, v \in T_1G$ the value $w(1) = [u, v]$ at 1 of the Lie bracket w of the unique left-invariant vector fields on G whose values at 1 are u and v.

Example 8.1. For any finite-dimensional real vector space V, the underlying additive group of V is a Lie group $G = V$ of class C^ω (cf. Example 4.1), and its Lie algebra $\mathfrak{g} = T_0V = V$ is Abelian. In fact, left-invariant vector fields on V are precisely the *constant* V-valued functions on V (Problem 5) and so their Lie brackets are all zero (Problem 10 in §6).

Example 8.2. Let G be the C^ω Lie group of all invertible elements of a finite-dimensional real associative algebra $\mathcal{A}$ with unit. The Lie algebra $\mathfrak{g}$ of G then is the vector space $\mathcal{A}$ (as $\mathfrak{g} = T_1G = T_1\mathcal{A} = \mathcal{A}$) with the commutator multiplication $[a, b] = ab - ba$ (cf. Example 7.2). See also Problem 7 below and Problem 10 in §6.

Example 8.3. In particular, the Lie algebra of the C^ω Lie group $\mathrm{GL}(V)$ for a finite-dimensional real or complex vector space V (Example 4.3) coincides with the Lie algebra $\mathfrak{gl}(V)$ of all real/complex linear operators of V into itself, with the commutator bracket (Example 7.3).

Example 8.4. For the matrix Lie group $\mathrm{GL}(n, \mathbf{K})$ with $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$, the Lie algebra $\mathfrak{g} = \mathfrak{gl}(V) = \mathfrak{gl}(n, \mathbf{K})$ consists of all $n \times n$ matrices over $\mathbf{K}$, with the matrix commutator.

EXAMPLE 8.5. Any group G, viewed as a discrete Lie group ($\dim G = 0$) has the trivial Lie algebra $\mathfrak{g} = \{0\}$.

A mapping $F : G \to H$ between groups is a homomorphism if and only if

$$F \circ L_x \,=\, L_{F(x)} \circ F \tag{8.7}$$

for all $x \in G$. If we now apply the differentials of both sides at $1 \in G$ to any $w \in T_1G$ and use the chain rule (5.18) and the notation of (8.2), we obtain

$$dF_x(xw) \,=\, [F(x)](dF_1 w) \tag{8.8}$$

whenever $x \in G$ and $w \in T_1G$.

An easy consequence of (8.8) is the following

LEMMA 8.6. *Any C^1 homomorphism $F : G \to H$ between C^s Lie groups G and H, $1 \le s \le \infty$, is automatically of class C^s.*

PROOF. Choose bases w_a and v_λ of the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ for the Lie groups G and H, and let the constants C_a^λ be characterized by $dF_1(w_a(1)) = C_a^\lambda v_\lambda(1)$. In any local coordinates x^j for G and y^α for H, $w_a = w_a^j p_j$ and $v_\lambda = v_\lambda^\alpha p_\alpha$ (formula (5.21)), with some C^{s-1} component functions w_a^j, v_λ^α. Also, $p_j = \Phi_j^a w_a$ for some C^{s-1} functions Φ_j^a on the domain U of the coordinates x^j; in fact, as matrices, $[\Phi_j^a(x)] = [w_a^j(x)]^{-1}$ at any $x \in U$ (and the inverse exists, since the w_a form a basis of each tangent space). Now, for any x that is both in U and in the F-preimage of the y^α coordinate domain,

$$\begin{aligned} &dF_x(p_j(x)) = dF_x(\Phi_j^a(x)w_a(x)) = \Phi_j^a(x)dF_x(w_a(x)) = \Phi_j^a(x)dF_x(xw_a(1)) \\ &= \Phi_j^a(x)[F(x)]dF_1(w_a(1)) = \Phi_j^a(x)[F(x)](C_a^\lambda v_\lambda(1)) = C_a^\lambda \Phi_j^a(x)[F(x)](v_\lambda(1)) \\ &= C_a^\lambda \Phi_j^a(x) v_\lambda(F(x))\,, \end{aligned}$$

and hence

$$dF_x(p_j(x)) \,=\, C_a^\lambda \Phi_j^a(x) v_\lambda^\alpha(F(x)) p_\alpha(F(x))\,,$$

where we have used (8.3), (8.8), as well as the definitions of the Φ_j^a and C_a^λ and linearity of the differentials of C^1 mappings. However, in view of (5.17) and (5.21), the partial derivatives $(\partial_j F^\alpha)(x)$ are the coefficients in the expansion of $dF_x(p_j(x))$ as a combination of the $p_\alpha(F(x))$. The equality just established thus reads

$$\partial_j F^\alpha \,=\, C_a^\lambda \Phi_j^a (v_\lambda^\alpha \circ F)\,. \tag{8.9}$$

We can now show, by induction on q, that the F^α are of class C^q for each finite q with $1 \le q \le r$. In fact, assuming that the F^α are of class C^q for a fixed $q < r$, we see from (8.9) that the $\partial_j F^\alpha$ must be of class C^q as well, and so the F^α are of class C^{q+1}. This completes the proof. □

Another easy consequence of (8.8) is

LEMMA 8.7. *Given C^1 Lie groups G and H, a left-invariant vector field w on G, and a C^1 homomorphism $F : G \to H$,*

a. *w is F-projectable.*
b. *There exists a unique left-invariant vector field v on H with $(dF)w = v$ on $F(G)$.*

In fact, the vector field v required in (b) is, by (8.8), the unique $v \in \mathfrak{h}$ (the Lie algebra of H) with $v(1) = dF_1(w(1))$.

For a C^2 homomorphism $F : G \to H$ of Lie groups, we will denote by $F_* : \mathfrak{g} \to \mathfrak{h}$ the mapping between the corresponding Lie algebras, assigning to each $w \in \mathfrak{g}$ the unique v described in Lemma 8.7(b). As we just observed, under the identifications (8.6) for both G and H, F_* is nothing else than dF_1, the differential of F at $1 \in G$:

$$F_* \,=\, dF_1 : \mathfrak{g} = T_1 G \,\to\, T_1 H \,=\, \mathfrak{h}\,. \tag{8.10}$$

On the other hand, by Theorem 6.1, F_* is a Lie-algebra homomorphism $\mathfrak{g} \to \mathfrak{h}$. Since

$$(F' \circ F)_* \,=\, F'_* \circ F_* \tag{8.11}$$

for two C^2 homomorphisms $F : G \to G'$ and $F' : G' \to G''$, and $\mathrm{Id}_* = \mathrm{Id}$, it follows that *$C^2$-isomorphic Lie groups must have isomorphic Lie algebras.*

Example 8.8. For finite-dimensional real vector spaces V, W, regarded as "additive" C^ω Lie groups $G = V$, $H = W$ with the Abelian Lie algebras $\mathfrak{g} = T_0 V = V$ and $\mathfrak{h} = T_0 W = W$ (Example 8.1), the continuous Lie group homomorphisms from G to H are precisely the linear operators $F : V \to W$ (and so they are automatically real-analytic). For such a homomorphism F, we then have $F_* = F$. See Problem 9.

Example 8.9. Let V be a fixed finite-dimensional vector space over a field $\mathbf{K}$ (with $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$). For the homomorphism $\det : \mathrm{GL}(V) \to \mathbf{K} \setminus \{0\}$ (Problem 6 in §4), the corresponding Lie-algebra homomorphism is

$$\det{}_* \,=\, \mathrm{Trace} : \mathfrak{gl}(V) \to \mathbf{K} \tag{8.12}$$

(Here the vector space $\mathbf{K}$ is treated as an Abelian Lie algebra.) For details, see Problems 10 and 13.

Example 8.10. Every element a of any group G gives rise to the *inner automorphism* $\mu(a)$ of G, that is, the group homomorphism $\mu(a) : G \to G$ with

$$[\mu(a)](x) \,=\, axa^{-1} \tag{8.13}$$

for all $x \in G$. When G happens to be a Lie group of class C^s, $0 \le s \le \omega$, $\mu(a)$ obviously is a C^s homomorphism. Under the identification (8.7), for $s \ge 1$, the linear isomorphism $[\mu(a)]_* = d\,[\mu(a)]_1 : T_1 G \to T_1 G$ is given by

$$[\mu(a)]_* v \,=\, ava^{-1} \tag{8.14}$$

whenever $v \in T_1 G$. (We use the notation $avb = (av)b = a(vb)$, as in Problem 3 below.) Thus, in the case where $s \ge 3$, $[\mu(a)]_*$ is a Lie-algebra isomorphism of $\mathfrak{g} = T_1 G$ onto itself.

Problems

1. Given a Lie group G and elements $a, b \in G$, verify that

$$L_{ab} \,=\, L_a \circ L_b\,, \qquad R_{ab} \,=\, R_b \circ R_a\,. \tag{8.15}$$

2. For a C^s Lie group G, $1 \le r \le \omega$, and a C^1 curve $t \mapsto x(t) \in G$, show that, at any parameter value t,

$$a\dot{x} \,=\, (ax)^{\boldsymbol{\cdot}}\,, \tag{8.16}$$

that is, $a[\dot{x}(t)] = (ax)^{\boldsymbol{\cdot}}(t)$, with $(ax)(t) = a(x(t))$.

3. Given a Lie group G of class C^s, $1 \le r \le \omega$, prove that the "multiplication" defined by 8.2) is *associative* and *distributive* in the sense that

$$(8.17)\qquad \begin{aligned} &(ab)v \,=\, a(bv)\,, \qquad v(ab) \,=\, (va)b\,, \qquad (av)b \,=\, a(vb)\,,\\ &a(v+w) \,=\, av \,+\, aw\,, \qquad (v+w)a \,=\, va \,+\, wa \end{aligned}$$

for all elements $x, a, b \in G$, and vectors $v, w \in T_xG$. (Thus, we may skip the parentheses and write abv for $(ab)v$ or $a(bv)$, etc.) (Hint below.)

4. For G, x, v as in Problem 3, verify that $1v = v1 = v$, where $1 \in G$ is the identity element.

5. Given a finite-dimensional real vector space V, treated as an Abelian "additive" Lie group $G = V$ of class C^ω (Example 4.1), verify that the products av, va with 8.2) both coincide with $v \in V = T_{a+x}V = T_xV$.

6. Let G be obtained as in Example 8.2. Show that the "products" av and va defined by 8.2) for $a \in G \subset \mathcal{A}$ and $v \in T_xG = \mathcal{A}$ coincide with the ordinary products in the algebra $\mathcal{A}$.

7. For G as in Example 8.2, a left-invariant vector field w on G regarded as an $\mathcal{A}$-valued function with $w(x) = xv$ and $v \in \mathcal{A}$, and a vector $u \in T_yG = \mathcal{A}$, verify that $d_uw = uv \in \mathcal{A}$.

8. Prove the statement in Example 8.2 above.

9. Prove the assertion of Example 8.8. (Hint below.)

10. Prove (8.12). (Hint below.)

11. Show that

$$(8.18)\qquad \omega(u_1, \ldots, u_k) = \det \mathfrak{B} \cdot \omega(v_1, \ldots, v_k)$$

whenever ω is a k-linear skew-symmetric mapping $V \times \ldots \times V \to W$ between real or complex vector spaces, and vectors $u_1, \ldots, u_k \in V$ are combinations of $v_1, \ldots, v_k \in V$ with the coefficient matrix $\mathfrak{B} = [B_\alpha^\beta]$, so that $u_\alpha = B_\alpha^\beta v_\beta$, $\alpha, \beta \in \{1, \ldots, k\}$. (Hint below.)

12. Prove that

$$(8.19)\qquad \omega(Fv_1, \ldots, Fv_n) = (\det F) \cdot \omega(v_1, \ldots, v_n)$$

and

$$(8.20)\qquad \begin{aligned} &\omega(Fv_1, v_2 \ldots, v_n) + \omega(v_1, Fv_2, v_3 \ldots, v_n) + \ldots + \omega(v_1, \ldots, v_{n-1}, Fv_n)\\ &= (\operatorname{Trace} F) \cdot \omega(v_1, \ldots, v_n) \end{aligned}$$

whenever ω is an n-linear skew-symmetric mapping $V \times \ldots \times V \to W$ between real or complex vector spaces V, W with $\dim V = n < \infty$ and $v_1, \ldots, v_k \in V$, while $F : V \to V$ is a linear operator. (Hint below.)

13. Let $t \mapsto F = F(t) \in \mathrm{GL}(V)$ be a C^1 curve of linear automorphisms of a finite-dimensional real or complex vector space V. Prove the equality

$$(8.21)\qquad (\det F)^{\boldsymbol{\cdot}} = (\det F) \operatorname{Trace}(F^{-1}\dot{F})$$

with $(\ )^{\boldsymbol{\cdot}} = d/dt$, that is, $\frac{d}{dt} \det F(t) = [\det F(t)] \cdot \operatorname{Trace}[(F(t))^{-1} \circ \dot{F}(t)]$ for all t. (Hint below.)

Hint. In Problem 3, write $v = \dot{x}(t)$ and use Problem 2.

Hint. In Problem 7, let $u = \dot{x}(t)$ with $x(t) = x$, so that $(d_uw)(x) = \frac{d}{dt}\, x(t)v = uv$.

Hint. In Problem 9, a group homomorphism $F : V \to W$ must satisfy $F(kx) = kF(x)$ for $k \in \mathbf{Z}$ and $x \in V$ (by additivity), so, if $k \ne 0$ and $x = y/k$, we have $F(y/k) = F(y)/k$. Thus, F is linear over the field $\mathbf{Q}$ of rational numbers, and its $\mathbf{R}$-linearity follows from continuity. Relation $F_* = F$ is immediate from (8.10) and Problem 8 in §5.

Hint. In Problem 10, use (8.10) and the fact that $\det(\mathrm{Id} + tA)$ is a polynomial in $t \in \mathbf{R}$ with

$$\det(\mathrm{Id} + tA) = 1 + (\operatorname{Trace} A)\, t + \ldots, \tag{8.22}$$

where $\ldots$ stands for terms of order higher than one in t.

Hint. In Problem 11, note that

$$\begin{aligned} \omega(B_1^{\alpha_1} v_{\alpha_1}, \ldots, B_k^{\alpha_k} v_{\alpha_k}) &= B_1^{\alpha_1} \ldots B_k^{\alpha_k}\, \omega(v_{\alpha_1}, \ldots, v_{\alpha_k}) \\ &= \varepsilon_{\alpha_1 \ldots \alpha_k} B_1^{\alpha_1} \ldots B_k^{\alpha_k}\, \omega(v_1, \ldots, v_k)\,, \end{aligned}$$

$\varepsilon_{\alpha_1 \ldots \alpha_k}$ being the *Ricci symbol* (equal to the signum of the permutation $(\alpha_1, \ldots, \alpha_k)$, if $\alpha_1, \ldots, \alpha_k$ are all distinct, and to 0, if they are not), while $\varepsilon_{\alpha_1 \ldots \alpha_k} B_1^{\alpha_1} \ldots B_k^{\alpha_k} = \det[B_\alpha^\beta]$.

Hint. In Problem 12, note that both sides of either equality are skew-symmetric in $v_1, \ldots, v_n$, and so we may assume that $v_1, \ldots, v_n$ form a basis of V. Setting $Fv_\alpha = F_\alpha^\beta v_\beta$ we may now use (8.18) for $k = n$ and $B_\alpha^\beta = F_\alpha^\beta$, since $\det F = \det[F_\alpha^\beta]$ and $\operatorname{Trace} F = F_\alpha^\alpha$.

Hint. In Problem 13, fix a basis v_α of V, $\alpha = 1, \ldots, n = \dim V$, and a nonzero n-linear skew-symmetric scalar-valued function ω in V. From (8.19) and (8.20) we get $(\det F)^{\boldsymbol{\cdot}}\, \omega(v_1, \ldots, v_n) = [\omega(Fv_1, \ldots, Fv_n)]^{\boldsymbol{\cdot}} = \sum_\alpha \omega(Fv_1, \ldots, \dot F v_\alpha, \ldots, Fv_n) = \sum_\alpha \omega(Fv_1, \ldots, FF^{-1}\dot F v_\alpha, \ldots, Fv_n) = (\det F) \cdot \sum_\alpha \omega(v_1, \ldots, F^{-1}\dot F v_\alpha, \ldots, v_n) = (\det F)(\operatorname{Trace} F^{-1}\dot F)\, \omega(v_1, \ldots, v_n)$, and we may use Problem 12.

CHAPTER 3

Immersions and Embeddings

9. The rank theorem, immersions, submanifolds

Topics: The rank of a mapping at a point; openness of the maximum-rank subset; the rank theorem; submersions; immersions; embeddings; submanifolds; submanifolds with the subset topology; continuity versus differentiability for submanifold-valued mappings; uniqueness of submanifold structure with the subset topology; critical and regular points and values of mappings; submanifolds defined by equations, their dimensions and tangent spaces; tangent spaces of Cartesian-product manifolds.

For a C^1 mapping $F : M \to N$ between C^s manifolds, $s \geq 1$, and a point $x \in M$, the *rank* of F at x is defined by $(\operatorname{rank} F)(x) = \dim [dF_x(T_xM)]$. Thus, $\operatorname{rank} F$ is a function on M valued in the finite set $\{0, 1, \ldots, k\}$, where we have set $k = \min(\dim M, \dim N)$.

Given local coordinates x^j in M and y^α in N, we have $(dF_x v)^\alpha = v^j(\partial_j F^\alpha)(x)$ for all points x in the F-preimage of the y^α coordinate domain and all $v \in T_xM$ (formula (5.17)). Therefore, $(\operatorname{rank} F)(x)$ equals the rank of the matrix $[(\partial_j F^\alpha)(x)]$, i.e., the maximum size of its nonzero subdeterminants. Consequently, $\operatorname{rank} F$ is constant in a neighborhood of each point where $\operatorname{rank} F$ assumes its maximum value, i.e., the set $U \subset M$ of all such points is both *nonempty* and *open*.

The following classical result is known as *the rank theorem*.

THEOREM 9.1. *Let $F : M \to N$ be a C^s mapping between C^s manifolds, $s \geq 1$, whose rank has a constant value*

$$\operatorname{rank} F = r \tag{9.1}$$

in a neighborhood of some given point $z \in M$. Then there exist local coordinates x^j at z in M and y^α at $F(z)$ in N, such that, near z, the components $F^\alpha = y^\alpha \circ F$ of F are given by

$$F^A = x^A \quad \text{for } A \leq r\,, \qquad F^\lambda = 0 \quad \text{for } \lambda > r\,. \tag{9.2}$$

REMARK 9.2. In other words, the rank theorem states that for any C^s mapping $F : M \to N$ having a constant rank r near z, the composite $\varphi' \circ F \circ \varphi^{-1}$ of F with suitable coordinate mappings defined near z and $F(z)$ has the "standard form"

$$(x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^r, 0, \ldots, 0)\,, \tag{9.3}$$

where $m = \dim M$ (and the number of zeros is $\dim N - r$).

PROOF OF THEOREM 9.1. Let us set $m = \dim M$, $n = \dim N$ and introduce the following ranges for indices:

$$1 \leq j, k \leq m\,, \quad 1 \leq \alpha, \beta \leq n\,, \quad 1 \leq A, B \leq r\,, \quad r < \lambda, \mu \leq n\,. \tag{9.4}$$

We start from arbitrary local coordinates x^j at z in M and y^α at $F(z)$ in N, and modify them in three successive steps (keeping the notation x^j, y^α for the new coordinates obtained after each modification).

First step. We may assume that $\det[\partial_A F^B] \neq 0$, i.e., a nonzero $r \times r$ subdeterminant sits in the left-upper corner of the $n \times m$ matrix $[\partial_j F^\alpha]$. This is achieved by suitably permuting the x^j and the y^α.

Second step. We may require that $F^A = x^A$ and $F^\lambda = \Psi^\lambda(x^1, \dots, x^r)$ for some C^s functions Ψ^λ of r variables. To this end, we replace $x^1, \dots, x^m$ by the new coordinates $F^1, \dots, F^r, x^{r+1}, \dots, x^m$ near z. That this actually is a coordinate system (in a suitable neighborhood of z) follows from the fact that the mapping

$$(x^1, \dots, x^m) \mapsto (F^1, \dots, F^r, x^{r+1}, \dots, x^m) \tag{9.5}$$

is a diffeomorphism of a neighborhood of $(z^1, \dots, z^m) = (x^1(z), \dots, x^m(z))$ in $\mathbf{R}^m$ onto an open set in $\mathbf{R}^m$, which in turn is immediate from the inverse mapping theorem (see Appendix B) and the determinant condition obtained in step 1. In these new coordinates, now denoted just by x^j, we thus have $F^A = x^A$. That F^λ depend only on the x^A is clear as $\partial_j F^A = 0$ if $j > r$ and $\partial_B F^A = \delta^A_B$, so that, otherwise, the matrix $[\partial_j F^\alpha]$ would have a nonzero subdeterminant of size $r+1$.

Third step. To achieve $F^A = x^A$, $F^\lambda = 0$ we now change the coordinates y^α (i.e., y^A and y^λ) in N, replacing the y^λ by $y^\lambda - \Psi^\lambda(y^1, \dots, y^r)$ with Ψ^λ introduced in step 2, and leaving the y^A unchanged. As in step 2, the new functions form a coordinate system in view of the inverse mapping theorem (Appendix B). This completes the proof. $\square$

A mapping $F : M \to N$ between C^s manifolds, $s \geq 1$, is called an *immersion* if it is of class C^s and, at each point $x \in M$, the differential $dF_x : T_x M \to T_{F(x)} N$ is injective (or, equivalently, $\operatorname{rank} F = \dim M$ everywhere in M). In suitable local coordinates, F then has the form (9.3) with $r = m$, that is,

$$(x^1, \dots, x^m) \mapsto (x^1, \dots, x^m, 0, \dots, 0) \tag{9.6}$$

with $m = \dim M$ (and with $\dim N - m$ zeros). Thus, any immersion $F : M \to N$ is *locally injective* in the sense that each $x \in M$ has a neighborhood in M the restriction of F to which is one-to-one. By an *embedding* $F : M \to N$ we mean an immersion that is *globally* injective, i.e., one-to-one on the whole of M.

A *submanifold* of a C^s manifold M, $s \geq 1$, is a subset K of M endowed with its own C^s manifold structure such that the inclusion mapping $F : K \to M$, $F(x) = x$, is an embedding. (In particular, F must be of class C^s.) A submanifold K of M is said to have *the subset topology* if this F is a homeomorphism of the manifold K onto the subset K of M (§3), i.e., $x_l \to x$ in the manifold K, as $l \to \infty$, whenever x_l is a sequence in K and $x \in K$ with $x_l \to x$ in M.

LEMMA 9.3. *Suppose that M, N are C^s manifolds, $s \geq 1$, K is a submanifold of M, and $F : N \to M$ is a C^l mapping, $0 \leq l \leq s$. If $F(N) \subset K$ and $F : N \to K$ is continuous as a mapping into the manifold K, then $F : N \to K$ is C^l-differentiable.*

PROOF. Fix $y \in N$ and choose local coordinates x^j in K, y^α in M, both at the point $z = F(y) \in K \subset M$, in which the inclusion mapping $K \to M$ has the form (9.6). By continuity of $F : N \to K$, there is a neighborhood U of y in N with $F(U)$ contained in the domain of the x^j. The components $F^\alpha = y^\alpha \circ F$ of

F are, by hypothesis, C^l-differentiable on U, and hence so are the $F^j = x^j \circ F$ as they coincide with the first $\dim K$ of the F^α. This completes the proof. $\square$

THEOREM 9.4. *If M, N are C^s manifolds, $s \geq 1$, K is a submanifold of M endowed with the subset topology, and $F : N \to M$ is a C^l mapping, $0 \leq l \leq s$, such that $F(N) \subset K$, then $F : N \to K$ is of class C^l, that is, F is a C^l mapping into the manifold K.*

This is clear from Lemma 9.3, since $F : N \to K$ is continuous.

COROLLARY 9.5. *If a subset K of a manifold M admits a structure of a submanifold of M having the subset topology, then such a structure is unique.*

PROOF. Let K', K'' denote by K endowed with two such manifold structures. Applying Theorem 9.4 to the identity mapping of K, acting in either direction between K' and K'', we conclude that it is a diffeomorphism between K' and K''. This proves our assertion (see Problem 1). $\square$

Without the assumption about the subset topology, a submanifold structure on a subset of a manifold may fail to be unique. This is obviously illustrated by the case of a manifold treated as its own discrete (zero-dimensional) submanifold. However, even connectedness of a submanifold structure on a given set does not, in general, guarantee its uniqueness (see Problem 12):

Fig. 5. Two different *connected* submanifold structures on a set

Whenever K is a submanifold of a manifold M, we write

$$T_x K \subset T_x M\,, \tag{9.7}$$

identifying $T_x K$, at any $x \in K$, with its image under the differential of the inclusion mapping $K \to M$.

Given manifolds M, N and a C^1 mapping $F : M \to N$, we say that $x \in M$ is a *regular point* of F if the differential $dF_x : T_x M \to T_{F(x)} N$ is surjective, i.e., "onto" (or, equivalently, $\operatorname{rank} F = \dim N$). Otherwise, $x \in M$ is called a *critical point* of F. Let $\operatorname{Crit}_F \subset M$ be the set of all critical points of F. The image $F(\operatorname{Crit}_F) \subset N$ and its complement $N \setminus F(\operatorname{Crit}_F) \subset N$ are known as the sets of *critical* and *regular values* of F, respectively. (Thus, a 'regular' value is not always a value.)

THEOREM 9.6. *Suppose that M, N are C^s manifolds and $F : M \to N$ is a C^s mapping, $s \geq 1$. If $y \in N$ is a regular value of F and $K = F^{-1}(y)$ is nonempty, then K is a C^s submanifold of M endowed with the subset topology, of dimension*

$$\dim K \,=\, \dim M \,-\, \dim N \tag{9.8}$$

and, for each $x \in K$, we have

$$T_x K = \operatorname{Ker} dF_x \subset T_x M \,. \tag{9.9}$$

Proof. Fix $z \in K$. Using the rank theorem, we may choose local coordinates x^j at z in M, with some coordinate domain U, and y^α at y in N, such that $F^1 = x^1, \dots, F^n = x^n$, $n = \dim N$. The functions $x^{n+1}, \dots, x^m$, $m = \dim M$, then form an $(m-n)$-dimensional coordinate system in the set K, with the coordinate domain $K \cap U$. Two such coordinate systems in K are automatically compatible due to analogous compatibility for the maximal atlas of M, while convergence $z_k \to z$ in the resulting topology on K means noting else than $x^j(z_k) \to x^j(z)$ for all j, i.e., convergence $z_k \to z$ in M (as $x^1, \dots, x^n$ are constant along K). The atlas on K thus obtained therefore leads to the subset topology, which is necessarily Hausdorff. The inclusion $T_z K \subset \operatorname{Ker} dF_z$ (which must be an equality for dimensional reasons) follows from the definition of dF_z in terms of velocities (formula (5.16)), since $F \circ \gamma$ is constant for any curve γ in K. This completes the proof. □

Problems

1. Let a set K admit two C^r manifold structures (i.e., maximal atlases), denoted by K', K''. Verify that these manifold structures coincide if and only if the identity mapping of K is a C^r diffeomorphism $K' \to K''$.

2. Let $F : M \to K$ be a one-to-one mapping of a C^r manifold M onto a set K. Show that K then carries a unique manifold structure (i.e., maximal atlas) such that $F : M \to K$ is a C^r diffeomorphism.

3. Let $F : M \to N$ be an embedding between C^r manifolds, and define a C^r manifold structure on $F(M)$ so as to make $F : M \to F(M)$ a C^r diffeomorphism. Verify that, with this structure, $F(M)$ is a submanifold of N.

4. Find an embedding of an open interval into $\mathbf{R}^2$ whose image is homeomorphic (as a subset of $\mathbf{R}^2$; see §3) to a "figure eight" (i.e., to a union of two circles having one point in common, cf. Fig. 5 above).

5. Verify that an open submanifold of a manifold M (Problem 11 in §2) is also a submanifold in the sense defined above. Does it have the subset topology?

6. Show that any affine subspace M in a finite-dimensional real affine space N is a submanifold of N endowed with the subset topology.

7. Prove that the unit sphere $S(V) = \{v \in V : |v| = 1\}$ of any Euclidean space $(V, \langle\,,\rangle)$, with the manifold structure defined as in §2, is a submanifold of V carrying the subset topology.

8. Let $T : V \to W$ be a linear operator between finite-dimensional vector spaces V, W. Verify that T is injective (surjective) if and only if there exists a linear operator $T' : W \to V$ with $T'T = \operatorname{Id}_V$ (or, respectively, $TT' = \operatorname{Id}_W$).

9. For a Euclidean space V, let $\nu : V \setminus \{0\} \to S(V)$ be the normalization mapping with $\nu(w) = w/|w|$ (Problem 7 in §3), and let $\psi : S(V) \to V \setminus \{0\}$ be the inclusion mapping ($\psi(u) = u$). Show that

(a) ψ is real-analytic,

(b) At each $v \in V \setminus \{0\}$, the linear operator $d\nu_v : T_v(V \setminus \{0\}) \to T_{\nu(v)}(S(V))$ is surjective.

(c) At each $u \in S(V)$, the linear operator

$$d\psi_u : T_u(S(V)) \to T_{\psi(u)}(V \setminus \{0\})$$

is injective, and, under the identifications $T_{\psi(u)}(V \setminus \{0\}) = T_{\psi(u)}V = V$ described in Examples 5.1 and 5.3, the image of $d\psi_u$ coincides with the orthogonal complement $u^\perp \subset V$.

10. Given manifolds M and N and points $x_0 \in M$, $y_0 \in N$, verify that the mappings $M \ni x \mapsto (x, y_0) \in M \times N$, $N \ni y \mapsto (x_0, y) \in M \times N$ are embeddings, and that their images $M \times \{y_0\}$, $\{x_0\} \times N$, with the manifold structures defined as in Problem 3, are submanifolds of $M \times N$ and have the subset topology.

11. For a subspace W of a finite-dimensional real or complex vector space V, let the mapping $F : P(W) \to P(V)$ between the corresponding projective spaces (§2) assign to each $L \in P(W)$ the same line $F(L) = L$ treated as a 1-dimensional vector subspace of V. Show that F is an embedding, making $P(W)$ (as in Problem 3) into a submanifold of $P(V)$ endowed with the subset topology.

12. Suppose that a subset M of a manifold N admits the structure of a *connected* manifold that makes it a submanifold of N. Is such a structure always unique? (Hint below.)

13. Verify that, for manifolds M, N, a continuous surjective mapping $F : M \to N$ and a dense set $K \subset M$ (Problem 8 in §6), the image $F(K)$ is dense in N.

14. Show that a sequence (x_k, y_k) of points in a product manifold $M \times N$ converges to a point $(x, y) \in M \times N$ as $k \to \infty$ if and only if $x_k \to x$ in M and $y_k \to y$ in N.

15. Verify that, for dense subsets K, K' of manifolds M, M', respectively, the product $K \times K'$ is dense in the product manifold $M \times M'$.

16. Show that a subset K of a manifold M is dense in M if and only if it intersects every nonempty open set in M.

17. Verify that, for open subsets U, U' of manifolds M, M', respectively, the product $U \times U'$ is open in $M \times M'$ and, conversely, every open subset of $M \times M'$ is a union of such product sets $U \times U'$.

18. Prove that any subgroup G of the real line $\mathbf{R}$ (with addition) is either cyclic (generated by a single element, i.e., $G = \mathbf{Z} \cdot a$ for some $a \in \mathbf{R}$), or dense in $\mathbf{R}$. (Hint below.)

19. Show that any subgroup Γ of the circle S^1 (the latter being an Abelian group when endowed with the complex multiplication), is either finite and cyclic, or infinite and dense in S^1. (Hint below.)

20. Let T^2 be the 2-dimensional torus $T^2 = S^1 \times S^1$, where the circle (1-dimensional sphere) S^1 consists of all complex numbers $z \in \mathbf{C} = \mathbf{R}^2$ with $|z| = 1$ (§2). Any fixed vector $(a, b) \in \mathbf{R}^2$ gives rise to the mapping (curve) $\gamma : \mathbf{R} \to T^2$ given by $\gamma(t) = (e^{iat}, e^{ibt})$. Verify that γ is a real-analytic (C^ω) Lie-group homomorphism, and that it is an immersion unless $a = b = 0$. Show that γ is an embedding if and only if neither of a, b is a rational multiple of the other.

21. Let $a, b \in \mathbf{R}$ be chosen so that the curve $\gamma : \mathbf{R} \to T^2$ defined in Problem 20 is an embedding.
   (a) Show that the image $\gamma(\mathbf{R})$, with the manifold structure defined as in Problem 3, does not have the subset topology.
   (b) Prove that the set $\gamma(\mathbf{R})$ is dense in T^2. (Hint below.)

22. Given linear functions $f^1, \ldots, f^m \in V^*$ on a finite-dimensional real vector space V, show that the linear mapping $(f^1, \ldots, f^m) : V \to \mathbf{R}^m$ formed by them is
   (a) Injective if and only if $f^1, \ldots, f^m$ span V^*.
   (b) Surjective if and only if $f^1, \ldots, f^m$ are linearly independent in V^*.

23. Let $F : M \to N$ be a C^∞ mapping between manifolds. Verify that F is an immersion if and only if, for any coordinate system y^α in N, with a coordinate domain U, the differentials dF_x^α of the component functions of F span T_x^*M at each point $x \in F^{-1}(U)$. (Thus, immersions $F : M \to \mathbf{R}^k$ are precisely those k-tuples of functions $F^1, \ldots, F^k : M \to \mathbf{R}$ for which $dF_x^1, \ldots, dF_x^k$ span T_x^*M at every $x \in M$.)

24. Show that, given a manifold M with $n = \dim M$ and a point $x \in M$, there exists a C^∞ mapping $F : M \to \mathbf{R}^n$ and a neighborhood U of x in M such that F restricted to U is an immersion. (Hint below.)

25. A mapping $F : M \to N$ between C^s manifolds, $s \ge 1$, is called a *submersion* if it is of class C^s and the differential $dF_x : T_xM \to T_{F(x)}N$ is surjective at each point $x \in M$ (or, equivalently, $\operatorname{rank} F = \dim N$ everywhere in M). Show that any submersion $F : M \to N$ is an *open* mapping in the sense that the F-image $F(U)$ of any open set $U \subset M$ is an open subset of N. (Hint below.)

26. Recall that a vector space W is said to be the *direct sum* of its subspaces V and V', which one writes as

$$W \,=\, V \oplus V' \,, \tag{9.10}$$

if each $w \in W$ can be uniquely expressed as a sum $w = v + v'$ with $v \in V$ and $v' \in V'$. Verify that, for subspaces V, V' of W, relation (9.10) holds if and only if the set $V \cup V'$ spans W and $V \cap V' = \{0\}$. Assuming (9.10), show that the assignment $V \times V' \ni (v, v') \mapsto v + v' \in W$ establishes an isomorphism between the *direct product* $V \times V'$ of V and V' with the componentwise operations (which is often also called the direct sum and denoted by $V \oplus V'$), and W. (For subspaces V, V' of a vector space W satisfying (9.10), we usually identify vectors $w \in W$ with the corresponding pairs $(v, v') \in V \times V'$ satisfying $w = v + v'$. Thus, the standard *inclusions* $V \to W$, $V' \to W$ and *projections* $W \to V$, $W \to V'$ can be characterized by $v \mapsto (v, 0)$, $v' \mapsto (0, v')$, $w = (v, v') \mapsto v$ and, respectively, $w = (v, v') \mapsto v'$.)

27. Given C^s manifolds M, N, verify that the mappings $M \ni x \mapsto (x, y) \in M \times N$ (with a fixed $y \in N$) and $N \ni y \mapsto (x, y) \in M \times N$ (with a fixed $x \in M$) are of class C^s and, if $s \ge 1$, their differentials at any $x \in M$ and $y \in N$, respectively, are injective. (From now on, we will regard T_xM and T_yM as subspaces of the tangent space $T_{(x,y)}(M \times N)$, by identifying them with their images.) (Hint below.)

28. For M, N, x, y as in Problem 27 with $s \ge 1$, show that

$$T_{(x,y)}(M \times N) = T_xM \oplus T_yN \tag{9.11}$$

in the sense of (9.10), and that the corresponding inclusions and projections (Problem 26) are the differentials at x and y of the mappings mentioned above and, respectively, the differentials at (x, y) of the obvious C^s Cartesian-product projection mappings $M \times N \to M$ and $M \times N \ni (x', y') \to N$. Verify that the identification $w = (u, v)$ of vectors $w \in T_{(x,y)}(M \times N)$ with pairs $(u, v) \in T_xM \times T_yN$ then takes the form $(x, y)\dot{}(t) = (\dot{x}(t), \dot{x}(t))$ for C^1 curves $t \mapsto x(t)$ in M and $t \mapsto y(t)$ in N, where (x, y) is the curve in $M \times N$ defined by $(x, y)(t) = (x(t), y(t))$. (Hint below.)

29. Suppose that M, N, P are manifolds and we are given an arbitrary C^1 mapping $M \times N \to P$, which we write as a "multiplication" $M \times N \ni (x, y) \mapsto xy \in P$. For any $x \in M$ and $y \in N$, let us denote by $T_xM \ni v \mapsto vy \in T_{xy}P$ and

$T_yN \ni w \mapsto xw \in T_{xy}P$ the differentials at x and y of the C^1 mappings $M \ni \tilde{x} \mapsto \tilde{x}y \in P$ and $N \ni \tilde{y} \mapsto x\tilde{y} \in P$. Given an interval $I \subset \mathbf{R}$ and C^1 curves $t \mapsto x(t) \in M$ and $t \mapsto y(t) \in N$, defined on I, prove the *Leibniz rule*

$$(xy)^{\cdot} = \dot{x}y + x\dot{y}\,, \tag{9.12}$$

that is, $d[x(t)y(t)]/dt = \dot{x}(t)y(t) + x(t)\dot{y}(t)$ for all $t \in I$. (Hint below.)

Hint. In Problem 12, consider a figure-eight curve (see Fig. 5 above).

Hint. In Problem 18, let $G \neq \{0\}$ and set $a = \inf(G \cap (0,\infty))$. If $a > 0$, then the infimum is a minimum (as any two distinct elements x, y of G are at a distance $|x - y| \geq a$), and $G = \mathbf{Z}\cdot a$. If $a = 0$, for any $\varepsilon > 0$, G contains a number r with $0 < r < \varepsilon$, and the subgroup $\mathbf{Z}\cdot r$ of G comes closer to any real number than ε.

Hint. In Problem 19, let $\Gamma \neq \{1\}$ and let G be the additive subroup of $\mathbf{R}$ with $G = \Phi^{-1}(\Gamma)$, $\Phi : \mathbf{R} \to S^1$ being the group homomorphism $\Phi(t) = e^{it}$. If G is dense in $\mathbf{R}$, so is Γ in S^1 (Problem 13). Otherwise, $G = \mathbf{Z}\cdot a$ for some $a > 0$ (Problem 18), and then Γ (generated by e^{ia}) must be *finite*. In fact, if it were infinite, compactness of S^1 would imply the existence of a sequence of pairwise distinct elements $z_k \in \Gamma$ that converges in S^1, so that the ratios $z_k/z_l \in \Gamma$ may assume the form $e^{i\theta}$ with arbirarily small $\theta > 0$, contradicting the obvious relation $a = \min\{\theta > 0 : e^{i\theta} \in \Gamma\}$.

Hint. In Problem 21, $ab \neq 0$. The torus is an Abelian group (as the direct sum of two copies of S^1, the latter endowed with the complex multiplication), and $\gamma : \mathbf{R} \to T^2$ is a group homomorphism (from the additive group of real numbers). The subgroups $\Gamma = \{z \in S^1 : (z,1) \in \gamma(\mathbf{R})\}$ $\Gamma' = \{z \in S^1 : (1,z) \in \gamma(\mathbf{R})\}$ of S^1 are both *infinite* (otherwise Γ or Γ' would be generated by its element z having the smallest possible positive argument and, by finiteness, $z^n = 1$ for some integer $n \geq 1$, thus making a/b rational.) Hence Γ, Γ' are dense in S^1 (Problem 19) and $\gamma(\mathbf{R})$ contains the dense subgroup $\Gamma \times \Gamma'$ generated by $\Gamma \times \{1\}$ and $\{1\} \times \Gamma'$. See also Problem 15.

Hint. In Problem 24, let $F = (F^1, \ldots, F^m)$, where the F^j are obtained by extending arbitrary coordinate functions x^j at x from a suitable neighborhood of x to the whole of M (Problem 20 in §6).

Hint. In Problem 25, note that, in suitable local coordinates, (9.3) becomes

$$(x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^n) \tag{9.13}$$

with $\dim M = m \geq n = \dim N$. We can now use the openness property of surjective linear operators between finite-dimensional real vector spaces.

Hint. In Problem 27, use the chain rule to conclude that the differentials of the projection mappings $M \times N \to M$, $M \times N \to N$ are one-sided inverses of the differentials in question.

Hint. In Problem 28, use the hint for Problem 27.

Hint. In Problem 29, fix $t_0 \in I$ and set $x_0 = x(t_0)$, $y_0 = y(t_0)$, $v = \dot{x}(t_0)$, $w = \dot{y}(t_0)$ and let $F : M \times N \to P$ be our "multiplication mapping" $F(x,y) = xy$. Then, at $t = t_0$, $\frac{d}{dt}[x(t), y(t)] = \frac{d}{dt}F(x(t), y(t)) = dF_{(x,y)}(v,w)$ (Problem 28) and, since $(v,w) = (v,0) + (0,w)$, this equals $dF_{(x,y)}(v,0) + dF_{(x,y)}(0,w) = \frac{d}{dt}[x(t)y_0] + \frac{d}{dt}[x_0\chi(t)] = vy_0 + x_0w$.

10. More on tangent vectors

Topics: *Ad hoc* descriptions of tangent spaces of real and complex projective spaces; tangent spaces of Grassmannians; more general cases of submanifolds defined by equations.

Let $P(V)$ be the projective space for a given finite-dimensional real or complex vector space V (§2). We then have a natural isomorphic identification

$$T_L[P(V)] \,=\, \mathrm{Hom}(L, V/L) \tag{10.1}$$

at any projective point $L \in P(V)$. (The symbol 'Hom' stands for "the space of $\mathbf{K}$-linear operators", where $\mathbf{K}$ is the scalar field.)

To define a natural linear isomorphism

$$\varPhi : \mathrm{Hom}(L, V/L) \,\to\, T_L[P(V)]\,. \tag{10.2}$$

let us fix $L \in P(V)$ and $F \in \mathrm{Hom}(L, V/L)$. We now choose any vector

$$u \in L \setminus \{0\}\,, \tag{10.3}$$

any *lift* $\tilde F$ of F to V, that is, an operator $\tilde F : L \to V$ whose composite with the projection operator $V \to V/L$ equals F:

$$\tilde F \in \mathrm{Hom}(L, V)\,, \qquad \pi \circ \tilde F \,=\, F\,, \tag{10.4}$$

and any C^1 curve $I \ni t \mapsto x(t) \in V \setminus \{0\}$ along with a parameter $a \in I$ such that

$$x(a) \,=\, u\,, \qquad \dot x(a) \,=\, \tilde F u\,. \tag{10.5}$$

With the aid of these additional data, we declare the image of F under $\varPhi$ to be

$$\varPhi F \,=\, \dot y(a) \in T_L[P(V)]\,, \qquad \text{where} \quad y(t) \,=\, \pi(x(t))\,, \tag{10.6}$$

$\pi : V \setminus \{0\} \to P(V)$ being this time the standard projection mapping (Problem 13 in §3).

We need to prove that this definition is correct, that is, independent of the choice of the quadruple u, $\tilde F$, $x(t)$, a with (10.3) – (10.5). To this end, let us first note that, for fixed u and $\tilde F$ with (10.3) and (10.4), the vector $\dot y(a)$ in (10.6) is the image $d\pi_u[\tilde F u]$ of the vector $\dot x(a) = \tilde F u \in T_u[V \setminus \{0\}] = T_u V = V$ (cf. Examples 5.1, 5.3 under the differential at u of the mapping $\pi : V \setminus \{0\} \to P(V)$ (see (5.16)) and, consequently, $\dot y(a)$ does not depend on the choice of the curve $t \mapsto x(t)$ and a (satisfying (10.5) for the given u and $\tilde F$).

Secondly, let u and $\tilde F$ now be replaced with another vector $w \in L \setminus \{0\}$ and another lift F' of F. Thus, $\tilde F$ and F' differ by an L-valued operator $L \to V$ and, in particular, $F'u = \tilde F u + \mu u$ for some scalar $\mu \in \mathbf{K}$, while, as $\dim L = 1$, we have $w = cu$ for some scalar $c \neq 0$. Choosing a C^1 curve $I \ni t \mapsto x(t) \in V \setminus \{0\}$ and $a \in I$ with (10.5), and any C^1 curve $I \ni t \mapsto \lambda(t) \in \mathbf{K} \setminus \{0\}$ with $\lambda(a) = c$ and $\dot\lambda(a) = c\mu$, we now easily verify that the new curve $I \ni t \mapsto \lambda(t)x(t)$ and the parameter a satisfy the analogue of (10.5) with w, F' instead of u, $\tilde F$. However, $\pi(\lambda(t)x(t)) = \pi(x(t))$ due to the definition of π (Problem 13 in §3). Consequently, the value of $\varPhi F$ obtained from (10.6) will be the same, whether we use u and $\tilde F$, or w and F'.

Thus, $\varPhi F$ is defined correctly, i.e., it depends on F (and L), but not on the quadruple u, $\tilde F$, $x(t)$, a with (10.3) – (10.5).

To show that the mapping (10.2) is linear, let us fix u with (10.3) and choose a subspace $\tilde V$ of V with $V = \tilde V \oplus L$ (cf. Problem 26 in §9. We thus have a natural isomorphism $\tilde V \to V/L$ obtained by restricting the projection operator $V \to V/L$ to $\tilde V$. Denoting $\varPsi : V/L \to \tilde V$ the inverse of that isomorphism, we can associate with every $F \in \mathrm{Hom}(L, V/L)$, a particular lift $\tilde F$ given by $\tilde F = \varPsi \circ F$. (In other words, this is the unique $\tilde{F}$ with (10.4) which is, at the same time, $\tilde{V}$-valued.) By (10.5), (10.6) and (5.16), we now have

$$\varPhi F \,=\, d\pi_u[\varPsi(Fu)]\,, \tag{10.7}$$

which obviously shows that $\varPhi$ is linear.

Relation (10.7) also implies that $\varPhi$ is injective: if $\varPhi F = 0$, it follows from (10.7) that $\varPsi \circ F$ is valued both in $L = \mathbf{K}u$ (Problem 1(a)) and in $\tilde{V}$ (due to our choice of $\varPsi$), so that $F = 0$ as $L \cap \tilde{V} = \{0\}$. Finally, $\varPhi$ is an isomorphism, since both spaces in (10.1) are easily seen to be of the same dimension.

We conclude this section by discussing two important generalizations of Theorem 9.6 (cf. also Problem 5 below.

Theorem 10.1. *Suppose that M, N are C^s manifolds and $F : M \to N$ is a C^s mapping, $s \ge 1$. If the rank of F is constant on M, while $y \in N$ is a point such that $K = F^{-1}(y)$ is nonempty, then K is a C^s submanifold of M endowed with the subset topology, of dimension*

$$\dim K \,=\, \dim M \,-\, \operatorname{rank} F\,, \tag{10.8}$$

and with the tangent spaces given by (9.9).

Proof. See Problem 4. □

Given C^1 manifolds M and N, a C^1 mapping $F : M \to N$, and a C^1 submanifold P of N, one says that F is *transversal* to P if, for every $x \in M$ such that $y = F(x) \in P$, the tangent space T_yN is spanned by the subset $T_yP \cup dF_x(T_xM)$. For instance, F is automatically tranversal to P if it is a submersion, as well as in the case where $F(M)$ does not intersect P.

Theorem 10.2. *Suppose that M, N are C^s manifolds, $F : M \to N$ is a C^s mapping, $s \ge 1$, and P is a C^s submanifold of N endowed with the subset topology. If F is tranversal to P and the set $K = F^{-1}(P)$ is nonempty, then K is a C^s submanifold of M, carrying the subset topology, while its dimension and its tangent spaces are given by*

$$\dim K \,=\, \dim M \,-\, \dim N \,+\, \dim P\,, \tag{10.9}$$

$$T_xK \,=\, (dF_x)^{-1}(T_{F(x)}P) \quad \text{whenever} \quad x \in K\,. \tag{10.10}$$

Proof. See Problem 6. □

Problems

1. Let $\pi : V \setminus \{0\} \to P(V)$ denote the standard projection mapping (Problem 13 in §3).

(a) Verify that, for any $u \in V \setminus \{0\}$, denoting $\mathbf{K}$ the scalar field, we have

$$\operatorname{Ker} d\pi_u \,=\, \mathbf{K}u \,\subset\, T_u[V \setminus \{0\}] = T_uV = V\,. \tag{10.11}$$

(b) Show that π is a submersion. (Hint below.)

2. Generalize the construction of the natural isomorphism (10.1) to the case of arbitrary Grassmann manifolds $\mathrm{Gr}_q(V)$ (Problem 2 in §81), in the sense of providing a natural isomorphic identification

$$T_W[\mathrm{Gr}_q(V)] \,=\, \operatorname{Hom}(W, V/W) \tag{10.12}$$

for any $W \in \mathrm{Gr}_q(V)$. (Hint below.)

3. Verify that any C^1 homomorphism between C^1 Lie groups has constant rank. (Hint below.)
4. Prove Theorem 10.1. (Hint below.)
5. Explain why Theorem 10.1 is a generalization of Theorem 9.6. (Hint below.)
6. Prove Theorem 10.2. (Hint below.)

Hint. In Problem 1, let us choose a basis $e_1, \dots, e_n$ of V with $u = e_1$ and let $f = e^1$ be the first element of the dual basis in V^*. Using the linear coordinates in V associated with the basis $e_1, \dots, e_n$ and the coordinate system in $P(V)$ obtained from (U_f, φ_f) (§2) by replacing φ_f with φ_f followed by $(e^2, \dots, e^n) : f^{-1}(1) \to \mathbf{K}^{n-1}$, we see that in such coordinates π appears as

$$(x^1, \dots, x^n) \mapsto (x^2/x^1, \dots, x^n/x^1)\,. \tag{10.13}$$

(Note that $x^1 \neq 0$ throughout the coordinate domain U_f.) Differentiating the right-hand side with respect to a parameter t at a value of t for which $x^1 = 1$ and $x^2 = \ldots = x^n = 0$ (which corresponds to the point $u = e_1$), we easily obtain the following description of $d\pi_u$ in terms of the components of tangent vectors in the coordinate systems selected above (which proves both (a) and (b)):

$$(\dot{x}^1, \dots, \dot{x}^n) \mapsto (\dot{x}^2, \dots, \dot{x}^n)\,. \tag{10.14}$$

Hint. In Problem 2, let us introduce the manifold $\mathcal{B}_q(V)$ consisting of all linearly independent q-tuples of vectors in V. (This is an open subset of the vector space V^q, the qth Cartesian power of V.) We have the *standard projection* mapping $\pi : \mathcal{B}_q(V) \to \mathrm{Gr}_q(V)$ given by $\pi(u_1, \dots, u_q) = \mathrm{Span}\{u_1, \dots, u_q\}$. We can now define a linear isomorphism

$$\varPhi : \mathrm{Hom}(W, V/W) \to T_W[\mathrm{Gr}_q(V)] \tag{10.15}$$

by setting, for $F \in \mathrm{Hom}(W, V/W)$,

$$\varPhi F = \dot{y}(a) \in T_L[P(V)]\,, \qquad \text{with} \quad y(t) = \pi(x_1(t), \dots, x_q(t))\,, \tag{10.16}$$

where we have fixed a basis $u_1, \dots, u_q$ of W and a lift $\tilde{F} : W \to V$ of F, and used any C^1 curve $I \ni t \mapsto (x_1(t), \dots, x_q(t))$ with $x_j(a) = u_j$ and $\dot{x}_j(a) = \tilde{F}u_j$ for a specific $a \in I$ and all $j = 1, \dots, q$. Proceeding as before, we now show that $\varPhi$ is independent of the choice of the additional data.

Hint. In Problem 3, use (8.8) to observe that, for a C^1 homomorphism $F : G \to H$ and any $x \in G$, dF_x is the composite $v \mapsto [F(x)](dF_1(x^1 v))$ involving two linear isomorphisms, and so it must have the same rank as dF_1.

Hint. In Problem 4, use an obvious modification of the proof of Theorem 9.6.

Hint. In Problem 5, note that Theorem 9.6 is a special case of Theorem 10.1 obtained by replacing M with the open submanifold $(\operatorname{rank} F)^{-1}(\dim N)$.

Hint. In Problem 6, fix $x \in K$ and let $y = F(x)$. The rank theorem guarantees the existence of a neighborhood U of y in N and a C^s submersion $H : U \to U'$ onto a neighborhood U' of 0 in $\mathbf{R}^{n-p}$ (where we have set $m = \dim M$, $n = \dim N$, $p = \dim P$) such that $U \cap P = H^{-1}(0)$. Thus, $F^{-1}(U) \cap K = (H \circ F)^{-1}$, while the transversality assumption makes 0 a regular value of $H \circ F$. The rest of the argument is the same as in the proof of Theorem 9.6.

11. Lie subgroups

Topics: Lie subgroups; Lie subgroups with the subset topology; the identity component; discrete subgroups; S^1 and S^3 as Lie subgroups of $\mathbf{C}\setminus\{0\}$ and $\mathbf{H}\setminus\{0\}$; other simple examples; orientation in real vector spaces.

Given a group G whose operation is written as a multiplication, an element $a \in G$ and subsets $K, L \subset G$, let us set $aK = \{ax : x \in K\}$, $Ka = \{xa : x \in K\}$, $KL = \{xy : x \in K\,,\, y \in L\}$ and $K^{-1} = \{x^{-1} : x \in K\}$. Thus, a subgroup of G is any nonempty subset $H \subset G$ with $HH \subset H$ and $H^{-1} \subset H$. By a *normal subgroup* of G we mean, as usual, a subgroup $H \subset G$ such that $aHa^{-1} \subset H$ whenever $a \in G$ (and hence, $aHa^{-1} = H$ for all $a \in G$); in other words, a subgroup is called normal if it is closed under all inner automorphisms of G (Example 8.10).

For a fixed subgroup H of a group G, the *left cosets* of H are defined to be all sets of the form aH for some $a \in G$. They form a disjoint decomposition of G into a union of subsets, one of which is H itself. The same is true for the *right cosets* of H, which are the sets of the form Ha with $a \in G$. One easily sees that H is a normal subgroup if and only if the families of its left and right cosets coincide.

Suppose now that G is a Lie group of class C^r, $0 \le r \le \omega$. By the *identity component* of G we mean the connected component G° of the manifold G, containing 1 (1 is often called the *identity* of G). We then have $G^\circ G^\circ \subset G^\circ$, $(G^\circ)^{-1} \subset G^\circ$, and $aG^\circ a^{-1} \subset G^\circ$ for all $a \in G$, as $G^\circ G^\circ$, $(G^\circ)^{-1}$ and $aG^\circ a^{-1}$ are connected subsets of G containing 1 (Problem 8 in 3). The connected components of G are nothing else than the left (or right) cosets of G°; this is immediate from Problem 1 below. Furthermore, G° regarded as an open submanifold of G is a Lie group of class C^r (with the group operation inherited from G; see Problem 3). The identity inclusion mapping $G^\circ \to G$ now is a C^r homomorphism of Lie groups, inducing the familiar identification $T_1 G^\circ = T_1 M$, so that, when $r \ge 3$, the Lie algebras of G and G° are naturally isomorphic (and will from now on be identified).

In other words, the algebraic structure (i.e., isomorphism type) of the Lie algebra $\mathfrak{g}$ of any Lie group G depends solely on the connected Lie group G°. Thus, the only conclusions about G that may be expected to follow from assumptions about $\mathfrak{g}$ are those that pertain to G° alone.

Example 11.1. The C^ω Lie group $\mathrm{GL}(V)$ for a finite-dimensional vector space V over the field $\mathbf{K}$ of real or complex numbers (Example 4.3), is connected when $\mathbf{K} = \mathbf{C}$, and has two connected components when $\mathbf{K} = \mathbf{R}$ and $\dim V > 0$. In the latter case, the identity component of $\mathrm{GL}(V)$, denoted by $\mathrm{GL}^+(V)$, consists of all linear isomorphisms $A : V \to V$ with $\det A > 0$. See Problem 10.

Example 11.2. In particular, the matrix Lie group $\mathrm{GL}\,(n, \mathbf{C})$ is connected, while $\mathrm{GL}\,(n, \mathbf{R})$ with $n \ge 1$ has two components, the one containing the identity being the group $\mathrm{GL}^+(n, \mathbf{R})$ of all real $n \times n$ matrices having positive determinants.

Let G be a Lie group of class C^r, $r \ge 0$. By a *Lie subgroup* of class C^r in G we mean any subgroup H of G with a fixed structure of a C^r manifold which makes H both a C^r submanifold of G and a C^r Lie group (for the group operation H inherits from G). We will call H a *Lie subgroup of G with the subset topology* if its manifold structure represents the subset topology. (In the latter case, such a manifold structure is unique, cf. Corollary 9.5, and so does not have to be specified; also, the final clause in the definition of a Lie subgroup follows automatically, and hence need not be verified; see Remark 11.8 below.) Similarly, a *normal Lie subroup* of G is a Lie subroup of G which is simultaneously a normal subgroup.

EXAMPLE 11.3. Every open subgroup of a Lie group is a Lie subgroup with the subset topology. (See Problem 3.)

EXAMPLE 11.4. The kernel of any C^1 homomorphism $G \to H$ between C^r Lie groups G and H, $r \geq 1$, is a C^r normal Lie subgroup of G with the subset topology, which is also closed as a subset of G. (See Problem 4.)

EXAMPLE 11.5. As a special case of Example 11.4, for any finite-dimensional real or complex vector space V, the automorphism group $\mathrm{GL}(V)$ has a C^ω normal Lie subgroup $\mathrm{SL}(V)$ with the subset topology, consisting of all $A : V \to V$ with $\det A = 1$. (cf. Problem 6 in §4.)

EXAMPLE 11.6. One easily finds examples of Lie subgroups which do not have the subset topology. For instance, every Lie group G contains as a Lie subgroup the same group G treated as *discrete* (Example 4.6). More generally, G contains as a Lie subgroup the disjoint union of all cosets of any given Lie subgroup H. However, there also more interesting examples of such subgroups, namely, *connected* ones; for instance, the torus T^2 contains a 1-dimensional dense Lie subgroup which does not have the subset topology (see Problem 21(a) in §9).

EXAMPLE 11.7. The circle S^1 and 3-sphere S^3 are Lie subgroups of $\mathbf{C}\setminus\{0\}$ and $\mathbf{H}\setminus\{0\}$. Namely, they are kernels of the *norm homomorphisms* valued in $\mathbf{R}\setminus\{0\}$.

REMARK 11.8. Let a subgroup H of a C^r Lie group G be, at the same time, a C^r submanifold of G endowed with the subset topology. Then H is a Lie subgroup of class C^r in G. In fact the group operations in H (that is, (4.2) restricted to $H \times H$ or H) are of class C^r as mappings into H, which is an immediate consequence of Theorem 9.4.

Problems

1. Show that the family of all connected components of a given manifold M is the unique collection of pairwise disjoint, nonempty connected open subsets of M whose union is M.

2. Show that $f(G^\circ) \subset H^0$ for a continuous Lie-group homomorphism $f : G \to H$.

3. Let H be a subgroup of a Lie group G of class C^r, $r \geq 0$, which at the same time is an open subset of G. Verify that H is a C^r Lie subgroup of G with the subset topology.

4. Prove the claim made in Example 11.4. (Hint below.)

Hint. In Problem 4, use Lemma 8.6, Problem 3 in §10 and Theorem 10.1.

12. Orthogonal and unitary groups

Topics: Lie-group actions on manifolds; transitive and free actions; isotropy groups; (special) orthogonal and (special) unitary groups; their Lie algebras; explicit descriptions in low dimensions; orbits of group actions.

By a *left action* of a group G on a set M we mean a mapping $G \times M \to M$, denoted by $(a, x) \mapsto ax$, such that, with ab denoting the product in G,

i. $a(bx) = (ab)x$ whenever $x \in M$ and $a, b \in G$,

ii. $1x = x$ for all $x \in M$, where 1 stands for the neutral element of G.

If such an action is given, we say, informally, that G *acts on* M (from the left).

Similarly, a *right action* of a group G on a set M is a mapping $M \times G \ni (x,a) \mapsto xa \in M$ with $(xa)b = x(ab)$ for all $x \in M$ and $a, b \in G$. Every left/right action can be naturally converted into a right/left one if one sets $xa = a^{-1}x$ or, respectively, $ax = xa^{-1}$. In other words, a left/right action of G on M may be thought of as a right/left action on M of the group G obtained from G by "reversing the multiplication" (see Problem 1).

By a C^r *action from the left/right* of a C^r Lie group G on a C^r manifold M we mean an action of G on the underlying set of M which is also a C^r mapping $G \times M \to M$ (or, respectively, $M \times G \to M$). When r is clear from the context and $r \geq 1$ (e.g., under the blanket assumption that all objects considered are of class C^∞), we will then simply say that G *acts on* M *differentiably* from the left or from the right.

Example 12.1. Every (Lie) group G admits the *trivial action* on any set/manifold M, with $ax = x$ (or $ax = x$) for all $a \in G$ and $x \in M$.

Example 12.2. Every subgroup H of a group G acting from the left/right on a set M also acts on M via the restriction to H of the original action. Similarly, for every subset N of M *invariant under* the given left/right action of G (in the sense that $ax \in N$ or $xa \in N$ whenever $a \in G$ and $x \in N$), the restriction of the original action is a left/right action of G on N. Analogous statements are obviously valid for C^r actions of Lie groups on manifolds, provided that H is a C^r Lie subgroup of G or, respectively, the G-invariant subset $N \subset M$ is a C^r submanifold of M endowed with the subset topology. (In the latter case, our claim is immediate from Theorem 9.4.)

Example 12.3. Every (Lie) group G acts on itself (differentiably), both from the left and from the right, via the group multiplication. Similarly, $G \times G$ acts on G from the left via $((a,b),x) \mapsto axb^{-1}$.

Given a left or right action $(a,x) \mapsto ax$ or $(x,a) \mapsto xa$ of a group G on a set M and any $a \in G$, $x \in M$, let us now generalize the notations introduced in (8.1), denoting $L_a : M \to M$ and $R_x : G \to M$ (or, $R_a : M \to M$ and $L_x : G \to M$) the mappings given by

$$L_a(y) = ay\,, \quad R_x(b) = bx \quad \text{or} \quad R_a(y) = ya\,, \quad L_x(b) = xb \tag{12.1}$$

(whichever applies), for $y \in M$, $b \in G$. Again, each L_a (or R_a) is a bijection, with the inverse $L_{a^{-1}}$ (or, $R_{a^{-1}}$) and, in the case of a C^r-differentiable Lie-group action, every L_a (or R_a) is a C^r-diffeomorphism, while every R_x (or, L_x) is of class C^r.

From now on, 'actions' will stand for left actions; all the statements made about (left) actions have obvious counterparts for right actions, the details of which are left to the reader.

Note that, by the above definition, an action of G on a set M amounts to any fixed group homorphism $a \mapsto L_a$ from G into the group of all permutations of M (i.e., bijections $M \to M$), with composition as the group operation. For instance, the trivial homomorphism (sending all of G to $\mathrm{Id} : G \to G$) corresponds to the *trivial action* of G on M with $ax = x$ for all $a \in G$ and $x \in M$.

An action $(a,x) \mapsto ax$ of a group G on a set M is called *transitive* if, for any $x, y \in M$, there is $a \in G$ with $ax = y$, and is called *free* if $ax \neq x$ for any $a \in G \setminus \{1\}$ and any $x \in M$. Thus, the given action is free if and only if, for any $a \in G$ other than the identity, L_a has no fixed points (in M). An action which is both free and transitive is referred to as *simply transitive*. For instance, the action of G on itself by left/right translations is simply transitive; the action of $G \times G$ on G in Example 12.3 is transitive but not free unless G is trivial (consider (x,x) and $(1,1)$ acting on the element x).

Example 12.4. A real or complex affine space $(M, V, +)$ (§2, appendix) is nothing else than a set M with a fixed "additively written" simply transitive action on M of the additive group $G = V$ of a given real/complex vector space V. If V is finite-dimensional, we may treat M as a C^ω manifold (§2) and $G = V$ as a C^ω Lie group (Example 4.1); the action in question then is, obviously, also of class C^ω.

Lemma 12.5. *For any C^1 action $(a,x) \mapsto ax$ of a C^1 Lie group G on a C^1 manifold M and any given point $x \in M$, the mapping $R_x : G \to M$ defined by* (12.1) *is of constant rank.*

Proof. Let us set $F = R_x$. Given $a \in G$, using the multiplicative notation of Problem 29 in §9 we have $dF_a v = vx$ and $dF_1 u = ux$ for any $v \in T_a G$ and $u \in T_1 G$. On the other hand, the "associativity" of the group action (condition (i) above) gives $(au)y = a(uy)$ for any $a \in G$, $y \in M$ and any vector u tangent to G at any point. (To see this, realize u as the velocity of a C^1 curve.) Thus, writing aux (without parentheses) and choosing $u = a^{-1}v \in T_1 G$ for $a \in G$ and $v \in T_a G$, we now have $vx = (aa^{-1}v)x = a((a^{-1}v)x)$, i.e., dF_a is the isomorphism $T_a G \ni v \mapsto a^{-1}v \in T_1 G$, followed by dF_1, followed by the isomorphism $T_x M \ni w \mapsto aw \in T_{ax} M$. Thus, dF_a and dF_1 have the same rank, as required. □

Given an action $(a,x) \mapsto ax$ of a group G on a set M and a point $x \in M$, by the *isotropy group* of the action at x we mean the subset H_x of G given by

$$H_x = \{a \in G : ax = x\}. \tag{12.2}$$

It is easy to see that H_x is always a *subgroup* of G. Moreover, since $H_x = R_x^{-1}(x)$, combining Lemma 12.5 with Theorem 10.1 we obtain

Corollary 12.6. *Given a C^r action of a C^r Lie group G on a C^r manifold M, $r \geq 1$, the isotropy group H_x of every point $x \in M$ is a Lie subgroup of G, endowed with the subset topology and closed as a subset of G.*

Let V be a vector space over a scalar field $\mathbf{K}$ (where $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$). Suppose that $B : V \times V \to \mathbf{K}$ is either bilinear ($\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$), or sesquilinear ($\mathbf{K} = \mathbf{C}$). Recall that sesquilinearity of B means that it is linear in the first, and antilinear in the second variable. Any vector space W over $\mathbf{K}$ with any linear operator $F : W \to V$ then gives rise to the *pullback* of B under F, which is the bilinear/sesquilinear form on W with

$$(F^*B)(w, w') = B(Fw, Fw') \qquad \text{for} \quad w, w' \in W. \tag{12.3}$$

Note that, if there is also a third vector space W' over $\mathbf{K}$ and another operator $H : W' \to W$, we can form the composite $FH : W' \to V$ and then, obviously,

$$(FH)^*B = H^*(F^*B). \tag{12.4}$$

Suppose now that our vector space V over the field $\mathbf{K}$ is finite-dimensional, and let $\mathcal{Q}$ stand for the vector space of all forms on V which are of a fixed type (bilinear or sesquilinear). In view of (12.4), the assignment

$$(B,F) \mapsto F^*B \tag{12.5}$$

is a right C^ω action on $\mathcal{Q}$ of the group $\mathrm{GL}(V)$ (notation of Example 4.3). We denote by $\mathrm{Aut}(V,B)$ the *automorphism group* of B, that is, the subgroup of $\mathrm{GL}(V)$ consisting of those linear automorphisms $F: V \to V$ which satisfy the condition $F^*B = B$ (i.e., preserve B). We have

COROLLARY 12.7. *The automorphism group* $\mathrm{Aut}(V,B)$ *of any bilinear/sesquilinear form* B *on a finite-dimensional real or complex vector space* V *is a* C^ω *Lie subgroup of* $\mathrm{GL}(V)$*, carrying the subset topology, and it is a closed subset of* $\mathrm{GL}(V)$*.*

This is clear from Corollary 12.6, as $\mathrm{Aut}(V,B)$ is the isotropy group of the point $B \in \mathcal{Q}$ for the action of $\mathrm{GL}(V)$ on $\mathcal{Q}$ given by (12.5).

Suppose that we are given a fixed finite-dimensional real (or complex) vector space V and a fixed form B which is bilinear and symmetric (or, respectively, sesquilinear and Hermitian) and positive definite. (In other words, B is an inner product in V.) The automorphism group $\mathrm{Aut}(V,B)$ then is usually denoted by $\mathrm{O}(V)$ (in the real case) or $\mathrm{U}(V)$ (in the complex case) and called the *orthogonal group* or the *unitary group* of the inner-product space V. (The form B is usually omitted from the notation.) Furthermore, one denotes by $\mathrm{SO}(V)$ (in the real case) or $\mathrm{SU}(V)$ (in the complex case) the Lie subgroup of $G = \mathrm{O}(V)$ or $G = \mathrm{U}(V)$ obtained as the kernel of the determinant homomorphism (4.19) restricted to G. Those groups are referred to as the *special orthogonal group* or the *special unitary group* of V. In the case where V is $\mathbf{R}^n$ or $\mathbf{C}^n$ with the standard inner product, one denotes these groups by $\mathrm{O}(n)$, $\mathrm{SO}(n)$ (or, $\mathrm{U}(n)$, $\mathrm{SU}(n)$), and regards them as consisting of $n \times n$ matrices.

The Lie algebras of the groups just described are denoted by $\mathfrak{o}(V)$, $\mathfrak{so}(V)$ (the real case) or $\mathfrak{u}(V)$, $\mathfrak{su}(V)$ (the complex case). For $V = \mathbf{K}^n$, these become the matrix Lie algebras $\mathfrak{o}(n)$, $\mathfrak{so}(n)$ or $\mathfrak{u}(n)$, $\mathfrak{su}(n)$. Explicitly, we have

$$\begin{aligned} &\mathfrak{o}(V) = \mathfrak{so}(V) = \mathfrak{u}(V) = \{A \in \mathfrak{gl}(V) : A^* = -A\}, \\ &\mathfrak{su}(V) = \{A \in \mathfrak{gl}(V) : A^* = -A \text{ and } \operatorname{Trace} A = 0\}. \end{aligned} \tag{12.6}$$

In fact, if V is real, $\mathrm{SO}(V)$ is the identity component of $\mathrm{O}(V)$ (Problem 14), and so $\mathfrak{o}(V) = \mathfrak{so}(V)$ (see §11). Both in the real and complex case, the first line of (12.6) is an obvious special case of Problem 4. To obtain the second line, it now suffices to apply Problem 5 to the restriction $\det : \mathrm{U}(V) \to \mathbf{C} \setminus \{0\}$ of the determinant homomorphism (4.19) (using the description of the corresponding Lie-algebra homomorphism $\det_*$ provided by (8.12)).

The dimensions of the Lie algebras in (12.6) are easily computed using matrix representation of operators; we thus have

$$\begin{aligned} &\dim \mathrm{O}(V) = \dim \mathrm{SO}(V) = \binom{n}{2} = \frac{n(n-1)}{2}, \qquad n = \dim V, \\ &\dim \mathrm{U}(V) = n^2, \qquad \dim \mathrm{SU}(V) = n^2 - 1. \end{aligned} \tag{12.7}$$

For a finite-dimensional inner-product space V over the field $\mathbf{K}$ of real or complex numbers, the determinant homomorphism $\det : G \to \mathbf{K} \setminus \{0\}$ restricted to $G =$

$\mathrm{O}(V)$ or $G = \mathrm{U}(V)$ takes values in $\{1, -1\}$ (when $\mathbf{K} = \mathbf{R}$), or in the unit circle $S^1 \subset \mathbf{C}$ (when $\mathbf{K} = \mathbf{C}$); see Problem 17. Moreover, one easily sees that det maps $\mathrm{O}(V)$ onto $\{1, -1\}$ and $\mathrm{U}(V)$ onto S^1. This reduces the question of understanding the structure of $\mathrm{O}(V)$ or $\mathrm{U}(V)$ to the corresponding question for $\mathrm{SO}(V)$ or $\mathrm{SU}(V)$. Of course, each of the groups involved may also be identified with the corresponding group of $n \times n$ matrices, $n = \dim_{\mathbf{K}} V$.

On the other hand, if V is complex, we have an *n-to-one mapping*

$$S^1 \times \mathrm{SU}(V) \ni (z, F) \mapsto zF \in \mathrm{U}(V)\,, \tag{12.8}$$

that is, a surjective mapping whose every value has exactly n preimages. (See Problem 21.) Moreover, (12.8) is clearly a Lie-group homomorphism of class C^ω, and its kernel is the cyclic group $\mathbf{Z}_n$ of all nth roots of unity (treated as the corresponding multiples of the identity operator $V \to V$). Thus, we can identify $\mathrm{U}(V)$ with the quotient of $S^1 \times \mathrm{SU}(V)$ over the normal subroup $\mathbf{Z}_n$.

In the lowest real or complex dimensions, we have some explicit descriptions.

Example 12.8. When $\dim_{\mathbf{K}} V = 1$, we have the natural isomorphic identification $\mathrm{GL}(V) = \mathbf{K} \setminus \{0\}$, and then $\mathrm{O}(V) = \{1, -1\}$, $\mathrm{SO}(V) = \{1\}$ (if $\mathbf{K} = \mathbf{R}$) and $\mathrm{U}(V) = S^1$, $\mathrm{SU}(V) = \{1\}$ (if $\mathbf{K} = \mathbf{C}$).

Example 12.9. When V is a 2-dimensional real inner-product space, $\mathrm{SO}(V)$ is isomorphic, as a C^ω Lie group, to the circle S^1, and so the underlying manifold of $\mathrm{O}(V)$ is diffeomorphic to a disjoint union of two circles.

Example 12.10. When V is a 2-dimensional complex inner-product space, $\mathrm{SU}(V)$ is isomorphic, as a C^ω Lie group, to the 3-sphere group S^3 of unit quaternions. (See Problem 20.) Thus, according to the preceding discussion, the group $\mathrm{U}(V)$ then is isomorphic to a $\mathbf{Z}_2$ quotient of the direct product $S^1 \times S^3$, with $\mathbf{Z}_2$ generated by $(-1, 1)$.

Denoting $S^3 \subset \mathbf{H}$ the sphere of unit quaternions and using quaternion multiplication (see §4), we now define two C^ω Lie group homomorphisms:

$$S^3 \ni p \mapsto \{x \mapsto px\overline{p}\} \in \mathrm{SO}(1^\perp)\,, \tag{12.9}$$

where $1^\perp$ is the 3-dimensional Euclidean space (4.7) of pure quaternions, and

$$S^3 \times S^3 \ni (p, q) \mapsto \{x \mapsto px\overline{q}\} \in \mathrm{SO}(\mathbf{H})\,. \tag{12.10}$$

Note that $\overline{p} = p^{-1}$ for $p \in S^3$ (by (4.11)), and each homomorphism is valued in the *special* orthogonal group as a consequence of Problem 2 in §11 (since S^3 is connected; see Problem 15 in §3). Computing the differentials of (12.9) at 1 and of (12.10) at $(1, 1)$, we find that they send any $\dot{p}$ (or, any $(\dot{p}, \dot{q})$) to the operator $x \mapsto \dot{p}x - x\dot{p}$ (or, $x \mapsto \dot{p}x - x\dot{q}$). (Note that $\dot{p}, \dot{q} \in 1^\perp$ differ from their quaternion conjugates by sign.) Thus, both differentials are injective (according to Problems 15 and 16 in §4, as $\mathbf{R} \cap 1^\perp = \{0\}$). On the other hand, both groups in (12.9) (or, (12.10)) are of the same dimension 3 (or, 6), the differentials in question are isomorphisms. Thus, the homomorphisms (12.9) and (12.10) are submersions (Problem 3 in §10), and hence, in this particular case, must both be surjective (see Problem 22). They are also both two-to-one, since their kernels are copies of $\mathbf{Z}_2$, generated by -1 or, respectively, $(-1, -1)$.

Example 12.11. For a real inner-product space V with $\dim V = 3$, the underlying manifold of $\mathrm{SO}(V)$ is diffeomorphic to $\mathbf{R}\mathrm{P}^3$. (See Problem 20.)

EXAMPLE 12.12. For a real inner-product space V with $\dim V = 4$, the group $\mathrm{SO}(V)$ is isomorphic to a $\mathbf{Z}_2$ quotient of the direct product $S^3 \times S^3$, with $\mathbf{Z}_2$ generated by $(-1,-1)$. (This is immediate from our discussion of (12.10).)

Problems

1. Given a group G, let $*$ denote a new binary operation in the set G, with $a * b = ba$ for $a, b \in G$, where ab is the product in G. Verify that $*$ is a group operation and, denoting $\overline{G}$ the group formed by the set G with the multiplication $*$, we have a group isomorphism $G \ni x \mapsto x^{-1} \in \overline{G}$ (inverse taken in G). Show that left/right actions of G coincide with right/left actions of $\overline{G}$. (cf. Problem 9 in §4.)

2. Show that, given a C^1 curve $t \mapsto x = x(t) \in G$ in a Lie group G of class C^r, $r \ge 1$, for the corresponding curve $t \mapsto x^{-1} = [x(t)]^{-1} \in G$ one has

$$(x^{-1})^{\cdot} = -x^{-1}\dot{x}x^{-1}\,, \tag{12.11}$$

that is, $d[x(t)]^{-1}/dt = -[x(t)]^{-1}\dot{x}(t)[x(t)]^{-1}$. (Hint below.)

3. Given a C^2 Lie subgroup H of a Lie group G, verify that the Lie-algebra homomorphism $\mathfrak{h} \to \mathfrak{g}$ induced by the inclusion homomorphism $H \to G$ maps $\mathfrak{h}$ isomorphically onto a Lie subalgebra of $\mathfrak{g}$. (From now on we will always identify $\mathfrak{h}$ with that subalgebra of $\mathfrak{g}$, and write $\mathfrak{h} \subset \mathfrak{g}$.)

4. Verify that, for any bilinear/sesquilinear form B on a finite-dimensional real or complex vector space V, the Lie algebra of the automorphism group $\mathrm{Aut}(V, B)$, viewed as its tangent space at 1 and thus identified with a Lie subalgebra of $\mathfrak{gl}(V) = \mathrm{Hom}(V, V)$ (cf. Problem 3 and Example 8.3) consisting of all operators $F : V \to V$ with $B(Fv, w) = -B(v, Fw)$ for all $v, w \in V$. (Hint below.)

5. For any C^2 homomorphism $F : G \to H$ of Lie groups, verify that the normal Lie subgroup $K = \mathrm{Ker}\, F$ of G (Example11.4) has the Lie algebra $\mathfrak{k} \subset \mathfrak{g}$ (cf. Problem 3) given by $\mathfrak{k} = \mathrm{Ker}\, F_*$, with F_* as in the paragraph preceding formula (8.10).

6. Given a finite-dimensional real or complex vector space V, verify that the Lie subgroup $\mathrm{SL}(V)$ of the automorphism group $\mathrm{GL}(V)$ (Example 11.5) has the Lie algebra $\mathfrak{sl}(V) \subset \mathfrak{gl}(V)$ defined as in Problem 3 in §7.

7. Verify that, for any finite-dimensional complex vector space and any linear operator $F : V \to V$, there exists a basis e_a of F that makes F *upper triangular* (i.e., each Fe_a is in the span of the e_c with $c \le a$). (Hint below.)

8. Given a complex vector space V with $\dim V < \infty$, prove that the Lie group $\mathrm{GL}(V)$ is connected. Denoting $\mathcal{B}(V)$ the set of all bases of V, verify that $\mathcal{B}(V)$ is connected. (Hint below.)

9. Verify that, for any finite-dimensional complex inner-product space and any *linear isometry* $F : V \to V$ (that is, a linear operator $F \in \mathrm{U}(V)$), all eigenvalues of F are of modulus 1 and there exists an orthonormal basis of V consisting of eigenvectors of F.

10. For any complex inner-product space V with $\dim V < \infty$, prove that the Lie groups $\mathrm{U}(V)$ and $\mathrm{SU}(V)$ are both connected. (Hint below.)

11. Verify that, for any real (or, complex) $n \times n$ matrix $\mathfrak{M}$, the following three conditions are equivalent:

(i) $\mathfrak{M} \in \mathrm{O}(n)$ (or, respectively, $\mathfrak{M} \in \mathrm{U}(n)$).

(ii) The rows (or, columns) of $\mathfrak{M}$ form an orthonormal basis of $\mathbf{R}^n$ (or, $\mathbf{C}^n$) with the standard inner product.

(iii) $\mathfrak{M}$ is invertible and its inverse coincides with its transpose (or, with the entrywise complex conjugate of its transpose).

12. Verify that, for all finite-dimensional real/complex inner-product spaces V, the Lie groups $\mathrm{O}(V)$, $\mathrm{SO}(V)$, $\mathrm{U}(V)$, $\mathrm{SU}(V)$) are all compact.

13. Given a real or complex inner-product space V with $\dim V = n$, $1 \le n < \infty$, let $M \subset V^n$ be the set of all orthonormal bases of V (see also Problem 5 in §11. Verify that M is a compact C^ω submanifold of V^n carrying the subset topology; in addition, M is connected (if V is complex), or has two connected components (if V is real).

14. Show that, for any finite-dimensional real inner-product space V, $\mathrm{SO}(V)$ is the identity component of $\mathrm{O}(V)$.

15. Given a complex vector space V with $1 \le \dim V < \infty$, verify that the underlying real space of V has a "distinguished" orientation, containing the real basis e_1, $ie_1 \ldots, e_n$, ie_n for every basis $e_1, \ldots, e_n$ of V. (Hint below.)

16. A bilinear/sesquilinear form B in a real or complex vector space V is called *nondegenerate* if for every $v \in V \setminus \{0\}$ there exists $w \in V$ with $B(v, w) \neq 0$. Prove that, if $\dim V < \infty$, nondegeneracy of B is equivalent to the requirement that $\det \mathfrak{B} \neq 0$ for the matrix representing B in some (or, any) basis of V.

17. Let B be a bilinear/sesquilinear form in a real or complex vector space V. We say that a vector subspace W of V is *nondegenerate* (relative to B) if the restriction of B to W is nondegenerate. (Note that B itself is not assumed nondegenerate.) Show that, if $\dim V < \infty$ and $W \subset V$ is nondegenerate, then $V = W \oplus W^\perp$, where $W^\perp$, called the *B-orthogonal complement* of W, is the space of all $v \in V$ such that $B(v, w) = 0$ for every $w \in W$. (Hint below.)

18. Let a bilinear/sesquilinear form B in a finite-dimensional real or complex vector space V be nondegenerate, and let $A \in \mathrm{Aut}(V, B)$. Prove that in the bilinear case $\det A \in \{1, -1\}$ and in the sesquilinear case $\det A \in S^1 \subset \mathbf{C}$. (Hint below.)

19. Verify that the group $\mathrm{SU}(2)$ consists of all matrices of the form

$$\begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} \tag{12.12}$$

with $a, b \in \mathbf{C}$ such that $|a|^2 + |b|^2 = 1$.

20. Show that the real algebra of quaternions is isomorphic to the subalgebra of $\mathfrak{gl}(2, \mathbf{C})$ formed by all matrices of the form (12.12) with arbitrary $a, b \in \mathbf{C}$, in such a way that the norm squared of the quaternion corresponding to the matrix (12.12) equals $|a|^2 + |b|^2$. (Hint below.)

21. Prove the claims made in Examples 12.10 and 12.11. (Hint below.)

22. Prove that, for any finite-dimensional complex inner-product space V, formula (12.8) defines an n-to-one homomorphism $S^1 \times \mathrm{SU}(V) \to \mathrm{U}(V)$ with the kernel $\mathbf{Z}_n \times \{1\}$.

23. Let $F : M \to N$ be a C^1 mapping between manifolds M, N such that M is compact, N is connected, and F has no critical points, i.e., is a submersion (as defined in §9). Prove that F is surjective, that is, $F(M) = N$. (Hint below.)

Hint. In Problem 2, use the fact that $xx^{-1} = 1$ is constant in t, so by (9.12), $0 = (xx^{-1})^{\boldsymbol{\cdot}} = \dot{x}x^{-1} + x(x^{-1})^{\boldsymbol{\cdot}}$, as required.

Hint. In Problem 4, use Lemma 12.5 for the action (12.5), so that the final clause of Theorem 10.1 can be applied to find the tangent space of $\mathrm{Aut}(V, B)$ at the

identity 1. This leads to differentiating $B(Av, Aw)$ with respect to a parameter t such that $A \in \mathrm{GL}(V)$ is a C^1 function of t, arriving at 1 for some fixed value of t. Now it is clear that $B(Fv, w) = -B(v, Fw)$.
Hint. In Problem 7, choose an eigenvector e_1 of F and proceed by induction on $\dim V$, noting that F descends to an operator $V/W \to V/W$, there $W = \mathbf{C}e_1$.
Hint. In Problem 8, given $A \in \mathrm{GL}(V)$ find operators $F, \varPhi : V \to V$ such that $A = F + \varPhi$ and, for some basis e_a, $Fe_1 = e^{z(1)}e_1, \ldots, Fe_n = e^{z(n)}e_n$ $(n = \dim V)$, while the matrix representing $\varPhi$ is upper triangular with zeros on the diagonal. (See Problem 7.) The curve $[0,1] \ni t \mapsto F_t + t\varPhi$ with $F_t e_a = e^{tz(a)}e_a$ (no summation!) now connects 1 to A in $\mathrm{GL}(V)$.
Hint. In Problem 10, use Problem 9, as in the hint for Problem 8.
Hint. In Problem 15, note that the assignment

$$(e_1, \ldots, e_n) \mapsto (e_1,\, ie_1 \ldots, e_n,\, i\, e_n) \tag{12.13}$$

is continuous, and use Problem 10 in §11.
Hint. In Problem 17, $W \cap W^{\perp} = \{0\}$ due to nondegeneracy of W, and so its suffices to show that $\dim W^{\perp} = n - k$, where $n = \dim V$ and $k = \dim W$. This is achieved by fixing a basis w_j of W and noting that the operator $V \to \mathbf{R}^k$ sending v to $(B(v, e_1), \ldots, B(v, e_k))$ is surjective (since its restriction to W is injective, and hence constitutes an isomorphism $W \to \mathbf{R}^k$).
Hint. In Problem 18, apply det to the matrix relation $\tilde{\mathfrak{B}} = \mathfrak{A}\mathfrak{B}\mathfrak{A}'$, where $\mathfrak{B}$ and $\tilde{\mathfrak{B}}$ represent B in bases e_a and Ae_a (so that $\tilde{\mathfrak{B}} = \mathfrak{B}$), while $\mathfrak{A}$, $\mathfrak{A}'$ are the matrix of A and its (Hermitian) transpose. Note that $\det \mathfrak{B} \neq 0$ (Problem 16).
Hint. In Problem 20, treat the left multiplication by any quaternion as a $\mathbf{C}$-linear operator in $\mathbf{H}$ viewed as a complex 2-space in which the multiplication by scalars (real combinations of 1, $\mathbf{i} \in \mathbf{H}$) is the quaternion multiplcation *from the right*.
Hint. In Problem 21, use Problems 19 and 20 (for Example 12.10), or combine the two-to-one homomorphism (12.9) with Problem 16 in §3 (for Examplesorpt).
Hint. In Problem 23, note that $F(M)$ is both open (Problem 25 in §9) and closed (as it is compact; see Problem 4(a) in §3). Then use Problem 11 in §3.

13. Orbits of Lie-group actions

Topics: Orbits of group actions on sets and of Lie-group actions on manifolds; vector fields tangent to submanifolds.

Given an action $(a, x) \mapsto ax$ of a group G on a set M, let us define a relation $\sim$ in M by declaring that $x \sim y$ if and only if $y = ax$ for some $a \in G$. One easily verifies that $\sim$ is an equivalence relation. The equivalence classes of $\sim$ are called the *orbits* of the action in question. For instance, the action is transitive if and only if it has only one orbit.

Problems

1. Let a group G act on a set M. Prove that the isotropy groups of two points $x, y \in M$ lying in the same orbit are *conjugate* in G, that is, one is the other's image under an inner automorphism of G.
2. Find the orbits of the *evaluation action* $(F, v) \mapsto Fv$ of the group $\mathrm{GL}(V)$ on V, where V is any finite-dimensional real or complex vector space. The same for the pullback action $(B, F) \mapsto F{*}B$ of $\mathrm{GL}(V)$ on the space of all symmetric bilinear forms on V, where V is this time a finite-dimensional *real* vector space.

3. Given a C^∞ right action $(x, a) \mapsto xa$ of a Lie group G on a manifold M and a vector $v \in T_1G$, let us define a vector field v_* on M by $v_*(x) = dF_1 v$, where F stands for the mapping $L_x : G \to M$ defined by (12.1). (In other words, $v_*(x) = xv$, with the same multiplicative notation as in the proof of Lemma 12.5.) Show that v_* then is of class C^∞ and that, for every fixed point $x \in M$, we have $(dF)w = v_*$ on $F(G)$ (notation of (6.9), with $F = L_x$ as above), where w is the unique left-invariant vector field on G such that $w(1) = v$. (Hint below.)

4. Under the assumptions of Problem 3, prove that $[u_*, v_*] = [u, v]_*$ for any $u, v \in T_1G$, the bracket in T_1G being the Lie bracket in the Lie algebra $\mathfrak{g}$ of G, transferred onto T_1G via the evaluation isomorphism $\mathfrak{g} \to T_1G$. In this way, identifying $\mathfrak{g}$ with T_1G, we may regard the assignment $v \mapsto v_*$ as a Lie-algebra homomorphism from $\mathfrak{h}$ into the Lie algebra of all C^∞ vector fields on M. (Hint below.)

5. Verify that every constant-rank injective C^r mapping between manifolds, $r \ge 1$, is a C^r embedding. (Hint below.)

6. Let us define a *free orbit* of a right action $(x, a) \mapsto xa$ of a group G on a set M to be any orbit N of the action such that, for some $x \in N$, the isotropy group H_x (cf. (12.2)) is trivial. Verify that, for *every* $x \in N$, we then have $H_x = \{1\}$ and the mapping $L_x : G \to N$ defined by (12.1) is bijective. Show that any free orbit N of a right C^r action of a Lie group G on a manifold M, $r \ge 1$, carries a unique structure of a C^r submanifold of M such that $L_x : G \to N$ is a C^r-diffeomorphism for every $x \in N$ and, if G is compact, the submanifold N of M is endowed with the subset topology. (Hint below.)

7. Given a simply transitive C^r right action $(x, a) \mapsto xa$ of a Lie group G on a manifold M, $r \ge 1$, verify that, for every $x \in M$, the mapping $L_x : G \to M$ defined by (12.1) is a bijective C^r embedding. Does it always have to be a C^r-diffeomorphism? (Hint below.)

8. Given a manifold N, a submanifold M of N and a vector field v on N, we say that v is *tangent to* M (along M) if $v(x) \in T_xM \subset T_xN$ for each $x \in M$. Restricting v to M, we then obtain a vector field v_M on the manifold M. Verify that each vector field on any manifold M is tangent along U to every open submanifold U of M.

9. Given a C^r right action of a Lie group G on a manifold M, $r \ge 1$, and a vector $v \in T_1G$, verify that the vector field v_* on M (see Problem 3) is tangent to every free orbit of the action (cf. Problem 6).

10. Let a vector field v on a manifold N be of class C^l and tangent (along M) to a submanifold M of N. Prove that the restriction v_M then is of class C^l as a vector field on the manifold M. (Hint below.)

11. Let M be a submanifold of a manifold N. Prove that for any two vector fields u, v on N which are tangent to M along M, the bracket $[u, v]$ is also tangent to M along M, and $[u, v]_M = [u_M, v_M]$. (Hint below.)

12. Let B be a nonzero symmetric bilinear form in a finite-dimensional real or complex vector space V, and let a be a scalar (real or, respectively, complex). Define M_a to be the set $\{x \in V : B(x, x) = a\,,\ B(x, \cdot) \ne 0\}$, $B(x, \cdot) : V \to V^*$ being the linear operator $y \mapsto B(x, y)$. (Thus, the condition $B(x, \cdot) \ne 0$ may be omitted if $a \ne 0$.) Verify that M_a is empty if and only if V is real and aB is negative definite. Show that, if M_a is nonempty, then it is a submanifold of $V \setminus \{0\}$, carrying the subset topology, and that its tangent space $T_xM_a \subset V =$

$T_x(V\setminus\{0\})$ at any $x\in M_a$ coincides with the B-orthogonal complement $x^\perp$ of x in V, that is, $T_xM_a=\{y\in V: B(x,y)=0\}$. (Hint below.)

13. Given V, B as in Problem 12, we say that a linear operator $F:V\to V$ is *skew-adjoint* relative to B if $B(Fx,y)+B(x,Fy)=0$ for all $x,y\in V$. Verify that this is the case if and only if $B(Fx,x)=0$ for all $x\in V$. Show that the linear vector field $w=F$ on V (Problem 11 in §6), restricted to the open submanifold $V\setminus\{0\}$, then is tangent along M_a to the submanifold M_a defined as in Problem 12.

Hint. In Problem 3 use the "associative law" $(xa)v=x(av)$, $a\in G$.
Hint. In Problem 4, use Problem 3 and Theorem 6.1.
Hint. In Problem 5, note that injectivity of (9.3) implies $m=r$.
Hint. In Problem 6, use Lemma 12.5, Problem 5 above, and Problem 4(b) in §3.
Hint. In Problem 7, last question: no. Consider G (treated as discrete) acting by translations on $M=G$ with $\dim G>0$.
Hint. In Problem 10, use the rank theorem (§9) to observe that in suitable local coordinates for M and N the component functions of $v_{_M}$ are just the restrictions to M of some component functions of v.
Hint. In Problem 11, note that $(dF)v_{_M}=v$ on $M=F(M)$ (notation as in (6.9)), $F:M\to N$ being the inclusion mapping, and use Theorem 6.1.
Hint. In Problem 12, apply Theorem 9.6 to $F:V\setminus\{0\}\to\mathbf{R}$ given by $F(x)=B(x,x)$.

14. Whitney's embedding theorem

Topics: The countability axiom; the Borel-Heine theorem; Whitney's embedding theorem for compact manifolds; specific embeddings of projective spaces.

From now on we will assume, whenever needed, that the manifolds in question satisfy the following condition (which happens to be equivalent to metrizability, as well as paracompactness, and also to separability of every connected component).

> **The countability axiom:** *Every manifold admits a countable atlas, that is, forms the union of a countable family of coordinate domains.*

This implies that M has at most countably many connected components. (See Problem 6.) Thus, the countability axiom fails for a disjoint union of an uncountable family of manifolds of any given dimension, and so, from now on, we will restrict the disjoint-union construction to countable families. However, there also exist numerous examples of *connected* manifolds, in the sense used so far, that do not satisfy the countability axiom. They all are, however, quite "pathological" in many respects, and do not come up naturally in geometry or physics. The only reason why the countability axiom was not introduced sooner, or made a part of our definition of a manifold, is that it was not needed before.

To state and prove some important results that *do* require the above assumption, let us first define an *open covering* of a subset K of a manifold M to be a family $\mathcal{U}$ of open sets in M whose union contains K. A subfamily of $\mathcal{U}$ which itself a covering of K then is called a *subcovering*. A family $\mathcal{U}$ of open subsets of M is called a *basis of open sets* if every open subset of M is the union of a subfamily of $\mathcal{U}$. Finally, we say that a manifold M is *separable* if it admits a countable dense subset.

Lemma 14.1. *As a consequence of the countability axiom, every manifold M is separable, and admits a countable basis $\mathcal{U}$ of open sets.*

Proof. Fix a countable atlas on M and let $\mathcal{U}$ be the family of all preimages under the coordinate mappings of open balls in $\mathbf{R}^n$, $n = \dim M$, that are contained in images of the coordinate domains and have rational radii and centers with rational coordinates. The set of all centers of these balls is dense and countable. $\square$

A family $\mathcal{U}$ of sets is said to be *subordinate* to a family $\mathcal{U}'$ if every $U \in \mathcal{U}$ is a subset of some $U' \in \mathcal{U}'$.

The next result is known as the *Borel-Heine theorem*.

Theorem 14.2. *Given a manifold M satisfying the countability axiom, and a set $K \subset M$, the following two conditions are equivalent:*

a. *K is compact.*
b. *Every open covering of K has a finite subcovering.*

Proof. Let $K \subset M$ be compact. By Lemma 14.1, M admits a countable basis $\mathcal{U}$ of open sets. For any given open covering $\mathcal{U}'$ of K, we may thus choose a (countable) subfamily $\{U_s\}_{s=1}^{\infty}$ of $\mathcal{U}$ which itself a covering of K and is subordinate to $\mathcal{U}'$. Now one of the sets $Y_l = \bigcup_{s=1}^{l} U_s$ contains K, which shows that (a) implies (b). In fact, if none of the Y_l contained K (while $K \subset \bigcup_{l=1}^{\infty} Y_l$), then choosing $x_l \in K \setminus Y_l$ we would get a sequence in K with no subsequence that converges in K (since a limit of such a subsequence would lie in some Y_s along with x_l for infinitely many $l \geq s$).

Conversely, assume that K is not compact, and fix a sequence x_l of points in K that has no subsequence converging in M to a point of K. The set K' formed by all the x_l is then infinite, and each $x \in K$ has a neighborhood U_x in M such that $(U_x \setminus \{x\}) \cap K'$ is empty. The sets U_x, for all $x \in K$, clearly form an open covering of K without a finite subcovering: since each of them contains at most one point ok K', any finite family of them contains only a finite subset of the infinite set K'. This completes the proof. $\square$

Remark 14.3. Note that (b) implies (a) in the Borel-Heine theorem even for manifolds that do not satisfy the countability axiom.

Proposition 14.4. *Every compact C^∞ manifold M satisfying the countability axiom admits an immersion $h : M \to \mathbf{R}^k$ for some integer $k \geq 1$.*

Proof. Set $m = \dim M$. The family of all open sets $U \subset M$ for which there exists a diffeomorphism $U \to U'$ onto an open set $U' \subset \mathbf{R}^m$ that has a C^∞ extension $M \to \mathbf{R}^m$ forms an open covering of M (Problem 24 in §9). By the Borel-Heine theorem, we can cover M with *finitely many* of these sets, say, $U_1, \ldots, U_s$, with the respective immersions $U_\lambda \to \mathbf{R}^m$ having C^∞ extensions F_λ to M. Then $F = (F_1, \ldots, F_s) : M \to \mathbf{R}^{ms}$ (Problem 23 in §9). $\square$

Now we can prove *Whitney's embedding theorem for compact manifolds*:

Theorem 14.5. *For every compact C^∞ manifold M that satisfies the countability axiom, there exists an embedding $F : M \to \mathbf{R}^l$ into a Euclidean space of some dimension l.*

Proof. By Proposition 14.4, there exists an immersion $h = (h^1, \dots, h^k)$ of M in $\mathbf{R}^k$. Since h is locally injective (§9), there exists an open set U in $M \times M$ containing the diagonal subset $\{(x, x) : x \in M\}$ and such that $h(x) \neq h(y)$ whenever $(x, y) \in U$. Set $K = (M \times M) \setminus U$. For each $(x, y) \in K$ select a C^∞ function $f_{x,y} : M \to \mathbf{R}$ with $f_{x,y}(x) = 1$, $f_{x,y}(y) = 0$ (Problem 19 in §6). Since K is compact, it is covered by a finite number s of the open sets $f_{x,y}^{-1}((1/2, \infty)) \times f_{x,y}^{-1}((-\infty, 1/2))$. Letting $f^1, \dots, f^s$ be the corresponding functions $f_{x,y}$, we see that for $l = k + s$, the mapping $F = (h^1, \dots, h^k, f^1, \dots, f^s) : M \to \mathbf{R}^l$ is an embedding, which completes the proof. □

Problems

1. Show that a compact subset of a manifold M can intersect only finitely many connected components of M.

2. Prove that a submanifold K of a manifold M that is compact with respect to its own manifold structure must have the subset topology (and be a compact subset of M). Does the subset-topology property follow just from the assumption that K is a compact subset of M? (Hint below.)

3. Let V be a finite-dimensional real or complex inner-product space, and let W be the real vector space

$$W = \{A \in \mathfrak{gl}(V) : A^* = A \text{ and } \operatorname{Trace} A = 0\} \tag{14.1}$$

of all traceless self-adjoint operators in V. Prove that the projective space $P(V)$ admits a C^ω embedding $F : P(V) \to W$, which sends every $L \in P(V)$ to the operator A having the eigenspaces L and $L^\perp \subset V$ with the corresponding eigenvalues $n - 1$ and -1, where $n = \dim V$. (Hint below.)

4. For a finite-dimensional real/complex inner-product space V, verify that the formula

$$\langle A, A' \rangle = \operatorname{Trace} AA' \tag{14.2}$$

(involving the real/complex trace) defines an inner product in the real space W given by (14.1), and that the image $F(P(V))$ of the embedding described in Problem 3 is contained in the sphere $\Sigma \subset V$ of radius $\sqrt{n(n-1)}$, centered at 0. Show that $F(P(V)) = \Sigma$ whenever $\dim V = 2$, and that F then is a C^ω diffeomorphism between $P(V)$ and the sphere Σ of dimension 1 (V real) or 2 (V complex); cf. also Problems 17 – 19 in §3.

5. For V, W, F and Σ as in Problem 4, assume that $\dim V \geq 3$. Prove that the image $F(P(V))$ then is a proper subset of the sphere Σ. Show that, for any integer $n \geq 2$, the real projective space $\mathbf{R}\mathrm{P}^n$ can be embedded in $\mathbf{R}^m$, with $m = (n^2 + 3n - 2)/2$, and the complex projective space $\mathbf{C}\mathrm{P}^n$ can be embedded in $\mathbf{R}^q$, with $q = n^2 + 2n - 1$. (Hint below.)

6. Show that, as a consequence of the countability axiom, every manifold M has at most countably many connected components. (Hint below.)

Hint. In Problem 2, use Problem 4(b) in §3; the answer to the last question is 'no' (Problem 4 in §9).

Hint. In Problem 3, note that, for any nonzero vector $v \in V$, we have

$$F(\mathbf{K}v) = n \frac{\langle \,\cdot\, , v \rangle}{\langle v, v \rangle} v - 1, \tag{14.3}$$

where the dot $\cdot$ is the "blank space" (for the argument), and 1 stands for the identity operator $V \to V$. Thus, F is of class C^ω, since it appears so in the standard coordinates for $P(V)$ (see §2). The fact that F is an immersion follows since it has local "one-sided inverses" of class C^ω, as $A = F(\mathbf{K}v)$ determines $v/f(v)$ (with f as in §2) via $v/f(v) = (A+1)u/[f((A+1)u)]$ for any vector $u \in V$ with $(A+1)u \neq 0$. Similarly, $F(L)$ determines L (its only eigenspace for a positive eigenvalue), so that F is an embedding.

Hint. In Problem 5, use Problem 4 and the fact that the sphere S^k minus a point is diffeomorphic to $\mathbf{R}^k$ (e.g., via a stereographic projection, defined as in §2).

Hint. In Problem 5, note that the connected components form a family of disjoint open sets, and such a set must be countable in view of the separability clause in Lemma 14.1.

CHAPTER 4

Vector Bundles

15. Real and complex vector bundles

Topics: Vector bundles over sets; line bundles; sections over subsets; trivializations; transition functions; atlases; compatibility; vector bundles of given regularity over manifolds; local/global regular sections and trivializations; product bundles; tangent bundles; vector fields; tautological line bundles over projective spaces; tautological vector bundles over the Grassmann manifolds; affine bundles.

A real or complex *vector bundle* η over a set B is a family

$$B \ni x \mapsto \eta_x\,, \tag{15.1}$$

parametrized by B, of real or complex vector spaces η_x of some finite dimension q (independent of x). One then calls q the *fibre dimension* (or *rank*) of η, while B is referred to as the *base* of η. For $x \in B$, the space η_x is known as the *fibre* of η over $x \in B$. If $q = 1$, η is called a *line bundle*.

By a *section* of a vector bundle η as above over a set $K \subset B$ we mean any mapping ψ that assigns to each $x \in K$ an element of the fibre η_x. The set K then is referred to as the *domain* of ψ. When $K = B$, the section ψ is called *global*.) Sections ψ of η with a fixed domain K can be added valuewise, and multiplied by functions f on K valued in the scalar field according to the rule $(f\psi)(x) = f(x)\psi(x) \in \eta_x$, and so they form a module over the algebra of functions on K.

Let η be a vector bundle over a set B, of some fibre dimension q. A *trivialization of η over a set* $K \subset B$ is a q-tuple $e_1, \dots, e_q$ of sections of η over K whose values $e_1(x), \dots, e_q(x)$ form, at each $x \in K$, a basis of the fibre η_x. Instead of $e_1, \dots, e_q$, we then simply write e_a, where a varies in the fixed range $\{1, \dots, q\}$. For $x \in K$ and $\phi \in \eta_x$, we define the *components* ϕ^a of ϕ relative to the trivialization e_a over K to be the scalar coefficients of the expansion $\phi = \phi^a e_a(x)$, while the corresponding scalar-valued *component functions* $\psi^a : K \to \mathbf{R}$ or $\psi^a : K \to \mathbf{C}$ of any section ψ of η over K then are given by $\psi^a(x) = [\psi(x)]^a$. Thus

$$\psi = \psi^a e_a\,. \tag{15.2}$$

Another such trivialization (over a set $K' \subset B$) will be denoted by $e_{a'}$, so that different trivializations will be distinguished by using mutually disjoint ranges of indices (just like in the case of coordinate systems on a manifold; see §5), usually without specifying the domain K or K'. The scalar-valued *transition functions* $e_{a'}^a$ on $K \cap K'$ then are defined to be just the component functions of the $e_{a'}$ relative to the e_a, i.e., are characterized by

$$e_{a'} \;=\; e_{a'}^a e_a\,, \tag{15.3}$$

and satisfy

$$\phi^{a'} = e_a^{a'}\phi^a \tag{15.4}$$

for any $x \in K \cap K'$ and $\phi \in \eta_x$, as well as

$$e_a^{a''} = e_a^{a'} e_{a'}^{a''} \tag{15.5}$$

in the intersection of all three trivialization domains, whenever three coordinate systems are involved. In particular, $e_a^b = \delta_a^b$.

Again, a trivialization e_a of η is called *global* if its domain K coincides withe the base set B.

Consider now a vector bundle η whose base set, denoted by M, carries a fixed structure of a C^r *manifold*, $r \geq 1$. By a *local section* (*trivialization*) of η we then mean a section ψ (or, trivialization e_a) whose domain is an *open* set $U \subset M$. Two local trivializations e_a, $e_{a'}$ with domains U, U' are called C^s*-compatible* $(0 \leq s \leq r)$ if the scalar-valued transition functions $e_{a'}^a$ on $U \cap U'$ are of class C^s. The C^s-compatibility relation is symmetric (Problem 1). A C^s *atlas* $\mathcal{B}$ for η is a collection of local trivializations which are pairwise C^s compatible and whose domains cover M. Such an atlas is said to be *maximal* if it is not contained in any other C^s atlas. Every C^s atlas $\mathcal{B}$ for η is contained in a unique maximal C^s atlas $\mathcal{B}_{\max}$, formed by all local trivializations of η that are C^s compatible with each of the local trivializations constituting $\mathcal{B}$. We define a C^s *vector bundle* over a C^r manifold M $(0 \leq s \leq r)$ to be any vector bundle η over M endowed with a fixed maximal C^s atlas $\mathcal{B}_{\max}$. Note that, to describe a C^s vector bundle η over M, it suffices to provide just one C^s atlas $\mathcal{B}$ contained in its maximal C^s atlas $\mathcal{B}_{\max}$.

A local section ψ of a C^s vector bundle η is said to be *of class* C^l, $0 \leq l \leq s$, if its components ψ^a relative to all local trivializations e_a forming the maximal atlas $\mathcal{B}_{\max}$ of η, are C^l functions. This is a local geometric property (§2): to verify that ψ is C^l, we only need to use, instead of $\mathcal{B}_{\max}$, just any C^s atlas $\mathcal{B}$ contained in $\mathcal{B}_{\max}$. (Problem 2.) Obviously, the set of local C^l sections of η with a fixed domain $U \subset M$ is closed under addition and multiplication by scalar-valued C^l functions on U, thus forming a module over the algebra of such functions. Instead of C^0-regularity for local sections of a vector bundle one often uses the term *continuity*.

Among all local trivializations e_a of a given C^s vector bundle η, those forming the maximal atlas $\mathcal{B}_{\max}$ of η are characterized by the condition that their constituent local sections e_a are all of class C^s (Problem 3). From now on, such local trivializations will simply be called C^s local trivializations of η.

EXAMPLE 15.1. A fixed finite-dimensional real or complex vector space $\mathbf{F}$ gives rise to the *product bundle* over any base set B, denoted by $\eta = B \times \mathbf{F}$, which is the "constant" family

$$B \ni x \mapsto \eta_x = \mathbf{F}\,. \tag{15.6}$$

Its sections over any $K \subset B$ are just mappings $K \to \mathbf{F}$, which gives rise to the distinguished classes of *constant* sections (and trivializations). If $B = M$ is a C^r manifold, $\eta = B \times \mathbf{F}$ thus becomes a C^r vector bundle with the (non-maximal) atlas formed by all constant global trivializations.

Example 15.2. The *tangent bundle* of any C^r manifold M, $r \ge 1$, is the vector bundle $\eta = TM$ over M given by

$$M \ni x \mapsto \eta_x = T_xM\,. \tag{15.7}$$

The local sections of TM are called (*local*) *vector fields* on M (see §6). Each local coordinate system x^j in M defines a local trivialization p_j of TM over the coordinate domain U, with $p_j(x) \in T_xM$ for $x \in U$ defined by (5.20). Since $p_{j'} = p_{j'}^j p_j$ with $p_j^{j'} = \partial_j x^{j'}$ (see (5.4), such local trivializations make TM into a C^{r-1} vector bundle over M.

Example 15.3. For any finite-dimensional real or complex vector space V, the *tautological line bundle* λ over the projective space $P(V)$ (the set of all 1-dimensional vector subspaces of V; see §2), is defined by

$$P(V) \ni L \mapsto \lambda_L = L\,. \tag{15.8}$$

Every nonzero scalar-valued linear function $f \in V^*\setminus\{0\}$ gives rise to the local trivialization for λ formed by the nowhere-zero local section e_f with the domain $U_f = \{L \in P(V) : f(L) \ne \{0\}\}$ (§2), defined by $e_f(L) = v/f(v) \in L$ if $L = \mathrm{Span}(v)$, $v \in V\setminus\{0\}$. (This happens to be the same formula as for the coordinate mapping $\varphi_f : U_f \to f^{-1}(1)$ in §2.) If $f, f' \in V^*\setminus\{0\}$, we have $e_{f'} = e_{f'}^f e_f$ (not really summed over f, since either of f, f' forms a *one-element* index set), with $e_{f'}^f(L) = f(v)/f'(v)$ if $L \in U_f \cap U_f'$ is spanned by $v \in V$. Thus, each transition function $e_{f'}^f$ is real-analytic, as its coordinate representation $e_{f'}^f \circ \varphi_f$ has the form $f^{-1}(1) \ni v \mapsto f(v)/f'(v)$, which is a *rational* function on $V\setminus \operatorname{Ker} f'$ restricted to the open subset $f^{-1}(1)\setminus \operatorname{Ker} f'$ of the affine subspace $f^{-1}(1)$. The resulting atlas, parametrized by $f \in V^*\setminus\{0\}$, turns λ into a C^ω vector bundle over $P(V)$.

For a more general construction, see Problem 8.

Problems

1. Show that the C^s compatibility relation between local trivializations of a vector bundle over a manifold is symmetric.

2. Verify that C^l regularity for local sections of a C^s vector bundle, $0 \le l \le s$, is a local geometric property.

3. Show that a local trivialization e_a of a C^s vector bundle η over a manifold M belongs to the maximal atlas $\mathcal{B}_{\max}$ of η if and only if the local sections e_a are all C^s-differentiable.

4. Show that C^l-differentiable local sections of a product vector bundle $\eta = B \times \mathbf{F}$ over a manifold M are just C^l-differentiable mappings of open subsets of M into $\mathbf{F}$.

5. Suppose that λ is the tautological line bundle over the projective space $P(V)$ of a given finite-dimensional real or complex vector space V. Let us assign, to each global section ψ of λ, the scalar-valued function h on $V\setminus\{0\}$ characterized by $\psi(L) = h(v)v$ whenever $L \in P(V)$ is spanned by $v \in V\setminus\{0\}$. Verify that this establishes a bijective correspondence between the set of all global sections ψ of λ and the collection of all functions h on $V\setminus\{0\}$, valued in the scalar field, which are *homogeneous of degree* -1 in the sense that $h(av) = h(v)/a$ for any $v \in V\setminus\{0\}$ and any nonzero scalar a.

6. Show that a global section ψ of the tautological line bundle λ over the projective space $P(V)$ of any finite-dimensional real or complex vector space V is of class C^l, $0 \le l \le \infty$, if and only if so is the function h corresponding to ψ as in Problem 5.

7. Let V be a real vector space with $2 \le \dim V < \infty$. Prove that every continuous global section ψ of the tautological line bundle λ over $P(V)$ has to vanish somewhere, i.e., the value of ψ at some point is the zero vector of the corresponding fibre. (Hint below.)

8. Let V be a real or complex vector space of real/complex dimension $n < \infty$, and let q be an integer with $0 \le q \le n$. Generalize the construction in Example 15.3 above by introducing the *tautological vector bundle* κ over the Grassmann manifold $\mathrm{Gr}_q(V)$ (Problem 2 in §81), with

$$\mathrm{Gr}_q(V) \ni W \mapsto \kappa_W = W . \tag{15.9}$$

Verify that the coordinate mappings for $\mathrm{Gr}_q(V)$ introduced in Problem 2 in §81, when regarded as local trivializations of κ, form a C^ω atlas, parametrized by the set of all surjective linear mappings $f : V \to \mathbf{K}^q$ (where $\mathbf{K}$ is the scalar field). This turns κ into a C^ω real/complex vector bundle of fibre dimension q over $P(V)$.

Hint. In Problem 7, the nowhere-zero scalar-valued function h on the connected set $V \setminus \{0\}$, corresponding to ψ as in Problems 5 and 6, assumes values of both signs due to homogeneity.

16. Vector fields on the 2-sphere

Topics: Retractions; linear subvarieties of projective spaces; projective lines in projective spaces; nonexistence of C^2 retractions of real projective spaces onto projective lines in them; nonexistence of C^2 tangent vector fields without zeros on the 2-sphere; nonexistence of global C^2 sections without zeros in the tautological line bundles over complex projective spaces.

We will use the symbols $D, S^1 \subset \mathbf{C}$ for the unit disk centered at 0 in $\mathbf{C}$ and its boundary circle:

$$D = \{z \in \mathbf{C} : |z| \le 1\}, \qquad S^1 = \{z \in \mathbf{C} : |z| = 1\}. \tag{16.1}$$

Given intervals I, I' of any kind, we will say that a function $I \times I' \to \mathbf{C}$ is *of class* C^s if it has continuous partial derivatives up to order s which, in the case where one or both of I, I' have endpoints, includes one-sided derivatives. Similarly, we will refer to a function $f : D \to \mathbf{C}$ as being *of class* C^s *in polar coordinates* if $(r, \theta) \mapsto f(re^{i\theta}) \in \mathbf{C}$ is of class C^s on $I \times I' = [0,1] \times \mathbf{R}$.

Lemma 16.1. *Suppose that we are given an interval* $I \subset \mathbf{R}$ *and a* C^2 *mapping* $I \times \mathbf{R} \ni (r, \theta) \mapsto z(r, \theta) \in \mathbf{C} \setminus \{0\}$ *such that* $z(r, \theta + a) = z(r, \theta)$ *for some real* $a > 0$ *and all* $r \in I$ *and* $\theta \in \mathbf{R}$. *Then the integral*

$$\int_0^a \frac{1}{z} \frac{\partial z}{\partial \theta} \, d\theta \tag{16.2}$$

is constant as a function of $r \in I$.

Proof. Using a subscript notation for partial derivatives, we have $(z_\theta/z)_r = (z_r/z)_\theta$, since both are equal to $z_{r\theta}/z - z_r z_\theta/z^2$. Thus, $d[\int_0^a (z_\theta/z)\, d\theta]/dr = \int_0^a [z_\theta/z]_r \, d\theta = \int_0^a [z_r/z]_\theta \, d\theta$, which equals the increment of z_r/z between $\theta = 0$ and $\theta = a$, and hence vanishes due to periodicity. This completes the proof. $\square$

LEMMA 16.2. *Given an integer* $k \neq 0$, *there exists no continuous mapping* $F : D \to \mathbf{C} \setminus \{0\}$ *of the unit disk* $D \subset \mathbf{C}$, *as in* (16.1), *whose restriction to the boundary circle is given by* $F(z) = z^k$.

PROOF. First, if such a mapping existed, there would also exist a C^∞ mapping with the same property. In fact, applying Corollary 75.2 (in Appendix B) to $M = V = \mathbf{C}$ and $K = D$, we can select a C^∞ function $h : \mathbf{C} \to \mathbf{C}$ with $|h - F| < \varepsilon$, for $\varepsilon > 0$ chosen so that $|F| \geq 3\varepsilon$ on D. Thus, $|h| > 2\varepsilon$ on D by the triangle identity. Next, there must exist $a \in (0,1)$ such that $|z^k - h(z)| \leq \varepsilon$ for all $z \in \mathbf{C}$ with $a \leq |z| \leq 1$. (Otherwise, a sequence of complex number z with $|z|$ approaching 1 from below would have $|z^k - h(z)| \geq \varepsilon$, and choosing a convergent subsequence we would get a contradiction with the fact that $\varepsilon > |h - F| = |z^k - h|$ at all z with $|z| = 1$.) Now let us set $F'(z) = h(z) + [z^k - h(z)][1 - \varphi(|z|)]$, with φ chosen as in Problem 18 of §6 for any fixed $b \in (a,1)$. That $F'(z) \neq 0$ at evere $z \in D$ is clear: $F'(z) = h(z) \neq 0$ if $|z| \leq a$, while $|F' - h| \leq |z^k - h| \leq \varepsilon$ and $|h| > 2\varepsilon$ at all $z \in D$ with $|z| \geq a$.

However, a C^∞ mapping F with the stated property does not exist: if it did, Lemma 16.1 applied to $z(r,\theta) = F(re^{i\theta})$, for $r \in I = [0,1]$, would imply that the corresponding integral (16.2) is the same at $r = 0$ as at $r = 1$, whereas the two integrals are easily seen to have the values 0 and, respectively, $2k\pi i$. This completes the proof. $\square$

By a *retraction* of a set X onto a subset $Y \subset X$ we mean any mapping $X \to Y$ whose restriction to Y equals the identity mapping of Y. Retractions $X \to Y$ are usually of interest only in those cases where they satisfy some regularity conditions such as continuity or differentiability (or, in the case of vector spaces X, Y, also linearity).

EXAMPLE 16.3. The Cartesian product $X = M \times N$ of C^r manifolds M, N admits C^r retractions, given by $(x,y) \mapsto (x,y_0)$ or $(x,y) \mapsto (x_0,y)$ onto all "factor submanifolds" Y of the form $M \times \{y_0\}$ or $\{x_0\} \times N$, with any fixed $y_0 \in N$ or $x_0 \in M$. Note that all these Y are C^r submanifolds, endowed with the subset topology, of $X = M \times N$ (Problem 10 in §9).

EXAMPLE 16.4. When Y is a vector subspace of a vector space X, a retraction $X \to Y$ which is also a linear operator is usually called a *projection operator* of X to Y, and amounts to the ordinary Cartesian-product projection $Y \times Y' \to Y$ under some isomorphic identification of X with a direct-product space $Y \times Y'$ which makes each $y \in Y$ correspond to $(y,0)$. In other words, if $\dim X < \infty$, this may be viewed as a special case of Example 16.3.

The fact that a retraction $X \to Y$ of a specific regularity class *does not exist* means that Y is positioned in X in some "topologically nontrivial" way (in particular, not as a factor submanifold in a Cartesian product). For instance, the case $k = 1$ in Lemma 16.2 can be restated as

COROLLARY 16.5. *There exists no continuous retraction* F *of the unit disk* D *in* $\mathbf{C}$ *onto its boundary circle* S^1.

We know (see Problem 11 in §9) that, for any vector subspace W of a finite-dimensional real or complex vector space V, the projective space $P(W)$ (which is obviously a subset of $P(V)$) is also a C^ω submanifold of $P(V)$ endowed with the subset topology. Any such submanifold of $P(V)$ will be called a *linear submanifold* of $P(V)$ or a *linear variety* in $P(V)$. When $W \subset V$ is of dimension 2 over the respective scalar field, we will refer to $P(W)$ as a *projective line* in $P(V)$. Note that projective lines are a natural analogue of "lines" in classical elementary geometry; see, for example, Problem 1.

LEMMA 16.6. *Let $\pi : W \setminus \{0\} \to P(W)$ be the standard projection mapping defined as in* Problem 13 *of* §3 *for a 2-dimensional real vector space W. Then, for any fixed real-isomorphic identification $W = \mathbf{C}$, there exists a C^ω-diffeomorphic identification between the projective line $P(W)$ and the circle S^1 with* (16.1) *such that both identifications together make π appear as the mapping $\mathbf{C} \setminus \{0\} \to S^1 \subset \mathbf{C}$ given by $z \mapsto z^2/|z|^2$.*

In fact, for any $\mathbf{C} \setminus \{0\}$, the number $z^2/|z|^2$ determines its square root $z/|z|$ up to a sign, and hence it determines z up to a nonzero real factor. A fixed real-isomorphic identification $W = \mathbf{C}$ thus gives rise to a bijection $h : P(W) \to S^1$ with $h(\mathbf{R}z) = z^2/|z|^2$, which is a C^ω-diffeomorphism, as one easily verifies using a projective coordinate system (cf. the hint for Problem 17 in §3).

LEMMA 16.7. *A real projective plane admits no continuous retraction onto a projective line in it.*

PROOF. Assume, on the contrary, that $\Phi : P(V) \to P(W)$ is such a retraction, where W is a subspace of a real vector space V with $\dim V = 3$ and $\dim W = 2$. Let us fix a real-isomorphic identification $W = \mathbf{C}$ and a vector $u \in V \setminus W = V \setminus \mathbf{C}$. Choosing a C^ω-diffeomorphic identification $P(W) = S^1$ as in Lemma 16.6, we now define $F : D \to S^1$ (with D, S^1 as in (16.1)) by $F(z) = \Phi(\mathbf{R}[z + (1 - |z|^2)u])$. (Here $z \in D \subset \mathbf{C} = W$, so that $z + (1 - |z|^2)u \in V$, $\mathbf{R}[z + (1 - |z|^2)u] = \pi(z + (1 - |z|^2)u) \in P(V)$ and $F(z) \in P(W) = S^1$.) Since Φ is continuous, so is F. Finally, if $z \in D$ lies on the boundary circle S^1, that is, $|z| = 1$, we have $F(z) = \Phi(\mathbf{R}z) = \Phi(\pi(z)) = \pi(z) = z^2$, since $\pi(z) \in P(W)$ as $z \in D \subset \mathbf{C} = W$ and Φ restricted to $P(W)$ equals the identity, while $\pi(z) = z^2/|z|^2 = z^2$ by Lemma 16.6. The existence of such a mapping F contradicts the case $k = 2$ in Lemma 16.2, which completes the proof. $\square$

COROLLARY 16.8. *Let M be a real projective space of any dimension $n \geq 2$. Then M does not admit a continuous retraction onto any projective line in M.*

PROOF. If there was such a retraction $\Phi : M \to P(W)$, where $M = P(V)$ for some real vector space of dimension $n+1$ and W is a 2-dimensional subspace of V, then, choosing a subspace V' with $W \subset V' \subset V$ and $\dim V' = 3$, and restricting Φ to the linear subvariety $P(V')$ of $P(V)$, we would obtain a C^2 retraction of the real projective plane $P(V')$ onto the projective line $P(W)$ in it, whose existence would directly contradict Lemma 16.7. $\square$

The *level sets* of any mapping $F : M \to N$ between sets M and N are, by definition, the subsets of M having the form $F^{-1}(y)$ with $y \in N$. Thus, M is the disjoint union of the (nonempty) level sets $F^{-1}(y)$ with $y \in F(M)$.

Let M be the set of all orthonormal bases in a given oriented 3-dimensional Euclidean vector space V which are also *positive-oriented*, that is, compatible with the orientation of V. Clearly, M is an orbit of the Lie group $\mathrm{SO}(V)$ acting on bases of V as a Lie subgroup of $\mathrm{GL}(V)$ (see (12.5) and Example 12.2). Therefore, M is a compact, connected C^ω submanifold, endowed with the subset topology, of the third Cartesian power V^3 of V, and for every fixed $(u,v,w) \in M$ the mapping $\mathrm{SO}(V) \ni A \mapsto (Au, Av, Aw) \in M$ is a C^ω-diffeomorphism. (All of this is immediate from Problem 6 in §13.) In particular, $\dim M = 3$ (cf. (12.7)). We now denote by $F : M \to S(V)$ the mapping given by $F(u,v,w) = u$, where $S(V) = \{x \in V : \langle x, x\rangle = 1\}$ is the (2-dimensional) unit sphere in V.

Lemma 16.9. *Let $V = 1^\perp$ be the Euclidean 3-space of pure quaternions, oriented through the basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and let M and $F : M \to S(V)$ correspond to this V as in the preceding paragraph. Then there exists a C^ω-diffeomorphic identification between M and the 3-dimensional real projective space $P(\mathbf{H})$, where $\mathbf{H}$ is the real 4-space of quaternions, which makes all level sets of F appear as projective lines in $P(\mathbf{H})$.*

Proof. The mapping $\mathbf{H}\setminus\{\mathbf{0}\} \ni p \mapsto (p\mathbf{i}p^{-1}, p\mathbf{j}p^{-1}, p\mathbf{k}p^{-1})$ (with quaternion multiplication and the quaternion inverse $p^{-1} = \overline{p}/|p|^2$) is constant on $L\setminus\{\mathbf{0}\}$ for any real line through $\mathbf{0}$ in $\mathbf{H}$, i.e., its value does not change if p is replaced by λp for any $\lambda \in \mathbf{R}\setminus\{0\}$. Therefore, it descends to a mapping $\varPhi : P(\mathbf{H}) \to M$ with $\varPhi)(\mathbf{R}p) = (p\mathbf{i}p^{-1}, p\mathbf{j}p^{-1}, p\mathbf{k}p^{-1})$. (Note that the image $\varPhi(P(\mathbf{H}))$ coincides with our manifold M, since every $\mathbf{R}p$ is spanned by some $p \in S^3$, while the homomorphism (12.9) maps S^3 onto $\mathrm{SO}(1^\perp)$, and so $\varPhi(P(\mathbf{H}))$ is the $\mathrm{SO}(1^\perp)$-orbit of the basis $(\mathbf{i},\mathbf{j},\mathbf{k})$.) Also, two elements of S^3 have the same image under (12.9) only if they differ at most by sign, while (12.9) is locally C^ω-diffeomorphic (see §12); therefore, $\varPhi : P(\mathbf{H}) \to M$ is a C^ω-diffeomorphism.

The fact that the identification $M = P(\mathbf{H})$ provided by $\varPhi$ has the required properties amounts to saying that, for each $u \in S(V)$, the the real vector subspace $W = \{p \in \mathbf{H} : p\mathbf{i} = up\}$ of $\mathbf{H}$ (that is, the union of all the lines forming the $\varPhi$-preimage of u) is 2-dimensional. However, by (4.10), the left or right multiplication by any fixed pure quaternion is a skew-adjoint operator in the Euclidean 4-space $\mathbf{H}$. (This is clear from Problem 13 in §13, as $\langle px, x\rangle = \mathrm{Re}\, px\overline{x} = 0$ and $\langle x, xp\rangle = \mathrm{Re}\,\overline{x}xp = 0$ whenever $\mathrm{Re}\, p = 0$, by (4.10) and (4.11).) Hence, being the kernel of a skew-adjoint operator, W must be of some dimension $m \in \{0,2,4\}$ (Problem 3; see also formula (16.4)). Now $m \neq 4$, or else $p = 1$ would be in W (so that $u = \mathbf{i}$), but then $p = \mathbf{j}$ cannot be in W (as $p\mathbf{i} = \mathbf{j}\mathbf{i} = -\mathbf{k}$, while $up = \mathbf{i}\mathbf{j} = \mathbf{k}$, cf. (4.3)). Also, $m \neq 0$ since $u + \mathbf{i} \in W$ (by (4.20)) and either $u + \mathbf{i} \neq \mathbf{0}$, or $u = -\mathbf{i}$ and then $p = \mathbf{j}$ is in W. Thus, $m = 2$, which completes the proof. □

By a *zero* of a local section ψ of a vector bundle η over a manifold M we mean any point $x \in M$ lying in the domain of ψ and such that $\psi(x) = 0$. In particular, we can speak of *zeros of vector fields* in a C^1 manifold M, since they are (local) sections of TM.

Theorem 16.10. *Every global continuous vector field on the 2-dimensional sphere S^2 must have at least one zero.*

Proof. Suppose, on the contrary, that u is a global continuous vector field without zeros on the unit sphere $N = S(V)$ in an oriented Euclidean 3-space V. Since $T_xN = x^\perp \subset V$ (Problem 12 in §13), u may be treated as a continuos mapping $u : N \to V\setminus\{0\}$. (In fact, the components of u relative to linear coordinates in V corresponding to an orthonormal basis e_a are $d_u f_a$, with $f_a(x) = \langle x, e_a\rangle$ for $x \in N$.) Replacing u with $u/|u|$ we may assume that $|u| = 1$.

Let M and $F : M \to S(V) = N$ be defined, for our space V, as in the paragraph preceding Lemma 16.9. The existence of u now leads to a homeomorphic identification $M = S^2 \times S^1$ which makes F appear as the standard projection $S^2 \times S^1 \to S^2$. The idea is to set $w(x) = x \times u(x)$ (vector product), and then assign to any pair $(x, z) = (x, a+bi) \in S^2 \times S^1$ (with S^1 as in (16.1)) the orthonormal basis (x, u', w') obtained by rotating the basis $x, u(x), w(x)$ about the axis $\mathbf{R}x$ by the angle θ with $z = e^{i\theta}$. Explicitly, $(x, u', w') = (x, au(x) - bw(x), bu(x) + aw(x))$. This assignment is a homeomorphism, since its inverse is given by $(x, u', w') \mapsto (x, z)$ with $z = \langle u', u(x)\rangle + i\langle w', u(x)\rangle$. Thus, according to Example 16.3, M would admit a continuous retraction onto any given level set of F, which (in view of Lemma 16.9) would give rise to a continuous retraction of the 3-dimensional real projective space $P(\mathbf{H})$ onto a projective line in $P(\mathbf{H})$. The latter, however, cannot exist as a consequence of Corollary 16.8. This contradiction completes the proof. $\square$

A vector bundle η over a set B can obviously be *restricted* to any subset B' of B, yielding a vector bundle η' over B' with the same fibres η_x, $x \in B'$. A section (or trivialization) of η defined on any set $K \subset B$ can similarly be restricted to B, giving rise to a section (or trivialization) of η' defined on $K' = K \cap B'$, and the transition functions (§15) in η' between two such restricted trivializations are nothing else than restrictions to B' of the corresponding transition functions between the original trivializations in η'. Consequently, the restriction of a C^s vector bundle η over a C^s manifold M' to a C^s submanifold N of M is a C^s vector bundle η' over M', with an atlas of local trivializations obtained by restricting to N' such an atlas in η. Also, any local/global C^s section of η, when restricted to M', produces a local/global C^s section of η.

An analogue of Problem 7 in §15 also holds for *complex* projective spaces:

THEOREM 16.11. *Let V be a complex vector space with $2 \le \dim V < \infty$. Every global continuous section ψ of the tautological line bundle λ over $P(V)$ then must have at least one zero.*

PROOF. First, we may assume that $\dim V = 2$. In fact, if the assertion is known to be true for 2-dimensional complex vector spaces, it follows for all complex spaces V with $\dim V \ge 2$. To see this, we pick a 2-dimensional subspace W of V and restrict the given global continuous section ψ of λ over $P(V)$ to the submanifold $P(W)$ of $P(V)$, which produces a global continuous section of the tautological line bundle over $P(W)$ (see the paragraph preceding Theorem 16.11 and Problem 4). Therefore, $\psi(L) = 0$ for some $L \in P(W)$, as required.

Let us now assume that $\dim V = 2$ and, contrary to our assertion, ψ is a global continuous section of λ without zeros. Thus, ψ assigns to every line $L \in P(V)$ a nonzero element $\psi(L) \in L$. Using a fixed nonzero skew-symmetric bilinear form $\Omega : V \times V \to \mathbf{C}$, we can now choose, for each $L \in P(V)$, a vector $v \in V$ with $\Omega(\psi(L), v) = 1$ (Problems 5, 6). The coset $\chi(L) = v + L \in V/L \setminus \{L\}$ then is uniquely determined by L (Problem 7). Since $\psi(L)$ and $\chi(L)$ form bases of the 1-dimensional vector spaces L and V/L, there exists a unique linear isomorphism $w(L) \in \mathrm{Hom}(L, V/L)$ sending $\psi(L)$ to $\chi(L)$. However, the natural isomorphic identification $\mathrm{Hom}(L, V/L) = T_L[P(V)]$ (see formula (10.1)) allows us to treat $w(L)$ as a (nonzero) vector tangent to $P(V)$ at L. In other words, the assignment $L \mapsto w(L)$ is a global vector field w without zeros on $P(V)$.

Our claim is thus reduced to showing that this vector field w must be continuous. In fact, since the complex projective line $P(V)$ is C^ω-diffeomorphic to the

2-sphere S^2 (see Problem 4 in §14, or Problem 18 in §3), the existence of a global vector field w without zeros on $P(V)$ is precluded by Theorem 16.10. Thus, our assumption that λ admits a global continuous section of without zeros leads to a contradiction.

To show that w is a continuous vector field, we will use the coordinate system (U_f, φ_f) (§2) for any fixed $f \in V^* \setminus \{0\}$, and verify that the push-forward $(d\varphi_f)w$ of w (restricted to U_f) under the coordinate diffeomorphism $\varphi_f : U_f \to f^{-1}(1)$ is continuous as a mapping $f^{-1}(1) \to \operatorname{Ker} f$, and hence also as a vector field on the affine space $f^{-1}(1)$ with the translation vector space $\operatorname{Ker} f$ (cf. Example 5.2 and Problems 16 – 17 in §2). Specifically, for any fixed $u \in \operatorname{Ker} f \setminus \{0\}$,

$$[(d\varphi_f)w](v) = [\Omega(v,u)]^{-1}[h(v)]^{-2}u\,, \tag{16.3}$$

where $h : V \setminus \{0\} \to \mathbf{C}$ is the continuous function such that $\psi(L) = h(v)v$ whenever $v \in V \setminus \{0\}$ spans $L \in P(V)$ (see Problems 5 – 6 in §15). In fact, if $v \in f^{-1}(1)$ spans L, the identification (10.2) makes any $F \in \operatorname{Hom}(L, V/L)$ correspond to the image under the differential of φ_f^{-1} (i.e., of π) at v of the unique $\dot v \in \operatorname{Ker} f$ with $Fv = \dot v + \mathbf{C}v \in V/\mathbf{C}v$ (see §10). For $\dot v$ equal to the right-hand side of (16.3), the corresponding F has $F\psi(L) = h(v)\dot v + L$, with $L = \mathbf{C}v$, and so $\Omega(\psi(L), F\psi(L)) = 1$, i.e., F corresponds under (10.2) to $w(L) \in T_L[P(V)]$. This completes the proof. □

Problems

1. Given a finite-dimensional real or complex vector space V, verify that, for any two distinct projective points L, L' in the projective space $P(V)$, there exists a unique projective line N in $P(V)$ with $L, L' \in N$.

2. Let V be a finite-dimensional, real or complex inner-product space. Verify that, for any linear operator $F : V \to V$ which is self-adjoint ($F^* = F$) or skew-adjoint ($F^* = -F$), the subspaces $\operatorname{Ker} F$ and $F(V)$ are mutually orthogonal and $V = \operatorname{Ker} F \oplus F(V)$ (cf. Problem 26 in §9). (Hint below.)

3. Let $F : V \to V$ be a skew-adjoint linear operator in a finite-dimensional real inner-product space V. Prove that the rank of F (that is, the dimension of its image $F(V)$) then must be even. (Hint below.)

4. Given a vector subspace W of a finite-dimensional real or complex vector space V, verify that the restriction of the tautological line bundle over $P(V)$ to the C^ω submanifold $P(W)$ (Problem 11 in §9) coincides, as a C^ω vector bundle over $P(W)$ (Example 15.3) with the tautological line bundle over $P(W)$.

5. Show that, given a 2-dimensional real or complex vector space V, the vector space of all scalar-valued skew-symmetric bilinear forms Ω on V is 1-dimensional, and that, for any given basis u, v of V, the assignment $\Omega \mapsto \Omega(u,v)$ provides an isomorphism between that space and the scalar field.

6. Let $\Omega : V \times V \to \mathbf{K}$ be a nonzero skew-symmetric bilinear form on a 2-dimensional vector space V over a scalar field $\mathbf{K}$ (which is $\mathbf{R}$ or $\mathbf{C}$). Verify that any two vectors $u, v \in V$ then are linearly independent if and only if $\Omega(u,v) \neq 0$.

7. For V and Ω as in Problem 6, show that for every $u \in V \setminus \{0\}$ there exists $v \in V$ with $\Omega(u,v) = 1$, and that any two vectors v with this property differ by a multiple of u.

Hint. In Problem 2, orthogonality follows since $\langle Fv, w\rangle = \pm\langle v, Fw\rangle = 0$ whenever $v \in V$ and $w \in \operatorname{Ker} F$. The direct-sum decomposition now is immediate for dimensional reasons, in view of the *rank-nullity theorem*, which states that

$$\dim \operatorname{Ker} F \,+\, \dim F(V) \,=\, \dim V \tag{16.4}$$

for any linear operator $F : V \to V'$ between arbitrary vector spaces V, V'.
Hint. In Problem 3, the image $W = F(V)$ is an F-invariant subspace of V. If $\dim W$ were odd, W would contain an eigenvector of F, that is, we would have $Fw = \lambda w$ for some nonzero $w \in W$ and $\lambda \in \mathbf{R}$. However, skew-adjointness of F then would give $0 = \langle Fw, w\rangle = \lambda\langle w, w\rangle$, so that $\lambda = 0$ and $w \neq 0$ would simultaneously be in $F(V)$ and $\operatorname{Ker} F$, contradicting the orthogonality property described in Problem 2.

17. Operations on bundles and vector-bundle morphisms

Topics: Operations on vector bundles; the dual; total spaces of vector bundles; the conjugate vector bundle; direct sum; pullback; (base-fixing) morphisms; C^l morphisms.

Any algebraic operation that makes new finite-dimensional real or complex vector spaces out of old ones, gives rise to the corresponding *fibrewise construction* for vector bundles. If, in addition, the operation in question is applicable in a sufficiently "regular" manner to *bases* of the spaces involved, it can be applied to local trivializations, which in turn leads to an analogous operation in the category of C^s vector bundles over C^r manifolds ($s \leq r$).

Example 17.1. The *dual* vector bundle η^* of a vector bundle η over a set B is given by

$$B \ni x \mapsto \eta^*_x \tag{17.1}$$

with $\eta^*_x = (\eta^*)_x$ defined to be the dual space $(\eta_x)^*$ of the original fibre η_x (and we skip the parentheses as their location is of no consequence). Sections ξ of η^* and ψ of η, with a common domain $K \subset B$, can be *paired* to produce the scalar-valued function $\xi(\psi)$ on K with $[\xi(\psi)](x) = [\xi(x)]((\psi)(x))$. Thus, any trivialization e_a of η over K defines a trivialization of η^* over K, denoted by e^a, and characterized by $e^a(\psi) = \psi^a$ for ψ as before (that is, assigning to each $x \in K$ the basis $e^a(x)$ of η^*_x *dual* to the basis $e_a(x)$ of η_x in the sense that $e^a(e_b) = \delta^a_b$).

Let η again be a real/complex vector bundle over a set B. By the *total space* of η we then mean the set (also denoted by η)

$$\eta \,=\, \{(x, \phi) : x \in M\,,\ \phi \in \eta_x\}\,. \tag{17.2}$$

In other words, η is a disjoint union of the fibres η_x :

$$\eta \,=\, \bigcup_{x\in B} (\{x\} \times \eta_x)\,. \tag{17.3}$$

The mapping $\pi : \eta \to B$ given by

$$\pi(x, \phi) \,=\, x$$

then is called the *bundle projection* of η. Furthermore, we will identify each fibre η_x of η with the subset $\{x\} \times \eta_x$ of the total space η. In this way, η is decomposed into the preimages

$$\eta_x \,=\, \pi^{-1}(x)\,,$$

$x \in B$. Any section ψ of η, with a domain $K = \mathrm{dom}(\psi) \subset B$, may be identified with the mapping

$$(\mathrm{Id}, \psi) : K \to \eta \tag{17.4}$$

given by

$$x \mapsto (x, \psi(x)), \tag{17.5}$$

and then

$$\pi \circ (\mathrm{Id}, \psi) = \mathrm{Id} : K \to K. \tag{17.6}$$

The *graph* of ψ then is defined to be the image set of (17.4), i.e.,

$$\mathrm{graph}(\psi) = \{(x, \psi(x)) : x \in \mathrm{dom}(\psi)\}.$$

We will also treat the base set B as a subset of the total space η, via the inclusive mapping

$$B \ni x \mapsto (x, 0_x) \in \eta,$$

where, this time, 0_x stands for the zero vector of the vector space η_x. (Normally, we just write $0 \in \eta_x$.) In other words,

$$B \subset \eta$$

is identified with the graph of the *zero section* of η.

Fig. 6. The total space of a vector bundle

In the case where $B = M$ is a C^r manifold and η is a C^s vector bundle over M $(0 \leq s \leq r)$, the total space η carries a natural structure of a C^s manifold of dimension $n+q$, where $n = \dim M$ and q is the fibre dimension of η. In fact, any local coordinate system x^j for M along with a C^s local trivialization e_a for η, both defined on the same coordinate-and-trivialization domain $U \subset M$, give rise to the $(n+q)$-dimensional coordinate system x^j, e^a for η, with the domain $\pi^{-1}(U)$; more precisely, the coordinate mapping is given by $(x, \phi) \mapsto (x^1, \dots, x^n, \phi^1, \dots, \phi^q)$. The bundle projection $\pi : \eta \to M$ then becomes is called a mapping of class C^s between C^s manifolds, in fact, a submersion (cf. Problem 25 in §9). The fibres η_x, $x \in M$, as well as the graphs of all local C^s sections of η, then are submanifolds of η carrying the subset topology; in particular the *zero section* $M \subset \eta$ is such a submanifold. (See Problem 7.)

Fig. 7. The Möbius strip as the total space of a line bundle (Problems 4 and 5)

Example 17.2. Given a *complex* vector space V with the multiplication of vectors by scalars denoted, as usual, by $\mathbf{C} \times V \ni (z, v) \mapsto zv \in V$, we define the (complex) *conjugate* of V to be the complex vector space $\overline{V}$ with the same underlying set V and the same addition of vectors, but with a different multiplication of vectors by scalars, namely

$$\mathbf{C} \times V \ni (z, v) \mapsto \overline{z}v \in V .$$

The *conjugate* of a complex vector bundle η over a set B now is the complex vector bundle $\overline{\eta}$ over B with

$$B \ni x \mapsto \overline{\eta}_x ,$$

$\overline{\eta}_x = \overline{\eta_x}$ being the conjugate of the fibre η_x. Since $\overline{\eta}_x$ and η_x coincide as sets (and as real vector spaces), sections ξ of η over any set K naturally constitute sections of $\overline{\eta}$. Any trivialization e_a of η over K thus becomes a trivialization of $\overline{\eta}$ over K.

Example 17.3. Given vector bundles $\eta^1, \dots, \eta^p$ over the same set B, their *direct sum* is the vector bundle

$$\eta = \eta^1 \oplus \ldots \oplus \eta^p$$

over B with $\eta_x = \eta^1_x \oplus \ldots \oplus \eta^p_x$ for all $x \in B$. Sections ψ_1 of η_1, ..., ψ_p of η^p with any domain $K \subset B$ can be treated as sections of η by regarding each summand η^l_x of η_x as a subspace of η_x via the l th-place embedding $\phi \mapsto (0, \dots, 0, \phi, 0, \dots, 0)$. Thus, trivializations $e_{a_1}, \dots, e_{a_p}$ of $\eta^1, \dots, \eta^p$, respectively, with a common domain K, can be *juxtaposed* to form the trivialization $e_{a_1}, \dots, e_{a_p}$ of η over K, indexed by the disjoint union of the respective index sets.

For a vector bundle η over a set B and a set B' along with a mapping $h : B' \to B$, the *pullback* of η under h is defined to be the vector bundle $h^*\eta$ over B' given by

$$B' \ni y \mapsto \eta_{h(y)} ,$$

that is, $(h^*\eta)_y = \eta_{h(y)}$. Every section ψ of η over a set $K \subset B$ can now be *pulled back* to $h^*\eta$, leading to the section $h^*\psi$ of $h^*\eta$ over $h^{-1}(K) \subset B'$ with $(h^*\psi)(y) = \psi(h(y)) \in \eta_{h(y)} = (h^*\eta)_y$. Applied to a trivialization e_a of η over K, this results in the trivialization h^*e_a of $h^*\eta$ over $h^{-1}(K)$.

An important special case of the pullback construction is the restriction η_B of a vector bundle η over a set B to a subset $B' \subset B$, defined in §16. (Here $h : B' \to B$ is the inclusion mapping.)

The above constructions of the dual, conjugate, direct sum and pullback can now be applied to C^s vector bundles over any fixed manifold M and, respectively, C^s mappings $h : M' \to M$ between C^r manifolds ($s \le r$). The corresponding operations applied to their maximal atlases of C^s local trivializations e_a yield atlases of local trivializations in each resulting bundle, thus endowing the latter with a natural structure of a C^s vector bundle. (Mutual compatibility is immediate from the transformation rules stated in Problems 1, 2.) Again, a special case of the pullback is provided by the *restriction* η_B of a C^s vector bundle η over a C^s manifold M to a C^s submanifold $N \subset M$.

In the case where $\eta = TM$ for a C^s manifold M ($s \ge 1$), the dual η^* thus is a C^{s-1} vector bundle over M which we call the *cotangent bundle* of M and denote by T^*M (rather than $(TM)^*$).

By a (*vector-bundle*) *morphism* $F : \eta \to \zeta$ from a vector bundle η into a vector bundle ζ, both over the same base set B, we mean a family

$$B \ni x \mapsto F_x$$

of linear operators $F_x : \eta_x \to \zeta_x$, indexed by $x \in B$. Such a morphism F sends any section ψ of η over a set $K \subset B$ onto a section $F\psi$ of ζ defined by $(F\psi)(x) = F_x(\psi(x))$. Trivializations e_a for η and e_λ for ζ with the same domain $K \subset B$ thus lead to the scalar-valued *component functions* F_a^λ of F, defined on K by the relation $Fe_a = F_a^\lambda e_\lambda$. We say that a morphism between C^s vector bundles over a C^r manifold is *of class* C^l $(0 \le l \le s \le r \le \omega)$ if so are its component functions. (This is a local geometric property; see Problem 4.)

Problems

1. Show that the transformation rule $e_{a'} = e_{a'}^a e_a$ in a vector bundle η over a set B implies $e^{a'} = e_a^{a'} e^a$ in η^* and $e_{a'} = \overline{e_{a'}^a} e_a$ in $\overline{\eta}$.
2. Verify that, given a vector bundle η over a set B, and a mapping $h : B' \to B$, we have the transformation rule $h^* e_{a'} = (e_{a'}^a \circ h) h^* e_a$ in $h^*\eta$ whenever $e_{a'} = e_{a'}^a e_a$ in η.
3. Establish the transformation rule $F_{a'}^{\lambda'} = e_{a'}^a e_\lambda^{\lambda'} F_a^\lambda$ for the component functions of any vector-bundle morphism $F : \eta \to \zeta$, assuming that $e_{a'} = e_{a'}^a e_a$ in η and $e_{\lambda'} = e_{\lambda'}^\lambda e_\lambda$ in ζ.
4. Verify that C^l regularity for morphisms between C^s vector bundles over a C^r manifold $(0 \le l \le s \le r \le \omega)$ is a local geometric property.
5. For complex vector spaces V, W, denote by $\mathrm{Hom}(V,W)$, $\overline{\mathrm{Hom}}\,(V,W)$ the complex vector spaces of all mappings $V \to W$ that are complex linear (or, respectively, antilinear in the sense that $T(\lambda z) = \overline{\lambda} Tz$ for all $\lambda \in \mathbf{C}$, $v \in V$), with the valuewise addition and multiplication by scalars. Verify that the identity transformation provides natural isomorphic identifications $\mathrm{Hom}(\overline{V}, W) = \overline{\mathrm{Hom}}(V, \overline{W}) = \overline{\mathrm{Hom}}(V,W) = \overline{\mathrm{Hom}(V,W)}$, as well as $\overline{\overline{V}} = V$ and $\overline{V}^{\,*} = \overline{V^*} = \overline{\mathrm{Hom}}(V, \mathbf{C})$.
6. For finite-dimensional complex vector spaces V, exhibit natural isomorphisms $\overline{V} = \overline{\mathrm{Hom}}(V^*, \mathbf{C}) = [\,\overline{\mathrm{Hom}}(V, \mathbf{C})]^*$. (This provides alternative definitions of $\overline{V}$ when $\dim V < \infty$.)
7. Prove the claims made in the paragraph preceding Example 17.2.
8. Let η be a vector bundle over a set B. Verify that formula (17.5) establishes a bijective correspondence between sections of η with any given domain $K \subset B$, and mappings (17.4) satisfying condition (17.6). State and prove the analogous statement for local C^s sections of C^s vector bundles over C^s manifolds.

18. Vector bundle isomorphisms and triviality

Topics: VecTor-bundle morphisms; isomorphisms of vector bundles; regularity of the inverse isomorphism; trivial bundles; triviality and global trivializations; nontriviality of TS^2 and of tautological line bundles over projective spaces.

Morphisms $\eta \to \zeta$ between fixed vector bundles having a common base set B can be added and multiplied by scalar-valued functions φ on B, so that $(F+G)_x = F_x + G_x$ and $(\varphi F)_x = \varphi(x) F_x$ for $x \in B$ and morphisms $F, G : \eta \to \zeta$. (Thus, they form a module over the algebra of functions on B.) Similarly, the *composite* of the morphisms $F : \eta \to \eta'$, $F' : \eta' \to \eta''$ of vector bundles over B is the morphism $F'F : \eta \to \eta''$ given by $(F'F)_x = F'_x F_x$. The *zero morphism* $0 : \eta \to \zeta$ and the *identity morphism* $\mathrm{Id} : \eta \to \zeta$ are the "neutral elements" of addition and composition, respectively.

Given vector bundles η and ζ over the same base set B, a vector-bundle morphism $F : \eta \to \zeta$ is called a (*vector-bundle*) *isomorphism* if all the linear operators $F_x : \eta_x \to \zeta_x$, $x \in B$, constituting F, are isomorphisms. The *inverse morphism* $F^{-1} : \zeta \to \eta$ then is defined by $(F^{-1})_x = (F_x)^{-1} : \zeta_x \to \eta_x$ for $x \in B$.

Recall that a vector-bundle morphism $F : \eta \to \zeta$ between C^s vector bundles η, ζ over a C^r manifold M is said to be *of class* C^l, $0 \le l \le s \le r$, if so are the component functions F_a^λ of F relative to all C^s local trivializations e_a of η and e_λ of ζ; this is a local geometric property. Sums and composites of C^l morphisms are again of class C^l (Problem 1). Furthermore, if $F : \eta \to \zeta$ is a C^l *isomorphism*, i.e., a vector-bundle isomorphism that is a morphism of class C^l, then the inverse isomorphism $F^{-1} : \zeta \to \eta$ is of class C^l as well (Problem 2). We will say that η, ζ are C^l-*isomorphic* if there is a C^l isomorphism between them.

A C^s vector bundle η over a C^s-manifold M is called C^s-*trivial* (or just *trivial*) if it C^s-isomorphic to a product bundle over M. Also, a C^s-manifold M, $s \ge 1$, is said to be C^{s-1}-*parallelizable* if its tangent bundle TM is C^{s-1}-trivial.

PROPOSITION 18.1. *For a C^s vector bundle η over a C^r manifold M, $s \le r$, the following two conditions are equivalent*:

i. *η is C^s-trivial.*
ii. *η admits a C^s trivialization e_a which is global, that is, defined on the whole of M.*

PROOF. Set $\mathbf{F} = \mathbf{R}^q$ or $\mathbf{F} = \mathbf{C}^q$, where q is the fibre dimension of η over $\mathbf{R}$ or, respectively, $\mathbf{C}$. A global C^s trivialization e_a for η leads to the C^s vector-bundle isomorphism $F : M \times \mathbf{F} \to \eta$, with $F_x : \mathbf{F} \to \eta_x$ sending each $(\phi^1, \ldots, \phi^q)$ onto $\phi^a e_a(x)$. Thus, (ii) implies (i). The inverse implication is clear since $M \times \mathbf{F}$ admits global (e.g., constant) C^s trivializations e_a, which any C^s-isomorphism $F : M \times \mathbf{F} \to \eta$ will send onto global C^s trivializations Fe_a for η (notation of §17). Combining the above proposition with Theorem 16.10, Problem 7 in §15 and Theorem 16.11, we obtain the following results for TS^2 and tautological line bundles over projective spaces (defined in Example 15.3). $\square$

COROLLARY 18.2. *The 2-dimensional sphere is not continuously parallelizable.*

COROLLARY 18.3. *For any real vector space V with $2 \le \dim V < \infty$, the tautological line bundle λ over the projective space $P(V)$ is not C^0-trivial.*

COROLLARY 18.4. *For any complex vector space V with $2 \le \dim V < \infty$, the tautological line bundle λ over the projective space $P(V)$ is not C^0-trivial.*

Problems

1. Given vector bundles η, η', η'' over the same base set B and two morphisms, $F : \eta \to \eta'$ and $G : \eta' \to \eta''$, along with trivializations e_a, e_λ, e_A for the respective bundles (all having the same domain), verify that $(GF)_a^A = G_\lambda^A F_a^\lambda$.

2. Let $F : \eta \to \zeta$ be an isomorphism between vector bundles η and ζ over the same base set B. Show that the component functions G_λ^a of the inverse isomorphism $G = F^{-1} : \zeta \to \eta$ are related to those of F by $G^b_\lambda F^\lambda_a = \delta^b_a$, i.e., as matrix-valued functions, $[G^a_\lambda] = [F^\lambda_a]^{-1}$.

3. Let $h : M \to N$ be a C^r mapping between C^r manifolds, $1 \le r \le \omega$. Verify that the differentials dh_x of h at all points $x \in M$ then constitute a vector-bundle morphism $dh : TM \to h^*TN$ of class C^{r-1} and

$$(dh)p_j = (\partial_j h^\alpha) h^* p_\alpha \,, \tag{18.1}$$

i.e., dh has the component functions $(dh)^\alpha_j = \partial_j h^\alpha$ in the local trivializations p_j for TM and h^*p_α for h^*TN, provided by arbitrary local coordinates x^j for M and y^α for N.

4. Let λ be the tautological line bundle over the projective space $P(V)$ of any finite-dimensional vector space V over the field $\mathbf{K}$ of real or complex numbers. (Thus, $P(V)$ is the set of all 1-dimensional vector subspaces of V, while λ is defined by $P(V) \ni L \mapsto \lambda_L = L$.) Prove that the total space of the dual bundle λ^* of λ is C^ω-diffeomorphic to the manifold

$$P(V \times \mathbf{K}) \setminus \{\{0\} \times \mathbf{K}\} \tag{18.2}$$

obtained by removing a point from the projective space $P(V \oplus \mathbf{K})$. (Hint below.)

5. Verify that the Möbius strip is diffeomorphic to the real projective plane $\mathbf{R}\mathrm{P}^2$ minus a point. (Hint below.)

Hint. In Problem 4, note that every line L' through zero in $V \oplus \mathbf{K}$, with the exception of $\{0\} \times \mathbf{K}$, is the graph of a linear function from a unique line L through zero in V (namely, the V-projection of L') into $\mathbf{K}$.

Hint. In Problem 5, use Problems 4 above and Problem 20 in §3.

19. Subbundles of vector bundles

Topics: Subbundles of vector bundles; quotient bundles; C^k-subbundles; the image and kernel of a constant-rank bundle morphism; the tangent and normal bundles of an immersion.

A *vector subbundle* or simply *subbundle* of a real/complex vector bundle η over a set B is any real/complex vector bundle ζ over B such that, for every $x \in B$, ζ_x is a vector subspace of η_x. To express that ζ is a subbundle of η, we will write

$$\zeta \subset \eta$$

This notation is consistent with our use of the same symbol η for both the bundle in question and its total space (17.2); in fact, the total space of a subbundle ζ obviously is a subset of the total space of the given bundle η. (See also Problem 1.)

If $\zeta \subset \eta$, we can form the *quotient* vector bundle η/ζ over the same base B, with the fibres η_x/ζ_x, $x \in B$, and with the obvious projection morphism

$$\pi : \eta \to \eta/\zeta \,.$$

Let η be a vector bundle of class C^k over a C^k manifold M. By a *C^k-subbundle* of η we then mean any vector subbundle ζ of η with the property that, for every $x \in M$ and $\phi \in \zeta_x$, there exists a local section ψ of ζ defined on a neighborhood of x which is of class C^k as a local section of η, and has $\psi(x) = \phi$. It follows that in this case ζ is automatically a C^k vector bundle over M with the C^k-bundle structure uniquely characterized by the property that the local C^k sections of ζ are precisely those local sections of ζ which are C^k as local sections of η. (In fact, local trivializations of ζ obtained using ψ as above for ϕ running through a basis of ζ_x form a C^k atlas of local trivializations for ζ, as defined in §15; this is in turn clear since, for two such local trivializations, the transition functions are of class C^k as they form, locally, a submatrix of a matrix of transition functions for two C^k local trivializations of η.)

Similarly, given a C^k-subbundle ζ of a C^k vector bundle η over a C^k manifold M, the quotient vector bundle η/ζ carries a natural structure of a C^k vector bundle over M, uniquely characterized by requiring that the π-image of any local C^k section of η be a local C^k section of η/ζ. (To see this, note that, near any $x \in M$, the π-images of a suitable subcollection of any system of local C^k sections of η, trivializing η in a neighborhood of x, form a local trivialization of η/ζ, and two such local trivializations are C^k-compatible by a submatrix argument similar to that in the preceding paragraph.)

Let $\varPhi : \eta \to \zeta$ now be a vector-bundle morphism between two vector bundles η, ζ over the same base set B. The *rank* of $\varPhi$ at a point $x \in B$ is, by definition, the rank of the linear operaton $\varPhi_x : \eta_x \to \zeta_x$, i.e., the dimension of its image. If the point x is not specified, the term 'rank of $\varPhi$' means the function $B \to \mathbf{Z}$ assigning to every x the rank of $\varPhi$ at x, and we will say that $\varPhi$ is *of constant rank* if that function is constant on B. For every constant-rank morphism $\varPhi : \eta \to \zeta$ of vector bundles over B, the *image* $\varPhi(\eta)$ of $\varPhi$, with the fibres $\varPhi_x(\eta_x)$, $x \in B$, forms a vector subbundle of ζ. Also, in view of the rank-nullity formula (16.4), the *kernel* $\operatorname{Ker}\varPhi$ of $\varPhi$, with the fibres $\operatorname{Ker}\varPhi_x$, $x \in B$, forms a vector subbundle of η.

Proposition 19.1. *The image and the kernel of any constant-rank C^k morphism $\varPhi : \eta \to \zeta$ between C^k vector bundles η, ζ over a C^k manifold M are C^k-vector subbundles of ζ and η, respectively.*

Proof. All we need to do is find local trivializations of $\varPhi(\eta)$ (or, $\operatorname{Ker}\varPhi$) defined in a neighborhood of any given point $x \in M$ and consisting of C^k local sections of ζ (or, η). To this end, let us choose the ranges of indices to be $a, b \in \{1, \dots, r\}$ and $\lambda, \mu \in \{r+1, \dots, q\}$, where r is the rank of $\varPhi$ and q is the fibre dimension (rank) of η. By suitably reordering the sections $e_1, \dots, e_q$ forming any fixed local trivialization of η in a neighborhood of x, we may assume that the $\varPhi e_a$ constitute, at x, a basis of $\varPhi_x(\eta_x)$, and hence that they form a local trivialization of $\varPhi(\eta)$ near x, as required for the $\varPhi(\eta)$ case. As for $\operatorname{Ker}\varPhi$, note that $\varPhi e_\lambda = H_\lambda^a \varPhi e_a$ with some C^k functions H_λ^a, so that $e_\lambda - H_\lambda^a e_a$ are C^k local sections of $\operatorname{Ker}\varPhi$, which also form a local trivialization (since their linear independence at every point is immediate from that of $e_1, \dots, e_q$). This completes the proof. □

Every C^k mapping $F : M \to N$ between C^k manifolds, $k \ge 1$, leads to the C^{k-1} bundle morphism $dF : TM \to F^*(TN)$ whose action on each T_xM, $x \in M$, is nothing else than $dF_x : T_xM \to T_{F(x)}N = [F^*(TN)]_x$. (Note that dF is of class C^{k-1} since so are its component functions $\partial_j F^\alpha$.) If, in addition, F is an immersion, the morphism dF is injective, i.e., of constant rank $\dim M$. Its image

$$\tau = dF(TM) \subset F^*(TN) \tag{19.1}$$

is called the *tangent bundle of the immersion* F. (Of course, τ is isomorphic to TM, via the isomorphism $dF : TM \to \tau$.) The quotient bundle

$$\nu = F^*(TN)/\tau \tag{19.2}$$

is called the *normal bundle* of the immersion $F : M \to N$. In the case where M is a submanifold of N and F is the inclusion mapping, we clearly have $\tau = TM$, while $\nu = F^*(TN)/TM$ then is referred to as the *normal bundle of* M *in* N.

According to Proposition 19.1, τ and ν as above both carry natural structures of C^{k-1}-vector bundles over M.

Given real vector space V and W, we will denote by $\mathcal{L}(V,V,W)$ the vector space of all bilinear mappings $V \times V \to W$. There is an obvious natural isomorphism $\mathrm{Hom}(V, \mathrm{Hom}(V,W)) \to \mathcal{L}(V,V,W)$, defined by $\Phi \mapsto B$ with $B(v,u) = (\Phi v)u$ for $v, u \in V$. We will use the symbols $\mathcal{S}(V,V,W)$ and $\mathcal{A}(V,V,W)$ for the subspaces of $\mathcal{L}(V,V,W)$ formed by those bilinear mappings $V \times V \to W$ with are also symmetric (or, respectively, skew-symmetric).

Let η and ζ be real vector bundles of class C^k over a manifold M. A fibrewise application of the three operations just described leads to vector bundles $\mathcal{L}(\eta,\eta,\zeta)$, $\mathcal{S}(\eta,\eta,\zeta)$ and $\mathcal{A}(\eta,\eta,\zeta)$ over M. Furthermore, $\mathcal{L}(\eta,\eta,\zeta)$ is naturally a C^k vector bundle (due to the isomorphic identification $\mathrm{Hom}(\eta, \mathrm{Hom}(\eta,\zeta)) \to \mathcal{L}(\eta,\eta,\zeta)$, cf. the last paragraph). Finally, $\mathcal{S}(\eta,\eta,\zeta)$ and $\mathcal{A}(\eta,\eta,\zeta)$ are C^k vector subbundles of $\mathcal{L}(\eta,\eta,\zeta)$. Namely, $\mathcal{L}(\eta,\eta,\zeta) = \mathcal{S}(\eta,\eta,\zeta) \oplus \mathcal{A}(\eta,\eta,\zeta)$ and the projection morphisms are of class C^k as vector-bundle morphisms $\mathcal{L}(\eta,\eta,\zeta) \to \mathcal{L}(\eta,\eta,\zeta)$. See Problems 3 and 4.

Problems

1. Verify that, for a C^k vector subbundle ζ of a C^k vector bundle η over a manifold M, the total space of ζ is contained in the total space of η as a C^k submanifold with the subset topology.

2. Let $S(V) = \{x \in V : \langle x, x \rangle = 1\}$ be the unit sphere in a Euclidean space V. Verify that the normal bundle of $S(V)$ in V is trivial.

3. Given real vector bundles η and ζ of class C^k over a manifold M, verify that $\mathcal{L}(\eta,\eta,\zeta) = \mathcal{S}(\eta,\eta,\zeta) \oplus \mathcal{A}(\eta,\eta,\zeta)$ and the projection morphisms are C^k vector-bundle morphisms $\mathcal{L}(\eta,\eta,\zeta) \to \mathcal{L}(\eta,\eta,\zeta)$ of constant rank.

4. Let η, ζ be C^k vector bundles over a manifold M, and let ζ' be a C^k vector subbundle of ζ. Verify that $\mathrm{Hom}(\eta,\zeta')$ then is a C^k vector subbundle of $\mathrm{Hom}(\eta,\zeta)$.

CHAPTER 5

Connections and Curvature

20. The curvature tensor of a connection

Topics: Connections in vector bundles; germ-dependence of the covariant derivative; the component functions of a connection; curvature; curvature components; the standard flat connection in a product bundle.

Let η be a real or complex C^s vector bundle over a C^r manifold M, $1 \le s \le r$, and let $z \in M$.

As in §5, let us say that two local C^l sections of η $(l \le s)$, defined on neighborhoods of z, are *equivalent at* z if they coincide on some (smaller) neighborhood of z. The equivalence classes of this relation are known as C^l *germs* of local sections of η at z, and they form a vector space as well as a module over the algebra of the germs of scalar-valued functions in M at z.

By a *connection* at z in η we mean a mapping

$$(v, \psi) \mapsto \nabla_v \psi \in \eta_z \tag{20.1}$$

associating an element $\nabla_v \psi$ of the fibre η_z of η over z with any tangent vector $v \in T_z M$ and any local C^1 section ψ of η defined on any open set containing z, in such a way that

a. $T_z M \ni v \mapsto \nabla_v \psi \in \eta_z$ is (real) linear when ψ as above is fixed,
b. For a fixed $v \in T_z M$, $\nabla_v \psi$ depends only on the C^1 germ of ψ at z,
c. $\nabla_v(\psi + \phi) = \nabla_v \psi + \nabla_v \phi$ for local C^1 sections ψ, ϕ of η defined near z,
d. The *Leibniz rule* (or *product rule*)

$$\nabla_v(f\psi) \,=\, (d_v f)\psi(z) + f(z)\nabla_v \psi \tag{20.2}$$

holds for all vectors $v \in T_z M$ whenever ψ is a local C^1 section of η and f is a scalar-valued C^1 function, both defined near z.

Intuitively, a connection at z in η represents an analogue of the directional differentiation with respect to vectors tangent to M at z (see §6), applied to local sections of η rather than real-valued functions; since no such operation is naturally distinguished by the vector-bundle structure of η, we have to select it as an additional feature of the geometry in question.

Let us denote by $\mathcal{C}_z(\eta)$ the set of all connections at z in η. When η and z are fixed, the set $\mathcal{C}_z(\eta)$ carries a natural structure of a real or complex affine space of dimension nq^2, where q is the (real or complex) fibre dimension of η and $n = \dim M$. The translation vector space is $\mathrm{Hom}(T_z M, \mathrm{Hom}(\eta_z, \eta_z))$; see Problem 1 below.

If x^j is a local coordinate system in M, and e_a is a local trivialization of η, both having the same domain $U \subset M$ containing z, we define the *components of the connection* (20.1) relative to the x^j and the e_a to be the scalars Γ_{ja}^b uniquely characterized by

$$\nabla_{p_j(z)} e_a \;=\; \Gamma^b_{ja} e_b(z)\,. \tag{20.3}$$

The Γ^b_{ja} then uniquely determine the connection (20.1) at z via the formula

$$\nabla_v \psi \;=\; v^j[(\partial_j \psi^a)(z) + \Gamma^a_{jb}\psi^b(z)]e_a(z)\,. \tag{20.4}$$

Briefly,

$$\nabla_v \psi \;=\; v^j(\partial_j \psi^a \,+\, \Gamma^a_{jb}\psi^b)\, e_a\,, \qquad \text{i.e.} \qquad (\nabla_v\psi)^a \;=\; v^j(\partial_j \psi^a \,+\, \Gamma^a_{jb}\psi^b)\,, \tag{20.5}$$

where the dependence on z is suppressed for clarity. In particular, note that $\nabla_v\psi$ involves ψ only through the $\psi^a(z)$ and $(\partial_j\psi^a)(z)$.

The definition of a vector bundle in §15 can be naturally generalized to that of an *affine bundle*, that is a bundle of affine spaces, with the analogous concepts of fibre dimension (rank), base, fibres, sections (and their domains), as well as local trivializations, compatibility, and C^s affine bundles over C^r manifolds, with local C^l sections ($l \le s \le r$). The *associated vector bundle* of such an affine bundle ζ with base set B is the vector bundle η over B whose each fibre η_x ($x \in B$) is the translation vector space of the fibre ζ_x of ζ. For details, see the appendix following §15.

Every vector bundle naturally constitutes an affine bundle. As our primary example of affine bundles other than those, note that every C^s vector bundle η over a C^r manifold M ($1 \le s \le r$) gives rise to a C^{s-1} affine bundle $\mathcal{C}(\eta)$ over M whose fibre over each $z \in M$ is defined to be $\mathcal{C}_z(\eta)$, the affine space of all connections at z in η. The associated vector bundle of $\mathcal{C}(\eta)$ thus is $\mathrm{Hom}(TM, \mathrm{Hom}(\eta,\eta))$.

By a *C^l connection in a C^s vector bundle* η over a C^r manifold M ($0 \le l \le s-1 \le r-1$) we then mean a global C^l section of the C^{s-1} affine bundle $\mathcal{C}(\eta)$. Without introducing affine bundles, we could just define a C^l connection in η to be a mapping ∇ assigning to each point $z \in M$ a connection $\nabla^z \in \mathcal{C}_z(\eta)$ whose components $\Gamma^a_{jb}(z)$ with any x^j, e_a as in (20.3) are C^l functions of z. The assignments $z \mapsto \Gamma^a_{jb}(z)$ then are called the *component functions* of the connection ∇ relative to the x^j and the e_a, and denoted by Γ^a_{jb}. Note that C^l regularity of connections is a local geometric property, e.g., in view of Problem 2 below. Furthermore, for a C^l connection in a vector bundle η over M, a local vector field w in M, and a local C^1 section ψ of η, both defined on the same open set $U \subset M$, a continuous local section $\nabla_w\psi$ of η can be defined on U by

$$(\nabla_w\psi)(x) \;=\; \nabla^x_{w(x)}\psi\,. \tag{20.6}$$

We then have the following versions of (20.2), (20.3) and (20.5) for vector *fields*:

$$\nabla_w(f\psi) \;=\; (d_w f)\psi + f\nabla_w\psi\,, \qquad \nabla_{p_j} e_a \;=\; \Gamma^b_{ja} e_b\,,$$

and

$$(\nabla_w\psi)^a \;=\; w^j(\partial_j\psi^a + \Gamma^a_{jb}\psi^b)\,. \tag{20.7}$$

From now on all connections, vector bundles and manifolds will be assumed C^∞; most statements can be generalized to much lower regularity, which is left as an easy exercise for the reader.

Given a connection ∇ in a vector bundle η over a manifold M, an open set $U \subset M$, local C^1 vector fields v, w in M, defined on U, and a local C^2 section

ψ of η, also with the domain U, let $R(v,w)\psi$ denote the continuous local section of η, given (on U) by

$$R(v,w)\psi \,=\, \nabla_w\nabla_v\psi \,-\, \nabla_v\nabla_w\psi \,+\, \nabla_{[v,w]}\psi\,. \tag{20.8}$$

Then, for any local coordinate system x^j in M, and any local trivialization e_a of η, both having the same domain $U \subset M$, we have (see Problem 4)

$$R(v,w)\psi \,=\, v^j w^k \psi^a R_{jka}{}^b e_b\,, \tag{20.9}$$

with the scalar-valued C^∞ functions $R_{jka}{}^b$ on U given by

$$R_{jka}{}^b \,=\, \partial_k \varGamma_{ja}^b \,-\, \partial_j \varGamma_{ka}^b \,+\, \varGamma_{kc}^b \varGamma_{ja}^c \,-\, \varGamma_{jc}^b \varGamma_{ka}^c\,. \tag{20.10}$$

In view of (20.9), the operation defined in (20.8) has the remarkable property that the value $[R^\nabla(v,w)\psi](x)$ of $R^\nabla(v,w)\psi$ at any point $x \in M$ depends *only* on the values $v(x), w(x)$ and $\psi(x)$ of v, w and ψ at x. We can thus define the *curvature* tensor field $R = R^\nabla$ of any connection ∇ in a C^∞ vector bundle over a manifold M, to be the the assignment $M \ni x \mapsto R_x^\nabla = R^\nabla(x)$, associating with each x the skew-symmetric bilinear mapping

$$R^\nabla(x) : T_xM \times T_xM \,\to\, \mathrm{Hom}(\eta_x, \eta_x)\,,$$

also written as $(v,w) \mapsto R_x^\nabla(v,w)$, which is valued in linear operators $\eta_x \to \eta_x$, and characterized by (20.8) for (local) C^1 vector fields v, w on M and (local) C^1 sections ψ of η, all three with the same domain. Specifically, $R^\nabla(v,w)\psi$ then is the local section of η with $[R^\nabla(v,w)\psi](x) = R_x^\nabla(v(x), w(x))[\psi(x)]$.

By (20.9), the component functions $R_{jka}{}^b$ of R^∇ relative to any local coordinate system x^j in M and any local trivialization e_a of η, both having the same domain $U \subset M$, thus may be expressed as

$$R^\nabla(p_j, p_k)e_a \,=\, R_{jka}{}^b e_b\,.$$

Equivalently,

$$R_{jka}{}^b \,=\, e^b\,[R^\nabla(p_j, p_k)e_a]\,.$$

Remark 20.1. The curvature tensor R is, clearly, a C^∞ section of the vector bundle $\mathcal{A}(TM, TM, \mathrm{Hom}(\eta,\eta)))$, defined as at the end of §19.

Example 20.2. Every product bundle $\eta = M \times \mathbf{F}$ carries the *standard flat connection*, often denoted by d (instead of the generic symbol ∇ for connections), such that $d_v\psi$ is the ordinary directional derivative of any local C^1 section of η viewed as an $\mathbf{F}$-valued function.

Problems

1. Let η be a real/complex C^s vector bundle over a C^r manifold M, $1 \le s \le r$, and let $z \in M$. For $\nabla \in \mathcal{C}_z(\eta)$ and $F \in \mathrm{Hom}(T_zM, \mathrm{Hom}(\eta_z, \eta_z))$ set

$$(\nabla + F)_v\psi \,=\, \nabla_v\psi \,+\, F_v(\psi(z))$$

whenever $v \in T_zM$ and ψ is a local C^1 section of η defined near z, with $F_v = F(v) \in \mathrm{Hom}(\eta_z, \eta_z))$. Verify that then $\nabla + F \in \mathcal{C}_z(\eta)$. Prove that this addition turns $(\mathcal{C}_z(\eta), \mathrm{Hom}(T_zM, \mathrm{Hom}(\eta_z, \eta_z)), +)$ into a real/complex affine space. (Hint below.)

2. Verify the *transformation rule*

$$\Gamma_{j'a'}^{b'} = p_{j'}^{j} e_{a'}^{a} e_{b}^{b'} \Gamma_{ja}^{b} + e_{c}^{b'} \partial_{j'} e_{a'}^{c}$$

for the component functions Γ_{ja}^{b} of a connection ∇ in a C^∞ vector bundle η over a manifold M, characterized by (20.2), under any change of the coordinates x^j in M and the local trivialization e_a in η, with the transition functions $p_j^{j'} = \partial_j x^{j'}$ and $e_{a'}^a = e^a(e_{a'})$.

3. Let ∇ be a connection in a C^∞ vector bundle η over a manifold M. A local C^1 section ψ of η is said to be ∇-*parallel* (or just *parallel*) if $\nabla_v\psi = 0$ for each vector $v \in T_xM$ tangent to M at any point x in the domain of ψ. Prove that every ∇-parallel section ψ of η is C^∞ differentiable. What are the ∇-parallel sections for the standard flat connection $\nabla = D$ in a product bundle $\eta = M \times \mathbf{F}$? (Hint below.)

4. Verify that (20.8) and (20.10) imply (20.9).

5. Given a connection in a vector bundle η over a manifold M, verify that in a given coordinate-and-trivialization domain we have $\Gamma_{ja}^b = 0$ identically if and only if all the e_a are parallel.

6. A connection ∇ in a C^∞ vector bundle η over a manifold M is called *flat* if its curvature R^∇ is identically zero. Verify that under any of the following three assumptions, flatness of ∇ follows:

(a) $\dim M = 1$.

(b) M can be covered by coordinate-and-trivialization domains with $\Gamma_{ja}^b = 0$, where Γ_{ja}^b are the corresponding component functions of ∇.

(c) For each $x \in M$ and each $\phi \in \eta_x$ there exists a ∇-parallel local section ψ of η (see Problem 3), defined in a neighborhood of x and such that $\psi(x) = \phi$.

7. Verify that the standard flat connection $\nabla = D$ in a product bundle $\eta = M \times \mathbf{F}$ is actually flat.

8. Let ∇ be a connection in a complex line bundle η over a manifold M (that is, a C^∞ complex vector bundle of fibre dimension 1 over M). Verify that, at each $x \in M$, the curvature $R^\nabla = R^\nabla(x)$ may be regarded as a skew-symmetric real-bilinear mapping $T_xM \times T_xM \to \mathbf{C}$, so that, given $v, w \in T_xM$ and $\phi \in \eta_x$, the element $R^\nabla(v,w)\phi$ of η_x is obtained by multiplying the complex scalar $R^\nabla(v,w)$ by $\phi \in \eta_x$.

9. Consider a connection ∇ in a complex line bundle η over a manifold M (Problem 8), and a fixed local coordinate system x^j in M with some domain $U \subset M$, which also is the domain of a C^∞ local trivializing section e_1 of η. Departing from our usual notational convention, let us use the symbol Γ_j (rather than Γ_{j1}^1) for the complex-valued component function of ∇ relative to such a section e_1. Show that

(a) $R^\nabla(p_j, p_k) = \partial_k\Gamma_j - \partial_j\Gamma_k$, where $R^\nabla(\partial_j, \partial_k)$ is regarded as a complex-valued function on U (Problem 8).

(b) $\Gamma_j' = \Gamma_j + (\partial_j f)/f$, where Γ_j' stands for $\Gamma_{j1'}^{1'}$ relative to another C^∞ local trivializing section $e_{1'}$ of η over U and the same coordinates x^j, and $f : U \to \mathbf{C} \setminus \{0\}$ is the ratio of $e_{1'}$ and e_1, i.e., $e_{1'} = fe_1$ (in other words, $f = e_{1'}^1$).

10. Verify that the above *ad hoc* definition of a C^l *connection in a* C^s *vector bundle* η over a C^r manifold M $(0 \le l \le s-1 \le r-1)$ describes precisely a C^l section of the C^{s-1} affine bundle $\mathcal{C}(\eta)$.

Hint. In Problem 1, given $\nabla, \nabla' \in \mathcal{C}_z(\eta)$, show that $\nabla_v\psi - \nabla'_v\psi$ depends only on $v \in T_zM$ and $\psi(z)$ (rather than on the whole local C^1 section ψ), by applying the Leibniz rule to $\psi = \psi^a e_a$. Thus, $\nabla' = \nabla + F$ with $F \in \operatorname{Hom}(T_zM, \operatorname{Hom}(\eta_z, \eta_z))$.
Hint. In Problem 3, a ∇-parallel section ψ of η satisfies

$$\partial_j\psi^a = -\Gamma^a_{jb}\psi^b$$

(by (20.7)), and induction on k shows that its component functions ψ^a are C^k-differentiable for each integer $k \ge 1$. In the case of the standard flat connection $\nabla = D$ in a product bundle $\eta = M \times \mathbf{F}$, the D-parallel local sections are just the locally constant mappings $\psi : U \to \mathbf{F}$ defined on open sets $U \subset M$.

21. Connections in the tangent bundle

Topics: Connections on a manifold; the torsion tensor and its component functions; torsionfree (symmetric) connections; the case of an affine space; twice-covariant tensor fields; the Ricci tensor of a connection on a manifold; pullbacks of twice-covariant tensors.

Let ∇ be a *connection on the manifold* M, by which we means a connection in the tangent bundle TM. For any open set $U \subset M$, and local C^1 vector fields v, w in M, defined on U, we denote by $\mathrm{T}(v,w)$ the continuous vector field

$$\mathrm{T}(v,w) = \nabla_v w - \nabla_w v - [v,w] \tag{21.1}$$

on U. In any local coordinates x^j for M with a domain $U' \subset U$, we then have (see Problem 1)

$$\mathrm{T}(v,w) = v^j w^k \mathrm{T}^l_{jk}\, p_l\,, \tag{21.2}$$

with the scalar-valued C^∞ functions T^l_{jk} on U' given by

$$\mathrm{T}^l_{jk} = \Gamma^l_{jk} - \Gamma^l_{kj}\,. \tag{21.3}$$

The Γ^l_{jk} stand here for the component functions of ∇ relative to the x^j, i.e., involving the local trivialization p_j of TM, so that

$$\nabla_{p_j} p_k = \Gamma^l_{jk}\, p_l\,.$$

In view of (21.2), the operation defined by (21.1) has the same "tensorial" property as the curvature in (20.9); namely, the value $[\mathrm{T}(v,w)](z)$ of $\mathrm{T}(v,w)$ at any point $z \in M$ depends just on the values $v(z), w(z)$ of v and w at z. We can thus define the *torsion tensor field* $\mathrm{T} = \mathrm{T}^\nabla$ of any connection ∇ in a TM to be the mapping assigning to each $x \in M$ the skew-symmetric bilinear mapping $T_xM \times T_xM \ni (v,w) \mapsto \mathrm{T}_x(v,w) \in T_xM$ given by (21.1) for (local) C^1 vector fields v, w defined near x in M. By (21.2), the component functions T^l_{jk} of T relative to any local coordinate system x^j in M, given by (21.3), can also be characterized by

$$\mathrm{T}(p_j, p_k) = \mathrm{T}^l_{jk} p_l$$

Equivalently,

$$\mathrm{T}^l_{jk} = dx^l(\mathrm{T}(p_j, p_k))\,.$$

A connection ∇ in TM is called *torsionfree* (or *symmetric*) if its torsion tensor field is identically zero. This is the case if and only if, in any local coordinates x^j,

$$\Gamma_{jk}^l = \Gamma_{kj}^l. \tag{21.4}$$

The torsion tensor field T is, obviously, a C^∞ section of $\mathcal{A}(TM, TM, TM)$ (see the end of §19).

Example 21.1. The natural isomorphic identification $T_xM = V$ for a finite-dimensional real affine space $(M, V, +)$ viewed as a manifold (Example 5.2) amounts to a natural vector-bundle isomorphism between TM and $M \times V$; from now on, we will write $TM = M \times V$. The standard flat connection $\nabla = D$ in TM (Example 20.2) is torsionfree; in fact, $\mathrm{T}(v, w) = 0$ for constant vector fields v, w. (Or, $\Gamma_{jk}^l = 0$ in affine coordinates.)

By a *twice-covariant tensor field* on a manifold M we mean any mapping b assigning to every point $x \in M$ a real-valued bilinear form b_x on the tangent space T_xM. We will usually skip the word 'field' and simply speak of a twice-covariant tensor b. Such b is called *symmetric* or, respectively, *skew-symmetric*, if so is b_x at every point $x \in M$.

Twice-covariant tensors b on M are nothing else than all possible sections of a specific C^∞ vector bundle over M, namely, $\mathcal{L}(TM, TM, M \times \mathbf{R})$ (see the end of §19), or, in other words, $\mathrm{Hom}\,(TM, T^*M)$. Explicitly, each b_x may be viewed as an operator $T_xM \to T_x^*M$, given by $v \mapsto b_x(v, \cdot\,))$. It therefore makes sense to speak of their C^k-differentiability for $k = 0, 1, \ldots, \infty$, which is the same as the differentiability class of their *component functions* $b_{jk} = b(p_j, p_k)$ relative to any local coordinates x^j in M. Note that the b_{jk} coincide with the components of b as a section of $\mathrm{Hom}\,(TM, T^*M)$ in the local trivialization corresponding to the x^j.

Similarly, twice-covariant tensors which are symmetric (or, skew-symmetric) coincide with arbitrary sections of the vector bundle $\mathcal{S}(TM, TM, M \times \mathbf{R})$ (or, respectively, $\mathcal{A}(TM, TM, M \times \mathbf{R})$), defined in §19.

Let ∇ now be a connection on a manifold M. Its *Ricci tensor* is the twice-covariant tensor field $\mathrm{Ric} = \mathrm{Ric}^\nabla$ on M, defined by requiring, for every $x \in M$, that the bilinear mapping $\mathrm{Ric}_x : T_xM \times T_xM \to \mathbf{R}$ send vectors v, w to

$$\mathrm{Ric}_x(v, w) = \mathrm{Trace}\,[u \mapsto R(v, u)w]\,, \tag{21.5}$$

the trace being that of an operator $T_xM \to T_xM$. The component functions of Ric in local coordinates x^j are traditionally written as R_{jk}. Obviously,

$$R_{jk} = R_{jlk}{}^l\,. \tag{21.6}$$

Problems

1. Verify that (21.1) and (21.3) imply (21.2).

2. For a C^∞ connection ∇ in the tangent bundle TM of a manifold M and a global C^∞ section F of the vector bundle $\mathrm{Hom}(TM, \mathrm{Hom}(TM, TM))$, show that the torsion tensor field of the connection $\nabla + F$ in TM is given by $\mathrm{T} + A$, where T is the torsion of ∇ and $A(v, w) = F_v w - F_w v$ for all $x \in M$ and $v, w \in T_xM$.

3. Given a C^∞ connection ∇ in the tangent bundle TM, write a natural formula for a (new) connection ∇' in TM that is torsionfree. (Hint below.)

4. Let η and ζ be C^∞ vector bundles over the same C^∞ manifold M, with local trivializations e_λ (for η) and e_a (for ζ), both having the same domain $U \subset M$. Given a local C^s section ϕ of η a local C^s section Ψ of $\mathrm{Hom}(\eta, \zeta)$, both defined on U, verify that the local section $\Psi\phi$ of ζ obtained by "valuewise evaluation" of Ψ on ϕ is also of class C^s and has the component functions

$$[\Psi\phi]_a = \Psi^a_\lambda \phi^\lambda .$$

5. Let b be a twice-covariant tensor field on a manifold M, and let $F : N \to M$ be a C^1 mapping between manifolds. Verify that the *pullback* F^*b of b under F, defined by $(F^*b)_y(v, w) = b_{F(y)}(dF_y v, dF_y w)$ for $y \in N$ and $v, w \in T_yN$, is a twice-covariant tensor field on and that F^*b is symmetric whenever b is.

6. For b, F, M, N as in Problem 4, verify the local-coordinate formula $(F^*b)_{\lambda\mu} = (\partial_\lambda F^j)(\partial_\mu F^k) b_{jk}$, and conclude that F^*b is of class C^∞ whenever b and F are.

Hint. In Problem 3, write $\nabla' = \nabla - \frac{1}{2}\mathrm{T}$, where T is the torsion of ∇.

22. Parallel transport and geodesics

Topics: Sections of a vector bundle along a curve; covariant differentiation of sections along curves; the component formula; the case of the standard flat connection in a product bundle; sections parallel along a curve in a vector bundle with a connection; spaces of parallel sections; evaluation isomorphisms; parallel transport; the case of the standard flat connection in a product bundle; velocity and acceleration; geodesics; straight lines in affine spaces.

Let η be a C^∞ real or complex vector bundle over a manifold M. Given an interval $I \subset \mathbf{R}$ and a C^r curve $\gamma : I \to M$, by a C^s *section* ϕ *of* η *along* γ, $0 \le s \le r \le \infty$, we mean a mapping sending each $t \in I$ to $\phi(t) \in \eta_{\gamma(t)}$ whose component functions $t \mapsto \phi^a(t)$ relative to all local trivializations e_a of η, defined on suitable subsets of I and characterized by $\phi(t) = \phi^a(t) e_a(\gamma(t))$, are of class C^s. (This C^s regularity is obviously a local geometric property, as described in §2.)

From now on, dealing with a fixed curve γ, we will often use the generic symbol $x(t)$ instead of $\gamma(t)$.

Let ∇ be a connection in a vector bundle η over M as above. Given a C^1 section ϕ of η along a C^1 curve $I \ni t \mapsto x(t) \in M$, parametrized by an interval $I \subset \mathbf{R}$, we define the *covariant derivative* of ϕ to be the C^0 section $\nabla_{\dot{x}}\phi$ of η along the same curve, with the component functions

$$(\nabla_{\dot{x}}\phi)^a = \dot{\phi}^a + \Gamma^a_{jb}(x)\dot{x}^j \phi^b , \tag{22.1}$$

where $(\ )^\cdot = d/dt$, that is, $(\nabla_{\dot{x}}\phi)^a(t) = \dot{\phi}^a(t) + \Gamma^a_{jb}(x(t))\dot{x}^j(t)\phi^b(t)$ for all $t \in I$. This operation is uniquely characterized by the following "axioms" (see Problems 1 and 2):

a. $\phi \mapsto \nabla_{\dot{x}}\phi$ is real/complex linear (for a fixed C^1 curve $I \ni t \mapsto x(t)$ on a fixed interval I);
b. $\nabla_{\dot{x}}(f\phi) = \dot{f}\phi + f\nabla_{\dot{x}}\phi$ for ϕ, I and a curve $t \mapsto x(t)$ as above, and any scalar-valued C^1 function f on I, where $\dot{f} = df/dt$;
c. The operation in question is *local*, that is, when applied to the restrictions of ϕ and the curve as above to any subinterval I' of I, it yields the restriction to I' of the original covariant derivative $\nabla_{\dot{x}}\phi$ (on I);
d. $\nabla_{\dot{x}}[\psi(x)] = \nabla_{\dot{x}(t)}\psi$ for local C^1 sections ψ of η, with $\psi(x)$ being the composite $t \mapsto \psi(x(t))$.

EXAMPLE 22.1. The standard flat connection $\nabla = D$ in a product bundle $\eta = M \times \mathbf{F}$ (Example 20.2) satisfies $D_v\psi = d_v\psi$, for local C^1 sections of η (which are nothing else than $\mathbf{F}$-valued C^1 functions) and vectors v tangent to M. Given γ and ϕ along γ as above, we now have (see Problem 3)

$$D_{\dot\gamma}\phi = \dot\phi\,, \tag{22.2}$$

where $(\ )^{\boldsymbol{\cdot}} = d/dt$ is the ordinary differentiation of $\mathbf{F}$-valued C^1 functions

$$I \ni t \mapsto \phi(t) \in \eta_{\gamma(t)} = \mathbf{F}\,.$$

Let ∇ be a connection in a C^∞ real or complex vector bundle η over a manifold M. Given an interval $I \subset \mathbf{R}$ and a C^l curve $I \ni t \mapsto x(t) \in M$, we will say that a C^1 section ϕ of η along the curve is *parallel* if

$$\nabla_{\dot x}\phi = 0 \tag{22.3}$$

identically. If the curve is fixed, sections parallel along it form a vector space $\mathcal{W}$ ("axiom" (a) in §22). If, moreover, $a \in I$ and $\chi \in \eta_{x(a)}$ are fixed, the global existence and uniqueness theorem for (systems of) linear ordinary differential equations (Appendix C), applied to the system

$$\dot\phi^a = -\Gamma^a_{jb}(x)\dot x^j\phi^b\,. \tag{22.4}$$

implies that there is a unique parallel section ϕ with the *initial value* $\phi(a) = \chi$. (See also Problem 8.) The *evaluation mapping* $\mathrm{ev}_a : \mathcal{W} \to \eta_{x(a)}$ given by

$$\mathcal{W} \ni \phi \mapsto \phi(a) \in \eta_{x(a)}$$

is, therefore, a linear isomorphism. For $a, b \in I$, the *parallel transport* (or *displacement*) from a to b along any C^1 curve $\gamma : I \to M$ is the linear isomorphism

$$\tau_a^b(\gamma) : \eta_{\gamma(a)} \to \eta_{\gamma(b)}$$

with

$$\tau_a^b(\gamma) = \mathrm{ev}_b \circ (\mathrm{ev}_a)^{-1}\,.$$

Thus, $\tau_a^b(\gamma)$ sends each $\chi \in \eta_{\gamma(a)}$ onto $\phi(b)$, where ϕ is the unique parallel section of η along γ with $\phi(a) = \chi$.

EXAMPLE 22.2. Consider the standard flat connection $\nabla = D$ in a product bundle $\eta = M \times \mathbf{F}$ (Example 20.2). For γ, I, a, b and ϕ along γ as above, we see from (22.4) that ϕ is parallel if and only if it is constant as a $\mathbf{F}$-valued C^1 function, so that $\tau_a^b(\gamma) : \eta_{\gamma(a)} \to \eta_{\gamma(b)}$ is nothing else than $\mathrm{Id} : \mathbf{F} \to \mathbf{F}$.

Let ∇ now be a connection on the manifold M. Every C^2 curve $\gamma : I \to M$ the gives rise, besides the velocity $\dot\gamma$, to its *acceleration vector field*, also known as the *geodesic curvature* of γ, which is the C^0 vector field $\nabla_{\dot\gamma}\dot\gamma$ along γ. In local coordinates x^j, the component functions of $\nabla_{\dot\gamma}\dot\gamma$ are

$$\ddot\gamma^l + (\Gamma^l_{jk} \circ \gamma)\dot\gamma^j\dot\gamma^k\,. \tag{22.5}$$

A C^2 curve $I \ni t \mapsto x(t) \in M$ then is said to be a *geodesic* for ∇ if $\nabla_{\dot x}\dot x = 0$ identically. In local coordinates x^j, geodesics are characterized by

$$\ddot x^l + \Gamma^l_{jk}(x)\dot x^j\dot x^k = 0\,. \tag{22.6}$$

Thus, for a fixed $a \in \mathbf{R}$, and any $x \in M$ and $v \in T_xM$, there exists an open interval I containing a and a unique geodesic $\gamma : I \to M$ with $\gamma(a) = x$ and $\dot\gamma(a) = v$. Moreover, geodesics are automatically curves of class C^∞ (Problem 7).

Given a connection ∇ on a manifold M, the *exponential mapping*

$$\exp_x : U_x \to M \tag{22.7}$$

of ∇ at a point $x \in M$ is defined as follows. Its domain U_x is the subset of T_xM consisting of all those $v \in T_xM$ for which there exists a geodesic $t \mapsto x(t)$ of ∇ defined on the whole interval $[0,1]$, such that $x(0) = x$ and $\dot{x}(0) = v$. For such v and $x(t)$, we set $\exp_x v = x(1)$. (One traditionally omits the parentheses around v.) It is obvious from the dependence-on-parameters theorem for ordinary differential equations that the set U_x is open in T_xM and contains 0, while the mapping $\exp_x$ is of class C^∞. Furthermore,

(22.8) the geodesic with $x(0) = x$ and $\dot{x}(0) = v$ is given by $x(t) = \exp_x tv$,

as one sees fixing $t \in [0,1]$ and noting that the assignment $s \mapsto x(ts)$ then is a geodesic defined on $[0,1]$, with the value and velocity at $s = 0$ equal to x and, respectively, tv. In particular, $d[\exp_x tv]/dt$ at $t = 0$ equals v and, obviously, $\exp_x 0 = x$. Thus, the differential of $\exp_x$ at the point $0 \in U_x$ coincides with the identity mapping $T_xM \to T_xM$. (Cf. (5.16) and Examples 5.1, 5.3.) According to the inverse mapping theorem, there exist a neighborhood U of x in M and a neighborhood U' of 0 in T_xM such that $U' \subset U_x$ and $\exp_x : U' \to U$ is a C^∞ diffeomorphism. Its inverse diffeomorphism may be thought of as a coordinate system x^j with the domain U (after one has identified T_xM with $\mathbf{R}^n$, for $n = \dim M$, using any fixed linear isomorphism). A coordinate system obtained in this way is called a *geodesic* coordinate system centered at x.

Example 22.3. The tangent bundle TM of any finite-dimensional real affine space M may be regarded as (that is, is naturally isomorphic to) the product bundle $M \times V$, where V is the translation vector space of M (cf. Example 5.2). The standard flat connection $\nabla = D$ in $M \times V$ thus becomes a connection on M, called the *standard flat connection of the affine space* M. It is clear from Example 22.2 that the geodesics of this connection D are precisely all line segments in M with uniform (constant-speed) parametrizations, along with, all constant curves. For $x \in M$, the set U_x coincides with $T_xM = V$, and $\exp_x : V \to M$ is given by $\exp_x v = x + v$.

Problems

1. Given a C^∞ real/complex vector bundle η over a manifold M, an interval $I \subset \mathbf{R}$, a C^r curve $\gamma : I \to M$, and an integer s with $1 \le s \le r \le \infty$, verify that the set of all C^s sections ϕ of η along γ forms a real/complex vector space (with valuewise operations), which is also a module over the algebra of all scalar-valued C^s functions f on I.

2. Given a connection ∇ in a C^∞ vector bundle η over a manifold M, prove that there is a unique operation $\phi \mapsto \nabla_{\dot\gamma}\phi$ satisfying conditions (a) – (d) above, and that its local component description is provided by (22.1). (Hint below.)

3. Verify (22.2) for the standard flat connection D in any product bundle. (Hint below.)

4. Prove that

$$\nabla_{[x(f)]'}(\phi \circ f) \;=\; f'\,[(\nabla_{\dot x}\phi) \circ f]$$

whenever ∇ is a connection in a C^∞ vector bundle η over a manifold M, $I, I' \subset \mathbf{R}$ are intervals, $I \ni t \mapsto x(t) \in M$ is a C^1 curve, $f : I' \to I$ is a C^1

function, the prime $'$ stands for the derivative d/ds, where $s \in I'$, and ϕ is a C^1 section of η along $t \mapsto x(t)$.

5. Given a connection ∇ on a manifold M, a geodesic $t \mapsto x(t)$, and real constants p, q, verify that $s \mapsto x(ps+q)$ is also a geodesic.

6. For a connection ∇ in a C^∞ vector bundle η over a manifold M, an interval $I \subset \mathbf{R}$, a C^r curve $\gamma : I \to M$, $1 \le r \le \infty$, and a C^1 section ϕ of η along γ which is parallel, show that ϕ is C^r-differentiable.

7. Verify that, for a C^∞ connection ∇ on a C^∞ manifold M, any geodesic (which is, by definition, C^2 differentiable) is actually of class C^∞.

8. Let η be a C^∞ vector bundle, endowed with a fixed connection ∇, over a manifold M. Given an interval $I \subset \mathbf{R}$ and a C^1 curve $\gamma : I \to M$, show that the isomorphisms $\tau_a^b(\gamma)$ of parallel transport along γ satisfy the relations

$$\tau_b^c(\gamma)\tau_a^b(\gamma) = \tau_a^c(\gamma)\,, \qquad \tau_a^a(\gamma) = \mathrm{Id}\,, \qquad \tau_b^a(\gamma) = [\tau_a^b(\gamma)]^{-1}$$

for any $a, b, c \in I$.

9. Prove that

$$\tau_c^d(\gamma \circ f) = \tau_{f(c)}^{f(d)}(\gamma)$$

whenever ∇ is a connection in a C^∞ vector bundle η over a manifold M, $I, I' \subset \mathbf{R}$ are intervals, $\gamma : I \to M$ is a C^1 curve, $f : I' \to I$ is a C^1 function, f' stands for the derivative of f with respect to the parameter $s \in I'$, and ϕ is a C^1 section of η along γ.

Hint. In Problem 2, uniqueness: writing $\phi(t) = \phi^a(t)e_a(x(t))$ and using the "axioms" (a) – (d), we obtain (22.1). Existence: defining $\nabla_{\dot{x}}\phi$ locally by (22.1), we see that, in view of its uniqueness, this formula produces the same operation in the intersection of any two coordinate-and-trivialization domains.
Hint. In Problem 3, use formula (22.1) or, equivalently, the "axioms" (a) – (d).

23. The "comma" notation for connections

Topics: The homomorphism-bundle construction; bundle morphisms as sections of the homomorphism bundle; tensor multiplication of sections; the case of differentiable vector bundles over manifolds; covariant derivative $\nabla\psi$ as a (local) section of a homomorphism bundle; the "nabla" and "comma" notational conventions for covariant-derivative components.

Suppose that V and W are two finite-dimensional real/complex vector spaces. For any $\xi \in V^*$ and $w \in W$ we define the *tensor product* $\xi w \in \mathrm{Hom}(V, W)$ to be the linear operator $V \to W$ given by

$$[\xi w](v) \;=\; \xi(v)\, w \tag{23.1}$$

for all $v \in V$. The *tensor multiplication* thus defined is clearly a bilinear operation $V^* \times W \to \mathrm{Hom}(V, W)$. Given a basis e_λ of V and a basis e_a of W, let e^λ denote (as in §17) the basis of V^* dual to the basis e_λ in the sense that $e^\lambda(e_\mu) = \delta_\mu^\lambda$. The tensor products $e^\lambda e_a$ then form a basis of $\mathrm{Hom}(V, W)$. In fact, denoting by F_λ^a the components of any given $F \in \mathrm{Hom}(V, W)$ relative to the bases e_λ and e_a, (that are characterized by $Fe_\lambda = F_\lambda^a e_a$, i.e., $F_\lambda^a = e^a(Fe_\lambda)$), we thus have

$$F \;=\; F_\lambda^a e^\lambda e_a$$

for each F (since both sides applied to any e_μ yield the same value $Fe_\mu = F_\mu^a e_a$). On the other hand, if for some coefficients F_λ^a the combination $F = F_\lambda^a e^\lambda e_a$ is the zero operator, we have $F^a_\mu e_a = Fe_\mu = 0$ and so the F^a_μ are all zero as the e_a are linearly independent.

Let η and ζ be vector bundles over the same base set B. By the *morphism bundle* of η and ζ we then mean the vector bundle $\chi = \mathrm{Hom}(\eta,\zeta)$ over B with the fibres $\chi_x = \mathrm{Hom}(\eta_x,\zeta_x)$, $x \in B$. For instance, vector-bundle morphisms $\eta \to \zeta$ (§17) are nothing else than global sections of $\mathrm{Hom}(\eta,\zeta)$. Given trivializations e_λ for η and e_a for ζ, both having the same domain $K \subset B$, we can now use valuewise tensor multiplication to obtain the trivialization $e^\lambda e_a$ for $\mathrm{Hom}(\eta,\zeta)$, also with the domain K. Since tensor multiplication is bilinear, we have the transformation rule

$$e^{\lambda'} e_{a'} = e^{\lambda'}_\lambda e^a_{a'} e^\lambda e_a \tag{23.2}$$

(sf. Problem 1 in §17). Given a section Ψ of $\mathrm{Hom}(\eta,\zeta)$, defined on K, we will denote by Ψ^a_λ its component functions relative to the trivialization $e^\lambda e_a$, so that

$$\Psi = \Psi^a_\lambda e^\lambda e_a\,.$$

Consequently, in the case where η and ζ are C^∞ vector bundles over a C^∞ manifold M, $\mathrm{Hom}(\eta,\zeta)$ naturally becomes a C^∞ vector bundle over M, with the atlas of local trivializations $e^\lambda e_a$ obtained using local trivializations e_λ for η and e_a for ζ. The tensor product $\xi\psi$ of a local C^s section ξ of η^* and a local C^s section ψ of ζ, both defined on the same open set $U \subset M$, then obviously is a local C^s section of $\mathrm{Hom}(\eta,\zeta)$, with the component functions

$$[\xi\psi]^a_\lambda = \xi_\lambda \psi^a\,,$$

that is, $\xi\psi = \xi_\lambda\psi^\lambda e^\lambda e_\lambda$, which is due to bilinearity of the tensor multiplication.

Finally, let ∇ be a connection in a C^∞ vector bundle η over a manifold M. For any a local C^1 section ψ of η, we define the *covariant derivative* $\nabla\psi$ of ψ to be the local section of $\mathrm{Hom}(TM,\eta)$, with the same domain as ψ, whose value $(\nabla\psi)_z \in \mathrm{Hom}(T_zM,\eta_z)$ at any z in the domain of ψ is given by $(\nabla\psi)_z(v) = \nabla_v\psi \in \eta_z$ for all $v \in T_zM$. In any local trivialization e_a and coordinates x^j at z, the covariant derivative $\nabla\psi$ then has the component functions $(\nabla\psi)^a_j$ characterized by

$$(\nabla\psi)^a_j = [(\nabla\psi)(p_j)]^a = e^a(\nabla_{p_j}\psi)\,.$$

Instead of $(\nabla\psi)^a_j$, it is usually convenient to use the symbol $\nabla_j\psi^a$ or, when the connection in question is fixed, simply $\psi^a{}_{,j}$. Thus, we write

$$\psi^a{}_{,j} = \nabla_j\psi^a = (\nabla\psi)^a_j\,, \tag{23.3}$$

and so, for v as above,

$$(\nabla_v\psi)^a = v^j\psi^a{}_{,j}\,, \qquad \nabla_v\psi = v^j\psi^a{}_{,j}e_a\,.$$

Consequently (see Problem 2)

$$\nabla\psi = \psi^a{}_{,j}\,dx^j e_a\,. \tag{23.4}$$

Furthermore, by (20.7),

$$\psi^a{}_{,j} = \partial_j\psi^a + \Gamma^a_{jb}\psi^b. \tag{23.5}$$

In the special case where ∇ is a connection in the tangent bundle TM and w is a (local) C^1 vector field, (23.5) becomes

$$w^k{}_{,j} = \partial_j w^k + \Gamma^k_{jl}w^l. \tag{23.6}$$

REMARK 23.1. It must be emphasized that the symbol $\psi^a{}_{,j} = \nabla_j \psi^a$ in (23.3) does *not* stand for the result of applying some operator ∇_j (or $(\ )_{,j}$) to the scalar-valued function ψ^a. In fact, it is clear from (23.5) that $\psi^a{}_{,j}$ also depends on the functions ψ^b with $b \neq a$.

Problems

1. Establish (23.4).

2. For a C^∞ connection ∇ in a vector bundle η over a manifold M and a global C^∞ section F of the vector bundle $\mathrm{Hom}(TM, \mathrm{Hom}(\eta,\eta))$, verify that the component functions of the connection $\nabla + F$ in η equal $\Gamma^a_{jb} + F^a_{jb}$, where F^a_{jb} denote the component functions of F with $F^a_{jb} = e^a(F(p_j, e_b))$.

3. Let η be a real or complex C^∞ vector bundle over a manifold M, and let η^U denote the *restriction* of η to a fixed open submanifold U of M, i.e., the vector bundle

$$U \ni x \mapsto \eta^U_x = \eta_x\,.$$

(Thus, η^U is nothing else than the pullback of η under the inclusion mapping $U \to M$.) Verify that every connection ∇ in η gives rise to a unique connection ∇^U in η^U with $[\nabla^U]_v\psi = \nabla_v\psi$ for any $x \in U$, any $v \in T_xU$, and any local C^1 section ψ of η^U defined on a neighborhood of x (so that ψ is, obviously, also a local C^1 section of η). We then call ∇^U the *restriction* of ∇ to U. Show that

(a) The component functions relative to any coordinates x^j in M and any local trivialization e_a in η^U with the coordinate-and-trivialization domain contained in U, coincide with the respective component functions Γ^b_{ja} of ∇.

(b) Given a family $\mathcal{B} \ni \beta \mapsto U_\beta$ of nonempty open sets in M whose union is M and a connection $\nabla^{(\beta)}$ in the restriction of η to each U_β, such that the restrictions of $\nabla^{(\beta)}$ and $\nabla^{(\beta')}$ to $U_\beta \cap U_{\beta'}$ coincide whenever $\beta, \beta' \in \mathcal{B}$ and $U_\beta \cap U_{\beta'}$ is nonempty, there exists a unique connection ∇ in η whose restriction to each U_β equals $\nabla^{(\beta)}$. (Hint below.)

Hint. In Problem 3, define $\nabla_v\psi$ for $x \in M$, $v \in T_xM$, and a local C^1 section ψ of η defined on a neighborhood U of x to be $\nabla^{(\beta)}_v\psi$ for any β with $x \in U_\beta$ and ψ restricted to $U_\beta \cap U$. This is independent of the choice of β in view of the germ-dependence of the covariant derivative ((b) in §20).

24. The Ricci-Weitzenböck identity

Topics: The Hom operation for connections; the special case of the dual connection in the dual bundle; higher-order covariant derivatives; the Ricci-Weitzenböck identity.

In most of our discussion we will use the generic symbol ∇ for all connections we encounter, the only exceptions being the case where more than one connection in the given vector bundle η is studied, and the case of the standard flat connection in a product bundle $\eta = M \times \mathbf{F}$ (denoted by D rather than ∇, cf. Example 20.2). Since sections to which two such connections ∇ are applied then live in different bundles (and so their components are labeled with different sorts of indices), no confusion is likely to arise. The same convention applies to the "comma" notation (§23).

Let there be given two connections, both denoted by ∇, in two C^∞ real/complex vector bundles η and ζ over a manifold M. Recall (§23) that we can form the homomorphism bundle $\mathrm{Hom}(\eta,\zeta)$, which is also a C^∞ vector bundle over M, and has the fibres $\mathrm{Hom}(\eta_x,\zeta_x)$, $x\in M$. Two local trivializations, e_a for η and e_λ for ζ, both with the same domain $U\subset M$, give rise to the "tensor-product" local trivialization $e^a e_\lambda$ for $\mathrm{Hom}(\eta,\zeta)$, defined on U. (See §23 for details.)

Dealing with a local trivialization $e^a e_\lambda$ as above, we obviously must modify our notational conventions. Namely, the role of the superscript $_a$ in the "generic" natation e_a for local trivializations of vector bundles (§15) now is played by the two-tiered double index ${}^a_\lambda$. The corresponding upper indices ^{a} (such as those in $\psi=\psi^a e_a$, for sections ψ) now have to be replaced with the inverted double indices ${}^\lambda_a$, appearing in the analogous expansion $F=F_a^\lambda e^a e_\lambda$ of sections F of $\mathrm{Hom}(\eta,\zeta)$ (with a double summation, over a and λ). Any connection ∇ in $\mathrm{Hom}(\eta,\zeta)$ thus has the component functions $\Gamma_{j\lambda b}^{a\mu}$ with

$$\nabla_{p_j}(e^a e_\lambda)\,=\,\Gamma_{j\lambda b}^{a\mu}\,e^b e_\mu\,. \tag{24.1}$$

Proposition 24.1. *Given two connections, both denoted by ∇, in C^∞ real or complex vector bundles η and ζ over a manifold M, there exists a unique connection in the real/complex vector bundle $\mathrm{Hom}(\eta,\zeta)$, also denoted by ∇, which is characterized by the requirement that, writing ϕ, F instead of $\phi(x)$ and $F(x)$, we have*

$$\nabla_v(F\phi)\,=\,(\nabla_v F)\phi\,+\,F(\nabla_v\phi) \tag{24.2}$$

for every open set $U\subset M$, every point $x\in U$ and vector $v\in T_xM$, and any C^1 local sections F of $\mathrm{Hom}(\eta,\zeta)$ and ϕ of η, both defined on U. The component functions $\Gamma_{j\lambda b}^{a\mu}$ of this unique connection ∇, defined by (24.1), are given by

$$\Gamma_{j\lambda b}^{a\mu}\,=\,\delta_b^a\Gamma_\lambda^\mu\,-\,\delta_\lambda^\mu\Gamma_{ja}^b\,, \tag{24.3}$$

while, for any F as above, ∇F has the component functions

$$F_{a,j}^\lambda\,=\,\partial_j F_a^\lambda\,+\,\Gamma_{j\mu}^\lambda F_a^\mu\,-\,\Gamma_{ja}^b F_b^\lambda\,. \tag{24.4}$$

Proof. The uniqueness part and formula (24.3) are obvious, since (24.2) determines the action of each ∇_v on all local trivializing sections $F=e^a e_\lambda$. Namely, using this F and $\phi=e_b$, we obtain $F\phi=\delta_b^a e_\lambda$, and so (24.2) with $v=p_j$ at x (for any fixed local coordinates x^j) gives (24.1) with $\Gamma_{j\lambda b}^{a\mu}$ as in (24.3).

To establish existence of ∇, let us define ∇ by (24.1) with (24.3) (replacing the bundles with their restrictions to the coordinate-and-trivialization domain). Now (24.4) is immediate from (24.3) and the general formula (23.5), which in turn easily gives (24.2). The fact that two connections defined (locally) in this way coincide in the intersection of the respective coordinate-and-trivialization domains is obvious from the just-proven uniqueness assertion. This completes the proof. $\square$

A particularly interesting special case of the Hom operation occurs when the target bundle ζ is the product bundle $M\times\mathbf{K}$, $\mathbf{K}$ being the scalar field. Obviously, $\mathrm{Hom}(\eta,\zeta)$ then is nothing else than the dual η^* of the given bundle η. Any given connection ∇ in η, combined with the standard flat connection $\nabla=d$ in the product bundle $\zeta=M\times\mathbf{K}$ (Example 20.2), gives rise to the connection (also denoted by ∇) in the dual bundle $\eta^*=\mathrm{Hom}(\eta,\zeta)$, called the *dual* of the original connection ∇ in η. Given C^1 local sections ξ, ϕ and v of η^*, η and, respectively, TM, all defined on the same open set, we then have

$$(\nabla_v\xi)(\phi)\,=\,d_v[\xi(\phi)]\,-\,\xi(\nabla_v\phi)\,,$$

which is nothing else than (24.2) for $F = \xi$. Also, since $\zeta = M \times \mathbf{K}$ has the global trivialization consisting of the constant function 1, the tensor-product local trivializations $e^a e_\lambda$ for $\eta^* = \mathrm{Hom}(\eta, \zeta)$ contain a naturally distinguished subatlas formed by those $e^a e_\lambda$ in which e_λ is the constant function 1; such $e^a e_\lambda$ are in turn nothing else than the e^a, i.e., the duals (see Example 17.1). of all local C^∞ trivializations e_a of η. (In fact, the tensor product $\xi w \in \mathrm{Hom}(V, W)$ of $\xi \in V^*$ and $w \in W$, defined by (23.1), is nothing else than the valuewise product of the scalar-valued linear function ξ on V with w treated as a constant W-valued function on V.) As the component functions Γ_λ^μ of the standard flat connection d then are identically zero (see Example 20.2 and formula (20.3)), relations (24.1) and (24.4) (with $F = \xi$) become

$$\xi_{a,j} \;=\; \partial_j \xi_a \,-\, \Gamma_{ja}^b \xi_b\,, \qquad \nabla_{p_j} e^a \;=\; -\,\Gamma_{jb}^a e^b\,.$$

In other words, the components (relative to e^a) of the dual connection in η^* are obtained from those of the original connection in η by changing the sign and, in addition, switching the roles of upper and lower indices.

Suppose now that ∇ is a connection in the given vector bundle η over a manifold M and, in addition, we also have a fixed connection (also denoted by ∇) in the tangent bundle TM. A local C^s section ψ of η, $s \ge 1$, defined on an open set $U \subset M$, gives rise to the covariant derivative $\nabla\psi$, which is a local C^{s-1} section of $\mathrm{Hom}(TM, \eta)$, with the same domain U (§23). However, the bundle $\mathrm{Hom}(TM, \eta)$ now carries the connection obtained as in Proposition 24.1 using the original connections in TM and η. Thus, as long as $s \ge 2$, we can define the *second covariant derivative* of ψ, denoted by $\nabla^2\psi$, which is nothing else than the local section $\nabla(\nabla\psi)$ of $\mathrm{Hom}(TM, \mathrm{Hom}(TM, \eta))$. This can be repeated again, up to s times, leading to the *covariant derivatives of orders* $r = 1, 2, \dots, s$, with $\nabla^{(r)}\psi = \nabla(\nabla^{(r-1)}\psi)$, which live in Hom-bundles of increasing complexity (depending on r).

Let us now examine in some detail the second covariant derivative of ψ as above (with $s \ge 2$). At any given point $x \in U$, $\nabla(\nabla\psi)$ sends any vector $v \in T_x M$ to an operator, which in turn assigns to every $w \in T_x M$ the element

$$[\nabla_v(\nabla\psi)]w \,\in\, \eta_x\,. \tag{24.5}$$

However, from (24.2) (with $F = \nabla\psi$ and $\phi = w$), $[\nabla_v(\nabla\psi)]w = \nabla_v[(\nabla\psi)w] - (\nabla\psi)[\nabla_v w]$, i.e., for C^1 vector fields $, v, w$,

$$[\nabla_v(\nabla\psi)]w \;=\; \nabla_v\nabla_w\psi \,-\, \nabla_{\nabla_v w}\psi\,. \tag{24.6}$$

We can now establish the *Ricci-Weitzenböck identity*

$$[\nabla_v(\nabla\psi)]w \,-\, [\nabla_w(\nabla\psi)]v \;=\; R(w, v)\psi\,, \tag{24.7}$$

valid for any local C^2 section ψ of a C^∞ vector bundle η over a manifold M endowed with any connection ∇, with the curvature tensor R, and any fixed connection in TM which, in addition, is *torsionfree.* (About the non-torsionfree case, see Problem 6.) In fact, (24.7) is immediate from (24.6), (20.8) and the assumption that the right-hand side of (21.1) vanishes.

Since the dependence of (24.5) on v, w is real-bilinear, we have

$$[\nabla_v(\nabla\psi)]w \;=\; v^k w^j \psi^a{}_{,jk} e_a\,, \tag{24.8}$$

where $\psi^a{}_{,jk} = \psi^a{}_{,jk}(x)$ is the value at x of the function $[\nabla_{p_k}(\nabla\psi)]p_j$. Thus, from (24.6) and (20.4) (or, equivalently, from (24.4) applied to $F = \nabla\psi$),

$$\psi^a{}_{,jk} \,=\, \partial_k\psi^a{}_{,j} \,+\, \Gamma^a_{kb}\psi^b{}_{,j} \,-\, \Gamma^l_{kj}\psi^a{}_{,l}\,, \tag{24.9}$$

which can be further rewritten as

$$\psi^a{}_{,jk} = \partial_k\partial_j\psi^a + (\partial_k\Gamma^a_{jb})\psi^b + \Gamma^a_{kb}\partial_j\psi^b + \Gamma^a_{kb}\Gamma^b_{jc}\psi^c - \Gamma^l_{kj}\partial_l\psi^a - \Gamma^l_{kj}\Gamma^a_{lb}\psi^b.$$

From (24.8) we also easily obtain the component version of the Ricci-Weitzenböck identity (24.7):

$$\psi^a{}_{,jk} - \psi^a{}_{,kj} = R_{jkb}{}^a\psi^b. \tag{24.10}$$

In the case where $\eta = TM$ and the same tosionfree connection ∇ in TM is used in both roles, (24.10) becomes

$$w^l{}_{,jk} - w^l{}_{,kj} = R_{jks}{}^l w^s \tag{24.11}$$

for local C^2 vector fields w. We then also have (see Problem 7)

$$R_{js}w^s = w^s{}_{,js} - w^s{}_{,sj}\,. \tag{24.12}$$

Problems

1. Let ∇ be a connection in the tangent bundle of a manifold M, and let $f : U \to \mathbf{R}$ be a C^2 function on an open subset U of M. The second covariant derivative ∇df of f then can be defined as in the general case (with f treated as a local section of the product bundle $\eta = M \times \mathbf{R}$ endowed with the standard flat connection d). Verify that ∇df then coincides with the covariant derivative of the local section df of the cotangent bundle T^*M (§17) relative to the connection in T^*M dual to ∇, while relations (24.6) and (24.9) become

$$\text{i)}\;\; [\nabla_v(df)]w \,=\, d_v d_w f \,-\, d_{\nabla_v w} f\,, \qquad \text{ii)}\;\; f_{,jk} \,=\, \partial_k\partial_j f \,-\, \Gamma^l_{kj} f_{,l}\,. \tag{24.13}$$

2. Let M, ∇, U and f be as in Problem 1. The value of ∇df at every point $x \in U$ may be treated as a bilinear form, sending tangent vectors $v, w \in T_xM$ to $(\nabla df)(v,w) = [\nabla_v(df)]w = v^j w^k f_{,jk}$. If, in addition, ∇ is torsionfree, then, at every $x \in U$, the form $(\nabla df)(x)$ is symmetric. Derive this fact from:
 (a) formula (24.13.i),
 (b) relation (24.13.ii),
 (c) the Ricci-Weitzenböck identity (24.7) (or (24.10)).

3. Prove that the covariant differentiations of C^1 sections along any C^1 curve $t \mapsto x(t) \in M$ relative to a connection ∇ in a vector bundle η over a manifold M and its dual connection ∇ in η^* are related by

$$(\nabla_{\dot{x}}\xi)(\phi) \,=\, [\xi(\phi)]^{\cdot} \,-\, \xi(\nabla_{\dot{x}}\phi) \tag{24.14}$$

for such sections ϕ of η and ξ of η^*. (Hint below.)

4. Given a connection ∇ in the tangent bundle of a manifold M and a C^2 function $f : U \to \mathbf{R}$ on an open set $U \subset M$, verify that, for every geodesic $t \mapsto x(t) \in U$,

$$[f(x)]^{\cdot\cdot} \,=\, (\nabla df)(\dot{x},\dot{x})\,, \tag{24.15}$$

that is, $d^2[f(x(t))]/dt^2 = [(\nabla df)(x(t))](\dot{x}(t),\dot{x}(t))$. (Hint below.)

5. Verify that the connection in $\mathrm{Hom}(TM,\eta)$, described in Proposition 24.1 can, instead of (24.2), be also uniquely characterized by

$$\nabla_v(\xi\psi) \,=\, (\nabla_v\xi)\psi \,+\, \xi(\nabla_v\psi)$$

for local C^1 sections ξ of η^* and ψ of ζ. (Hint below.)

6. Generalize the Ricci-Weitzenböck identity (24.7) to the case where the connection in TM has an arbitrary (not necessarily vanishing) torsion tensor field T. Write the corresponding component version (as in (24.10)). (Hint below.)

7. Prove (24.12). (Hint below.)

Hint. In Problem 3, we may use formula $(\nabla_{\dot{x}}\xi)_a = \dot{\xi}_a - \Gamma^b_{ja}(x)\dot{x}^j\xi_b$, (that is, (22.1) for the dual connection) or, equivalently, the "axioms" (a) – (d) in §22.

Hint. In Problem 4, use (24.14) for $\eta = TM$, $\xi = (df)(x)$ and $\phi = \dot{x}$. Another option is the following direct argument: from the chain rule $d[f(x(t))]/dt = \dot{x}^j(t)(\partial_j f)(x(t))$ we obtain

$$\frac{d^2}{dt^2}\,f(x(t)) = \ddot{x}^j(t)(\partial_j f)(x(t)) + \dot{x}^j(t)\dot{x}^k(t)(\partial_j\partial_k f)(x(t))$$

for any C^2 function f and any C^2 curve $t\mapsto x(t)$ in a manifold M, and arbitrary local coordinates x^j in M. Now (24.15) is immediate from (24.13.ii) and (22.6).

Hint. In Problem 5, use (24.4) and the fact that

$$(\xi\psi)_a^\lambda = \xi_a\psi^\lambda\,.$$

Hint. In Problem 6, the formulae are

$$[\nabla_v(\nabla\psi)]w \,-\, [\nabla_w(\nabla\psi)]v \,=\, R(w,v)\psi \,+\, \nabla_{\mathrm{T}(w,v)}\psi\,.$$

$$\psi^a{}_{,jk} - \psi^a{}_{,kj} = R_{jkb}{}^a\psi^b \,+\, \mathrm{T}^l_{jk}\psi^a{}_{,l}\,,$$

Hint. In Problem 7, obtain (24.12) by contracting in $k = l$ the Ricci-Weitzenböck identity (24.11).

25. Variations of curves and the meaning of flatness

Topics: Sections of vector bundles defined along mappings from a rectangle into the base manifold; their partial covariant derivatives relative to a connection in the vector bundle; a component formula; partial derivatives of a rectangle mapping, treated as as vector fields along the mapping; a curvature formula of the Ricci-Weitzenböck type; manifolds with connections as configuration spaces of mechanical systems with constraints; geodesics as trajectories of free pointlike particles; the curvature as the infinitesimal shape-deformation factor for a geodesic segment set in free motion.

A C^k mapping F of a rectangle $X = [a,b]\times[c,d]$ into a manifold M may be referred to as a *variation of curves* $[a,b]\ni t\mapsto F^s(t) = F(t,s)\in M$, each of which corresponds to a fixed value of the *variation parameter* $s\in[c,d]$. When $F(a,s) = y$ and $F(b,s) = y$ for some $y,z\in M$ and all $s\in[c,d]$, F is also called a C^k *homotopy* with fixed endpoints between the curves F^c and F^d; if such a homotopy exists, one says that the curves F^c and F^d connecting y to z are C^k*-homotopic with fixed endpoints.*

Let η be a vector bundle over a manifold M, and let ϕ be a *section of* η *along a* C^k *mapping* $F: X\to M$, where $X = [a,b]\times[c,d]$, that is, an assignment of an element $\phi(t,s)$ of the fibre $\eta_{F(t,s)}$ to each $(t,s)\in X$. (In other words, ψ is a section of the pullback bundle $F^*\eta$.) We say that ϕ is *of class* C^k if its components ϕ^a relative to any local trivialization e_a of η are C^k-differentiable functions of (t,s). If $k \ge 1$, we can now define the *partial covariant derivatives* ϕ_t and ϕ_s relative to a fixed connection ∇ in η to be the C^{k-1} sections of η along F, obtained by covariant differentiation of ϕ treated as a section along the curve $F(\cdot,s)$ or $F(t,\cdot)$ (while s or t is kept fixed). Writing $x(t,s)$ instead of $F(t,s)$, we see that

$$\phi_t(t,s) = [\nabla_{[x(\cdot,s)]^\cdot}(\phi(\cdot,s))](t)\,, \qquad \phi_s(t,s) = [\nabla_{[x(t,\cdot)]^\cdot}(\phi(t,\cdot))](s)\,, \tag{25.1}$$

so that

$$\begin{aligned} \phi_t &= \phi_t^a(e_a\circ F)\,, \quad \text{where} \quad \phi_t^a = \frac{\partial\phi^a}{\partial t} + (\Gamma_{jb}^a\circ F)\frac{\partial F^j}{\partial t}\phi^b\,, \\ \phi_s &= \phi_s^a(e_a\circ F)\,, \quad \text{where} \quad \phi_s^a = \frac{\partial\phi^a}{\partial s} + (\Gamma_{jb}^a\circ F)\frac{\partial F^j}{\partial s}\phi^b\,. \end{aligned} \tag{25.2}$$

We call ϕ_t^a and ϕ_s^a the *component functions* of ϕ_t and ϕ_s. Let us also set (still with $x(t,s) = F(t,s)$)

$$x_t(t,s) = [x(\cdot,s)]^\cdot(t)\,, \qquad x_s(t,s) = [x(t,\cdot)]^\cdot(s)\,, \tag{25.3}$$

so that

$$x_t = x_t^j(p_j\circ x)\,, \qquad x_t^j = \frac{\partial x^j}{\partial t}\,, \qquad x_s = x_s^j(p_j\circ x)\,, \qquad x_s^j = \frac{\partial x^j}{\partial s}\,. \tag{25.4}$$

Taking in turn the partial covariant derivatives of ϕ_t and ϕ_s (when $k \ge 2$), we obtain the second-order partial covariant derivatives $\phi_{tt} = (\phi_t)_t$, $\phi_{ts} = (\phi_t)_s$, $\phi_{st} = (\phi_s)_t$ and $\phi_{ss} = (\phi_s)_s$. It is now easy to verify that, if $k \ge 2$,

$$R^\nabla(x_t,x_s)\phi = \phi_{ts} - \phi_{st}\,. \tag{25.5}$$

where both sides are C^{k-2} sections of η along the mapping $(t,s) \mapsto x(t,s)$. (In fact, (20.10) and (25.2) easily yield $R_{jkb}{}^a x_t^j x_s^k \phi^b = \phi_{ts}^a - \phi_{st}^a$.)

LEMMA 25.1. *Suppose that ∇ is a flat connection in a vector bundle η over a manifold M, while $x,y \in M$ and $F^0, F^1 : [a,b] \to M$ are C^2 curves in M that connect x to y. If F^0 and F^1 are C^2-homotopic with fixed endpoints, then they give rise to the same ∇-parallel transport $\eta_x \to \eta_y$.*

PROOF. Choose a fixed-endpoints C^2 homotopy $F : [a,b]\times[0,1] \to M$ between F^0 and F^1. For any given $\psi \in \eta_x$, let $\phi(t,s) \in \eta_{F(t,s)}$ be the image of ψ under the parallel transport along the curve $[a,t] \ni t' \mapsto F^s(t') = F(t',s)$. Since $R^\nabla = 0$ and $\phi_t = 0$, (25.5) yields $\phi_{st} = 0$, that is, ϕ_s is parallel in the t direction. Therefore $\phi_s = 0$, as $\phi_s(a,s) = 0$ (due to our initial conditions $F(a,s) = x$, $\phi(a,s) = \psi$). Setting $t = b$, we now obtain constancy of the curve $[0,1] \ni s \mapsto \phi(b,s) \in \eta_y$. $\square$

The following basic classification result states that any flat connection looks, locally, like the standard flat connection in a product bundle:

LEMMA 25.2. *Any flat connection ∇ in a vector bundle η over a manifold M admits, locally, a local trivialization e_a consisting of parallel sections. In other words, every point of M has a neighborhood U such that for each $y \in U$ and any $\phi \in \eta_y$ there exists a unique parallel local section ψ of η, defined on U, with $\psi(y) = \phi$.*

PROOF. Fix $x \in M$ and identify a neighborhood U of x with an open convex subset of $\mathbf{R}^n$, $n = \dim M$. For any given $\phi \in \eta_x$ we can construct a parallel section ψ of η restricted to U with $\psi(x) = \phi$ be defining $\psi(y)$, for $y \in U$, to be the parallel translate of ϕ along any C^2 curve connecting x to y in U; by Lemma 25.1, this does not depend on the choice of the curve, as two such curves admit an obvious fixed-endpoints C^2 homotopy due to convexity of U. Our e_a now may to be chosen to be the parallel sections of η restricted to U with $e_a(x)$ forming any prescribed basis of η_x. □

Problems

1. Prove (25.5) for C^2 sections ϕ, along C^2 mappings $[a, b] \times [c, d] \ni (t, s) \mapsto x(t, s) \in M$, in a C^∞ vector bundle η with a fixed connection ∇ over a manifold M, where the partial covariant derivatives ϕ_t, ϕ_s of ϕ and partial-derivative vector fields x_t, x_s are given by (25.1), (25.3), while $\phi_{ts} = (\phi_t)_s$, $\phi_{st} = (\phi_s)_t$. (Hint below.)

2. Let T be the torsion tensor field of a connection ∇ on a manifold M. Prove that, for any C^2 mapping $(t, s) \mapsto x(t, s) \in M$,

$$x_{ts} - x_{st} = T(x_s, x_t)\,,$$

where $t \in [a, b]$ and $s \in [c, d]$, x_t, x_s are the partial-derivative vector fields given by (25.3) or (25.4), and $x_{ts} = (x_t)_s$, $x_{st} = (x_s)_t$.

3. Let ∇ be a torsionfree connection on a manifold M. Verify that

$$x_{tst} = x_{tts} - R^\nabla(x_t, x_s)x_t\,, \qquad x_{sst} = x_{sts} - R^\nabla(x_t, x_s)x_s \tag{25.6}$$

for any C^3 mapping $(t, s) \mapsto x(t, s) inM$.

4. Given a manifold M with a torsionfree connection ∇ in TM, interpreted as the configuration space of a mechanical system with constraints, let a C^∞ mapping of a rectangle $[a, b] \times [c, d]$ into M consist of time-parametrized trajectories $[a, b] \ni t \mapsto x(t, s)$ of pointlike particles labeled with real numbers $s \in [c, d]$. Suppose that each particle is moving freely, i.e., its trajectory is a geodesic, so that

$$x_{tt} = 0$$

identically in K, and that at the initial moment $t = a$ the particles form a geodesic segment (a "straight bar"):

$$x_{ss}|_{t=a} = 0\,,$$

and are set in motion so as to have "the same" initial velocity:

$$x_{ts}|_{t=a} = 0\,.$$

Show that, if ∇ is flat in the sense that its curvature R^∇ is identically zero, the bar will retain its straight shape at all times, that is,

$$x_{ss} = 0$$

everywhere in K.

5. Let ∇ be a connection on a manifold M (i.e., in the tangent bundle TM). Recall that the *geodesic curvature* of a C^2 curve $\gamma : I \to M$ is the vector field $\nabla_{\dot\gamma}\dot\gamma$ along γ (which is identically zero if and only if γ is a geodesic). Verify that, given another interval I' and a C^2 function $\varphi : I' \to I$, the geodesic curvature of the composite curve $\gamma \circ \varphi : I' \to M$ is given by

$$\nabla_{[\gamma\circ\varphi]^{\cdot}}[\gamma \circ \varphi]^{\cdot} = (\nabla_{\dot\gamma}\dot\gamma \circ \varphi)\dot\varphi^2 + (\dot\gamma \circ \varphi)\ddot\varphi\,. \tag{25.7}$$

(Hint below.)

6. Let ∇ be a connection on a manifold M, and let $\gamma : I \to M$ be a C^2 curve defined on an interval $I \subset \mathbf{R}$. Prove that the following two conditions are equivalent:
(a) $\nabla_{\dot\gamma}\dot\gamma = h\dot\gamma$ for some continuous function $h : I \to \mathbf{R}$.
(b) γ is an *unparametrized geodesic* in the sense that there exists an interval I' and a C^2 function $\varphi : I' \to I$ such that $\varphi(I') = I$, $\dot\varphi \neq 0$ everywhere in I', and $\gamma \circ \varphi : I' \to M$ is a geodesic. (Hint below.)

7. Is the word 'continuous' necessary in (a) of Problem 6? (Hint below.)

Hint. In Problem 1, use (25.2) to obtain $\phi_{ts} - \phi_{st} = R_{jka}{}^b x^j_t x^k_s \phi^a e_b$ with $R_{jka}{}^b$ given by (27.10).

Hint. In Problem 5, note that the components of $\nabla_{\dot\gamma}\dot\gamma$ are given by the left-hand side of 22.5).

Hint. In Problem 6, (b) implies (a) in view of (25.7). On the other hand, assuming (a), choose an antiderivative $H : I \to \mathbf{R}$ for h and an antiderivative $\Psi : I \to \mathbf{R}$ for e^{-H}, and use the inverse mapping $\varphi = \Psi^{-1} : I' \to I$, where $I' = \Psi(I)$.

Hint. In Problem 8, yes: in $\mathbf{R}^2$ with the standard flat connection, consider $\gamma : \mathbf{R} \to \mathbf{R}^2$ given by $\gamma(t) = (f(t), f(-t))$, where $f : \mathbf{R} \to \mathbf{R}$ is the C^∞ function with $f(s) = e^{-1/s}$ for $s > 0$ and $f(s) = 0$ for $s \leq 0$ (Problem 12 in §5).

26. Bianchi identities

Topics: Sections parallel at a point; their existence for any prescribed value at the point; vanishing of connection components at a point; vanishing of connection components at a point; the effect of the Hom operation on curvature; the case of torsionfree connections in the tangent bundle; the first Bianchi identity for torsionfree connections in tangent bundles; the second Bianchi identity for arbitrary connections in vector bundles (when the connection in TM is torsionfree).

Let ∇ be a connection in a C^∞ vector bundle η over a manifold M and let $z \in M$. A local C^1 section ψ of η is said to be ∇-*parallel at* z (or just *parallel at* z) if its domain contains z and $(\nabla\psi)(z) = 0$, that is, $\nabla_v\psi = 0$ for each vector $v \in T_zM$. Similarly, a local trivialization e_a of η defined on a neighborhood of z is called *parallel at the point* z if so are all the e_a. Note that the latter condition means that $(\nabla e_a)(z) = 0$, i.e.,

$$\Gamma^b_{ja}(z) \,=\, 0 \tag{26.1}$$

in *every* coordinate system x^j at z. Assuming (26.1) at the given point z, we obtain

$$\psi^a{}_{,j} \,=\, \partial_j\psi^a \qquad \text{at } z$$

for any local C^1 section ψ of η defined near z, while, from (20.10),

$$R_{jka}{}^b \,=\, \partial_k\Gamma^b_{ja} \,-\, \partial_j\Gamma^b_{ka} \qquad \text{at } z\,. \tag{26.2}$$

Lemma 26.1. *For any connection ∇ in a C^∞ vector bundle η over a manifold M, any point $z \in M$, and any $\phi \in \eta_z$, there exists a local C^∞ section ψ of η defined near z, parallel at z, and such that $\psi(z) = \phi$.*

PROOF. We need (23.5) to be zero just at the point z, with $\psi^a(z) = \phi^a$, which may be achieved e.g. by choosing the ψ^a be suitable (nonhomogeneous) linear functions of the coordinates used. $\square$

Applying Lemma 26.1 to each vector of any fixed basis of η_z, we obtain

COROLLARY 26.2. *Let ∇ be a connection in a C^∞ vector bundle η over a manifold M, and let $z \in M$. Then there exists a local trivialization e_a of η defined on a neighborhood of z and parallel at z.*

Lemma 26.1 (as well as its more refined version for torsionfree connections in tangent bundles, cf. Problem 2) may lead to significant simplifications in various calculation-type arguments, as we will see later in §26. Right now we will use it to calculate the curvature R^{Hom} the connection ∇ in $\mathrm{Hom}(\eta, \eta')$ corresponding (as in Proposition 24.1) to two given connections (both also denoted by ∇), with curvature tensors R, R', in vector bundles η, η' over a manifold M. Namely, we have, for any $x \in M$, $v, w \in T_xM$ and $F \in \mathrm{Hom}(\eta_x, \eta'_x)$,

$$R^{\mathrm{Hom}}(v, w)F \,=\, R'(v, w) \circ F \,-\, F \circ R(v, w)\,. \tag{26.3}$$

In fact, by (20.8), relation (26.3) amounts to

$$\nabla_w \nabla_v F \,-\, \nabla_v \nabla_w F \,+\, \nabla_{[v,w]} F \,=\, R'(v, w) \circ F \,-\, F \circ R(v, w) \tag{26.4}$$

for any local C^2 section F of $\mathrm{Hom}(\eta, \eta')$ and C^1 vector fields v, w in M, all defined on the same open set. Equality (26.4) is in turn easily verified if one applied both sides to any local C^1 section ϕ of η, using (24.2) and assuming that both F and $,\phi$ are parallel at the point in question.

In the case where $\eta = \eta' = \eta$ and the connection ∇ in η coincides with ∇ in η', (26.3) (with $R = R' = R$) becomes

$$R^{\mathrm{Hom}}(v, w)F \,=\, [R(v, w), F]\,,$$

$[\,,\,]$ being the ordinary fibrewise commutator of bundle morphisms.

The curvature R of every torsionfree C^∞ connection ∇ in the tangent bundle TM of a C^∞ manifold M satisfies the relation

$$R_{jkl}{}^m + R_{klj}{}^m + R_{ljk}{}^m \,=\, 0 \tag{26.5}$$

known as the *first Bianchi identity*. (For its analogue in the non-torsionfree case, see Problem 3.) As in the case of relation (24.10), the content of (26.5) does not depend on the local coordinates x^j, since (26.5) simply states that

$$R(u, v)w + R(v, w)u + R(w, u)v \,=\, 0 \tag{26.6}$$

for any vectors u, v, w tangent to M at any point x. A more detailed version of (26.6) would read

$$R_x(u, v)w + R_x(v, w)u + R_x(w, u)v \,=\, 0\,.$$

To prove (26.5), use (20.10) along with the assumption that ∇ is torsionfree, i.e., $\Gamma^l_{kj} = \Gamma^l_{jk}$. (We could also fix $z \in M$ and choose coordinates with $\Gamma^l_{jk}(z) = 0$, as in Problem 2 of §26, making the argument even shorter.)

Suppose now that we are given a C^∞ vector bundle η over a manifold M endowed with an arbitrary C^∞ connection ∇. The curvature tensor field $R = R^\nabla$ of ∇ then has the component functions $R_{jkb}{}^a$ given by (20.10). Any fixed connection in TM then can be used (along with ∇) to form the covariant derivative ∇R of R; as R is a section of $\mathrm{Hom}(TM, \mathrm{Hom}(TM, \mathrm{Hom}(\eta, \eta)))$, ∇R then will be a section of $\mathrm{Hom}(TM, \mathrm{Hom}(TM, \mathrm{Hom}(TM, \mathrm{Hom}(\eta, \eta))))$, so that at each point $x \in M$ its value is a trilinear mapping

$$(\nabla R)_x : T_xM \times T_xM \times T_xM \to \mathrm{Hom}(\eta_x, \eta_x)$$

with $(\nabla R)(u, v, w) = (\nabla_u R)(v, w)$ for $u, v, w \in T_xM$. In a local trivialization e_a and coordinates x^j, sharing the same domain $U \subset M$, ∇R has the component functions $R_{jka}{}^b{}_{,l} = e^b[(\nabla R)(p_l, p_j, p_k)e_a]$, so that

$$(\nabla R)(u, v, w)\phi = u^l v^j w^k \phi^a R_{jka}{}^b{}_{,l}(x) e_b(x)$$

whenever $x \in U$, $u, v, w \in T_xM$ and $\phi \in \eta_x$. In terms of the connection component functions,

$$R_{jka}{}^b{}_{,l} = \partial_l R_{jka}{}^b - \Gamma_{lj}^m R_{mka}{}^b - \Gamma_{lk}^m R_{jma}{}^b - \Gamma_{la}^c R_{jkc}{}^b + \Gamma_{lc}^b R_{jka}{}^c . \tag{26.7}$$

In the case where the connection in TM is *torsionfree*, ∇R satisfies the so-called *second Bianchi identity*

$$R_{jka}{}^b{}_{,l} + R_{kla}{}^b{}_{,j} + R_{lja}{}^b{}_{,k} = 0 . \tag{26.8}$$

(The non-torsionfree case is discussed in Problem 4.) As before, the meaning of (26.8) does not depend on the local trivialization e_a and coordinates x^j; in fact, (26.8) can be rewritten as

$$(\nabla R)(u, v, w) + (\nabla R)(v, w, u) + (\nabla R)(w, u, v) = 0$$

or, equivalently,

$$(\nabla_u R)(v, w) + (\nabla_v R)(w, u) + (\nabla_w R)(u, v) = 0$$

for any point $x \in M$ and any vectors $u, v, w \in T_xM$.

To prove (26.8), fix $z \in M$ and choose e_a at z with (26.1) (for any x^j), which is possible in view of Corollary 26.2. Combining (20.10) and (26.7) and using (26.1) we obtain

$$R_{jka}{}^b{}_{,l} = \partial_l \partial_k \Gamma_{ja}^b - \partial_l \partial_j \Gamma_{ka}^b - \Gamma_{lj}^m R_{mka}{}^b - \Gamma_{lk}^m R_{jma}{}^b \qquad \text{at } z .$$

Relation (26.8) now is a trivial consequence of the (skew)symmetry of the Γ_{jk}^l and $R_{jka}{}^b$ in j, k. (Again, this proof can be made even simpler with the aid of Problem 2.)

Problems

1. Let ∇ be a C^∞ connection in the tangent bundle of a C^∞ manifold M, and let x^j, $x^{j'}$ be two coordinate systems in M, both defined near a point $z \in M$. Verify that, for the local trivialization $p_{j'}$ of TM to be parallel at z, it is necessary and sufficient that (Hint below.)

$$\partial_j \partial_k x^{k'} = \Gamma_{jk}^l p_l^{k'} \qquad \text{at } z . \tag{26.9}$$

2. Let T be the torsion tensor field of a C^∞ connection ∇ in the tangent bundle of a manifold M. For any point $z \in M$, prove that $\mathrm{T}(z) = 0$ if and only if there exists a coordinate system x^j at z such that $\Gamma_{jk}^l(z) = 0$, i.e., the local trivialization p_j of TM is parallel at z. (Hint below.)
3. Generalize the first Bianchi identity (26.5) to the case where the connection in TM is not necessarily torsionfree (with the torsion tensor field T appearing on the right-hand side). State the corresponding generalization of the coordinate-free formula (26.6). (Hint below.)

4. The same for the second Bianchi identity. (Hint below.)

Hint. In Problem 1, $\nabla_{p_j} p_{j'} = \nabla_{p_j}(p_{j'}^m p_m) = (\Gamma_{jm}^l p_{j'}^m + \partial_j p_{j'}^l) p_l$, which vanishes at z if and only if so does $p_k^{j'} p_l^{k'} (\Gamma_{jm}^l p_{j'}^m + \partial_j p_{j'}^l) = \Gamma_{jk}^l p_l^{k'} - p_k^{j'} p_{j'}^l \partial_j p_l^{k'}$ (as $p_l^{k'} p_{j'}^l = \delta_{j'}^{k'}$ is constant), while $p_k^{j'} p_{j'}^l \partial_j p_l^{k'} = \partial_j p_k^{k'} = \partial_j \partial_k x^{k'}$.

Hint. In Problem 2, necessity is obvious as $\mathrm{T}_{jk}^l = \Gamma_{jk}^l - \Gamma_{kj}^l$; for sufficiency, fix some coordinates x^j at z and find the new ones, $x^{j'}$, with the required property, by choosing them so as to satisfy (26.9) at z, which is possible due to symmetry of $\Gamma_{jk}^l(z)$ in j, k. We may, e.g., let $x^{j'}$ be (nonhomogeneous) quadratic functions of the x^j.

Hint. In Problems 3 and 4, the formulae are

$$R_{jkl}{}^m + R_{klj}{}^m + R_{ljk}{}^m + \mathrm{T}_{jk,l}^m + \mathrm{T}_{kl,j}^m + \mathrm{T}_{lj,k}^m = 0\,,$$

$$R(u,v)w + R(v,w)u + R(w,u)v + (\nabla_u \mathrm{T})(v,w) + (\nabla_v \mathrm{T})(w,u) + (\nabla_w \mathrm{T})(u,v) = 0\,,$$

$$R_{jka}{}^b{}_{,l} + R_{kla}{}^b{}_{,j} + R_{lja}{}^b{}_{,k} + \mathrm{T}_{jk}^m R_{lma}{}^b + \mathrm{T}_{kl}^m R_{jma}{}^b + \mathrm{T}_{lj}^m R_{kma}{}^b = 0\,,$$

$$\begin{aligned}(\nabla_u R)(v,w) &+ (\nabla_v R)(w,u) + (\nabla_w R)(u,v)\\ &+ R(u, \mathrm{T}(v,w)) + R(v, \mathrm{T}(w,u)) + R(w, \mathrm{T}(u,v)) = 0\,.\end{aligned}$$

27. Further operations on connections

Topics: Pullbacks of connections: expression in terms of component functions, covariant derivatives along curves, and the effect on curvature; direct sums of connections; projections of connections in a direct-sum bundle; generalized convex combinations of connections; more on the dual; existence of connections in vector bundles over compact manifolds; on connections: the effect of selected operations on curvature; the description of operations in terms of component functions and of the corresponding covariant differentiation of C^1 sections defined only along C^1 curves.

Recall that, given manifolds M, N, a C^∞ mapping $F: M \to N$, and a C^∞ vector bundle η over M, we denote by $F^*\eta$ the corresponding pullback bundle over M with the fibres $(F^*\eta)_x = \eta_{F(x)}$, $x \in M$. (See §17.) Any connection ∇ in η then gives rise to the *pullback connection* $F^*\nabla$ in $F^*\eta$, uniquely characterized by the requirement that

$$(F^*\nabla)_v (F^*\psi) = \nabla_{dF_x v}\psi \in \eta_{F(x)} = (F^*\eta)_x \tag{27.1}$$

whenever $x \in M$, $v \in T_x M$ and ψ is a local C^1 section of η defined in a neighborhood of $F(x)$ (while $F^*\psi$ is the pullback of ψ to $F^*\eta$, with $(F^*\psi)_x = \psi_{F(x)}$). The component functions Γ_{ja}^b of $F^*\nabla$ relative to any local coordinates x^j in M and the pullback trivialization F^*e_a of $F^*\eta$ obtained from any C^∞ local trivialization e_a of η, can be expressed as

$$\Gamma_{ja}^b = (\partial_j F^\lambda)(\Gamma_{\lambda a}^b \circ F) \tag{27.2}$$

in the intersection of the respective domains, where y^λ are any local coordinates in N, and $\Gamma_{\lambda a}^b$ are the corresponding component functions of ∇. Thus, any C^1 section ϕ along a C^1 curve $\gamma: I \to M$ of the pullback bundle $F^*\eta$ is an assignment $I \ni t \mapsto \phi(t) \in (F^*\eta)_{\gamma(t)} = \eta_{F(\gamma(t))}$, so that it is also a C^1 section along the image C^1 curve $F \circ \gamma : I \to N$ of the original bundle η. Then

$$(F^*\nabla)_{\dot\gamma}\phi = \nabla_{[F\circ\gamma]^\cdot}\phi\,. \tag{27.3}$$

(See Remark 27.1 and Problem 2.) Furthermore, the curvatures $R^{F^*\nabla}$ and R^∇ of $F^*\nabla$ and ∇ are related by

$$R^{F^*\nabla}(v,w) \,=\, R^\nabla(dF_xv, dF_xw) \tag{27.4}$$

for any $x \in M$ and $v, w \in T_xM$, both sides being linear operators sending the fibre $(F^*\eta)_x = \eta_{F(x)}$ into itself (see Problem 3).

REMARK 27.1. The operation of covariant differentiation of C^1 sections ϕ of η along any C^1 curve $\gamma : I \to M$ in a C^∞ real or complex vector bundle η with a connection ∇ over a manifold M determines ∇ uniquely via condition (d) in §22. Thus, $F^*\nabla$ can, equivalently, be characterized as the unique connection with (27.3). For a similar characterization of the dual connection in the dual bundle, see Problem 3 in §24.

We begin with more constructions leading from given connections to new ones.

EXAMPLE 27.2. Given connections $\nabla^{(1)}, \dots, \nabla^{(p)}$ in vector bundles $\eta^1, \dots, \eta^p$, all with the same base manifold M, we define the *direct-sum connection* $\nabla = \nabla^{(1)} \oplus \nabla^{(p)}$ in the direct-sum vector bundle $\eta = \eta^1 \oplus \ldots \oplus \eta^p$ (Example 17.3) by

$$\nabla_v(\psi_1, \dots, \psi_p) \,=\, (\nabla_v^{(1)}\psi_1, \dots \nabla_v^{(p)}\psi_p)$$

for any local C^1 section $\psi = (\psi_1, \dots, \psi_p)$ of η and any vector v tangent to M at a point of the domain of ψ. (See also Problems 6 and 13.)

EXAMPLE 27.3. An arbitrary connection ∇ in a direct-sum vector bundle $\eta = \eta^+ \oplus \eta^-$ can be *projected* onto connections in the summand bundles $\eta^\pm$, denoted by $\nabla^\pm$ and defined by

$$\nabla_v^\pm \psi \,=\, (\nabla_v\psi)^\pm \tag{27.5}$$

for local C^1 sections ψ of $\eta^\pm$. Note that, since $\eta^\pm \subset \eta$, ψ then also is a local C^1 section of η, so that $\nabla_v\psi$ on the right-hand side makes sense. Also, for any section ϕ (or an element ϕ of a fibre) of η, we use the symbol $\phi^\pm$ for the $\eta^\pm$ component of ϕ. In (27.5) this is applied to $\phi = \nabla_v\psi$. See Problems 6 and 12.

EXAMPLE 27.4. A *generalized convex combination* of connections $\nabla^{(1)}, \dots, \nabla^{(p)}$ in a real/complex vector bundle η over a manifold M has the form

$$\nabla \,=\, f_1\nabla^{(1)} \,+\, \ldots \,+\, f_p\nabla^{(p)} \tag{27.6}$$

with any fixed real/complex valued C^∞ functions $f_1, \dots, f_p$ on M such that

$$f_1 \,+\, \ldots \,+\, f_p \,=\, 1\,.$$

As an operator associating $\nabla_w\psi$ with a local C^1 section ψ of η and a local vector field w in M, this ∇ clearly is a connection, and in particular $(\nabla_w\psi)(x)$ depends on w only through $w(x)$. See also Problem 14.

The operation of covariant differentiation of C^1 sections ϕ of η along any C^1 curve $\gamma : I \to M$ in a C^∞ real or complex vector bundle η with a connection ∇ over a manifold M determines ∇ uniquely (condition (d) in §22), and is often much easier to work with than the connection itself. Therefore it is useful to have a description of this operation for the connection obtained from some given connection(s) through a specific constructions. In the case of the pullback (or, dual-bundle) construction, such a description is provided by formula (27.3) (or, respectively, (24.14)). For some of the other constructions we have discussed, see Problem 6.

Problems

1. Prove the existence and uniqueness of the pullback connection $F^*\nabla$ with (27.1). (Hint below.)

2. Prove (27.3). (Hint below.)

3. Prove the curvature formula (27.4) for pullbacks of connections. (Hint below.)

4. Let ϕ be a C^1 section along a C^1 curve $\gamma : I \to M$ of the bundle $F^*\eta$ over a manifold M obtained by pulling back a C^∞ vector bundle η over a manifold N via a C^∞ mapping $F : M \to N$. Verify that ϕ then may be viewed as a C^1 section along the curve $F \circ \gamma : I \to N$ of the bundle η and, for any connection ∇ in η we have (27.3). (Hint below.)

5. Given η, M, N, γ, I and ∇ as in Problem 4, and any real numbers $a, b \in I$, show that the ∇-parallel transport along $F \circ \gamma$ from a to b (§22) coincides with the $F^*\nabla$-parallel transport along γ from a to b.

6. Verify that the covariant derivative of any C^1 section of the vector bundle in question along a C^1 curve $I \ni t \mapsto x(t) \in M$ in the base manifold, relative to the connection ∇ obtained as in Example 27.2, Proposition 24.1 or Example 27.3 is given, respectively, by $\nabla_{\dot x}(\phi_1, \dots, \phi_p) = (\nabla_{\dot x}^{(1)}\phi_1, \dots \nabla_{\dot x}^{(p)}\phi_p)$, $(\nabla_{\dot x}F)(\phi) = [F\phi]^{\cdot} - F\nabla_{\dot x}\phi$, and $\nabla_{\dot x}^{\pm}\phi = (\nabla_{\dot x}\phi)^{\pm}$. (In the second equality, F and ϕ are C^1 sections of $\mathrm{Hom}(\eta, \zeta)$ and η along the curve.) (Hint below.)

7. The *adjoint* F^* of a linear operator $F : V \to W$ between real or complex vector spaces V, W is the linear operator $F^* : W^* \to V^*$ defined by $F^*(f) = f \circ F$. Show that, if V are finite-dimensional, with bases e_a for V, e_λ for W, in which F is represented by the matrix $[F_a^\lambda]$ in (i.e., $F(e_a) = F_a^\lambda e_\lambda$), then the matrix of F^* relative to the dual bases e^a of V^* and e^λ of W^* is the transpose of $[F_a^\lambda]$ in the sense that $F^*(e^\lambda) = F_a^\lambda e^a$.

8. Let us denote by ∇^* (instead of the customary ∇) the connection in η^* dual to the given connection ∇ in a vector bundle η over a manifold M. Verify that the curvatures R of ∇ and R^* of ∇^* are related by

$$R^*(v, w) \;=\; -\,[R(v, w)]^* \tag{27.7}$$

for any $x \in M$ and $v, w \in T_xM$. Here $[R(v, w)]^*$ denotes the adjoint $\eta_x^* \to \eta_x^*$ of $R(v, w) : \eta_x \to \eta_x$ (Problem 7), and the point x is, as usual, suppressed from the notation (so that $R(v, w)$ stands for $[R(x)](v, w)$). Show that the component version of (27.7) is

$$R_{jk}{}^{b}{}_{a} \;=\; -\,R_{jka}{}^{b}\,,$$

$R_{jk}{}^{b}{}_{a}$ being the component functions of R^*, with $R_{jk}{}^{b}{}_{a} = [R^*(p_j, p_k)e^b](e_a)$.

9. Let η be a C^∞ real/complex vector bundle of some fibre dimension q over a C^∞ manifold M. Show that, for any $x \in M$, there exist global C^∞ sections $\psi_1, \dots, \psi_q$ of η whose restrictions to some neighborhood of x form a local trivialization of η. (Hint below.)

10. Suppose we are given a C^∞ real/complex vector bundle η over a compact C^∞ manifold M satisfying the countability axiom (§14). Prove that there exists a finite collection $\psi_1, \dots, \psi_l$ of global C^∞ sections of η such that the values $\psi_1(x), \dots, \psi_l(x)$ span the fibre η_x at every point $x \in M$. (Hint below.)

11. For η and M as in Problem 10, show that there exist a product vector bundle $M \times V$ and a C^∞ vector-bundle morphism $F : M \times V \to \eta$ which is *surjective* in the sense that so is $F_x : V \to \eta_x$ for each $x \in M$. (Hint below.)

12. Generalize the construction of Example 27.3 to the case of direct sums with an arbitrary finite number of summands.

13. Describe what happens with the curvature under the direct-sum operation for connections (Example 27.2).

14. Given an affine space $(M, V, +)$, points $x_1, \ldots, x_r \in M$ and scalars $\lambda_1, \ldots, \lambda_r$ with $\lambda_1 + \ldots + \lambda_r = 1$, one defines the *affine combination* of $x_1, \ldots, x_r$ with the coefficients $\lambda_1, \ldots, \lambda_r$ to be the point, denoted by

$$x = \lambda_1 x_1 + \ldots + \lambda_r x_r \,, \tag{27.8}$$

and obtained by identifying M with its translation space V (with the aid of a fixed "origin" $o \in M$) and then forming the corresponding combination of vectors. In other words,

$$x = o + \lambda_1(x_1 - o) + \ldots + \lambda_r(x_r - o) \,.$$

Verify that this definition is correct (i.e., independent of the choice of o), and that formula (27.6) represents, at any given point of the base manifold, a special case of (27.8). (Another important application of (27.8) is in the case where $\dim M = 3$ and $x_1, \ldots, x_r$ are locations of pointlike particles with masses $m_1, \ldots, m_r$. The point (27.8) with $\lambda_j = m_j/m$ for $j = 1, \ldots, r$, where $m = m_1 + \ldots + m_r$, then is called the system's *center of mass.*)

Hint. In Problem 1, note that (27.1) implies

$$(F^*\nabla)_{p_j}(F^*e_a) = (\partial_j F^\lambda)(\Gamma^b_{\lambda a} \circ F)(F^*e_b) \qquad \text{whenever} \qquad \nabla_{p_\lambda} e_a = \Gamma^b_{\lambda a} e_b \,, \tag{27.9}$$

and hence uniqueness of $F^*\nabla$. Using (27.2) as a definition of $F^*\nabla$, we obtain the existence assertion.

Hint. In Problem 2, use (27.2) and the component formula (22.1), i.e., $(\nabla_{\dot\gamma}\phi)^a = \dot\phi^a + (\Gamma^a_{jb} \circ \gamma)\dot\gamma^j \phi^b$ for $\nabla_{\dot\gamma}\phi$.

Hint. In Problem 3, use (27.2) to establish the component version

$$R_{jka}{}^b = (\partial_j F^\lambda)(\partial_k F^\mu)(R_{\lambda\mu a}{}^b \circ F) \tag{27.10}$$

of (27.9) relative to local coordinates x^j in M, local coordinates y^λ in N, a C^∞ local trivialization e_a of η, and the corresponding F^*e_a for $F^*\eta$.

Hint. In Problem 4, use formulae (22.1) and (27.2).

Hint. In Problem 6: see the hint for Problem 3 in §24.

Hint. In Problem 9, set $\psi_a = \phi e_a$ for a local trivialization e_a of η defined on a neighborhood U of x and a C^∞ cut-off function $\phi : M \to \mathbf{R}$ chosen as in Problem 19 of §6 for x and the closed subset $K = M \setminus U$.

Hint. In Problem 10, use Problem 9 and the Borel-Heine theorem.

Hint. In Problem 11, choose the ϕ_λ with $\lambda = 1, \ldots, l$ as in Problem 10 and, for $V = \mathbf{R}^n$ or $V = \mathbf{C}^n$ define $F_x : V \to \eta_x$ by $F_x(c^1, \ldots, c^l) = c^\lambda \phi_\lambda(x)$.

CHAPTER 6

Riemannian Distance Geometry

28. Fibre metrics

Topics: Pseudo-Riemannian and pseudo-Hermitian fibre metrics in vector bundles; the positive-definite case; Riemannian and pseudo-Riemannian manifolds; vector and affine spaces; immersion pullbacks of Riemannian metrics; isometries; isometric immersions/embeddings and submanifold metrics; Euclidean spheres; hyperbolic spaces.

Let η be a real (or, complex) vector bundle over a manifold M. By a *pseudo-Riemannian* (or, respectively, *pseudo-Hermitian*) *fibre metric* in η we mean any mapping g assigning to every point $x \in M$ a scalar-valued bilinear symmetric (or, sesquilinear Hermitian) nondegenerate form g_x on the fibre η_x. (For a definition of nondegeneracy, see Problem 16 in §12.) If g_x happens to be positive definite for every x, one drops the prefix 'pseudo' and refers to the fibre metric g as *Riemannian* or *Hermitian.* In addition, we will always require such a fibre metric g to be C^∞-differentiable; this makes sense as g is a section of a specific C^∞ vector bundle, namely, $\mathrm{Hom}\,(\eta, \eta^*)$ or $\mathrm{Hom}\,(\eta, \overline{\eta}^*)$. (In fact, g associates with every $x \in M$ the operator $\eta_x \ni \phi \mapsto g_x(\phi, \cdot\,)$ valued in η_x^* or, respectively, $\overline{\eta}_x^*$.)

For η, M, g as above, $x \in M$, and $\psi, \phi \in \eta_x$, we will often simplify the notation writing $g(\psi, \phi)$, or just $\langle \psi, \phi \rangle$, instead of $g_x(\psi, \phi)$. The symbols $g(\psi, \phi)$ and $\langle \psi, \phi \rangle$ will be also used for the scalar-valued function $U \ni x \mapsto g_x(\psi_x, \phi_x)$, whenever ψ, ϕ are local *sections* of η with the domain $U \subset M$.

A pseudo-Riemannian fibre metric g in the tangent bundle TM of a manifold M is referred to as a *pseudo-Riemannian metric* on the manifold M, and the pair (M, g) is then called a *pseudo-Riemannian manifold.* Again, one omits 'pseudo' if g is positive definite at every point; when M is connected, 'every point' may be replaced here by 'some point' (see Problem 10).

Example 28.1. In a product vector bundle $\eta = M \times \mathbf{F}$ one has a special class of *constant* fibre metrics g, obtained by setting $g_x = \langle\,,\rangle$ for $x \in M$, where $\langle\,,\rangle$ is any fixed scalar-valued bilinear symmetric (or, sesquilinear Hermitian) nondegenerate form on $\mathbf{F}$. Similarly, if M a finite-dimensional real affine space (or a nonempty open subset thereof), the tangent bundle TM is naturally isomorphic to $M \times V$, where V is the translation vector space (Example 5.2), and so we have a distinguished class of *constant* pseudo-Riemannian metrics on M. Such a constant metric on an affine space is called a *pseudo-Euclidean metric*, and, if it is positive definite, a *Euclidean metric.*

Let $g = \langle\,,\rangle$ now be a pseudo-Riemannian fibre metric in a C^∞ real vector bundle η over a manifold M. Any local trivialization e_a of η gives rise to the *component functions* of g, defined (on the trivialization domain) by

$$g_{ab} \,=\, g(e_a, e_b) \,=\, \langle e_a, e_b \rangle \tag{28.1}$$

Symmetry of g means that $g_{ab} = g_{ba}$. Clearly, for fibre elements or local sections ψ, ϕ as before,

$$g(\psi, \phi) = \langle \psi, \phi \rangle = g_{ab}\psi^a \phi^b . \tag{28.2}$$

Suppose now that, in addition to a pseudo-Riemannian fibre metric $g = \langle\,,\rangle$ in a C^∞ real vector bundle η over a manifold M, we are also given a C^∞ connection ∇ in η. Any local trivialization e_a of η and local coordinates x^j in M, with the same domain U, give rise to the *g-modified component functions* of ∇, given by

$$\varGamma_{jba} = \langle \nabla_{p_j} e_b, e_a \rangle , \qquad \text{that is,} \quad \varGamma_{jba} = \varGamma_{jb}^c g_{ca} . \tag{28.3}$$

(Note that $g_{ca} = \langle e_c, e_a \rangle$.) By themselves, the $\varGamma_{jba}$ do *not* constitute the component functions of any invariant object; their usefulness lies in simplifying some local component expressions. For instance, when g is treated as a section of the vector bundle $\mathrm{Hom}(\eta, \eta^*)$, its covariant derivative ∇g relative to the connection in η^*, dual to ∇, is itself a section of $\mathrm{Hom}(TM, \mathrm{Hom}(\eta, \eta^*))$ with the local component functions $g_{ba,j} = \partial_j g_{ba} - \varGamma_{jb}^c g_{ca} - \varGamma_{ja}^c g_{bc}$, that now can be rewritten as

$$g_{ba,j} = \partial_j g_{ba} - \varGamma_{jba} - \varGamma_{jab} . \tag{28.4}$$

A connection ∇ in a real vector bundle η over M is called be *compatible with a pseudo-Riemannian fibre metric* $g = \langle\,,\rangle$ in η if $\nabla g = 0$, that is, if g is parallel as a section of $\mathrm{Hom}(\eta, \eta^*)$. One then also says that ∇ is a *metric connection*, or a *Riemannian connection*, especially when g is fixed (and clear from the context). The local-component version of this condition is, by (28.4),

$$\partial_j g_{ab} = \varGamma_{jab} + \varGamma_{jba} . \tag{28.5}$$

A condition necessary and sufficient in order that a connection ∇ in a real vector bundle η over M be compatible with a pseudo-Riemannian fibre metric $g = \langle\,,\rangle$, is the *Leibniz rule*

$$d_v \langle \psi, \phi \rangle = \langle \nabla_v \psi, \phi(x) \rangle + \langle \psi(x), \nabla_v \phi \rangle \tag{28.6}$$

whenever $x \in M$, $v \in T_x M$ and ψ, ϕ are local C^1 sections of η defined in a neighborhood of x. This is in turn equivalent to

$$d_v(|\psi|^2) = 2\langle \nabla_v \psi, \psi(x) \rangle \tag{28.7}$$

for x, v, ψ as above. In fact, (28.6) and (28.7) mean that

$$d_w \langle \psi, \phi \rangle = \langle \nabla_w \psi, \phi \rangle + \langle \psi, \nabla_w \phi \rangle \tag{28.8}$$

and

$$d_w(|\psi|^2) = 2\langle \nabla_w \psi, \psi \rangle \tag{28.9}$$

for local C^1 sections w of TM and ψ, ϕ of η, all three with the same domain. Now, (28.8) implies (28.5) as a special case (with $w = p_j,\ \psi = e_a,\ \phi = e_b$) and the converse implication is immediate if one expands $w = w^j p_j,\ d_w = w^j \partial_j,\ \psi = \psi^a e_a$ and $\phi = \phi^b e_b$.

Let ∇ again be a connection in a real vector bundle η over M. The curvature tensor R^∇ of ∇ then assigns to each $x \in M$ a bilinear skew-symmetric mapping $T_x M \times T_x M \ni (v, w) \mapsto R^\nabla(v, w)$ valued in real-linear operators $\eta_x \to \eta_x$. If, in addition, η carries a fixed pseudo-Riemannian fibre metric $g = \langle\,,\rangle$, each of these mappings may be regarded as a real-valued bilinear form on η_x (Problem 5). Specifically, that form is given by

$$\eta_x \times \eta_x \ni (\psi, \phi) \mapsto \langle R^\nabla(v, w)\psi, \phi \rangle . \tag{28.10}$$

Instead of $\langle R^\nabla(v,w)\psi,\phi\rangle$ we will usually write $R(v,w,\psi,\phi)$ (when ∇ and g are fixed).

The *g-modified curvature* of ∇ is defined to be the assignment, to each $x\in M$, of the real-quadrilinear function $T_xM\times T_xM\times\eta_x\times\eta_x\ni(v,w,\psi,\phi)\mapsto R(v,w,\psi,\phi)$, skew-symmetric in v,w. The modified curvature (also denoted by R) thus is a C^∞ section of the vector bundle $\mathcal{A}(TM,TM,\mathrm{Hom}(TM,\mathrm{Hom}(\eta,\eta^*)))$, while the original R is a section of $\mathcal{A}(TM,TM,\mathrm{Hom}(\eta,\eta)))$ (see Remark 20.1).

Any local trivialization e_a of η and coordinates x^j in M now give rise to the *g-modified curvature-component functions*

$$R_{jkab}=R(p_j,p_k,e_a,e_b)=\langle R^\nabla(p_j,p_k)e_a,e_b\rangle\,,$$

which, obviously, can also be expressed as

$$R_{jkab}=R_{jka}{}^c g_{cb}\,. \tag{28.11}$$

Proposition 28.2. *Let a connection ∇ in a real vector bundle over M be compatible with a pseudo-Riemannian fibre metric $g=\langle\,,\rangle$. Then the bilinear form* (28.10) *is skew-symmetric, for any fixed $x\in M$ and $v,w\in T_xM$.*

Proof. We just need to verify that

$$R_{jkab}=-R_{jkba}.$$

This is done by choosing a local trivialization e_a of η defined on a neighborhood of any given $z\in M$ so that $\Gamma_{ja}^b(z)=0$ (formula (26.1)); then, $R_{jkab}=\partial_k\Gamma_{jab}-\partial_j\Gamma_{kab}$ at z (cf. (26.2)), and so $R_{jkab}+R_{jkba}=0$ at z by (28.5). $\square$

See also Problem 6.

Given manifolds M,N, a C^∞ immersion $F:N\to M$ and a Riemannian metric g on M, the pullback F^*g, defined as in Problem 5 of §21, is obviously a Riemannian metric on N. When N is a submanifold of M and F denotes the inclusion mapping, we refer to F^*g as the *submanifold metric* on N induced by g. A unit sphere centered at 0 in a Euclidean vector space, with its submanifold metric, is referred to as a *standard* (Euclidean) *sphere*.

Remark 28.3. It is now immediate from Whitney's embedding theorem that *on every compact manifold there exists a Riemannian metric*. For more general conclusions, see Problem 7(c) below and Problem 2 in §84 of Appendix D.

By an *isometry* between pseudo-Riemannian manifolds (N,h) and (M,g) we mean any C^∞ diffeomorphism $F:N\to M$ with $F^*g=h$. Two pseudo-Riemannian manifolds are said to be *isometric* if an isometry between them exists.

Pullbacks of *pseudo-Riemannian* metrics under C^∞ immersions $F:N\to M$ are twice-covariant symmetric tensor fields of class C^∞ (see Problems 5 – 6 in §21). They may, however, fail to be nondegenerate. If F^*g is nondegenerate for a given pseudo-Riemannian metric g on M, we call F a *nondegenerate* immersion of N into the pseudo-Riemannian manifold (M,g). One similarly defines a *nondegenerate submanifold* N of (M,g) (that is, one for which the inclusion mapping F is nondegerate). In both cases, (N,F^*g) is a pseudo-Riemannian manifold.

The classical differential geometry of surfaces, originated by Gauss, studies the submanifold metrics of 2-dimensional submanifolds of a Euclidean 3-space.

The submanifold metric of a nondegenerate submanifold N in a pseudo-Riemannian manifold (M,g) may be positive definite even if g is not. One prominent example of this kind is the n-dimensional *hyperbolic space* of any positive "radius"

a, here denoted by N, and defined as follows. Let (M, g) be the space $\mathbf{R}^{n+1} = \mathbf{R}\times\mathbf{R}^n$ with the pseudo-Euclidean metric corresponding to the bilinear form sending $(t, v), (s, w) \in \mathbf{R} \times \mathbf{R}^n$ to $-ts + \langle v, w\rangle$, where $\langle\,,\rangle$ is the standard inner product of $\mathbf{R}^n$, and let N be the connected component, containing $(1, 0)$, of the two-sheeted hyperboloid given by $-t^2 + |v|^2 = -a^2$, where $|\;|$ is the Euclidean norm of $\mathbf{R}^n$. (In other words, N is the graph of the function $\mathbf{R}^n \to \mathbf{R}$ defined by $v \mapsto t = (a^2 + |v|^2)^{1/2}$.) The *hyperbolic-space metric* on N is its (positive-definite) submanifold metric. See Problems 11 – 12.

Problems

1. Given a pseudo-Riemannian fibre metric g in a real vector bundle η over a manifold M, we say that a vector subbundle ζ of η is *nondegenerate* if so is the restriction of the form g_x to ζ_x for every $x \in M$. (Cf. Problem 16 in §12.) Show that, for any nondegenerate C^∞ vector subbundle ζ of η, its *g-orthogonal complement* $\eta^\perp$, with the fibre $\eta^\perp_x$ over each $x \in M$ defined to be the g_x-orthogonal complement of ζ_x in η_x (see Problem 17 in §12), is also a nondegenerate C^∞ vector subbundle of η. (Hint below.)

2. For a connection ∇ in a C^∞ real vector bundle with a pseudo-Riemannian fibre metric g, prove that condition $\nabla g = 0$ is equivalent to the Leibniz rule (28.8) without using the component functions. (Hint below.)

3. Verify that (28.9) implies (28.8).

4. Let a connection ∇ in a real vector bundle η over a manifold M be compatible with a pseudo-Riemannian metric $g = \langle\,,\rangle$ in η. Show that

$$\langle\psi, \phi\rangle^\cdot = \langle\nabla_{\dot{x}}\psi, \phi\rangle + \langle\psi, \nabla_{\dot{x}}\phi\rangle,$$

with $(\;)^\cdot = d/dt$, whenever ψ, ψ are C^1 sections of η along a C^1 curve $t \mapsto x(t) \in M$. (Hint below.)

5. Given a finite-dimensional real vector space V with a fixed pseudo-Euclidean inner product $g = \langle\,,\rangle$, show that the assignment, to each linear operator $F : V \to V$, of the bilinear form $B : V \times V \to \mathbf{R}$ given by $B(v, w) = \langle Fv, w\rangle$ constitutes a linear isomorphism $\mathrm{Hom}(V, V) \to \mathcal{L}(V, V, \mathbf{R})$, where $\mathcal{L}(V, V, \mathbf{R})$ is the vector space of all bilinear forms $V \times V \to \mathbf{R}$ (cf. §19).

6. Prove skew-symmetry of $R(v, w, \psi, \phi)$ in ψ, ϕ for metric connections without using local component functions. (Hint below.)

7. Given a C^∞ real/complex vector bundle η over a compact C^∞ manifold M satisfying the countability axiom, prove that

(a) η is C^∞ isomorphic to a C^∞ vector subbundle of a suitable product vector bundle $M \times V$,

(b) there exists a C^∞ vector bundle ζ over M such that the direct sum $\eta\oplus\zeta$ is trivial,

(c) η admits a C^∞ (positive-definite) Riemannian/Hermitian metric g and a connection ∇ compatible with g. (Hint below.)

8. State and prove a version of Problem 5 for complex spaces, pseudo-Hermitian inner products and sesquilinear (rather than bilinear) forms.

9. For V and $g = \langle\,,\rangle$ as in Problem 5, show that $\operatorname{Trace} F = 0$ whenever $F : V \to V$ is linear and *skew-adjoint* in the sense that the corresponding bilinear form $B : V \times V \to \mathbf{R}$ is skew-symmetric.

10. Let g be a pseudo-Riemannian or pseudo-Hermitian fibre metric in a real or, complex vector bundle η over a connected manifold. Show that the algebraic type of g, that is, its signature, is the same in every fibre.

11. Prove that the submanifold metric of a hyperbolic space is in fact positive definite. (Hint below.)

12. Verify that, up to an isometry, an n-dimensional hyperbolic space of "radius" $a > 0$ could also be defined to be either connected component of the submanifold $\{x \in V : \langle x, x\rangle = -a^2\}$ of an $(n+1)$-dimensional real vector space V with a nondegenerate symmetric bilinear form $\langle\,,\rangle$ of the *Lorentzian* sign pattern $- + \ldots +$.

Hint. In Problem 1, apply Proposition 19.1 to the morphism $\varPhi : \eta \to \zeta^*$ sending $\phi \in \eta_x$, for any $x \in M$, to $g_x(\phi, \cdot\,) : \zeta_x \to \mathbf{R}$.

Hint. In Problem 2, note that $(\nabla_w g)(\phi, \psi) = d_w\langle\psi, \phi\rangle - \langle\nabla_w\psi, \phi\rangle - \langle\psi, \nabla_w\phi\rangle$ in view of the definitions of the dual and tensor-product connections.

Hint. In Problem 4, use (28.2), (22.1) and (28.5).

Hint. In Problem 6, the Leibniz rule (28.8) implies, after obvious cancellations, $\langle R^\nabla(v, w)\psi, \psi\rangle = \langle\nabla_w\nabla_v\psi - \nabla_v\nabla_w\psi + \nabla_{[v,w]}\psi, \psi\rangle = d_w\langle\nabla_v\psi, \psi\rangle - d_v\langle\nabla_w\psi, \psi\rangle + \langle\nabla_{[v,w]}\psi, \psi\rangle$ and so, by (28.9) and (6.8), $-2\langle R^\nabla(v, w)\psi, \psi\rangle = d_v d_w|\psi|^2 - d_w d_v|\psi|^2 - d_{[v,w]}|\psi|^2 = 0$.

Hint. In Problem 7, choose V, F as in Problem 11 of §27 and define ζ to be the subbundle of $M \times V$ with $\zeta = \operatorname{Ker} F$. Let $\zeta^\perp$ be the orthogonal complement of ζ for some fixed (e.g., constant) metric in $M \times V$. Then $F : \zeta^\perp \to \eta$ is a C^∞ vector-bundle isomorphism; to obtain (c), fix a metric connection in $M \times V$ (e.g., the standard flat one) and project it onto $\zeta^\perp$.

Hint. In Problem 11, use Problem 12 of §13.

29. Raising and lowering indices

Topics: A fibre metric as a vector-bundle isomorphism; the inverse (reciprocal) metric; identifying vector fields on a pseudo-Riemannian manifold with 1-forms; the gradient operator; identification of twice-covariant tensor fields on a pseudo-Riemannian manifold with vector-bundle morphisms acting in the tangent bundle; the inner product of twice-covariant tensors.

Let g be a pseudo-Riemannian fibre metric in a real vector bundle η over a manifold M. Any local trivialization e_a of η, defined on an open set U, gives rise to the component functions g_{ab} of g, characterized by (28.1). By the *reciprocal components* of g relative to the e_a we mean the functions $g^{ab} : U \to \mathbf{R}$ such that $[g^{ab}(x)]$ is the matrix inverse of $[g_{ab}(x)]$, for every $x \in U$. In other words,

$$g_{ac}g^{cb} = g_{ca}g^{bc} = \delta_a^b. \tag{29.1}$$

For instance, if there is also given a connection ∇ in η, the relation $\varGamma_{jba} = \varGamma_{jb}^c g_{ca}$ in (28.3) can be rewritten as $\varGamma_{ja}^b = \varGamma_{jac}g^{cb}$.

As another example, in a pseudo-Riemannian manifold (M, g), a local coordinate system x^j leads to the reciprocal components g^{jk} of the metric g, with $[g^{jk}] = [g_{jk}]^{-1}$ (that is, $g^{jl}g_{lk} = \delta_k^j$) at every point of the coordinate domain.

The reciprocal components g^{ab} clearly depend on the choice of the local trivialization e_a (since so do the original components g_{ab}). However, the g_{ab} are the components, relative to the e_a, of a trivialization-independent object (namely, g), and a similar interpretation exists for the g^{ab}. Specifically, we may treat g as a vector-bundle morphism $g : \eta \to \eta^*$ which, at any point $x \in M$, sends $\phi \in \eta_x$

to the linear functional $g_x(\phi,\,\cdot\,) \in \eta_x^*$. Nondegeneracy of g (which is a part of the definition of a fibre metric, cf. §28) means that $g : \eta \to \eta^*$ is an isomorphism. Furthermore, g_{ab} are the components of the morphism $g : \eta \to \eta^*$ relative to the local trivialization e_a for η and its dual e^a for η^* (see Problem 3). The inverse isomorphism $g^{-1} : \eta^* \to \eta$ then has the components g^{ab}, as one easily sees using (29.1).

The isomorphisms $g : \eta \to \eta^*$ and $g^{-1} : \eta^* \to \eta$ are traditionally referred to as *index lowering* and *index raising*. This terminology reflects the common practice of using the same symbol both for a fibre element $\phi \in \eta_x$ (or, a local section ϕ of η) and its image $g(\phi,\,\cdot\,)$. In a local trivialization e_a we thus have $\phi = \phi^a e_a$ and $\phi = \phi_a e^a$, with $\phi_a = g_{ac}\phi^c$ and $\phi^a = g^{ac}\phi_c$. thus

An important special case is the *gradient operator* of any given pseudo-Riemannian manifold (M, g). It associates with every C^1 function $f : U \to \mathbf{R}$ defined on an open set $U \subset M$ the tangent vector field ∇f on U, also denoted by $\operatorname{grad} f$, which is the result of applying the index-raising operation to the differential df. In other words, ∇f is the unique vector field w on M with $g(w, v) = d_v f$ for all (local) vector fields v (that is, $w^j = g^{jk}\,\partial_k f$). The component functions of ∇f relative to any local coordinates x^j are denoted by $\nabla^j f$ or $f^{,j}$. Thus,

$$f^{,j} \;=\; \nabla^j f \;=\; g^{jk}\,\partial_k f \;=\; g^{jk} f_{,j}\,, \tag{29.2}$$

where, to be consistent, one also uses the symbol $f_{,j} = \partial_j f$ for the component functions of df. Note that

$$d_v f \;=\; g(v, v) \quad \text{if} \quad v = \nabla f\,. \tag{29.3}$$

The use of the symbol ∇ to represent the gradient is due to a long-standing tradition and has virtually nothing to do with our notation ∇ for *connections*.

To describe further examples of index raising/lowering, let us assume for simplicity that we are given a pseudo-Riemannian manifold (M, g) (even though the following discussion remains valid, with obvious modifications, for Riemannian fibre metrics in arbitrary real vector bundles).

Let b be a twice-covariant tensor on a pseudo-Riemannian manifold (M, g), that is, a vector-bundle morphism $TM \to T^*M$ (acting by $v \mapsto b(v,\,\cdot\,)$. Replaced by a composite morphism, in which the original g is followed by $g^{-1} : T^*M \to TM$, our b thus becomes a bundle morphism $TM \to TM$, with the components ${b_j}^k$ related to the components b_{jk} of the original tensor by ${b_j}^k = b_{jl}g^{lk}$, or $b_{jk} = {b_j}^l g_{lk}$. Some confusion may arise due to the possibility of treating b as a morphism $TM \to T^*M$ acting in a different (but equally natural) way, namely, by $v \mapsto b(\,\cdot\,, v)$. When the composite of this other morphism with $g^{-1} : T^*M \to TM$ is used, one writes its components as ${b^k}_j$, with ${b^k}_j = g^{kl}b_{lj}$ and $b_{jk} = g_{jl}{b^l}_k$. In other words, leaving blank spaces above subscripts and below superscripts allows us to keep track of which index was actually raised or lowered.

The situation is simpler when b is symmetric. The two vector-bundle morphisms $TM \to T^*M$ represented by b then coincide, and so both ways of index raising lead to the same morphism $TM \to TM$, the components of which may, without risk of ambiguity, be written as b_j^k.

The *trace of a vector-bundle morphism* $A : TM \to TM$ is the function $\operatorname{Trace} A : M \to \mathbf{R}$, assigning to $x \in M$ the trace of the operator $A_x : T_xM \to T_xM$. A fixed pseudo-Riemannian metric g allows us to define the *g-trace* of a twice-covariant tensor b on M to be the function $\operatorname{Trace}_g b : M \to \mathbf{R}$ equal to the trace of the vector-bundle morphism $TM \to TM$ obtained from b by index raising. (Which is the two indices is raised makes no difference, as the resulting morphism are, at each point, each other's adjoints, so that they have the same trace.) Clearly,

$$b^j{}_j \,=\, \mathrm{Trace}_g b \,=\, \langle g, b\rangle \,=\, g^{jk} b_{jk}\,. \tag{29.4}$$

Here $\langle\,,\rangle$ is the *g-inner product of twice-covariant tensors*, defined by $\langle a, b\rangle = \mathrm{Trace}\, AB^*$, where $A, B : TM \to TM$ are obtained from a, b by raising an index and the "product" is the composite of vector-bundle morphism, while B^* is the (pointwise) g-adjoint of B. In local coordinates,

$$\langle a, b\rangle \,=\, a^{jk} b_{jk} \quad \text{with} \quad a^{jk} = g^{jp} g^{kq} a_{pq}\,. \tag{29.5}$$

See Problem 4.

Problems

1. Verify that the reciprocal components of a pseudo-Riemannian fibre metric in a real vector bundle satisfy the symmetry condition $g^{ba} = g^{ab}$.
2. Show that, for n-dimensional pseudo-Riemannian manifold, $\langle g, g\rangle = n$, or, in local coordinates, $g^{jk} g_{jk} = \delta_k^k = n$.
3. Verify that $g(e_a, \cdot\,) = g_{ac} e^c$ for any pseudo-Riemannian fibre metric g in a real vector bundle η and any local trivialization e_a of η.
4. Establish formula (29.5).

30. The Levi-Civita connection

Topics: The existence and uniqueness of the Levi-Civita connection on a pseudo-Riemannian manifold; the Christoffel symbols; geodesics; the curvature tensor; the modified curvature tensor; the Ricci tensor and scalar curvature; symmetry of the Ricci tensor; a further symmetry of the modified curvature tensor.

For a pseudo-Riemannian manifold (M, g), a connection ∇ in TM, and local coordinates x^j in M, the g-modified component functions Γ_{jkl} of ∇ relative to the x^j , i.e., involving the local trivialization p_j, are given by

$$\Gamma_{jkl} = g(\nabla_{p_j} p_k, p_l) = \Gamma_{jk}^s g_{sl}\,, \qquad \Gamma_{jk}^l = \Gamma_{jks} g^{sl} \tag{30.1}$$

(cf. §29), where g_{jk} and Γ_{jk}^l as usual denote the corresponding component functions of g and ∇ with $g_{jk} = g(p_j, p_k)$ and $\nabla_{p_j} p_k = \Gamma_{jk}^l p_l$, while g^{jk} are the component functions of the reciprocal g^{-1} of g with $g^{jl} g_{lk} = \delta_k^j$ so that, at any point of the coordinate domain $[g^{jk}] = [g_{jk}]^{-1}$ as matrices. (See §29.)

Thus, by (28.5) and (21.4), the requirement that ∇ be compatible with g and torsionfree, expressed in local coordinates, reads

$$\partial_j g_{kl} \,=\, \Gamma_{jkl} \,+\, \Gamma_{jlk}\,, \qquad \Gamma_{jkl} \,=\, \Gamma_{kjl}\,. \tag{30.2}$$

THEOREM 30.1. *For any pseudo-Riemannian manifold (M, g) there exists a unique connection ∇ in TM which is torsionfree and compatible with g. The component functions of ∇ then are given by*

$$\Gamma_{jk}^l \,=\, \frac{1}{2}\, g^{ls}(\partial_j g_{ks} + \partial_k g_{js} - \partial_s g_{jk})\,, \tag{30.3}$$

that is,

$$2\,\Gamma_{jkl} \,=\, \partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}\,. \tag{30.4}$$

Proof. Uniqueness: If ∇ is torsionfree and compatible with g, (30.4) is immediate from (30.2). Existence: Define ∇ by (30.3), i.e., (30.4) and the last formula in (30.1), which easily implies (30.2). The independence of the resulting connection of the coordinates used follows from its uniqueness. (See also Problem 3 in §23.) $\square$

The unique torsionfree metric connection ∇ in the tangent bundle of a given pseudo-Riemannian manifold (M,g) is called its *Levi-Civita connection* of (M,g). Its component functions (30.3) are known as the *Christoffel symbols* of g, and the g-modified components (30.4) are sometimes referred to as the *Christoffel symbols of the second kind*.

All objects normally associated with a connection ∇ on a manifold M will from now on be also associated with a pseudo-Riemannian metric g on M, via its Levi-Civita connection ∇. Thus, we will speak of *geodesics* in (M,g), its *curvature tensor* R (as well as the *g-modified* curvature tensor, with the components R_{jklp}), and its *Ricci tensor* Ric. We also define the *scalar curvature* of a pseudo-Riemannian manifold (M,g) to be the C^∞ function $\mathrm{s} : M \to \mathbf{R}$ given by

$$\mathrm{s} \;=\; \mathrm{Trace}_g\,\mathrm{Ric}\,, \tag{30.5}$$

where Trace_g is the g-trace defined in §29. In other words, $\mathrm{s}(x)$, at any $x \in M$, equals the trace of the g-modified Ricci tensor of (M,g) at x. In local coordinates (cf. (29.4)),

$$\mathrm{s} \;=\; R_j^j \;=\; g^{jk}R_{jk}\,. \tag{30.6}$$

Lemma 30.2. *The Ricci tensor of any pseudo-Riemannian manifold (M,g) is symmetric and. in local coordinates, $g^{pq}R_{jpkq} = R_{jk} = g^{pq}R_{pjqk}$.*

Proof. For $x \in M$ and $v,w \in T_xM$, the operator sending $u \in T_xM$ to $R(v,u)w - R(w,u)v$ is traceless, since, by the first Bianchi identity (26.6), it coincides with the operator $u \mapsto R(v,w)u$, and the latter is skew-adjoint (cf. Problems 6 and 9 in §28). The local-coordinate now follows from Proposition 28.2. $\square$

At any point x of a pseudo-Riemannian manifold (M,g), the modified curvature tensor $R = R_x$ constitutes a quadrilinear function on the tangent space T_xM with $R(v,v',w,w') = -R(v',v,w,w') = -R(v,v',w',w)$ and $R(u,v,w,u') + R(v,w,u,u') + R(w,u,v,u') = 0$ for all $v,v',w,w',u \in V$. These properties alone imply an additional symmetry:

Proposition 30.3. *The modified curvature tensor R of any pseudo-Riemannian manifold (M,g) satifies the relation*

$$R_{jklm} \;=\; R_{lmjk}\,, \tag{30.7}$$

that is, $R(v,v',w,w') = R(w,w',v,v')$ whenever $x \in M$ and $v,v',w,w' \in T_xM$.

Proof. See Problem 7. $\square$

Problems

1. Verify that, if $\gamma : I \to M$ is a geodesic for the Levi-Civita connection ∇ on a pseudo-Riemannian manifold (M,g), then the function $g(\dot\gamma,\dot\gamma) : I \to \mathbf{R}$ is constant. (Hint below.)

2. Given a finite-dimensional real affine space $(M, V, +)$ and a pseudo-Euclidean inner product $\langle\,,\rangle$ in V treated as a (constant) pseudo-Riemannian metric g on the manifold M, show that the Levi-Civita connection of the pseudo-Riemannian manifold (M, g) coincides with the standard flat connection D in $TM = M \times V$. (Hint below.)

3. Let g be a pseudo-Riemannian metric on a manifold M. Any fixed C^∞ function $f : M \to \mathbf{R}$ then gives rise to a new metric

$$\tilde{g} = e^{2f} g\,. \tag{30.8}$$

One then says that g and $\tilde{g}$ are *conformally related.* Verify that, for the Christoffel symbols $\tilde{\Gamma}^l_{jk}$ of $\tilde{g}$ and Γ^l_{jk} of g, one then has

$$\tilde{\Gamma}^l_{jk} = \Gamma^l_{jk} + \delta^l_k \partial_j f + \delta^l_j \partial_k f - g_{jk} g^{ls} \partial_s f\,.$$

4. Let two pseudo-Riemannian metrics g and $\tilde{g}$ on a manifold M be conformally related, with (30.8) for some C^∞ function $f : M \to \mathbf{R}$. Show that the Levi-Civita connections ∇ of g and $\tilde{\nabla}$ of $\tilde{g}$ then are related by

$$\tilde{\nabla}_v w = \nabla_v w + (d_v f) w + (d_w f) v - g(v, w) \nabla f\,,$$

where v, w are any C^1 vector fields, d_v is the directional derivative corresponding to v, and ∇f stands for the g-gradient of f, defined by (29.2).

5. Given pseudo-Riemannian manifolds (M, g) and (N, h), by the *Riemannian product* of (M, g) and (N, h) we mean the pseudo-Riemannian manifold $(M \times N,\, g \times h)$, where the *product metric* $g \times h$ assigns to each $(x, y) \in M \times N$ the pseudo-Euclidean inner product $(g \times h)_{(x,y)}$ defined to be the orthogonal direct sum of g_x and h_y in the tangent space $T_{(x,y)}(M \times N) = T_x M \oplus T_y N$ (see Problem 28 in §9); in other words,

$$(g \times h)_{(x,y)}((v, v'), (w, w')) = g_x(v, w) + h_y(v', w')$$

(notation as in Problem 28 of §9). Show that, in product coordinates x^j, y^α for $M \times N$ obtained using local coordinates x^j in M and y^α in N (Problems 1,3 in §9) the components of the product metric, along with its reciprocal components, Christoffel symbols, as well as the components of its curvature and Ricci tensors, all have the following property: the components involving both kinds of indices (Roman and Greek) are zero, while those involving just one kind of indices are equal to the respective components corresponding to g or h.

6. Let $F : M \to N$ be an isometry between pseudo-Riemannian manifolds (M, g) and (N, h). Verify that, if $t \mapsto x(t)$ is a geodesic of (M, g), then $t \mapsto F(x(t))$ is a geodesic of (N, h). (Hint below.)

7. Prove Proposition 30.3. (Hint below.)

Hint. In Problem 1, use Problem 4 of §28.

Hint. In Problem 2, either use formula (30.3) in affine coordinates (Problem 8 in §5), or note that D is torsionfree and compatible with every constant metric.

Hint. In Problem 6, choose local cordinates in which F appears as the identity mapping and then use (30.3) along with (22.6).

Hint. In Problem 7, write for simplicity $abcd = R(a, b, c, d)$ for $a, b, c, d \in V$, so $abcd = -abdc = dabc + bdac = -dacb - bdca = (acdb + cdab) + (dcba + cbda) = -acbd + (cdab + cdab) - cbad = 2cdab - acbd - cbad = 2cdab + bacd = 2cdab - abcd$, as required.

31. The lowest dimensions

Topics: Pseudo-Riemannian manifolds of dimensions 0 and 1; the curvature and Ricci tensors of pseudo-Riemannian surfaces; the Gaussian curvature.

In a zero-dimensional pseudo-Riemannian manifold, the metric, curvature and Ricci tensors, as well as the scalar curvature are all equal to zero by definition (and they must be so, if we want the densors in question to represent multilinear mappings in the zero-dimensionaltangent spaces).

In dimension 1, the curvature tensor R of any pseudo-Riemannian metric is zero, due to skew-symmetry of $R(u,v)w$ in u,v. Hence $\mathrm{Ric}=0$ and $\mathrm{s}=0$.

Finally, let (M,g) be a pseudo-Riemannian surface, that is, a pseudo-Riemannian manifold with $\dim M=2$. We define the Gaussian curvature of (M,g) to be the function $K=\mathrm{s}/2 : M\to\mathbf{R}$, where s is the scalar curvature of (M,g).

Proposition 31.1. *The Ricci and curvature tensors of every pseudo-Riemannian surface (M,g) satisfy the equalities $\mathrm{Ric}=Kg$ and $R(u,v)w=K[g(u,w)v-g(v,w)u]$, as well as $\mathrm{s}=2K$, where u,v,w are any vector fields or vectors tangent to M at any point, and K is the Gaussian curvature. In local coordinates,*

$$\text{(31.1)} \qquad \text{a)}\quad R_{jlp}{}^q \;=\; K(g_{jp}\delta_l^q-g_{lp}\delta_j^q)\,, \qquad \text{b)}\quad R_{jl}\;=\;Kg_{jl}\,.$$

Proof. For any fixed point $x\in M$ we may treat R_x as a skew-symmetric bilinear mapping from $T_xM\times T_xM$ into the vector space of skew-adjoint operators $T_xM\to T_xM$. (See Proposition 28.2.) Since the latter space is 1-dimensional (cf. (12.7)) and $\dim M=2$, such skew-symmetric mappings form a 1-dimensional space as well, and so R_x much be a multiple of the mapping taking the vectors $v,w\in T_xM$ to the operator $g_x(u,w)v-g_x(v,w)u$. (In fact, the latter mappings is nonzero and has all the required properties). This yields (31.1.a) and, by contraction, we also obtain (31.1.b) and $\mathrm{s}=2K$. □

Problems

1. Let (M,g) be a 1-dimensional pseudo-Riemannian manifold. Verify that the geodesics of (M,g) are precisely those C^1 curves $I\ni t\mapsto x(t)\in M$, where $I\subset\mathbf{R}$ is an interval, which have constant speed. (The latter means that $g(\dot x,\dot x)$ is constant as a function of $t\in I$.) (Hint below.).

2. Let ∇ be a connection on a manifold M. One says that a submanifold N of M is *totally geodesic* (relative to ∇ if, given any $x\in N$ and $v\in T_xN\subset T_xM$, we have $\exp_x tv\in N$ for all $t\in\mathbf{R}$ sufficiently close to 0. Show that
   (a) the image of any nonconstant injective geodesic of ∇ is a 1-dimensional totally geodesic submanifold of M,
   (b) any affine subspace of an affine space $(M,V,+)$ is totally geodesic relative to the standard flat connection in $TM=M\times V$ (Example 21.1),
   (c) all zero-dimensional submanifolds of M and open submanifolds of M are totally geodesic.

Hint. In Problem 1, one implication is clear from Problem 1 of §30. Next, assuming constant speed and leaving aside the obvious case $g(\dot x,\dot x)=0$ we get $g(\nabla_{\dot x}\dot x,\dot x)=0$ (see Problem 4 in §28), and so $\nabla_{\dot x}\dot x=0$. Namely, as $g(\dot x,\dot x)$ now is a (nonzero) constant, $\dot x(t)$ spans $T_{x(t)}M$ at every $t\in I$.

32. Riemannian manifolds as metric spaces

Topics: Curve length; the distance function; the exponential mapping; Gauss's Lemma; the injectivity radius; a distance-preserving property of the exponential mapping; the metric-space axioms.

By the *length* of a piecewise C^1 curve $[a,b] \ni t \mapsto x(t) \in M$ in a Riemannian manifold (M,g) we mean the number

$$\mathsf{L} = \int_a^b |\dot{x}(t)|\, dt\,. \tag{32.1}$$

Let (M,g) now be a *connected* Riemannian manifold. The *distance function* $\mathrm{d} : M \times M \to \mathbf{R}$ of (M,g) assigns to $(x,y) \in M \times M$ the infimum $\mathrm{d}(x,y)$ of the lengths of all piecewise C^1 curves in M joining x and y.

The following lemma uses the subscript conventions of §25.

Lemma 32.1. *Suppose that we are given a Riemannian manifold (M,g) a C^2 mapping $(t,s) \mapsto x(t,s) \in M$, defined on a rectangle $[a,b] \times [c,d]$ in the ts-plane. If $x_{tt} = 0$ identically and $|x_t|$ is constant on the rectangle, then $\langle x_t, x_s\rangle_t = 0$ identically on the rectangle.*

In fact, $\langle x_t, x_s\rangle_t = \langle x_{tt}, x_s\rangle + \langle x_t, x_{st}\rangle = 0$, since $x_{tt} = 0$ and $x_{st} = x_{ts}$ (Problem 2 in §25), so that $\langle x_t, x_{st}\rangle = \langle x_t, x_{ts}\rangle = \langle x_t, x_t\rangle_s/2$.

The *exponential mapping* of a Riemannian manifold (M,g) at a point $x \in M$ is the mapping $\exp_x : U_x \to M$ defined as in the lines following (22.7) for the Levi-Civita connection ∇ of (M,g). Lemma 32.1 easily implies the following classical result known as the *Gauss lemma.*

Lemma 32.2. *Given a Riemannian manifold (M,g), a point $x \in M$, and $v \in U_x$, let $H : T_xM \to T_yM$ be the differential of $\exp_x : U_x \to M$ at v, where $y = \exp_x v$ and T_vU_x is identified with T_xM as in* Examples 5.1 *and* 5.3. *Then, for $w \in T_xM = T_vU_x$ such that $g_x(v,w) = 0$,*

$$g_y(Hv,Hv) = g_x(v,v)\,, \qquad g_y(Hv,Hw) = 0\,. \tag{32.2}$$

Proof. See Problem 1. □

For a Riemannian manifold (M,g) and $x \in M$, let $[a,b] \ni t \mapsto v(t) \in U_x$ be a piecewise C^1 curve. Then, with L as in (32.1) for $x(t) = \exp_x v(t)$,

$$\mathsf{L} \ge |r(b) - r(a)|\,, \qquad \text{where} \quad r(t) = |v(t)|\,. \tag{32.3}$$

In fact, we may assume that $r(a) \ne r(b)$. Problem 2 thus allows us to replace $[a,b]$ with a subinterval $[c,d]$ such that $r(c) = r(a)$, $r(d) = r(b)$ and $r(t) > 0$ for all $t \in (c,d)$. Then r is a C^1-differentiable function of $t \in (c,d)$, and so we may use the argument presented below; however, once (32.3) is proved for $[c,d]$ rather than $[a,b]$, it will clearly follow for $[a,b]$ as well. Namely, suppressing the dependence of $\dot{x}, v$ and r on t, and writing H for the differential of $\exp_x$ at $v(t)$, we have $\dot{x} = H\dot{v}$ (by the definition of the differential, cf. (5.16)). Hence $\mathsf{L} \ge \int_c^d |\dot{x}|\, dt = \int_c^d |H\dot{v}|\, dt \ge \int_c^d |\langle v, \dot{v}\rangle|/|v|\, dt = \int_c^d |\dot{r}|\, dt \ge \left|\int_c^d \dot{r}\, dt\right| = |r(d) - r(c)|$, with the last two inequalities provided by Problems 3 and 4, and $\langle\,,\rangle$ denoting the Euclidean inner product g_x in T_xM. (When one of $r(a), r(b)$ is zero, this still makes sense for *improper* integrals.)

If, in addition, the differential of $\exp_x$ at $v(t)$ is injective for every $t \in [a, b]$, the case where the inequality (32.3) is actually an *equality* is characterized as follows:

(32.4) $\mathsf{L} = |r(b) - r(a)|$ if and only if all $v(t)$ lie in a single line segment emanating from 0 in T_xM and the function $t \mapsto r(t)$ is (weakly) monotone.

In fact, this is clear since we understand the equality case in each of the three inequality steps involved in the above proof of (32.3). (See Problems 3 and 4; the replacement of $[a, b]$ with $[c, d]$ makes L smaller except when the curve $x(t)$ is constant on both $[a, c]$ and and $[d, b]$.)

Let (M, g) again be a connected Riemannian manifold. The *injectivity radius* of (M, g) is the function $\mathrm{r_{inj}} : M \to (0, \infty]$ assigning to every $x \in M$ the supremum $\mathrm{r_{inj}}(x)$ of those $\varepsilon \in (0, \infty)$ for which the domain U_x of $\exp_x$ contains the open ball U_ε of radius ε centered at 0 in the Euclidean space T_xM, and $\exp_x$ maps U_ε diffeomorphically onto an open set in M. (That $\mathrm{r_{inj}}(x) > 0$ is clear as ε with the named property always exists; see the lines following (22.7).)

LEMMA 32.3. *Let there be given a point x of a Riemannian manifold (M, g), a real number ε with $0 < \varepsilon < \mathrm{r_{inj}}(x)$, and a continuous curve $[a, c] \ni t \mapsto x(t) \in M$ with $x(a) \in \exp_x(U_\varepsilon)$ and $x(c) \notin \exp_x(U_\varepsilon)$, where U_ε is the open ball of radius ε around 0 in T_xM. Then there exists $b \in (a, c]$ such that $x(t) \in \exp_x(U_\varepsilon)$ for all $t \in [a, b)$ and $x(b) \in \exp_x(S_\varepsilon)$, with S_ε denoting the sphere of radius ε in T_xM, centered at 0.*

PROOF. Let b be the supremum of those $t' \in [a, c]$ with $x(t) \in \exp_x(U_\varepsilon)$ for all $t \in [a, t']$. We may choose sequences $t_k \in [a, b]$ and $v_k \in U_\varepsilon$ with $t_k \to b$ as $k \to \infty$ and $x(t_k) = \exp_x v_k$. Passing to a subsequence, we may also assume that $v_k \to v$ as $k \to \infty$, for some $v \in T_xM$ with $|v| \le \varepsilon$. Hence, by continuity, $x(b) = \exp_x v$. If we had $|v| < \varepsilon$, the point $x(b)$ would lie in the open set $\exp_x(U_\varepsilon)$, and, consequently, so would $x(t)$ for all $t > b$ sufficiently close to b, contrary to how b was defined. Hence $|v| = \varepsilon$ and $x(b) \in \exp_x(S_\varepsilon)$, as required. $\square$

PROPOSITION 32.4. *Let x be a point in a connected Riemannian manifold (M, g), and let $v \in T_xM$ be a vector with $|v| < \mathrm{r_{inj}}(x)$. Then*

$$\mathrm{d}(x, \exp_x v) \;=\; |v|\,, \tag{32.5}$$

with $|v|$ denoting, as before, the Euclidean norm of v, so that $|v|^2 = g_x(v, v)$.

If $v \ne 0$, the piecewise C^1 curves $[a, b] \ni t \mapsto x(t) \in M$ with $x(a) = x$ and $x(b) = y$, where $y = \exp_x v$, that have the minimum length $|v|$ are those and only those having the form $x(t) = \exp_x[s(t)v/|v|]$ with any nondecreasing piecewise C^1 surjective function $s : [a, b] \to [0, |v|]$.

PROOF. That $[0, 1] \ni t \mapsto \exp_x tv$ is a curve of length $|v|$ is a trivial exercise. Next, let L be the length of any piecewise C^1 curve $[a, b] \ni t \mapsto x(t) \in M$ with $x(a) = x$ and $x(b) = y$ which is *not* of the special form described in the final clause of Proposition 32.4. Then $\mathsf{L} > |v|$. In fact, we may choose the maximum c of those $t \in [a, b]$ for which the restriction of the curve to $[a, t]$ lies in the $\exp_x$-image of the closed ball in T_xM centered at 0 and having the radius $|v|$. Thus, $c > a$, since $x(a) = x$ and $\exp_x$ is diffeomorphic on a neighborhood of 0 in T_xM (see §22), while, as it is a *closed* ball that we use, $x(c) = \exp_x u$ for some unique $u \in T_xM$ with $|u| \le |v|$, and, necessarily, $|u| = |v|$, since the strict inequality $|u| < |v|$ would contradict maximality of c.

If our curve *is not* constant on $[c,b]$, its length L is greater than the length L' of its restriction to $[a,c]$, so that $\mathsf{L} > \mathsf{L}' \geq |v|$ from (32.3) (for L', a, c rather than L, a, b, with $r(a) = 0$ and $r(c) = |u| = |v|$). If, however, it *is* constant on $[c,b]$, the entire curve lies in the $\exp_x$-image of the closed ball mentioned above, and $\mathsf{L} > |v|$ by (32.3) and assertion (32.4). This completes the proof. □

Corollary 32.5. *Suppose that (M,g) is a connected Riemannian manifold, $x \in M$, and ε is a real number with $0 < \varepsilon < \mathrm{r_{inj}}(x)$. Then the radius ε open metric ball around x in M is the $\exp_x$-diffeomorphic image of the radius ε Euclidean open ball U_ε centered at 0 in T_xM.*

Proof. In view of Proposition 32.4 it suffices to show that every $y \in M$ such that $\mathrm{d}(x,y) < \varepsilon$ lies in $\exp_x(U_\varepsilon)$. To this end, let us fix a piecewise C^1 curve $[a,c] \ni t \mapsto x(t) \in M$ of length $\mathsf{L} < \varepsilon$ with $x(a) = x$ and $x(c) = y$. If we had $y \notin \exp_x(U_\varepsilon)$, choosing b as in Lemma 32.3 we would have $\mathsf{L} \geq \mathrm{d}(x, x(b))$, and so $\mathsf{L} \geq \varepsilon$ by Proposition 32.4, contrary to how the curve was chosen. □

We will now show that the distance function d satisfies the usual axioms of a metric space (cf. §73 in Appendix B), namely: $\mathrm{d}(x,y) = \mathrm{d}(y,x)$ (symmetry), $\mathrm{d}(x,z) \leq \mathrm{d}(x,y) + \mathrm{d}(y,z)$ (the triangle inequality), $\mathrm{d}(x,x) = 0$, and $\mathrm{d}(x,y) > 0$ if $x \neq y$ (positivity), for any $x,y,z \in M$.

Theorem 32.6. *Let d be the distance function of a connected Riemannian manifold (M,g). Then*

a. *(M,d) is a metric space,*
b. *the metric-space topology of (M,d) coincides with the manifold topology of M.*

Proof. The metric-space axioms other than positivity are completely straightforward. To prove positivity, suppose that $x,y \in M$ and $x \neq y$. Choosing $\varepsilon \in \mathbf{R}$ with $0 < \varepsilon < \mathrm{r_{inj}}(x)$ such that the open ball U_ε of radius ε in T_xM, centered at 0, is contained in the $\exp_x$-diffeomorphic pre-image of a neighborhood of x in M, not containing y, we see from Corollary 32.5 that y does not lie in the radius ε open metric ball around x in M, and so $\mathrm{d}(x,y) \geq \varepsilon > 0$. This proves (a).

Assertion (b) is in turn immediate from Corollary 32.5. □

Problems

1. Prove the Gauss lemma. (Hint below.)

2. Let $t \mapsto r(t)$ be a nonnegative continuous function on a closed interval $[a,b]$, and let $r(a) \neq r(b)$. Prove that $[a,b]$ contains a nontrivial subinterval $[c,d]$ such that $r(c) = r(a)$, $r(d) = r(b)$ and $r(t) > 0$ for all $t \in (c,d)$.

3. Suppose that $|Hv| = |v|$ and $\langle Hv, Hw\rangle = 0$ for a linear operator $H : V \to V'$ between Euclidean spaces, a fixed vector $v \in V \setminus \{0\}$, and all vectors $w \in V$ with $\langle v, w\rangle = 0$ (where $\langle\,,\rangle$ denotes both inner products). Prove the inequality $|Hw| \geq |\langle v,w\rangle|/|v|$ for *every* $w \in V$, and, if H is also injective, the inequality is strict unless w is a scalar multiple of v.

4. Given a continuous function $f : [a,b] \to \mathbf{R}$ on a closed interval, verify that $\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt$, and the inequality is strict unless $f \geq 0$ on $[a,b]$ or $f \leq 0$ on $[a,b]$.

5. Verify that $\int_a^b |\dot{x}(t)|\,dt \geq |x(b) - x(a)|$ whenever $[a,b] \ni t \mapsto x(t)$ is a piecewise C^1 curve in an affine Euclidean space.

6. Show that the claim made in Problem 4 remains true even if one replaces $<$ by $=$, provided that an open ball of infinite radius in M is defined to be M.

7. Let d, d' be the distance functions of a connected Riemannian manifold (M, g) and, respectively, a connected submanifold M' of M endowed with the submanifold metric g'. Show that $\mathrm{d} \le \mathrm{d}'$ on $M' \times M'$.

Hint. In Problem 1, apply Lemma 32.1 to $x(t,s) = \exp_x tv(s)$, where $s \mapsto v(s)$ is a C^1 curve in U_x, contained in a sphere centered at 0 in T_xM.

Hint. In Problem 2, the last inequality will follow if $r(t)$ lies between $r(a)$ and $r(b)$ and is different from both of them, for every $t \in (c,d)$. For instance, choose $d = \min\{t \in [a,b] : r(t) = r(b)\}$, and then $c = \max\{t \in [a,d] : r(t) = r(a)\}$.

33. Completeness

Topics: Positive lower bounds for the injectivity radius; geodesics as distance-minimizing curves; geodesic completeness versus metric completeness; the Hopf-Rinow theorem.

LEMMA 33.1. *For any compact subset Y of a connected Riemannian manifold (M, g) there exist $\varepsilon > 0$ and an open set U containing Y such that $\mathrm{r_{inj}} > \varepsilon$ on U, where $\mathrm{r_{inj}}$ is the injectivity radius of (M, g).*

In fact, the Borel-Heine theorem allows us to assume that Y consists of a single point. The assertion is now obvious from the inverse mapping theorem (see §74 in Appendix B). Namely, the mapping

$$(x, v) \mapsto (x, \exp_x v) \tag{33.1}$$

restricted to a suitable neighborhood in TM of any given point $(x, 0)$ lying in the zero section $M \subset TM$, sends that neighborhood diffeomorphically onto an open set in $M \times M$. Note that (33.1) is defined and C^∞-differentiable on some open set in TM containing the zero section, due to the regularity theorem for ordinary differential equations with parameters; see §80 in Appendix C.

One says that a piecewise C^1 curve $[a,b] \ni t \mapsto x(t) \in M$ is *minimizing* if its length L equals $\mathrm{d}(x(a), x(b))$. For any $c \in (a,b)$ the restrictions of the curve to $[a,c]$ and $[c,b]$ then are minimizing as well. To prove this, note that $\mathsf{L} = \mathrm{d}(x(a), x(b)) \le \mathrm{d}(x(a), x(c)) + \mathrm{d}(x(c), x(b)) \le \mathsf{L}' + \mathsf{L}'' = \mathsf{L}$, where $\mathsf{L}', \mathsf{L}''$ are the lengths of the restrictions, and so all inequalities used here must in fact be equalities. Applying this principle twice in a row, we see that the restriction of a minimizing curve to any closed subinterval of its parameter interval is also a minimizing curve. It its therefore natural to agree that a piecewise C^1 curve $I \ni t \mapsto x(t) \in M$ defined on an arbitrary interval I should be called *minimizing* if its restriction to every closed subinterval of I is a minimizing curve.

In view of (32.5), every geodesic $[a,b] \ni t \mapsto x(t) \in M$ of length L which is less than $\mathrm{r_{inj}}(x(a))$ or $\mathrm{r_{inj}}(x(b))$ is necessarily a minimizing curve.

Conversely, every minimizing curve $I \ni t \mapsto x(t) \in M$ "is a geodesic" in the sense that it is obtained from some geodesic by a reparameterization, or, more precisely, that $x(t) = \exp_x [s(t)v]$ for some (weakly) monotone piecewise C^1 function $I \ni t \mapsto s(t) \in \mathbf{R}$, some $x \in M$, and some $v \in T_xM$ with $s(t)v \in U_x$ for all $t \in I$.

To see this, first assume that $I = [a,b]$ is a closed interval and $\mathrm{d}(x(a), x(b))$ is less than $\mathrm{r_{inj}}(x(a))$ or $\mathrm{r_{inj}}(x(b))$. Our claim then is immediate from the final clause of Proposition 32.4. The general case now easily follows from Lemma 33.1.

A connection ∇ on a manifold M is said to be *geodesically complete* if its every geodesic can be extended to a geodesic defined on the whole real line $\mathbf{R}$. This amounts to requiring that $U_x = T_xM$ for every $x \in M$ (notation of §22).

Similarly, a Riemannian manifold (M, g) is called *geodesically complete* if so is its Levi-Civita connection ∇.

The following theorem is due to Hopf and Rinow [**HR**].

Theorem 33.2. *In a geodesically complete Riemannian manifold (M, g), any two points can be joined by a minimizing geodesic.*

Proof. Let us fix a point $x \in M$ and define $A \subset [0, \infty)$ to be the set of all $r \geq 0$ such that every point $y \in M$ with $\mathrm{d}(x, y) \leq r$ can be joined to x by a minimizing geodesic (or, equivalently, equals $\exp_x v$ for some $v \in T_xM$ with $|v| = \mathrm{d}(x, y)$). Since x was arbitrary, we just need to show that $A = [0, \infty)$.

By Corollary 32.5, A contains every r with $0 \leq r < \mathrm{r_{inj}}(x)$. Also, if $r \in A$, then, clearly, $r' \in A$ for every $r' \in [0, r]$. Thus, A is an interval and the equality $A = [0, \infty)$ will follow once we show that $r_0 = \sup A$ is infinite.

Let us suppose that, on the contrary, $r_0 < \infty$. Then $r_0 \in A$ (that is, $r_0 = \max A$). In fact, let $y \in M$ be a point with $\mathrm{d}(x, y) = r_0$, and let $[a, b] \ni t \mapsto x_k(t) \in M$, $k = 1, 2, \ldots$, be a sequence of piecewise C^1 curves of lengths L_k such that $x_k(a) = x$, $x_k(b) = y$ and $\mathsf{L}_k \to r_0$ as $k \to \infty$. As $\mathsf{L}_k \geq r_0 = \mathrm{d}(x, y)$, we may find a sequence of parameters $c_k \in [a, b]$ such that the length of the curve $x_k(t)$ restricted to $[a, c_k]$ is $r_0 - 1/k$. The sequence $y_k = x(c_k)$ then converges to y in M, since $\mathrm{d}(y_k, y)$ does not exceed the length $\mathsf{L}_k - r_0 + 1/k$ of the restriction of the curve $x_k(t)$ to $[c_k, b]$, while $\mathrm{d}(x, y_k) \leq r_0 - 1/k < r_0 = \sup A$, and so $y_k = \exp_x v_k$ for some $v_k \in T_xM$ with $|v_k| = \mathrm{d}(x, y_k) < r_0$. Boundedness of the sequence v_k implies that it has a subsequence convergent to some limit $v \in T_xM$. Thus, $y = \exp_x v$ due to continuity of $\exp_x$, which shows that $r_0 \in A$.

The closed metric ball $Y = \{y \in M : \mathrm{d}(x, y) \leq r_0\}$ is a compact subset of M, as it is the $\exp_x$-image of the closed Euclidean ball $\{v \in T_xM : |v| \leq r_0\}$. Choosing ε for Y as in Lemma 33.1, we will now show that A contains $(r_0, r_0 + \varepsilon)$. (This contradicts the maximality of r_0 in A, and hence shows that r_0 cannot be finite, completing the proof.) Namely, let $y \in M$ and $r_0 < \mathrm{d}(x, y) < r_0 + \varepsilon$. Compactness of Y allows us to find a point $z \in Y$ having the minimum distance $\delta > 0$ from y. It follows that $\delta < \varepsilon$. (In fact, let us choose a curve joining x to y and having length L with $r_0 < \mathsf{L} < r_0 + \varepsilon$. Using a point z' on the curve to partition it into two segments of lengths r_0 and $\mathsf{L} - r_0$ we see that $z' \in Y$ and $\mathrm{d}(y, z') \leq \mathsf{L} - r_0 < \varepsilon$.) Thus, both x and y can be joined to z by minimizing geodesics, which for x follows from the definition of A (as $r_0 \in A$), and for z from Corollary 32.5 applied to z instead of x (note our choice of ε). The lengths of the two geodesics are r_0 and, respectively, our δ. (We have $\mathrm{d}(x, z) = r_0$, since the inequality $\mathrm{d}(x, z) < r_0$ would also remain true at some points near z and different from z on the length δ geodesic, contradicting the minimum property of δ.) The two geodesics together form a piecewise C^1 curve joining x to y, which is also minimizing (and hence a geodesic, for reasons explained earlier in this section). Namely, if there existed a curve connecting x to y and having some length L' with $r_0 < \mathsf{L}' < r_0 + \delta$, some point z'' on it would lie at the distance r_0 from x (as the distance function is continuous), and so the length of the curve segment from z'' to y would be less than or equal to $\mathsf{L}' - r_0 < \delta$, which, as $z'' \in Y$, would contradict the choice of δ. □

A connected Riemannian manifold naturally constitutes a metric space (§32), which may or may not be (metrically) complete, in the sense that every Cauchy sequence in it converges. (See §73 in Appendix B.) Both kinds of completeness mean exactly the same:

THEOREM 33.3. *For a connected Riemannian manifold* (M,g)*, with the distance function* d*, the following three conditions are equivalent*:

a. (M,g) *is geodesically complete,*
b. *every bounded closed subset of* (M,d) *is compact,*
c. (M,d) *is a complete as a metric space.*

PROOF. Assuming (a) we get (b): a bounded set is contained in a metric ball, which (by Theorem 33.2) is in turn contained in the $\exp_x$-image, for some $x \in M$, of a closed Euclidean ball (and the latter is compact). Next, (b) implies (c): one easily verifies (see Problems 3 and 4 in §73 of Appendix B) that a Cauchy sequence is necessarily bounded; thus, by (b), it has a convergent subsequence; and hence the Cauchy sequence is itself convergent. Finally, let us assume (c). To prove (a), we suppose that, on the contrary, there exists a maximal geodesic $t \mapsto x(t)$, with some constant speed $c > 0$, defined on an interval $I \neq \mathbf{R}$. Using the parameter change $t \mapsto -t$, if necessary, we may also assume that I has a finite *upper* endpoint b. For any sequence t_k in I such that $t_k \to b$ as $k \to \infty$, the values $x(t_k)$ form a Cauchy sequence, which, by (c), converges to some point $x(b) \in M$. (Note that $\mathrm{d}(x(t), x(s)) \leq c|t-s|$.) The limit $x(b)$ does not depend on how t_k were chosen (as one sees considering the "union" of two such sequences), and so it equals the limit of $x(t)$ as $t \to b^-$. Let us now fix a coordinate domain U containing $x(b)$ and contained, along with its compact closure, in another coordinate domain, along with $a \in I$ such that $x(t) \in U$ for all $t \in [a,b]$. Using the coordinate identification, we may treat $[a,b] \ni t \mapsto x(t)$ as a continuous curve in an open subset of $\mathbf{R}^n$, $n = \dim M$. As it has a constant g-speed $c > 0$, its Euclidean speed (that is, the Euclidean norm of $\dot{x}(t)$) is bounded, and so, in view of the geodesic equations (22.6), the Euclidean norm of $\ddot{x}(t)$ is bounded as well; this, however, implies that $\dot{x}(t)$ has a limit in $\mathbf{R}^n$ as $t \to b^-$. (See Problem 3 in §79 of Appendix C.) Using that limit as the initial velocity for a g-geodesic $[b, b+\varepsilon) \ni t \mapsto x(t)$, we thus extend the original geodesic past b, contrary to its maximality. (The extension, being of class C^1, is necessarily also of class C^∞, by (22.6).) This contradiction completes the proof. $\square$

The *diameter* of a connected Riemannian manifold (M,g) is defined by

$$\operatorname{diam}(M,g) \;=\; \sup\{\mathrm{d}(x,y) : x,y \in M\} \;\in\; [0,\infty]. \tag{33.2}$$

Thus, M is bounded (as a set in (M,g)) if and only if $\operatorname{diam}(M,g) < \infty$.

As an obvious consequence of Theorem 33.3, we have

COROLLARY 33.4. *A complete Riemannian manifold is bounded, that is, has a finite diameter, if and only if it is compact.*

Problems

1. Show that, given a point $x \in M$, every neighborhood of $(x,0)$ in TM contains a neighborhood of the form $\{(y,v) \in TM : y \in U \text{ and } |v| < \varepsilon\}$ for some $\varepsilon > 0$ and some neighborhood U of x in M.

2. For any manifold M and any point $x \in M$, describe a natural isomorphic identification $T_{(x,0)}(TM) = T_xM \times T_xM$.

3. Show that the differential of (33.1) at every $(x, 0)$ in the zero section is, under the identification of Problem 2, given by $(v, w) \mapsto (v, v + w)$.

4. Verify that the square of the distance function d of any connected Riemannian manifold (M, g) is of class C^∞ on some open set containing the *diagonal* $\{(x, x) : x \in M\}$ in $M \times M$. (Hint below.)

5. Let h be the submanifold metric of a totally geodesic submanifold N in a Riemannian manifold (M, g). Prove that the geodesics of (N, h) are precisely those geodesics of (M, g) which are contained in N. (Hint below.)

6. Given two Riemannian metrics g, h on a manifold M and a compact subset Y of M, show that there exist positive constants ε, C such that

$$\varepsilon h(v, v) \leq g(v, v) \leq C h(v, v)$$

for all $x \in Y$ and all $v \in T_xM$. (Hint below.)

7. Generalize Problem 6 to Riemannian/Hermitian fibre metrics in vector bundles.

Hint. In Problem 4, cover the diagonal with diffeomorphic images, under (33.1), of neighborhoods of $(x, 0)$, for $x \in M$.

Hint. In Problem 5, note that the geodesics of (M, g) contained in N are minimizing curves in (N, h) (Problem 7 in §32), and realize all initial conditions in N. (We are free to assume that M, N are both connected, by restricting our discussion to one connected component of N at a time, and the connected component of M which contains it.)

Hint. In Problem 6, argue by contradiction: if the infimum (or supremum) of $g(v, v)/h(v, v)$ over nonzero vectors v tangent to M at points of Y were 0 (or, respectively, ∞), choosing a convergent sequence of points in a coordinate domain, realizing such a limit, and vectors v which are unit relative to the Euclidean norm, we get a contradiction with positive definiteness of g or h.

34. Convexity

Topics: Strongly convex sets; the strong local convexity theorem.

Given a connected Riemannian manifold (M, g) and an open set $U \subset M$, we say that U is *strongly convex* if any two points $y, z \in U$ can be joined by a unique minimizing geodesic in M, and, in addition, that unique minimizing geodesic

a. lies entirely in U, and
b. depends C^∞-differentiably on y and z, that is, has a parameterization of the form $[0, 1] \ni t \mapsto \exp_y tv$ with a vector $v \in U_y$ depending on y, z so as to form a C^∞ differentiable mapping $U \times U \ni (y, z) \mapsto v \in V$.

The following fact may be called the *strong local convexity theorem.*

Theorem 34.1. *Let Y be a compact subset of a connected Riemannian manifold (M, g). Then there exists $\varepsilon > 0$ such that every open ball of radius less than ε in M, centered at any point of Y, is strongly convex.*

Proof. According to Lemma 33.1, there exists an open ball in M, centered at x, on which $\mathrm{r_{inj}} > \varepsilon$ for some $\varepsilon > 0$. Any sufficiently small concentric open ball U of radius less than both $\varepsilon/2$ and the radius of this ball will automatically satisfy all of our claims except, possibly, assertion (a) in the above definition; this is clear from the final clause of Proposition 32.4 and the last paragraph of §11 along with the fact that Exp must be diffeomorphic (cf. Problem 3 in §33) on some neighborhood of $(x,0)$ in TM having the form described in Problem 1 in §33.

Next, for every point y in an even smaller, and sufficiently small, open ball U centered at x and every unit vector $u \in T_yM$, we have

$$\frac{d^2}{dt^2}\,[\mathrm{d}(x, \exp_y tu)]^2\bigg|_{t=0} \;>\; 0\,. \tag{34.1}$$

In fact, the left-hand side of (34.1), with any fixed x, is a C^∞ function of (y,u) on a suitable open set in TM (where u is not assumed to be unit; see Problem 4 in §33). If the inequality (34.1) failed for y arbitrarily closed to x and some unit vectors u, choosing a sequence of such "counterexamples" and replacing it with a subsequence for which the sequence of the u converges, while the y (necessarily) tend to x, we would get nonpositivity of the left-hand side of (34.1) for $y = x$ and some unit vector $u \in T_xM$, which clearly contradicts the fact that, by (32.5), $\mathrm{d}(x, \exp_x tu) = t$.

Relation (34.1) will still hold if, instead of $t = 0$, the second derivative is evaluated at any value c of t in an interval $I = [a,b]$ containing 0 and such that $\exp_y tu \in U$ for every $t \in I$. (In fact, $[a-c, b-c] \ni t \mapsto \exp_y(c+t)u$ then is also a unit-speed geodesic in U.) We now obtain (a) from Problem 1 applied to $f(t) = [\mathrm{d}(x, \exp_y tu)]^2$. This completes the proof. $\square$

Corollary 34.2. *For every compact connected Riemannian manifold (M,g) there exists $\varepsilon > 0$ such that every open ball in M of radius not exceeding ε is strongly convex.*

Problems

1. Let a C^2 function $f : [a,b] \to \mathbf{R}$ have a positive second derivative. Show that $f \le \max\{f(a), f(b)\}$ everywhere in $[a,b]$.

2. Let $\mathcal{H}$ be a nonempty set of isometries of a given Riemannian manifold (M,g) onto itself. Show that the set

$$Y \;=\; \{x \in M : F(x) = x \text{ for every } F \in \mathcal{H}\} \tag{34.2}$$

is either empty, or it is a disjoint union of totally geodesic submanifolds of (M,g), each of which is closed as a subset of M and carries the subset topology. (Hint below.)

Hint. In Problem 2, for any given point $x \in Y$, choose ε with $0 < \varepsilon < \mathrm{r_{inj}}(x)$ and, letting U_ε stand for the open ball of radius ε in T_xM, centered at 0, note that the preimage of $Y \cap \exp_x(U_\varepsilon)$ under the diffeomorphism $\exp_x : U_ve \to \exp_x(U_\varepsilon)$ is the intersection of U_ε and the vector subspace $\{v \in T_xM : dF_xv = v \text{ for every } F \in \mathcal{H}\}$ of T_xM.

35. Myers's theorem

Topics: The unit sphere bundle for a vector bundle with a Riemannian or Hermitian fibre metric; the unit tangent bundle of a Riemannian manifold; the Ricci curvature function; the Myers theorem.

Let $\langle\,,\rangle$ be a Riemannian (or, Hermitian) fibre metric in a real (or, complex) vector bundle η over a manifold M. By its *unit sphere bundle* we mean the subset η^1 of the total space η formed by all (x,ϕ) with $\langle\phi,\phi\rangle = 1$. Since $(x,\phi) \mapsto \langle\phi,\phi\rangle$ is a continuous function $\eta \to \mathbf{R}$ (Problem 1), η^1 is a closed subset of η. Moreover,

η^1 is compact whenever M is (Problem 2). Actually, η^1 is also a codimension-one submanifold of η endowed with the subset topology (see Problem 3), although we do not need that fact in the present discussion. When η is the tangent bundle TM of a Riemannian manifold (M,g), we will call η^1 the *unit tangent bundle of* (M,g) and denote it by T^1M.

The *Ricci curvature* of a Riemannian manifold (M,g) is the real-valued function on T^1M, denoted by Ric (just like the Ricci tensor), which sends any $(x,u) \in T^1M$ to $\mathrm{Ric}(u,u)$. Since the Ricci tensor is continuous, so is $\mathrm{Ric} : T^1M \to \mathbf{R}$ (due to the local-coordinate formula $\mathrm{Ric}(u,u) = R_{jk}u^j u^k$).

The following classical result of Myers [**M**] establishes a relation between the diameter of a complete Riemannian manifold and its Ricci curvature.

THEOREM 35.1. *Let a complete Riemannian manifold* (M,g) *satisfy the Ricci-curvature lower bound*

$$\mathrm{Ric} \ \geq \ (n-1)\,\delta \ > \ 0, \quad \text{where } \ n = \dim M, \tag{35.1}$$

with a constant δ. *Then*

i. M *is compact,*
ii. $\operatorname{diam}(M,g) \ \leq \ \pi/\sqrt{\delta}$.

PROOF. Let $[a,b] \ni t \mapsto w(t) \in T_{x(t)}M$ be any C^∞ vector field with $w(a) = w(b) = 0$, tangent to M along any minimizing geodesic $[a,b] \ni t \mapsto x(t)$ of (M,g). Let us also choose a C^∞ mapping $(t,s) \mapsto x(t,s) \in M$, defined on a rectangle $[a,b] \times [c,d]$ in the ts-plane, such that $x(t,c) = x(t)$, $x(a,s) = x(a)$, $x(b,s) = x(b)$ and $x_s(t,c) = w(t)$ for all t,s, with the subscript conventions of §25. (For instance, we may set $x(t,s) = \exp_{x(t)}[(s-c)w(t)]$.) Defining $\mathsf{L}(s)$ and $\mathsf{A}(s)$ for $s \in [c,d]$ by $\mathsf{L}(s) = \int_a^b |x_t(t,s)|\,dt$ and $2\mathsf{A}(s) = \int_a^b |x_t(t,s)|^2\,dt$, we obtain

$$2(b-a)\mathsf{A}(c) \ = \ [\mathsf{L}(c)]^2 \ \leq \ [\mathsf{L}(s)]^2 \ \leq \ 2(b-a)\mathsf{A}(s) \quad \text{for all } \ s. \tag{35.2}$$

In fact, the three relations, from left to right, follow since the function $t \mapsto |\dot{x}(t)|$ is constant (Problem 1 in §30), the original geodesic $t \mapsto x(t) = x(t,c)$ is minimizing, and, respectively, from the Schwarz inequality for the L^2 inner product of functions $[a,b] \to \mathbf{R}$. Thus, $\mathsf{A}(s)$ assumes its minimum value at $s = c$, leading to the second-derivative relation $\mathsf{A}''(c) \ \geq \ 0$. However, writing $2\mathsf{A}(s) = (x_t, x_t)$, where $(\ ,\ )$ is the appropriate L^2 inner product, we obtain the derivative formula $2\mathsf{A}'(s) = (x_t, x_{ts}) = (x_t, x_{st}) = -(x_{tt}, x_s)$ (integration by parts, with the boundary term vanishing as $w(a) = w(b) = 0$), and so $\mathsf{A}''(c) = -(x_{tts}, x_s)$ (at $s = c$), in view of the geodesic equation $x_{tt}(t,c) = 0$. Since $x_{tts} = x_{stt} - R(x_s, x_t)x_t$ (see (25.6)) and $x_s(t,c) = w(t)$, relation $\mathsf{A}''(c) \geq 0$ thus reads $0 \geq (w, \nabla_{\dot{x}}\nabla_{\dot{x}}w - R(w,\dot{x})\dot{x}) = (R(\dot{x},w)\dot{x}, w) - (\nabla_{\dot{x}}w, \nabla_{\dot{x}}w)$, that is, after integration by parts, with $(\ ,\ )$ as above, $(R(\dot{x},w)\dot{x}, w) \leq (\nabla_{\dot{x}}w, \nabla_{\dot{x}}w)$ for any w such that $w(a) = w(b) = 0$.

Let $\dot{\varphi}$ now stand for the derivative of any C^∞ function $\varphi : [a,b] \to \mathbf{R}$ with $\varphi(a) = \varphi(b) = 0$. Given a parallel unit vector field $t \mapsto u(t)$ along our geodesic $t \mapsto x(t)$, we have $(\varphi^2 R(\dot{x},u)\dot{x}, u) \leq \int_a^b \dot{\varphi}^2\,dt$, as one sees applying the last inequality to $w = \varphi u$. Summing this over n orthonormal fields u with this property, one of which is tangent to the geodesic, we get $\int_a^b \varphi^2\,\mathrm{Ric}(\dot{x},\dot{x})\,dt \leq (n-1)\int_a^b \dot{\varphi}^2\,dt$, which, combined with (35.1), gives $\delta\mathsf{L}^2 \int_a^b \varphi^2\,dt \leq (b-a)^2 \int_a^b \dot{\varphi}^2\,dt$, with L denoting the length of the original minimizing geodesic.

So far the C^∞ function φ with $\varphi(a) = \varphi(b) = 0$ was arbitrary. Now, choosing $\varphi(t) = \sin[\pi(t-a)/(b-a)]$, we can rewrite the last inequality as $\delta\mathsf{L}^2 \le \pi^2$, that is, $\mathsf{L} \le \pi/\sqrt{\delta}$. This yields (ii), and, in view of Corollary 33.4, also implies (i), completing the proof. □

Problems

1. Verify that, for any pseudo-Riemannian or pseudo-Hermitian fibre metric $g = \langle\,,\rangle$ in a C^∞ real or complex vector bundle η over a manifold, the assignment $(x, \phi) \mapsto \langle\phi, \phi\rangle$ defines a C^∞ function $\eta \to \mathbf{R}$.

2. Show that the unit sphere bundle η^1 is compact if so is the base manifold M. (Hint below.)

3. Prove that the unit sphere bundle η^1 is a codimension-one submanifold of η, carrying the subset topology. (Hint below.)

Hint. In Problem 2, M is covered by finitely many open sets whose closures Y are compact and contained in local trivialiation domains, while η^1 is the union of the compact sets $\eta^1 \cap \pi^{-1}(Y)$, with $\pi : \eta \to M$ denoting the bundle projection.

Hint. In Problem 3, note that nonzero real numbers are regular values of the function in Problem 1.

CHAPTER 7

Integration

36. Finite partitions of unity

Topics: Support of a continuous section of a vector bundle; compactly supported sections; finite partitions of unity; small supports.

Let ψ be a global section of a vector bundle η over a manifold M. The *support* of ψ, denoted $\operatorname{supp}\psi$, is the closure in M of the set $\{x \in M : \psi(x) \neq 0\}$. In other words, $\operatorname{supp}\psi$ is the complement $M \setminus U$ of the largest open subset U of M on which ψ vanishes ('largest' meaning the union of all such open subsets). We also say that ψ is *compactly supported* if $\operatorname{supp}\psi$ is compact, and that it is *supported in an open set* $U \subset M$ if $\operatorname{supp}\psi \subset U$.

Suppose that Y is a subset of a manifold M. By a *finite partition of unity* for Y we mean any finite family $\varphi_1, \dots, \varphi_k$ of compactly supported C^∞ functions $\varphi_q : M \to \mathbf{R}$ such that $0 \leq \varphi_q \leq 1$ for all $q = 1, \dots, k$, and $\varphi_1 + \dots + \varphi_k = 1$ on some open set containing Y. If, in addition, $\mathcal{U}$ is an open covering of Y (§14), that is, a family of open sets in M whose union contains Y, we will say that the partition of unity $\varphi_1, \dots, \varphi_k$ for Y is *subordinate to* $\mathcal{U}$ if for every $q \in \{1, \dots, k\}$ there exists $U \in \mathcal{U}$ with $\operatorname{supp}\varphi_q \subset U$.

Lemma 36.1. *Let $\mathcal{U}$ be an open covering of a compact subset Y of a manifold M. Then there exists a finite partition of unity for Y, subordinate to $\mathcal{U}$.*

Proof. Every manifold M satisfies, by definition, the countability axiom (§14). Given $x \in M$, let us choose open sets $U'(x)$, $U(x)$, U^x and a compactly supported C^∞ function $f_x : M \to [0, \infty)$ with $U^x \in \mathcal{U}$, $x \in U'(x) \subset U(x) \subset U^x$ and $f_x > 0$ on $U'(x)$. (See Problem 19 in §6.) In view of the Borel-Heine Theorem (§14) there is a finite collection of points $x_1, \dots, x_k \in M$ such that $Y \subset \bigcup_{q=1}^k U'(x_q)$. Let us write U_q, U'_q and f_q instead of $U(x_q)$, $U'(x_q)$ and f^{x_q}. Now $U' = U'_1 \cup \ldots \cup U'_k$ is an open set containing Y and $f = f_1 + \ldots + f_k$ is positive on U'. Choosing a C^∞ function $\varphi : M \to \mathbf{R}$ with $\varphi = 1$ on an open set containing Y and $\operatorname{supp}\varphi \subset U'$ (see Problem 3), we may define the required functions φ_q, $q = 1, \dots, k$, by $\varphi_q = \varphi f_q / f$. This completes the proof. ☐

We will say that a subset Y of a manifold M is *small* if it is compact and contained in an open set diffeomorphic to an open ball in $\mathbf{R}^n$, $n = \dim M$. The components x^j of such a diffeomorphism then form a coordinate system, whose domain contains Y.

Applying Lemma 36.1 to the family of all open sets U diffeomorphic to an open ball in $\mathbf{R}^n$, $n = \dim M$, we obtain the following result.

Proposition 36.2. *For any compact subset Y of a manifold M, there exists a finite partition of unity $\varphi_1, \dots, \varphi_k$ for Y such that the support of each φ_q is small.*

Here is a further consequence.

COROLLARY 36.3. *Every compactly supported* C^l *section* ψ *of any vector bundle* η *over a manifold, where* $l = 0, 1, \ldots, \infty$*, can be written as a finite sum* $\psi = \psi_1 + \ldots + \psi_k$ *of* C^l *sections of* η *whose supports are all small.*

In fact, we may set $\psi_q = \varphi_q \psi$, with φ_q chosen as in Proposition 36.2.
The next result will not be needed until §58:

LEMMA 36.4. *For every compact connected* n*-dimensional manifold* M *there exist an integer* $k \geq 1$*, open sets* $U_q \subset M$ *and* C^∞ *functions* $\varphi_q : M \to \mathbf{R}$*,* $q = 1, \ldots, k$*, such that*

a. *Each* U_q *is diffeomorphic to an open ball in* $\mathbf{R}^n$*,*
b. *The support of each* φ_q *is compact and contained in* U_q*,*
c. $\sum_{q=1}^{k} \varphi_q = 1$*,*
d. $U_q \cap U_{q+1}$ *is nonempty for all* $q = 1, \ldots, k-1$*.*

PROOF. For every $x \in M$, let us choose open sets $U(x)$, $U'(x)$ with $x \in U'(x) \subset U(x)$ and such that there is a diffeomorphism of $U(x)$ onto an open ball in $\mathbf{R}^n$ which sends $U'(x)$ onto a smaller concentric open ball. Thus, there is a C^∞ function $f^x : M \to [0, \infty)$ with $f^x > 0$ on $U'(x)$, whose support is a compact subset of $U(x)$. By the Borel-Heine Theorem (§14) we can find a finite set $\varGamma \subset M$ such that $M = \bigcup_{x \in \varGamma} U'(x)$ and $\varGamma$ is a minimal set with that property, i.e., $\bigcup_{x \in \varGamma'} U'(x) \neq M$ for each proper subset $\varGamma'$ of $\varGamma$. For every $x \in \varGamma$ we may thus select $z(x) \in M$ with

$$z(x) \in U'(x) \setminus \bigcup_{x' \in \varGamma \setminus \{x\}} U'(x') . \tag{36.1}$$

Since M is connected, we can find a continuous curve $\gamma : [a, b] \to M$ whose image $\gamma([a, b])$ contains all $z(x)$ with $x \in \varGamma$, that is,

$$\{z(x) : x \in \varGamma\} \subset \gamma([a, b]) . \tag{36.2}$$

Compactness of $[a, b]$ now implies (Problem 2) that there are an integer $k \geq 1$ and $t_0, \ldots, t_k \in \mathbf{R}$ such that $a = t_0 < t_1 < \ldots < t_k = b$ and each of the curve segments $\gamma([t_{q-1}, t_q])$, $q = 1, \ldots, k$, is contained in the set $U'(x_q)$ for some $x_q \in \varGamma$. From now on we will write U_q, U'_q and f_q instead of $U(x_q)$, $U'(x_q)$ and f^{x_q}. Now (a) is obvious, and so is (d) since $\gamma(t_q) \in U_q \cap U_{q+1}$. Furthermore, the collection $U_1, \ldots, U_k$ contains (possibly with repetitions) all the $U(x)$ for $x \in \varGamma$, since the U_q cover $\gamma([a, b])$, while no proper subfamily of the $U(x)$ does (by (36.1), (36.2)). Hence $M = U_1 \cup \ldots \cup U_k$ and so $f = f_1 + \ldots + f_k$ is positive everywhere on M. Setting $\varphi_q = f_q / f$, we now obtain (b) and (c), which completes the proof. $\square$

Problems

1. Let $\mathcal{U}$ be a family of open sets in $\mathbf{R}$ covering (that is, containing in their union) a closed interval $[a, b]$. Show the existence of $\varepsilon > 0$ such that for every subinterval I of $[a, b]$ whose length $|I|$ is less than ε, there is $U \in \mathcal{U}$ with $I \subset U$. (Hint below.)

2. Let $\mathcal{U}$ be a family of open sets in a manifold M and let $\gamma : [a, b] \to M$ be a continuous curve whose image set $\gamma([a, b])$ is contained in the union of $\mathcal{U}$. Prove the existence of an integer $k \geq 1$ and $t_0, \ldots, t_k \in \mathbf{R}$ such that

$a = t_0 < t_1 < \ldots < t_k = b$ and each of the curve segments $\gamma([t_{q-1}, t_q])$, $q = 1, \ldots, k$, is contained in some $U \in \mathcal{U}$. (Hint below.)

3. *Separation of sets by functions.* Let there be given a manifold M (satisfying, by definition, the countability axiom, cf. §14), a closed subset K of M and a compact subset K' of M. Prove that, if K and K' are disjoint, then there exists a C^∞ function $f : M \to \mathbf{R}$ with $0 \le f \le 1$ and $f = 1$ on an open set containing K, as well as $f = 0$ on an open set containing K'. (Hint below.)

Hint. In Problem 1, if no such $\varepsilon > 0$ existed, we could find two sequences x_k, y_k in $[a, b]$ with $|x_k - y_k| \to 0$ as $k \to \infty$, while, for each k, $[x_k, y_k]$ would not be contained in any $U \in \mathcal{U}$. Choosing a convergent subsequence of the x_k, we then obtain a contradiction.

Hint. In Problem 2, extend γ continuously to an open interval containing $[a, b]$ (e.g., making it constant beyond a and beyond b), and then apply Problem 1 to the family of γ-preimages of the sets forming $\mathcal{U}$.

Hint. In Problem 3, for any fixed $x \in K'$ there is a function $\phi = \phi_x$ satisfying the conditions in Problem 19 of §6, including $\phi_x = 0$ on some neighborhood U_x of x. By the Borel-Heine theorem, finitely many of the U_x cover K', and we can let f be the product of the corresponding ϕ_x.

37. Densities and integration

Topics: Densities in real vector spaces; components; positivity; the bundle of densities; densities on manifolds; the volume element of a Riemannian manifold; a generalization to the pseudo-Riemannian case; integration of compactly supported densities on manifolds; the volume of a compact manifold with a fixed positive density.

Let V be a real vector space of dimension n, with $0 \le n < \infty$, and let $\mathcal{B}(V)$ be the set of all (ordered) bases of V. By a *density* in V we mean any function $\mu : \mathcal{B}(V) \to \mathbf{R}$ with the property that

$$\mu_{1' \ldots n'} \,=\, |\mathcal{J}| \mu_{1 \ldots n}\,, \quad \text{with } \mathcal{J} \,=\, \det [e^a_{a'}], \tag{37.1}$$

for any two bases e_a and $e_{a'}$ of V. Here we write $\mu_{1 \ldots n}$ instead of $\mu(e_1, \ldots, e_n)$, while $\mathcal{J}$ is the determinant of the transition matrix $[e^a_{a'}]$ with the entries characterized by $e_{a'} = e^a_{a'} e_a$.

A density thus is uniquely determined by the value it assigns to any single fixed basis; in other words, densities in V form a line (that is, a 1-dimensional real vector space). The line of densities in V is naturally oriented, since a nonzero density is clearly either positive or negative as a real valued function. A fixed basis e_a of V naturally distinguishes a positive density μ characterized by $\mu_{1 \ldots n} = 1$.

When $n = 0$, the set $\mathcal{B}(V)$ has just one element (the empty basis), so that densities in $V = \{0\}$ are nothing else than real numbers, with the unique distinguished density correponding to the number 1.

Applying this fibre-by-fibre to any C^∞ real vector bundle η over a manifold M, we obtain the C^∞ real-line *bundle of densities* in η, with the local trivializing C^∞ sections distinguished as above (at each point of the trivialization domain) by all possible local C^∞ trivializations of η. (Differentiability of the resulting transition functions is obvious from (37.1).) In the case of the tangent bundle $\eta = TM$, global sections of its bundle of densities will from now on be called *densities on the manifold* M. A density μ on M is represented, in any local coordinates, by its *component function* $\mu_{1 \ldots n}$, defined, at any point x of the coordinate domain. by

$\mu_{1\ldots n}(x) = \mu(p_1(x), \ldots, p_n(x))$, where p_j are the coordinate vector fields. A density on M is positive (at every point) if and only if so are its component functions in all local coordinate systems.

A Riemannian metric g on a manifold M canonically defines a positive C^∞ density μ on M, known as the *volume element* of the Riemannian manifold (M, g). Namely, given $x \in M$, we declare $\mu(x)$ to be the positive density in T_xM naturally distinguished, as descibed above, by any g-orthonormal basis of T_xM. This definition is correct, that is, $\mu(x)$ does not depend on the choice of such a basis (which is clear from (37.1) since, according to Problem 18 in §12, the transition matrix $\mathfrak{A}$ between two orthonormal bases has $\det \mathfrak{A} = \pm 1$). Instead of using the generic symbol μ, we will denote the volume element of a Riemannian metric g by dg. In local coordinates x^j, the component function $\mu_{1\ldots n}$ of $\mu = dg$ is given by

$$(dg)_{1\ldots n} = \sqrt{\det g}\,, \qquad n = \dim M \tag{37.2}$$

(see Problems 1 and 2), so that dg is C^∞-differentiable; here $\det g$ is the function, depending on the choice of the coordinates, which assigns to each point x of the coordinate domain the determinant of the matrix $[g_{jk}(x)]$.

Suppose now that μ is a compactly supported continuous density on a manifold M. We define the *integral* of μ to be the real number $\int_M \mu$ obtained as follows. First, let us assume that $\operatorname{supp}\mu$ is small in the sense of §36, and choose a coordinate system x^j containing $\operatorname{supp}\mu$ in its domain. We then declare $\int_M \mu$ to be the ordinary integral of the component function $\mu_{1\ldots n}$ of μ treated, with the aid of the coordinate diffeomorphism, as a compactly supported continuous function on $\mathbf{R}^n$, where $n = \dim M$. The latter integral does *not* depend on the coordinates used (Problem 4).

Next, let μ be an arbitrary compactly supported continuous density on M. By Proposition 36.2, there exists a finite partition of unity $\varphi_1, \ldots, \varphi_k$ for the compact set $\operatorname{supp}\omega$ such that each $\operatorname{supp}\varphi_q$, $q = 1, \ldots, k$, is small. We then set $\int_M \mu = \sum_{q=1}^k \int_M \varphi_q\mu$, where the small-support integrals $\int_M \varphi_q\mu$ are defined as above. Finally. the number thus obtained does not depend on the partition of unity used, since for another such partition $f_1, \ldots, f_l$ the value $\sum_{a=1}^l \int_M f_a\mu$ coincides with $\sum_{q=1}^k \int_M \varphi_q\mu$ as they both equal $\sum_{a=1}^l \sum_{q=1}^k \int_M f_a\varphi_q\mu$ (here we use the obvious linearity of the small-support integral in μ).

Of particular importance is the case where the manifold M is compact and the continuous density μ on M (often assumed C^∞-differentiable) is also *positive* everywhere on M. (This is, for instance, the case with the volume element dg of any compact Riemannian manifold.) If such μ is fixed, we refer to the integral

$$\operatorname{Vol} M = \int \mu \in \mathbf{R}\,, \tag{37.3}$$

as the *volume* of M. In dimensions 1 and 2 one employs the terms *length* and *area* rather than 'volume' and uses the symbol $\operatorname{Area} M$ for $\operatorname{Vol} M$ when $\dim M = 2$.

Problems

1. Verify (37.2). (Hint below.)

2. Let there be given a positive C^∞ density μ on a manifold M, and a positive C^∞ function $\varphi : M \to \mathbf{R}$. Prove that M admits an atlas in whose every chart $\mu_{1\ldots n} = \varphi$, where $n = \dim M$. (Hint below.)

3. Show that the volume element can be similarly defined in the more general case of a *pseudo-Riemannian* metric. How must formula (37.2) be modified? (Hint below.)

4. Verify that the integral of a continuous density with a small support, defined above, does not depend on the choice of the coordinate system. (Hint below.)

5. Given a Riemannian manifold (M,g) and local coordinates x^j in M, show that

$$\text{(37.4)} \qquad \text{a)}\quad \Gamma^j_{jk} = \partial_k \log\sqrt{\det g}\,, \qquad \text{b)}\quad g^{jl}\partial_k g_{jl} = \partial_k \log\det g\,,$$

where Γ^l_{jk} are the Christoffel symbols of g, and $\det g$ is the same coordinate-dependent function as in (37.2). (Hint below.)

Hint. In Problem 1, fix a point x in the coordinate domain and treat both sides of (37.2) as functions $\mathcal{B} \to \mathbf{R}$. Then note that they agree on orthonormal bases, and obey the same transformation rule under a change of basis.

Hint. In Problem 2, fix a coordinate system x^j at any given point y and use it to find new coordinates at y that have the required property. In view of (37.1) and the inverse mapping theorem, this amounts to finding n functions $F^1,\dots,F^n$ of the variables x^j that are of class C^∞ and $\mu_{1\dots n} = \varphi\,\det[\partial_j F^k]$. We may choose $F^2 = x^2,\dots,F^n = x^n$ and let F^1 be any function with $\partial_1 F^1 = \mu_{1\dots n}/\varphi$.

Hint. In Problem 3, note that Problem 18 in §12 can still be applied. Under the square root symbol in (37.2), the determinant must be replaced by its absolute value.

Hint. In Problem 4, use the transformation rule (37.1) and the change-of-variables formula for n-dimensional Riemann integrals.

Hint. In Problem 5, (b) is immediate from (8.21) with $t = x^k$ and $F = [g_{jl}]$, while (a) then easily follows if one sums (30.3) over $j = l$, noting that two of the three resulting terms cancel each other due to symmetry of g^{jk} (Problem 1 in §29).

38. Divergence operators

Topics: The divergence operator corresponding to a differentiable positive density; the Laplacian of a Riemannian manifold; more general divergence operators; divergences of the curvature and Ricci tensors; the Bianchi identity for the Ricci tensor.

Any fixed positive C^∞ density μ on a manifold M gives rise to the *divergence operator*, which associates with every C^1 vector field w on M the function $\operatorname{div} w$ defined, in local coordinates, by

$$\text{(38.1)} \qquad \mu_{1\dots n}\operatorname{div} w = \partial_j(w^j\mu_{1\dots n})\,, \quad \text{where} \quad n = \dim M\,.$$

THEOREM 38.1. *For any positive C^∞ density μ on a manifold M, the divergence operator* div *with* (38.1) *is well defined, that is, independent of the choice of the coordinate system. Furthermore, if $\mu = dg$ is the volume element of a Riemannian metric g on M, then, for every C^1 vector field w,*

$$\text{(38.2)} \qquad \operatorname{div} w = \operatorname{Trace}\nabla w\,, \quad \text{that is,} \quad \operatorname{div} w = w^j{}_{,j}\,,$$

where ∇ is the Levi-Civita connection of g, and Trace *is the pointwise trace of the vector-bundle morphism $\nabla w : TM \to TM$.*

PROOF. If we *define* div by (38.2), for a fixed metric g, the relation $w^j{}_{,j} = \partial_j w^j + \Gamma^j_{jk}w^k$, obvious from (23.6), will yield (38.1) for $\mu = dg$, as one sees using (37.4) and (37.2). Our assertion now follows since, by (37.2), every positive C^∞ density μ on a coordinate domain U is the volume element of some Riemannian metric on U (for instance, one obtained when one multiplies a prescribed metric by a suitable positive function). This completes the proof. $\square$

For a fixed positive C^∞ density μ on a manifold M, a C^1 function f and a C^1 vector field w, we have

$$\operatorname{div}(fw) = f \operatorname{div} w + d_w f, \tag{38.3}$$

as one easily sees using (38.1) and (5.14).

Let (M,g) be a Riemannnian manifold. The *Laplace operator*, or *Laplacian*, of (M,g) is the operator sending every (local) C^2 function f on M to the function

$$\Delta f = \operatorname{div} \nabla f, \tag{38.4}$$

where ∇f denotes the g-gradient of f. Thus, by (29.4),

$$\Delta f = f^{,j}{}_j = \operatorname{Trace}_g \nabla df = \langle g, \nabla df \rangle = g^{jk} f_{,jk}, \tag{38.5}$$

Note that, by (38.4) and (38.3), with $w = \nabla f$,

$$f \Delta f = \operatorname{div}(f \nabla f) - |\nabla f|^2 \tag{38.6}$$

for any C^2 function f on a Riemannnian manifold (M,g).

Suppose now that we are given a Riemannnian manifold (M,g) and a real vector bundle η over M, along with a connection ∇ in η. We now define a *divergence operator* which sends every C^1 section Ψ of $\operatorname{Hom}(\eta, TM)$ to the section $\operatorname{div}\Psi$ of η^*, given by

$$[\operatorname{div} \Psi]\phi = \operatorname{Trace}\{v \mapsto [\nabla_v \Psi]\phi\}, \quad \text{that is,} \quad [\operatorname{div} \Psi]_a = \Psi^j{}_{a,j}, \tag{38.7}$$

for any $x \in M$ and $phi \in \eta_x$, where v varies in $T_x M$, and ∇ stands also for the connection in $\operatorname{Hom}(\eta, TM)$ induced by the connection ∇ in η and the Levi-Civita connection of g. (The components refer to local coordinates x^j in M and a local trivialization e_a of η.) This operator generalizes div for vector fields on a Riemannnian manifold (Problem 1).

Some other special cases of (38.7) are of interest. First, given a C^2 vector field w on a Riemannnian manifold (M,g), we may apply (38.7) to $\Psi = \nabla w$, which is a section $\operatorname{Hom}(TM, TM)$ (and, in $\eta = TM$, the Levi-Civita connection is used). The resulting divergence $\operatorname{div} \nabla w$ is a cotangent vector field, and appears in the identity

$$\operatorname{Ric}(w, \cdot\,) = \operatorname{div} \nabla w - d(\operatorname{div} w), \tag{38.8}$$

which is nothing else than a coordinate-free version of (24.12). As another example, we can form the divergence $\operatorname{div} R$ of the curvature tensor field of any Riemannian manifold (M,g), with the component functions

$$(\operatorname{div} R)_{jkl} = R_{jkl}{}^s{}_{,s}. \tag{38.9}$$

As ∇ is torsionfree, we may use the second Bianchi identity (26.8), that is, $R_{jkl}{}^p{}_{,s} + R_{ksl}{}^p{}_{,j} + R_{sjl}{}^p{}_{,k} = 0$. Summed over $p = s$, it yields the relation

$$R_{jkl}{}^s{}_{,s} = R_{jl,k} - R_{kl,j}. \tag{38.10}$$

The coordinate-free version of (38.10) reads

$$(\operatorname{div} R)(u, v, w) = (\nabla_v \operatorname{Ric})(u, w) - (\nabla_u \operatorname{Ric})(v, w) \tag{38.11}$$

for any vectors $u, v, w \in T_x M$ and any $x \in M$.

Next, since contractions commute with the covariant differentiation, we have $g^{jl}R_{jkl}{}^{s}{}_{,s} = (g^{jl}R_{jkl}{}^{s})_{,s} = R^s_{k,s}$, the $R^s_k = g^{sl}R_{kl}$ being as usual the components of the g-modified Ricci tensor. (Cf. Lemma 30.2.) Similarly,

$$g^{jl}R_{jl,k} \,=\, (g^{jl}R_{jl})_{,k} \,=\, \mathrm{s}_{,k}\,. \tag{38.12}$$

s being the scalar curvature. Therefore, "multiplying" (38.10) by g^{jl} we obtain the following equality, called *the Bianchi identity for the Ricci tensor*:

$$2\,\mathrm{div\ Ric} \,=\, d\mathrm{s}\,, \quad \text{that is,} \quad 2g^{jl}R_{kl,j} \,=\, \mathrm{s}_{,k}\,. \tag{38.13}$$

where the divergence is applied to the modified Ricci tensor Ric, treated as a section of $\mathrm{Hom}(TM, TM)$.

Problems

1. How does the divergence operator for vector fields on a Riemannnian manifold (M, g) arise as a special case of (38.7)? (Hint below.)

2. Verify that $\Delta[H(f)] = H'(f)\Delta f + H''(f)|\nabla f|^2$ whenever $f : M \to \mathbf{R}$ is a C^2 function on a Riemannnian manifold (M, g), assuming values in some interval $I \subset \mathbf{R}$, and $H : I \to \mathbf{R}$ is of class C^2, while $H(f)$ stands for the composite $H \circ f$, and H' is the derivative of H.

2. Show that $\mathrm{Ric}(w, w) = (\mathrm{div}\,w)^2 - \mathrm{Trace}\,(\nabla w)^2 + \mathrm{div}\,[\nabla_w w - (\mathrm{div}\,w)w]$ for any C^2 vector field w on a Riemannnian manifold (M, g). Here $(\nabla w)^2$ is the vector-bundle morphism $TM \to TM$ obtained by composing ∇w with itself. (Hint below.)

3. A Riemannian manifold (M, g) is said to have *harmonic curvature* if $\mathrm{div}\,R = 0$ identically in M. Verify that this is the case if and only if the Ricci tensor satisfies the *Codazzi equation* $R_{jl,k} = R_{kl,j}$.

4. Let a connected Riemannian manifold (M, g) have harmonic curvature (Problem 3). Show that its scalar curvature is constant. (Hint below.)

Hint. In Problem 1, use the product bundle $\eta = M \times \mathbf{R}$ with the standard flat connection, noting the natural identification $\mathrm{Hom}(\eta, TM) = TM$.

Hint. In Problem 3, use (24.12) to verify that $R_{jk}w^k = w^j{}_{,kj}w^k - w^j{}_{,jk}w^k = w^j{}_{,j}w^k{}_{,k} - w^j{}_{,k}w^k{}_{,j} + (w^j{}_{,k}w^k)_{,j} - (w^j{}_{,j}w^k)_{,k}$.

Hint. In Problem 4, "multiply" the equality $R_{jl,k} = R_{kl,j}$ (see Problem 3) by g^{jl} and use (38.12) – (38.13).

39. The divergence theorem

Topics: The divergence theorem; Bochner's Lemma; Bochner's integral formula.

We begin with a classical result, usually referred to as the *divergence theorem*:

THEOREM 39.1. *For any compactly supported C^1 vector field w on a manifold any fixed positive C^∞ density μ on a manifold M, we have*

$$\int_M (\mathrm{div}\,w)\mu \,=\, 0\,. \tag{39.1}$$

PROOF. See Problem 1. □

Thus, for any compactly supported C^2 function f on a Riemannnian manifold (M,g), combining (39.1) with (38.4) and (38.6), we obtain the integral formulae

$$\text{(39.2)} \qquad \text{i)} \quad \int_M \Delta f \, dg \,=\, 0\,, \qquad \text{ii)} \quad \int_M f\Delta f \, dg \,=\, -\int_M |\nabla f|^2 \, dg\,.$$

The following consequence of the divergence theorem is known as *Bochner's Lemma*:

COROLLARY 39.2. *A C^2 function f on a compact connected Riemannian manifold, such that $\Delta f \ge 0$, is necessarily constant.*

PROOF. See Problem 2. □

A further consequence is *Bochner's integral formula*

$$\text{(39.3)} \qquad \int_M \mathrm{Ric}(w,w)\, dg \,=\, \int_M (\operatorname{div} w)^2 \, dg \,-\, \int_M \operatorname{Trace}(\nabla w)^2 \, dg\,,$$

valid whenever w is a compactly supported C^2 vector field on a Riemannian manifold (M,g). See Problem 3.

Problems

1. Prove the divergence theorem. (Hint below.)
2. Prove Bochner's Lemma. (Hint below.)
3. Establish Bochner's integral formula. (Hint below.)
4. Given a Riemannnian manifold (M,g), a real vector bundle η over M, and a connection ∇ in η, show that, for arbitrary C^1 sections ψ of η and Φ of $\mathrm{Hom}(\eta,TM)$,

$$\text{(39.4)} \qquad \int_M \operatorname{Trace}(\Phi\nabla\psi)\, dg \,=\, \int_M (\delta\Phi)\psi \, dg\,.$$

Hint. In Problem 1, use Corollary 36.3, noting that (39.1) is obvious when the support of w is small: the left-hand side of (39.1) equals the Euclidean integral of the right-hand side of (38.1), in which each of the n summands clearly vanishes.
Hint. In Problem 2, (39.2.i) gives $\int_M \Delta f \, dg = 0$. Since Δf is nonnegative, it must thus vanish identically, and so f is constant by (39.2.ii).
Hint. In Problem 3, use Problem 3 in §38.

40. Theorems of Bochner and Lichnerowicz

Topics: Killing fields and harmonic 1-forms on Riemannian manifolds; Bochner's theorem; eigenvalues and eigenfunctions of the Laplacian; the theorem of Lichnerowicz.

By a *Killing field* on a pseudo-Riemannian manifold (M,g) we mean any C^∞ vector field w on M such that the vector-bundle morphism $\nabla w : TM \to TM$ is skew-adjoint at every point of M.

On the other hand, a *harmonic* 1-*form* on a compact Riemannian manifold (M,g) is defined to be any C^∞ section ξ of $T^*!M$ such that, for the vector field w corresponding to ξ under the index-raising operation (§29), ∇w is self-adjoint at every point and $\operatorname{div} w = 0$ everywhere in M.

Both Killing fields and harmonic forms are of fundamental importance in Riemannian geometry, and will be discussed in more detail later. (See §86 in Appendix D and §69 in Chapter 13.) Here we will establish just one result about them, due to Bochner [**Bo**]:

THEOREM 40.1. *Let* (M,g) *be a compact Riemannian manifold with the Ricci curvature function* $\mathrm{Ric}: T^1M \to \mathbf{R}$.

a. *If* $\mathrm{Ric} > 0$*, then* (M,g) *admits no nontrivial harmonic* 1*-form.*
b. *If* $\mathrm{Ric} \ge 0$*, then every harmonic* 1*-form on* (M,g) *is parallel.*
c. *If* $\mathrm{Ric} < 0$*, then* (M,g) *admits no nontrivial Killing field.*
d. *If* $\mathrm{Ric} \le 0$*, then every Killing field on* (M,g) *is parallel.*

PROOF. This is immediate from (39.3), as in either case the right-hand side of (39.3) has a specific sign, while $\mathrm{Trace}\,(\nabla w)^2$ equals $\pm|\nabla w|^2$. (Note that $\operatorname{div} w = 0$ for any Killing field w.) □

From now on we will use the symbols $(\ ,\ )$ and $\|\ \|$ for the L^2 inner product, and the L^2 norm corresponding to it, for functions $f, \varphi : M \to \mathbf{R}$, vector fields v, w on M, and twice-covariant tensor fields a, b on a Riemannian manifold (M,g), all of which are assumed continuous and compactly supported. Specifically,

$$(f,\varphi) = \int_M f\varphi\, dg\,, \quad (v,w) = \int_M g(v,w)\, dg\,, \quad (a,b) = \int_M \langle a,b\rangle\, dg\,, \tag{40.1}$$

while $\|f\|^2 = (f,f)$ (and similarly in other cases), $\langle\,,\rangle$ being the pointwise inner product with $\langle a,b\rangle = g^{jp}g^{kq}a_{jk}b_{pq}$ (see §28).

For instance, given a compactly supported C^2 function f on a Riemannian manifold (M,g), we have

$$\text{a)}\quad (f,\Delta f) \;=\; -\|\nabla f\|^2, \qquad \text{b)}\quad \int_M \mathrm{Ric}(\nabla f, \nabla f)\, dg \;=\; \|\Delta f\|^2 \,-\, \|\nabla df\|^2, \tag{40.2}$$

as one sees using (39.2.ii) and, respectively, Bochner's integral formula (39.3) with $w = \nabla f$ (so that $\mathrm{Trace}\,(\nabla w)^2 = |\nabla df|^2$).

Let (M,g) now be a *compact* Riemannian manifold. As usual, Δ denotes its Laplacian. We call a real number λ an *eigenvalue of* $-\Delta$ if $\Delta f = -\lambda f$ for some C^2 function $f : M \to \mathbf{R}$ which is not identically zero. Any such f is said to be an *eigenfunction of* $-\Delta$ for the eigenvalue λ (or, simply an *eigenfunction of* $-\Delta$, if λ is not specified). By the *eigenspace of* $-\Delta$ for the eigenvalue λ we mean the vector space of all C^2 functions $f : M \to \mathbf{R}$ with $\Delta f = -\lambda f$.

LEMMA 40.2. *For any compact connected Riemannian manifold* (M,g),

a. 0 *is an eigenvalue of* $-\Delta$ *and the corresponding eigenspace consists of constant functions*;
b. *all nonzero eigenvalues of* $-\Delta$ *are positive.*

PROOF. See Problem 1. □

The next result is due to Lichnerowicz.

THEOREM 40.3. *Let a compact* n*-dimensional Riemannian manifold* (M,g) *satisfy the following lower bound on the Ricci curvature*:

$$\mathrm{Ric} \;\ge\; (n-1)\,\delta \;>\; 0 \tag{40.3}$$

with a constant δ*, and let* λ *be a nonzero eigenvalue of* $-\Delta$*. Then*

$$\lambda \;\ge\; n\delta. \tag{40.4}$$

Proof. Let $f : M \to \mathbf{R}$ be a C^2 function. As $\langle g, g\rangle = n$ (see Problem 2 in §29), the Schwarz inequality $(\Delta f)^2 = \langle g, \nabla df\rangle^2 \le n|\nabla df|^2$ shows that sum of the integrands on the right-hand side of (40.2.b) does not exceed $(n-1)(\Delta f)^2/n$. Therefore, $n\delta\|\nabla f\|^2 \le n(n-1)^{-1}\int_M \mathrm{Ric}(\nabla f, \nabla f)\, dg \le \|\Delta f\|^2$ due to the assumption (40.3). If we now choose f with $\Delta f = -\lambda f$ and $\lambda\|f\| > 0$ (cf. Lemma 40.2(b)), formula (40.2.a) will give $\lambda\|f\|^2 = -(f, \Delta f) = \|\nabla f\|^2$, and so $n\delta\lambda\|f\|^2 = n\delta\|\nabla f\|^2 \le \|\Delta f\|^2 = \lambda^2\|f\|^2$. As $\lambda\|f\| > 0$, this yields (40.4). □

Problems

1. Establish Lemma 40.2. (Hint below.)

2. Under the assumptions of Theorem 40.3, suppose that (40.4) holds as an *equality*, that is, $n\delta$ is a (nonzero) eigenvalue of $-\Delta$. Prove that every eigenfunction f of $-\Delta$ for the eigenvalue $\lambda = n\delta$ then satisfies the equation $n\nabla df = -\lambda f g$. (Hint below.)

Hint. In Problem 1, (a) is obvious from Bochner's Lemma (Corollary 39.2), while (40.2.a) with $\Delta f = -\lambda f$ and $\|f\| \ne 0$ gives $\lambda = \|\nabla f\|^2/\|f\|^2 \ge 0$.

Hint. In Problem 2, consider the equality case in the Schwarz inequality.

41. Einstein metrics and Schur's theorem

Topics: Einstein manifolds; spaces of constant curvature; flat pseudo-Riemannian manifolds; Schur's theorem.

A pseudo-Riemannian manifold (M, g) of any dimension n is called an *Einstein manifold*, and its metric g is said to be an *Einstein metric*, if its Ricci tensor Ric is a multiple of g, that is,

$$\mathrm{Ric} = \kappa g \tag{41.1}$$

for some constant κ. One then refers to κ as the *Einstein constant* of (M, g).

Taking the g-trace §29 of both sides of (41.1) we obtain

$$\kappa = \frac{1}{n}\,\mathrm{s}\,, \qquad \text{where} \quad n = \dim M\,. \tag{41.2}$$

In particular, the scalar curvature s of any Einstein manifold is constant.

Examples of Einstein manifolds will be described in §42 (Example 42.2).

We say that a pseudo-Riemannian manifold (M, g) is a *space of constant curvature* K, where K is a real number, if

$$R_{jlp}{}^{q} = K(g_{jp}\delta_l^q - g_{lp}\delta_j^q)\,, \tag{41.3}$$

R being the curvature tensor of (M, g), that is, if

$$R(u,v)w = K[g(u,w)v - g(v,w)u] \tag{41.4}$$

for all points $x \in M$ and all vectors $u, v, w \in T_xM$. On the other hand, one calls (M, g) a *flat pseudo-Riemannian manifold* if $R = 0$ identically. In dimensions $n \ge 2$, flatness amounts to being a space of constant curvature 0. On the other hand, every pseudo-Riemannian manifold of dimension $n = 1$ is necessarily flat (§31), and relation (41.3) then holds for any given choice of K, since both sides then equal zero; this is why we declare that, by definition, $K = 0$ when $n = 1$.

Lowering an index, we can equivalently rewrite (41.3) in terms of the g-modified curvature tensor:

$$R_{jlpq} = K(g_{jp}g_{lq} - g_{lp}g_{jq}))\,. \tag{41.5}$$

Contracting (41.3) in $l = q$, we obtain $R_{jp} = (n-1)Kg_{jp}$, where $n = \dim M$. Thus, a space of constant curvature K is also an Einstein manifold with the Einstein constant $\kappa = (n-1)K$.

According to Proposition 31.1, equality (41.3) is always satisfied in dimension 2, except that K then stands for the Gaussian curvature function and need not be a constant. On the other hand, the following result, known as *Schur's theorem*, states that in dimensions $n > 2$ the constancy assumption about κ in (41.1) (or K in (41.3)) is redundant:

THEOREM 41.1. *Let a connected pseudo-Riemannian manifold (M, g) of dimension $n \neq 2$ satisfy* (41.1) *with some function $\kappa : M \to \mathbf{R}$. Then κ is constant, that is, (M, g) is an Einstein manifold.*

PROOF. Formula (41.1) implies (41.2) and so $nR_j^k = \mathrm{s}\delta_j^k$. Hence $nR_{j,l}^k = \mathrm{s}_{,l}\delta_j^k$ and $nR_{j,k}^k = \mathrm{s}_{,j}$. However, by (38.13), $2R_{j,k}^k = \mathrm{s}_{,j}$. Hence $(n-2)\,\mathrm{s}_{,j} = 0$, so that s and κ are constant. □

COROLLARY 41.2. *If a connected pseudo-Riemannian manifold (M, g) with $\dim M \neq 2$ satisfies* (41.3) *with a function $K : M \to \mathbf{R}$, then K is constant, that is, (M, g) is a space of constant curvature.*

Problems

1. Verify that if a connected Einstein manifold (M, g) admits a local C^1 vector field w which is parallel (Problem 3 in §20), then either $w = 0$ identically, or $\mathrm{Ric} = 0$ everywhere. (Hint below.)

2. Show that the Riemannian product of two Einstein manifolds with the same Einstein constant (or, or two flat pseudo-Riemannian manifolds), is also an Einstein manifold with the same Einstein constant (or, respectively, is also flat). (Hint below.)

3. Can the Riemannian product of two spaces of constant curvature, which are not both flat, ever be a space of of constant curvature? (Hint below.)

4. Verify that a Riemannian manifold (M, g) is an Einstein manifold if and only if its Ricci-curvature function $\mathrm{Ric} : T^1M \to \mathbf{R}$ is constant.

5. Let (M, g) be a connected pseudo-Riemannian manifold, and let $f : M \to \mathbf{R}$ be a C^2 function such that $\nabla df = Cfg$, that is, $f_{,jk} = Cfg_{jk}$ for some constant C.
   (a) Verify that the function $\varphi = g^{jk}f_{,j}f_{,k} - Cf^2$ is constant.
   (b) If, moreover, (M, g) is a space of constant curvature K and f is not constant, then $C = -K$.
   (Hint below.)

Hint. In Problem 1, use (24.12).
Hint. In Problem 2, use Problem 5 of §30.
Hint. In Problem 3, no: if it were, it would be non-flat (as so is at least one factor metric, cf. Problem 5 in §30), while, in coordinates x^j, y^α chosen as in Problem 5 of §30, we have $R_{j\alpha j\alpha} = 0 \neq g_{jj}h_{\alpha\alpha}$.
Hint. In Problem 5, to obtain (a), differentiate φ. As for (b), we can use the Ricci-Weitzenböck identity

$$\xi_{j,kl} - \xi_{j,lk} = -R_{klj}{}^s\xi_s \tag{41.6}$$

for local cotangent vector fields ξ of class C^2, easily derived from (24.10) by lowering an index. We apply (41.6) to $\xi = df$, then use the relation $f_{,jk} = Cfg_{jk}$ and (41.5), followed by a suitable contraction.

42. Spheres and hyperbolic spaces

Topics: Geodesics of standard spheres and hyperbolic spaces; the second order equation satisfied by linear functions restricted to them; constancy of their curvatures; the eigenspace of the minus-Laplacian of a sphere for the lowest positive eigenvalue; examples of Einstein manifolds.

Let V be a real vector space of dimension $n+1$, where $1 \le n < \infty$, endowed with a fixed a nondegenerate symmetric bilinear form $\langle\,,\rangle$ of the sign pattern $\pm + \ldots +$, so that $\langle\,,\rangle$ is Euclidean or Lorentzian, depending on whether the sign $\pm$ is $+$ or $-$. As in §28, let us consider the Riemannian manifold (M,g) defined to be the sphere or hyperbolic space of a fixed radius $a > 0$ in V, so that $M = \{x \in V : \langle x, x\rangle = 1\}$ (if $\pm = +$), or M is one of the two connected components of the hyperboloid $\{x \in V : \langle x, x\rangle = -1\}$ (if $\pm = -$). (Cf. also Problems 11 and 12 in §28.) In both cases, g is the submanifold metric: $g_x(v,w) = \langle v, v\rangle$ for $x \in M$ and $v, w \in T_xM = x^\perp \subset V$ (see Problem 12 in §13).

Given $x \in M$ and $v \in T_xM = x^\perp$ with $\langle v, v\rangle = a^2$ (that is, $g_x(v,v) = a^2$), let $x(t) = \exp_x tv$ be the geodesic of (M,g) with $x(0) = x$ and $\dot{x}(0) = v$, defined on a maximal possible interval containing zero. That interval then is the whole real line and, explicitly,

$$x(t) \;=\; \begin{cases} (\cos t)x \,+\, (\sin t)v\,, & \text{if } \pm \text{ stands for } +,\\ (\cosh t)x \,+\, (\sinh t)v\,, & \text{if } \pm \text{ stands for } -\,. \end{cases} \tag{42.1}$$

In fact, the image set of the curve $x(t)$ *defined* by (42.1) is great circle or one component of a hyperbola, obtained by intersecting M with the plane $P = \mathrm{Span}\,(x, v)$, and so it is the fixed point set of the isometry of (M,g), obtained by restricting to M the reflection in P, that is, the linear isomorphism $V \to V$ equal to Id on P and to $-\mathrm{Id}$ on $P^\perp$. (Note that the reflection preserves the form $\langle\,,\rangle$.) Since $g(\dot{x},\dot{x}) = a^2$ is constant as a function of t, the conclusion that (42.1) defines a geodesic is now immediate if one combines Problem 2 in §34 with Problem 5 in §33 and Problem 1 in §31. Using affine change of parameter (cf. Problem 5 in §22), we can easily modify (42.1) so as to obtain a formula for every geodesic in (M,g). In particular, (M,g) is complete. (For the sphere, this is also clear from compactness, cf. Theorem 33.3.)

Let $f : M \to \mathbf{R}$ now denote the restriction to our manifold M of a linear (homogeneous) function $V \to \mathbf{R}$. As a function on the Riemannian manifold (M,g), any such f satisfies the equation

$$\nabla df \;=\; -Kfg\,, \quad \text{that is,} \quad f_{,jl} \;=\; -Kfg_{jl}\,, \tag{42.2}$$

for the constant $K = \pm 1/a^2$. In fact, since (42.1) defines a geodesic of (M,g), formula (24.15) gives equality in (42.2) when both sides are applied to a pair (v,v), with $v \in T_xM$, for any $x \in M$, such that $g(v,v) = a^2$. Now (42.2) follows from bilinearity and symmetry of ∇df (see Problem 2 in §24). "Multiplying" (42.2) by g^{jl} we now obtain

$$\Delta f \;=\; -nKf\,. \tag{42.3}$$

Thus, every nonzero linear function $V \to \mathbf{R}$, restricted to M, is an eigenfunction of $-\Delta$ for the eigenvalue $\lambda = nK$.

If $n \geq 2$, formula (42.2) also implies that (M, g) *is a space of constant curvature* $K = \pm 1/a^2$. In fact, differentiating (42.2) and then applying the Ricci-Weitzenböck identity (41.6) to $\xi = df$, we obtain the equality in (41.3): both sides agree when "multiplied" by f_q, while every cotagent vector at any point $x \in M$ equals the value at x of df, for some linear function f.

In the case of the sphere, we also have the following theorem about eigenvalues and eigenfunctions of $-\Delta$.

THEOREM 42.1. *Let (M, g) be the n-dimensional standard sphere of radius $a > 0$ around 0 in a Euclidean space V. Then the lowest nonzero eigenvalue of $-\Delta$ for (M, g) is $\lambda = nK$, for $K = 1/a^2$, and the corresponding eigenspace consists precisely of all linear homogeneous functions $V \to \mathbf{R}$, restricted to M.*

PROOF. See Problem 3. $\square$

EXAMPLE 42.2. The results of this and the previous section lead to the following examples of Einstein manifolds.

a. Spaces of nonzero constant curvature K, including standard spheres ($K > 0$), hyperbolic spaces ($K < 0$), and flat manifolds ($K = 0$). Spheres are compact, hyperbolic spaces are noncompact but complete.
b. Flat manifolds, that is, spaces of constant curvature 0. Among them, pseudo-Euclidean (and Euclidean) spaces are noncompact but geodesically complete (cf. Problem 2 in §30). Since 1-dimensional pseudo-Riemannian manifolds are flat (§41) and flatness is preserved by the Riemannian-product operation (Problem 2 in §41), examples of *compact* flat manifolds, in all dimensions, are provided by tori with product metrics.
c. Combining (a) with Problem 2 in §41, we obtain examples of Einstein manifolds in every dimension $n \geq 4$ which are *not* spaces of constant curvature. On the other hand, no such examples exist in dimensions $n \leq 3$ (see Problem 4).

Problems

1. Explain why formula (42.2) fails to imply that (M, g) is a space of constant curvature $K = \pm 1/a^2$ when $n = 1$.

2. Show that, for the n-dimensional sphere (M, g) of radius $a > 0$ centered at 0 in a Euclidean space V, the C^2 functions $f : M \to \mathbf{R}$ satisfying (42.2) are precisely the restrictions to M of linear homogeneous functions on V. (Hint below.)

3. Prove Theorem 42.1. (Hint below.)

4. Prove that every Einstein manifold (M, g) of dimension $n \leq 3$ is a space of constant curvature. (Hint below.)

5. For (M, g) as in Problem 2, show that the distance function is given by

$$\mathrm{d}(x, y) \,=\, a \arccos[\langle x, y\rangle / a^2]\,,$$

where $\arccos : [-1, 1] \to [0, \pi]$ is the inverse of $\cos : [0, \pi] \to [-1, 1]$. (Hint below.)

6. Verify that the distance function of a radius a hyperbolic space of any dimension n, is given by

$$\mathrm{d}(x, y) \,=\, a \cosh^{-1}[|\langle x, y\rangle| / a^2]\,,$$

$\cosh^{-1} : [1, \infty) \to [0, \infty)$ being the inverse function of $\cosh : [0, \infty) \to [1, \infty)$. (Hint below.)

7. Show that $\arccos\langle x, z\rangle \le \arccos\langle x, y\rangle + \arccos\langle y, z\rangle$ for unit vectors x, y, z in a real vector space with a positive-definite inner product $\langle\,,\rangle$. (Hint below.)

Hint. In Problem 2, note that both spaces are of the dimension $n + 1$. (With the obvious inclusion between them, this will prove their equality.) That $\dim W = n+1$ for the space W of all solutions f to (42.2) can be seen by fixing a point $x \in M$ and considering the operator $W \to \mathbf{R} \times T_x^*M$ given by $f \mapsto (f(x), df_x)$. Its injectivity follows from uniqueness of solutions for second-order ordinary differential equations, applied to $f(x(t))$, with $x(t)$ as in (42.1) (and the equation provided by (24.15)).
Hint. In Problem 3, use Problem 2 above and Problem 2 in §40.
Hint. In Problem 4, the cases $n = 1$ and $n = 2$ are settled by the discussion in §41 and Proposition 31.1. If $n = 3$, fix $x \in M$ and a coordinate system at x such that, at x, the the coordinate vector fields form a g-orthonormal basis of T_xM. Then directly verify the equality (41.5) with $K = \kappa/2$ (where κ is the Einstein constant of (M, g)).
Hint. In Problem 5 and Problem 6, use (42.1) and the Hopf-Rinow theorem.
Hint. In Problem 7, use Problem 4 and the triangle inequality for d.

43. Sectional curvature

Topics: The sectional curvature function of a pseudo-Riemannian manifold.

Let (M, g) pseudo-Riemannian manifold of dimension $n \ge 2$. By the *Grassmannian of nondegenerate planes* in (M, g) we mean the set G_2 of all pairs (x, P) formed by a point $x \in M$ and a two-dimensional vector subspace $P \subset T_xM$ such that the restriction of the metric g_x to P is nondegenerate. (Cf. Problem 16 in §12.) The dependence of G_2M on the metric g is, for simplicity, suppressed in our notation, just as it was in the case of T^1M in §35. Note that, however, G_2 is the same for all positive-definite metrics g on M.

Given $(x, P) \in G_2$, we let ε_P stand for 1 or -1, depending on whether the restriction of $\langle\,,\rangle$ to P is definite or not. The *sectional curvature* of (M, g) is the function $K : G_2 \to \mathbf{R}$ given by

$$K(x, P) = \varepsilon_P R(v, w, v, w) \tag{43.1}$$

for $(x, P) \in G_2$ and any basis v, w of P which is *orthonormal*, that is, $|g(v, v)| = |g(w, w)| = 1$ and $g(v, w) = 0$. In view of (8.18) and Problem 1, this definition is correct, that is, independent of the choice of such a basis v, w.

Problems

1. Given a finite-dimensional real or complex vector space V and a nondegenerate symmetric bilinear form $\langle\,,\rangle$ on V, let us call a basis e_α of V *orthonormal* if $\langle e_\alpha, e_\beta\rangle = 0$ for $\alpha \ne \beta$ and $\langle e_\alpha, e_\alpha\rangle = \varepsilon_\alpha = \pm 1$ for each α. Show that, for any two orthonormal bases e_α and $e_{\alpha'}$ of V, the transition matrix $[A^\alpha_{\alpha'}]$, defined by $e_{\alpha'} = A^\alpha_{\alpha'} e_\alpha$, satisfies

$$\det[A^\alpha_{\alpha'}] = \pm 1\,.$$

(Hint below.)

2. Verify that, for a space of constant curvature K of dimension greater than 1, the sectional curvature function is constant and equal to K.

3. Explain how the sectional curvature of a pseudo-Riemannian surface may be identified with its Gaussian curvature.

4. Let v_q, $q = 1, \dots, n$, be an orthonormal basis of the tangent space T_xM at a point x in an n-dimensional Riemannian manifold (M, g), and let us set $K_{qr} = K(x, P)$ whenever $q, r \in \{1, \dots, n\}$, $q \neq r$, and $P = \mathrm{Span}\,(v_q, v_r)$. Show that $\mathrm{Ric}\,(v_q, v_q) = \sum_r K_{qr}$, with r ranging in $\{1, \dots, n\} \setminus \{q\}$.

5. Let (M, g) be a four-dimensional Riemannian manifold. Prove that (M, g) is an Einstein manifold if and only if $K(x, P) = K(x, P^\perp)$ for every pair $(x, P) \in G_2$, where $P^\perp$ is the orthogonal complement of P in T_xM. (Hint below.)

6. By an *algebraic curvature tensor* in a real vector space V we mean any quadrilinear function $R : V \times V \times V \times V \to \mathbf{R}$ with $R(v, v', w, w') = -R(v', v, w, w') = -R(v, v', w', w)$ and $R(u, v, w, u') + R(v, w, u, u') + R(w, u, v, u') = 0$ for all $v, v', w, w', u \in V$. (Thus, R has an additional symmetry: $R(v, v', w, w') = R(w, w', v, v')$, cf. Problem 7 in §30.) Prove that any algebraic curvature tensor R in a real vector space V is uniquely determined by its *biquadratic function*

$$V \times V \ni (v, w) \to R(v, w, v, w). \tag{43.2}$$

(Hint below.)

Hint. In Problem 1, $\langle v_\alpha, v_\beta \rangle = B_\alpha^\rho B_\beta^\sigma \langle e_\rho, e_\sigma \rangle = \sum_\rho \varepsilon_\rho B_\alpha^\rho B_\beta^\rho$, i.e., $\mathfrak{G} = \mathfrak{B}^T \mathfrak{D} \mathfrak{B}$ with $\mathfrak{D} = \mathrm{diag}\,(\varepsilon_1, \dots, \varepsilon_n)$.

Hint. In Problem 5, let us fix a point $x \in M$ and an orthonormal basis e_j of T_xM. Setting $R_{jlpq} = R(e_j, e_l, e_p, e_q)$ and $R_{jl} = \mathrm{Ric}(e_j, e_l)$, we have $R_{jl} = \sum_q R_{jqlq}$. Let $\{j, l, p, q\} = \{1, 2, 3, 4\}$. If $K(x, P) = K(x, P^\perp)$ for all planes P in T_xM, it follows from (30.7) that $0 = R(e_j + e_l, e_p, e_j + e_l, e_p) - R(e_j - e_l, e_q, e_j - e_l, e_q) = R_{jplp} + R_{jqlq} = R_{jl}$, and $0 = (R_{jpjp} - R_{lqlq}) + (R_{jqjq} - R_{lplp}) = R_{jj} - R_{ll}$, so that Ric_x is a multiple of g_x, and, as x was arbitrary, g is an Einstein metric by Theorem 41.1. Conversely, let g be an Einstein metric, and, with x and e_j as above, let us fix j, p, q with $\{j, p, q\} = \{2, 3, 4\}$. Setting $A_l = R_{1l1l} - R_{pqpq}$ for $l = 2, 3, 4$, we have $D_p + D_q = R_{11} - R_{jj} = 0$ and, similarly, $D_j + D_p = D_j + D_q = 0$, so that $D_2 = D_3 = D_4$. Thus, for instance, $K(x, P) = K(x, P^\perp)$ for the plane $P = \mathrm{Span}\,(e_1, e_2)$, and hence for all planes P in T_xM (since the orthonormal basis e_j was arbitrary).

Hint. In Problem 6, write $abcd = R(a, b, c, d)$ whenever $a, b, c, d \in V$. Since a symmetric bilinear form is determined by its quadratic function, the function (43.2) determines, via the quadratic function $b \mapsto abab$, also the symmetric form $(b, d) \mapsto abad$. This form in turn uniquely determines (for the same reason) the form $(a, c) \mapsto abcd + cbad$ and, consequently, also the substitution version $(a, d) \mapsto abdc + dbac$. Subtracting the last two forms (treated as functions of (a, b, c, d)), we see that (43.2) determines $(abcd + cbad) - (abdc + dbac) = (abcd - abdc) + cbad - dbac = (abcd + abcd) - cbda - bdca = 2abcd + dcba = 2abcd + badc = 3abcd$, as required.

CHAPTER 8

Geometry of Submanifolds

45. Projected connections

Topics: The second fundamental forms of direct-summand subbundles of a vector bundle with a connection; the van der Waerden-Bortolotti connection; the Gauss formula and the Codazzi equation for projected connections.

Let ∇ be a given connection in a real or complex C^∞ vector bundle η over a manifold M endowed with a fixed direct-sum decomposition

$$\eta = \eta^+ \oplus \eta^- \tag{45.1}$$

into C^∞ subbundles $\eta^\pm$. Note that ∇ is *not* assumed to be a direct-sum connection.

Recall (Example 27.3) that ∇ then can be projected onto connections $\nabla^\pm$ in $\eta^\pm$ with $\nabla_v^\pm \psi = (\nabla_v^\pm \psi)^\pm$ for any $x \in M$, $v \in T_xM$ and a local C^1 section ψ of $\eta^\pm$ defined near x, where $\phi = \phi^+ + \phi^-$ is the decomposition of any $\phi \in \eta_x$ relative (45.1), while the $\eta^\pm$ components of any local section ψ of η are the local sections $\psi^\pm$ of $\eta^\pm$ given by $(\psi^\pm)(x) = [\psi(x)]^\pm$. We will skip the extra parentheses and simply write $\psi^\pm(x)$.

The *second fundamental form* of the decomposition (45.1) relative to ∇ is the section b of the vector bundle $\mathrm{Hom}(TM, \mathrm{Hom}(\eta,\eta))$, defined by requiring that it assign to any vector field v on M the morphism $b(v,\,\cdot\,) : \eta \to \eta$ sending any C^1 section ψ of η to the section $b(v,\psi)$ characterized by

$$\nabla_v \psi^\pm = \nabla_v^\pm \psi^\pm + b(v,\psi). \tag{45.2}$$

A crucial fact about b is that it actually is a section of $\mathrm{Hom}(TM, \mathrm{Hom}(\eta,\eta))$. In other words, the value at any point $x \in M$ of the section $b(v,\psi)$ of η, defined by (45.2), depends on v and ψ only through their values at x. For v this is clear since both ∇ and $\nabla^\pm$ are connections (cf. (20.6); for ψ, it follows, according to the hint for Problem 1 in §20, from the equality

$$\nabla = \overline{\nabla} + b\,, \tag{45.3}$$

which is in turn immediate from (45.2), $\overline{\nabla}$ being the direct-sum connection

$$\overline{\nabla} = \nabla^+ \oplus \nabla^-, \tag{45.4}$$

in $\eta = \eta^+ \oplus \eta^-$, known (in this particular context) as the *van der Waerden-Bortolotti connection*. (Note that $\overline{\nabla}_v \psi = \nabla_v^+ \psi^+ + \nabla_v^- \psi^-$ and $\nabla_v \psi = \nabla_v^+ \psi^+ + \nabla_v^- \psi^- + b(v,\psi)$ for local C^1 sections ψ of η and tangent vectors v.)

Clearly, for any $x \in M$, $v \in T_xM$ and $\phi^\pm \in \eta^\pm$,

$$b(v,\phi^\pm) \in \eta_x^\mp, \qquad \text{that is,} \quad b(T_xM \times \eta_x^\pm) \subset \eta_x^\mp. \tag{45.5}$$

Instead of b, we will often use its restriction $b^\mp : T_xM \times \eta_x^\pm \to \eta_x^\mp$, which we will call the *second fundamental form* of the summand $\eta^\pm$ in the decomposition (45.1). Thus, $b^\mp$ is a section of the vector bundle $\mathrm{Hom}(TM, \mathrm{Hom}(\eta^\pm, \eta^\mp))$.

The curvatures R of ∇ and $R^\pm$ of $\nabla^\pm$ are related by the *Gauss-Codazzi identity*

$$\begin{aligned} R(v,w)\psi \,=\, & R^+(v,w)\psi^+ + R^-(v,w)\psi^- - b(v,b(w,\psi)) + b(w,b(v,\psi)) \\ & + (\overline{\nabla}_w b)(v,\psi) - (\overline{\nabla}_v b)(w,\psi) - b(\mathrm{T}(v,w),\psi)\,, \end{aligned} \tag{45.6}$$

for $v, w \in T_xM$, $\psi \in \eta_x$ and $x \in M$. Here T is the torsion tensor field of a fixed connection in the tangent bundle TM. (Note that b is a section of $\mathrm{Hom}(TM, \mathrm{Hom}(\eta,\eta))$, and so, to form the covariant derivative $\overline{\nabla}_w b$, we first need to fix a connection in TM.) In fact, since (45.6) is linear in ψ, we may assume that $\psi = \psi^\pm \in \eta_x^\pm$, and then (45.6) becomes the requirement that

$$\begin{aligned} R(v,w)\psi^\pm \,=\, & R^\pm(v,w)\psi^\pm - b^\pm(v,b^\mp(w,\psi^\pm)) + b^\pm(w,b^\mp(v,\psi^\pm)) \\ & + (\overline{\nabla}_w b^\mp)(v,\psi^\pm) - (\overline{\nabla}_v b^\mp)(w,\psi^\pm) - b^\mp(\mathrm{T}(v,w),\psi^\pm) \end{aligned} \tag{45.7}$$

for both choices of the sign $\pm$. For a proof of (45.7), see Problem 2.

Note that the operation of projecting connections onto summands of a direct-sum decomposition treats the curvature tensor in a more complicated way than the other operations on connections, discussed before: $R^\pm$ may be nonzero, even if the original connection ∇ is flat.

The right-hand side of each of the equalities (45.6) – (45.7) is written as the sum of its $\eta^\pm$ component (the first line) and its $\eta^\mp$ component (the second line). The η^+ and η^- component versions of (45.7), treated as separate identities, are known as the *Gauss formula*

$$R^\pm(v,w)\psi^\pm \,=\, [R(v,w)\psi^\pm]^\pm + b^\pm(v,b^\mp(w,\psi^\pm)) - b^\pm(w,b^\mp(v,\psi^\pm))\,, \tag{45.8}$$

and the *Codazzi equation*

$$[R(v,w)\psi^\pm]^\mp \,=\, (\overline{\nabla}_v b^\mp)(w,\psi^\pm) - (\overline{\nabla}_w b^\mp)(v,\psi^\pm) + b^\mp(\mathrm{T}(v,w),\psi^\pm)\,. \tag{45.9}$$

A case of particular interest arises when η carries a fixed pseudo-Riemannian fibre metric $g = \langle\,,\rangle$ compatible with the original connection ∇, such that η^+ and η^- are mutually g-orthogonal. (Thus, either of $\eta^\pm$ is g-nondegenerate, and coincides with the other's g-orthogonal complement; cf. Problem 1 in §28.) It then follows that the projected connections $\nabla^\pm$ are compatible with the metrics in $\eta^\pm$ obtained by restricting g. In addition,

$$\langle b(v,\psi^+),\psi^-\rangle + \langle \psi^+, b(v,\psi^-)\rangle \,=\, 0 \tag{45.10}$$

for any $x \in M$, $v \in T_xM$ and $\psi^\pm \in \eta_x^\pm$. In other words, $b^+(v,\cdot)$ and $b^-(v,\cdot)$ are each other's negative adjoints. Moreover, g then gives rise to the restricted metrics in the subbundles $\eta^\pm$ (also denoted $g = \langle\,,\rangle$), either of which is automatically compatible with the corresponding projected connection. We can thus form the g-modified versions of the curvatures of ∇ and $\nabla^\pm$ and use notations such as $R(v,w,\psi,\phi) = \langle R(v,w)\psi,\phi\rangle$ (see §28). In view of (45.10), the Gauss formula (45.8) now takes the form

$$\begin{aligned} & R^\pm(v,w,\psi^\pm,\phi^\pm) \\ & = R(v,w,\psi^\pm,\phi^\pm) + \langle b^\mp(v,\psi^\pm), b^\mp(w,\phi^\pm)\rangle - \langle b^\mp(v,\phi^\pm), b^\mp(w,\psi^\pm)\rangle\,. \end{aligned} \tag{45.11}$$

Problems

1. For a C^∞ connection ∇ in a vector bundle η over a manifold M and a global C^∞ section F of the vector bundle $\mathrm{Hom}(TM, \mathrm{Hom}(\eta,\eta))$, show that the curvature tensor fields R of ∇ and R' of the connection $\nabla' = \nabla + F$ are related by

$$\begin{aligned} R'(v,w)\psi \,&=\, R(v,w)\psi \,+\, R^-(v,w)\psi \,-\, F(v,F(w,\psi)) \,+\, F(w,F(v,\psi)) \\ &\quad+\, (\nabla_w F)(v,\psi) \,-\, (\nabla_v F)(w,\psi) \,-\, F(\mathrm{T}\,(v,w),\psi)\,, \end{aligned} \tag{45.12}$$

where T is the torsion tensor field of any fixed C^∞ connection in the tangent bundle of M, used to form the covariant derivative $\nabla_w F$.

2. Derive (45.6) from (45.3) and (45.12). (Hint below.)

3. Establish (45.10).

4. Let η and ζ be C^∞ real/complex vector bundles over a C^∞ manifold M, and let $h : \eta \to \zeta$ be a C^∞ vector-bundle isomorphism. Verify that h then can be used to transport (push-forward) any connection ∇ in η onto a connection $h\nabla$ in ζ, given by

$$[h\nabla]_v\psi \,=\, h_x(\nabla_v(h^{-1}\psi))$$

for any $x \in M,\ v \in T_xM$ and a local C^1 section ψ of ζ defined near x, where h^{-1} is the inverse of h. (One often says that $h\nabla$ is obtained from ∇ by a *gauge transformation.*) Prove that the curvature tensor hR of $h\nabla$ can be expressed in terms of the curvature tensor R of ∇ as

$$[hR]_x(v,w)\phi \,=\, h_x[R_x(v,w)(h_x^{-1}\phi_x)] \tag{45.13}$$

for $x \in M,\ v,w \in T_xM$ and $\phi \in \zeta_x$. (Hint below.)

5. For $\eta,\ \zeta,\ h$ as in Problem 33.1 and a pseudo-Riemannian/Hermitian fibre metric $g = \langle\,,\rangle$ in η, define the *push-forward* hg of g under h so that it is a fibre metric in ζ, and show that if g and a connection ∇ in η are compatible, then so are hg and $h\nabla$. (Hint below.)

6. Verify that any C^∞ diffeomorphism $F : M \to N$ between C^∞ manifolds M and N can be used to transport (push-forward) any connection ∇ in TM onto a connection $(dF)\nabla$ in TN with

$$[(dF)\nabla]_{df_xv}[(dF)w] \,=\, dF_x(\nabla_v w) \tag{45.14}$$

for any $x \in M,\ v \in T_xM$ and a local C^1 tangent vector field w in M defined near x, where $(dF)w$ is the push-forward of w under the diffeomorphism F, defined as in (6.9). Show that the curvature and torsion tensors $(dF)R$ and $(dF)\mathrm{T}$ of $(dF)\nabla$ satisfy the condition

$$[(dF)R]_{F(x)}(dF_xv, dF_xw)dF_xu \,=\, dF_x(R_x(v,w)u)\,, \tag{45.15}$$

$$[(dF)\mathrm{T}]_{F(x)}(dF_xv, dF_xw)dF_xu \,=\, dF_x(\mathrm{T}_x(v,w)u) \tag{45.16}$$

for $x \in M$ and $v,w,u \in T_xM$, where R and T are the curvature and torsion tensors of ∇. (Hint below.)

7. Given a C^∞ diffeomorphism $F : M \to N$ between C^∞ manifolds M and N and a pseudo-Riemannian metric g on M, verify that the *push-forward* $(dF)g = (F^{-1})^*g$ of g under F is a pseudo-Riemannian metric on N whose Levi-Civita connection and curvature tensor are and $(dF)\nabla$ and $(dF)R$ (notation of Problem 6), where ∇ and R are the analogous objects for g. (Hint below.)

8. Let $F : M \to M'$ be an isometry between the pseudo-Riemannian manifolds (M, g) and (M', g'). Verify that $(dF)g = g'$, $(dF)\nabla = \nabla'$ and $(dF)R = R'$, where ∇, R, ∇' and R' are the Levi-Civita connection and curvature tensor of (M, g) and (M', g'), respectively (notation of Problem 6).

Hint. In Problem 2, extend ψ, v, w to local C^∞ sections of $\eta^\pm$ and TM defined near x. Then use (20.8) and (24.2).

Hint. In Problems 4 – 7, use local trivializations e_a in η and Fe_a in ζ, or local coordinates y^j in N and F^j in M (to make h or F appear as Id), and note that the components of ∇ and g then coincide, as functions of the coordinates, with those of their push-forwards. Then apply (20.10), (23.2) and (30.3).

46. The second fundamental form

Topics: The second fundamental form of an immersion into a Riemannian manifold; the Gauss and Codazzi equations for immersions; totally geodesic and totally umbilical immersions.

Let $F : M \to N$ be a C^∞ immersion of a manifold M in a Riemannian manifold (N, h). We use the symbols τ and $g = F^*h$ for the tangent bundle TM and the pullback metric on M, identifying τ, as in (19.1), with a subbundle of the pullback bundle $F^*(TN)$ over M. The *normal bundle* ν of F is defined to be the orthogonal complement of τ in $F^*(TN)$ relative to the pullback fibre metric $\langle\,,\rangle$. Note that ν is naturally isomorphic to the normal bundle defined, in the absence of a metric, by (19.2).

The pullback fibre metric $F^*(TN)$ is naturally induced by h, as the fibre of $F^*(TN)$ over $x \in M$ is, by definition, $T_{F(x)}N$ (see §17). We denote it by $\langle\,,\rangle$ rather than F^*h since the latter symbol is already used in a different meaning (the pullback metric on M).

We thus have a situation of the type described at the beginning of §45: the real vector bundle $F^*(TN)$ over M is endowed both with a direct-sum decomposition, namely, $F^*(TN) = \tau \oplus \nu$, and with the connection F^*D obtained by pulling back from TN the Levi-Civita connection of h (which we denote by D to distinguish it from Levi-Civita connection ∇ of g). For the connections in τ and ν obtained by projecting F^*D we now use the symbols ∇^{tng} and ∇^{nrm}, rather than $\nabla^\pm$, and we refer to them as the *tangent* and *normal connections* of the immersion F. Simlarly, we denote by ψ^{tng} and ψ^{nrm} the components of any local section of $F^*(TN)$, or an element of a fibre of $F^*(TN)$, relative to the decomposition $F^*(TN) = \tau \oplus \nu$.

LEMMA 46.1. *Let $F : M \to N$ be a C^∞ immersion of a manifold M in a Riemannian manifold (N, h), and let $g = F^*h$ be the pullback metric on M. The tangent connection ∇^{tng} in $\tau = TM$ then coincides with the Levi-Civita connection ∇ of g.*

in preparation

in preparation

47. Hypersurfaces in Euclidean spaces

Topics: The second fundamental tensor of a codimension-one immersion; Euclidean versions of the Gauss and Codazzi equations; local classification of totally geodesic and totally umbilical submanifolds of Euclidean spaces.

in preparation

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48. Bonnet's theorem

Topics: Reconstructing the ambient connection from immersion data; Bonnet's theorem.

in preparation

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CHAPTER 9

Differential Forms

49. Tensor products

Topics: Tensor products, symmetric and exterior powers of vector spaces; symmetric powers and polynomial functions.

Given an integer $k \geq 1$ and real or complex vector spaces $V_1, \ldots, V_k$ and W, let the symbol $L(V_1, \ldots, V_k; W)$ denote the vector space of all k-linear mappings $B : V_1 \times \ldots \times V_k \to W$ (with the valuewise operations). In the case where the spaces $V_1, \ldots, V_k$ all coincide, i.e., $V_1 = \ldots = V_k = V$, we denote by $L_{\mathrm{sym}}(V, \ldots, V; W)$ and $L_{\mathrm{skew}}(V, \ldots, V; W)$ the subspaces of $L(V, \ldots, V; W)$ (where $V, \ldots, V$ stands for V repeated k times) consisting of all k-linear mappings $V \times \ldots \times V \to W$ which are symmetric or, respectively, skew-symmetric (for more details, see Problem 1). Natural surjective linear operators $\mathfrak{S}$, $\mathfrak{A}$ of $L(V, \ldots, V; W)$ onto $L_{\mathrm{sym}}(V, \ldots, V; W)$ and $L_{\mathrm{skew}}(V, \ldots, V; W)$, known as the *symmetrization* and *skew-symmetrization projections* then can be defined by

$$(49.1) \qquad \begin{aligned} (\mathfrak{S}B)(v_1, \ldots, v_k) &= \frac{1}{k!} \sum_\tau B(v_{\tau(1)}, \ldots, v_{\tau(k)})\,, \\ (\mathfrak{A}B)(v_1, \ldots, v_k) &= \frac{1}{k!} \sum_\tau \operatorname{sgn}(\tau) B(v_{\tau(1)}, \ldots, v_{\tau(k)})\,, \end{aligned}$$

where τ runs through all permutations of $\{1, \ldots, k\}$. See also Problem 2.

When $V_1, \ldots, V_k$ as above are all finite-dimensional and W is the scalar field $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$, one defines the *tensor product* of $V_1, \ldots, V_k$ to be the vector space

$$(49.2) \qquad V_1 \otimes \ldots \otimes V_k = L(V_1^*, \ldots, V_k^*; \mathbf{K})\,.$$

The *tensor multiplication* is the natural k-linear mapping

$$(49.3) \qquad V_1 \times \ldots \times V_k \ni (v_1, \ldots, v_k) \mapsto v_1 \otimes \ldots \otimes v_k \in V_1 \otimes \ldots \otimes V_k\,,$$

with $(v_1 \otimes \ldots \otimes v_k)(\xi^1, \ldots, \xi^k) = \xi^1(v_1) \ldots \xi^k(v_k)$. If, moreover, $V_1 = \ldots = V_k = V$, the tensor product $V \otimes \ldots \otimes V = L(V^*, \ldots, V^*; \mathbf{K})$ of k copies of V is referred to as the kth *tensor power* of V and denoted by $V^{\otimes k}$. The kth *symmetric power* $V^{\odot k}$ of V and the kth *exterior power* $V^{\wedge k}$ of V then are defined to be the vector subspaces of $V^{\otimes k}$ given by

$$V^{\odot k} = L_{\mathrm{sym}}(V^*, \ldots, V^*; \mathbf{K})\,, \qquad V^{\wedge k} = L_{\mathrm{skew}}(V^*, \ldots, V^*; \mathbf{K})\,.$$

It is also convenient to extend these definitions to all integers k by setting

$$(49.4) \qquad V^{\otimes 0} = V^{\odot 0} = V^{\wedge 0} = \mathbf{K}\,, \quad V^{\otimes k} = V^{\odot k} = V^{\wedge k} = \{0\} \quad \text{for } k < 0\,.$$

The projections

$$\mathfrak{S} : V^{\otimes k} \to V^{\odot k}\,, \qquad \mathfrak{A} : V^{\otimes k} \to V^{\wedge k}$$

given by (49.1) thus give rise to the k-linear mappings $V \times \ldots \times V \to V^{\odot k}$, $V \times \ldots \times V \to V^{\wedge k}$ of *symmetric* and *exterior multiplication*, assigning the values

$$v_1 \odot \ldots \odot v_k = \mathfrak{S}(v_1 \otimes \ldots \otimes v_k), \qquad v_1 \wedge \ldots \wedge v_k = \mathfrak{A}(v_1 \otimes \ldots \otimes v_k)\,. \tag{49.5}$$

to $(v_1, \ldots, v_k)$. These multiplications can be expressed in terms of familiar operations: valuewise multiplication of linear functions and the determinant of an $k \times k$ matrix viewed as a "multiplication" of its row or column vectors; see Proposition 49.1 and Problem 5.

Let V be a finite-dimensional real or complex vector space. A function f on V valued in the scalar field $\mathbf{K}$ ($\mathbf{R}$ or $\mathbf{C}$) is called a *monomial of degree* $k \geq 0$ if it can be written as a valuewise product $\xi^1 \ldots \xi^k$ of linear functions $\xi^1, \ldots, \xi^k \in V^*$. (When $k = 0$, this means that f is constant.) We call $f : V \to \mathbf{K}$ a *polynomial* if f is a sum of monomials, and say that a polynomial f on V is *homogeneous of degree* $k \geq 0$ if it can be written as a sum of monomials all of which are of degree k. (Thus, the zero function is a homogeneous polynomial, and a monomial, of all degrees.) Denoting by $\mathcal{P}_k(V)$ the vector space of all degree k homogeneous polynomials on V, and letting $\mathcal{P}(V)$ be the algebra of all polynomials on V (with the valuewise multiplication of functions), we now have

$$\mathcal{P}_0(V) = \mathbf{K}\,, \qquad \mathcal{P}_1(V) = V^*, \qquad \mathcal{P}(V) = \bigoplus_{k \geq 0} \mathcal{P}_k(V\,,)$$

sothat every polynomial can be *uniquely* expressed as a sum of homogeneous polynomials (see Problem 7). As before, we set $\mathcal{P}_k(V) = \{0\}$ if $k < 0$.

Given a homogeneous polynomial $f \in \mathcal{P}_k(V)$ on a finite-dimensional real or complex vector space V and a vector $v \in V$, we have $d_v f \in \mathcal{P}_{k-1}(V)$, as one easily sees by considering monomials. Therefore, we may define a liinear operator $\Psi : \mathcal{P}_k(V) \to (V^*)^{\odot k} = L_{\text{sym}}(V, \ldots, V; \mathbf{K})$ by

$$(\Psi f)(v_1, \ldots, v_k) \;=\; \frac{1}{k!}\, d_{v_1} \ldots d_{v_k} f \;\in\; \mathcal{P}_0(V) = \mathbf{K}\,. \tag{49.6}$$

On the other hand, formula

$$(\Phi B)(x) = B(x, \ldots, x) \tag{49.7}$$

defines a linear operator $\Phi : (V^*)^{\odot k} = L_{\text{sym}}(V, \ldots, V; \mathbf{K}) \to \mathcal{P}_k(V)$ since, for any fixed basis e_j of V, ΦB is a sum of degree k monomials:

$$\Phi B = B(e_{j_1}, \ldots, e_{j_k})\, e^{j_1} \ldots e^{j_k}\,. \tag{49.8}$$

Proposition 49.1. *For any finite-dimensional real or complex vector space V and any integer $k \geq 0$, the operator $\Phi : (V^*)^{\odot k} = L_{sym}(V, \ldots, V; \mathbf{K}) \to \mathcal{P}_k(V)$ defined by* (49.7) *is a linear isomorphism and its inverse $\Psi = \Phi^{-1}$ is given by* (49.6). *Furthermore, the symmetric multiplication* (49.5) *of linear functions $\xi^1, \ldots, \xi^k \in V^*$ corresponds under Φ to their valuewise multiplication, that is,*

$$\Phi(\xi^1 \odot \ldots \odot \xi^k) = \xi^1 \ldots \xi^k\,. \tag{49.9}$$

Proof. Relation (49.9) is immediate from formula (49.10) below. Thus, Φ is surjective, since $\mathcal{P}_k(V)$ is spanned by monomials. On the other hand, using (49.8) with $d_v \xi = \xi(v)$ for $v \in V$ and $\xi \in V^*$ we obtain $(k-s)!\, d_{v_1} \ldots d_{v_s}(\Phi B) = k!\, \Phi[B(v_1, \ldots, v_s, \cdot, \ldots, \cdot)]$ (induction on $s \leq k$), where $B(v_1, \ldots, v_s, \cdot, \ldots, \cdot)$ is the element of $(V^*)^{\odot k-s} = L_{\text{sym}}(V, \ldots, V; \mathbf{K})$ obtained from $B \in (V^*)^{\odot k}$ by fixing the first s arguments. Setting $s = k$, we thus have $\Psi(\Phi B) = B$. Therefore Φ is also injective and $\Phi^{-1} = \Psi$, which completes the proof. $\square$

Problems

1. A k-linear mapping $B : V \times \ldots \times V \to W$ of real or complex vector spaces V and W is called *symmetric* (or, *skew-symmetric*) if $B(v_1, \ldots, v_k)$ remains unchanged (or, respectively, changes the sign) whenever two of the arguments $v_1, \ldots, v_k \in V$ are interchanged. Verify that

(a) The phrase 'two of the arguments $v_1, \ldots, v_k \in V$' can be replaced by 'two neighboring arguments among $v_1, \ldots, v_k \in V$'.

(b) B is symmetric (or, skew-symmetric) if and only if, for all $v_1, \ldots, v_k \in V$ and all permutations τ of $\{1, \ldots, k\}$, one has $B(v_{\tau(1)}, \ldots, v_{\tau(k)}) = B(v_1, \ldots, v_k)$ (or, respectively, $B(v_{\tau(1)}, \ldots, v_{\tau(k)}) = \mathrm{sgn}(\tau)\, B(v_1, \ldots, v_k)$).

(c) B is skew-symmetric if and only if $B(v_1, \ldots, v_k) = 0$ whenever two neighboring arguments among $v_1, \ldots, v_k \in V$ coincide.

2. Let V be a real or complex vector space. A linear operator $S : V \to V$ is called a *projection* if $S^2 = S$. Verify that, for a linear operator $S : V \to V$, the following three conditions are equivalent:

(a) S is a projection.

(b) V admits a direct-sum decomposition $V = V_0 \oplus V_1$ such that $S = 0$ on V_0 and $S = \mathrm{Id}$ on V_1.

(c) The restriction of S to the image subspace $S(V)$ coincides with the identity transformation.

3. A linear operator $S : V \to V$ in a real or complex vector space V is called an *involution* if $S^2 = \mathrm{Id}$. Verify that a linear operator $S : V \to V$ is an involution if and only if V admits a direct-sum decomposition $V = V_+ \oplus V_-$ such that $S = \pm\mathrm{Id}$ on $V_\pm$. Show that, for an involution $S : V \to V$, the summands $V_\pm$ are given by $V_\pm = \mathrm{Ker}\,(S \mp \mathrm{Id})$, and that the direct-sum projections $V = V_+ \oplus V_- \to V_\pm$ coincide with $\frac{1}{2}(S \pm \mathrm{Id})$.

4. Show that, for any finite-dimensional real or complex vector space V one has $V^{\otimes 2} = V^{\odot 2} \oplus V^{\wedge 2}$, with $\mathfrak{S}$, $\mathfrak{A}$ playing the roles of the direct-sum projections. Verify that the corresponding decomposition of $v \otimes w$ for $v, w \in V$ is $v \otimes w = v \odot w + v \wedge w$. Give an explicit description of $\mathfrak{S}B$, $\mathfrak{A}B$ for any $B \in V^{\otimes 2}$ (that is, a bilinear function $B : V \times V \to \mathbf{K}$).

5. Verify that, given a finite-dimensional real or complex vector space V and any $v_1 \odot \ldots \odot v_k \in V$, $\xi^1, \ldots, \xi^k \in V^*$, we have

$$(v_1 \odot \ldots \odot v_k)(\xi^1, \ldots, \xi^k) = \frac{1}{k!} \sum_\tau \xi^{\tau(1)}(v_1) \ldots \xi^{\tau(k)}(v_k)\,, \tag{49.10}$$

$$(v_1 \wedge \ldots \wedge v_k)(\xi^1, \ldots, \xi^k) = \frac{1}{k!} \sum_\tau \mathrm{sgn}(\tau)\, \xi^{\tau(1)}(v_1) \ldots \xi^{\tau(k)}(v_k) = \frac{1}{k!} \det_{\alpha,\beta} [\xi^\alpha(v_\beta)]\,,$$

where τ runs through all permutations of $\{1, \ldots, k\}$.

6. Let V be a real or complex vector space. Show that, given a linear operator $S : V \to V$ and vectors $v_1, \ldots, v_k \in V$ such that $v_1 + \ldots + v_k = 0$ and $Sv_1 = \lambda_1 v_1, \ldots, Sv_k = \lambda_k v_k$ with pairwise distinct scalars $\lambda_1, \ldots, \lambda_k$, we must have $v_1 = \ldots = v_k = 0$; in other words, nonzero eigenvectors of S corresponding to mutually distinct eigenvalues are always linearly independent. (Hint below.)

7. Let V be a finite-dimensional real or complex vector space. Verify that every polynomial on V can be *uniquely* expressed as a sum of homogeneous polynomials. (Hint below.)

8. (See also Problem 1(c) in §88, Appendix D.) Let V be a finite-dimensional real vector space and let a function $f : V \to \mathbf{R}$ be C^k-differentiable and positively homogeneous of degree k for some integer $k \geq 0$. Show that $f \in \mathcal{P}_k(V)$.

Hint. In Problem 6, set $T_\alpha = S_1 \dots \widehat{S_\alpha} \dots S_k$, where $\widehat{\ }$ means 'delete' and $S_\alpha = S - \lambda_\alpha \cdot \mathrm{Id}$. Then $0 = T_\alpha(v_1 + \ldots + v_k) = c_\alpha v_\alpha$ with $c_\alpha = \prod_{\beta \neq \alpha}(\lambda_\alpha - \lambda_\beta) \neq 0$, as the S_α all commute and $S_\alpha v_\alpha = 0$ while $S_\beta v_\alpha = (\lambda_\alpha - \lambda_\beta) v_\alpha$ for $\alpha \neq \beta$.

Another argument: Let s be the number of nonzero vectors among the v_α, and let $n = \dim W$ with $W = \mathrm{Span}\,(v_1, \dots, v_k)$. Thus, $n \leq s$. However, since S has at least s distinct eigenvalues in W, and n is the degree of its characteristic polynomial in W, we obtain $s \leq n$, so that $s = n$ and those s nonzero vectors are linearly independent as they span an s-dimensional space, which contradicts our assumption unless $s = 0$.

Hint. In Problem 7, apply Problem 6 to $S = d_{\mathrm{Id}}$ in the space of all C^∞ functions $V \to \mathbf{K}$. Cf. Problem 11 in §6 and Problem 2(a) in §88, Appendix D.

50. Exterior and symmetric powers

Topics: Natural bases in tensor products, exterior powers, and symmetric powers of vector spaces; the universal factorization properties; the tensor, symmetric and exterior (graded) algebras of a vector space.

Suppose that we are given finite-dimensional real or complex vector spaces $V_1, \dots, V_k$, $k \geq 1$, and bases e_{α_1} for V_1, $\dots$, e_{α_k} for V_k.

A basis of the tensor product space $V_1 \otimes \ldots \otimes V_k$ then is provided by the collection of all tensor products

$$e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_k}\,, \tag{50.1}$$

where each index α_s, $s = 1, \dots, k$, varies independently in $\{1, \dots, \dim V_s\}$ (notations of (49.2), (49.3)). Therefore

$$\dim(V_1 \otimes \ldots \otimes V_k) = \dim V_1 \ldots \dim V_k\,.$$

To see that the products (50.1) form a basis (and are mutually distinct), note that, treating them as k-linear functions, we have

$$(e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_k})(e^{\beta_1}, \dots, e^{\beta_k}) = \delta^{\beta_1}_{\alpha_1} \ldots \delta^{\beta_k}_{\alpha_k}\,,$$

each e^{α_s} denoting, as usual (see Example 5.1) the basis of V_s^* dual to the e_{α_s}. Hence the equality $B = \lambda^{\alpha_1 \dots \alpha_k} e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_k}$ implies $\lambda^{\alpha_1 \dots \alpha_k} = B(e^{\alpha_1}, \dots, e^{\alpha_k})$, and so the products (50.1) are linearly independent (the case $B = 0$) and, for any $B \in V_1 \otimes \ldots \otimes V_k$, we have the expansion

$$B = B^{\alpha_1 \dots \alpha_k} e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_k} \tag{50.2}$$

with

$$B^{\alpha_1 \dots \alpha_k} = B(e^{\alpha_1}, \dots, e^{\alpha_k}) \tag{50.3}$$

(as both sides of (50.2) coincide on each $(e^{\beta_1}, \dots, e^{\beta_k})$).

Let us now assume, in addition, that $V_1 = \ldots = V_k = V$, and a single basis e_a is chosen in V. The families

$$e_{\alpha_1} \odot \ldots \odot e_{\alpha_k}\,, \qquad 1 \leq \alpha_1 \leq \ldots \leq \alpha_k \leq \dim V \tag{50.4}$$

and

$$e_{\alpha_1} \wedge \ldots \wedge e_{\alpha_k}\,, \qquad 1 \leq \alpha_1 < \ldots < \alpha_k \leq \dim V \tag{50.5}$$

then consists of mutually distinct elements, forming a basis of the kth symmetric power $V^{\odot k}$ of V and the kth exterior power $V^{\wedge k}$ of V, respectively. (Unlike the previous case, the indices $\alpha_1, \ldots, \alpha_k$ now all vary in *the same* range $\{1, \ldots, \dim V\}$.) In fact, from (49.5) and (50.2) we obtain the expansions

$$B = B^{\alpha_1 \ldots \alpha_k} e_{\alpha_1} \odot \ldots \odot e_{\alpha_k}$$

with (50.3) for each $B \in V^{\odot k}$ and

$$B = B^{\alpha_1 \ldots \alpha_k} e_{\alpha_1} \wedge \ldots \wedge e_{\alpha_k}$$

with (50.3) for each $B \in V^{\wedge k}$. Although the summations are over the unrestricted range of all k-tuples $(\alpha_1, \ldots, \alpha_k)$, they clearly represent combinations of (50.4) or (50.5) in view of the (skew)symmetry of the multiplications involved. Finally, the products (50.4) or (50.5) are linearly independent (and pairwise distinct) since, by (49.5) and (49.1),

$$(e_{\alpha_1} \wedge \ldots \wedge e_{\alpha_k})(e^{\beta_1}, \ldots, e^{\beta_k}) = \delta_{\alpha_1}^{\beta_1} \ldots \delta_{\alpha_k}^{\beta_k}$$

whenever $1 \le \alpha_1 < \ldots < \alpha_k \le \dim V$ and $1 \le \beta_1 < \ldots < \beta_k \le \dim V$, while

$$(e_{\alpha_1} \odot \ldots \odot e_{\alpha_k})(e^{\beta_1}, \ldots, e^{\beta_k}) = C\delta_{\alpha_1}^{\beta_1} \ldots \delta_{\alpha_k}^{\beta_k}$$

whenever $1 \le \alpha_1 \le \ldots \le \alpha_k \le \dim V$ and $1 \le \beta_1 \le \ldots \le \beta_k \le \dim V$, where C is a positive integer depending on the indices involved.

Consequently,

$$\dim V^{\wedge k} = \binom{n}{k} \qquad \text{if} \qquad 0 \le k \le n = \dim V\,, \tag{50.6}$$

(see also (49.4), while

$$V^{\wedge k} = \{0\} \qquad \text{for} \qquad k > \dim V\,. \tag{50.7}$$

The last relation is clear since the set (50.5) is empty when $k > \dim M$, but also follows directly as $B(e^{\beta_1}, \ldots, e^{\beta_k}) = 0$ if B is skew-symmetric and $k > \dim M$, since two or more of the arguments e^{β_s} then must coincide. About the dimension of $V^{\odot k}$, see Problem 8.

Tensor products and symmetric/exterior powers of finite-dimensional real or complex vector spaces $V_1, \ldots, V_k$ (or V) have the following *universal factorization properties*. Given any vector space W and a k-linear mapping $\varphi : V_1 \times \ldots \times V_k \to W$ or a k-linear symmetric/skew-symmetric mapping $\varphi : V \times \ldots \times V \to W$, there exists a *unique* linear operator $F : V_1 \otimes \ldots \otimes V_k \to W$, or $F : V^{\odot k} \to W$, or, respectively, $F : V^{\wedge k} \to W$, such that, for all $v_1 \in V_1, \ldots, v_k \in V_k$ (or $v_1, \ldots, v_k \in V$),

$$\varphi(v_1, \ldots, v_k) = \begin{cases} F(v_1 \otimes \ldots \otimes v_k)\,, & \text{or} \\ F(v_1 \odot \ldots \odot v_k)\,, & \text{or} \\ F(v_1 \wedge \ldots \wedge v_k)\,, & \text{whichever applies.} \end{cases}$$

In fact, such F can be defined (uniquely!) on basis elements of the form (50.1), (50.4) or (50.5), which then easily implies the required property for all $v_1, \ldots, v_k$.

In other words, to define linear operators from tensor-product or symmetric/exterior-power spaces it suffices to prescribe their values on product elements, and such a definition is automatically correct as long as the dependence of the value on the factors is k-linear, or k-linear and (skew)symmetric, respectively.

Let $V_1, \dots, V_{k+l}$ and V, with integers $k, l \ge 1$, be arbitrary finite-dimensional real or complex vector spaces. There exist unique bilinear mappings

$$(V_1 \otimes \dots \otimes V_k) \times (V_{k+1} \otimes \dots \otimes V_{k+l}) \underset{\otimes}{\to\!\to} V_1 \otimes \dots \otimes V_{k+l} \tag{50.8}$$

and

$$V^{\odot k} \times V^{\odot l} \underset{\odot}{\to\!\to} V^{\odot(k+l)}, \qquad V^{\wedge k} \times V^{\wedge l} \underset{\wedge}{\to\!\to} V^{\wedge(k+l)}, \tag{50.9}$$

known as the *tensor, symmetric* and *exterior multiplications*, and characterized by

$$(v_1 \dots v_k)(v_{k+1} \dots v_{k+l}) = v_1 \dots v_{k+l} \tag{50.10}$$

where, for simplicity, we use ordinary multiplicative notation (without a dot), omitting the symbols $\otimes$, $\odot$ or $\wedge$. See Problem 1. These multiplications can also be defined directly, without invoking the universal factorization properties (Problems 2 and 3).

For $V_1, \dots, V_k$ as before, any permutation τ of $\{1, \dots, k\}$ gives rise to a unique isomorphism

$$V_1 \otimes \dots \otimes V_k \to V_{\tau(1)} \otimes \dots \otimes V_{\tau(k)}$$

sending each $v_1 \otimes \dots \otimes v_k$ to $v_{\tau(1)} \otimes \dots \otimes v_{\tau(k)}$.

For any finite-dimensional real or complex vector space V, there is a unique isomorphism $\mathbf{K} \otimes V \to V$ with

$$\lambda \otimes v \mapsto \lambda v \tag{50.11}$$

for $\lambda \in \mathbf{K}$ and $v \in V$ ($\mathbf{K}$ being the scalar field). In fact, since $(\lambda, v) \mapsto \lambda v$ is bilinear, a linear operator with (50.11) exists and is unique in view of the universal factorization property for $\otimes$, and its inverse is given by $v \mapsto 1 \otimes v$.

The multiplications (50.8), (50.9) are associative (Problem 4) and so they turn each of the direct sums

$$V^{\otimes} = \bigoplus_{k \ge 0} V^{\otimes k}, \qquad V^{\odot} = \bigoplus_{k \ge 0} V^{\odot k}, \qquad V^{\wedge} = \bigoplus_{k \ge 0} V^{\wedge k}$$

into an associative algebra with the unit 1 (which belongs to the 0th summand, cf. (49.4)), called the *tensor, symmetric* and *exterior algebras* of V. See also Problem 6.

Problems

1. Given finite-dimensional real or complex vector spaces $V_1, \dots, V_{k+l}$ and V, prove the existence and uniqueness of the bilinear mappings (50.8), (50.9) with (50.10). (Hint below.)

2. Show that (50.8) can be explicitly defined by

$$(B \otimes B')(\xi^1, \dots, \xi^{k+l}) = B(\xi^1, \dots, \xi^k) B'(\xi^{k+1}, \dots, \xi^{k+l}) \tag{50.12}$$

for $B \in V_1 \otimes \dots \otimes V_k$, $B' \in V_{k+1} \otimes \dots \otimes V_{k+l}$ and $\xi^s \in V_s^*$, $s = 1, \dots, k+l$.

3. Prove that

$$B \odot B' = \mathfrak{S}(B \otimes B'), \qquad B \wedge B' = \mathfrak{A}(B \otimes B')$$

for all $B \in V^{\odot k}$ and $B' \in V^{\odot l}$ or, respectively, $B \in V^{\wedge k}$ and $B' \in V^{\wedge l}$. (Hint below.)

4. Verify associativity of the multiplications (50.8), (50.9).

5. Given linear operators $F_1 : V_1 \to W_1,\ \ldots,\ F_k : V_k \to W_k$, show that there is a unique linear operator $F_1 \otimes \ldots \otimes F_k : V_1 \otimes \ldots \otimes V_k \to W_1 \otimes \ldots \otimes W_k$ (called the *tensor product* of $F_1, \ldots, F_k$) with

$$(F_1 \otimes \ldots \otimes F_k)(v_1 \otimes \ldots \otimes v_k) = (F_1 v_1) \otimes \ldots \otimes (F_k v_k)$$

for all $v_s \in V_s,\ s = 1, \ldots, k$.

6. Verify that

$$\dim V^{\wedge} \,=\, 2^n$$

whenever V is a real or complex vector space with $\dim V = n < \infty$. (Hint below.)

7. Given integers $n \ge 1$ and $k \ge 0$, let $\Pi_{n,k}$ be the set of all ordered n-tuples $(\alpha_1, \ldots, \alpha_n)$ of integers $\alpha_j \ge 0$ with $\sum_{j=1}^n \alpha_j = k$. Prove that $\Pi_{n,k}$ has $\binom{n+k-1}{k}$ elements. (Hint below.)

8. Given a finite-dimensional real or complex vector space V, show that, for integers $k \ge 0$,

$$\dim \mathcal{P}_k(V) = \binom{n+k-1}{k}, \qquad n = \dim V \tag{50.13}$$

with $\mathcal{P}_k(V)$ defined in §49. (Hint below.)

9. Verify that

$$\dim\, V^{\odot k} = \binom{n+k-1}{k}, \qquad n = \dim V$$

for any finite-dimensional real or complex vector space V and any integer $k \ge 0$. (Hint below.)

10. Show that, when the isomorphism of Problem 5 in §28 is regarded as an isomorphism $V^* \otimes V \to V^* \otimes V^*$, it coincides with the tensor product (Problem 5) of the operators $\mathrm{Id} : V^* \to V^*$ and $V \ni v \mapsto \langle \cdot, v \rangle \in V^*$.

Hint. In Problem 1, fix $v_1, \ldots, v_k$ and define $(v_1 \ldots v_k)B'$ for B' in the appropriate product/power space of the last l factors, combining (50.10) with the universal factorization property, and note that the result is $(k+1)$-linear in $v_1, \ldots, v_k, B'$. Then use the same device to define BB' for B in the product/power of the first k factors.

Hint. In Problem 3, combine (50.12) with (49.1).

Hint. In Problem 6, use (50.6) and (50.7).

Hint. In Problem 7, use induction on n along with the formula $\binom{n+k}{k} = \sum\limits_{s=0}^{k} \binom{n+s-1}{s}$, obtained from $\binom{n+k}{k} = \binom{n+k-1}{k-1} + \binom{n+k-1}{k}$ by induction on k.

Hint. In Problem 8, note that, for a fixed basis $f_1, \ldots, f_n$ of V^*, the monomials $(f_1)^{\alpha_1} \ldots (f_n)^{\alpha_n}$ form a basis of $P_k(V)$ indexed by $\Pi_{n,k}$ defined in Problem 7.

Hint. In Problem 9, use (50.13) and Proposition 49.1.

51. Exterior forms

Topics: Exterior forms; differential forms; exterior derivative; closed and exact differential forms; the Poincaré lemma.

Given a manifold M, a point $x \in M$, and an integer $r \ge 0$, by *exterior forms of degree* r (or, *exterior* r*-forms*) at x we mean elements of the vector space

$$\Lambda^r_x M \,=\, [T^*_x M]^{\wedge r}\,,$$

which is nothing else than the fibre over x of the vector bundle

$$\Lambda^r M \,=\, [T^*M]^{\wedge r}\,.$$

Sections of $\Lambda^r M$ then are called *differential r-forms* on M. We have $\Lambda^r M = M \times \{0\}$ when $r < 0$ (by definition) as well as when $r > \dim M$ (by (50.7)). Also, $\Lambda^0 M = M \times \mathbf{R}$ and $\Lambda^1 M = T^*M$. Thus, differential 0-forms are just real-valued functions on the manifold, while differential 1-forms are nothing else than cotangent-vector fields. Any differential r-form ω on M gives rise to the r-linear skew-symmetric mapping sending vector fields $v_1, \dots, v_r$ on M to the function

$$\omega(v_1, \dots, v_r) : M \to \mathbf{R}$$

given by

$$[\omega(v_1, \dots, v_r)](x) \,=\, [\omega(x)](v_1(x), \dots, v_r(x))\,. \tag{51.1}$$

Any local coordinate system x^j in M defines a local trivialization of each $\Lambda^r M$ formed by all $dx^{j_1} \wedge \ldots \wedge dx^{j_r}$ with $1 \le j_1 < \ldots < j_r \le \dim M$ (cf. (50.5). A (local) differential r-form ω defined on the coordinate domain then can be expanded as

$$\omega = \omega_{j_1 \dots j_r} dx^{j_1} \wedge \ldots \wedge dx^{j_r} \tag{51.2}$$

(summation over *all* r-tuples $j_1, \dots, j_r$), where

$$\omega_{j_1 \dots j_r} \,=\, \omega(p_{j_1}, \dots, p_{j_r}) \tag{51.3}$$

are called the *component functions* ω relative to the local coordinates x^j in M. They are skew-symmetric in $j_1 \dots j_r$ in the sense that $\omega_{j_{\tau(1)} \dots j_{\tau(r)}} \,=\, (\operatorname{sgn} \tau)\, \omega_{j_1 \dots j_r}$ for any permutation τ of $\{1, \dots, r\}$. Consequently,

$$\omega \,=\, r! \sum_{j_1 < \ldots < j_r} \omega_{j_1 \dots j_r} dx^{j_1} \wedge \ldots \wedge dx^{j_r}\,, \tag{51.4}$$

and, by (51.3),

$$\omega(v_1, \dots, v_r) \,=\, \omega_{j_1 \dots j_r} v_1^{j_1} \dots v_r^{j_r}\,,$$

Note that, given ω and the x^j, the coefficients $\omega_{j_1 \dots j_r}$ of the expansions (51.2), (51.4) are made unique by the requirement of skew-symmetry or, respectively, the restriction of the range of summation; in general, it may happen that

$$\omega = f_{j_1 \dots j_r} dx^{j_1} \wedge \ldots \wedge dx^{j_r}$$

with functions $f_{j_1 \dots j_r}$ other than the $\omega_{j_1 \dots j_r}$, and then

$$\omega_{j_1 \dots j_r} = \frac{1}{r!} \sum_\tau (\operatorname{sgn} \tau) f_{j_{\tau(1)} \dots j_{\tau(r)}}\,,$$

the summation being over all permutations τ of $\{1, \dots, r\}$.

Applying the exterior multiplication (see (50.9)) to exterior/differential forms ω, θ of degrees r and s on M, we obtain the $(r+s)$-form $\omega \wedge \theta$, and (Problem10.13)

$$\omega \wedge \theta = (-1)^{rs} \theta \wedge \omega\,. \tag{51.5}$$

In local coordinates x^j, $\omega \wedge \theta$ has the components

$$(\omega \wedge \theta)_{j_1 \dots j_{r+s}} = \frac{1}{(r+s)!} \sum_\tau (\operatorname{sgn} \tau)\, \omega_{j_{\tau(1)} \dots j_{\tau(r)}}\, \theta_{j_{\tau(r+1)} \dots j_{\tau(r+s)}}\,, \tag{51.6}$$

where τ this time varies through all permutations of $\{1, \dots, r+s\}$ (Problem 1). Due to skew-symmetry of the components of ω and θ, each term on the right-hand side of (51.6) is repeated $r!\,s!$ times, so that the number of terms can be reduced from $(r+s)!$ to $\binom{r+s}{r} = \frac{(r+s)!}{r!\,s!}$. In particular (see Problem 2)

$$(51.7)\quad (p \wedge q)_{jk} = \frac{1}{2}(p_j q_k - p_k q_j)\,, \qquad (p \wedge \omega)_{jkl} = \frac{1}{3}(p_j\,\omega_{kl} + p_k\,\omega_{lj} + p_l\,\omega_{jk})\,,$$

$$(51.8)\qquad (\omega \wedge \omega)_{jklm} = \frac{1}{3}(\omega_{jk}\,\omega_{lm} + \omega_{jl}\,\omega_{mk} + \omega_{jm}\,\omega_{kl})$$

for 1-forms p, q and 2-forms ω.

Theorem 51.1. *There exists a unique operation d, called the* exterior derivative*, which associates with each differential r-form ω of class C^l on any manifold M, for all integers $r \geq 0$ and $l \geq 1$, a differential $(r+1)$-form $d\omega$ of class C^{l-1} on M, in such a way that*

a. *d is linear when r and l are fixed.*
b. *d sends any C^l function (0-form) f to its differential df.*
c. *$d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^r\, \omega \wedge d\theta$ for C^l forms ω, θ of any degrees r, s.*
d. *$d^2 = 0$, i.e., $d(d\omega) = 0$ for C^l forms ω, $l \geq 2$, of all degrees.*
e. *d is local. i.e., the restriction of $d\omega$ to any open set $U \subset M$ depends only on the restriction of ω to U.*

Furthermore, this unique operation d can be written as

$$(51.9)\qquad d(f_{j_1 \ldots j_r} dx^{j_1} \wedge \ldots \wedge dx^{j_r}) = df_{j_1 \ldots j_r} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_r}$$

for any local coordinates x^j and any C^l functions $f_{j_1 \ldots j_r}$, $l \geq 1$, which need not be skew-symmetric in $j_1, \ldots, j_r$.

Proof. Uniqueness of d is obvious from (51.9), which in turn follows immediately from (a) – (e). Existence in any fixed coordinate domain is immediate if we define d by (51.9) with $f_{j_1 \ldots j_r} = \omega_{j_1 \ldots j_r}$. Its independence of the coordinates used is obvious from uniqueness. This completes the proof. □

A local-component description of the exterior derivative is

$$(51.10)\qquad (d\omega)_{j_0 \ldots j_r} \;=\; \frac{1}{r+1} \sum_{q=0}^{r} (-1)^q \partial_{j_q} \omega_{j_0 \ldots \widehat{j_q} \ldots j_r}\,,$$

where $\widehat{\ }$ stands for 'delete'. In fact, the right-hand side of (51.10) is clearly skew-symmetric in $j_0, \ldots, j_r$, while from formula (51.9) with $f_{j_1 \ldots j_r} = \omega_{j_1 \ldots j_r}$ we obtain, using (51.2) and skew-symmetry of the $dx^{j_0} \wedge \ldots \wedge dx^{j_r}$ in the factors dx^j, and grouping terms, $(r+1)\, d\omega = (r+1)\, \partial_{j_0} \omega_{j_1 \ldots j_r}\, dx^{j_0} \wedge \ldots \wedge dx^{j_r} = \left[\sum_{q=0}^{r} (-1)^q \partial_{j_q} \omega_{j_0 \ldots \widehat{j_q} \ldots j_r}\right] dx^{j_0} \wedge \ldots \wedge dx^{j_r}$.

The exterior derivative can also be easily described in terms of degree r differential forms treated as r-linear mappings sending r vector fields to functions as in (51.1); see Problem 3.

Let ω be a local differential r-form in a manifold M, defined on an open subset U of M. One says that ω is *closed* if ω is of class C^1 and $d\omega = 0$ identically in U. On the other hand, ω is called *exact* if $\omega = d\theta$ for some $(r-1)$-form θ of class C^2 having the same domain U. Condition (d) of the theorem simply states that *every exact differential form is closed*. The converse statement obviously fails for 0-forms, i.e., (local) functions: closedness then means that the function is locally constant in its domain, while there is no nonzero exact 0-form. It also fails for 1-forms; one easily verifies (see Problem 10) that the 1-form

$$\omega_0 \,=\, \frac{x}{x^2+y^2}\,dy \,-\, \frac{y}{x^2+y^2}\,dx\,, \tag{51.11}$$

on $\mathbf{R}^2 \setminus \{0\}$ (where x, y are the standard Cartesian coordinate functions) is closed but not exact.

The following fundamental result is known as the *Poincaré lemma.* It shows that the existence of such counterexamples in degrees $r > 0$ indicates some topological complexity of the domain U in question.

THEOREM 51.2. *Let ω be a closed differential r-form of class C^l, $r, l \geq 1$, on a manifold M which is diffeomorphic to a star-shaped open subset U of a finite-dimensional real vector space V. Then ω is exact. More precisely, $\omega = d\theta$ for some global $(r-1)$-form θ on M which is of class C^{l+1}.*

PROOF. Set $n = \dim M$ and choose global coordinates x^j on M that constitute such a diffeomorphism onto U; using a translation in $\mathbf{R}^n$, we may assume that U is star-shaped relative to 0, i.e., with every point U also contains the whole line segment connecting that point to 0. Let $\Omega_{j_1\dots j_r}$ be the functions of n variables defined on $U \subset \mathbf{R}$ and such that, applied to the coordinate functions, they yield the components of ω, that is $\omega_{j_1\dots j_r} = \Omega_{j_1\dots j_r}(x^1,\dots,x^n)$. Now define $\theta = \theta_{j_2\dots j_r}dx^{j_2}\wedge\dots\wedge dx^{j_r}$ through its component functions by setting

$$\theta_{j_2\dots j_r} \,=\, \int_0^1 t^{r-1}x^j\,\Omega_{jj_2\dots j_r}(tx^1,\dots,tx^n)\,dt\,. \tag{51.12}$$

Formula (51.10) with $d\omega = 0$ yields

$$\partial_{j_1}\Omega_{jj_2\dots j_r} - \partial_{j_2}\Omega_{jj_1j_3\dots j_r} + \dots + (-1)^{r+1}\,\partial_{j_r}\Omega_{jj_1\dots j_{r-1}} \,=\, \partial_j\Omega_{j_1j_2\dots j_r}$$

and $r(d\theta)_{j_1\dots j_r} = \partial_{j_1}\theta_{j_2\dots j_r} - \partial_{j_2}\theta_{j_1j_3\dots j_r} + \dots + (-1)^{r+1}\,\partial_{j_r}\theta_{j_1\dots j_{r-1}}$. Since we have $\partial_{j_1}\theta_{j_2\dots j_r} = \int_0^1 t^{r-1}\Omega_{j_1\dots j_r}(tx^1,\dots,tx^n)\,dt + \int_0^1 t^r x^j\,\partial_{j_1}\Omega_{jj_2\dots j_r}(tx^1,\dots,tx^n)\,dt$ (by (51.12)), this and skew-symmetry of the $\Omega_{j_1\dots j_r}$ in $j_1,\dots,j_r$ implies

$$\begin{aligned} r(d\theta)_{j_1\dots j_r} \,&=\, r\int_0^1 t^{r-1}\Omega_{j_1\dots j_r}(tx^1,\dots,tx^n)\,dt \\ &\quad+ \int_0^1 t^r x^j\,\partial_j\Omega_{j_1\dots j_r}(tx^1,\dots,tx^n)\,dt\,, \end{aligned} \tag{51.13}$$

On the other hand, $\omega_{j_1\dots j_r} = \Omega_{j_1\dots j_r}(x^1,\dots,x^n) = \int_0^1 \frac{d}{dt}[t^r\Omega_{j_1\dots j_r}(tx^1,\dots,tx^n)]\,dt$, so $\omega_{j_1\dots j_r} = r(d\theta)_{j_1\dots j_r}$ by (51.13). Thus, $\omega = d\theta$ □

52. Cohomology spaces

Topics: The de Rham cohomology spaces; cohomology of the Euclidean spaces; pullbacks under C^∞ mappings of manifolds; restrictions to submanifolds; cohomology of the circle, computed via integration.

Let M be a C^∞ manifold. We denote by $\Omega^r M$ the real vector space of all global differential r-forms of class C^∞ defined on M. Thus, $\Omega^r M$ is trivial when $r < 0$ or $r > \dim M$, and *infinite-dimensional* otherwise. The subspaces $Z^r M$ and $B^r M$ of $\Omega^r M$ are, by definition, the kernel of $d : \Omega^r M \to \Omega^{r+1} M$ and, respectively, the image $d(\Omega^{r-1}M)$ of $d : \Omega^{r-1}M \to \Omega^r M$. Since $d^2 = 0$, we have $B^r M \subset Z^r M$. The quotient vector space

$$H^r(M, \mathbf{R}) \;=\; Z^r M / B^r M$$

is called the rth *de Rham cohomology space* of M. For simplicity, we will often write $H^r M$ instead of $H^r(M, \mathbf{R})$. As before,

$$H^r M \;=\; \{0\} \qquad \text{if} \quad r < 0 \quad \text{or} \quad r > \dim M\,. \tag{52.1}$$

We will also use the symbol

$$[\omega] \in H^r M$$

for the *cohomology class* of any closed form $\omega \in Z^r M$, that is, its equivalence class $\omega \bmod B^r M \in Z^r M/B^r M$. In other words, $[\omega]$ is the coset $\omega + B^r M$ of $B^r M$ in $Z^r M$.

EXAMPLE 52.1. $Z^0 M$ consists of all functions locally constant in M, and $B^0 M = \{0\}$. Thus, if M is connected, we have an isomorphism

$$H^0 M \;\approx\; \mathbf{R}\,. \tag{52.2}$$

See also Problem 4.

EXAMPLE 52.2. By the Poincaré lemma, $H^r M = \{0\}$ if M is diffeomorphic to a star-shaped open subset of some $\mathbf{R}^n$. Thus,

$$H^r \mathbf{R}^n \;\approx\; \begin{cases} \mathbf{R}\,, & \text{if } r = 0 \\ \{0\}\,, & \text{if } r \neq 0\,. \end{cases}$$

Every C^1 mapping $F : M \to N$ between manifolds can be used to *pullback* differential forms from N to M, preserving the degree, so that it associates with every r-form ω on N the r-form $F^*\omega$ on M given by

$$(F^*\omega)_x(v_1, \ldots, v_r) \;=\; \omega_{F(x)}(dF_x v_1, \ldots, dF_x v_r)\,.$$

The component functions $(F^*\omega)_{j_1 \ldots j_r} = (F^*\omega)(p_{j_1}, \ldots, p_{j_r})$ of $F^*\omega$ relative to local coordinates x^j in M thus can be expressed with the aid of the component functions $\omega_{\alpha_1 \ldots \alpha_r} = \omega(p_{\alpha_1}, \ldots, p_{\alpha_r})$ of ω relative to local coordinates y^α in N as

$$(F^*\omega)_{j_1 \ldots j_r} \;=\; (\partial_{j_1} F^{\alpha_1}) \ldots (\partial_{j_r} F^{\alpha_r})(\omega_{\alpha_1 \ldots \alpha_r} \circ F)\,. \tag{52.3}$$

Therefore, $F^*\omega$ is of class C^∞ if so are ω and F. Any C^∞ mapping $F : M \to N$ thus gives rise to linear mapping $F^* : \Omega^r N \to \Omega^r M$ sending each ω to $F^*\omega$. Furthermore,

$$d \circ F^* \;=\; F^* \circ d\,, \tag{52.4}$$

that is,

$$F^*(d\omega) \;=\; d(F^*\omega) \tag{52.5}$$

for $\omega \in \Omega^r N$ with any degree r (Problem 5). Therefore, $F^*(Z^r N) \subset Z^r M$ and $F^*(B^r N) \subset B^r M$. As a consequence, $F^* : Z^r N \to Z^r M$ descends to a unique linear operator

$$F^* : H^r N \to H^r M$$

between the cohomology spaces in every dimension r, with $F^*[\omega] = [F^*\omega]$. In the case where M is a submanifold of a manifold N and $F : M \to N$ is the inclusion mapping, we call $F^*\omega$ and $F^*[\omega]$ the *restriction to* M of the differential form ω on N or the cohomology class $[\omega]$, with the notation

$$\omega_M = F^*\omega\,, \qquad \sigma_M = F^*\sigma = [\omega_M]$$

for $\omega \in \Omega^r N$ and $\sigma = [\omega] \in H^r N$ (if ω is closed). See also Problems 6 and 7.

EXAMPLE 52.3. For the circle $S^1 = \{z \in \mathbf{C} : |z| = 1\}$, we have a natural isomorphic identification

$$H^1 S^1 = \mathbf{R}\,, \tag{52.6}$$

so that, by (52.1) and Problem 4, $H^0 S^1$ and $H^1 S^1$ are 1-dimensional, while $H^r S^1$ is trivial for $r \notin \{0,1\}$. In fact, the surjective C^∞ mapping

$$\mathbf{R} \ni t \mapsto e^{it} \in S^1 \tag{52.7}$$

is, locally, a diffeomorphism (by Theorem 74.2) and any two local inverses of (52.7) differ by a multiple of 2π in any connected component of the intersection of their domains. Thus, even though t is not a single-valued function of $z = e^{it} \in S^1$, the differential dt is a well-defined global C^∞ 1-form on S^1, forming a global trivialization of T^*S^1. Every C^∞ 1-form ω on S^1 can be written as a function times dt, and since functions on S^1, when composed with (52.7), become precisely all possible functions of $t \in \mathbf{R}$ that are periodic with period 2π, we have $\omega = f(t)\,dt$ with a unique C^∞ function f on $\mathbf{R}$ such that $f(t+2\pi) = f(t)$ for all t. Then, by (51.9) $d\omega = \dot f(t)\,dt$. Let us define the *integral* of any such $\omega \in Z^1 S^1 = \Omega^1 S^1$ to be the number

$$\int \omega = \int_0^{2\pi} f(t)\,dt \in \mathbf{R}\,. \tag{52.8}$$

Note that $\omega = f(t)\,dt \in B^1 S^1$ if and only if $f = \dot h$ for some C^∞ function h on $\mathbf{R}$ that is *also periodic with period* 2π, which is in turn equivalent to $\int \omega = 0$ (as h is obtained from f by integration). Thus, $B^1 S^1$ is the kernel of the nonzero linear function $Z^1 S^1 \ni \omega \mapsto \int \omega \in \mathbf{R}$, which establishes the required isomorphism (52.6).

Problems

1. Establish (51.6). (Hint below.)

2. Verify (51.7), (51.8).

3. Given C^1 vector fields $v_0, \ldots, v_r$ on a manifold M and a differential r-form ω of class C^1 on M, show that

$$\begin{aligned}(r+1)(d\omega)(v_0,\ldots,v_r) &= \sum_{q=0}^{r} (-1)^q d_{v_q}[\omega(v_0,\ldots,\widehat{v_q},\ldots,v_r)] \\ &+ \sum_{0\le p<q\le r} (-1)^{p+q}\,\omega([v_p,v_q],v_0,\ldots,\widehat{v_p},\ldots,\widehat{v_q},\ldots,v_r)\,,\end{aligned}$$

where d_v stands for directional differentiation. (Hint below.)

4. For a manifold M with a finite number k of connected components, describe an isomorphism

$$H^0 M \approx \mathbf{R}^k\,.$$

5. Prove (52.4) (or (52.5)). (Hint below.)

6. Given C^∞ mappings $F : M \to N$, $G : N \to P$ between manifolds, verify the *composite rule*

$$(G \circ F)^* = F^* \circ G^* \tag{52.9}$$

for the pullbacks both of differential r-forms ($\Omega^r P \to \Omega^r M$) and r-dimensional cohomology classes ($H^r P \to H^r M$), and an analogous *identity rule*

$$\mathrm{Id}^* = \mathrm{Id} \tag{52.10}$$

for $M = N$ and $F = \mathrm{Id}$.

7. Show that C^∞-diffeomorphic manifolds M, N must have isomorphic cohomology spaces, i.e., for all integers r,

$$H^r M \approx H^r N.$$

8. Let $\gamma : [a, b] \to M$ be a piecewise C^1 curve in a manifold M. By the *line integral* over γ of any continuous 1-form ξ in M (defined on an open set containing $\gamma([a, b])$) one means the number $\int_\gamma \xi = \int_a^b \xi_{\gamma(t)}(\dot\gamma(t))\, dt$. Assuming that γ is a closed curve in the sense that $\gamma(b) = \gamma(a)$, show that, for any 1-dimensional cohomology class $\sigma \in H^1 M$, the number

$$\int_\gamma \sigma = \int_\gamma [\xi] = \int_\gamma \xi\, dt\,,$$

where $\xi \in Z^1 M$ is any closed C^∞ 1-form with $[\xi] = \sigma$, is well-defined (i.e., independent of the choice of such ξ), and that the function

$$H^1 M \ni \sigma \mapsto \int_\gamma \sigma \in \mathbf{R} \tag{52.11}$$

is linear; one calls (52.11) the *period mapping* corresponding to γ.

9. Given a C^1 mapping $F : M \to N$ between manifolds a piecewise C^1 curve in the source manifold M, and a continuous 1-form ξ in the target manifold M (defined on an open set containing $F(\gamma([a, b]))$), prove that

$$\int_\gamma F^* \xi = \int_{F \circ \gamma} \xi\,. \tag{52.12}$$

Also, verify that, if F is of class C^∞, we have, for any $\sigma \in H^1 N$,

$$\int_\gamma F^* \sigma = \int_{F \circ \gamma} \sigma\,. \tag{52.13}$$

10. Verify that, for any C^1 function f on a manifold M and a piecewise C^1 curve $\gamma : [a, b] \to M$,

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a))\,. \tag{52.14}$$

(Notation as in Problem 8.) Show that the 1-form (51.11) on $\mathbf{R}^2 \setminus \{0\}$ is closed but not exact.

11. Construct a 2-dimensional connected manifold M such that $H^1 M$ is infinite-dimensional. (Hint below.)

Hint. In Problem 1, note that the right-hand side of (51.6) is skew-symmetric in $j_1, \ldots, j_{r+s}$, while, summed against $dx^{j_1} \wedge \ldots \wedge dx^{j_{r+s}}$ it gives $\omega \wedge \theta$ in view of (50.10) and (51.2).

Hint. In Problem 3, use (51.10), (6.6) and (51.1).

Hint. In Problem 5, write (51.10) with ω replaced by $F^*\omega$, using (52.3). The terms on the right-hand side involving second-order derivatives of the F^α will undergo cancellations due to symmetry properties of second-order partial derivatives. The assertion now follows since, by the chain rule, $\partial_j(\omega_{\alpha_1\ldots\alpha_r}\circ F) = [\partial_{\alpha_0}(\omega_{\alpha_1\ldots\alpha_r})\circ F]\,\partial_j F^{\alpha_0}$.

Hint. In Problem 10, note that ω_0 is, locally, the differential of $\arctan(y/x)$ or $-\arctan(x/y)$. Also, $\int_\gamma \omega_0 = 2\pi$ for $\gamma : [0,2\pi]\to \mathbf{R}^2\setminus\{0\}$ with $\gamma(t) = (\cos t, \sin t)$, and so, by (52.14), ω_0 cannot be exact.

Hint. In Problem 11, set, for instance, $M = \mathbf{C}\setminus\mathbf{Z}$. To prove our assertion, use the family ω_k of closed 1-forms labeled with integers k, given by $\omega_k = F^*_{-k}\omega_0$, where $F_k : M \to M$ is the translation by $k\in\mathbf{Z}$ with $F_k(z) = z+k$ and ω_0 is the closed form (51.11) (restricted to the open submanifold M). The closed curves curves $\gamma_k : [0,2\pi]\to M$ with $\gamma_k(t) = k + e^{it}/2$ now satisfy $\gamma_k = F_k\circ\gamma_0$, so by (52.12), (52.13) with $F_{k+l} = F_k\circ F_l$, we have

$$\int_{\gamma_k}[\omega_l] = \int_{\gamma_k} F^*_{-l}\omega_0 = \int_{\gamma_{k-l}}\omega_0 = 2\pi\delta_{kl}\,. \tag{52.15}$$

(Note that $\int_{\gamma_s}\omega_0 = 0$ for all $s\neq 0$, since the image of γ_s then lies in the open set $U = \{z\in M : z = x+iy, x,y\in\mathbf{R}, x\neq 0\}$, while $\omega_0 = df$ in U with $f(z) = f(x+iy) = \arctan(y/x)$.) In view of (52.15), the system $[\omega_k]$, $k\in\mathbf{Z}$, of vectors in H^1M is linearly independent: if any finite combination of these vectors is zero, so is each coefficient as one sees by taking line integrals.

CHAPTER 10

De Rham Cohomology

53. Homotopy invariance of the cohomology functor

Topics: Betti numbers; the Poincaré polynomial; the graded de Rham cohomology algebra; piecewise C^r homotopies between C^r mappings of manifolds; homotopy invariance of the induced cohomology-algebra homomorphism; extension of the cohomology functor to continuous mappings between compact C^∞ manifolds.

For a manifold M and an integer r, the rth *Betti number* of M, denoted by $b_r(M)$, is the dimension of the cohomology space H^rM. We set $b_r(M) = \infty$ if H^rM is infinite-dimensional. By (52.1), $b_r(M) = 0$ if $r < 0$ or $r > \dim M$.

For any manifold M whose Betti numbers are all finite, one defines the *Poincaré polynomial* of M to be the polynomial $\mathbb{P}[M]$ in the variable t given by

$$\mathbb{P}[M] \,=\, \sum_{r=0}^{n} b_r t^r, \quad \text{where} \quad b_r = b_r(M) \quad \text{and} \quad n = \dim M. \tag{53.1}$$

For instance, according to Examples 52.2 and 52.3,

$$\mathbb{P}[\mathbf{R}^n] \,=\, 1, \qquad \mathbb{P}[S^1] \,=\, 1 + t. \tag{53.2}$$

In this chapter we will introduce an object called the *Mayer-Vietoris sequence* and use it to prove that the Betti numbers of a compact manifold are all finite, and determine them for all spheres, tori, projective spaces and closed orientable surfaces of any genus.

A *graded algebra* is a real or complex associative algebra $\mathcal{A}$ along with a fixed direct-sum decomposition $\mathcal{A} = \bigoplus_{r\in\mathbf{Z}} \mathcal{A}_r$ of the underlying vector space such that $\mathcal{A}_r\mathcal{A}_s \subset \mathcal{A}_{r+s}$ for all $r, s \in \mathbf{Z}$ (where the algebra multiplication of $\mathcal{A}$ is written multiplicatively). Examples of interest for us are, for a given manifold M, the graded algebra Ω^*M and its (graded) subalgebras Z^*M and B^*M, with the summands Ω^rM, Z^rM and B^rM, respectively. By (c) in Theorem 51.1, B^*M is a two-sided ideal in Z^*M (but only a subalgebra in Ω^*M). We can thus form the quotient (associative) algebra

$$H^*M \,=\, Z^*M/B^*M\,,$$

called the *cohomology algebra* of M. Its multiplication is referred to as the *cup product* and denoted by $\cup$. Explicitly, H^*M is the direct sum

$$H^*M \,=\, H^0M \oplus \ldots \oplus H^nM\,, \qquad\qquad n = \dim M$$

of all cohomology spaces of the given manifold M, and the cup product is given by

$$[\omega] \cup [\theta] \,=\, [\omega \wedge \theta]\,.$$

Thus, whenever $\rho \in H^rM$ and $\sigma \in H^sM$, (51.5) gives

$$\rho \cup \sigma \,=\, (-1)^{rs}\sigma \cup \rho.$$

Any C^∞ mapping $F : M \to N$ gives rise to the pullbacks $F^* : \Omega^*N \to \Omega^*M$ and $F^* : H^*N \to H^*M$ (see the end of §51), which are algebra homomorphisms (Problem 1). Thus, C^∞-diffeomorphic manifolds have isomorphic cohomology algebras, and the isomorphism in question is *graded*, that is, compatible with the direct-sum decompositions of the algebras.

Let $\varPhi, \varPsi : M \to N$ be C^k mappings between C^k manifolds, $0 \le k \le \omega$. A C^k *homotopy* between $\varPhi$ and $\varPsi$ is any continuous mapping $F : [a,b] \times M \to N$, with $-\infty < a < b < \infty$, such that $F(a,x) = \varPhi(x)$ and $F(b,x) = \varPsi(x)$ for every $x \in M$, while F is of class C^k in the sense that it appears so in any given local coordinates for M and N. (This last requirement is, clearly, a geometric property.) If such F exists, $\varPhi$ and $\varPsi$ are said to be C^k*-homotopic*. When $k = 0$, one drops the prefix C^0 and speaks simply of homotopies and homotopic mappings.

Given manifolds M, N, let $F : [a,b] \times M \to N$ be a C^∞ homotopy between C^∞ mappings $\varPhi, \varPsi : M \to N$. Then, for each integer r, there exists a linear operator $\mathcal{H} : \Omega^r N \to \Omega^{r-1}M$ such that

$$\varPsi^* - \varPhi^* \,=\, d \circ \mathcal{H} \,+\, \mathcal{H} \circ d \tag{53.3}$$

(which, for each r, is an equality between operators $\mathcal{H} : \Omega^r N \to \Omega^r M$, while the dependence of all operators involved on r is suppressed in our notation). Specifically, for $\omega \in \Omega^r N$, we define $\mathcal{H}\omega$ by

$$(\mathcal{H}\omega)_x(v_2, \ldots, v_r) \,=\, \int_a^b [\omega(F(t,x))](\dot F(t,x), dF_x^t v_2, \ldots, dF_x^t v_r)\, dt\,.$$

whenever $x \in M$, that is, in local coordinates x^j for M and y^λ for N,

$$(\mathcal{H}\omega)_{j_2 \ldots j_r} \,=\, \int_a^b (\omega_{\lambda\lambda_2 \ldots \lambda_r} \circ F)\, \frac{\partial F^\lambda}{\partial t} (\partial_{j_2} F^{\lambda_2}) \ldots (\partial_{j_r} F^{\lambda_r})\, dt\,.$$

In fact, (53.3) is easily obtained by combining the last formula with (51.10) and (52.3), due to pairwise cancellations of terms involving second-order partial derivatives of the components of F, and the relation $\varPsi^*\omega - \varPhi^*\omega = \int_a^b (d[F_t^*\omega]/dt)\, dt$, with $F_t = F(t, \cdot\,)$.

We can now prove the fact stated in the title of this section.

THEOREM 53.1. *If two C^∞ mappings $\varPhi, \varPsi : M \to N$ between manifolds are C^∞-homotopic, they induce the same graded algebra homomorphism $H^*N \to H^*M$.*

PROOF. By (53.3), for any closed r-form $\omega \in Z^r N$, the pullbacks $\varPsi^*\omega$ and $\varPhi^*\omega$ differ by the exact form $d(\mathcal{H}\omega) \in B^{r-1}M$. $\square$

Given continuous mappings $\varPhi, \varPsi : N \to M$ from a manifold N into a connected Riemannian manifold (M, g), we define their *uniform distance* to be

$$\operatorname{dist}(\varPhi, \varPsi) \,=\, \sup\{\mathrm{d}(\varPhi(y), \varPsi(y)) : y \in N\}\,. \tag{53.4}$$

Thus, $0 \le \operatorname{dist}(\varPhi, \varPsi) \le \infty$, and $\operatorname{dist}(\varPhi, \varPsi) < \infty$ if one of M, N is compact. We will say that a continuous mapping $\varPsi : N \to M$ is the *uniform limit* of a sequence $\varPhi_k$ of continuous mappings $N \to M$ if $\operatorname{dist}(\varPhi_k, \varPsi) \to 0$ as $k \to \infty$. See also Problem 2.

THEOREM 53.2. *Every continuous mapping $\varPsi$ from a compact manifold N into a compact connected Riemannian manifold (M, g) is the uniform limit of a sequence of C^∞ mappings $N \to M$.*

Proof. Whitney's embedding theorem for compact manifolds allows us to treat M as a submanifold of a Euclidean space V. We may choose an open set U in V, containing M, and a C^∞ retraction $\pi : U \to M$. (Namely, U is the image of a tubular-neighborhood diffeomorphism, cf. Problem in §..., under which π corresponds to the bundle projection of the normal bundle of M in V.) In view of the Stone-Weierstrass theorem (or, more precisely, Corollary 75.2 in Appendix B), Ψ is the uniform limit, relative to the Euclidean metric of V, of a sequence F_k of C^∞ mappings $N \to V$. The image $F_k(N)$ must, for all but finitely many k, be contained in U, or else there would exist points in $V \setminus U$ arbitrarily close to points in M, and so, taking a convergent sequence of the latter points we would get a contadiction (a limit lying in M and, simultaneously, in $V \setminus U$). The sequence $\Phi_k = \pi \circ F_k$ of C^∞ mappings $N \to M$ (with "large" k) now converges to Ψ uniformly relative to the metric g. In fact, otherwise there would exist $\varepsilon > 0$ and a sequence of "large" integers k with points $y_k \in N$ such that $\mathrm{d}(\Phi_k(y_k), \Psi(y_k)) \ge \varepsilon$, and, choosing a subsequence of the y_k that converges to some $y \in N$ (see Problem 3), we would get $0 = \mathrm{d}(\Psi(y), \Psi(y)) \ge \varepsilon$. This contradiction completes the proof. □

Lemma 53.3. *For every compact connected Riemannian manifold (M, g) there exists a constant $\varepsilon > 0$ such that any two C^k mappings $\Phi, \Psi : N \to M$ from any manifold N with $\operatorname{dist}(\Phi, \Psi) < \varepsilon$ are C^k-homotopic.*

Proof. With ε as in Corollary 34.2, we define a homotopy $F : [0,1] \times N \to M$ by $F(t, y) = \exp_x tv$, where $x = \Phi(y)$ and $v \in T_xM$ is the vector such that $\exp_x v = \Psi(y)$, depending C^∞-differentiably on x and $\Psi(y)$. □

We will now extend the cohomology functor, for compact manifolds, to a larger category of all continuous mappings, rather than just C^∞-differentiable ones.

Theorem 53.4. *A graded algebra homomorphism $\Psi^* : H^*M \to H^*N$ can be uniquely assigned to every continuous mapping $\Psi : N \to M$ between compact C^∞ manifolds in such a way that for C^∞ mappings Ψ it is the homomorphism defined at the end of* §51*, while $\Phi^* = \Psi^*$ if two continuous mappings $\Phi, \Psi : N \to M$ are homotopic.*

Proof. Let $\Psi : N \to M$ be continuous. Choosing a Riemannian metric g on M (see Remark 28.3), a constant $\varepsilon > 0$ for (M, g) as in Lemma 53.3, and a C^∞ mapping $\Phi : N \to M$ with $\operatorname{dist}(\Phi, \Psi) < \varepsilon/2$ (which exists in view of Theorem 53.2), we nay now define Ψ^* by setting $\Psi^* = \Phi^*$. This does not depend on how Φ was chosen: for another such choice Ξ, we have $\operatorname{dist}(\Phi, \Xi) < \varepsilon$ due to the triangle inequality for dist (Problem 2), we get $\Xi^* = \Phi^*$ from Lemma 53.3 for $k = \infty$ and Theorem 53.1.

We have thus extended the functor to continuous mappings, and the extension will clearly be unique once we establish the remaining property of homotopy invariance. However, the prceding argument clearly gives $\Phi_1^* = \Phi_2^*$ for two continuous mappings $\Phi_1, \Phi_2 : N \to M$ with $\operatorname{dist}(\Phi_1, \Psi_2) < \varepsilon/2$. Homotopy invariance now is immediate from Problem 5. □

Problems

1. Verify that $F^* : \Omega^*N \to \Omega^*M$ and $F^* : H^*N \to H^*M$ are algebra homomorphisms whenever $F : M \to N$ is a C^∞ mapping.

2. Given a manifold N and a compact connected Riemannian manifold (M, g), show that the set $\mathrm{C}(N, M)$ of all continuous mappings $N \to M$, with the distance function (53.4), forms a metric space. Is it complete?

3. For a compact manifold N, a connected Riemannian manifold (M, g), a sequence $F_k \in \mathrm{C}(N, M)$ with the uniform limit $F \in \mathrm{C}(N, M)$, and a sequence $y_k \in N$ with a limit y, prove that $F_k(y_k) \to F(y)$ in M as $k \to \infty$.

4. Verify that the "functorial" properties (52.9) – (52.10) remain valid in the category of compact C^∞ manifolds and continuous mappings.

5. For a compact manifold N and a compact connected Riemannian manifold (M, g), show that if two continuous mappings $\Phi, \Psi : N \to M$ are homotopic, then, for any $\varepsilon > 0$, there exist an integer $k \geq 1$ and continuous mappings $\Phi_j : N \to M$, $j = 0, 1, \ldots, k$, with $\Phi_0 = \Phi$, $\Phi_k = \Psi$ and $\mathrm{dist}(\Phi_{j-1}, \Psi_j) < \varepsilon$ for all $j \in \{1, \ldots, k\}$. (Hint below.)

Hint. In Problem 5, note that otherwise, with a fixed homotopy $F : [a, b] \times N \to M$ between Φ and Ψ, we could find $\varepsilon > 0$ and sequences $t_l, s_l \in [a, b]$ and $y_l, z_l \in N$, $l = 1, 2, \ldots$, such that $|t_l - s_l| \to 0$ as $l \to \infty$ and $\mathrm{d}(\Phi(t_l, y_l), \Phi(s_l, z_l)) \geq \varepsilon$. Choosing "simultaneous" convergent subsequences, we would get a contradiction.

54. The homotopy type

Topics: Homotopy equivalences and algebra isomorphisms; homotopy type; examples.

One calls a continuous mapping $\Phi : M \to N$ between manifold a *homotopy equivalence* there exists a continuous mapping $\Psi : N \to M$ such that both composites of Φ and Ψ are homotopic to identity mappings $\Phi \circ \Psi$ to Id_N, and $\Psi \circ \Phi$ to Id_M. In this case, Ψ is also a homotopy equivalence, called a *homotopy inverse* of Φ. If a homotopy equivalence $M \to N$ exists, the manifolds M and N are said to be *homotopy equivalent*. One similarly defines (a) C^∞*-homotopy equivalence* by requiring, in addition, that Φ and Φ be of class C^∞.

One also expresses the fact that two manifolds are homotopy equivalent, or C^∞-homotopy equivalent, by saying that they have the same *homotopy type*, or C^∞*-homotopy type*.

Remark 54.1. According to Problem 4 in §53 (and Theorem 53.1), two compact manifolds having the same homotopy type, or two arbitrary manifolds of the smae C^∞-homotopy type, must also have isomorphic graded cohomology algebras.

Lemma 54.2. *Let M be a submanifold of a manifold N and let $\Psi : N \to M$ is a C^∞ mapping which, as a mapping $N \to N$, is homotopy equivalent to the identity* $\mathrm{Id} : N \to N$. *Then the inclusion mapping $M \to N$ is a homotopy equivalence, for which Ψ is a homotopy inverse.*

In fact, for the inclusion mapping $\Phi : M \to N$, we have $\Phi \circ \Psi = \Psi : N \to N$, and $\Psi \circ \Phi = \mathrm{Id}_M$.

Example 54.3. For any $n \geq 1$ and points $o \in \mathbf{R}^n$, $y, z \in S^n$ with $y \neq z$,

a. $S^n \setminus \{y\}$ is diffeomorphic to $\mathbf{R}^n$.
b. $\mathbf{R}^n \setminus \{o\}$ is homotopy equivalent to S^{n-1} (where S^0 is a two-point space).
c. $S^n \setminus \{y, z\}$ is homotopy equivalent to S^{n-1}.
d. Any star-shaped open subset U of $\mathbf{R}^n$ has the homotopy type of a one-point space (which is a 0-dimensional manifold).

See Problem 1.

Problems

1. Prove the claims made in Example 54.3. (Hint below.)

2. Show that the total space of a C^∞ vector bundle over a manifold M is C^∞-homotopy equivalent to M. (Hint below.)

3. Verify that $M \times \mathbf{R}^k$ has, for any manifold M and any integer $k \geq 0$, the C^∞-homotopy type of M.

4. Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$. Prove that the projective space $\mathbf{K}\mathrm{P}^n$ minus a point has, for each $n \geq 1$, the homotopy type of $\mathbf{K}\mathrm{P}^{n-1}$. (Hint below.)

5. Let W be a codimension-one subspace of a finite-dimensional real/complex vector space V, and let $P(W)$ be the corresponding linear subvariety of the projective space $P(V)$. Prove that $P(V) \setminus P(W)$ is diffeomorphic to a Euclidean space. (Hint below.)

6. Let U be a strongly convex, nonempty open set in a connected Riemannian manifold. Verify that U has the homotopy type of a point. (Hint below.)

Hint. In Problem 1, for (a): use the stereographic projection from y (§2), for (b): replace o with 0 (using a translation diffeomorphism) and apply Lemma 54.2 with $N = \mathbf{R}^n \setminus \{0\}$, $M = \{x \in N : |x| = 1\} \approx S^{n-1}$ and $\varPhi(x) = x/|x|$, the homotopy $[0,1] \ni (t,x) \mapsto F(t,x)$ between Id and $\varPhi$ being $F(t,x) = (1-t)x + tx/|x|$. Finally, (c) is immediate from (a) and (b), while in (d) we may assume that U is star-shaped relative to 0 and apply Lemma 54.2 to $N = U$, $M = \{0\}$, $\varPhi(x) = x$ and the homotopy $F(t,x) = (1-t)x$, $0 \leq t \leq 1$.

Hint. In Problem 2, use Lemma 54.2.

Hint. In Problem 4, use Problem 4 in §18 and Problem 2.

Hint. In Problem 5, note that $P(V) \setminus P(W)$ is the domain of a projective coordinate system (§2).

Hint. In Problem 6, choose $x \in U$ and apply Lemma 54.2 to $N = U$, $M = \{x\}$, the constant mapping $\varPhi = x$ and the homotopy $F(t,x) = (1-t)x$, $0 \leq t \leq 1$.

55. The Mayer-Vietoris sequence

Topics: The connecting homomorphism; exact sequences; the Mayer-Vietoris sequence; finiteness of Betti numbers of a compact manifold.

We now turn to a method of determining the Betti numbers of various specific manifolds.

Finding dimensions of vector spaces, and bounds on those dimensions, is often made possible by exhibiting linear operators between pairs of such spaces that are injective and/or surjective, that is, have a trivial or improper kernel or image.

In many cases the information that is readily available has the seemingly weaker form of equality between the image of one operator and the kernel of another; namely, some natural sequences of operators are *exact*, in the sense that defined below. As we will see in the next section, exactness of such a sequence is in turn often sufficient to determine the dimensions of the spaces involved.

Let U, U' be open subsets of a manifold M such that $U \cup U' = M$. The (disjoint) closed sets $K = M \setminus U$ and $K' = M \setminus U'$ can be separated by a C^∞ function $f : M \to \mathbf{R}$ in the sense that $f = 1$ on an open set containing K and $f = 0$ on an open set containing K'. (See Problem 1 in §84 of Appendix D.) Thus, whenever s is an integer and $\omega \in \Omega^s(U \cap U')$, there exist θ, θ' with

$$\omega = \theta - \theta' \quad \text{on} \quad U \cap U', \quad \text{while} \quad \theta \in \Omega^s U \text{ and } \theta' \in \Omega^s U'. \tag{55.1}$$

(In fact, we may set $\theta = f\omega$ and $\theta' = (f-1)\omega$.) For any given $s \in \mathbf{Z}$ we define the *connecting homomorphism* $\delta^* : H^s(U \cap U') \to H^{s+1}M$ to be the linear operator such that, with θ, θ' chosen for $\omega \in Z^s(U \cap U')$ as in (55.1),

$$\delta^*[\omega] = [\sigma], \quad \text{where } \sigma = \begin{cases} d\theta & \text{on } U \\ d\theta' & \text{on } U'. \end{cases} \tag{55.2}$$

That δ^* is well-defined and linear can be seen as follows. First, if the closed form ω representing the given cohomology class $[\omega]$ is fixed, the right-hand side of (55.2) does not depend on the choice of θ, θ' with (55.1): letting $\tilde{\theta}, \tilde{\theta}'$ be another such choice, we have $\tilde{\theta} - \theta = \tilde{\theta}' - \theta'$ on $U \cap U'$, and so there exists $\chi \in \Omega^s M$ with $\chi = \tilde{\theta} - \theta$ on U and $\chi = \tilde{\theta}' - \theta'$ on U'. Hence, for $\tilde{\sigma}$ defined as in (55.2) using $\tilde{\theta}, \tilde{\theta}'$ instead of θ, θ', we have $\tilde{\sigma} = \sigma + d\chi$, that is, $[\tilde{\sigma}] = [\sigma]$. Next, let ω is replaced by another closed form $\tilde{\omega} = \omega + d(\chi - \chi')$ representing the given cohomology class; here we wrote an arbitrary $(s-1)$-form of class C^∞ on $U \cap U'$ as a difference $\chi - \chi'$ of one such form on U and another on U'. Now, if θ, θ' are chosen as in (55.1) for ω, then analogous forms $\tilde{\theta}, \tilde{\theta}'$ for $\tilde{\omega}$ may be defined by $\tilde{\theta} = \theta + d\chi$ and $\tilde{\theta}' = \theta' + d\chi'$. Thus, σ in (55.2) for $\tilde{\omega}$ is the same as for ω. Finally, linearity of δ^* is obvious.

A sequence $\ldots \to V_{k-1} \to V_k \to V_{k+1} \to \ldots$, infinite in both directions and consisting of vector spaces and linear operators (or, more generally, groups and homomorphisms) is called *exact* if, for every integer k, the image of $V_{k-1} \to V_k$ coincides with the kernel of $V_k \to V_{k+1}$.

An important example of an exact sequence is the *Mayer-Vietoris sequence* associated with any pair U, U' of open subsets of a manifold M such that $U \cup U' = M$. (See Problem 2.) It is given by

$$\ldots \xrightarrow[\text{conn.}]{\delta^*} H^sM \xrightarrow[\text{rstr.}]{} H^sU \times H^sU' \xrightarrow[\text{sbtr.}]{} H^s(U \cap U') \xrightarrow[\text{conn.}]{\delta^*} H^{s+1}M \xrightarrow[\text{rstr.}]{} \ldots . \tag{55.3}$$

Here 'conn.', 'rstr.' and 'sbtr.' mean "connecting", "restriction" and "subtraction," while δ^* or 'conn.' is defined by (55.2), 'rstr.' sends $[\omega]$ to $([\chi], [\chi'])$, where χ, χ' are the restrictions of ω to U, U', and 'sbtr.' assigns to $([\chi], [\chi'])$ the difference $[\theta] - [\theta']$, where θ, θ' are the restrictions of χ, χ' to $U \cap U'$.

In addition, we allow $U \cap U'$ in (55.3) to be *empty.* More precisely, the sequence (55.3) remains exact also when we agree to treat the empty set as a manifold, perhaps of dimension -1, and set $H^s\emptyset = \{0\}$, that is, $b_s(\emptyset) = 0$, for all $s \in \mathbf{Z}$.

Note that $\delta^* : H^0(U \cap U') \to H^1M$ is the *zero operator* if $U \cap U'$ is connected. In fact, a closed 0-form ω on $U \cap U'$ then is a constant function, and θ, θ' with (55.1) may be chosen constant as well.

LEMMA 55.1. *If a manifold M is the union of two open sets U, U' such that $H^1U = H^1U' = \{0\}$ and $U \cap U'$ is connected, then $H^1M = \{0\}$.*

PROOF. The restriction operator in (55.3) for $s = 1$ is trivial, since so is its target space; hence its kernel is H^1M. In view of exactness, H^1M is also the image of the zero operator mentioned in the last paragraph. $\square$

As our second application of the Mayer-Vietoris sequence, we will now show that some simple conditions, such as compactness of a manifold M, are suffcient for finite-dimensionality of its cohomology algebra H^*M, that is, finiteness of all Betti numbers $b_r(M)$.

PROPOSITION 55.2. *Let a manifold M admit a finite open covering $\mathcal{U}$ such that $\dim H^*U < \infty$ for the cohomology algebra H^*U of the intersection U of every nonempty subfamily of $\mathcal{U}$. Then $\dim H^*M < \infty$.*

PROOF. Let $\mathcal{U} = \{U_1, \ldots, U_k\}$. We use induction on k. When $k = 1$, our claim follows as $M = U_1$ is the intersection of the subfamily $\{U_1\}$. Next, suppose that $k \geq 2$ and our assertion holds whenever the open covering with the stated property consists of fewer than k sets. The Mayer-Vietoris sequence (55.3) with $U = U_1 \cup \ldots \cup U_{k-1}$ and $U' = U_k$ now satisfies the finite-dimensionality assumption, and hence conclusion, of Problem 4(a). □

COROLLARY 55.3. *The Betti numbers of any compact manifold are all finite.*

This is immediate from Proposition 55.2 in view of Corollary 34.2 combined with the Borel-Heine theorem and Problem 6 in §54. (Intersections of strongly convex are, obviously, strongly convex.)

Problems

1. Verify that the Mayer-Vietoris sequence (55.3) is exact (also when $U \cap U' = \emptyset$).

2. Show that whenever an exact sequence contains a portion that has the form $\ldots \to \{0\} \to V \to W \to \{0\} \to \ldots$, the arrow $V \to W$ appearing in it must be an isomorphism.

3. Given an exact sequence $\ldots \to V_{k-1} \to V_k \to V_{k+1} \to \ldots$ of vector spaces, let us set $n_k = \dim V_k$ and define r_k to be the rank, i.e., dimension of the image, of the operator $V_{k-1} \to V_k$. (Thus, all n_k, r_k lie in the set $\{0, 1, 2, \ldots, \infty\}$, consisting of all nonnegative integers and the symbol ∞.) Verify that $n_k = r_k + r_{k+1}$ and $n_k \leq n_{k-1} + n_{k+1}$ for every integer k. (Hint below.)

4. Let us agree to say that, in an exact sequence $\ldots \to V_{k-1} \to V_k \to V_{k+1} \to \ldots$, *two out of every three spaces* satisfy some given condition if there exist two distinct numbers $q, q' \in \{0, 1, 2\}$ such that this condition holds for V_k whenever $k - q$ or $k - q'$ is divisible by 3.
  (a) Show that if two out of every three spaces in an exact sequence are finite-dimensional, then all spaces in it are finite-dimensional.
  (b) Verify that if two out of every three spaces in an exact sequence are trivial, then all of its spaces are trivial. (Hint below.)

Hint. In Problem 3, the first relation $n_k = r_k + r_{k+1}$ follows from the rank-nullity theorem (since r_k is the dimension of the kernel of $V_k \to V_{k+1}$), and the second one is immediate from the first.
Hint. In Problem 4, use the inequality $n_k \leq n_{k-1} + n_{k+1}$ (Problem 3).

56. Explicit calculations of Betti numbers

Topics: The cohomology spaces of spheres, real and complex projective spaces, tori, connected sums, and and closed orientable surfaces of any genus.

In view of Corollary 55.3, the Poincaré polynomial $\mathbb{P}[M]$ is well defined whenever the manifold M is compact. Using exactness of the Mayer-Vietoris sequence (55.3) and Remark 54.1, we obtain the following relations. Some specific manifolds appearing in them are compact; others, denoted by M and N, are assumed to have finite Betti numbers (and the same then follows for their disjoint sum, connected sum, and Cartesian product). The relatons in question are

i. $\mathbb{P}[\emptyset] = 0$ and $\mathbb{P}[\{0\}] = 1$, where $\{0\}$ is a one-point space,
ii. $\mathbb{P}[S^n] = 1 + t^n$,
iii. $\mathbb{P}[\mathbf{C}\mathrm{P}^n] = 1 + t^2 + t^4 + \ldots + t^{2n} = (1 - t^{2n+2})/(1 - t^2)$,
iv. $\mathbb{P}[\mathbf{R}\mathrm{P}^n] = 1$ for even n, and $\mathbb{P}[\mathbf{R}\mathrm{P}^n] = 1 + t^n$ for odd n,
v. $\mathbb{P}[T^n] = (1+t)^n$,
vi. $\mathbb{P}[M \cup N] = \mathbb{P}[M] + \mathbb{P}[N]$ if M, N are disjoint and $\dim M = \dim N$,
vii. $\mathbb{P}[M \# N] = \mathbb{P}[M] + \mathbb{P}[N] - \mathbb{P}[S^n]$, where $n = \dim M = \dim N$,
viii. $\mathbb{P}[M \times N] = \mathbb{P}[M] \cdot \mathbb{P}[N]$,
ix. $\mathbb{P}[S^1 \times M] = (1+t)\,\mathbb{P}[M]$,
x. $\mathbb{P}[\Sigma_k] = 1 + 2kt + t^2$, where Σ_k is a closed orientable surface of genus k.

Relation (viii), known as the *Künneth formula*, will not be proved (or used) in this text, and we include it only for the reader's infomation. We do, however, prove its special case (ix).

The first equality in (i) is our convention about the empty set (see a comment following (55.3)), the second one is immediate from Example 52.2. We will now derive (ii) – (iv), (vi), and (ix) from exactness of the Mayer-Vietoris sequence (55.3), in which the role of M is played by the manifold appearing on the left-hand sise of each equality. In most cases, instead of the open sets U, U', we will describe their (disjoint, closed) complements Y, Y'. First, in (vi), Y, Y' are M and N, and we can either use the trivial empty-intersection case of (55.3), or use a simple direct argument, analogous to that in Problem 1. In (ii), we choose Y, Y' to be one-point sets, so that, in view of Example 54.3(a),(c), $U \approx U' \approx \{0\}$ and $U \cap U' \approx S^{n-1}$. (Here and in the sequel we let $\approx$ stand for homotopy equivalence.) Thus, by (53.2) and Problem 2 in §55, exactness of (55.3) gives $b_{r-1}(S^{n-1}) = b_r(S^n)$ whenever $r \geq 2$, and hence $b_r(S^n) = b_{r-1}(S^{n-1}) = \ldots = b_1(S^{n-r+1})$ for $r = 2, \ldots, n$. Thus, in view of (53.2) and Lemma 55.1, $b_r(S^n) = 1$ if $r = n$ and $b_r(S^n) = 0$ if $1 \leq r < n$, so that (ii) follows.

In (iii) and (iv), let our Y, Y' be a one-point set and a linear subvariety of codimension one over the scalar field $\mathbf{K}$, as in Problem 5 of §54. Thus, $U \approx \mathbf{K}\mathrm{P}^{n-1}$, $U' \approx \{0\}$, and, in (iii), $U \cap U' \approx S^{2n-1}$, while, in (iv), $U \cap U' \approx S^{n-1}$. (See Problem 5 in §54 and Example 54.3(b).) For any $n \geq 1$ we have $b_{2n-1}(\mathbf{C}\mathrm{P}^n) = 0$, since (55.3) contains the fragment $\{0\} \to H^{2n-1}\mathbf{C}\mathrm{P}^n \to \{0\}$ (by (ii) and (52.1)). Another fragment is $\{0\} \to H^{2n-1}S^{2n-1} \to H^{2n}\mathbf{C}\mathrm{P}^n \to \{0\}$ (cf. (52.1)), and so $b_{2n}\mathbf{C}^n = b_{2n-1}S^{2n-1} = 1$ from Problem 2 in §55 and (ii). Similarly, the fragment $\{0\} \to H^s\mathbf{C}\mathrm{P}^n \to H^s\mathbf{C}\mathrm{P}^{n-1} \to \{0\}$ in (55.3), for $s = 2, \ldots, 2n-2$, shows that $b_s(\mathbf{C}\mathrm{P}^n)$ is the same for all $n \geq s/2$, with any fixed $s \geq 2$. Also, Lemma 55.1 implies, by induction on n, that $b_1(\mathbf{C}\mathrm{P}^n) = 0$ for all $n \geq 1$. These formulae, combined, easily give (iii), and a similar argument, with S^{2n-1} replaced by S^{n-1}, yields (iv) (see Problem 2).

Proofs of (v), (ix) and (x) are left to the reader in the form of Problems 3, 4, 5 and 6.

Relations (vii) will be proved later, in §58, under the assumption that M and N are compact.

Problems

1. For a manifold M having a finite number k of connected components $U_1, \ldots, U_k$, exhibit for each $r \in \mathbf{Z}$ a natural isomorphism $H^rM \approx H^rU_1 \oplus \ldots \oplus H^rU_k$. (Hint below.)

2. Verify (iv).
3. Prove (ix). (Hint below.)
4. Prove (v). (Hint below.)
5. Prove (x). (Hint below.)

Hint. In Problem 1, use the assignment

$$H^rM \ni [\omega] \mapsto ([\omega_1], \ldots, [\omega_k]) \in H^rU_1 \oplus \ldots \oplus H^rU_k \,,$$

ω_q being the restriction of ω to U_q.
Hint. In Problem 3, choose the closed sets Y, Y' to be $\{p\} \times M$ and $\{q\} \times M$ for two distinct points $p, q \in S^1$, and use Problem 3 in §54 (for $k = 1$) along with (vi).
Hint. In Problem 4, use (ii) for $n = 1$, (ix) and induction on n.
Hint. In Problem 5, use (v) for $n = 2$ and (vii).

57. Stokes's formula

Topics: Oriented integral of a compactly supported continuous highest-degree forms on oriented manifolds; the Stokes formula (without a boundary term).

By an *orientation in a real vector bundle* η with a positive fibre dimension over a manifold M we mean an assignment that associates with every $x \in M$ an orientation of the fibre η_x (cf. §70 in Appendix A) and is continuous in the sense of being represented, in a suitable neighborhood of any point of M, by a local trivialization of η. One calls η *orientable* if an orientation in η exists.

Similarly, an *orientation of a manifold* M with $\dim M \geq 1$ is any orientation in the tangent bundle TM, and M is said to be an *orientable manifold* if TM is an orientable vector bundle.

In addition, by an *oriented* bundle (or, manifold) we mean a pair consisting of an orientable real vector bundle (or, manifold) and a fixed orientation for it.

A local coordinate system x^j in an oriented manifold M is said to be *compatible with the orientation* if, for every x in the coordinate domain, the basis $p_j(x)$ represents the orientation chosen in T_xM. Note that every coordinate system x^j can be "made" compatible with the orientation by changing the sign of x^1 in suitable connected components of the coordinate domain.

Let ω be a continuous differential n-form with a compact support on an n-dimensional oriented manifold M. We now define a number

$$\int_M \omega \in \mathbf{R} \tag{57.1}$$

called the *oriented integral* of ω. To do this, we note that the fixed orientation of M naturally determines a C^∞ vector-bundle isomorphism between Λ^nM and the real-line bundle of densities on M, which assigns to an exterior n-form at a point $x \in M$ the density μ sending any (ordered) basis $e_1, \ldots, e_n$ of T_xM to $\pm\mu(e_1, \ldots, e_n)$, with the sign indicating whether or not $e_1, \ldots, e_n$ agrees with the fixed orientation. We now declare (57.1) to be $\int_M \mu$.

For instance, if $\operatorname{supp}\omega$ is small in the sense of §36, and we choose a coordinate system x^j compatible with the orientation, which contains $\operatorname{supp}\omega$ in its domain, then

$$\int_M \omega \,=\, \int_{\mathbf{R}^n} \Omega_{1\ldots n}(x^1, \ldots, x^n)\, dx^1 \ldots dx^n. \tag{57.2}$$

where $\Omega_{1\dots n}$ is a function of n real variables with $\omega_{1\dots n} = \Omega_{1\dots n}(x^1, \dots, x^n)$, for the only "essential" component function $\omega_{1\dots n}$ of ω (see Problem 1), characterized by

$$\omega_{1\dots n} \,=\, \omega(p_1, \dots, p_n)\,,$$

that is

$$\omega \,=\, \omega_{1\dots n}\, dx^1 \wedge \ldots \wedge dx^n$$

(cf. (51.2), (51.3)). The right-hand side of (57.2) makes sense as $\Omega_{1\dots n}$ may be trivially extended to a compactly supported continuous function on $\mathbf{R}^n$.

The following classical result is know as *Stokes's theorem.*

THEOREM 57.1. *Let θ be a compactly supported differential $(n-1)$-form of class C^1 on an n-dimensional oriented manifold M. Then*

$$\int_M d\theta \,=\, 0\,.$$

PROOF. See Problem 7. □

Given a n-dimensional manifold M which is both *compact* and *oriented*, we can now define a linear function

$$\int_M : H^n M \to \mathbf{R} \tag{57.3}$$

called the *integration of top cohomology classes*, and assigning to each $\sigma \in H^n M$ the value

$$\int_M \sigma \,=\, \int_M [\omega] \,=\, \int_M \omega\,,$$

with $\omega \in \Omega^n M = Z^n M$ such that $\sigma = [\omega]$. By the Stokes theorem, $\int_M \sigma$ is well defined (that is, independent of the choice of ω in the given cohomology class).

REMARK 57.2. The integration (57.3) is always surjective (i.e., nonzero), as we may choose ω with a small support and such that $\Omega_{1\dots n}$ in (57.2) is nonnegative but not identically zero.

Problems

1. Verify that the components (51.3) of any exterior (or differential) n-form ω in an n-dimensional manifold M satisfy $\omega_{j_1\dots j_n} = \varepsilon_{j_1\dots j_n}\,\omega_{1\dots n}$, where $\varepsilon_{j_1\dots j_n}$ is the Ricci symbol (see the hint for Problem 11 in §8).

2. Show that
$$\omega(u_1, \dots, u_k) = \det \mathfrak{B} \cdot \omega(v_1, \dots, v_k)$$
whenever ω is a k-linear skew-symmetric mapping $V \times \ldots \times V \to W$ between real or complex vector spaces, and the vectors $u_1, \dots, u_k \in V$ are combinations of $v_1, \dots, v_k \in V$ with the coefficient matrix $\mathfrak{B} = [B_\alpha^\beta]$, so that $u_\alpha = B_\alpha^\beta v_\beta$, $\alpha, \beta \in \{1, \dots, k\}$. (Hint below.)

4. Show that the integral in (57.1) is a linear function of ω.

5. Verify that $\int_{-\infty}^{\infty} f'(t)\, dt = 0$ for any compactly supported C^1 function on $\mathbf{R}$.

6. Show that $\int_{\mathbf{R}^n} [\partial f / \partial x^j]\, dx^1 \dots dx^n = 0$ for any compactly supported C^1 function on $\mathbf{R}^n$ and any $j = 1, \dots, n$. (Hint below.)

7. Prove Theorem 57.1. (Hint below.)

8. Verify that the integrals (57.1), (57.3) both change sign when the orientation of the manifold in question is reversed.

9. Verify that formula (52.8) expresses the oriented integral (57.1) in the case where the manifold in question is S^1 with the orientation that assigns to each $z \in S^1$ the set of bases (i.e., nonzero vectors v) in $T_z S^1$ given by $(dt)(v) > 0$, with dt defined according to (52.7).

10. Show that every C^1 diffeomorphism F between oriented connected manifolds M, N with $\dim M = \dim N = n \geq 1$ is either *orientation-preserving* or *orientation-reversing* in the sense that, at every $x \in M$, the differential dF_x sends the orientation chosen in $T_x M$ onto that chosen in $T_{F(x)} N$ or, respectively, its opposite (and the choice of one or the other option is the same for all x). Is the connectedness assumption necessary?

Hint. In Problem 2, note that

$$\begin{aligned} \omega(B_1^{\alpha_1} v_{\alpha_1}, \ldots, B_k^{\alpha_k} v_{\alpha_k}) \;&=\; B_1^{\alpha_1} \ldots B_k^{\alpha_k}\, \omega(v_{\alpha_1}, \ldots, v_{\alpha_k}) \\ &=\; \varepsilon_{\alpha_1 \ldots \alpha_k} B_1^{\alpha_1} \ldots B_k^{\alpha_k}\, \omega(v_1, \ldots, v_k), \end{aligned}$$

where $\varepsilon_{\alpha_1 \ldots \alpha_k}$ is the Ricci symbol (Problem 1), while

$$\varepsilon_{\alpha_1 \ldots \alpha_k} B_1^{\alpha_1} \ldots B_k^{\alpha_k} \;=\; \det\,[B_\alpha^\beta]\,.$$

Hint. In Problem 6, use iterated integration and Problem 5.

Hint. In Problem 7, use a partition of unity to assume that θ is supported in a coordinate domain, and note that, by Problem 6, each term of $(d\theta)_{1\ldots n} = \partial_1 \theta_{2\ldots n} - \partial_2 \theta_{13\ldots n} + \ldots \pm \partial_n \theta_{1\ldots n-1}$ then contributes zero to the integral $\int_M d\theta$.

Hint. In Problem 9, note that due to linearity of either integral, we may assume (using a partition of unity) that ω is supported in an open subset U of S^1 that is a diffeomorphic image under (52.7) of an open set in $\mathbf{R}$. This makes t into a coordinate (system) with the domain U, compatible with the orientation (as $(dt)(\partial/\partial t) = 1 > 0$) and (52.8) is nothing else than (57.2) with $n = 1$, $x^1 = t$ and $\Omega_1 = f$.

58. The fundamental class and mapping degree

Topics: Integration of top cohomology classes as an isomorphism for spheres; compactly supported antiderivatives for compactly supported closed forms in Euclidean spaces; integration as an isomorphism between the top cohomology space of any compact oriented connected manifold and the real line; the fundamental class; the mapping degree; examples.

Remark 58.1. Given a compact oriented n-dimensional manifold M, the following three conditions are equivalent:

a. For every n-form ω of class C^∞ on M with $\int_M \omega = 0$ there exists a differential $(n-1)$-form θ of class C^∞ on M such that $\omega = d\theta$.
b. The oriented-integration operator (57.3) is an *isomorphism*.
c. $\dim H^n M = 1$.

In fact, (a) implies (b), and (b) implies (c). Now, assume (c), and let W be the space of all $\omega \in \Omega^n M$ with $\int_M \omega = 0$. By the Stokes theorem, $B^n M \subset W \subset Z^n M$, while $W \neq Z^n M$ (see Remark 57.2), so that $W = B^n M$ as the image of W under the projection into the 1-dimensional quotient space $Z^n M / B^n M$ now must be trivial.

Each of the equivalent three conditions (a) – (c) in Remark 58.1 is actually satisfied by every n-dimensional manifold M which is compact, oriented and, in addition, connected. (See Theorem 58.5 below.) We have already established that fact (namely, condition (b)) for all spheres; see (i) in §56.

We now need the following, more refined version of the Poincaré lemma.

LEMMA 58.2. *Suppose that K is a closed ball centered at 0 in $\mathbf{R}^n$, while $U \subset \mathbf{R}^n$ is an open set with $K \subset U$, and ω is a differential r-form of class C^∞ on $\mathbf{R}^n$ such that*

i. $\operatorname{supp}\omega \subset K$, *and*
ii. *Either $1 \le r \le n-1$ and ω is closed, or $r = n$ and $\int_{\mathbf{R}^n}\omega = 0$.*

Then $\omega = d\theta$ for some differential $(r-1)$-form θ of class C^∞ on $\mathbf{R}^n$ such that $\operatorname{supp}\theta \subset U$.

PROOF. Let K' be another closed ball centered at 0 with $K \subset K' \subset U$, and let $f : \mathbf{R}^n \to \mathbf{R}$ be a C^∞ function such that $f = 0$ on an open set containing K and $f = 1$ on $\mathbf{R}^n \setminus K'$ (Problem 18 in §6). Let us also identify $\mathbf{R}^n$ with $S^n \setminus \{z\}$ for some $z \in S^n$ as in Example 54.3(a). The center 0 of K and K' then becomes the antipodal point $-z$ of z, and the stereographic projection from $-z$ provides a diffeomorphism between both $S^n \setminus K$, $S^n \setminus K'$ and suitable open Euclidean balls. Regarding ω as an r-form of class C^∞ on S^n, we have $\omega \in B^r S^n$ (by Remark **??**, (a) in Remark 58.1 and (i) in §56). Thus, there exists a differential $(r-1)$-form θ' of class C^∞ on S^n such that $\omega = d\theta'$. Next, the form obtained by restricting θ' to the "ball" $S^n \setminus K$ is obviosly closed. If $r > 1$, Poincaré's Lemma for that restricted form implies the existence of $\chi \in \Omega^{r-2}(S^n \setminus K)$ with $\theta' = d\chi$ on $S^n \setminus K$. The form $\theta = \theta' - d(f\chi)$ then satisfies our assertion. On the other hand, if $r = 1$, the function θ' with $d\theta' = 0$ on $S^n \setminus K$ equals a constant c there (as on $S^n \setminus K$ is connected), and we may use $\theta = \theta' - c$. This completes the proof. □

PROPOSITION 58.3. *Let $\omega \in Z^r M$ be a closed differential r-form of class C^∞ on an n-dimensional manifold M such that* $\operatorname{supp}\omega$ *is small in the sense of* §36. *If, in addition, either*

i. $1 \le r \le n-1$, *or*
ii. $r = n$, *while M is orientable and $\int_M \omega = 0$,*

then $\omega \in B^r M$, that is, there exists a differential $(r-1)$-form θ of class C^∞ on M such that $\omega = d\theta$.

PROOF. Identifying U with an open ball centered at 0 in $\mathbf{R}^n$, we may treat ω as a compactly supported global form on $\mathbf{R}^n$. Let K be the smallest closed ball centered at 0 in $\mathbf{R}^n$ such that $\operatorname{supp}\omega \subset K$. Thus, $K \subset U$ and the radius of K equals the distance between $\operatorname{supp}\omega$ and $\mathbf{R}^n \setminus U$. Applying Lemma 58.2 to ω, the open ball U, and this K, we find the corresponding θ which has a compact support contained in U, and so it may be treated as a form on M. □

REMARK 58.4. Note that the $(r-1)$-form θ obtained in the above proof also has the property that $\operatorname{supp}\theta$ is small.

The following *isomorphism theorem* shows that the three equivalent conditions (a) – (c) in Remark 58.1 hold for all compact, oriented, connected manifolds.

THEOREM 58.5. *For every compact, connected, oriented n-dimensional manifold M, the integration of top cohomology classes is an isomorphism*

$$H^n M \ni \sigma \mapsto \int_M \sigma \in \mathbf{R}\,. \tag{58.1}$$

PROOF. We will prove condition (a) of Remark 58.1. Let $\omega \in \Omega^n M$ with $\int_M \omega = 0$. Choosing U_q and φ_q, $q = 1, \ldots, k$, as in Lemma 36.4 we obtain $\omega = \sum_q \omega'_q$ with $\omega'_q = \varphi_q \omega$. In addition, let us select compactly supported forms $\chi_q \in \Omega^n M$, $q = 1, \ldots, k$, with $\chi_k = 0$ and such that, for all $q = 1, \ldots, k-1$, $\operatorname{supp} \chi_q \subset U_q \cap U_{q+1}$ and $\int_M \chi_q = 1$. (This is possible as $U_q \cap U_{q+1}$ is nonempty.) If we now set $a_q = \sum_{j=1}^q \int_M \omega'_j$ for $q = 0, \ldots, k$, then $a_0 = a_k = 0$. Formula

$$\omega_q \,=\, a_{q-1}\chi_{q-1} \,+\, \omega'_q \,-\, a_q \chi_q\,,$$

for $q = 1, \ldots, k$, defines compactly supported n-forms ω_q of class C^∞, such that $\int_M \omega_q = 0$, while $\operatorname{supp} \omega_q \subset U_q$ for all $q = 1, \ldots, k$ and, finally, $\sum_q \omega_q = \sum_q \omega'_q = \omega$. Thus, each ω_q satisfies the hypotheses of Proposition 58.3, so that $\omega_q \in B^n M$. Consequently, $\omega \in B^n M$, which completes the proof. $\square$

Let us again consider a compact, connected, oriented manifold M of any dimension $n \geq 1$. The isomorphism (58.1) now singles out a distinguished nonzero element σ^M of the top cohomology space $H^n M$ $(n = \dim M)$, characterized by

$$\int_M \sigma^M \,=\, 1\,. \tag{58.2}$$

One calls σ^M the *orientation class* (or *fundamental class*) of the manifold M. Note that σ^M depends on the orientation, and is replaced with $-\sigma^M$ when the orientation is reversed. Furthermore, σ^M spans $H^n M$ and, for any $\sigma \in H^n M$,

$$\sigma \,=\, \left(\int_M \sigma\right) \sigma^M\,. \tag{58.3}$$

Let $F : M \to N$ be any C^∞ mapping between compact, connected, oriented manifolds M, N of the same dimension $n \geq 1$. The *degree* of F is by definition the real number

$$\deg F \,=\, \int_M F^* \sigma^N\,. \tag{58.4}$$

In view of (58.2) and (58.3), we have

$$F^* \sigma^N \,=\, \deg F \cdot \sigma^M \tag{58.5}$$

(since the integrals of both sides coincide) and

$$\int_M F^* \sigma \,=\, \deg F \cdot \int_M \sigma$$

for each $\sigma \in H^n N$. For $\sigma = [\omega]$ with any given $\omega \in \Omega^n N$, this gives

$$\int_M F^* \omega \,=\, \deg F \cdot \int_M \omega\,.$$

Thus, $\deg F$ can be found from the formula

$$\deg F \,=\, \frac{\int_M F^* \omega}{\int_M \omega} \tag{58.6}$$

which is valid for any differential n-form ω of class C^∞ on the n-dimensional manifold N such that $\int_M \omega \neq 0$.

In the following examples, F, M, N, n are as above.

EXAMPLE 58.6. We have $\deg F = 0$ if F is a constant mapping. In fact, then $\int_M F^* \omega = 0$ in (58.6).

EXAMPLE 58.7. More generally, $\deg F = 0$ if F is not surjective (that is, not "onto" N. Namely, we then have $\int_M F^*\omega = 0$ for any $\omega \in \Omega^n N$ supported in the (nonempty and open!) set $N \setminus F(M)$.

EXAMPLE 58.8. The degree of a mapping changes sign when the orientation of M or N is reversed, since so does $\int_M$ and σ^N, respectively. Reversing both orientations leaves the degree unchanged. This leads to an interesting situation when $M = N$ (and we choose to endow both "copies" of M with the same orientation). Then M needs only to be orientable (but not necessarily oriented) for the degree to be well-defined, by the formula $F^*\sigma^M = \deg F \cdot \sigma^M$ (or (58.4), or (58.6), with $M = N$). In other words, if $n = \dim M$ and $F : M \to M$ is a C^∞ mapping,

$$F^* \text{ acts on } H^n M \text{ via multiplication by } \deg F.$$

Applied to the identity mapping of M, this yields

$$\deg \mathrm{Id} \,=\, 1\,. \tag{58.7}$$

EXAMPLE 58.9. For the composites of C^∞ mappings between compact, connected, oriented manifolds of the same dimension we have

$$\deg(G \circ F) \,=\, \deg F \,\cdot\, \deg G\,,$$

by (58.4), (58.5) and (52.9). Now (58.7) yields

$$\deg F \,\cdot\, \deg F^{-1} \,=\, 1\,, \qquad \text{and so } \deg F \neq 0 \tag{58.8}$$

whenever F is a diffeomorphism.

EXAMPLE 58.10. From the homotopy invariance of F^* (see §53) it follows that homotopic mappings must have the same degree. In particular, if F is a homotopy equivalence, (58.8) remains valid with F^{-1} replaced by a homotopy inverse of F.

EXAMPLE 58.11. For every compact oriented n-dimensional manifold M there exists a C^∞ mapping $F : M \to S^n$ with $\deg F = 1$ for a suitable orientation of the sphere S^n. (See Problem 8.)

Problems

1. Verify that, for a C^∞ diffeomorphism $F : M \to N$ between compact, connected, oriented manifolds M, N with $\dim M = \dim N = n \geq 1$, we have $\deg F = 1$ if F is orientation-preserving (cf. Problem 10 in §48) and $\deg F = -1$ if F is orientation-reversing.

2. Let $F, F' : M \to N$ be C^∞ mappings between compact, connected, oriented manifolds M, N. Show that if F and F' are C^∞-homotopic (§53), then $\deg F = \deg F'$.

3. Show that for every Euclidean inner product $\langle\,,\rangle$ in a real vector space V of any even finite dimension, V may be identified with the underlying real space of some complex vector space in such a way that $\langle\,,\rangle$ becomes the real part of a Hermitian inner product. (Hint below.)

4. Let $F : S^n \to S^n$ be the antipodal mapping of the unit sphere in $\mathbf{R}^{n+1}$, that is, the restriction to S^n of the multiplication by -1. Prove that $\deg F = (-1)^{n+1}$. (Hint below.)

5. For a C^∞ mapping $F : S^1 \to S^1$ of the circle $S^1 = \{z \in \mathbf{C} : |z| = 1\}$, let us define the function $\mathbf{R} \ni t \mapsto z(t) \in \mathbf{C}$ by $z(t) = F(e^{it})$, and set $\dot z(t) = dz/dt$. Prove that
$$\deg F \,=\, \frac{1}{2\pi i}\int_0^{2\pi} \frac{\dot z(t)}{z(t)}\,dt\,.$$
(Hint below.)

6. Can a compact, connected, orientable manifold be homotopy equivalent to a compact, connected, orientable manifold of some other dimension? (Hint below.)

7. Given an integer $n \ge 1$, show thet there exists a C^∞ mapping $F : \mathbf{R}^n \to S^n$ and a point $z \in S^n$ such that $F : U \to S^n \setminus \{z\}$ is a diffeomorphism (where $U = \{\mathbf{x} \in \mathbf{R}^n : |\mathbf{x}| < 1\}$) and $F(\mathbf{y}) = z$ for all $\mathbf{y} \in \mathbf{R}^n \setminus U$. (Hint below.)

8. Prove the statement in Example 58.11. (Hint below.)

Hint. In Problem 3, fix a Hermitian inner product in a complex vector space W of the same real dimension as V and choose a linear isometry between its underlying real space of some and V.

Hint. In Problem 4, note that, when n is even, F is orientation-reversing (both in $\mathbf{R}^{n+1}$ and S^n), while, if n is odd, F is homotopic to the identity via the homotopy (curve of mappings) whose tth stage, for $t \in [0,\pi]$, is the multiplication by e^{it} in $\mathbf{R}^{n+1}$ treated as $\mathbf{C}^{(n+1)/2}$, cf. Problem 3.

Hint. In Problem 4, note that $\omega = dt$ is a well-defined 1-form on S^1, even though t is not a well-defined function, and use (58.6).

Hint. In Problem 6, note that $\dim M$ equals the largest r satisfying the condition $H^rM \ne \{0\}$, which in turn is invariant under homotopy equivalences.

Hint. In Problem 7, let $S^n = \{(t,\mathbf{x}) : t \in \mathbf{R},\ \mathbf{x} \in \mathbf{R}^n,\ t^2 + |\mathbf{x}|^2 = 1\}$ and then set $z = (-1, \mathbf{0})$ and
$$F(\mathbf{x}) \,=\, (\cos(\varphi(|\mathbf{x}|))\,,\ \sin(\varphi(|\mathbf{x}|)) \cdot \mathbf{x}/|\mathbf{x}|)\,,$$
where $\varphi : \mathbf{R} \to \mathbf{R}$ is a C^∞ function such that $\varphi(t) \le 0$ for $t \le 0$, $\varphi(0) = 0$, $\dot\varphi(0) = 1$, $\varphi(t) = 2\pi$ for $t \ge 1$, and $\dot\varphi(t) > 0$ for $t \in (0,1)$. (Take φ to be, e.g, a multiple of an antiderivative for χ in Problem 16 in §83 of Appendix D.)

59. Degree and preimages

Topics: Integration as an isomorphism between the top cohomology space of any compact oriented connected manifold and the real line; the fundamental class; the mapping degree; examples.

Given a linear operator $\Phi : V \to W$ between two oriented real vector spaces of the same positive, finite dimension, we define
$$\operatorname{sgn}\Phi \in \{-1, 0, 1\}$$
to be $+1$ if Φ is an isomorphism that preserves the orientation (i.e., maps the distinguished orientation in V onto the distinguished orientation in W, to be -1 if Φ is an isomorphism that reverses (does not preserve) the orientation, and to be 0 if Φ is not an isomorphism.

Lemma 59.1. *Let $F : M \to N$ be any C^∞ mapping from a compact manifold M into a manifold N, and let $y \in N$ be such that $dF_x : T_xM \to T_yN$ is injective for every $x \in F^{-1}(y)$. Then the set $F^{-1}(y)$ is finite.*

Proof. Suppose, on the contrary, that there is an infinite sequence of distinct points $x_k \in M$ with $F(x_k) = y$. Replacing it with a subsequence, we may assume that $x_k \to x$ as $k \to \infty$ for some $x \in M$. Then $x \in F^{-1}(y)$ and, by the rank theorem (cf. formula (9.6), F is injective on some neighborhood of x, which contradicts the fact that such a neighborhood must contain infinitely many preimages of y. This completes the proof. $\square$

We now prove what may be called the *mapping degree theorem.*

THEOREM 59.2. *Let $F : M \to N$ be any C^∞ mapping between compact, connected, oriented manifolds M, N of the same dimension $n \geq 1$, and let y be a regular value of F, that is, a point $y \in N$ such that $dF_x : T_xM \to T_yN$ is an isomorphism for every $x \in F^{-1}(y)$. Then the set $F^{-1}(y)$ is finite, and the degree of F can be written as*

$$\deg F \;=\; \sum_{x \in F^{-1}(y)} \operatorname{sgn} dF_x \,,$$

This is immediate from (58.6) applied to a differential n-form ω of class C^∞ supported in a sufficiently small neighborhood of y and such that $\int \omega \neq 0$.

Since regular values y as required in the above theorem always exists (see Sard's theorem in §57 of Appendix B), we obtain

COROLLARY 59.3. *The degree $\deg F$ of any C^∞ mapping F between compact, connected, oriented manifolds M, N of the same dimension $n \geq 1$, is an integer:*

$$\deg F \,\in\, \mathbf{Z}\,.$$

EXAMPLE 59.4. The fundamental theorem of algebra: every nonconstant polynomial P with complex coefficients, viewed as a mapping $\mathbf{C} \to \mathbf{C}$, is onto (and hence has a root). In fact, we may treat $\mathbf{C}$ as an open subset of the Riemann sphere S^2, namely the domain of one of the two coordinate systems introduced in Problem 8 of §2. Then $\mathbf{C} = S^2 \setminus \{\infty\}$ for a specific point $\infty \in S^2$, and the other coordinate system has the domain $U = S^2 \setminus \{0\}$, with $0 \in \mathbf{C} \subset S^2$, and the coordinate mapping $U \ni z \mapsto 1/z \in \mathbf{C}$ (where $1/\infty = 0$). Our polynomial $P : \mathbf{C} \to \mathbf{C} \subset S^2$ has an extension $P : S^2 \to S^2$ given by $P(\infty) = \infty$, which is easily seen to be a C^∞ mapping. Since the Riemann sphere is compact, connected and orientable (Problems 19 and 15 in §3), P has a well defined mapping degree $\deg P$. Furthermore, $\deg P$ coincides with the algebraic degree of P (the integer k with $P(z) = a_0 + a_1 z + \ldots + a_k z^k$, where $a_0, a_1, \ldots, a_k \in \mathbf{C}$ and $a_k \neq 0$). In fact, P, as a mapping $S^2 \to S^2$, is homotopic to each of the polynomials $P_t(z) = t(a_0 + a_1 z + \ldots + a_{k-1} z^{k-1}) + a_k z^k$, $t \in [0, 1]$, including P_0, and P_0 is in turn homotopic to F with $F(z) = z^k$ (write $a_k = re^{i\theta}$ and consider the homotopy $[0, 1] \ni t \mapsto F_t$ with $F_t(z) = (r - tr + t)e^{i\theta(1-t)} z^k$). Since $\deg F = k$ (by Theorem 59.2), we have $\deg P = k > 0$ and so P is surjective (Example 58.7), as required.

Problems

1. Given a C^∞ mapping $F : S^1 \to S^1$ of the circle

$$S^1 \;=\; \{z \in \mathbf{C} : |z| = 1$$

into itself, define the complex-valued function $\mathbf{R} \ni t \mapsto F(e^{it}) \in \mathbf{C}$. Prove that

$$\deg F \;=\; \frac{1}{2\pi i} \int_0^{2\pi} \frac{\dot z(t)}{z(t)}\, dt\,, \tag{59.1}$$

where, as usual, $(\ )^{\cdot} = d/dt$. Thus, the integral in (59.1) is automatically real. (Hint below.)

CHAPTER 11

Characteristic Classes

60. The first Chern class

Topics: The (first) Chern form of a connection in a complex vector bundle; the first Chern class for complex vector bundles; the first Chern class and operations on bundles.

Given a connection ∇ in a complex vector bundle η over a manifold M, we define the (first) *Chern form* of ∇ to be the differential form $\mathrm{c}^\nabla \in \Omega^2 M$ of degree 2 on M with

$$\mathrm{c}_x^\nabla(v,w) \;=\; \frac{1}{2\pi}\,\mathrm{Im}\,[\mathrm{Trace}\,(R_x^\nabla(v,w))] \tag{60.1}$$

for $x \in M$ and $v,w \in T_xM$ (where Im means 'the imaginary part of'). Thus, in terms of local coordinates x^j for M and a local trivialization e_a for η, the component functions of c^∇ are

$$\mathrm{c}_{jk}^\nabla \;=\; \frac{1}{2\pi}\,\mathrm{Im}\,R_{jka}{}^a\,. \tag{60.2}$$

Furthermore, by (20.10),

$$R_{jka}{}^a \;=\; \partial_k \Gamma_{ja}^a - \;\partial_j \Gamma_{ka}^a\,,$$

as the two terms quadratic in the Γ_{ja}^b cancel each other after contraction (which amounts to vanishing of the trace of the commutator of two square matrices). Thus, by (51.2) and (51.10) we have

$$\mathrm{c}^\nabla \;=\; -\,\frac{1}{\pi}\,d\,(\mathrm{Im}\,\Gamma_{ja}^a\,dx^j)\,, \tag{60.3}$$

so that c^∇ is locally exact, and therefore it is closed:

$$d\,\mathrm{c}^\nabla \;=\; 0\,,$$

i.e., $\mathrm{c}^\nabla \in Z^2M$. Any other connection $\tilde\nabla$ in η can be written as $\tilde\nabla = \nabla + F$ for some C^∞ section of the vector bundle $T^*M \otimes \mathrm{Hom}(\eta,\eta) \;=\; \mathrm{Hom}(TM,\,\mathrm{Hom}(\eta,\eta))$. (Problem 1 in §20.) Then, by (60.3), $\mathrm{c}^{\tilde\nabla} - \mathrm{c}^\nabla \;=\; d\theta/\pi$, with the *global* 1-form $\theta = \mathrm{Im}\,F_{ja}^a\,dx^j$. (In component-free language this says $\theta_x(v) = \mathrm{Im}\,[\mathrm{Trace}\,(F_x(v))]$ with $F_x(v) \in \mathrm{Hom}(\eta_x,\eta_x)$ for $x \in M$ and $v \in T_xM$.) The cohomology class

$$c_1(\eta) \;=\; [\mathrm{c}^\nabla] \in H^2M \tag{60.4}$$

thus depends on η alone, and not on the choice of the connection ∇. We call $c_1(\eta)$ the (real) *first Chern class* of the complex vector bundle η.

The first Chern class $c_1(\eta)$ is well-defined for any complex vector bundle η over a *compact* manifold M since, according to Problem 7(c) in §28, η then admits a connection. However, a connection in η must exist even without compactness of M. We will not prove that general fact, which is a consequence of the countability axiom (cf. §14). Since our applications deal exclusively with compact manifolds, we will simply treat the existence of a connection/metric as an extra assumption we choose to make in the noncompact case.

EXAMPLE 60.1. Combining (60.1), (60.2) and (60.4) with curvature-components formulae for connections obtained by applying natural operations to other connections (§27), we obtain the following results for complex vector bundles η, η' over a manifold M.

a. $c_1(\eta) = c_1(\eta')$ if η and η' are isomorphic (Problem 2).
b. $c_1(\eta) = 0$ if η is trivial (use (a) and a standard flat connection).
c. $c_1(\eta \oplus \eta') = c_1(\eta) + c_1(\eta')$.
d. $c_1(\eta^*) = -\,c_1(\eta)$.
e. $c_1(\eta \otimes \eta') = q' c_1(\eta) + q\, c_1(\eta)$, where q and q' are the fibre dimensions of η and η'.
f. $c_1(\eta^{\wedge q}) = c_1(\eta)$, if q is the fibre dimension of η.
g. $c_1(F^*\eta) = F^*(c_1(\eta))$, whenever $F : N \to M$ is a C^∞ mapping and $F^*\eta$ is the bundle over N obtained as the F-pullback of a bundle η over M.

Problems

1. Verify that, in a real vector bundle over a compact manifold, the 2-form $R_{jka}{}^a\, dx^j \wedge dx^k$ is always exact. Give a description of this form that does not involve local coordinates or a local trivialization. (Hint below.)
2. The same for the 2-form $\operatorname{Re} R_{jka}{}^a\, dx^j \wedge dx^k$ in a complex vector bundle.
3. Prove the claim made in Example 60.1(a). (Hint below.)
4. Show that a complex line bundle η over a compact manifold has $c_1(\eta) = 0$ if and only if η admits a flat connection. (Hint below.)

Hint. In Problem 1, first use a connection compatible with a Riemannian fibre metric, and then note that the cohomology class of the form in question does not depend on the connection.
Hint. In Problem 3, use a connection in η and its push-forward under an isomorphism $\eta \to \eta'$.
Hint. In Problem 4, assume exactness of c^∇ and then conclude that another connection in η, written as $\tilde{\nabla} = \nabla + F$. Let both connections be compatible with a fized Hermitian fibre metric, so that Im in relevant expressions can be omitted.

61. Poincaré's index formula for surfaces

Topics: Poincaré's index theorem.

PROPOSITION 61.1. *Let η be a real vector bundle of fibre dimension n over a compact n-dimensional manifold M. Then there exists a global C^∞ section ψ of η having only finitely many zeros.*

PROOF. Using Problem 7 in §28, we may choose a vector bundle ζ of some fibre dimension k over M, an $(n+k)$-dimensional real vector space V, and a C^∞ vector-bundle isomorphism $h : \eta \oplus \zeta \to M \times V$. Let us fix a positive-definite inner product $\langle\,,\rangle$ in V, and denote

$$\Sigma \,=\, \{u \in V : \langle u, u\rangle = 1\}$$

the corresponding unit sphere. Also, $\langle\,,\rangle$ gives rise to a Riemannian fibre metric g in ζ characterized by $g_x(\xi,\xi') = \langle h_x(\xi), h_x(\xi')\rangle$, and the set

$$N \,=\, \{(x,\xi) : x \in M,\ \xi \in \zeta_x,\ g_x(\xi,\xi) = 1\}$$

(called the *total space of the unit sphere bundle of* ζ) carries a natural structure of a compact $(n+k-1)$-dimensional manifold, such that the mapping

$$F : N \to \Sigma$$

given by

$$F(x,\xi) \,=\, h_x(\xi)$$

is of class C^∞. Note that every $u \in V$ gives rise to a C^∞ section $\psi^{[u]}$ of η, given by

$$\psi^{[u]}(x) \,=\, \pi_x(h_x^{-1}(u))\,,$$

where $\pi : \eta \oplus \zeta \to \eta$ is the obvious projection morphism. As $\dim N = \dim \Sigma$, we can use Sard's theorem, to select a regular value u of F. By Lemma 59.1, $F^{-1}(u)$ is finite, so that the section $\psi = \psi^{[u]}$ has finitely many zeros. This completes the proof. □

We can now prove *Poincaré's index theorem*:

THEOREM 61.2. *Let η be an oriented real vector bundle of fibre dimension n over a compact oriented n-dimensional manifold M. For any C^∞ section ψ of η which is defined and nonzero outside a finite set* $\mathrm{Sing}\,(\psi) \subset M$, *we then have*

$$\chi(\eta) \,=\, \sum_{\{x \in\, \mathrm{Sing}\,(\psi)\}} \mathrm{ind}_x \psi\,.$$

in preparation

in preparation

COROLLARY 61.3. *Given a C^∞ vector field w defined and nonzero outside a finite set* $\mathrm{Sing}\,(w) \subset M$ *in a compact orientable manifold M, we have*

$$\chi(M) \,=\, \sum_{\{x \in\, \mathrm{Sing}\,(w)\}} \mathrm{ind}_x w\,.$$

PROOF. This is Theorem 61.2 for $\eta = TM$.

□

Problems

in preparation

62. The Gauss-Bonnet theorem

Topics: The Gaussian curvature function of a Riemannian surface; the Gauss-Bonnet formula.

The Euler class becomes particularly useful in cases where the base manifold M is compact and oriented, and the fibre dimension q of the oriented real vector bundle η over M iparticular interest arises when coincides with $n = \dim M$. One then defines the *Euler number* of η to be $\chi(\eta) = \int e(\eta)$, i.e.,

$$\chi(\eta) = \int e(\eta) = \int \mathrm{E}^\nabla \in \mathbf{R}.$$

If, in addition, $\eta = TM$, this number is called the *Euler characteristic* of M and denoted $\chi(M)$, so that

$$\chi(M) = \chi(TM) = \int e(TM)$$

and

$$\chi(M) = \int \mathrm{E}^\nabla \tag{62.1}$$

for any connection ∇ in TM, compatible with a Riemannian metric g on M. Note that $\chi(\eta)$ changes sign when one of the orientations (of η or TM) is reversed, and so it remains the same if both orientations are changed. Therefore, the Euler characteristic $\chi(M)$ is well-defined for any compact manifold M which is just *orientable*, not necessarily oriented. Obviously, by (62.1) and (63.4),

$$\chi(M) = 0 \qquad \text{if} \quad \dim M \text{ is odd.}$$

By a *zero* of a local section ψ of a vector bundle η over a manifold M we mean any point $x \in M$ lying in the domain of ψ and such that $\psi(x) = 0$. For a compact orientable manifold M, we have

(62.2) $\chi(M) = 0$ if M admits a global C^∞ tangent vector field without zeros.

In fact, such a vector field w leads to the decomposition $TM = \eta \oplus \eta^\perp$, where η is the line subbundle spanned by w, and we can use Example 3(c), with $e(\eta) = 0$ in view of (63.4). In dimension 2, we have the following classical result, known as the *Gauss-Bonnet theorem.*

THEOREM 62.1. *Let (M, g) be a compact orientable Riemannian surface, that is, a Riemannian manifold of dimension 2. Denoting K the Gaussian curvature function of (M, g) (see (30.13)), we then have*

$$\frac{1}{2\pi} \int K\nu = \chi(M), \tag{62.3}$$

where the integration opeator $\int$ and the volume form ν refer to any fixed orientation of M.

PROOF. Using the Levi-Civita connection in TM, we obtain

$$R_x(v, w) = 2Kv \wedge w \tag{62.4}$$

in (63.2), in view of (30.6) (i.e., (30.9)) and (51.7), where the tangent vectors v, w are treated as 1-forms via lowering of indices with the aid of g (so that $v_j = g_{jk}v^k$). If v, w form an orthonormal basis of T_xM, we thus have $\nu = \pm 2v \wedge w$ by (**??**), so that, from (63.2), $B(v, w) = \pm K$, and hence $2\pi\, \mathrm{E}^\nabla = K\nu$. This completes the proof. □

Thus, for instance, the Euler characteristic of the 2-sphere is

$$\chi(S^2) = 2.$$

(To see this, we can use (62.4) with $K = 1/r^2$ for a sphere of radius r (see ...), while $\mathrm{Area}(S_r^2) = 4\pi r^2$ by (37.3) for $n = 2$.) Consequently, the tangent bundle TS^2 of the 2-sphere is not C^∞ trivial. In fact, by (62.2), S^2 does not admit a global C^∞ tangent vector field without zeros. By a *local tangent frame* in a surface M we mean a pair e_1, e_2 of C^∞ tangent vector fields on an open submanifold V of M that form, at each point of V, an orthonormal basis of the tangent plane. The corresponding *connection form* then is the 1-form ω on V with $\omega(v) = g(\nabla_v e_1, e_2)$ for C^∞ tangent vector fields v on V. Thus,

$$\nabla_v e_1 = \omega(v) e_2, \quad \nabla_v e_2 = -\omega(v) e_1,$$

in view of orthonormality. We then also have

$$K\sqrt{\det[g_{jl}]} = \partial_2\omega_1 - \partial_1\omega_2,$$

K being the Gaussian curvature of the surface M.

63. The Euler class

Topics: Volumes of spheres; the Euler form of a connection, compatible with a Riemannian fibre metric, in an oriented real vector bundle; the Euler class of an oriented real vector bundle; the Euler class and operations on bundles; equality between Chern and Euler classes for complex line bundles; the Euler number; the Euler characteristic of a compact orientable manifold; the Gauss-Bonnet theorem for compact Riemannian surfaces; the Euler characterisic for spheres; nonexistence of C^∞ vector fields without zeros, and on even-dimensional spheres; nonexistence of a nonzero proper C^∞ vector subbundle in the tangent bundle of an even-dimensional sphere; the Gauss-Bonnet-Chern formula.

The construction described below, require the existence in the given vector bundle of a connection along with a compatible Riemannian fibre metric. We know (Problem 7(c) in §28) that such objects exist whenever the base manifold M is compact. Actually, this is true for every base manifold M, even without compactness. We will not prove here that general existence result (which is a consequence of the countability axiom, cf. §14). Since our applications deal exclusively with compact manifolds, we will simply treat the existence of a connection/metric as an extra assumption we choose to make in the noncompact case.

Example 63.1. For an n-dimensional sphere

$$S_r^n = \{v \in V : |v| = r\}$$

of radius $r > 0$ in an $(n+1)$-dimensional Euclidean vector space V, with the induced Riemannian metric g, we have

$$\mathrm{Vol}\, S_r^n = \begin{cases} \dfrac{2^{n+1}\pi^{n/2}(n/2)!}{n!}\, r^n\,, & \text{if } n \text{ is even} \\[2ex] \dfrac{2\pi^{(n+1)/2}}{[(n-1)/2]!}\, r^n\,, & \text{if } n \text{ is odd.} \end{cases}$$

See Problems ......

Let us now consider an oriented real vector bundle η of some fibre dimension $q \geq 1$ over a manifold M (see §48). and let g be a Riemannian (that is, positive-definite) fibre metric in η. The volume form ν of g in η allows us to express any $\omega \in (\eta_x^*)^{\wedge q}$, or any section ω of $(\eta^*)^{\wedge q}$, as a scalar (functional) multiple of ν_x or

ν, with the coefficient that will be denoted $\langle\nu,\omega\rangle$; thus, suppressing x we have, in either case

$$\omega = \langle\nu,\omega\rangle\nu$$

and

$$\langle\nu,\omega\rangle = \omega(e_1,\ldots,e_k) \tag{63.1}$$

with e_a as in (...). Furthermore, suppose that we are given a connection ∇ in η which is compatible with g (see §28). Recall that the g-modified curvature of ∇ at x (§28) assigns, to each $x \in M$ and all $v, w \in T_xM$, the skew-symmetric bilinear form

$$R_x(v,w) \in \eta_x^{\wedge 2},$$

characterized by (28.10), i.e., $\eta_x \times \eta_x \ni (\psi,\phi) \mapsto \langle R^\nabla(v,w)\psi,\phi\rangle \in \mathbf{R}$. This allows us to introduce the tensor $B_x \in L(T_xM,\ldots,T_xM;\mathbf{R}) = [T_x^*M]^{\otimes q}$ defined by

$$B_x(v_1,\ldots,v_q) = \langle\nu_x, R_x(v_1,v_2)\wedge\ldots\wedge R_x(v_{q-1},v_q)\rangle \tag{63.2}$$

for $v_1,\ldots,v_q \in T_xM$ (if q is even), or $B_x = 0$ (if q is odd). Finally, one defines the *Euler form* E^∇ of ∇ (and g) to be the differential form $\mathrm{E}^\nabla \in \Omega^q M$ of degree q on M, with

$$\mathrm{E}_x^\nabla = \frac{(q!)^2}{(8\pi)^{q/2}(q/2)!}\,\mathfrak{S}B_x \tag{63.3}$$

for all $x \in M$, where $\mathfrak{S}$ is the skew-symmetrization projection (see .....), applied here to B_x. Note that, by definition,

$$\mathrm{E}^\nabla = 0 \qquad \text{identically if the fibre dimension } q \text{ is odd.} \tag{63.4}$$

In terms of local coordinates x^j for M and a local positive-oriented, orthonormal trivialization e_a for η, we thus have, by (63.1)

(63.5)
$$\mathrm{E}_x^\nabla = \frac{q!}{(8\pi)^{q/2}(q/2)!}\,\varepsilon^{a_1\ldots a_q}R_{a_1a_2}\wedge\ldots\wedge R_{a_{q-1}a_q} \quad \text{if the fibre dimension } q \text{ is even}$$

with the local 2-forms

$$R_{ab} = R_{jkab}\,dx^j\wedge dx^k \tag{63.6}$$

(i.e., $R_{ab}(v,w) = \langle R^\nabla(v,w)e_a,e_b\rangle$) and the Ricci symbol $\varepsilon^{a_1\cdots a_q}$ analogous to that in Problem 1 of §57.

For a fixed $z \in M$, choose the e_a at z with $\Gamma_{ja}^b(z) = 0$ (§26). By (20.10) and (28.11) with $g_{ab} = \delta_{ab}$, we then have $\partial_l R_{jkab} + \partial_j R_{klab} + \partial_k R_{ljab} = 0$ at z, so that, by (51.10), $(d\,R_{ab})(z) = 0$. Hence, by (63.4) and (63.5), $(d\,\mathrm{E}^\nabla)(z) = 0$. Since $z \in M$ was arbitrary, we obtain

$$d\,\mathrm{E}^\nabla = 0,$$

i.e., $\mathrm{E}^\nabla \in Z^qM$. Furthermore, the cohomology class of E^∇ does not depend on the choice of the connection ∇ or the compatible Riemannian fibre metric g, and so it is an invariant associated with the oriented real vector bundle η (of fibre dimension q) alone. It is denoted

$$e(\eta) = [\mathrm{E}^\nabla] \in H^qM$$

and called the *Euler class* of the oriented bundle η. Thus, by definition (see (63.4)),

$$e(\eta) = 0 \qquad \text{if the fibre dimension of } \eta \text{ is odd.}$$

To see the independence of $[\mathrm{E}^\nabla]$ of ∇ or g, we may assume that the fibre dimension q of η is even. First, let us fix g and consider two connections ∇ and $\tilde\nabla$ in η both compatible with g. We can always find a curve $[0,1]\ni t\mapsto \nabla^{(t)}$ of connections in η which are all compatible with g, and such that for any coordinate-and-trivialization domain U, the components $\varGamma_{ja}^b(t,x)$ of $\nabla^{(t)}$ at x are C^∞ functions of $(t,x)\ni(-\varepsilon,1+\varepsilon)\times U$ for some $\varepsilon>0$ and, finally, $\nabla^{(0)}=\nabla$, $\nabla^{(1)}=\tilde\nabla$. (For instance, set $\nabla^{(t)}=\nabla+tF$, where F is the C^∞ section of $T^*M\otimes\mathrm{Hom}(\eta,\eta)=\mathrm{Hom}(TM,\mathrm{Hom}(\eta,\eta))$ with $\tilde\nabla=\nabla+F$, cf. Problem 1 in §cc.) The Euler forms $\mathrm{E}^{(t)}$ of $\nabla^{(t)}$ then satisfy

$$\frac{d}{dt}\,\mathrm{E}^{(t)}\,=\,d\theta^{(t)}$$

for a suitable curve $[0,1]\ni t\mapsto\theta^{(t)}$ of $(q-1)$-forms on M whose components in any coordinate domain U are C^∞ functions of $(t,x)\ni(-\varepsilon,1+\varepsilon)\times U$ with some $\varepsilon>0$. (See Problem ..). Then $\mathrm{E}^{\tilde\nabla}-\mathrm{E}^{\nabla}=d\omega$, with 1-form $\omega=\int_0^1\theta^{(t)}\,dt$. Finally, for two Riemannian metrics g and $\tilde g$ in η and a connection ∇ in η, compatible with g, we can find a bundle automorphism (gauge transformation) $F:\eta\to\eta$ sending g onto g' (see Problem ..), and the push-forward $\tilde\nabla=F\nabla$ of ∇ under F then is compatible with $\tilde g$ (Problem 2 in §25), so that $\mathrm{E}^{\tilde\nabla}=\mathrm{E}^{\nabla}$ by Problem ... This proves the independence property stated above.

Example 63.2. Combining (63.3), (63.4) and (63.6) with curvature-components formulae for operations on connections (§17) (**Homework #17, #18, #24**), we obtain the following results for oriented real vector bundles η, η' over a manifold M.

a. $e(\eta)=e(\eta')$ if η, η' are isomorphic (Problem 2).
b. $e(\eta)=0$ if η is trivial (use a flat connection).
c. $e(\eta\oplus\eta')=e(\eta)\cup e(\eta')$. (See Problem ...)
d. $e(F^*\eta)=F^*(e(\eta))$ for pullbacks under C^∞ mappings $F:N\to M$.
e. $e(\tilde\eta)=-\,e(\eta)$ if $\tilde\eta$ is obtained from η by reversing the orientation.
f. $e(\eta)=c_1(\eta)$ if the fibre dimension of η equals 2 and so a Riemannian fibre metric along with the orientation make η into a complex line bundle. See Problems .....

A generalization of these properties of S^2 to spheres of higher (even) dimensions is immediate. Specifically, we have

$$\chi(S^n)\,=\,2\qquad\text{whenever}\quad n\geq 2\text{ is even.}$$

To see this, note that we have (62.4) for the sphere S^n_r of radius r with the submanifold metric (again, from (..) with $K=1/r$).

in preparation

Therefore, if n is even, TS^n does not admit any nonzero proper C^∞ vector subbundle η. (If it did, we could apply Example 3(c) to $TS^n=\eta\,oplus\eta^\perp$ and then use (**??**).) In particular, every global C^∞ tangent vector field on S^n must have at least one zero. For odd-dimensional spheres, see Problem ...

Another interesting relation is the *product formula*

$$\chi(M \times N) \,=\, \chi(M)\chi(N)$$

(Problem ...) for compact orientable manifolds M, N of arbitrary dimensions.

A generalization of the Gauss-Bonnet theorem to compact orientable Riemannian manifolds of higher (even) dimensions is provided by the *Gauss-Bonnet-Chern formula*

$$\chi(M) \,=\, \frac{1}{(8\pi)^{n/2}(n/2)!} \int \left[\nu^{j_1\ldots j_n}\nu^{k_1\ldots k_n} R_{j_1j_2k_1k_2}\ldots R_{j_{n-1}j_nk_{n-1}k_n}\right] \nu\,.$$

See Problem ...

Problems

1. Verify that, in a real vector bundle, the 2-form $R_{jka}{}^{a}\, dx^j \wedge dx^k$ is always exact.

2. Isomorphic.

3.

$$e(\eta) \,=\, 0 \quad \text{if } \eta \text{ admits a global } C^\infty \text{ section without zeros.}$$

(Hint below.)

3. (Hint below.)

3. (Problem ...Use a connection in η and its push-forward under an isomorphism $\eta \to \eta'$.) (Hint below.)

3. $\chi(M) = \#M$ if M is finite. Verify product. odd-dimensional spheres (Hint below.)

3.

$$\chi(\eta) \,=\, \int \frac{1}{(n!)^2} \ldots \varepsilon^{j_1\ldots j_n}\varepsilon^{a_1\ldots a_n} R_{j_1j_2a_1a_2} \wedge \ldots \wedge R_{j_{n-1}j_na_{n-1}a_n}$$

(Hint below.)

3. By the *cross product* of a differential r-form ω on a manifold M and a differential s-form θ on a manifold N we mean the differential $(r+s)$-form $\omega \times \theta$ on the product manifold $M \times N$ given by

$$\omega \times \theta \,=\, \pi_M^*\omega \,\cup\, \pi_N^*\theta\,,$$

where π_M and π_N are the projection mappings of $M \times N$ onto M and N, respectively. Verify that then $d(\omega \times \theta) = (d\omega) \times \theta + (-1)^r\, \omega \times d\theta$. Suppose now that M and N are both compact and oriented and $r = \dim M$, $s = \dim N$. Prove that, if ω and θ are both continuous and compactly supported, then so is $\omega \times \theta$, and

$$\int (\omega \times \theta) \,=\, \left(\int \omega\right) \cdot \left(\int \theta\right)\,.$$

(Hint below.)

3. Given manifolds M, N and cohomology classes $\alpha \in H^rM$, $\beta \in H^sN$, one defines their *cross product* $\alpha \times \beta \in H^{r+s}(M \times N)$ by

$$\alpha \times \beta \,=\, \pi_M^*\alpha \,\cup\, \pi_N^*\beta$$

with π_M and π_N as in Problem ... Verify that $[\omega] \times [\theta] = [\omega \times \theta]$ whenever $\omega \in Z^rM$ and $\theta \in Z^rM$.

$$e(\eta \times \zeta) \,=\, e(\eta) \times e(\zeta)\,.$$

$$\chi(\eta \times \zeta) \,=\, \chi(\eta)\chi(\zeta)\,.$$

$$T(M \times N) = TM \times TN\,.$$

(Hint below.)

3. Given a finite-dimensional real or complex vector space V and a nondegenerate symmetric bilinear form $\langle\,,\rangle$ on V, let us call a basis e_α of V *orthonormal* if $\langle e_\alpha, e_\beta\rangle = 0$ for $\alpha \neq \beta$ and $\langle e_\alpha, e_\alpha\rangle = \varepsilon_\alpha = \pm 1$ for each α. Show that, for any two orthonormal bases e_α and $e_{\alpha'}$ of V, the transition matrix $[A^\alpha_{\alpha'}]$, defined by $e_{\alpha'} = A^\alpha_{\alpha'} e_\alpha$, satisfies

$$\det[A^\alpha_{\alpha'}] = \pm 1\,.$$

(Hint below.)

5. Let e_α be an orthonormal basis for a nondegenerate symmetric bilinear form $\langle\,,\rangle$ in a finite-dimensional real or complex vector space V, $\alpha = 1, \dots, n = \dim V$. Prove that, for any system $v_1, \dots, v_n$ of vectors in V, one has

$$\det \mathfrak{G} = \varepsilon_V\, (\det \mathfrak{B})^2\,,$$

where $\mathfrak{G} = [\langle v_\alpha, v_\beta\rangle]$ is the Gram matrix of the v_α, and $\mathfrak{B} = [B^\beta_\alpha]$ denotes the *coefficient matrix* of the v_α characterized by $v_\alpha = B^\beta_\alpha e_\beta$, $\alpha, \beta \in \{1, \dots, n\}$, while $\varepsilon_V = (-1)^r$, r being the negative index of $\langle\,,\rangle$ (the number of minus signs in its signature $- \dots - + \dots +$), i.e., $\varepsilon_V = \varepsilon_1 \dots \varepsilon_n$ with $\varepsilon_\alpha = \langle e_\alpha, e_\alpha\rangle = \pm 1$. (Hint below.)**50.4.**] Verify that, under any change of the coordinates x^j in M and the local trivialization e_a in η,

$$\Gamma^{a'}_{j'a'} = A^j_{j'} \Gamma^a_{ja} + \partial_{j'} \log|\det[A^c_{c'}]|\,,$$

where η is a C^∞ vector bundle over a manifold M and Γ^b_{ja} are the component functions of any connection ∇ in η, with the transition functions $p^{j'}_j = \partial_j x^{j'}$ (p. 8) and $e^a_{a'} = e^a(e_{a'})$ (p. 23). (Hint below.)

Hint. In Problem 2, use a partition of unity to assume that θ is supported in a coordinate domain, and note that, by Problem ... in §57, each term of $(d\theta)_{1\dots n} = \partial_1\theta_{2\dots n} - \partial_2\theta_{13\dots n} + \dots \pm \partial_n\theta_{1\dots n-1}$ then contributes zero to the integral $\int d\theta$.

Hint. In Problem 2, note that

$$\omega(B^{\alpha_1}_1 v_{\alpha_1}, \dots, B^{\alpha_k}_k v_{\alpha_k}) = B^{\alpha_1}_1 \dots B^{\alpha_k}_k\, \omega(v_{\alpha_1}, \dots, v_{\alpha_k}) = \varepsilon_{\alpha_1\dots\alpha_k} B^{\alpha_1}_1 \dots B^{\alpha_k}_k\, \omega(v_1, \dots, v_k)\,,$$

$\varepsilon_{\alpha_1\dots\alpha_k}$ being the *Ricci symbol* (see the hint for Problem 11 in §8), so that $\varepsilon_{\alpha_1\dots\alpha_k} B^{\alpha_1}_1 \dots B^{\alpha_k}_k = \det[B^\beta_\alpha]$.

Hint. In Problem 3, note that the matrices $\mathfrak{G} = [\langle e_\alpha, e_\beta\rangle]$, $\mathfrak{G}' = [\langle e_{\alpha'}, e_{\beta'}\rangle]$ and $\mathfrak{A} = [A^\alpha_{\alpha'}]$ satisfy $\mathfrak{G}' = \mathfrak{A}^T \mathfrak{G} \mathfrak{A}$ (i.e., $\langle e_{\alpha'}, e_{\beta'}\rangle = A^\alpha_{\alpha'} A^\beta_{\beta'} \langle e_\alpha, e_\beta\rangle$), while $\det \mathfrak{G}' = \det \mathfrak{G} = \pm 1$, so that $(\det \mathfrak{A})^2 = 1$.

Hint. In Problem 5, $\langle v_\alpha, v_\beta\rangle = B^\rho_\alpha B^\sigma_\beta \langle e_\rho, e_\sigma\rangle = \sum_\rho \varepsilon_\rho B^\rho_\alpha B^\rho_\beta$, i.e., $\mathfrak{G} = \mathfrak{B}^T \mathfrak{D} \mathfrak{B}$ with $\mathfrak{D} = \mathrm{diag}\,(\varepsilon_1, \dots, \varepsilon_n)$.

Hint. In Problem50.4, use the transformation rule $\Gamma^{b'}_{j'a'} = A^j_{j'} A^a_{a'} A^{b'}_b \Gamma^b_{ja} + A^{b'}_c \partial_{j'} A^c_{a'}$ (formula (30.8)) and note that

$$A^{a'}_a \partial_{j'} A^a_{a'} = \partial_{j'} \log|\det[A^c_{c'}]|$$

in view of (8.21) with $t = x^{j'}$ and $F = [A^c_{c'}]$.

CHAPTER 12

Elements of Analysis

64. Sobolev spaces

Topics: Sobolev norms; Sobolev spaces of sections of a vector bundle; the localization principle.

Let there be given a real/complex vector bundle η over a Riemannian manifold (M,g), a Riemannian/Hermitian fibre metric $\langle\,,\rangle$ in η and a connection ∇ in η, compatible with $\langle\,,\rangle$. For any open set $U \subset M$, we denote by $C_0^\infty(U,\eta)$ the real/complex vector space of all C^∞ sections of η which have compact supports contained in U. Whenever $p \in [1,\infty]$ and r is a nonnegative integer, we define norms $\|\;\|_{p,r}$ in $C_0^\infty(U,\eta)$ by declaring the pth power $\|\psi\|_{p,r}^p$ of $\|\psi\|_{p,r}$ to be

$$\|\psi\|_{p,r}^p \,=\, \|\psi\|_p^p + \|\nabla\psi\|_p^p + \ldots + \|\nabla^r\psi\|_p^p \quad \text{if } p < \infty \tag{64.1}$$

where ∇^r and $\|\;\|_p$ denote the rth covariant derivative and the L^p norm, with

$$\|\phi\|_p \,=\, \left[\int_M |\phi|\, dg\right]^{1/p}, \tag{64.2}$$

and setting

$$\|\psi\|_{\infty,r} \,=\, \max\left(\|\psi\|_\infty, \|\nabla\psi\|_\infty, \ldots, \|\nabla^r\psi\|_\infty\right) \tag{64.3}$$

where $\|\;\|_\infty$ is the L^∞ (or supremum) norm, with $\|\phi\|_\infty = \sup_M |\phi|$. When no section ψ is mentioned, we will use the symbol L_r^p for the norm $\|\;\|_{p,r}$ with $p < \infty$, while $\|\;\|_{\infty,r}$ will be denoted by C^r. The terminology traditionally employed is: the *Sobolev norm* L_r^p *(with r derivatives in L^p)* and *the C^r norm.* If $r = 0$, we will write $\|\;\|_p$ and $\|\;\|_\infty$ rather than $\|\;\|_{p,0}$ or $\|\;\|_{\infty,0}$ for the L^p and C^0 norms.

That $\|\;\|_{p,r}$ is actually a norm follows from the Minkowski inequality (Problem 1). In the case where M is compact, we will use the symbols $L_r^p(U,\eta)$ and $C^r(U,\eta)$ for the completions of $C_0^\infty(U,\eta)$ relative to the L_r^p and C^r norms.

Problems

1. Given measurable functions f, h valued in $[-\infty,\infty]$ on a fixed measure space and $p \in [1,\infty]$, prove the *Hölder inequality*

$$\|fh\|_1 \,\le\, \|f\|_p \|h\|_q \tag{64.4}$$

$q \in [1,\infty]$ being uniquely determined by the condition

$$\frac{1}{p} + \frac{1}{q} \,=\, 1 \tag{64.5}$$

where $1/\infty = 0$, and the *Minkowski inequality*

$$\|f+h\|_p \,\le\, \|f\|_p + \|h\|_p\,. \tag{64.6}$$

(Hint below.)

Hint. In Problem 1, for (64.4) we may clearly assume that $1 < p < \infty$ and $\|f\|_p, \|h\|_q$ are both finite and positive, and then, normalizing, also assume that $\|\phi\|_p = \|\psi\|_q = 1$. On the other hand, $\lambda^a\mu^b \le a\lambda + b\mu$ for any $a, b, \lambda, \mu \in [0, \infty)$ with $a + b = 1$, as one verifies by applying $d/d\lambda$ to find the maximum of $\lambda^a\mu^b - a\lambda - b\mu$, for fixed μ, a, b. Now (64.4) follows, via integration, from this last inequality for $\lambda = |\phi|^p$, $\mu = |\psi|^q$, $a = 1/p$, $b = 1/q$. To prove (64.6) we may let $p > 1$, and then use (64.4) with h replaced by $|f + h|^{p-1}$.

Hint. In Problem 1, find the maximum of $\lambda_1^{a_1}\ldots\lambda_k^{a_k}-a_1\lambda_1-\ldots-a_k\lambda_k$ using $d/d\lambda_1$, with $\lambda_2,\ldots,\lambda_k$ and all a_j fixed.
Hint. In Problem

The *localization principle*:

The *Sobolev lemma*:

$$C^r \ll L_s^p \text{ if } M \text{ is compact and } s>\frac{n}{p}+r. \tag{64.7}$$

The *Sobolev inequality*:

$$L_{s-r}^{np/(n-rp)} \ll L_s^p \text{ if } M \text{ is compact, } s\ge r\ge 1 \text{ and } 1<p<\frac{n}{r}. \tag{64.8}$$

Any compactly supported C^1 function $\varphi:\mathbf{R}^n\to\mathbf{R}$, $n\ge 2$, satisfies the following inequality (due to Gagliardo and Nirenberg, 1958):

$$\|\varphi\|_{n/(n-1)} \le \frac{1}{2}\prod_{j=1}^{n}\|\partial_j\varphi\|_1^{1/n}. \tag{64.9}$$

In fact, $2|\int_{-\infty}^{a} f(t)\,dt|\le\|f\|_1$ for any $a\in\mathbf{R}$ and any L^1 function $f:\mathbf{R}\to\mathbf{R}$ with $\int_{-\infty}^{\infty} f(t)\,dt=0$, as one sees writing $f=f_+-f_-$ with $f_+=\max(0,f)$ and $f_-=-\min(0,f)$, so that $f_\pm\ge 0$ and $2\|f_\pm\|_1=\|f\|_1$, while $2|\int_{-\infty}^{a} f\,dt|\le|2\int_{-\infty}^{a} f_+\,dt-2\int_{-\infty}^{a} f_-\,dt|$ which, being the distance between two numbers in the interval $[0,\|f\|_1]$, cannot exceed $\|f\|_1$. Fixing all but the jth component x^j of a point $x\in\mathbf{R}^n$ and applying this to $f=\partial_j\varphi$ with $t=x^j$, we get $2|\varphi(x)|\le\int_{-\infty}^{\infty}|\partial_j\varphi|\,dx^j$ and hence $2^n|\varphi(x)|^n\le\prod_{j=1}^{n}\varphi_j$, where $\varphi_j=\int_{-\infty}^{\infty}|\partial_j\varphi|\,dx^j$. (No summing over j, in either relation!). Problem 4 now yields (64.9).

Problems

1. Verify that $\lambda_1^{a_1}\ldots\lambda_k^{a_k}\le a_1\lambda_1+\ldots+a_k\lambda_k$ whenever $a_j,\lambda_j\in[0,\infty)$, $j=1,\ldots,k$, and $a_1+\ldots+a_k=1$, with the convention that $0^0=0$. (Hint below.)
2.
3. *A generalized Hölder inequality.* $\|h_1\ldots h_k\|_q\le\|h_1\|_{p(1)}\ldots\|h_k\|_{p(k)}$, if $p(j)\in(1,\infty)$ for $j=1,\ldots,k$, and $q\in[1,\infty)$ is given by $q^{-1}=[p(1)]^{-1}+\cdots+[p(k)]^{-1}$.
4. Given measurable functions $\varphi_j:\mathbf{R}^n\to\mathbf{R}$, $j=1,\ldots,n$, with $n\ge 2$, such that each φ_j is independent of the jth coordinate x^j, show that $\|\varphi_1\ldots\varphi_n\|_p\le\prod_{j=1}^{n}\|\varphi_j\|_1$, for $p=1/(n-1)$. (The L^p "norm" is sometimes used, and defined by the usual formula, also when $0<p<1$. Here $\|\;\|_p$ is applied to a function $\mathbf{R}^n\to\mathbf{R}$, while each $\|\varphi_j\|_1$ stands for the L^1 norm of a function $\varphi_j:\mathbf{R}^{n-1}\to\mathbf{R}$, with coordinates x^k in $\mathbf{R}^{n-1}$, such that $k\in\{1,\ldots,n\}\setminus\{j\}$.) (Hint below.)
5.
6.
7.

Hint. In Problem 1, find the maximum of $\lambda_1^{a_1}\ldots\lambda_k^{a_k}-a_1\lambda_1-\ldots-a_k\lambda_k$ using $d/d\lambda_1$, with $\lambda_2,\ldots,\lambda_k$ and all a_j fixed.
Hint. In Problem
Hint. In Problem
Hint. In Problem
Hint. In Problem

65. Compact operators

Topics: Compact operators.

A linear operator $A : V \to W$ between real/complex normed vector spaces V, W is called *compact* if the A-image of every bounded sequence in V contains a Cauchy subsequence. If W is complete, this is equivalent to requiring that the A-image of every bounded subset of V have a compact closure in W.

Every linear operator $A : V \to W$ with $\dim A(V) < \infty$ is clearly compact. This is, for instance, the case if $\dim V < \infty$. Sums of compact operators $V \to W$ are compact, and so are composites in which one operator is compact and the other continuous.

Problems

1. Verify that, for normed vector spaces V, W, compact operators $V \to W$ form a closed vector subspace of the normed space $L(V, W)$ (consisting of all continuous linear operators $V \to W$ with the operator norm), and compact operators $V \to V$ form a two-sided ideal in the associative algebra $L(V, V)$.

2. Verify that the identity operator $V \to V$ in an inner-product space V is compact if and only if $\dim V < \infty$.

5.

6.

7.

Hint. In Problem 1, find

66. The Rellich lemma

Topics: Smoothing operators; the Rellich lemma.

Let η, ζ be real/complex vector bundles over compact manifolds M, N, and let $\mathcal{K}$ be a C^∞ section of the vector bundle $\mathrm{Hom}(\pi_M^*\eta, \pi_N^*\zeta)$ over $M \times N$, where $\pi_M : M \times N \to M$, $\pi_N : M \times N \to N$ are the projections. (Thus, $\mathcal{K}$ assigns to $(x, y) \in M \times N$ a linear operator $\mathcal{K}(x, y) : \eta_x \to \zeta_y$.) Also, let N carry a fixed positive C^∞ density ν. The *smoothing operator* $S_{\mathcal{K}}$ with the *kernel* $\mathcal{K}$ assigns to every L^1 section ψ of ζ the section $S_{\mathcal{K}}\psi$ of η with

$$(S_{\mathcal{K}}\psi)(x) \,=\, \int_N \mathcal{K}(x, \cdot\,)\psi\,\nu$$

for $x \in M$. Note that on the right-hand side we integrate a vector-valued function $N \to \eta_x$, given by $y \mapsto \mathcal{K}(x, y)\psi_y$. Clearly,

$$\|S_{\mathcal{K}}\psi\|_\infty \,\le\, \|\mathcal{K}\|_\infty \|\psi\|_1 \,,$$

and, for every nonnegative integer r,

$$\nabla^r \circ S_{\mathcal{K}} \,=\, S_{\nabla^r \mathcal{K}} \,.$$

Thus, $S_{\mathcal{K}}$ maps $L^1(M, \zeta)$ into $C^\infty(M, \eta)$ and, for every integer $r \ge 0$, the operator $S_{\mathcal{K}} : L^1(M, \zeta) \to C^\infty(M, \eta)$ is compact relative to the L^1 norm in the first space and the C^r norm in the second.

The simplest kind of smoothing operators over a compact manifold M are those sending functions $M \to \mathbf{R}$ to functions $M \to \mathbf{R}$. The kernel of such an operator is just a C^∞ function $\mathcal{K} : M \times M \to \mathbf{R}$. For instance, choosing a Riemannian metric g on M, we obtain both a positive C^∞ density on M (namely, dg), and a special class of smoothing operators of this kind. Specifically, we fix a C^∞ function $\varphi : [0,\infty) \to \mathbf{R}$ such that $\varphi = 0$ everywhere in $[\delta^2,\infty)$ for some real number $\delta > 0$ satisfying the injectivity-radius bound $\mathrm{r}_{\mathrm{inj}} > \delta$ on M, and then we set, for integrable functions $\psi : M \to \mathbf{R}$,

$$(\Phi\psi)(x) \,=\, \int_{T_xM} (\varphi \circ r^2)(\psi \circ \exp_x)\, dg_x\,,$$

$r : T_xM \to \mathbf{R}$ being the norm. Thus, on the right-hand side we integrate the function $T_xM \to \mathbf{R}$ given by $v \mapsto \varphi(|v|^2)\psi(\exp_x v)$ relative to the Lebesgue measure of the Euclidean space T_xM.

The following result is known as *the Rellich lemma*.

Theorem 66.1. *Let η be a vector bundle over a compact manifold M, and let r be a positive integer. Then the inclusion operator $L^p_r(M,\eta) \to L^p_{r-1}(M,\eta)$ is compact for each $p \in [0,\infty)$.*

Proof. We may assume that $s = r+1$, since the composite of a compact operator and a continuous one is compact. Given L^1 functions f, φ on $\mathbf{R}^n$, for $n = \dim M$, of which one is valued in $\mathbf{R}$ and the other in a Euclidean space V, we define their *convolution* $f * \varphi : \mathbf{R}^n \to V$ by

$$(f * \varphi)(x) \,=\, \int_{\mathbf{R}^n} f(x-y)h(y)\, dy\,, \tag{66.1}$$

so that $f*\varphi$ is integrable by Fubini's theorem: $\int\int |f(x-y)h(y)|\,dy\,dx \le \|f\|_1\|\varphi\|_1$. Also, $\|f*\varphi\|_\infty \le \|f\|_\infty\|\varphi\|_1$, and $f*\varphi = \varphi * f$.

If $B \subset \mathbf{R}^n$ is a fixed open ball centered at 0, and $2B$ denotes the concentric ball of twice the radius of B, while $\varphi \in C^\infty_0(B\times\mathbf{R})$ (that is, φ is a C^∞ function $\mathbf{R}^n \to \mathbf{R}$ with a compact support contained in B), then the formula $A_\varphi f = \varphi * f$ defines an operator $A_\varphi : L^1(B\times V) \to C^r_0(2B\times V)$ which is compact (relative to the L^1 and C^r norms).

In fact, by the dominated convergence theorem, $\varphi * f$ is of class C^r, for every $r \ge 0$, and $\|\varphi * f\|_{\infty,r} \le \|\varphi\|_{\infty,r}\|f\|_1$. (Note that $\partial_j(\varphi * f) = (\partial_j\varphi) * f$.)

$\square$

... eigenfunctions of the Laplacian ...

67. The regularity theorem

Topics: The regularity theorem for operators with an injective symbol.

... *the regularity theorem for operators with injective symbol.*

Theorem 67.1. *Let η, ζ be vector bundles over a manifold M, and let a differential operator $P : C^\infty_0(\eta) \to C^\infty_0(\zeta)$ of essential order k have an injective symbol. If $\psi \in \mathcal{D}(\eta)$ and $P\psi \in L^2_{r,\mathrm{loc}}(\zeta)$ for some $r \in \mathbf{Z}$, then $\psi \in L^2_{r+k,\mathrm{loc}}(\eta)$.*

Proof. $\square$

The *Poincaré inequality*

68. Solvability criterion for elliptic equations

Topics: The regularity theorem for operators with an injective symbol.

THEOREM 68.1. *Let η, ζ be vector bundles over a compact manifold M, and let a differential operator $P : C_0^\infty(\eta) \to C_0^\infty(\zeta)$ of essential order k have an injective symbol. The image of $P : L_k^2(\eta) \to L^2(\zeta)$ then coincides with the kernel of the formal adjoint P^*, that is, the space of those L^2 sections ϕ of ζ for which $P^*\phi = 0$ in the sense of distributions.*

PROOF. □

69. The Hodge-de Rham decomposition theorem

Topics: The Hodge lemma; the theorem of Hodge and de Rham.

The following result is known as *the Hodge-de Rham decomposition theorem.*

THEOREM 69.1. *Let (M, g) be an n-dimensional compact Riemannian manifold. For any $r \in \{0, 1, \ldots, n\}$ we then have an L^2-orthogonal decomposition*

$$Z^r M \,=\, B^r M \,\oplus\, \mathcal{H}^r(M, g),$$

which leads to an isomorphic identification $H^r M \,\approx\, \mathcal{H}^r(M, g)$.

PROOF. The Hodge Laplacian $dd * + d * d$ is a self-adjoint elliptic operator sending the space of all differential r-forms on M into itself. Its image is therefore the L^2-orthogonal complement of its kernel $\mathcal{H}^r(M, g)$. (See Theorem 68.1.)

□

As a consequence, we obtain the *Poincaré duality formula*:

COROLLARY 69.2. *The Betti numbers b_r of any compact orientable n-dimensional manifold M satisfy the relations $b_r = b_{n-r}$ for $r = 0, 1, \ldots, n$.*

The next two consequences are due to Bochner:

COROLLARY 69.3. *The first Betti number $b_1(M)$ is zero for any compact manifold M admitting a Riemannian metric with positive Ricci curvature.*

COROLLARY 69.4. *If a compact Riemannian manifold (M, g) has nonnegative Ricci curvature, then every harmonic 1-form on $(M,)$ is parallel.*

A much more sweeping conclusion was obtained by Gallot and Meyer under a similar assumption on the curvature operator rather than Ricci curvature:

COROLLARY 69.5. *The Betti numbers b_r with $0 < r < n$ are all zero for any compact n-dimensional Riemannian manifold with positive curvature operator.*

COROLLARY 69.6. *If a compact Riemannian manifold (M, g) has nonnegative curvature operator, then every harmonic form on $(M,)$ is parallel.*

Appendix A. Some Linear Algebra

69. Affine spaces

An *affine space* is a triple $(M, V, +)$ formed by a nonempty set M, a real vector space V, and a mapping $M \times V \ni (x, v) \mapsto x + v \in M$ such that:

a. $(x + v) + w = x + (v + w)$ whenever $x \in M$ and $v, w \in V$, the latter $+$ being the addition in V;
b. $x + 0 = x$ for all $x \in M$, where 0 stands for the zero vector of V; and
c. for any $x, y \in M$ there is a unique $v \in V$ (denoted by $y - x$) with $x + v = y$.

We then call V the *translation* (*vector*) *space* of the affine space (briefly denoted by M rather than $(M, V, +)$), while $\dim V$ is called the *dimension* of M and denoted by $\dim M$. Elements of the set M are referred to as *points*, as opposed to *vectors*, that is, elements of V.

Given affine spaces $(M, V, +)$, $(M', V', +)$, a mapping $f : M \to M'$ is called *affine* if there is a linear operator $\psi : W \to W'$ (called the *linear part* of f), with $f(x + v) = f(x) + \psi(v)$ for all $x \in M$ and $v \in V$. For instance, all constant mappings f are affine (with $\psi = 0$). An *affine isomorphism* is an affine mapping which is one-to-one and onto. Examples of affine isomorphisms $M \to M$ are the *translations* $x \mapsto x + v$ with $v \in V$.

A nonempty subset M' of M is called an *affine subspace* of the given affine space $(M, V, +)$ if there is a (vector) subspace V' of V with $x + v \in M'$ and $y - z \in V'$ whenever $x, y, z \in M'$, $v \in V'$.

Problems

12. For an affine subspace M' of $(M, V, +)$, with V' as above and $+$ restricted to $M' \times V'$, show that $(M', V', +)$ is an affine space.

13. Prove that an affine mapping f uniquely determines its linear part ψ, and that f is an isomorphism if and only if ψ is. Conversely, if two affine mapings $M \to M'$ have the same linear part, then either of them equals the other followed by a translation in M'.

14. Any vector space V may be thought of as the affine space $(V, V, +)$. Prove that every affine space is affinely isomorphic to its translation vector space.

15. Show that affine mappings between vector spaces are just linear operators followed by translations.

16. Prove that affine subspaces of vector spaces coincide with cosets (i.e., translation images) of vector subspaces.

17. Show that composites of affine mappings are affine, and images, as well as *nonempty* preimages, of affine subspaces under affine mappings are affine subspaces. (Note that, as a special case, the set of all solutions x to an equation $f(x) = y$, where y is fixed and $f : M \to N$ is affine, is either empty, or an affine subspace of M. In fact, one-point subsets of an affine space are 0-dimensional affine subspaces.) Verify that the inverse mapping of an affine isomorphism is an affine isomorphism.

18. Two affine subspaces of a given affine space $(M, V, +)$ are called *parallel* if they have the same translation vector space. Show that, for such a subspace $(M', V', +)$, the quotient set M/M' of all affine subspaces of M which are parallel to M' carries a natural structure of a *vector* space canonically isomorphic to V/V'. On the other hand, for any fixed vector subspace V' of V, one can form the *affine* quotient space M/V' by declaring $x, y \in M$ congruent modulo V' if $x - y \in V'$. Verify that in either case, the projection mapping of M onto the quotient is affine.

19. Prove that any given affine space M can be canonically realized as an affine subspace of a vector space V' with $\dim V' = \dim M + 1$, where $\infty + 1 = \infty$. (Hint below.)

Hint. In Problem 19, let V' be the dual space of the vector space W of all affine functions (mappings) $f : M \to \mathbf{R}$, with the injective affine mapping $M \to V'$ sending any x onto the functional $f \mapsto f(x)$.

70. Orientation in real vector spaces

The set of all bases $\mathcal{B}(V)$ of a given real vector space V with $0 < n = \dim V < \infty$ (which is an open subset of the nth Cartesian power of V, cf. Problem 5) has precisely two connected components, called the *orientations* of V. (Problem 10.) Moreover, two bases of V *represent* or *determine* (i.e., belong to) the same orientation if and only if their transition determinant is positive (Problem 6).

Problems

5. Given a real or complex vector space V with $\dim V = n < \infty$ let $\mathcal{B}(V)$ be the subset of the nth Cartesian power $V^n = V \times \ldots \times V$ consisting of all (ordered) bases of V. Show that, when V^n is treated as a vector space (the direct sum of n copies of V), the set $\mathcal{B}(V)$ is open in V^n and $\mathcal{B}(V)$, as an open submanifold of V^n, is C^ω-diffeomorphic to the underlying manifold of the Lie group $\mathrm{GL}(V)$. (Hint below.)

6. Let V be a real vector space V with $1 \le \dim V < \infty$. Call two (ordered) bases of V *equivalent* if the transition matrix between them has a positive determinant. Verify that this actually is an equivalence relation and it has exactly two equivalence classes. (These equivalence classes are called the *orientations* of V.) Show that each connected component of $\mathcal{B}(V)$ (Problem 5) is contained in a unique orientation of V.

7. Let V be a real or complex vector space with $1 \le \dim V = n < \infty$, carrying a fixed inner product $\langle\,,\rangle$ (that is, a positive-definite form which is bilinear and symmetric or, respectively, sesquilinear and Hermitian). The *orthonormalization* e_α of a basis v_α of V, $\alpha = 1, \ldots, n$, is defined recursively by

$$e_\alpha = w_\alpha / |w_\alpha|\,, \qquad w_\alpha = v_\alpha - \sum_{\beta<\alpha} \langle v_\alpha, e_\beta \rangle e_\beta\,.$$

Show that the e_α is the unique orthonormal basis of V with

$$\mathrm{Span}\,(e_1, \ldots, e_\alpha) \;=\; \mathrm{Span}\,(v_1, \ldots, v_\alpha)\,, \qquad \alpha = 1, \ldots, n\,, \tag{70.1}$$

$$(70.2) \qquad \langle e_\alpha, v_\alpha\rangle \in (0,\infty)\,, \qquad 1\le\alpha\le n\,.$$

8. For V, $\langle\,,\rangle$ as in Problem 7 and any basis v_α of V, prove that the basis v_α and its orthonormalization e_α lie in the same connected component of $\mathcal{B}(V)$ (notation of Problem 5). (Hint below.)

9. Given a finite-dimensional complex inner-product space V, show that any two orthonormal bases of V representing the same orientation can be joined by a continuous curve in $\mathcal{B}(V)$ consisting of orthonormal bases. (Hint below.)

10. For a real vector space V with $1\le\dim V<\infty$, verify that $\mathcal{B}(V)$ has exactly two connected components, which coincide with the orientations of V. (Hint below.)

Hint. In Problem 5, fix a basis $(e_1,\dots,e_n)$ of V and note that the n-fold Cartesian product of the corresponding linear coordinate system in V is a linear coordinate system in V^n associating with each basis a transition matrix, which sends $\mathcal{B}(V)$ onto the set of $n\times n$ matrices with $\det\ne 0$. Finally, using a fixed basis $(e_1,\dots,e_n)$ of V, identify each $A\in\mathrm{GL}(V)$ with $(Ae_1,\dots,Ae_n)\in\mathcal{B}(V)$.
Hint. In Problem 8, use the sequence $\mathfrak{e}_k=(e_1,\dots,e_k,v_{k+1},\dots,v_n)$, $k=0,\dots,n$, of $n+1$ bases of V, Note that $\mathfrak{e}_0=(v_1,\dots,v_n)$, $\mathfrak{e}_n=(e_1,\dots,e_n)$. Now, for any $k=0,\dots,n-1$, formula $[0,1]\ni t\mapsto\mathfrak{e}_k(t)=(e_1,\dots,e_k,(1-t)v_{k+1}+te_{k+1},v_{k+2},\dots,v_n)$ defines a continuous curve in $\mathcal{B}(V)$ connecting $\mathfrak{e}_k$ with $\mathfrak{e}_{k+1}$. The fact that each $\mathfrak{e}_k(t)$ (and $\mathfrak{e}_k$) is a basis follows since, from (70.1), the first $k+1$ vectors of $\mathfrak{e}_k(t)$ lie in $\mathrm{Span}\,(v_1,\dots,v_{k+1})$ and the $(k+1)$st vector is orthogonal to $e_1,\dots,e_k$ and nonzero (as its inner product with e_{k+1} is positive by (70.2)). A continuous curve in $\mathcal{B}(V)$ connecting $\mathfrak{e}_0$ with $\mathfrak{e}_n$ can be written in the form $[0,n]\ni s\mapsto\mathfrak{e}_s$ with $\mathfrak{e}_s=\mathfrak{e}_k(t)$, where $k=[s]$ is the integer part of s (the largest integer not exceeding s), $t=s-[s]$, and we set $\mathfrak{e}_n(0)=\mathfrak{e}_n$.
Hint. In Problem 9, denote by $\mathfrak{e}_0$ and $\mathfrak{e}_n=(e_1,\dots,e_n)$ two given orthonormal bases of V, and make them a part of a sequence $\mathfrak{e}_k$, $k=0,\dots,n$ of $n+1$ orthonormal bases, such that each $\mathfrak{e}_k$ shares the first k vectors $e_1,\dots,e_k$ with $\mathfrak{e}_n$, and each $\mathfrak{e}_{k-1}$, $1\le k\le n$, can be connected with $\mathfrak{e}_k$ by a continuous curve of orthonormal bases. To achieve this, use induction on k, assuming that $1\le k<n$ and $\mathfrak{e}_1,\dots,\mathfrak{e}_k$ with the stated properties have already been constructed. Thus, $\mathfrak{e}_k$ has the form $\mathfrak{e}_k=(e_1,\dots,e_k,v_{k+1},\dots,v_n)$.

First, suppose that $k=n-1$, so $v_n=\varepsilon e_n$ with $\varepsilon=\pm1$ (by orthonormality). If both original bases determine the same orientation, then so do the intermediate stages including $\mathfrak{e}_{n-1}$ (Problem 6); thus, $\varepsilon=1$ and $\mathfrak{e}_n=\mathfrak{e}_{n-1}$ can be connected with $\mathfrak{e}_0$.

Now let $k+1<n$. Thus, we can choose a 2-dimensional subspace W of V containing the vectors $u=e_{k+1}$ and $v=v_{k+1}$, and orthogonal to $e_1,\dots,e_k$. Let us now complete u to an orthonormal basis u,w of the plane W. Thus, $v=pu+qw$ with scalars p,q such that $p^2+q^2=1$, and so $p=\cos\theta$, $q=\sin\theta$ for some $\theta>0$. We can now define a continuous curve $[0,\theta]\ni t\mapsto A_t$ of inner-product preserving linear operators in V by $A_tu=(\cos t)u-(\sin t)w$, $A_tw=(\sin t)u+(\cos t)w$ (so that $A_t(W)\subset W$), and $A_t=\mathrm{Id}$ on the orthogonal complement of W. Consequently, $A_te_1=e_1\,,\dots,A_te_k=e_k$. A continuous curve of orthonormal bases connecting $\mathfrak{e}_k$ to a basis of the form $\mathfrak{e}_{k+1}=(e_1,\dots,e_k,e_{k+1},*,\dots,*)$ now can be defined by $[0,\theta]\ni t\mapsto(A_te_1,\dots,A_te_k,A_tv_{k+1},\dots,A_tv_n)$.
Hint. In Problem 10, use Problems 5, 6, 8 and 9.

71. Complex lines versus real planes

Topics:

72. Indefinite inner products

Topics: Orthogonal complements relative to symmetric bilinear forms; degenerate and nondegenerate subspaces; the inequality for the dimension of the complement of a subspace; equality when the form or the subspace is nondegenerate; timelike, spacelike, null vectors and subspaces; orthogonal and orthonormal bases; Sylvester's law of inertia; rank, nullity, positive and negative indices, signature for symmetric (0,2) tensors; Euclidean, pseudo-Euclidean and Lorentz inner products.

27.3.] Given a finite-dimensional real or complex vector space V and a nondegenerate symmetric bilinear form $\langle\,,\rangle$ on V, let us call a basis e_α of V *orthonormal* if $\langle e_\alpha, e_\beta\rangle = 0$ for $\alpha \neq \beta$ and $\langle e_\alpha, e_\alpha\rangle = \varepsilon_\alpha = \pm 1$ for each α. Show that, for any two orthonormal bases e_α and $e_{\alpha'}$ of V, the transition matrix $[A^\alpha_{\alpha'}]$, defined by $e_{\alpha'} = A^\alpha_{\alpha'} e_\alpha$, satisfies

$$\det[A^\alpha_{\alpha'}] = \pm 1\,.$$

(Hint below.)**27.4.**] Given a symmetric bilinear form $\langle\,,\rangle$ in a finite-dimensional real or complex vector space V, verify that the following three conditions are equivalent:

a. $\langle\,,\rangle$ is nondegenerate (i.e., $V^\perp = \{0\}$),
b. $\det\mathfrak{G} \neq 0$ for some (or any) basis v_α of V, where $\mathfrak{G} = [\langle v_\alpha, v_\beta\rangle]$ is called the *Gram matrix* of the v_α,
c. V admits a $\langle\,,\rangle$-orthonormal basis (as defined in Problem.3). (Hint below.)

Hint. In Problem.4, (a) is equivalent to (b) since $\langle v_\alpha, v_\beta\rangle w^\beta = \langle v_\alpha, w\rangle$ for any scalars w^β, where $w = w^\beta v_\beta$.
Hint. In Problem27.3, note that the matrices $\mathfrak{G} = [\langle e_\alpha, e_\beta\rangle]$, $\mathfrak{G}' = [\langle e_{\alpha'}, e_{\beta'}\rangle]$ and $\mathfrak{A} = [A^\alpha_{\alpha'}]$ satisfy $\mathfrak{G}' = \mathfrak{A}^T\mathfrak{G}\mathfrak{A}$ (i.e., $\langle e_{\alpha'}, e_{\beta'}\rangle = A^\alpha_{\alpha'} A^\beta_{\beta'}\langle e_\alpha, e_\beta\rangle$), while $\det\mathfrak{G}' = \det\mathfrak{G} = \pm 1$, so that $(\det\mathfrak{A})^2 = 1$.
Hint. In Problem27.4, (a) is equivalent to (b) since $\langle v_\alpha, v_\beta\rangle w^\beta = \langle v_\alpha, w\rangle$ for any scalars w^β, where $w = w^\beta v_\beta$.

Appendix B. Facts from Topology and Analysis

73. Banach's fixed-point theorem

Topics: Metric spaces; convergence; Cauchy sequences; completeness; normed spaces; Banach spaces; Banach's contraction theorem; fixed point in a subset.

A *metric space* is a pair (X, d) consisting of a set X and a *distance function* $\mathrm{d} : X \times X \to [0, \infty)$ such that $\mathrm{d}(x, x') = \mathrm{d}(x', x)$, $\mathrm{d}(x, x'') \leq \mathrm{d}(x, x') + \mathrm{d}(x', x'')$ for any $x, x', x'' \in X$, and $\mathrm{d}(x, x') > 0$ unless $x = x'$. A sequence $x_k \in X$, $k = 1, 2, \ldots$ of points in X then is said to *converge to a limit* $x \in X$ if $\mathrm{d}(x_k, x) \to 0$ as $j \to \infty$, and it is called a *Cauchy sequence*) if $\mathrm{d}(x_k, x_l) \to 0$ as j, k simultaneously tend to ∞. The metric space (X, d) is called *complete* if every Cauchy sequence in (X, d) converges.

Any subset $K \subset X$ of a metric space (X, d) forms a metric space (K, d) with d restricted to $K \times K$.

The *open ball* $B_z(r) \subset X$ (with the center $z \in X$ and the radius $r > 0$) in the metric space (X, d) is defined by $B_z(r) = \{x \in X : \mathrm{d}(x, z) < r\}$. Similarly, the *closed ball* $\overline{B}_z(r) \subset X$ is $\overline{B}_z(r) = \{x \in X : \mathrm{d}(x, z) < r\}$. A set $U \subset X$ is called *open* if it is the union of some (possibly empty, or infinite) collection of open balls. A *neighborhood* of a point $x \in X$ is any open set containing x; as for manifolds, a sequence x_k, $k = 1, 2, \ldots$ of points in X converges to a limit $x \in X$ if and only if each neighborhood of x contains the x_k for all but finitely many k.

A *norm* in a real or complex vector space V is a function $V \to [0, \infty]$, usually written as $v \mapsto \|v\|$ (or $v \mapsto |v|$, when $\dim V < \infty$), such that $\|v\| > 0$ if $v \neq 0$, and $\|v + w\| \leq \|v\| + \|w\|$, $\|\lambda v\| = |\lambda| \cdot \|v\|$ for $v, w \in V$ and all scalars λ. With a fixed norm, V is called a *normed vector space*, and it naturally becomes a metric space (V, d) with $\mathrm{d}(v, w) = \|v - w\|$. A normed vector space is called a *Banach space* if it is complete as a metric space.

The following result is known as *Banach's fixed-point theorem.*

Theorem 73.1. *Let $Y \subset X$ be a subset of a metric space (X, d) such that (Y, d) is complete and let $h : Y \to X$ be a mapping with $\mathrm{d}(h(x), h(x')) \leq C\,\mathrm{d}(x, x')$ for all $x, x' \in Y$ and some C with $0 \leq C < 1$. If, moreover,*

(a) *there is $z \in Y$ with $h^k(z) \in Y$ for all integers $k \geq 0$,*

or

(b) *$B_z(r) \subset Y$ for some $z \in Y$ and $r = (1 - C)^{-1}\,\mathrm{d}(z, h(z))$,*

with $B_z(0) = \emptyset$, then there exists a unique $x \in Y$ with $h(x) = x$.

Proof. Uniqueness of x is clear as $C < 1$. To establish its existence, set $z_k = h^k(z)$ as long as it makes sense for a given $z \in Y$ and integers $k \geq 0$. Then $\mathrm{d}(z_k, z_{k+1}) \leq C^k\,\mathrm{d}(z, h(z))$ (induction on $k \geq 0$), and so, for integers $l \geq 0$ such that z_{k+l} exists,

$$\text{(73.1)}\quad \mathrm{d}(z_k, z_{k+l}) \le \sum_{s=k}^{k+l-1} \mathrm{d}(z_s, z_{s+1}) \le \left[\sum_{s=k}^{\infty} C^s\right] \mathrm{d}(z, h(z)) = \frac{C^k}{1-C}\,\mathrm{d}(z, h(z))\,.$$

We may assume that $h(z) \ne z$. Then, for r as in (b), (73.1) yields $\mathrm{d}(z_k, z_{k+l}) < r$; setting $k = 0$, we thus see that (b) implies (a). On the other hand, for z as in (a), (73.1) shows that the z_k form a Cauchy sequence, and we can take $x = \lim_{k\to\infty} z_k$. □

COROLLARY 73.2. *Given a complete metric space* (X, d) *and* $h : X \to X$ *such that* $\mathrm{d}(h(x), h(x')) \le C\,\mathrm{d}(x, x')$ *for all* $x, x' \in X$ *and some* C *with* $0 \le C < 1$, *there exists a unique point* $x \in X$ *with* $h(x) = x$.

Problems

1. Prove that
$$|\|v\| - \|w\|| \le \|v - w\|$$
for any norm $\|\,\|$ in any vector space V and any vectors $v, w \in V$.

2. Defining closed subsets of metric spaces as in the case of manifolds (Problem 5 in §1), verify that, both for manifolds and metric spaces, a subset is closed if and only if it contains the limits of all sequences of its points that converge in the ambient space.

3. A sequence $x_k \in X$, $k = 1, 2, \dots$ of points in a metric space (X, d) is called *bounded* if it lies in a ball $B_z(r)$ with some center $z \in X$ and some radius $r > 0$. Show that z then may be replaced by any other point $z' \in X$, and that every convergent sequence is Cauchy, while each Cauchy sequence is bounded.

4. Verify that any Cauchy sequence in a metric space that has a convergent subsequence, is itself convergent.

5. Defining compactness for subsets of metric spaces as in the case of manifolds (§2), show that every compact metric space is complete.

6. Show that a subset of a metric space which is complete in the restricted distance function must be closed, and that any closed subset of a complete metric space is itself complete as a metric space.

7. Let $|\,|_{\mathrm{E}}$ be the standard Euclidean norm in $\mathbf{R}^n$, with $|v|_{\mathrm{E}}^2 = |v^1|^2 + \ldots + |v^n|^2$ for $v = (v^1, \ldots, v^n)$. Show that $|\,|_{\mathrm{E}}$ is actually a norm and, for *any* norm $|\,|$ in $\mathbf{R}^n$, we have the estimate $|v| \le C|v|_{\mathrm{E}}$ for all $v \in \mathbf{R}^n$ and $C \ge 0$ with $C^2 = |e_1|^2 + \ldots + |e_n|^2$, $e_1, \ldots, e_n$ being the standard basis of $\mathbf{R}^n$ with $e_j^l = \delta_j^l$. (Hint below.)

8. Verify that any norm $|\,|$ in $\mathbf{R}^n$ is a continuous function $\mathbf{R}^n \to \mathbf{R}$, i.e., $|v_k| \to |v|$ as $k \to \infty$ whenever $v_k \to v$ in $\mathbf{R}^n$ (the latter being the componentwise convergence). (Hint below.)

9. Show that any two norms $|\,|$, $|\,|'$ in a finite-dimensional real or complex vector space are *equivalent* in the sense that $|v| \le C|v|'$ and $|v|' \le C'|v|$ for all $v \in V$, with suitable constants $C, C' > 0$. (Hint below.)

10. A finite-dimensional real vector space V constitutes both a metric space (with any fixed norm $|\,|$), and a manifold (§1). Show that the resulting classes of open/closed subsets of V, and the notions of convergence and limit for sequences in V coincide for all these structures and, in particular, do not depend on the choice of the norm. (Hint below.)

Hint. In Problem 18, use the Schwarz inequality: $|v| = |v^j e_j| \le |v^j||e_j| \le C|v|_{\mathrm{E}}$.
Hint. In Problem 19, note that $v_k \to v$ implies $|v_k - v|_{\mathrm{E}} \to 0$, so that we have $||v_k| - |v|| \le |v_k - v| \to 0$ in view of Problems 10 and 17.
Hint. In Problem 20, identify V with $\mathbf{R}^n$ and let one of the norms be $|\ |_{\mathrm{E}}$. Then the other norm has a maximum and a positive minimum on the unit sphere S^{n-1} (Problems 5, 6, 19 in §3).
Hint. In Problem 21, use Problem 20.

74. The inverse mapping theorem

Topics: The operator norm in finite dimensions; Newton's method for approximating a solution x to $y = F(x)$ by the sequence $x_n = h^n(x_0)$ with $h(x) = x + dF_x^{-1}[y - F(x)]$; use of dF_0 instead of dF_x; the inverse mapping theorem; the implicit mapping theorem.

Let V, W be finite-dimensional real or complex vector spaces carrying fixed norms (both denoted by $|\ |$). Any linear operator $T : V \to W$ is *bounded* in the sense that $|Tv| \le C|v|$ for some constant $C \ge 0$ and all $v \in V$ (Problem 11). The smallest constant C with this property is called the *operator norm* of T and denoted by $|T|$. If, moreover, $U \subset V$ is an open set, $h : U \to V$ is a C^1 mapping and x, z are points in U such that U contains the whole segment $\overline{xz} = \{z + t(x - z) : 0 \le t \le 1\}$ connecting z to x, then we have the estimate (see Problem 19)

$$|h(x) - h(z)| \le |x - z| \cdot \sup_{u \in \overline{xz}} |dh_u| \tag{74.1}$$

involving the operator norm of $dh_u : V \to W$, the supremum (which, in fact, is a maximum) being finite since $\overline{xz}$ is compact and the function $u \mapsto |dh_u|$ is continuous.

LEMMA 74.1. *Let U, U' be open sets in finite-dimensional vector spaces V, W, respectively, and let a C^l mapping $F : U \to U'$ with $1 \le l \le \min(r, \infty)$ be one-to-one and onto, and such that, for each $x \in U$, the differential $dF_x : V \to W$ is a linear isomorphism. Then the inverse mapping $F^{-1} : U' \to U$ is also C^l-differentiable.*

PROOF. Fix norms in V, W (both denoted by $|\ |$). For any fixed $z \in V$, differentiability of F at z means that

$$F(x) - F(z) = dF_z(x - z) + \alpha(x, z)\,, \qquad \frac{\alpha(x, z)}{|x - z|} \to 0 \quad \text{as } x \to z\,. \tag{74.2}$$

Since $|dF_z(x - z)| \ge 2C|x - z|$ for some constant $C > 0$ (Problem 13), choosing $\varepsilon > 0$ with $|\alpha(x, z)| \le C|x - z|$ for all $x \in U$ with $|x - z| < \varepsilon$, we obtain, for such x, $|F(x) - F(z)| \ge |dF_z(x - z)| - |\alpha(x, z)|$ in view of (74.2) and Problem 10 in §5, i.e., $|F(x) - F(z)| \ge C|x - z|$ for all x sufficiently close to any fixed $z \in U$ and a suitable $C > 0$, depending on z. (Thus, F^{-1} is continuous.) Applying $(dF_z)^{-1}$ to both sides of (74.2) and writing $\zeta = F(z)$, $\xi = F(x)$, we obtain

$$F^{-1}(\xi) - F^{-1}(\zeta) = (dF_z)^{-1}(\xi - \zeta) + \beta(\xi, \zeta) \tag{74.3}$$

with $\beta(\xi, \zeta) = -(dF_z)^{-1}\alpha(x, z)$. Thus, F^{-1} is differentiable at ζ and $d(F^{-1})_\zeta = (dF_z)^{-1}$ since $|\xi - \zeta|^{-1}|\beta(\xi, \zeta)| \le C^{-1}|x - z|^{-1}|\beta(\xi, \zeta)| = C^{-1}|(dF_z)^{-1}(|x - z|^{-1}\alpha(x, z))| \to 0$ as $\xi \to \zeta$, due to the estimate $|\xi - \zeta| \ge C|x - z|$. Induction on s now shows that the mapping $\zeta \mapsto d(F^{-1})_\zeta = (dF_{F^{-1}(\zeta)})^{-1}$ is C^{s-1} differentiable for each $s = 1, \ldots, l$. This completes the proof. $\square$

The following fundamental result is known as the *inverse mapping theorem.*

THEOREM 74.2. *Let $F : M \to N$ be a C^l mapping between C^s manifolds, $1 \le l \le s < \infty$, and let $z \in M$. If the differential $dF_z : T_zM \to T_{F(z)}N$ is a linear isomorphism, then, for a suitable neighborhood U of z in M, the image $F(U)$ is an open subset of N and $F : U \to F(U)$ is a C^l diffeomorphism.*

PROOF. Using local coordinates, we may assume that M, N are open subsets of finite-dimensional real vector spaces V, W endowed with some fixed norms, both denoted by $|\,|$. For any fixed $y \in W$, define the C^l mapping $h : M \to V$ by $h(x) = x + (dF_z)^{-1}(y - F(x))$. As $dh_x = \mathrm{Id} - (dF_z)^{-1}dF_x$, we have $dh_z = 0$ and hence there is a closed ball $\overline{B}_z(\varepsilon)$ centered at z, of some radius $\varepsilon > 0$, with the operator-norm inequality $|dh_x| \le 1/2$ for all $x \in \overline{B}_z(\varepsilon)$. On the other hand, $\mathrm{d}(z, h(z)) = |z - h(z)| \le |(dF_z)^{-1}|\cdot|y - F(z)|$ and so, whenever $y \in U' = B_{F(z)}(\varepsilon')$ with $2\varepsilon'|(dF_z)^{-1}| < \varepsilon$, the assumptions of Banach's fixed-point theorem (§4, appendix) will be satisfied, according to (74.1), by $X = V$, $Y = B_z(\varepsilon)$, $C = 1/2$, our h (depending on y), and $r = \varepsilon$ in assumption (b). The existence of a unique $x \in Y$ with $h(x) = x$, i.e., $y = F(x)$, then means that $F : U \to U'$ is one-to-one and onto, where $U = Y \cap F^{-1}(U')$, and the assertion follows from Lemma 74.1. ☐

Problems

11. Prove boundedness for linear operators between finite-dimensional normed vector spaces.

12. Let V, W be finite-dimensional normed vector spaces (with both norms denoted by $|\,|$). Show that the operator norm of any linear operator $T : V \to W$ is given by

$$|T| = \sup\{|Tv| : v \in V\,,\ |v| = 1\} = \sup\{|Tv| : v \in V\,,\ |v| \le 1\} \tag{74.4}$$

(the supremum of an empty set of nonnnegative real numbers being 0 by definition). Can $\sup$ be replaced by $\max$?

13. Show that, if a linear operator $T : V \to W$ between normed vector spaces V, W is injective and $\dim V < \infty$, then there exists a constant $C > 0$ with $|Tv| \ge C|v|$ for all $v \in V$.

14. For finite-dimensional normed vector spaces V_1, V_2, V_3 and linear operators $T : V_1 \to V_2$, $S : V_2 \to V_3$, verify the operator-norm inequality $|ST| \le |S|\cdot|T|$.

15. (*Riemann integral for vector-valued functions.*) Let $\gamma : [a, b] \to V$ be a continuous curve in a finite-dimensional real vector space V, prove that there is a unique vector $v \in V$ with $\xi(v) = \int_a^b \xi(\gamma(t))\,dt$ for all $\xi \in V^*$. (One writes $v = \int_a^b \gamma(t)\,dt$ and calls v the *Riemann integral* of γ.) Verify that the integration acts componentwise, i.e., for any basis e_j of V, $\int_a^b \gamma(t)\,dt = \left[\int_a^b \gamma^j(t)\,dt\right] e_j$, where $\gamma(t) = \gamma^j(t)e_j$.

16. For γ as in Problem 15, show that γ has a C^1 antiderivative $\Gamma : [a, b] \to V$ with $\dot\Gamma(t) = \gamma(t)$ for all t, and for any such Γ, $\int_a^b \gamma(t)\,dt = \Gamma(b) - \Gamma(a)$.

17. *Riemann-sum approximations.* Given γ as in Problem 15 and any partition $\mathcal{P} = \{t_0, \ldots, t_m\}$ of $[a, b]$, with $a = t_0 < \ldots < t_m = b$, and any selection of numbers $t'_j \in [t_{j-1}, t_j]$, set $\delta(\mathcal{P}) = \max\{t_j - t_{j-1} : 1 \le j \le m\}$, $j = 1, \ldots, m$, and define the corresponding Riemann sum by the familiar formula $\sum_{j=1}^m \gamma(t'_j)(t_j - t_{j-1})$.

Verify that, for any sequence of partitions $\mathcal{P}_k$ and selections such that $\delta(\mathcal{P}_k) \to 0$ as $k \to \infty$, the resulting sequence of Riemann sums converges to $\int_a^b \gamma(t)\,dt$.

18. Given a continuous curve $\gamma : [a,b] \to V$ in a finite-dimensional real vector space V with a fixed norm $|\ |$, prove the following estimate: (Hint below.)

$$\left| \int_a^b \gamma(t)\,dt \right| \le \int_a^b |\gamma(t)|\,dt\,.$$

19. Prove the estimate (74.1). (Hint below.)

20. *The implicit mapping theorem.* Suppose that M, N, P are C^s manifolds, $\Phi : M \times N \to P$ is a C^l mapping, $1 \le l \le \min(s, \infty)$, and $x_0 \in M$, $y_0 \in N$, $z_0 \in P$ are points such that $\Phi(x_0, y_0) = z_0$ and the differential of the mapping $N \ni y \mapsto \Phi(x_0, y) \in P$ at $y = y_0$ is a linear isomorphism $T_{y_0}N \to T_{z_0}P$. Prove that x_0, y_0 have neighborhoods U, U' in M, N, respectively, such that

(a) For each $x \in U$ there is a unique $y = y(x) \in U'$ with $\Phi(x, y(x)) = z_0$.

(b) The mapping $U \ni x \mapsto y(x) \in U'$ in (a) is C^l differentiable. (Hint below.)

21. Show that, in the definition of a Lie group of class C^s, $s \ge 1$ (§4), the requirement that the group multiplication and the inverse be both of class C^s may be replaced by C^s regularity of the multiplication alone. (Hint below.)

Hint. In Problem 18, use a Riemann-sum approximation (Problem 17).

Hint. In Problem 19, note that

$$h(x) - h(z) = \int_0^1 \frac{d}{dt} h(z + t(x - z))\,dt = \left[\int_0^1 dh_{z+t(x-z)}\,dt \right](x - z)\,. \tag{74.5}$$

Hint. In Problem 20, apply the inverse mapping theorem (Theorem 74.2) to $F : M \times N \to M \times P$ given by $F(x, y) = (x, \Phi(x, y))$.

Hint. In Problem 21, define the mapping $a \mapsto a^{-1}$ via the implicit mapping theorem.

75. The Stone-Weierstrass theorem

By 'a compact set' we mean here either a compact subset of a manifold, or a compact metric space; for readers familiar with general topology, it may also be interpreted as a compact (Hausdorff) topological space. For a subset X of a manifold M, open sets in X and neighborhoods of points in X are to be understood as *relatively* open (intersections of open subsets of M with X). One trivially verifies that, if X is compact, every open covering of X still has a finite subcovering (cf. Theorem 14.2), while closed subsets of X are themselves compact, and continuous preimages of open/closed sets are also open/closed.

The following fundamental result is known as the *Stone-Weierstrass theorem.* Here $\mathrm{C}(X)$ is the algebra of all continuous functions $X \to \mathbf{R}$, and we say that a set $\mathcal{Y} \subset \mathrm{C}(X)$ *separates the points of* X if, for any $x, y \in X$ with $x \ne y$, there exists $f \in \mathcal{Y}$ with $f(x) \ne f(y)$.

THEOREM 75.1. *Let X be a compact set and let $\mathcal{Y}$ be any subalgebra of $\mathrm{C}(X)$ which contains all constant functions and separates the points of X. Then $\mathcal{Y}$ is uniformly dense in $\mathrm{C}(X)$.*

PROOF. It suffices to show that the only closed subalgebra $\mathcal{Z}$ of $\mathrm{C}(X)$ which contains all constants and separates the points of X is $\mathrm{C}(X)$ itself; our assertion will follow if we apply this to to $\mathcal{Z}$ defined to be the closure of $\mathcal{Y}$ relative to the supremum norm, that is, the set of all uniform limits of sequences in $\mathcal{Y}$. (Note that the closure of a subalgebra is also a subalgebra.)

First, $f^{1/2} \in \mathcal{Z}$ whenever $f \in \mathcal{Z}$ and $f > 0$ everywhere in X. In fact, multiplying f by a constant factor we may assume that $0 < f < 2$ and so $f = 1+h$ with $h \in \mathcal{Z}$ and $|h| < 1$. Therefore, as X is compact, $|h| \le r$ for some $r \in (0,1)$. Since the Taylor series $\sum_{j\ge 0} a_j t^j$ of the function $(-1,1) \ni t \mapsto (1+t)^{1/2}$ converges uniformly on compact subintervals of $(-1,1)$, such as $(-r,r)$, it now follows that $f^{1/2}$ is the uniform limit of the partial sums of the series $f_k = \sum_{0\le j\le k} a_j h^j$, while $f_k \in \mathcal{Z}$. Thus, $f^{1/2} \in \mathcal{Z}$.

Secondly, $|f| \in \mathcal{Z}$ whenever $f \in \mathcal{Z}$. In fact, $|f|$ is the limit of $(f^2+\varepsilon)^{1/2}$ as $\varepsilon \to 0^+$ and the convergence is uniform, since, for $\varepsilon > 0$,

$$0 \,<\, (f^2+\varepsilon)^{1/2} \,-\, |f| \,\le\, \frac{\varepsilon}{(f^2+\varepsilon)^{1/2} \,+\, |f|} \,\le\, \varepsilon^{1/2}.$$

As $2\max(s,t) = s+t+|s-t|$ and $2\min(s,t) = s+t-|s-t|$ for $s,t \in \mathbf{R}$, this also shows that $\max(f,h)$ and $\min(f,h)$ are in $\mathcal{Z}$ whenever f and h are.

Next, given $x,y \in X$ with $x \ne y$, there exists $h \in \mathcal{Y}$ such that $0 \le h \le 1$, while $h = 0$ on some neighborhood of x and $h = 1$ on some neighborhood of y. In fact, choosing $f \in \mathcal{Y}$ with $f(x) \ne f(y)$ and setting $\varphi = \alpha f + \beta$ for suitable $\alpha,\beta \in \mathbf{R}$ we get $\varphi(x) < 0$ and $\varphi(y) > 1$, so that we can define h by $h = \max(0,\, \min(h,1))$.

Similarly, given a closed set $K \subset X$ and $x \in X \setminus K$, there exists $f \in \mathcal{Y}$ such that $0 \le f \le 1$, while $f = 0$ on some neighborhood of x and $f = 1$ on some open set containing K. Namely, choosing $h = h_y$ as above for this fixed x and any given $y \in K$, so that $h_y = 1$ on a neighborhood U_y of y, and noting that a finite family $U_{y(1)},\ldots,U_{y(l)}$ will cover K due to its compactness, we may set $f = \max(h_{y(1)},\ldots,h_{y(l)})$.

Furthermore, for any two disjoint closed sets $K, K' \subset X$ and any $p,q \in \mathbf{R}$ with $p < q$ there exists $h \in \mathcal{Y}$ such that $\min(p,q) \le h \le \max(p,q)$, while $h = p$ on some open set containing K and $h = q$ on some open set containing K'. In fact, we may set $p = 1$ and $q = 0$ (since the general case then is easily obtained if one replaces h by a suitable linear function of h); a compactness argument just like above then is straightforward.

Finally, given $f \in \mathrm{C}(X)$, let us set $\mathcal{V}(f) = \max(f) - \min(f)$. We will now show that, for any $f \in \mathrm{C}(X)$, there exists $h \in \mathcal{Z}$ with $5\mathcal{V}(f-h) \le 2\mathcal{V}(f)$. This will imply our assertion since, if we repeat the step of replacing f by $f-h$ a sufficient number of times, each time multiplying the value of $\mathcal{V}$ by a factor of $2/5$ or less, we eventually get $\mathcal{V}(f-h') < \varepsilon$ for any prescribed $\varepsilon > 0$ and some $h' \in \mathcal{Z}$ (which means that $f-h'$ is uniformly closer than the distance $\varepsilon/2$ from some constant). To show that such $h \in \mathcal{Z}$ exists, we may assume that $\mathcal{V}(f) \ne 0$, for otherwise f is constant and we may choose $h = 0$. Let $a = \min(f)$, $b = \max(f)$ and $c = (b-a)/5$, so that $\mathcal{V}(f) = 5c > 0$, and let $h \in \mathcal{Z}$ be chosen as in the last paragraph for $p = a+c$, $q = a+4c$, $K = f^{-1}([a,a+2c])$ and $K' = f^{-1}([a+3c,b])$. Thus, on K we have $h = a+c$ and $a \le f \le a+2c$, so that $|f-h| \le c$. Similarly, $|f-h| \le c$ on K'. On $X \setminus (K \cup K')$, however, $a+c \le h \le a+4c$ and $a+2c \le f \le a+3c$, and so $|f-h| \le 2c$. Hence $|f = h| \le 2c$ on X, which completes the proof. $\square$

Corollary 75.2. *Let K be a compact subset of a C^s manifold M, with $s = 1, 2, \ldots, \infty$. Every continuous mapping $f : K \to V$, valued in a finite-dimensional normed real vector space V, then is the limit of a uniformly convergent sequence of mappings $K \to V$ which are restrictions to K of C^s mappings $M \to V$.*

In fact, in the case where $V = \mathbf{R}$ this follows since the restrictions to K of C^s functions on M form a subalgebra $\mathcal{Y}$ of $\mathrm{C}(K)$ satisfying the assumptions of Theorem 75.1 with $X = K$. (That $\mathcal{Y}$ separates the points of K is clear in view of Problem 19 in §6.) The general case is now also immediate, as $|f| \le |f^a||e_a|$ whenever a function $f : K \to V$ is expanded into a combination $f = f^a e_a$ for any fixed basis e_a of V.

76. Sard's theorem

A point $x \in M$ is called *critical* for a C^1 mapping $F : M \to N$ between manifolds, if $dF_x : T_xM \to T_{F(x)}N$ is not surjective.

Sard's Theorem. *If $F : M \to N$ is a C^1 mapping between manifolds of the same dimension, then the F-image of the set of all critical points of F is of measure zero in N.*

Problems

1. Given a (bilinear, symmetric, positive-definite) inner product $\langle\,,\rangle$ in a real vector space V and $x, y \in V$, prove the *Schwarz inequality*

$$|\langle x, y\rangle| \le |x||y|, \tag{76.1}$$

along with the conclusion about the equality case, without invoking the standard discriminant argument. Here $|\,|$ is, as usual, the norm in V corresponding to $\langle\,,\rangle$. (Hint below.)

2. Establish (76.1) for a (sesquilinear, Hermitian, positive-definite) inner product $\langle\,,\rangle$ in a complex vector space V and $x, y \in V$. (Hint below.)

3. Prove (76.1) for vectors x, y in a real (or, complex) vector space and a scalar-valued form $\langle\,,\rangle$ which is bilinear symmetric (or, respectively, sesquilinear Hermitian) and positive semidefinite. (Hint below.)

4. For $\langle\,,\rangle$ as in Problem 3, verify that a vector x which is *null* (in the sense of having $\langle x, x\rangle = 0$) is necessarily $\langle\,,\rangle$-orthogonal to the whole space.

Hint. In Problem 1, $|x|^4|y|^2 - \langle x, y\rangle^2|x|^2$ is nonnegative, as it equals $|z|^2$ for $z = |x|^2y - \langle x, y\rangle x$, and so (76.1) follows, since we may assume that $x \ne 0$.

Hint. In Problem 2, use Problem 1 for rm $\mathrm{Re}\,\langle\,,\rangle$.

Appendix C. Ordinary Differential Equations

78. Existence and uniqueness of solutions

Topics: Continuous, bounded and Lipschitz mappings between metric spaces; spaces of mappings with the uniform distance; ordinary differential equations; reduction of order to 1; existence and uniqueness of solutions; autonomous equations.

A mapping $f : K \to X$ from a set K into a metric space (X, d) is said to be *bounded* if its image $f(K)$ is a bounded subset of (X, d) in the sense that it lies in a ball $B_z(r)$ with some center $z \in X$ and some radius $r > 0$. Let us denote by $\mathcal{X} = B(K, X)$ the set of all bounded mappings $f : K \to X$ and define the *uniform distance* function $\mathrm{d}_{\sup} : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by $\mathrm{d}_{\sup}(f, f') = \sup\{\, \mathrm{d}\,(f(x), f'(x)) : x \in K\}$. Endowed with $\mathrm{d}_{\sup}$), the set $\mathcal{X}$ becomes a metric space (Problem 1); the convergence in $(\mathcal{X}, \mathrm{d}_{\sup})$ is called the *uniform convergence* of bounded mappings $K \to X$.

In the case where (X, d) is the underlying metric space of a normed vector space $(X, |\ |)$ (see §1, Appendix), and K is any set, it is clear that $(\mathcal{X}, \mathrm{d}_{\sup}) = (B(K, X), \mathrm{d}_{\sup})$ is the underlying metric space of the normed vector space $(\mathcal{X}, \|\ \|_{\sup})$ with the valuewise operations on X-valued functions f on K and the *supremum norm* $\|f\|_{\sup} = \sup\{|f(x)| : x \in K\}$.

If, moreover, K happens to be a manifold or a metric space, the set $\mathcal{X} = B(K, X)$ contains the subset $C_B(K, X)$ formed by all bounded mappings $K \to X$ which are also *continuous*. (In both cases, a mapping $f : K \to N$ is said to be *continuous* if $f(x_k) \to f(x)$ in X as $k \to \infty$ whenever x_k, $k = 1, 2, \ldots$, is a sequence of points in K that converges to a point $x \in K$.) When K is compact, we write $C(K, X)$ rather than $C_B(K, X)$, deleting the subscript 'B' as boundedness then follows from continuity (Problem 13 in §2). With the restriction of the distance function $\mathrm{d}_{\sup}$, the set $C_B(K, X)$ constitutes a metric space which is *complete* whenever so is (X, d) (Problems 2, 3 below and 15 in §5).

We say that a mapping $f : K \to X$ between metric spaces (with both distances denoted by d) satisfies the *Lipschitz condition* if there exists a constant $C \geq 0$ such that $\mathrm{d}(F(x), F(y)) \leq C\,\mathrm{d}(x, y)$ for all $x, y \in K$. For instance, Problem 12 in §5 states that any norm satisfies the Lipschitz condition with $C = 1$. Note that the Lipschitz condition implies continuity.

Let us now consider an open subset U of a finite-dimensional real vector space V. By an *ordinary differential equation* of order $k \geq 1$ in U we mean a mapping $F : I \times U \times V^{k-1} \to V$, where $I \subset \mathbf{R}$ is an open interval and $V^{k-1} = V \times \ldots \times V$ is the $(k-1)$st Cartesian power of V. A C^k-differentiable curve $\gamma : I' \to V$ defined on a subinterval I' of I (open or not) is called a *solution* to the equation if

$$\gamma^{(k)} = F(t, \gamma, \dot{\gamma}, \ldots, \gamma^{(k-1)}) \tag{78.1}$$

in the sense that $\gamma^{(k)}(t) = F(t, \gamma(t), \dot{\gamma}(t), \ldots, \gamma^{(k-1)}(t))$ for all $t \in I'$, where $\gamma^{(k)} = d^k\gamma/dt^k$. Such a solution is said to satisfy the *initial condition* $(t_0, x_0, v_1, \ldots, v_{k-1})$ if $(t_0, x_0, v_1, \ldots, v_{k-1}) \in I \times U \times V^{k-1}$ and

$$\gamma(t_0) = x_0 \,,\; \dot{\gamma}(t_0) = v_1 \,, \ldots, \; \gamma^{(k-1)}(t_0) = v_{k-1} \,. \tag{78.2}$$

The mapping F is usually referred to as the *right-hand side* of the equation (rather than being called the equation itself). Together, (78.1) and (78.2) are said to form a *kth order initial value problem*. An initial value problem (78.1), (78.2) of any order $k > 1$ can always be reduced to a first-order problem $\dot{\chi} = \Phi(t, \chi)$, $\chi(t_0) = z_0$ in the open set $U \times V^{k-1}$ of the higher-dimensional space V^k, by setting $\chi(t) = (\gamma(t), \dot{\gamma}(t), \ldots, \gamma^{(k-1)}(t)) \in U \times V^{k-1}$, $\Phi(t, x, w_1, \ldots, w_{k-1}) = (w_1, \ldots, w_{k-1}, F(t, x, w_1, \ldots, w_{k-1}))$ for $(t, x, w_1, \ldots, w_{k-1}) \in I \times U \times V^{k-1}$, and $z_0 = (x_0, v_1, \ldots, v_{k-1})$. The theorem proved below for $k = 1$ can therefore be easily extended to initial value problems of any order k (Problem 7).

We have the following *existence and uniqueness theorem.*

THEOREM 78.1. *Let $I \subset \mathbf{R}$ be an open interval, and let U be an open subset of a finite-dimensional real vector space V. If $F : I \times U \to V$ is continuous and satisfies the Lipschitz condition in $x \in U$ uniformly in $t \in I$, i.e., $|F(t, x') - F(t, x)| \le C|x' - x|$ for some fixed norm $|\;|$ in V, some constant $C \ge 0$, and all $t \in I$, $x, x' \in U$, then, for any initial condition $(t_0, x_0) \in I \times U$ there is $\varepsilon > 0$ such that the equation $\dot{\gamma} = F(t, \gamma)$ has a unique C^1 solution $\gamma : [t_0 - \varepsilon, t_0 + \varepsilon] \to U$ with $\gamma(t_0) = x_0$.*

REMARK 78.2. The condition imposed on ε is

$$\varepsilon s_\varepsilon < (1 - C\varepsilon)\delta \,, \tag{78.3}$$

with C, $|\;|$ as above, $s_\varepsilon = \sup\{|F(t, x_0)| : |t - t_0| \le \varepsilon\}$ and $\delta = \inf\{|y - x_0| : y \in V \setminus U\} \in (0, \infty]$ equal to the distance between x_0 and the complement (or boundary) of U, where $r = \infty$ if $U = V$. Since $s_\varepsilon \to |F(t_0, x_0)|$ as $\varepsilon \to 0$, (78.3) holds for all sufficiently small $\varepsilon > 0$.

PROOF. For $\gamma : [t_0 - \varepsilon, t_0 + \varepsilon] \to U$, the requirement that γ be C^1 and satisfy $\dot{\gamma} = F(t, \gamma)$ and $\gamma(t_0) = x_0$, is equivalent to continuity of γ along with

$$\gamma(t) = x_0 + \int_{t_0}^{t} F(\tau, \gamma(\tau))\, d\tau \tag{78.4}$$

for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$. Let $\mathcal{X}_\varepsilon$ be the Banach metric space $C([t_0 - \varepsilon, t_0 + \varepsilon], V)$ with the supremum norm $\|\;\|_{\mathrm{sup}}$ defined above using the norm $|\;|$ in V. The mapping $h_\varepsilon : \mathcal{K}_\varepsilon \to \mathcal{X}_\varepsilon$ from the subset $\mathcal{K}_\varepsilon = C([t_0 - \varepsilon, t_0 + \varepsilon], U)$ of $\mathcal{X}_\varepsilon$ into $\mathcal{X}_\varepsilon$, given by $[h_\varepsilon(\gamma)](t) = x_0 + \int_{t_0}^{t} F(\tau, \gamma(\tau))\, d\tau$ then satisfies $\|h_\varepsilon(\gamma') - h_\varepsilon(\gamma)\|_{\mathrm{sup}} \le C\varepsilon\|\gamma' - \gamma\|_{\mathrm{sup}}$, as the length of the integration interval is $|t - t_0| \le \varepsilon$ and C is a Lipschitz constant for F. Denoting by z the constant curve $x_0 \in \mathcal{K}_\varepsilon$ and setting $r_\varepsilon = (1 - C\varepsilon)^{-1}\|z - h_\varepsilon(z)\|_{\mathrm{sup}}$, we obtain $r_\varepsilon \le (1 - C\varepsilon)^{-1}\varepsilon s_\varepsilon$. Thus, for ε chosen as in (78.3), $r_\varepsilon < \delta$ and hence the ball $B_z(r_\varepsilon)$ in $\mathcal{X}_\varepsilon$ is contained in $\mathcal{K}_\varepsilon$. The assumptions of Banach's fixed-point theorem (§73) thus will be satisfied if we replace $X, \mathrm{d}, K, h, C, z, r$ in the statement of that theorem by $\mathcal{X}_\varepsilon$, $\mathrm{d}_{\mathrm{sup}}$, $\mathcal{K}_\varepsilon$, h_ε, $C\varepsilon$, $z = x_0$, and, respectively, r_ε, for any ε with (78.3). The resulting existence and uniqueness of $\gamma \in \mathcal{K}_\varepsilon$ with $h_\varepsilon(\gamma) = \gamma$, i.e., (78.4), now proves our assertion. $\square$

Problems

1. Given a set K and a metric space (X, d), show that the distance function $\mathrm{d}_{\sup}$ in $\mathcal{X} = B(K, X)$ is well-defined (i.e., the supremum is always finite), and $(\mathcal{X}, \mathrm{d}_{\sup})$ is a metric space.

2. Prove that $(\mathcal{X}, \mathrm{d}_{\sup})$ in Problem 1 is complete as a metric space whenever so is (X, d).

3. Show that, if K is a metric space or a manifold, and (X, d) is a metric space, then $C_B(K, X)$ is a closed subset of $B(K, X)$, i.e., taking uniform limits preserves continuity.

4. Given real numbers a, b, c with $a \le 0 \le b$ and $c > 1$, verify that $\gamma : \mathbf{R} \to \mathbf{R}$ given by $\gamma(t) = -|t-a|^c$ if $t \le a$, $\gamma(t) = 0$ if $a \le t$, and $\gamma(t) = |t-b|^c$ if $t \ge b$ is a C^1 solution to the equation $\dot{\gamma} = c|\gamma|^{1-1/c}$. Explain why this is not a counterexample to the existence and uniqueness theorem for ordinary differential equations.

5. Let V, W be finite-dimensional real vector spaces with fixed norms (both denoted by $|\ |$), and let U be an open subset of V. We say that a mapping $F : U \to W$ is *locally Lipschitz* if, for each $x \in U$ there exists a neighborhood U_x of x in U and a constant $C \ge 0$ satisfying $|F(x') - F(x'')| \le C|x' - x''|$ for all $x', x'' \in U_x$. Show that every C^1 mapping is locally Lipschitz. (Hint below.)

6. Verify that the above existence and uniqueness theorem for any $(t_0, x_0) \in I \times U$ (with a suitable $\varepsilon > 0$ depending on t_0 and x_0) remains valid under the weaker assumption that the continuous mapping $F : I \times U \to V$ is *locally Lipschitz in* $x \in U$, *locally uniformly in* $t \in I$, which means that, with some fixed norm $|\ |$ in V, for each $(t_0, x_0) \in I \times U$ there exist neighborhoods I' of t_0 in I and U' of x_0 in U and a constant $C \ge 0$ satisfying $|F(t, x') - F(t, x)| \le C|x' - x|$ for all $t \in I'$, $x, x' \in U'$. Prove that uniqueness then holds in every interval on which a C^1 solution can be defined. (Hint below.)

7. Extend the above existence and uniqueness theorem in the "locally Lipschitz" version as in Problem 6 (including the "global uniqueness" statement) to initial value problems (78.1), (78.2) of any order $k \ge 1$.

8. Those equations of type (78.1) for which the right-hand side F does not depend explicitly on $t \in I$, i.e.,

$$\gamma^{(k)} = F(\gamma, \dot{\gamma}, \ldots, \gamma^{(k-1)}) \tag{78.5}$$

with $F : U \times V^{k-1} \to V$ (where U is a fixed open subset of a finite-dimensional real vector space V), are called *autonomous* kth order equations. Show that every kth order equation (78.1) is equivalent to an autonomous first-order equation $\dot{\chi} = \Phi(\chi)$ in a suitable (possibly higher-dimensional) space. Verify that a solution γ to any autonomous equation, and any constant c, lead to a solution given by $t \mapsto \gamma(t + c)$ on a suitable interval. (Hint below.)

9. Verify that a solution γ to the autonomous first-order initial value problem $\dot{\gamma} = F(\gamma)$, $\gamma(t_0) = x_0$ in an open interval $U \subset \mathbf{R}$, where $F : U \to \mathbf{R}$ is continuous, $x_0 \in U$ and $t_0 \in \mathbf{R}$, can be defined by
   (a) $\gamma(t) = x_0$ for all $t \in \mathbf{R}$ if $F(x_0) = 0$.
   (b) $\gamma(t) = \Psi^{-1}(t - t_0)$ for all $t \in \Psi((a, b))$, where (a, b), $-\infty \le a < b \le \infty$, is the largest subinterval of U containing x_0 with $F \ne 0$ everywhere in (a, b) and $\Psi : (a, b) \to \mathbf{R}$ is the (strictly monotone) antiderivative of $1/F$ with $\Psi(x_0) = 0$ (if $F(x_0) \ne 0$). (Hint below.)

10. Let $\gamma : I \to \mathbf{R}$ be the solution to the initial value problem $\dot{\gamma} = F(\gamma)$, $\gamma(t_0) = x_0$, where $U \subset \mathbf{R}$ is an open interval, $F : U \to \mathbf{R}$ is locally Lipschitz, $x_0 \in U$, $t_0 \in \mathbf{R}$, and I is the largest interval in $\mathbf{R}$ containing t_0 on which such a solution γ exists. Prove that the length $|I|$ of I is $|I| = \infty$ if $F(x_0) = 0$, and is given by

$$|I| = \left| \int_a^b \frac{dx}{F(x)} \right| \in (0, \infty],$$

if $F(x_0) \neq 0$, where $\infty \leq a < b \leq \infty$ are the endpoints of the largest subinterval (a, b) of U containing x_0 with $F \neq 0$ everywhere in (a, b). (Hint below.)

11. Let γ be a solution to the autonomous order equation (78.5). Verify that, for any real number c, the assignment $t \mapsto \gamma(t + c)$ defines a solution to (78.5).

Hint. In Problem 5, use estimate (74.1) and Problem 1 in §2.

Hint. In Problem 6, uniqueness follows from a continuity argument: Given two C^1 solutions $\gamma, \gamma' : (a, b) \to U$, set $t_1 = \sup\{t \in [t_0, b) : \gamma = \gamma' \text{ on } [t_0, t)\}$, so $t_1 \leq b \leq \infty$ must be equal to b, or else γ, γ' would coincide in $[t_1, t_1 + \varepsilon)$ for some $\varepsilon > 0$ due to the local uniqueness of solutions with the initial condition $(t_1, \gamma(t_1)) = (t_1, \gamma'(t_1))$.

Hint. In Problem 8, set $\chi(t) = (t, \gamma(t), \dot{\gamma}(t), \ldots, \gamma^{(k-1)}(t))$.

Hint. In Problem 9(b), $\dot{\gamma}(t) = 1/\Psi'(\Psi^{-1}(t - t_0)) = F(\Psi^{-1}(t - t_0)) = F(\gamma(t))$.

Hint. In Problem 10, suppose that γ is different from a constant solution, i.e, x_0; thus, $\dot{\gamma} \neq 0$ everywhere in I. (Otherwise, with $\dot{\gamma}(t_1) = 0$ at some $t \in I$, the equation would yield $F(\gamma(t_1)) = 0$ and so γ would be the constant solution $\gamma(t_1)$ due to the uniqueness statement of Problem 6 for the initial condition $(t_1, \gamma(t_1))$.) Therefore $F(\gamma(t)) \neq 0$ for all $t \in I$ and so γ is a strictly monotone function on I valued in the interval (a, b) with a, b defined as in part (b) of Problem 9. Thus, γ has (one-sided) limits at the endpoints of I and the limit at each *finite* endpoint c of I must itself be infinite, or else it would provide an initial condition that would allow us to extend γ beyond c. We only need to show that $I = \Psi((a, b))$ with Ψ as (b) of Problem 9. According to Problem 9, $\Psi((a, b)) \subset I$. Since Ψ is strictly monotone on (a, b), its image $\Psi((a, b))$ is (c, d) or (d, c), where $c, d \in [-\infty, \infty]$ are the limits of Ψ at a, b. If we had $(c, d) \neq I$, for instance $c \in I$, then both c and the limit $\gamma(c)$ of γ at c would be finite, contradicting the previous conclusion.

79. Global solutions to linear differential equations

Topics: Uniform estimates for differential inequalities; global existence of solutions for linear ordinary differential equations.

Given an interval $I \subset \mathbf{R}$ containing more than one point and otherwise arbitrary (so that I may be open, closed, or half-open, bounded or unbounded), and a nonnegative continuous function $h : I \to [0, \infty)$, we set

$$\int_I h(t)\, dt = \sup_{a,b \in I} \int_a^b h(t)\, dt \in [0, \infty]. \tag{79.1}$$

Note that $\int_I h(t)\, dt$ equals the limit of $\int_a^b h(t)\, dt$ as $a \to \inf I(+)$ and, simultaneously, $b \to \sup I(-)$. (The limit always exists for reasons of monotonicity.)

The following results concerning *differential inequalities* will later be applied to linear ordinary differential equations and the local regularity theorem.

LEMMA 79.1. *Suppose that* $I \subset \mathbf{R}$ *is an interval,* $h : I \to [0,\infty)$ *is a continuous function, and* $\gamma : I \to V$ *is a* C^1 *mapping of* I *into a finite-dimensional real vector space* V *with an inner product* $\langle\,,\rangle$*. Furthermore, assume that*

$$C \,=\, \int_I h(t)\,dt \,<\, \infty\,, \tag{79.2}$$

and let

$$|\dot\gamma| \le h|\gamma| \tag{79.3}$$

everywhere in I*,* $|\;|$ *being the norm in* V *determined by* $\langle\,,\rangle$*. Then*

$$\sup_I |\gamma| \,\le\, e^C \inf_I |\gamma|\,. \tag{79.4}$$

REMARK 79.2. From (79.3) it is clear that, whenever $\gamma, h, I, V, \langle\,,\rangle$ satisfy the hypotheses of the lemma, then either $\gamma = 0$ identically, or $\gamma \ne 0$ everywhere in I. This fact, however, will have to be established separately in the course the following proof.

PROOF. We may assume that γ is not identically zero. By the Schwarz inequality and (79.3), $\varphi = \langle\gamma,\gamma\rangle : I \to [0,\infty)$ satisfies $|\dot\varphi| = 2|\langle\gamma,\dot\gamma\rangle| \le 2h\varphi$. Thus, if $a, b \in I$ and $\gamma \ne 0$ everywhere in the closed interval $\overline{ab}$ connecting a and b, we have

$$|\gamma(b)| \le |\gamma(a)| \cdot \exp\left[\int_{\overline{ab}} h(t)\,dt\right] \tag{79.5}$$

as $2\log|\gamma(b)| - 2\log|\gamma(a)| = \log\varphi(b) - \log\varphi(a) = \int_a^b \varphi^{-1}\dot\varphi\,dt \le 2\int_{\overline{ab}} h(t)\,dt$. Consequently, $\gamma \ne 0$ everywhere in I. In fact, otherwise, we could select a maximal open subinterval I' of I with $\gamma \ne 0$ everywhere in I', so that $\gamma(c) = 0$ for at least one endpoint $c \in I$ of I'; fixing $b \in I'$ and letting $a \in I'$ vary, we would obtain the contradiction $0 < |\gamma(b)| \le 0$ by taking the limit of (79.5) as $a \to c$ and noting that $\int_{\overline{cb}} h(t)\,dt < \infty$. Therefore, by (79.5), $|\gamma(b)| \le e^C|\gamma(a)|$ for all $a, b \in I$, with C as in (79.4), and we can take the supremum over b and infimum over a. $\square$

COROLLARY 79.3. *Let* $\gamma, h, I, V, \langle\,,\rangle$ *satisfy the hypotheses of* Lemma 79.1. *Then* γ *has a limit at each finite endpoint of* I*, while the endpoint itself does not have to belong to* I*.*

PROOF. By (79.3), (79.4) we have $\int_I |\dot\gamma(t)|\,dt \,\le\, Ce^C \inf_I |\gamma|$, and so we can use Problem 2. $\square$

Let U now be an open subset of a finite-dimensional real vector space V, and let $I \subset \mathbf{R}$ be an open interval. A kth order ordinary differential equation (78.1) in U, i.e.,

$$\gamma^{(k)} \,=\, F(t, \gamma, \dot\gamma, \dots, \gamma^{(k-1)})\,,$$

is called *linear* if its right-hand side

$$F : I \times U \times V^{k-1} \to V$$

has the form

$$F(t, x, w_1, \dots, w_{k-1}) \,=\, B_0(t)x + B_1(t)w_1 + \ldots + B_{k-1}(t)w_{k-1}$$

with some *coefficient functions* (curves) $B_0, B_1, \ldots, B_{k-1} : I \to \mathrm{Hom}(V, V)$ valued in the vector space of all linear operators $V \to V$. In other words, a linear kth order equation reads

$$\gamma^{(k)} \,=\, B_0(t)\gamma + B_1(t)\dot\gamma + \ldots + B_{k-1}(t)\gamma^{(k-1)}\,. \tag{79.6}$$

Note that, due to linearity of the $B_0(t), B_1(t), \ldots, B_{k-1}(t)$, we may always assume that $U = V$.

A linear equation (79.6) of any order $k > 1$ can always be reduced to a first-order linear equation $\dot\chi = A(t)\chi$ in a higher-dimensional space, with the coefficient curve $t \mapsto A(t)$ of the same regularity as the original $B_0, B_1, \ldots, B_{k-1}$ (Problem 6).

PROPOSITION 79.4. *Suppose that V is a finite-dimensional real vector space and $I \subset \mathbf{R}$ is an open interval. If $F : I \times V \to V$ is continuous and locally Lipschitz in $x \in U$, locally uniformly in $t \in I$* (Problem 6 in Appendix II above)*, and satisfies the inequality*

$$|F(t,x)| \le h(t)|x| \tag{79.7}$$

for all $(t,x) \in I \times V$, where $h : I \to [0,\infty)$ is a continuous function and $|\;|$ is a fixed norm in V, then for any $(t_0, x_0) \in I \times V$ the initial value problem

$$\dot\gamma = F(t,\gamma)\,, \qquad \gamma(t_0) = x_0 \tag{79.8}$$

has a unique solution $\gamma : I \to V$ defined everywhere *in I.*

PROOF. We may assume that $|\;|$ is the norm determined by an inner product $\langle\,,\rangle$ in V (Problem 18 in §5). Let $\gamma : (a,b) \to V$ be the (unique) solution to (79.8) defined on the largest possible interval $(a,b) \subset I$ with $t_0 \in (a,b)$ (Problem 5). To show that $(a,b) = I$, suppose on the contrary that, for instance, $b \in I$. Applying Corollary 79.3 to $[t_0, b]$ instead of I, we see that $\gamma(t)$ has a limit y_0 as $t \to b(-)$, and so from the existence theorem (see, e.g., Problem 6 in the preceding appendix), there is $\varepsilon > 0$ with $b+\varepsilon \in I$ and a C^1 curve $\gamma_1 : [b, b+\varepsilon) \to V$ with $\dot\gamma_1 = F(t,\gamma_1)$ and $\gamma_1(b) = y_0$. Combining γ with γ_1 as in Problem 1, we obtain a C^1 solution to (79.8) defined on $(a, b+\varepsilon)$, which contradicts maximality of (a,b) and thus completes the proof. □

We can now prove a *global existence theorem for linear ordinary differential equations*

THEOREM 79.5. *Every linear initial value problem*

$$\begin{aligned} \gamma^{(k)} \,&=\, B_0(t)\gamma + B_1(t)\dot\gamma + \ldots + B_{k-1}(t)\gamma^{(k-1)}\,,\\ \gamma(t_0) \,&=\, x_0\,,\, \dot\gamma(t_0) = v_1\,,\, \ldots\,,\, \gamma^{(k-1)}(t_0) = v_{k-1} \end{aligned} \tag{79.9}$$

of order $k \ge 1$ in a finite-dimensional real vector space V, with continuous coefficient functions B_0, $B_1, \ldots, B_{k-1} : I \to \mathrm{Hom}(V,V)$, where $I \subset \mathbf{R}$ is an open interval, has a unique solution $\gamma : I \to V$ defined on the whole *interval I.*

PROOF. Fix a norm $|\;|$ in V. We may assume that $k = 1$ (Problem 6), so that (79.9) becomes $\dot\gamma = B(t)\gamma$ with $\gamma(t_0) = x_0$. Thus, (79.7) is satisfied by $F(t,x) = B(t)x$ and $h(t) = |B(t)|$ (the operator norm; see Appendix II above), and $h : I \to [0,\infty)$ is continuous according to Problem 18 or Problem 20 in §5). The assertion is now immediate from Proposition 79.4. □

Problems

1. Let a, t_0, b be real numbers with $a < t_0 < b$ and let $\gamma_1 : (a, t_0] \to U$, $\gamma_2 : [t_0, b) \to U$, be C^1 solutions to the first-order initial value problem

$$\dot{\gamma} = F(t, \gamma)\,, \qquad \gamma(t_0) = x_0 \tag{79.10}$$

in an open subset U of a finite-dimensional real vector space V. Verify that the curve $\gamma : (a, b) \to U$ with

$$\gamma = \begin{cases} \gamma_1 & \text{on } (a, t_0] \\ \gamma_2 & \text{on } [t_0, b) \end{cases}$$

then is a C^1 solution to (79.10).

2. Let $\gamma : (a, b) \to V$, $-\infty \le a < b \le \infty$, be a C^1 curve in a finite-dimensional real vector space V with a fixed norm $|\ |$, such that $\int_a^b |\dot{\gamma}(t)|\, dt < \infty$ (notation of (79.1)). Prove that γ then has one-sided limits at a and b. (Hint below.)

3. Verify that for any C^1 curve $\gamma : (a, b) \to V$ in a finite-dimensional real vector space V with a norm $|\ |$, such that $-\infty < a < b < \infty$ and $\sup\{|\dot{\gamma}(t)| : t \in (a, b)\} < \infty$, there must exist one-sided limits of $\gamma(t)$ as $t \to a$ and $t \to b$.

4. Solve the initial value problem $\dot{\gamma} = \gamma$, $\gamma(0) = 1$ for $\gamma : (-\varepsilon, \varepsilon) \to \mathbf{R}$ with an appropriate $\varepsilon > 0$ by retracing the steps used in the proofs of the existence and uniqueness theorem (Appendix II above) and Banach's fixed-point theorem (§73), i.e., constructing the approximating sequence $\gamma_k = h_\varepsilon^k(z)$, choosing $z = \gamma_0$ to be the constant function 1. (Hint below.)

5. For F, I, U, t_0, x_0 satisfying the hypotheses of Problem 6 in Appendix II above, verify that there exists the *largest* open interval $(a, b) \subset I$ with $t_0 \in (a, b)$ such that the initial value problem $\dot{\gamma} = F(t, \gamma)$, $\gamma(t_0) = x_0$ in U has a solution $\gamma : (a, b) \to U$, and that this solution is unique. (Hint below.)

6. Let the coefficient curves $B_0, B_1, \ldots, B_{k-1}$ of a kth order linear equation (79.9) be all C^l-differentiable, $l = 0, 1, 2, \ldots, \infty$. Verify that the standard order-reduction procedure (Appendix II above) then transforms (79.9) into a *linear* first-order equation $\dot{\chi} = A(t)\chi$ in the higher-dimensional space V^k, with $t \mapsto A(t)$ of class C^l.

7. Given a finite-dimensional real or complex vector space V and a linear mapping $A \in \mathrm{Hom}(V, V)$, define $e^A \in \mathrm{Hom}(V, V)$ by $e^A = \Gamma(1)$, where $\Gamma : \mathbf{R} \to \mathrm{Hom}(V, V)$ is the unique (global) solution to the linear initial value problem

$$\dot{\Gamma}(t) = A\Gamma(t)\,, \qquad \Gamma(0) = \mathrm{Id}\,. \tag{79.11}$$

The assignment $A \to e^A$ is called the *exponential mapping*. Show that $\Gamma(t) = e^{tA}$ for all $t \in \mathbf{R}$, and so (Hint below.)

$$\frac{d}{dt}\, e^{tA} = Ae^{tA}\,. \tag{79.12}$$

8. For V, A as in Problem 7 and any $s, t \in \mathbf{R}$, verify that (Hint below.)

$$e^{(t+s)A} = e^{tA} e^{sA}\,. \tag{79.13}$$

9. For any linear operator $A \in \mathrm{Hom}(V, V)$ of a finite-dimensional real or complex vector space V, show that $e^A : V \to V$ is a linear isomorphism and $(e^A)^{-1} = e^{-A}$.

10. For V, A as in Problem 7, prove that e^A commutes with A, and that the exponential mapping $A \to e^A = \Gamma(1)$ could also be defined using the initial value problem $\dot{\Gamma}(t) = \Gamma(t)A$, $\Gamma(0) = \mathrm{Id}$ instead of (79.11). (Hint below.)

11. A kth order linear differential equation (79.6) in a finite-dimensional real or complex vector space V is said to have *constant coefficients* if its coefficient functions $B_0, B_1, \dots, B_{k-1} : I \to \mathrm{Hom}(V, V)$ are all constant. Verify that the order-reduction procedure (Appendix II above) then leads to a first-order linear equation with constant coefficients.

12. Show that the unique solution $\gamma : \mathbf{R} \to V$ to a first-order linear equation $\dot{\gamma}(t) = A\gamma(t)$ with constant coefficients in a finite-dimensional real or complex vector space V (so that $A \in \mathrm{Hom}(V, V)$ is independent of $t \in \mathbf{R}$), satisfying the initial condition $\gamma(t_0) = x_0 \in V$, is given by

$$\gamma(t) = e^{(t-t_0)A} x_0 \, .$$

13. Prove that, for V, A as in Problem 7,

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \, , \tag{79.14}$$

and derive the conclusions of Problems 7 through 11 and 13 from formula (79.14) treated as the *definition* of e^A, without using any theorems on differential equations. Also, verify that the exponential mapping $A \to e^A$ is continuous. (Hint below.)

14. Generalize Corollary 79.3 replacing (79.3), (79.2) with

$$|\dot{\gamma}| \,\le\, h|\gamma| \,+\, f$$

and

$$C' \,=\, \int_I \sqrt{[h(t)]^2 + [f(t)]^2} \, dt \,<\, \infty \, .$$

with an additional continuous function $f : I \to [0, \infty)$. (Hint below.)

15. Generalize Proposition 79.4 replacing (79.7) with

$$|F(t, x)| \,\le\, h(t)|x| \,+\, f(t) \, ,$$

where $f : I \to [0, \infty)$ is an additional continuous function. (Hint below.)

16. Generalize the Global Existence Theorem to the case of *nonhomogeneous* linear ordinary differential equations, with the equation in (79.9) replaced by

$$\gamma^{(k)} \,=\, u(t) \,+\, B_0(t)\gamma + B_1(t)\dot{\gamma} + \ldots + B_{k-1}(t)\gamma^{(k-1)} \, ,$$

with an additional continuous function $u : I \to V$. (Hint below.)

Hint. In Problem 2, set $F(c, d) = \int_c^d |\dot{\gamma}(t)| \, dt$ for any $c, d \in (a, b)$, so that $|\gamma(d) - \gamma(c)| \le |F(c, d)| \to 0$ as c, d simultaneously approach either a or b. Thus, whenever a sequence t_k in (a, b) converges to a or b, the values $\gamma(t_k)$ form a Cauchy sequence.

Hint. In Problem 4, condition (78.3) amounts to $0 < \varepsilon < 1$, and $\gamma_k(t) = \sum_{j=0}^{k} t^k/k!$ is the kth partial sum of the exponential series for $\gamma(t) = e^t$.

Hint. In Problem 5, let (a, b) be the union of all $(a', b') \subset I$ with $t_0 \in (a', b')$ on which a solution exists, and then use the uniqueness statement of Problem 6 in Appendix II above.

Hint. In Problem 7, fix $\lambda \in \mathbf{R}$ and note that Γ_λ with $\Gamma_\lambda(t) = \Gamma(\lambda t)$ satisfies $\dot{\Gamma}_\lambda = \lambda A \Gamma_\lambda$, $\Gamma_\lambda(0) = \mathrm{Id}$, so $e^{\lambda A} = \Gamma_\lambda(1) = \Gamma(\lambda)$.

Hint. In Problem 8, use (79.12) to observe that the curves $t \mapsto e^{(t+s)A}$, $t \mapsto e^{tA}e^{sA}$ are solutions to the same initial value problem, and so they must coincide.
Hint. In Problem 9, note that the curve $t \mapsto B(t) = Ae^{tA} - e^{tA}A$ satisfies $\dot{B}(t) = AB(t)$ with $B(0) = 0$ and hence $B(t) = 0$ for all t due to uniqueness of solutions.
Hint. In Problem 13, fix a norm $|\ |$ in V and note that (79.14), as a series of functions of $A \in \mathrm{Hom}(V, V)$, converges absolutely and uniformly on each bounded subset K of $\mathrm{Hom}(V, V)$ due to the operator-norm estimate $|A^k| \le |A|^k$ (Problem 14 in §8), which shows that the partial sums of (79.14) restricted to $A \in K$ form a $\mathrm{d}_{\mathrm{sup}}$-Cauchy sequence of continuous functions of A, and we may use Problems 2 – 3 in Appendix II above to establish convergence and continuity. Therefore, the same applies to the derivative series

$$\sum_{k=0}^{\infty} \frac{d}{dt}\left(\frac{1}{k!}t^k A^k\right),$$

where A is fixed and t varies in an interval of the form $[-t_0, t_0]$. We now derive both (79.12) and its "mirror version" in Problem 10 using integration over t. The statement of Problem 9 can be obtained by differentiating $e^{tA}e^{-tA}$ with respect to t, and so the solutions Γ to (79.11) and (79.13) to $\dot{\gamma}(t) = A\gamma(t)$, $\gamma(t_0) = x_0$ are both unique since by differentiating $e^{-tA}\Gamma(t)$ or $e^{(t_0-t)A}\gamma(t)$ we see that it must be constant. As for the assertion of Problem 8, it can be obtained either from the previous uniqueness conclusion applied to $t \mapsto e^{-sA}e^{(t+s)A}$, or from a standard series multiplication argument.
Hint. In Problem 14, use the new C^1 mapping $\Gamma : I \to W$ into the direct-sum vector space $W = V \times \mathbf{R}$ with the direct-sum inner product (involving the standard inner product in $\mathbf{R}$), given by

$$\Gamma(t) = (\gamma(t), 1). \tag{79.15}$$

By the Schwarz inequality and (79.3), $|\dot{\Gamma}| = |\dot{\gamma}| \le h|\gamma| + f \le \sqrt{h^2 + f^2} \cdot \sqrt{|\gamma|^2 + 1} = H|\Gamma|$, with $H = \sqrt{h^2 + f^2}$, so that we can apply Corollary 79.3 to the primed data.
Hint. In Problem 15, proceed as in the original proof of Proposition 79.4, replacing Corollary 79.3 by its generalized version (Problem 14).
Hint. In Problem 16, proceed as in the original proof of the global existence theorem, replacing Proposition 79.4 by its generalization given in Problem 15, or simply note that (79.15) then defines a solution to a *homogeneous* linear differential equation in the new vector space $W = V \times \mathbf{R}$ and apply to it the original theorem.

80. Differential equations with parameters

Topics: Ordinary differential equations with parameters; the regularity theorems; initial conditions as parameters.

Let $U \subset V$, $U' \subset V'$ be open subsets of finite-dimensional real vector spaces V and V'. A kth order ordinary differential equation in U is said to *depend on the parameter* $\xi \in U'$ if it has the form (cf. (78.1))

$$\gamma^{(k)} = F(\xi, t, \gamma, \dot{\gamma}, \ldots, \gamma^{(k-1)}) \tag{80.1}$$

with the right-hand side $F : U' \times I \times U \times V^{k-1} \to V$. Together with an initial condition (78.2), i.e., $\gamma(t_0) = x_0$, $\dot{\gamma}(t_0) = v_1, \ldots, \gamma^{(k-1)}(t_0) = v_{k-1}$, where

$(t_0, x_0, v_1, \dots, v_{k-1}) \in I \times U \times V^{k-1}$, (80.1) then is referred to as a *kth order initial value problem with a parameter.*

Lemma 80.1. *Let a C^1 curve $\gamma : I \to V$ in a finite-dimensional real vector space V be defined on an interval $I \subset \mathbf{R}$ of length $\Lambda < \infty$ and satisfy the estimate*

$$|\dot\gamma| \,\le\, p|\gamma| \,+\, q \tag{80.2}$$

with some constants $p > 0$ and $q \ge 0$, $|\ |$ being a fixed Euclidean norm in V. If $\gamma(t) = 0$ for some $t \in I$, then

$$|\gamma| \,\le\, \frac{q}{p}\,(e^{p\Lambda} \,-\, 1) \tag{80.3}$$

everywhere in I.

Proof. Let $[a, b]$ be any closed subinterval of I on which the function $f = |\gamma|$ is positive. From (80.2) and the Schwarz inequality, $|\dot f| \le pf + q$ on $[a, b]$. Since $f + q/p > 0$ on $[a, b]$, rewriting the last inequality as $|(f + q/p)^{\cdot}| \le p(f + q/p)$, then dividing it by $f + q/p$ and, finally, integrating over $[a, b]$ and using Problem 18 in §74 (Appendix B), we obtain $|\log[f(b) + q/p] - \log[f(a) + q/p]| \le p(b - a) \le p\Lambda$. Thus,

$$e^{-p\Lambda} \,\le\, \frac{f(b) \,+\, q/p}{f(a) \,+\, q/p} \,\le\, e^{p\Lambda}\,. \tag{80.4}$$

Let us denote (c, d) the maximal open subinterval of I containing $[a, b]$ and such that $\gamma \ne 0$ everywhere on (c, d). Since $\gamma = 0$ somewhere in I, at least one of c, d must be an element of I at which $\gamma = 0$. Now, if $c \in I$ and $f(c) = 0$ (or, $d \in I$ and $f(d) = 0$), taking the limit of (80.4) as $a \to c(+)$ (or, as $b \to d(-)$) and writing $t = b$ (or, $t = a$), we obtain $f(t) \le qp^{-1}(e^{p\Lambda} - 1)$. Thus, (80.3) holds wherever $\gamma \ne 0$ and, consequently, everywhere in I. This completes the proof. □

Lemma 80.2. *Given finite-dimensional real vector spaces V, V' and open subsets $U \subset V$, $U' \subset V'$, let*

$$\dot\gamma \,=\, F(\xi, t, \gamma)\,, \qquad \gamma(t_0) \,=\, x_0\,. \tag{80.5}$$

be the first-order initial value problem in U with a parameter $\xi \in U'$ obtained by fixing an open interval $I \subset \mathbf{R}$, a mapping $F : U' \times I \times U \to V$, and a pair $(t_0, x_0) \in I \times U$. If F is of class C^{l+1} on $U' \times I \times U$, then, for any $\xi_0 \in U'$ there exist a neighborhood U'' of ξ_0 in U' and a real number $\varepsilon > 0$ such that for every $\xi \in U''$ a solution $\gamma = \gamma_\xi$ to (80.5) can be defined on the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ of the variable t and the mapping $(t_0 - \varepsilon, t_0 + \varepsilon) \times U'' \ni (t, \xi) \mapsto \gamma_\xi(t) \in U$ is of class C^l.

Proof. We use induction on $l \ge 0$. The mapping

The induction step: Suppose that our assertion is true for a given $l \ge 0$, and let F be of class C^{l+2}. Let us now set $m = \dim V'$ and consider the initial value problem

$$\begin{aligned} &\dot\gamma \,=\, F(\xi, t, \gamma)\,, \qquad \dot\gamma_\lambda \,=\, \frac{\partial F}{\partial \xi^\lambda}(\xi, t, \gamma) \,+\, \gamma^a_\lambda\,\frac{\partial F}{\partial \gamma^a}(\xi, t, \gamma)\,, \\ &\gamma(t_0) \,=\, x_0\,, \qquad \gamma_\lambda(t_0) \,=\, 0\,, \qquad \lambda = 1, \dots, m\,, \end{aligned} \tag{80.6}$$

imposed on a curve $t \mapsto (\gamma(t), \gamma_1(t), \dots, \gamma_m(t))$ valued in $U \times V^m$ (where V^m is the mth Cartesian power of V), with γ^a and γ^a_λ, $\lambda = 1, \dots, m$, standing for the ath component of the curve γ (or γ_λ) relative to some fixed basis of V. The right-hand side of (80.6) thus is of class C^{l+1} in (ξ, t, γ). $\square$

We now obtain what is known as the *regularity theorem.*

THEOREM 80.3. *Let* (80.1) *be a system of ordinary differential equations with a parameter, of any order* $k \geq 1$, *whose right-hand side is of class* C^{l+1} *in all quantities involved,* $l = 0, 1, 2, \ldots, \infty$. *The solutions to (80.1) with any given initial conditions then depend* C^l*-differentiably on the parameter, the independent variable, and the initial conditions, i.e., on the* $(k+3)$*-tuple* $(\xi, t, t_0, x_0, v_1, \ldots, v_{(k-1)}) \in U' \times I^2 \times U \times V^{k-1}$, *wherever they are defined, while the subset of* $U' \times I^2 \times U \times V^{k-1}$ *on which they are defined is open.*

PROOF. Using any fixed initial condition (78.2), and proceeding as in the paragraph preceding the Existence and Uniqueness Theorem (§4, Appendix II), we may rewrite (80.1) as a first-order initial value problem with parameters. In other words, we may assume that $k = 1$, and so our initial value problem has the form

$$\dot{\gamma} = F(\xi, t, \gamma), \qquad \gamma(t_0) = x_0. \tag{80.7}$$

Denoting $t \mapsto \gamma_\xi(t)$ the unique solution to (80.7) for any given value of the parameter(s) ξ, let us now "pretend" that the assertion we are trying to prove holds in this particular case. Applying to (80.7) the partial derivatives of all orders up to l, relative to the components ξ^λ of ξ, using the chain rule, and setting

$$\gamma_{\lambda_1 \ldots \lambda_s} = \frac{\partial^s \gamma}{\partial \xi^{\lambda_1} \ldots \partial \xi^{\lambda_s}}$$

for any $s \in \{1, \ldots, l\}$, we thus get a system consisting of

$$\dot{\gamma} = F(\xi, t, \gamma), \qquad \dot{\gamma}_\lambda = \frac{\partial F}{\partial \xi^\lambda}(\xi, t, \gamma) + \gamma^a_\lambda \frac{\partial F}{\partial \gamma^a}(\xi, t, \gamma),$$

then,

$$\ddot{\gamma}_{\lambda\mu} = \frac{\partial^2 F}{\partial \xi^\mu \partial \xi^\lambda} + \gamma^a_\mu \frac{\partial^2 F}{\partial \gamma^a \partial \xi^\lambda} + \gamma^a_{\mu\lambda} \frac{\partial F}{\partial \gamma^a} + \gamma^a_\lambda \gamma^b_\mu \frac{\partial^2 F}{\partial \gamma^a \partial \gamma^b},$$

(with (ξ, t, γ) omitted for brevity), etc., the right-hand sides of which involve partial derivatives of F up to order l. These come along with the initial conditions

$$\gamma(t_0) = x_0 \qquad \text{and} \qquad \gamma_{\lambda_1 \ldots \lambda_s}(t_0) = 0 \qquad \text{for all} \quad s \in \{1, \ldots, l\}.$$

This new system with the parameter ξ has the unknown functions γ and $\gamma_{\lambda_1 \ldots \lambda_s}$, for all $s = 1, \ldots, l$, and its right-hand side depends continuously on (ξ, t, γ). By Lemma 80.2,

......

$\square$

Appendix D. Some More Differential Geometry

81. Grassmann manifolds

Let V be a n-dimensional real/complex vector space, $1 \le n < \infty$. For any integer q with $0 \le q \le n$ we denote by $\mathrm{Gr}_q(V)$ the set of all q-dimensional real/complex vector subspaces of V.

Let $d \in \{1,2\}$ now be the dimension of the scalar field $\mathbf{K}$ over $\mathbf{R}$ (so that the *real* dimension of V is dn). Denote by e_a, $a = 1,\ldots,q$ the standard basis of $\mathbf{K}^q$ with $e_a = (\delta_a^1,\ldots,\delta_a^q)$. For every surjective linear operator $f : V \to \mathbf{K}^q$, set $A_f = f^{-1}(e_1) \times \ldots \times f^{-1}(e_q)$ (this is an affine space whose the translation space is the direct sum of q copies of $\mathrm{Ker}\, f$, cf. Problem 21). Also, setting

$$U_f \;=\; \{W \in \mathrm{Gr}_q(V) : f(W) = \mathbf{K}^q\},$$

we define $\varphi_f : U_f \to A_f$ to be the mapping sending each W onto $(w_1,\ldots,w_k)$, where w_a is the unique intersection point of W with $f^{-1}(e_a)$. The family

$$\mathcal{A} = \{(U_f,\varphi_f) : f \text{ is a } \mathbf{K}\text{-linear mapping of } V \text{ onto } \mathbf{K}^q\} \tag{81.1}$$

then is an atlas on $\mathrm{Gr}_q(V)$ (involving affine model spaces, as in §2), making $\mathrm{Gr}_q(V)$ a C^ω manifold of dimension $qd(n-q)$, known as the *Grassmann manifold* (or *Grassmannian*) of q-planes in V. See Problem 22.

Problems

1. Show that the Cartesian product of a finite collection of affine spaces is naturally an affine space, whose translation vector space is the direct sum of those of the factors.

2. For V, n, q, $\mathrm{Gr}_q(V), \mathbf{K}$ and d as above, prove that (81.1) is a Hausdorff atlas on $\mathrm{Gr}_q(V)$ and that it makes $\mathrm{Gr}_q(V)$ the underlying set of a C^ω manifold of dimension $qd(n-q)$. (Hint below.)

3. Let V, V' be finite-dimensional real or complex vector spaces. Verify that, for any linear isomorphism $F : V \to V'$, the assignment $W \mapsto F(W)$ is a C^ω diffeomorphism $\mathrm{Gr}_q(V) \to \mathrm{Gr}_q(V')$. State and prove a similar result in the case where $F : V \to V'$ is merely linear and injective.

4. For V, n, q as in Problem 2, verify that
  (a) if $q = 1$, we have $\mathrm{Gr}_q(V) = P(V)$ and the above atlas coincides with the one described in §2,
  (b) setting $F(W) = \{h \in V^* : h = 0 \text{ on } W\}$, we obtain a C^ω diffeomorphism $F : \mathrm{Gr}_q(V) \to \mathrm{Gr}_{n-q}(V^*)$.

5. For V, n, q as in Problem 2, prove that the Grassmann manifold $\mathrm{Gr}_q(V)$ is compact. (Hint below.)

Hint. In Problem 2, let $(U_f,\varphi_f), U_h,\varphi_h) \in \mathcal{A}$, and let Z be the vector space of all $q \times q$ matrices over $\mathbf{K}$. A C^ω mapping $F : A_f \to Z$ can be defined by

letting $F(w_1,\dots,w_q)$ be the matrix with column vectors $h(w_1),\dots,h(w_q)$. Now $\varphi_f(U_f\cap U_h)\subset A_f$ is given by $\det\circ F\neq 0$ and so it is open in A_f. Compatibility follows since $(\varphi_h\circ\varphi_f^{-1})(w_1,\dots,w_k)=C_1^a w_a,\dots,C_q^a w_a)$, where $C\in Z$ is given by $C=[F(w_1,\dots,w_q)]^{-1}$ (matrix inverse). The Hausdorff axiom is clear as any two points lie in a common chart of our atlas.
Hint. In Problem 5, fix a Euclidean/Hermitian inner product in V and apply Problems 4(a),5 in §3 to the mapping $\mathrm{Span}:K\to\mathrm{Gr}_q(V)$, where K is the subset of the direct sum of q copies of V formed by all q-tuples of orthonormal vectors.

82. Affine bundles

As in the case of vector bundles, we can define a real or complex *affine bundle* ζ over a set B to be a family $B\ni x\mapsto\zeta_x$, parametrized by B, of real or complex affine spaces ζ_x of some finite dimension q, independent of x. The notions of *fibre dimension* (or *rank*), *base*, *fibre*, *section* (with a *domain* $K\subset B$) now can be introduced by repeating the corresponding definitions for vector bundles. Every affine bundle ζ over B gives rise to a vector bundle η over B (its *associated vector bundle*), the fibre η_x of which at any $x\in B$ is the translation vector space of ζ_x. We can thus add sections of η to those of ζ (if their domains agree), obtaining sections of ζ.

Let ζ be an affine bundle over a set B, of some fibre dimension q, and let η be its associated vector bundle. A *trivialization of* ζ *over a set* $K\subset B$ then consists of a section o of ζ, defined on K, and a trivialization $e_1,\dots,e_q$ of η over K. Such a trivialization will be written o, e_a, where a varies in the fixed range $\{1,\dots,q\}$. For $x\in K$ and $\xi\in\zeta_x$, we define the *components* ξ^a of ξ relative to the trivialization o,e_a over K to be the scalars characterized by $\xi = o(x)+\xi^a e_a(x)$. Similarly, the scalar-valued *component functions* $\chi^a:K\to\mathbf{R}$ or $\chi^a:K\to\mathbf{C}$ of any section ψ of ζ over K then are given by $\chi^a(x)=[\chi(x)]^a$. Thus, $\chi = o+\chi^a e_a$. Another such trivialization (over a set $K'\subset B$) will be denoted by $o', e_{a'}$; it leads to the scalar-valued *transition functions* $c^a, e^a_{a'}$ on $K\cap K'$ given by $o' = o + c^a e_a$, $e_{a'}=e^a_{a'}e_a$.

Consider now a affine bundle ζ whose base set, denoted by M, carries a fixed structure of a C^r *manifold*, $r\geq 1$. By a *local section* (*trivialization*) of ζ we then mean a section χ (or, trivialization o, e_a) whose domain is an *open* set $U\subset M$. (When $U=M$, the section or trivialization is called *global*.) Two local trivializations o, e_a and o', $e_{a'}$ with domains U,U' are called C^s *compatible* $(0\leq s\leq r)$ if the scalar-valued transition functions c_a and $e^a_{a'}$ on $U\cap U'$ are all of class C^s. Compatibility is, again, a symmetric relation. A C^s *atlas* $\mathcal{B}$ for ζ is a collection of local trivializations which are pairwise C^s compatible and whose domains cover M. Such an atlas is said to be *maximal* if it is not contained in any other C^s atlas. Every C^s atlas $\mathcal{B}$ for ζ is contained in a unique maximal C^s atlas $\mathcal{B}_{\max}$, formed by all local trivializations of ζ that are C^s compatible with each of the local trivializations constituting $\mathcal{B}$.

We define a C^s *affine bundle* over a C^r manifold M $(0\leq s\leq r)$ to be any affine bundle ζ over M endowed with a fixed maximal C^s atlas $\mathcal{B}_{\max}$. Note that, to describe a C^s affine bundle ζ over M, it suffices to provide just one C^s atlas $\mathcal{B}$ contained in its maximal C^s atlas $\mathcal{B}_{\max}$.

A local section χ of a C^s affine bundle ζ is said to be *of class* C^l, $0\leq l\leq s$, if its components χ^a relative to all local trivializations o, e_a forming the maximal atlas $\mathcal{B}_{\max}$ of η, are C^l functions. This is a local geometric property (§2): to verify that χ is C^l, we only need to use, instead of $\mathcal{B}_{\max}$, just any C^s atlas $\mathcal{B}$ contained in $\mathcal{B}_{\max}$. Cf. Problem 11 in §20.

Problems

1. In notations as above, verify the transformation rule

$$\chi^{a'} = e_a^{a'}(\chi^a - c^a)$$

for component functions of sections of affine bundles under a change of a trivialization.

83. Abundance of cut-off functions

Functions on manifolds that are C^∞-differentiable and vanish on a nonempty open set, but not identically, are used for a variety of purposes. The following problems establish the existence of a large supply of such functions.

Problems

13. Let $f : (a,b) \to \mathbf{R}$ be a differentiable function, $-\infty < a < b \le \infty$, such that $L = \lim_{x\to a(+)} f'(x)$ exists and is finite. Show that f has a differentiable extension to $[a,b)$, also denoted by f, with $f'(a) = L$.

14. Define $f : (0,\infty) \to \mathbf{R}$ by $f(x) = e^{-1/x}$. Verify that, for each integer $k \ge 1$, the kth derivative $f^{(k)}(x)$ is a finite combination, with constant coefficients, of terms having the form $x^{-s}e^{-1/x}$, where the s are positive integers.

15. Prove that the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = e^{-1/x}$ for $x > 0$ and $f(x) = 0$ for $x \le 0$ is of class C^∞.

16. For any real numbers a, b with $a < b$, show that there is a C^∞ function $\chi : \mathbf{R} \to \mathbf{R}$ with $\chi = 0$ on $(-\infty, a]$, $\chi > 0$ on (a,b), and $\chi = 0$ on $[b,\infty)$.

17. Given real numbers a, b with $a < b$, prove the existence of a C^∞ function $\phi : \mathbf{R} \to \mathbf{R}$ with $\phi = 0$ on $(-\infty, a]$, $0 < \phi < 1$ on (a,b), and $\phi = 1$ on $[b,\infty)$.

18. For real numbers a, b with $0 < a < b$ and an integer $n \ge 1$, show that there exists a C^∞ function $\phi : \mathbf{R}^n \to [0,1]$ such that $\phi(x) = 1$ if $|x| \le a$ and $\phi = 0$ whenever $|x| \ge b$.

19. *Existence of cut-off functions.* Given a closed subset K of a C^r manifold M, $0 \le r \le \infty$, and a point $x \in M \setminus K$, prove the existence of a C^r function $\phi : M \to \mathbf{R}$ with $0 \le \phi \le 1$ on M, $\phi = 1$ in a neighborhood of x, and $\phi = 0$ in some open set containing K.

20. *Global extensibility of germs.* For a point x in a C^r manifold M, $0 \le r \le \infty$, and any $l = 0, 1, 2, \ldots, r$, show that each germ of a C^l function at x, is realized by a C^l function defined on the whole of M.

84. Partitions of unity

Topics: Locally finite open coverings; locally finite partitions of unity; existence theorems.

LEMMA 84.1. *Every manifold M is the union $M = \bigcup_{j=1}^\infty U_j$ of open sets U_j such that each U_j has a compact closure contained in U_{j+1}.*

in preparation

PROOF. ☐

A family $\mathcal{S}$ of subsets of a manifold M is called *locally finite* if every point of M has a neighborhood intersecting only finitely manu sets from $\mathcal{S}$.

We can now establish an important consequence of the countability axiom (§14): the condition that, according to the following theorem, is satisfied by every manifold, is known as *paracompactness*.

THEOREM 84.2. *For every open covering $\mathcal{U}$ of a manifold M there exists a locally finite open covering of M subordinate to $\mathcal{U}$.*

PROOF. Let us choose the U_j as in Lemma 84.1, and let Y_j be their (compact) closures. In view of the Borel-Heine theorem, for each $j \geq 1$ some finite subfamily $\mathcal{U}_j$ of $\mathcal{U}$ covers Y_j, and so does the finite family $\mathcal{U}'_j$ of open sets given by $\mathcal{U}'_j = \{U \setminus Y_{j-1} : U \in \mathcal{U}_j\}$ (where we have set $Y_0 = \emptyset$). The union $\mathcal{U}' = \bigcup_{j=1}^{\infty} \mathcal{U}'_j$ clearly is an open covering of M subordinate to $\mathcal{U}$. To see that it is locally finite, let $x \in M$ and let us fix a neighborhood of x having a compact closure Y. The Borel-Heine theorem applied to Y and the sets U_j shows that $Y \subset U_j$ for some j, so that the only sets in the family $\mathcal{U}'$ that might possibly intersect Y are those belonging to $\mathcal{U}_k$ for $k \in \{1, \ldots, j\}$. The latter sets are finite in number, as required. ☐

By a *locally finite partition of unity* we mean

in preparation

We say that a locally finite partition of unity is *subordinate* to an open covering $\mathcal{U}$ if

in preparation

COROLLARY 84.3. *For every open covering $\mathcal{U}$ of a manifold M, there exists a locally finite partition of unity subordinate to $\mathcal{U}$.*

in preparation

PROOF. ☐

Problems

1. Using Corollary 84.3, generalize Problem 3 in §36 to the case where, instead of compactness of K', one just assumes both sets K, K' to be closed.

2. Show that every C^∞ real/complex vector bundle over a C^∞ manifold satisfying the countability axiom admits a C^∞ (positive-definite) Riemannian/Hermitian metric g and a connection ∇ compatible with g. (Hint below.)

3. Show that a positive C^∞ density exists on every manifold. (Hint below.)

4. Prove that every manifold admits an atlas that has *unimodular transition mapping* in the sense that $\det[p^j_{j'}] = 1$ for any two charts of this atlas. (Hint below.)

Hint. In Problem 1, apply Corollary 84.3 to the open covering of M formed by the two sets $M \setminus K$ and $M \setminus K'$, then select those functions from your locally finite partition of unity whose supports are contained in $M \setminus K'$. The sum f of the selected functions is still equal to 1 on K, but is 0 on K'. We may now replace f with $\tau \circ f$, for a C^∞ function $\tau : [0,1] \to [0,1]$ equal to 0 near 0 and to 1 near 1.
Hint. In Problem 2,
Hint. In Problem 3, use a locally finite partition of unity.
Hint. In Problem 4, use Problem 3 to select a positive C^∞ density on the given manifold, and then apply Problem 2 in §37 with $\varphi = 1$.

85. Flows of vector fields

Topics: Integral curves and the flow of a vector field; the homomorphic property.

Suppose that w is a fixed C^l vector field on a manifold M, $1 \le l \le \infty$.

A C^1 curve $\gamma : I \to M$ defined on an interval $I \subset \mathbf{R}$ is called an *integral curve* of w if $\dot\gamma(t) = w(\gamma(t))$ for each $t \in I$. If, in addition, $t \in I$ is fixed and $\gamma(t) = z \in M$, we say that γ is a *solution to the initial value problem*

$$\dot\gamma = w(\gamma)\,, \qquad \gamma(t) = z \tag{85.1}$$

in M. In any local coordinates x^j at z, (85.1) is clearly equivalent to the initial value problem

$$\dot\gamma^j = \Psi^j(\gamma^1,\dots,\gamma^n)\,, \qquad \gamma^j(t) = z^j\,, \quad j = 1,\dots,n = \dim M$$

in an open subset of $\mathbf{R}^n$, with the unknowns $\gamma^j = x^j \circ \gamma$, where Ψ^j are the functions of n variables characterized by $w^j = \Psi^j(x^1,\dots,x^n)$, and $z^j = x^j(z)$. Therefore, according to the the existence and uniqueness theorem on p. 32, for each $t \in \mathbf{R}$ and $z \in M$, (85.1) has a unique C^1 solution $\gamma : I \to M$ defined on some open interval I containing t. Furthermore, there is always a largest possible open interval $I = I_{\max}$ with this property, and a solution γ is also unique on the whole of $I_{\max}$ (Problem 3).

For each $x \in M$, let $I_x \subset \mathbf{R}$ with $-\infty \le \inf I_x < 0 < \sup I_x \le \infty$ be the largest interval on which an integral curve $\gamma_x : I_x \to M$ of w with $\gamma_x(0) = x$ can be defined. Denote Y_w the subset of $M \times \mathbf{R}$ given by $Y_w = \{(x,t) \in M \times \mathbf{R} : t \in I_x\}$, and define the *flow* of w to be the mapping $Y_w \ni (x,t) \mapsto e^{tw}x \in M$ with

$$e^{tw}x = \gamma_x(t)\,.$$

We will use the phrase "$e^{tw}x$ exists" to express that $(x,t) \in Y_w$, i.e., $t \in I_x$.

The above notation is consistent in the sense that $e^{tw}x$ depends only on x and the product vector field tw, rather than t and w separately. In fact, if $tw = 0$, we have $e^0 x = x$, since either $t = 0$, or $w = 0$ and all integral curves γ_x are constant. Futhermore, if $u = \lambda w$ with $\lambda \in \mathbf{R}$, then $\lambda^{-1} I_x \ni s \mapsto \widetilde\gamma_x(s) = \gamma_x(\lambda s) \in M$ is an integral curve of u with $\widetilde\gamma_x(0) = x$, defined on the largest possible interval (Problem 4). For any $x \in M$ and $t, s \in \mathbf{R}$ with $tw = su$, i.e., $t = \lambda s$, this shows that $e^{tw}x$ exists if and only if so does $e^{su}x$, and then $e^{tw}x = e^{su}x$.

Note that, since $I_x \ni t \mapsto e^{tw}x$ is an integral curve of w, it obeys the standard "exponential derivative" rule

$$\frac{d}{dt}\, e^{tw}x = w(e^{tw}x)\,, \qquad e^0 x = x\,, \tag{85.2}$$

the exponent 0 now being the zero vector field.

For each $t \in \mathbf{R}$, let $U_t \subset M$ be the set of all $x \in M$ for which $e^{tw}x$ exists, i.e., $(x,t) \in Y_w$. Thus, $x \in U_t$ if and only if w has an integral curve γ with $\gamma(0) = x$ defined on an interval containing both 0 and t. Note that U_t may be empty for some t, even though $U_0 = M$ and $U_s \subset U_t$ whenever $0 \le t \le s$ or $s \le t \le 0$. We denote $e^{tw} : U_t \to M$ the mapping sending each $x \in U_t$ onto $e^{tw}x$.

Lemma 85.1. *Let w be a C^l vector field on a manifold M, $1 \le l \le \infty$, and fix $x \in M$ and $t \in \mathbf{R}$ such that $e^{tw}x$ exists. Then, for any $s \in \mathbf{R}$, the conditions*

a. *$e^{sw}e^{tw}x$ exists,*
b. *$e^{(s+t)w}x$ exists,*

are equivalent, and either of them implies that

$$e^{(s+t)w}x = e^{sw}e^{tw}x\,. \tag{85.3}$$

Proof. Note that $I_x - t \ni s \mapsto \gamma(s) = e^{(s+t)w}x = \gamma_x(s+t) \in M$ is an integral curve of w with $\gamma(0) = \gamma_x(t) = e^{tw}x$, defined on the largest possible interval (Problem 4), and so $\gamma(s) = e^{sw}e^{tw}x$. □

Remark 85.2. If w is a C^l vector field on a manifold M, $1 \le l \le \infty$, and $t \in \mathbf{R}$, then e^{tw} is a bijective mapping of U_t onto U_{-t} and its inverse mapping is e^{-tw}. In fact, this is immediate from the above lemma for $s = -t$.

Proposition 85.3. *Suppose that w is a C^l vector field on a manifold M, $1 \le l \le \infty$. Then Y_w is an open subset of $M \times \mathbf{R}$ and the flow mapping $Y_w \ni (x,t) \mapsto e^{tw}x \in M$ is C^l-differentiable.*

Proof. We only need to show that, for each $(x, t_0) \in Y_w$,

(85.4) There exists an open set $U \subset M$ and an open interval $I \subset \mathbf{R}$ with $x \in U$ and $0, t_0 \in I$, such that $U \times I \subset Y_w$ and the mapping $U \times I \ni (z,t) \mapsto e^{tw}z \in M$ is C^l-differentiable.

Fix $x \in M$. By the local regularity theorem (p. 40), (85.4) holds whenever t_0 is sufficiently close to 0. To prove (85.4) for all $t_0 \in I_x$, suppose that for instance $t_0 > 0$ and let $t_1 > 0$ be the supremum of those $t_0 > 0$ which satisfy (85.4) with the given x. The assertion will follow if we now show that $t_1 = \sup I_x$. Let us assume, on the contrary, that $t_1 \in I_x$. According to the local regularity theorem, there exists a neighborhood U' of $e^{t_1 w}x$ in M, positive numbers ε, δ, and a C^l mapping $(y, t', t) \mapsto \gamma(y, t', t) \in M$ defined for $y \in U'$ and real t, t' with $|t' - t_1| < \delta$ and $|t - t'| < \varepsilon$, such that for any such y and t', the assignment $(t' - \varepsilon, t' + \varepsilon) \ni t \mapsto \gamma(y, t', t) \in M$ is an integral curve of w with $\gamma(y, t', t') = y$. Now let us fix t' with $0 < t_1 - t' < \min(\delta, \varepsilon)$ and such that $e^{t'w}x \in U'$ (the latter being possible by continuity of $t \mapsto e^{tw}x$ at $t = t_1$). Since (85.4) holds for some $t_0 > t'$, we may choose U in (85.4) so that, in addition, $e^{tw}(U) \subset U'$ for all t sufficiently close to t'. According to Problem13.1, formula

$$e^{tw}z = \gamma(e^{t'w}z, t', t)$$

now defines a C^l-differentiable extension of the flow mapping to $z \in U$ and $0 \leq t < t' + \varepsilon$; since $t' + \varepsilon > t_1$, this contradicts our choice of t_1, thus completing the proof. $\square$

Combining this result with Problem2.7 and the preceding remark, we obtain

Corollary 85.4. *For any C^l vector field w on a manifold M, $1 \leq l \leq \infty$, and $t \in \mathbf{R}$, the set $U_t = \{x \in M : (x,t) \in Y_w\}$ is open in M and, if it is nonempty, $e^{tw} : U_t \to U_{-t}$ is a C^l diffeomorphism with the inverse diffeomorphism $e^{-tw} : U_{-t} \to U_t$.*

Problems

1. Verify that, for any linear vector field $w = A$ on a finite-dimensional real or complex vector space V (Problem16.9) and any $t \in \mathbf{R}$, the flow mapping e^{tw} coincides with e^{tA} defined in Problem14.2.

2. Describe the flow mapping $(x,t) \mapsto e^{tw}x$ and its domain Y_w for the vector field w on $\mathbf{R}$ given by $w(x) = x^2 + 1 \in \mathbf{R} = T_x\mathbf{R}$, $x \in \mathbf{R}$.

3. Prove that, for any C^1 vector field w on a manifold M, any point $z \in M$ and a real number t, a C^1 solution $\gamma : I \to M$ to the initial value problem (85.1) on any open interval I containing t on which it can be defined, and that there exists the largest possible interval $I_{\max}$ with this property. (Hint below.)

4. Given two intervals $I, I' \subset \mathbf{R}$, a C^1 function $\varphi : I \to I'$, and a C^1 curve $\gamma : I' \to M$ in a manifold M, verify the *chain rule*

$$(\gamma \circ \varphi)^{\boldsymbol{\cdot}} = \dot{\varphi}(\dot{\gamma} \circ \varphi),$$

i.e., $\frac{d}{dt}\gamma(\varphi(t)) = \dot{\varphi}(t)\dot{\gamma}(\varphi(t))$, where $(\ )^{\boldsymbol{\cdot}}$ stands for the differentiation with respect to either parameter (in I or in I').

5. Let $f : U \to \mathbf{R}$ be a function defined on an open subset U of a manifold M. We say that f is *locally Lipschitz* if so is $f \circ \varphi^{-1} : \varphi(U \cap U') \to \mathbf{R}$ (in the sense defined in Problem 12.5, with some norm in $\mathbf{R}^n$, $n = \dim M$) for each coordinate system (U', φ) in M. Prove that this is a local geometric property, i.e, it can be verified using any particular set of coordinate systems covering U. Show that every locally Lipschitz function is continuous, and every C^1 function is locally Lipschitz. (Hint below.)

6. Show that the class of all locally Lipschitz functions $U \to \mathbf{R}$ on a fixed open subset U of a manifold M forms an algebra, i.e., is closed under addition and multiplication of functions. (Hint below.)

7. A vector field w on a manifold M is said to be *locally Lipschitz* if so are its component functions w^j relative to any local coordinates x^j in M. Verify that this is a local geometric property (p. 4). Prove that the initial value problem (85.1) then has a unique C^1 solution $\gamma : I \to M$ defined on the largest possible open interval $I \subset \mathbf{R}$ containing t.

8. A mapping $F : M \to N$ between manifolds is called *locally Lipschitz* if it is continuous and its components $F^\alpha = y^\alpha \circ F$, relative to any local coordinates y^α in M, are locally Lipschitz functions in M. Verify that this is a local geometric property (p. 4). Show that composites of locally Lipschitz mappings between manifolds are again locally Lipschitz.

Hint. In Problem 3, use the hints for Problems12.6 and 13.5.

Hint. In Problem 5, the geometric-property assertion follows from the obvious fact that composites $h \circ F$ of locally Lipschitz mappings between open sets in finite-dimensional normed vector spaces are again locally Lipschitz; this can in turn be applied to $h = f \circ \varphi^{-1}$ and $F = \varphi \circ \widetilde{\varphi}^{-1}$ for two coordinate mappings $\varphi, \widetilde{\varphi}$ in M.
Hint. In Problem 6, use local boundedness (or continuity) of locally Lipschitz functions f, h in a normed vector space, along with $|f(x)h(x) - f(z)h(z)| \le |f(x)||h(x) - h(z)| + |h(z)||f(x) - f(z)|$.

86. Killing fields

Topics: Killing vector fields on Riemannian manifolds and flows consisting of local isometries.

87. Lie brackets and flows

Topics: Commutation between the flows of two vector fields, versus the vanishing of their Lie bracket.

For any C^1 diffeomorphism $F : M \to N$ between manifolds and any vector field w on M, there exists a unique vector field v on N which is F-related to w in the sense that (16.6) holds (see Problem16.5). We then simply write $v = (dF)w$ and call v the *push-forward* of w under F. In view of (16.7), if w is of class C^l, $l = 0, 1, \ldots, \infty$ and F is C^{l+1}-differentiable, then $v = (dF)w$ is also of class C^l. Thus, if $l = 1$ and w' is another C^1 vector field on M, Theorem.... implies

$$(dF)[w, w'] = [(dF)w, (dF)w'] \,. \tag{87.1}$$

Let w be a fixed C^2 vector field on a manifold M, and let the open sets $I_x \subset \mathbf{R}$, $Y_w \subset M \times \mathbf{R}$ and $U_t \subset M$, for any $x \in M$ and $t \in \mathbf{R}$, be defined as on p. 47. If u is a C^1 vector field on M, the push-forward $(de^{tw})u$ of the restriction of u to U_t under the C^1 diffeomorphism $e^{tw} : U_t \to U_{-t}$ is a C^1 vector field on U_{-t} (provided that U_t is nonempty, which is guaranteed if t is sufficiently close to 0). Since, for any fixed $y \in M$, the set of all $t \in \mathbf{R}$ with $y \in U_{-t}$ is the open interval $-I_y$, and the resulting curve $-I_y \ni t \mapsto [(de^{tw})u](y) \in T_yM$ if of class C^1 (in view of the proposition on p. 48 and formula (16.7)), the derivative

$$\frac{d}{dt}\,[(de^{tw})u] \tag{87.2}$$

is a well-defined vector field on U_{-t}, assigning to each $y \in U_{-t}$ the vector $d\{[(de^{tw})u](y)\}/dt$. Also note that, for $u = w$ and any $t \in \mathbf{R}$ we have

$$[(de^{tw})w] = w \tag{87.3}$$

everywhere in $U_{-t} = e^{tw}(U_t)$. In fact, by (85.2) and (85.3), $w(x) = \frac{d}{ds}\, e^{sw}x\big|_{s=0}$ for $x \in U_t$, and so $[(de^{tw})w](e^{tw}x) = (de^{tw})_x[w(x)] = \frac{d}{ds}\, e^{tw}e^{sw}x\big|_{s=0} = \frac{d}{ds}\, e^{(t+s)w}x\big|_{s=0} = \frac{d}{ds}\, e^{sw}\big|_{s=t} = w(e^{tw}x)$ (see also Problem 4 in §85).

Proposition 87.1. *Given a C^2 vector field w and a C^1 vector field u on a manifold M, we have, for each $t \in \mathbf{R}$,*

$$\frac{d}{dt}\,[(de^{tw})u] = (de^{tw})[u, w] \,. \tag{87.4}$$

everywhere in the open set $U_{-t} = \{y \in M : e^{-tw}y \text{ exists}\}$.

PROOF. Denote v_t the vector field $(de^{tw})u$ on U_{-t} and, for any given t such that U_t is nonempty, let x^j and y^α be local coordinates at fixed points $x_0 \in U_t$ and, respectively, $y_0 = e^{tw}x_0 \in U_{-t}$ in the open submanifolds U_t, U_{-t} of M. Writing $v^\alpha(t,y)$ instead of $v_t^\alpha(y)$ and $\phi(t,x)$ instead of $e^{tw}x$, with $\phi^\alpha(t,x) = y^\alpha \circ \phi(t,x)$ for x near x_0 and y near y_0, we will apply to both resulting component functions the partial derivative $\partial/\partial t$. Thus, v_t is characterized by $v_t(\phi(t,x)) = d[\phi(t,\cdot)]_x[u(x)]$, i.e., in view of (16.7),

$$v^\alpha(t,\phi(t,x)) = u^j(x)(\partial_j\phi^\alpha)(t,x) \tag{87.5}$$

for x near x_0. On the other hand, by (85.2),

$$\frac{d}{dt}\phi^\alpha(t,x) = \frac{\partial\phi^\alpha}{\partial t}(t,x) = w^\alpha(\phi(t,x)), \tag{87.6}$$

so that the relation $\dfrac{d}{dt}[\partial_j\phi^\alpha(t,x)] = \dfrac{\partial^2\phi^\alpha}{\partial t\partial x^j}(t,x) = \dfrac{\partial}{\partial x^j}\dfrac{\partial\phi^\alpha}{\partial t}(t,x) = \dfrac{\partial}{\partial x^j}w^\alpha(\phi(t,x))$ and the chain rule yield

$$\frac{d}{dt}[\partial_j\phi^\alpha(t,x)] = (\partial_\beta w^\alpha)(\phi(t,x))(\partial_j\phi^\beta)(t,x). \tag{87.7}$$

Applying d/dt to (87.5) we now obtain, setting $y = \phi(t,x)$,

$$\frac{\partial v^\alpha}{\partial t}(t,y) + \frac{\partial v^\alpha}{\partial y^\beta}(t,y)\cdot\frac{\partial\phi^\beta}{\partial t}(t,x) = u^j(x)\frac{d}{dt}[(\partial_j\phi^\alpha)(t,x)]$$

and so it follows from (87.5) – (87.7) that, at points y near y_0, the component functions of (87.2) are given by

$$\frac{\partial v^\alpha}{\partial t}(t,\cdot) = v^\beta(t,\cdot)\partial_\beta w^\alpha - w^\beta\partial_\beta v^\alpha(t,\cdot).$$

In view of (16.4), this shows that $dv_t/dt = [v_t,w]$, i.e.,

$$\frac{d}{dt}[(de^{tw})u] = [(de^{tw})u, w],$$

and (87.4) is immediate from (87.1) and (87.3). $\square$

COROLLARY 87.2. *For a C^2 vector field w and a C^1 vector field u on a manifold M, the following two conditions are equivalent:*

a. $[u,w] = 0$ *identically on M;*

b. u *is* invariant under the flow of w *in the sense that*

$$(de^{tw})u = u$$

everywhere in the open set $U_{-t} = e^{tw}(U_t)$, for each $t \in \mathbf{R}$ such that $U_t = \{x \in M : e^{tw}x \text{ exists}\}$ is nonempty.

This is a direct consequence of (87.4) and the fact that $[(de^{tw})u]\big|_{s=0} = u$.

The Lie bracket $[w,u]$ of two vector fields w,u on a manifold M vanishes if and only if their flows *commute*. More precisely, we have the following result.

THEOREM 87.3. *Suppose that we are given a C^2 vector field w and a C^1 vector field u on a manifold M. Then $[w,u] = 0$ everywhere if and only if, for all real t,s,*

$$e^{tw}e^{su} = e^{su}e^{tw}$$

at all points in M where both composites make sense.

REMARK 87.4. Note that, for C^1 vector fields w, u on M, the set of those $(x,t,s) \in M \times \mathbf{R}^2$ for which $e^{tw}e^{su}x$ exists is open and contains $M \times \{(0,0)\}$. In fact, this is just the preimage of the open set Y_w under the C^1 mapping $\mathbf{R} \times Y_u \ni (t,(x,s)) \mapsto (t, e^{su}x)$ (see the proposition on p. 48). Therefore, the same holds for the set of (x,t,s) such that $e^{tw}e^{su}x$ and $e^{su}e^{tw}x$ both exist.

PROOF. Condition $[w,u] = 0$ means that u is invariant under the flow of w (the corollary on p. 52), which in turn is equivalent (see Problem 2) to requiring that the composite $s \mapsto e^{tw}e^{su}x$ of each flow mapping e^{tw} with any integral curve $s \mapsto e^{su}x$ of u contained in the domain U_t of e^{tw} (bottom of p. 47) be the integral curve of u having the value $e^{tw}x$ at $s = 0$, i.e., coincide with $s \mapsto e^{su}e^{tw}x$. This completes the proof. $\square$

Problems

1. Given C^1 mappings $\Phi : M \to N$, $\Psi : N \to P$ between manifolds and C^1 vector fields w on M, v on N and u on P such that $(d\Phi)w = v$ on $\Phi(M)$ and $(d\Psi)v = u$ on $\Psi(N)$, prove that $(d[\Psi \circ \Phi])w = u$ on $\Psi(\Phi(M))$.

2. Let $F : M \to N$ be a C^1 mapping between manifolds and let w, v be C^1 vector fields on M and N, respectively. Show that $(dF)w = v$ on $F(M)$ (p. 44) if and only if, for every integral curve γ of w, the composite $F \circ \gamma$ is an integral curve of v.

88. Completeness of vector fields

Topics: .

A C^1 vector field w on a manifold M is said to be *complete* if $Y_w = M \times \mathbf{R}$ (notation as on p. 47), i.e., if each maximal integral curve of w is defined on the whole real line $\mathbf{R}$. Removing from a manifold M a point x with $w(x) \neq 0$ for a given C^1 vector field w we are obviously left with a non-complete vector field on the open submanifold $U = M \setminus \{x\}$, namely, the restriction of w. In fact, $\gamma : (0, \varepsilon) \to U$ given by $\gamma(t) = e^{tw}x$ for a sufficiently small $\varepsilon > 0$ then is an integral curve of w restricted to U that cannot be extended (in U) to $t < 0$. For a more general class of examples, see Problem 6.

Examples of complete vector fields. Each of the following assumptions on the manifold M and the C^1 vector field w on M guarantees completeness of w:

a. $M \times (-\varepsilon, \varepsilon) \subset Y_w$ for some $\varepsilon > 0$, i.e., for each $x \in M$ there is an integral curve γ of w with $\gamma(0) = x$, defined on the interval $(-\varepsilon, \varepsilon)$ with ε independent of x. See Problem 7.

b. M is separable (p. 29) and w is *compactly supported*, i.e., vanishes outside a compact subset K of M. In fact, for each $x \in M$, there is a product neighborhood $U_x \times (-\varepsilon_x, \varepsilon_x)$ of $(x, 0)$ in $M \times \mathbf{R}$ contained in Y_w, and choosing a finite set $\Gamma \subset M$ with $K \subset \bigcup_{x \in \Gamma} U_x$ (Problem11.8) we obtain $M \times (-\varepsilon, \varepsilon) \subset Y_w$, as in (a), for $\varepsilon = \min\{\varepsilon_x : x \in \Gamma\}$.

c. M is a compact separable manifold (and w is any C^1 vector field on M); this is (b) with $K = M$.

d. $M = V$ is a finite-dimensional real or complex vector space and w is a linear vector field on V (Problem16.9). Completeness of w follows here from the global existence theorem for linear differential equations (p. 37); for an explicit description of the flow of w, see Problem 1 in §85.

e. More generally, a C^1 vector field w on a finite-dimensional real or complex vector space $M = V$ is complete whenever it has *linear growth* in the sense that $|w(x)| \le C|x|$ for some norm $|\ |$ in V, a constant $C \ge 0$ and all $x \in V$, where w is identified with a mapping $V \to V$ as in Problem16.8. This is clear from the proposition on p. 37 applied to $F(t,x) = w(x)$, $h(t) = C$ and $I = \mathbf{R}$. (For generalizations, see Problems 8 and 9.)

f. Another condition ensuring completeness of a C^1 vector field w on a finite-dimensional vector space $M = V$ is its *boundedness* as a mapping $V \to V$ in the sense that $|w(x)| \le C$ for some (or any) norm $|\ |$ in V, a constant $C \ge 0$ and all $x \in V$. (This terminology agrees with that introduced on p. 31, but not with the use of the term 'bounded' for linear operators on p. 14.) In fact, any integral curve $\gamma : (a,b) \to V$ of w with $a > -\infty$ or $b < \infty$ must have a limit at a or b (Problem13.3), and hence one can extend it beyond a or b as in Problem13.1, using such a limit as the initial value at a or b. (See also Problems 5, 8 and 9.)

Problems

1. Let V, W be real or complex vector spaces and let K be a subset of V. A mapping $F : K \to W$ is said to be *homogeneous of degree* $k \in \mathbf{Z}$ if $F(\lambda v) = \lambda^k F(v)$ for any nonzero scalar λ and any vector $v \in K$ with $\lambda v \in K$. We also say that $F : K \to W$ is *positively homogeneous of degree* $a \in \mathbf{R}$ if $F(\lambda v) = \lambda^a F(v)$ for all vectors $v \in K$ and all positive real scalars λ such that $\lambda v \in K$. Prove that, under the assumption that V, W are both finite-dimensional,

(a) If F is positively homogeneous of degree 0, K is nonempty and contains λv whenever $v \in K$ and $\lambda \in [0,1)$, and F is continuous at 0, then F is constant.

(b) If K is an open subset of V, F is C^1-differentiable and (positively) homogeneous of some degree a, and $v \in V$, then $d_v F$ is (positively) homogeneous of degree $a - 1$.

(c) If $K = V$ and F is C^1-differentiable and positively homogeneous of degree 1, then $F : V \to W$ is a real-linear mapping.

(Hint below.)

2. Suppose that V, W are finite-dimensional real or complex vector spaces, U is an open subset of V, and $F : U \to W$ is a C^1 mapping. The identity mapping $\mathrm{Id} : V \to V$ can naturally be identified with a vector field on V as in Problem16.8; it is then referred to as the *radial vector field* on V. Show that

(a) $d_{\mathrm{Id}} F = aF$ whenever F is positively homogeneous of degree $a \in \mathbf{R}$.

(b) If the intersection of U with every real half-line $[0,\infty)\cdot v$ emanating from 0 in V (where $v \in V \setminus \{0\}$) is connected, and $d_{\mathrm{Id}} w = aw$ for some $a \in \mathbf{R}$, then w is positively homogeneous of degree a.

(Hint below.)

3. Let V be a finite-dimensional real vector space and let U be an open subset of a V. Any given C^1 mapping $w : U \to V$, as well as the identity mapping $\mathrm{Id} : V \to V$ can be regarded as vector fields on U and V (Problem16.8). Verify that

(a) The Lie bracket $[\mathrm{Id}, w]$ is given by $[\mathrm{Id}, w] = d_{\mathrm{Id}} w - w$, i.e., $[\mathrm{Id}, w](x) = (d_x w)(x) - w(x)$ for $x \in U$.

(b) Under the same assumption about U as in (b) of Problem 2, positive homogeneity of degree $a \in \mathbf{R}$ for w is equivalent to the condition $[\mathrm{Id}\,, w] = (a-1)w$.

(Hint below.)

4. For a C^2 vector field w and a C^1 vector field u on a manifold M, prove that the following three conditions are equivalent:

(a) $[u, w] = \lambda u$ everywhere in M, for some real number λ;

(b) $(de^{tw})u = e^{\lambda t}u$ everywhere in $U_{-t} = e^{tw}(U_t)$, for some real number λ and each $t \in \mathbf{R}$ such that $U_t = \{x \in M : e^{tw}x \text{ exists}\}$ is nonempty;

(c) u is *invariant, up to constant factors, under the flow of* w in the sense that, everywhere in U_{-t},

$$(de^{tw})u = c(t)u$$

for each $t \in \mathbf{R}$ such that U_t is nonempty, and some $c(t) \in \mathbf{R}$.

(Hint below.)

5. Let w be a C^1 vector field on a manifold M, and let $\gamma : (a, b) \to M$, $-\infty \le a < b \le \infty$, be a maximal integral curve of w. Show that if a limit $\lim_{t\to a(+)} \gamma(t)$ or $\lim_{t\to b(-)} \gamma(t)$ exists, then the corresponding endpoint (a or b) is infinite. (Hint below.)

6. Let w be a C^1 vector field on a manifold M, and let K be a closed subset of M which is not a union of (images of) maximal integral curves of w. Show that the vector field on the open submanifold $U = M \setminus K$ obtained by restricting w to U is not complete. (Hint below.)

7. Let w be a C^1 vector field on a manifold M such that $M \times (-\varepsilon, \varepsilon) \subset Y_w$ for some $\varepsilon > 0$ (notation as on p. 47). Prove that w is complete. (Hint below.)

8. Prove completeness of any C^1 vector field w on a finite-dimensional real or complex vector space V that has *nonhomogeneous linear growth*, i.e., $|w(x)| \le C_0|x| + C_1$ for some norm $|\ |$ in V, some constants $C_0, C_1 \ge 0$ and all $x \in V$, where w is regarded as a mapping $V \to V$ (Problem16.8). (Hint below.)

9. Let w be a C^1 vector field on a finite-dimensional real or complex vector space V satisfying the estimate $|w(x)| \le C|x| + C'$ for some norm $|\ |$ in V, some compact subset K of V, some constants $C, C' \ge 0$, and all $x \in V \setminus K$. Show that w is complete. (Hint below.)

Hint. In Problem 1, to obtain (c) apply (a) to d_vF for each $v \in V$ using (b).
Hint. In Problem 2, note that $x = \mathrm{Id}\,x = \frac{d}{dt}\,tx\big|_{t=1}$.
Hint. In Problem 3, use Problem16.8.
Hint. In Problem 4, (a) implies (b) via (87.4) and the "global uniqueness" theorem for ordinary differential equations (Problem12.6), and (c) implies (a) with $\lambda = \dot c(0)$ again by (87.4), since the function $t \mapsto c(t)$ must be of class C^1 unless $u = 0$ identically (in view of the proposition on p. 48).
Hint. In Problem 5, if for instance γ had a limit x at b with $b < \infty$, the assignment $(a, b) \ni t \mapsto \gamma(t)$, $[b + \varepsilon) \ni t \mapsto e^{(t-b)w}x$ would define, for a suitable $\varepsilon > 0$, an integral curve of w (see Problem13.1), contradicting the maximality of b.
Hint. In Problem 6, note that the images of the maximal integral curves of w in M are pairwise disjoint and their union equals M, so by the assumption on K one of these images intersects both U and K, thus providing an integral curve of w restricted to U that cannot be extended in U to the whole real line.

Hint. In Problem 7, let $\gamma : (a, b) \to V$ be a maximal integral curve of w. To show that $(a, b) = \mathbf{R}$, suppose on the contrary that, for instance, $b < \infty$. Choosing c with $a < c < b$ and $b - c < \varepsilon$, we may extend γ to an integral curve of w defined $(a, c + \varepsilon)$ and given by $(a, c] \ni t \mapsto \gamma(t)$ and $[c, c + \varepsilon) \ni t \mapsto e^{(t-c)w}[\gamma(c)]$ (see Problem13.1), which contradicts the maximality of b.

Hint. In Problem 8, for any maximal integral curve $\chi : I \to V$ of w we have $|\dot{\chi}| \leq C(|\chi| + 1)$ with $C = \max(C_0, C_1)$, and so the curve $I \ni t \mapsto \gamma(t) = (\chi(t), 1)$ in $V \oplus \mathbf{R}$ satisfies $|\dot{\gamma}| \leq C|\gamma|$ for the norm $|(x, \lambda)| = |x| + |\lambda|$. If I were not the whole of $\mathbf{R}$, γ (and χ) would have a limit at a finite endpoint of I (by the corollary on p. 36), which contradicts the statement of Problem 5.

Hint. In Problem 9, apply Problem 8 to $C_0 = C$ and $C_1 = C' + \sup\{|w(x)| : x \in K\}$.

Appendix E. Measure and Integration

89. The Hölder and Minkowski inequalities

$$\|fh\|_1 \le \|f\|_p \|h\|_q . \tag{89.1}$$

For $p \in [1, \infty]$,

$$\|f + h\|_p \le \|f\|_p + \|h\|_p . \tag{89.2}$$

In fact, we may let $p > 1$, and then apply (89.1) to h replaced by $|f + h|^{p-1}$.

90. Convergence theorems

Topics: B. Levi's monotone convergence theorem; Fatou's lemma; Lebesgue's dominated convergence theorem.

$$f * h = \int_V f(y - z)h(z)\, dz$$
$$\|f * h\|_1 \le \|f\|_1 \|h\|_1 .$$

Appendix F. More on Lie Groups

96. The exponential mapping

Topics: The exponential mapping associated with a Lie group G; relation with C^1 homomorphisms $\mathbf{R} \to G$; the geodesic exponential mapping for a connection in the tangent bundle TM; normal (geodesic) coordinates; the standard left- and right-invariant connections on a Lie group.

For a Lie group G of class C^r, $r \ge 2$, and a C^1 curve $\gamma : I \to G$ defined on an interval I, the requirement that

$$\dot\gamma = \gamma v \tag{96.1}$$

with a fixed vector $v \in T_1 G$, that is, $\dot\gamma(t) = \gamma(t)v$ for each $t \in I$ (with the multiplication of elements of G by tangent vectors defined as in (11.3)) may be thought of as an *ordinary differential equation* of order 1 in G, since in any given coordinate system x^j for G, (96.1) becomes a system of such equations with the unknown functions $t \mapsto \gamma^j(t)$ defined on subintervals of I. In fact, choosing the Φ^j as in (11.8), we can rewrite (96.1) as

$$\dot\gamma^j(t) = v^\alpha \frac{\partial \Phi^j}{\partial y^\alpha}(\gamma^1(t), \dots, \gamma^n(t), u^1, \dots, u^n)\,, \tag{96.2}$$

where u stands for $1 \in G$. Consequently, given $t_0 \in \mathbf{R}$ and $x_0 \in G$, (96.1) will have a C^1 solution $t \mapsto \gamma(t)$ with $\gamma(t_0) = x_0$, defined on some open interval containing t_0. Furthermore, such a solution γ is unique on every interval I with $t_0 \in I$ on which it exists, and it is of class C^r (since the $\partial\Phi^j/\partial y^\alpha$ in (96.2) are of class C^{r-1}). Thus, (96.1) with the initial condition $\gamma(t_0) = x_0$ has a unique C^1 solution γ defined the maximal possible open interval containing t_0 (i.e., the union of all such intervals on which solutions exist).

The same conclusion remains valid when (96.1) is replaced by the equation

$$\dot\gamma = v\gamma$$

with $v \in T_1 G$.

LEMMA 96.1. *Given a Lie group G of class C^r, $r \ge 2$, and a C^1 curve $\gamma : I \to G$ defined on an interval I containing 0, the following three conditions are mutually equivalent:*

i. *γ is the restriction to I of a C^1 homomorphism $\mathbf{R} \to G$,*
ii. *γ is a solution to (96.1), for some $v \in T_1 G$, with the initial condition $\gamma(0) = 1$,*
iii. *γ satisfies (96.1) and $\gamma(0) = 1$ for some $v \in T_1 G$.*

PROOF. Assume (i). Then $\dot\gamma(t) = \frac{d}{ds}\big|_{s=0}\gamma(t+s) = \frac{d}{ds}\big|_{s=0}\gamma(t)\gamma(s) = \gamma(t)v$ with $v = \dot\gamma(0)$. This yields (ii), as well as (iii) (since (i) implies $\gamma(t+s) = \gamma(s+t) = \gamma(t)\gamma(s) = \gamma(s)\gamma(t)$). Now, if (ii) holds, and $s \in I$ is fixed, the curves γ and $t \mapsto$

$[\gamma(s)]^{-1}\gamma(s+t)$ both satisfy (96.1) with the same initial value $1 \in G$ at $t = 0$. The uniqueness of solutions (as mentioned above) now shows that $\gamma(s+t) = \gamma(s)\gamma(t)$ whenever s, t and $s+t$ are all in I. To extend γ to a C^1 homomorphism, choose $\varepsilon > 0$ with $[0, \varepsilon] \subset I$ or $[-\varepsilon, 0] \subset I$ and then set $\gamma(k\varepsilon + s) = [\gamma(\varepsilon)]^k \gamma(s)$, noting that every real number can be uniquely written as $k\varepsilon + s$ with $k \in \mathbf{Z}$ and $s \in [0, \varepsilon)$ or, respectively, $s \in (-\varepsilon, 0]$. Thus, (ii) implies (i). Similarly, (i) will follow from (iii) if we use the curve $t \mapsto \gamma(s+t)[\gamma(s)]^{-1}$. □

PROPOSITION 96.2. *Let G be a Lie group of class C^r, $r \geq 2$. Then*

a. *For every $v \in T_1 G$ there exists a unique C^1 homomorphism $\gamma_v : \mathbf{R} \to G$ with*
$$\dot{\gamma}_v(0) = v\,.$$
b. *$\gamma = \gamma_v$ is a solution to both (96.1) and (79.10) with the initial condition $\gamma(0) = 1$.*
c. *For all $s, t \in \mathbf{R}$ and $v \in T_1 G$ we have*
$$\gamma_{sv}(t) = \gamma_v(st)\,.$$
d. *The mapping $\mathbf{R} \times T_1 G \ni (t, v) \mapsto \gamma_v(t)$ is of class C^{r-1}, and each homomorphism γ_v is of class C^r.*

PROOF. (a) and (b) are immediate from the lemma and the preceding remarks on the existence and uniqueness of solutions to ordinary differential equations. Assertion (c) easily follows from the uniqueness of solutions. Finally, (d) is a consequence of the regular dependence of solutions to ordinary differential equations on parameters, the latter being in this case the vectors $v \in T_1 G$. □

For a Lie group G of class C^r, $r \geq 2$, we define the *exponential mapping*

$$\exp : T_1 G \to G\,, \tag{96.3}$$

also written as $v \mapsto e^v$, by
$$\exp v = e^v = \gamma_v(1)$$
with γ_v introduced in the above proposition. Assertion (c) with $t = 1$ now yields

$$\gamma_v(t) = e^{tv} \tag{96.4}$$

for all $t \in \mathbf{R}$ and $v \in T_1 G$. Thus,
$$e^{(s+t)v} \;=\; e^{sv} e^{tv}$$
and, by (b),
$$\frac{d}{dt}\, e^{tv} \;=\; v e^{tv} \;=\; e^{tv} v\,, \qquad e^0 = 1\,.$$
In particular,

$$\left.\frac{d}{dt}\, e^{tv}\right|_{\substack{\to \\ t=0}} \;=\; v\,. \tag{96.5}$$

Consider now two Lie groups G, H of class C^r, $r \geq 2$, and a C^1 homomorphism $f : G \to H$. For any $v \in T_1 G$, formula $\gamma(t) = f(e^{tv})$ defines a C^1 homomorphism $\gamma : \mathbf{R} \to H$ with $\dot{\gamma}(0) = f_* v$ (notation of (12.1)), i.e., $\gamma = \gamma_{f_* v}$. By (96.4), we must have $\gamma(t) = \exp(t f_* v)$, the symbol exp being also used for the exponential mapping of H. With $t = 1$, this becomes

$$f(e^v) \;=\; e^{f_* v} \tag{96.6}$$

for all $v \in T_1G$, that is

$$f \circ \exp \,=\, \exp \circ f_* \,.$$

Let ∇ now be a connection in TM. The (geodesic) *exponential mapping*

$$\exp_x : U_x \,\to\, M \tag{96.7}$$

of the connection ∇ at any point $x \in M$ is defined as follows. Its domain U_x is the subset of T_xM consisting of those $v \in T_xM$ for which there exists a geodesic $t \mapsto x(t)$, defined on the whole interval $[0,1]$, and such that $x(0) = x$, $\dot{x}(0) = v$. For such v and $x(t)$, we set $\exp_x v = x(1)$. (One traditionally writes $\exp_x v$, without parentheses, rather than $\exp_x(v)$.) It is obvious from the **Regularity Theorem** in §80 that the set U_x is open in T_xM (and contains 0), and the mapping $\exp_x$ is of class C^∞. Furthermore, the geodesic $x(t)$ with $x(0) = x$ and $\dot{x}(0) = v$ is given by $x(t) = \exp_x tv$, as one sees fixing $t \in [0,1]$ and noting that $[0,1] \ni t' \mapsto x(tt')$ then is a geodesic with the value and velocity at $t' = 0$ equal to x and, respectively, tv. In particular, $d[\exp_x tv]/dt$ at $t = 0$ equals v while, obviously, $\exp_x 0 = x$; in other words, the differential of the mapping (96.7) at the point $0 \in U_x$ is given by

$$d(\exp_x)_0 \,=\, \mathrm{Id} : T_xM \to T_xM \,. \tag{96.8}$$

According to Theorem 74.2, there exist a neighborhood U of y in M and a neighborhood U' of 0 in T_xM such that $U' \subset U_x$ and $\exp_x : U' \to U$ is a C^∞-diffeomorphism. Its inverse diffeomorphism may be thought of as a coordinate system $x^1, \dots, x^n$ with the domain U (after one has identified T_xM with $\mathbf{R}^n$, $n = \dim M$, using any fixed linear isomorphism). A coordinate system obtained as a local inverse of $\exp_x$ is called a *normal*, or *geodesic*, coordinate system at x, for the iven connection ∇ in TM. Note that if the connection ∇ is torsionfree, its component functions Γ^l_{jk} satisfy

$$\Gamma^l_{jk}(x) \,=\, 0 \quad \text{in normal coordinates at} \quad x \,. \tag{96.9}$$

To see this, note that under the identification $U' \approx U$ provided by $\exp_x$, geodesics emanating from $0 \approx x$ appear as the radial line segments $t \mapsto tv$, and so we have $\ddot{x}^j = 0$. For such a geodesic, the system (22.6) gives, at $t = 0$, $\Gamma^j_{kl}(x)v^kv^l = 0$ for all v, which in view of the symmetry (21.4) implies (96.9).

Problems

1. Given a Lie group G of class C^r, $r \geq 2$, verify that the exponential mapping (96.3) is of class C^{r-1}.

2. For a Lie group G of class C^r, $r \geq 2$, show that the differential $d(\exp)_0$ at $0 \in T_1G$ of the exponential mapping (96.3) coincides with the identity mapping $T_1G = T_0(T_1G) \to T_1G$. (Hint below.)

3. Let G be a Lie group of class C^r, $r \geq 2$. Prove that there exists an open set $U \subset T_1G$ such that $0 \in U$, $\exp(U)$ is an open subset of G and

$$\exp : U \to \exp(U) \tag{96.10}$$

is a C^{r-1} diffeomorphism. (Hint below.)

4. For Lie groups G, H of class C^r, $r \geq 2$, and a C^1 homomorphism $f : G \to H$, show that the restriction of f to a neighborhood $\exp(U)$ of 1 chosen as in Problem 3 is completely determined by $f_* : T_1G \to T_1H$ (that is, $f'_* = f''_*$ for two such homomorphisms f', f'' implies that $f' = f''$ on $\exp(U)$). (Hint below.)

5. Given C^1 curves $I \ni\mapsto a_t \in G$ $I \ni\mapsto b_t \in G$ in a a Lie group G of class C^r ($r \geq 2$), both defined on a common interval I containing 0 and satisfying $a_0 = 1$, $b_0 = 1$, verify that

$$\frac{d}{dt}\, a_t b_t \Big|_{\overrightarrow{t=0}} = \dot{a}_0 + \dot{b}_0 \,. \tag{96.11}$$

6. For a_t, b_t, I and G as in Problem 5, choose U satisfying the condition stated in Problem 3. Show that

$$\lim_{t\to 0} \frac{1}{t} \exp^{-1}(a_t b_t) = \dot{a}_0 + \dot{b}_0 \,,$$

where the limit may be one-sided if 0 is an endpoint of I, and t varies in a sufficiently small subinterval I' of I containing 0 and such that $a_t b_t \in \exp(U)$ for all $t \in I'$, while $\exp^{-1}$ is the inverse mapping of (96.10). (Hint below.)

7. With the same assumptions and notations as in Problem 6, verify that, for any vectors $v, w \in T_1 G$,

$$v + w = \lim_{t\to 0} \frac{1}{t} \exp^{-1}(e^{tv} e^{tw}) \,.$$

Hint. In Problem 2, use (96.5).
Hint. In Problem 3, combine Problem 2 with the inverse mapping theorem (Theorem 74.2).
Hint. In Problem 4, note that by (96.6) we have, on $\exp(U)$,

$$f = \exp \circ f_* \circ \exp^{-1} \,.$$

Hint. In Problem 6, note that the expression under the limit symbol is a difference quotient, and use (96.11) and the fact that

$$d(\exp^{-1})_1 = \mathrm{Id} \,,$$

according to Problem 2 and the statement preceding Problem4.1.

97.

Topics: .

Given a group G whose operation is written as a multiplication, an element $a \in G$ and subsets $K, L \subset G$, let us set $aK = \{ax : x \in K\}$, $Ka = \{xa : x \in K\}$, $KL = \{xy : x \in K \,, y \in L\}$ and $K^{-1} = \{x^{-1} : x \in K\}$. Thus, a subgroup of G is any nonempty subset $H \subset G$ with $HH \subset H$ and $H^{-1} \subset H$. By a *normal subgroup* of G we mean, as usual, a subgroup $H \subset G$ with $aHa^{-1} \subset H$ whenever $a \in G$ (that is, $aHa^{-1} = H$ for all $a \in G$); in other words, a subgroup is called normal if it is closed under all inner automorphisms of G ((iii) of **#12**).

For a fixed subgroup H of a group G, the *left cosets* of H are defined to be all sets of the form aH for some $a \in G$. They form a disjoint decomposition of G into a union of subsets, one of which is H. The same is true for the *right cosets* of H, which are the sets of the form Ha with $a \in G$. One easily sees that H is a normal subgroup if and only if the families of its left and right cosets coincide.

Suppose now that G is a Lie group of class C^r, $0 \leq r \leq \omega$. By the *identity component* of G we mean the connected component G^0 of the manifold G, containing 1 (1 is often called the *identity* of G). We then have $G^0 G^0 \subset G^0$, $(G^0)^{-1} \subset G^0$, and $aG^0 a^{-1} \subset G^0$ for all $a \in G$, as $G^0 G^0$, $(G^0)^{-1}$ and $aG^0 a^{-1}$

are connected subsets of G containing 1 (Problems 3.2, 3.3). The connected components of G are nothing else than the left (or right) cosets of G^0; this is immediate from Problem 13.5. Furthermore, G^0 regarded as an open submanifold of G is a Lie group of class C^r (with the group operation inherited from G; see Problem 14.1). The identity inclusion mapping $G^0 \to G$ now is a C^r homomorphism of Lie groups, inducing as in (12.1) the familiar identification $T_1 G^0 = T_1 M$, so that, when $r \geq 3$, the Lie algebras of G and G^0 are naturally isomorphic.

In other words, the algebraic structure (i.e., isomorphism type) of the Lie algebra $\mathfrak{g}$ of a Lie group G depends solely on the connected Lie group G^0. Thus, the only conclusions about G that may be expected to follow from assumptions about $\mathfrak{g}$ are those that pertain to G^0 alone. **Examples**

i. The C^ω Lie group $\mathrm{GL}\,(V)$ for a finite-dimensional vector space V over the field $\mathbf{K}$ of real or complex numbers ((iii) of **#5**), is connected when $\mathbf{K} = \mathbf{C}$, and has two connected components when $\mathbf{K} = \mathbf{R}$ and $\dim V > 0$. In the latter case, the identity component of $\mathrm{GL}\,(V)$, denoted $\mathrm{GL}^+(V)$, consists of all linear isomorphisms $A : V \to V$ with $\det A > 0$. See Problem 14.8.

ii. In particular, the matrix Lie group $\mathrm{GL}\,(n, \mathbf{C})$ is connected, while $\mathrm{GL}\,(n, \mathbf{R})$ with $n \geq 1$ has two components, the one containing the identity being the group $\mathrm{GL}^+(n, \mathbf{K})$ of all real $n \times n$ matrices having positive determinants.

PROBLEMS

14.1. Let H be a subgroup of a Lie group G of class C^r, $r \geq 0$, which at the same time is an open subset of G. Verify that the group multiplication of G restricted to the open submanifold formed by H turns H into a Lie group of class C^r, and the identity inclusion mapping $H \to G$ then is a C^r homomorphism of Lie groups whose differential at 1 is an isomorphism $T_1 H \to T_1 M$. **14.2.** For a continuous homomorphism $f : G \to H$ of Lie groups, show that $f(G^0) \subset H^0$. **14.3.** Given a real or complex vector space V with $\dim V = n < \infty$, let $\mathcal{B}(V)$ be the subset of the nth Cartesian power $V^n = V \times \ldots \times V$ consisting of all (ordered) bases of V. Show that, when V^n is treated as a vector space (the direct sum of n copies of V), the set $\mathcal{B}(V)$ is open in V^n and, as an open submanifold of V^n, $\mathcal{B}(V)$ is C^ω diffeomorphic to the underlying manifold of the Lie group $\mathrm{GL}\,(V)$. (Hint below.)**14.4.** Let V be a real vector space V with $\dim V = n$, $1 \leq n < \infty$. Call two (ordered) bases of V *equivalent* if the transition matrix between them has a positive determinant. Verify that this actually is an equivalence relation and it has exactly two equivalence classes. (These equivalence classes are called the *orientations* of V.) Show that each connected component of $\mathcal{B}(V)$ (Problem 14.3) is contained in a unique orientation of V. **14.5.** Let V be a real or complex vector space with $1 \leq \dim V = n < \infty$, carrying a fixed inner product $\langle\,,\rangle$ (that is, a positive-definite form which is bilinear and symmetric or, respectively, sesquilinear and Hermitian). The *orthonormalization* e_α of a basis v_α of V, $\alpha = 1, \ldots, n$, is defined by the recursive formula

$$e_\alpha = w_\alpha / |w_\alpha|\,, \qquad w_\alpha = v_\alpha - \sum_{\beta<\alpha} \langle v_\alpha, e_\beta \rangle e_\beta\,.$$

Show that the e_α is the unique orthonormal basis of V with (70.1) and

$$\langle e_\alpha, v_\alpha \rangle \in (0, \infty)\,, \qquad 1 \leq \alpha \leq n\,. \tag{97.1}$$

14.6. For V, $\langle\,,\rangle$ as in Problem 14.5 and any basis v_α of V, prove that the basis v_α and its orthonormalization e_α lie in the same connected component of $\mathcal{B}(V)$ (notation of Problem 14.3). (Hint below.)**14.7.** For V, $\langle\,,\rangle$ as in Problem 14.5, show that

i. If V is complex, any two orthonormal bases of V can be connected by a continuous curve in $\mathcal{B}(V)$ consisting of orthonormal bases.
ii. If V is real, any two orthonormal bases of V representing the same orientation can be joined by a continuous curve in $\mathcal{B}(V)$ consisting of orthonormal bases. (Hint below.)

14.8. Prove the statement of Example (i) above. (Hint below.)

Hint. In Problem 14.3, fix a basis $(e_1, \ldots, e_n)$ of V and identify each $A \in \mathrm{GL}\,(V)$ with $(Ae_1, \ldots, Ae_n) \in \mathcal{B}(V)$.

Hint. In Problem 14.6, use the sequence of $n{+}1$ bases $\mathfrak{e}_k = (e_1, \ldots, e_k, v_{k+1}, \ldots, v_n)$ of V, $k = 0, \ldots, n$. Note that $\mathfrak{e}_0 = (v_1, \ldots, v_n)$, $\mathfrak{e}_n = (e_1, \ldots, e_n)$. Now, for any $k = 0, \ldots, n-1$, formula $[0,1] \ni t \mapsto \mathfrak{e}_k(t) = (e_1, \ldots, e_k, (1-t)v_{k+1} + te_{k+1}, v_{k+2}, \ldots, v_n)$ defines a continuous curve in $\mathcal{B}(V)$ connecting $\mathfrak{e}_k$ with $\mathfrak{e}_{k+1}$. The fact that each $\mathfrak{e}_k(t)$ (and $\mathfrak{e}_k$) is a basis follows since, from (70.1), the first $k{+}1$

vectors of $\mathfrak{e}_k(t)$ lie in $\mathrm{Span}\,(v_1,\ldots,v_{k+1})$ and the $(k+1)$st vector is orthogonal to $e_1,\ldots,e_k$ and nonzero (as its inner product with e_{k+1} is positive by (97.1)). A continuous curve in $\mathcal{B}(V)$ connecting $\mathfrak{e}_0$ with $\mathfrak{e}_n$ can be written in the form $[0,n]\ni s\mapsto \mathfrak{e}_s$ with $\mathfrak{e}_s=\mathfrak{e}_k(t)$, where $k=[s]$ is the integer part of s (the largest integer not exceeding s), $t=s-[s]$, and we set $\mathfrak{e}_n(0)=\mathfrak{e}_n$.

Hint. In Problem 14.7, denote $\mathfrak{e}_0$ and $\mathfrak{e}_n=(e_1,\ldots,e_n)$ two given orthonormal bases of V, and make them a part of a sequence $\mathfrak{e}_k$, $k=0,\ldots,n$ of $n+1$ orthonormal bases, such that each $\mathfrak{e}_k$ shares the first k vectors $e_1,\ldots,e_k$ with $\mathfrak{e}_n$, and each $\mathfrak{e}_{k-1}$, $1\le k\le n$, can be connected with $\mathfrak{e}_k$ by a continuous curve of orthonormal bases. To achieve this, use induction on k, assuming that $1\le k<n$ and $\mathfrak{e}_1,\ldots,\mathfrak{e}_k$ with the stated properties have already been constructed. Thus, $\mathfrak{e}_k$ has the form $\mathfrak{e}_k=(e_1,\ldots,e_k,v_{k+1},\ldots,v_n)$.

First, suppose that $k=n-1$, so $v_n=zu_n$ for some scalar z with $|z|=1$ (by orthonormality). In the case where V is reaI, we have $z=\pm1$, and if both original bases determine the same orientation, then so do the intermediate stages including $\mathfrak{e}_{n-1}$ (Problem 14.4); thus, $z=1$ and $\mathfrak{e}_n=\mathfrak{e}_{n-1}$ can be connected with $\mathfrak{e}_0$. On the other hand, if V is complex, the curve $[0,\varphi]\ni s\mapsto(e_1,\ldots,e_k,e^{is}v_n)$, with $\varphi>0$ such that $z=e^{i\varphi}$, connects $\mathfrak{e}_{n-1}$ with $\mathfrak{e}_n$.

Now let $k+1<n$. Thus, we can choose a 2-dimensional subspace W of V containing the vectors $u=e_{k+1}$ and $v=v_{k+1}$, and orthogonal to $e_1,\ldots,e_k$. Let us also select a scalar z with $|z|=1$ and $\langle u,zv\rangle\in\mathbf{R}$ (in the real case, we may set $z=1$). There exists an orthonormal basis (u,w) of W with $\langle zv,w\rangle\in\mathbf{R}$; to obtain w, pick a unit vector in W orthogonal to u, and multiply it by a suitable unit scalar. Therefore, $zv=pu+qw$ with *real* scalars p,q. Since $p^2+q^2=|zv|=|v|=1$, we have $p=\cos\theta$, $q=\sin\theta$ for some $\theta>0$. We can now define a continuous curve $[0,\theta]\ni t\mapsto A_t\in\mathrm{GL}\,(V)$ of inner-product preserving linear operators in V by $A_tu=(\cos t)u-(\sin t)w$, $A_tw=(\sin t)u+(\cos t)w$ (so that $A_t(W)\subset W$), and $A_t=\mathrm{Id}$ on the orthogonal complement of W. Thus, $A_te_1=e_1,\ldots,A_te_k=e_k$. A continuous curve of orthonormal bases connecting $\mathfrak{e}_k$ to a basis of the form $\mathfrak{e}_{k+1}=(e_1,\ldots,e_k,e_{k+1},*,\ldots,*)$ consists of two stages (segments). The former is trivial in the real case (a constant curve) and, in the complex case, is given by

$$[0,\varphi]\ni s\mapsto(e_1,\ldots,e_k,e^{is}v_{k+1},\ldots,e^{is}v_n)\,,$$

where $\varphi>0$ is chosen so that $z=e^{i\varphi}$. The latter (connecting $(e_1,\ldots,e_k,zv_{k+1},\ldots,zv_n)$ to $(e_1,\ldots,e_k,e_{k+1},*,\ldots,*)$) is defined by

$$[0,\theta]\ni t\mapsto(A_te_1,\ldots,A_te_k,A_t(zv_{k+1}),\ldots,A_t(zv_n))\,.$$

Hint. In Problem 14.8, use Problems 14.3, 14.4, 14.6 and 14.7.

98.

Topics: .

Given real or complex vector spaces V, W, a mapping $F:V\to W$ is called *affine* if it is the composite of a linear operator followed by a translation, i.e., there exist a linear operator $A:V\to W$ and a vector $b\in W$ with

$$F(x)=Ax+b \tag{98.1}$$

for all $x\in V$. Such a mapping F is called an *affine isomorphism* if it is one-to-one and onto.

Let V be a finite-dimensional real or complex vector space. The set $\mathrm{GA}(V)$ of all affine isomorphisms $V \to V$ then is a group (Problem 15.1(ii),(iii)). Moreover, the assignment

$$\mathrm{GA}(V) \ni F \mapsto (A, b) \in \mathrm{GL}(V) \times V \tag{98.2}$$

characterized by (98.1) is one-to-one and onto (Problem 15.1(i)), and so it identifies $\mathrm{GA}(V)$ with an open subset of the vector space $\mathfrak{gl}(V) \times V$. This provides $\mathrm{GA}(V)$ with the structure of a C^ω manifold. The group structure in $\mathrm{GL}(V) \times V$ obtained from that of $\mathrm{GA}(V)$ via the identification (98.2) then is given by the multiplication

$$(A, b)(A', b') = (AA', Ab' + b)\,, \tag{98.3}$$

which is an analytic (in fact, polynomial) function of (A, b) and (A', b'). The manifold $\mathrm{GA}(V)$ thus becomes a C^ω Lie group. According to (98.2), Example (i) of **#14** and Problems 3.3, 13.5, $\mathrm{GA}(V)$ is connected if V is complex, and has two connected components when V is real and $V \neq \{0\}$. In the latter case we denote $\mathrm{GA}^+(V)$ the identity component of $\mathrm{GA}^+(V)$; the group $\mathrm{GA}^+(V)$ thus consists of all affine isomorphisms $F : V \to V$ whose "linear part" A has a positive determinant.

The 2-dimensional Lie group $\mathrm{GA}^+(\mathbf{R})$ turns out to be the "simplest" (e.g., lowest–dimensional) example of a Lie group that is connected but non-Abelian. See Problem 15.2 and **#17**.

PROBLEMS

15.1. Let $F : V \to W$ be an affine mapping between real or complex vector spaces V, W, given by (98.1). Verify that

i. A and b in (98.1) are uniquely determined by F;
ii. F is an affine isomorphism if and only if A is a linear isomorphism, and then F^{-1} is also affine;
iii. Composites of affine mappings are affine;
iv. Affine mappings $V \to W$ with valuewise operations form a vector space.

15.2. Show that the Lie group $\mathrm{GA}^+(V)$ is non-Abelian unless $V = \{0\}$. **15.3.** Under (98.2), the 2-dimensional Lie group $\mathrm{GA}^+(\mathbf{R})$ is identified with the half-plane $\mathbf{R}^2_+ = (0, \infty) \times \mathbf{R} = \{(a, b) : a > 0\,,\, b \in \mathbf{R}\}$, and the multiplication formula (98.3) becomes

$$(a, b)(a', b') = (aa', ab' + b)\,. \tag{98.4}$$

(Note that $\mathrm{GL}^+(\mathbf{R}) = \mathrm{GL}^+(1, \mathbf{R}) = (0, \infty)$.) The Lie group $\mathbf{R}^2_+$ is an open subset of $\mathbf{R}^2$, and so its Lie algebra (i.e., tangent space at the unit element) is canonically identified with the vector space $\mathbf{R}^2$ itself. Write an explicit formula for the resulting Lie-algebra multiplication $[\,,\,]$ in $\mathbf{R}^2$. (Hint below.)**15.4.** Given a a finite-dimensional real or complex vector space V, note that the set $\mathfrak{ga}(V)$ all affine mappings $V \to V$ is a vector space closed under the composition (Problem 15.1(iii),(iv)). Do these operations turn $\mathfrak{ga}(V)$ into an algebra? (Hint below.)**15.5.** Let V be a finite-dimensional real or complex vector space with a fixed skew-symmetric bilinear mapping $V \times V \ni (u, v) \mapsto [u, v] \in V$. Show that any of the following assumptions implies that V and $[\,,\,]$ constitute a Lie algebra, i.e., the

Jacobi identity $[[u,v],w]+[[v,w],u]+[[w,u],v]=0$ (formula (10.1)) is satisfied by all $u,v,w\in V$:

i. $\dim V\le 2$;

ii. $\dim V=3$ and (10.1) holds for *just one basis* (u,v,w) of V .

15.6. Show that in dimension 2 there exist exactly two isomorphism types of Lie algebras: one Abelian, one non-Abelian. The multiplication $[\,,\,]$ of the latter is given, in a suitable basis (u,v) , by $[u,v]=u$. Verify that the non-Abelian 2-dimensional real Lie algebra is isomorphic to the Lie algebra of the connected Lie group $\mathrm{GA}^+(\mathbf{R})$. **15.7.** Let V be an *oriented Euclidean* 3*-space*, that is, a real vector space with $\dim V=3$, carrying a fixed orientation (Problem 14.4) and a fixed inner product $\langle\,,\rangle$. The *vector product* $[u,v]\in V$ of vectors $u,v\in V$ (sometimes also called the *cross product* and denoted $u\times v$) is uniquely characterized by

i. $\langle u,[u,v]\rangle=\langle v,[u,v]\rangle=0$.

ii. $\langle [u,v],[u,v]\rangle=\langle u,u\rangle\langle v,v\rangle-\langle u,v\rangle^2$. (This is nonnegative by the Schwarz inequality, and vanishes only if u,v are linearly dependent.)

iii. If u,v are linearly independent, then the basis $(u,v,[u,v])$ of V belongs to the distinguished orientation. (That this is then a basis follows from (ii) and (i).)

Verify that

$$[u,v]=w \tag{98.5}$$

whenever (u,v,w) is an orthonormal basis of V compatible with the orientation. **15.8.** Show that any oriented Euclidean 3-space V, with the corresponding vector multiplication, is a Lie algebra. (Hint below.)**15.9.** For V as in Problem 15.8, prove that the adjoint representation $\mathrm{Ad}:V\to D(V)$ of the Lie algebra V ((vii) of **#10**) sends V isomorphically onto the Lie subalgebra $\mathfrak{so}\,(V)$ defined in Problem 10.3. (Hint below.)**15.10.** Let V be an oriented Euclidean 3-space. Prove that every skew-adjoint linear operator $A:V\to V$ consists in a rotation by the right angle about some axis (a 1-dimensional subspace L of V) followed (or preceded) by a dilation (multiplication by a scalar $\lambda\in\mathbf{R}$). (Hint below.)**15.11.** Given an oriented Euclidean 3-space V with the standard norm $|\;|$, so that $|v|=\sqrt{\langle v,v\rangle}$, show that $|[u,v]|$ coincides, for any $u,v\in V$, with the area (i.e., base times height) of the parallelogram spanned by the vectors u and v. (Hint below.)

Hint. In Problem 15.3, note that $(1,0)$ is the unit element for (98.4), and the inverse of any $(a,b)\in\mathbf{R}^2_+$ is

$$(a,b)^{-1}=(1/a,-b/a)\,.$$

Also, for $(a,b),(c,d)\in\mathbf{R}^2_+$ and $(p,q)\in\mathbf{R}^2=T_{(a,b)}\mathbf{R}^2_+$, we have, according to (11.3),

$$(a,b)(p,q)=(ap,aq)\,,\qquad (p,q)(a,b)=(pa,pb+q)\,.$$

Applying (13.4) to a C^2 curve $t\mapsto(a_t,b_t)\in\mathbf{R}^2_+$ defined on an open interval containing 0 with $a_0=1$, $b_0=0$, we obtain, for any $(p,q)\in\mathbf{R}^2=T_{(1,0)}\mathbf{R}^2_+$, $[(\dot a_0,\dot b_0),(p,q)]=\frac{d}{dt}\big|_{t=0}(a_t,b_t)(p,q)(a_t,b_t)^{-1}=\frac{d}{dt}\big|_{t=0}(p,a_t(q-pb_t))=(0,\dot a_0q-\dot b_0p)$. In other words,

$$[(x,y),(x',y')]\,=\,(0,xy'-x'y)$$

for all $(x,y),(x',y')\in\mathbf{R}^2$.
Hint. In Problem 15.4, no: distributivity fails (from one side).
Hint. In Problem 15.8, the unique operation $[\,,\,]$ with (i) – (iii) is bilinear and skew-symmetric since (i) – (iii) are easily verified when $[\,,\,]$ is given by the formal-determinant expression

$$[u,v]=\begin{vmatrix}u^1 & u^2 & u^3\\ v^1 & v^2 & v^3\\ e_1 & e_2 & e_3\end{vmatrix},$$

where e_j is a fixed orthonormal basis of V compatible with the orientation and $u=u^je_j$, $v=v^je_j$. The Jacobi identity now follows from (98.5) and Problem 15.5(ii).
Hint. In Problem 15.9, Ad is injective by (98.5) and is valued in skew-adjoint linear operators $V\to V$ since $\langle v,(\mathrm{Ad}\,u)v\rangle=0$ for all $u,v\in V$ (Problem 15.7(i)). Now $\mathrm{Ad}(V)=\mathfrak{so}\,(V)$ since both are 3-dimensional.
Hint. In Problem 15.10, let $A:V\to V$ be a nonzero skew-adjoint linear operator $A:V\to V$. By Problem 15.9, there is a unique (nonzero) vector $u\in V$ with $Av=[u,v]$ for all $v\in V$. Set $L=\mathrm{Span}\,(u)$. Thus, $A=0$ on L and, by skew-adjointness, A leaves the plane $L^\perp$ invariant. Considering the matrix of A restricted to $L^\perp$ in any orthonormal basis we find that it has the skew-symmetric form $\begin{bmatrix}0 & -\lambda\\ \lambda & 0\end{bmatrix}$ with some real λ, as required.
Hint. In Problem 15.11, we may assume $u\ne 0\ne v$. The area in question then equals $|u||v|\sin\theta$, where θ is the *angle* between u and v, characterized by

$$\langle u,v\rangle=|u||v|\cos\theta\,,\qquad 0\le\theta\le\pi\,.$$

Our assertion now follows from condition (ii) of Problem 15.7.

99.

Topics: .

Given a Lie group G of class C^r, $r\ge 2$, an open set $D\subset\mathbf{R}^2$ in the plane $\mathbf{R}^2=\{(s,t):s,t\in\mathbf{R}\}$ with the coordinates denoted s,t, and a C^1 mapping $F:D\to H$, let us set

$$S\,=\,F^{-1}\,\frac{\partial F}{\partial s}\,,\qquad T\,=\,F^{-1}\,\frac{\partial F}{\partial t}\,,\tag{99.1}$$

that is, $S,T:D\to T_1G$ are the mappings given by $S(s,t)=[F(s,t)]^{-1}\,\frac{d}{ds}\,F(s,t)$ (and similarly for T), where $\frac{d}{ds}\,F(s,t)\in T_{F(s,t)}G$ is the velocity at any given s of the curve $I\ni s\mapsto F(s,t)\in G$ (and I is any interval with $I\times\{t\}\subset D$). If F is of class C^2, we can also form partial derivatives such as

$$\frac{\partial S}{\partial t}\,,\,\frac{\partial T}{\partial s}\,:D\to T_1G\,.$$

On the other hand, $[S,T]:D\to T_1G$ will denote the mapping with

$$[S,T](s,t)=[S(s,t),T(s,t)]\,,\tag{99.2}$$

$[\,,\,]$ on the right-hand side being the Lie algebra multiplication in the tangent space $T_1G=\mathfrak{g}$ (**#11**).

Proposition. *Suppose that* G *is a Lie group of class* C^r, $r \geq 3$, *and* F *is a* C^2 *mapping from an open set in the* s,t*-plane into* G. *In the notations of (99.1) and (99.2), we then have*

$$\frac{\partial S}{\partial t} - \frac{\partial T}{\partial s} = [S,T]. \tag{99.3}$$

Proof. We need to show that

Corollary. *Given a Lie group* G *of class* C^r, $r \geq 3$, *and a* C^3 *mapping* F *from an open rectangle* $(\delta,\eta)\times(-\varepsilon,\varepsilon)$ *in the* s,t*-plane into* G. *If, for each* $s \in (\delta,\eta)$, *the curve* $t \mapsto F(s,t)$ *is the restriction to* $(-\varepsilon,\varepsilon)$ *of a group homomorphism* $\mathbf{R} \to G$, *then we have the* Jacobi equation

$$\partial^2 S/\partial t^2 = [\,\partial S/\partial t\,,\, T\,].$$

Proof. The homomorphism assumption means that $\partial S/\partial t = 0$ (and $F(s,0) = 1$ for all s). Thus, applying $\partial/\partial t$ to (99.3), we obtain (....).

Consider the function $Q : \mathbf{R} \to \mathbf{R}$ given by

$$Q(\lambda) = \begin{cases} \dfrac{1-e^{-\lambda}}{\lambda}\,, & \text{if } \lambda \neq 0 \\ 1\,, & \text{if } \lambda = 0, \end{cases}$$

Clearly, Q is analytic and can be expanded into the Taylor series

$$Q(\lambda) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!}\,\lambda^k$$

convergent everywhere in $\mathbf{R}$. Therefore (see Problems 21...) we can apply Q to linear operators $A : V \to V$ of any finite-dimensional real vector space V into itself, obtaining the linear operator $Q(A) : V \to V$ with

$$Q(A) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!}\,A^k \tag{99.4}$$

Lemma. *Let G and F satisfy the same assumptions as in the Corollary of* **#20**. *Then, setting* $v(s) = T(s,t)$ *and* $w(s) = (\partial S/\partial t)|_{t=0}$, *and using the notation of (99.4), we have*

$$S(s,t) = t\,[Q(t\,\mathrm{Ad}\,v(s))]w(s)\,. \tag{99.5}$$

Proof. Equation (...) reads

$$\frac{\partial^2}{\partial t^2}\,S(s,t) = -[\mathrm{Ad}\,v(s)]\,\frac{\partial}{\partial t}\,S(s,t)\,.$$

Solving this for $\partial S/\partial t$ (Problem ...), we obtain

$$\frac{\partial}{\partial t}\,S(s,t) = e^{-[\mathrm{Ad}\,v(s)]}\,w(s)\,.$$

As $S(s,0) = 0$, the latter equation can be solved for $S(s,t)$ as in Problem 21...., which proves (99.5).

Corollary. *Given a Lie group* G *of class* C^r, $r \geq 3$, *and any vector* $v \in T_1G$, *let* $\Phi_v : T_1G \to T_1G$ *be the linear operator given by*

$$\Phi_v = d\,(L_{e^{-v}} \circ \exp)_v$$

(with the identification $T_v(T_1G) = T_1G$). Then

$$\Phi_v \,=\, Q(\operatorname{Ad} v)\,.$$

In other words, the differential $d(\exp)_v : T_1G \to T_{e^v}G$ of the exponential mapping at any $v \in T_1G$ is given by

$$d(\exp)_v w \,=\, e^v[Q(\operatorname{Ad} v)w]$$

for all $w \in T_1G$.

Proof. As $w = \frac{d}{ds}\big|_{s=0}(v+sw)$, (...) implies $\Phi_v w = e^{-v}\frac{d}{ds}\big|_{s=0}e^{v+sw}$. The above lemma now can be applied to $F(s,t) = e^{t(v+sw)}$. Since $v(s) = T(s,t) = v + sw$ and $w(s) = (\partial T/\partial s + [S,T])|_{t=0} = w$, (by (99.3) with $S|_{t=0} = 0$), we obtain, from (...), $S(s,t) = t[Q(t\operatorname{Ad} v)]w$. Now (...) follows as $\Phi_v w = S(0,1)$.

Remark. Note that, according to (...), the differential of $\exp$ at v is an isomorphism if and only if so is $Q(\operatorname{Ad} v)$.

Proposition. *Let G be a Lie group of class C^r, $r \ge 3$, and let $U \subset T_1G$ be a neighborhood of 0 that $\exp$ maps diffeomorphically onto a neighborhood of 1 in G. For any $u \in T_1G$, denote $\widetilde{u}$ the unique left-invariant vector field on G whose value at 1 is u. Then, the push-forward $(d\exp^{-1})\widetilde{u}$ under $\exp^{-1}$ of the restriction of $\widetilde{u}$ to $\exp(U)$ is the vector field on U given by*

$$U \ni v \mapsto [Q(\operatorname{Ad} v)]^{-1}u\,.$$

(*Note that the inverse of $Q(\operatorname{Ad} v)$ exists, according to the preceding remark.*)

Proof. Fix $v \in U$ and set $w = [(d\exp^{-1})\widetilde{u}]v$. By the chain rule for composite mappings (p. 10), $d(\exp)_v w = \widetilde{u}(e^v)$ and so, by (11.5), $d(\exp)_v w = e^v u$. Thus, (...) implies $u = Q(\operatorname{Ad} v)w$, as required.

Theorem. *For every Lie group G of class C^r, $r \ge 3$, the maximal C^r atlas of the underlying manifold of G contains a unique maximal C^ω atlas making G a Lie group of class C^ω.*

Proof.

PROBLEMS

1. Given a finite-dimensional real associative algebra $\mathcal{A}$ with unit, a fixed element $a \in \mathcal{A}$, and an interval I containing 0, show that formula

$$x(t) \,=\, tQ(ta)$$

defines the unique C^1 curve $I \ni t \mapsto x(t) \in \mathcal{A}$ with

$$\dot{x}(t) = e^{-ta}\,, \qquad x(0) = 0\,.$$

2. Given a finite-dimensional real vector space V, a vector $w \in V$, a linear operator $A : V \to V$, and an interval I containing 0, verify that

$$u(t) \,=\, t\,[Q(t\operatorname{Ad} v)]w$$

is the unique C^1 solution $I \ni t \mapsto u(t) \in \mathcal{A}$ to

$$\dot{u}(t) = e^{-t\operatorname{Ad} v}w\,, \qquad u(0) = 0\,.$$

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