Pre-metric Electromagnetism

By

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For every good man chopping at the roots of evil there are a thousand more hacking at the leaves.

Henry David Thoreau

Leaf-hacking, being more labor-intensive than root-chopping, is more likely to get funding.

David Henry Delphenich

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Introduction

The term "pre-metric electromagnetism" refers to the formulation of the mathematical theory of electromagnetism in a manner that not only does not assume the existence of a Lorentzian metric on the spacetime manifold to begin with, but also exhibits the appearance of such a geometrical structure as a natural consequence of investigating the manner in which electromagnetic waves propagate through that medium. Since the Lorentzian metric that appears – at least, under restricted conditions – is commonly taken to account for the presence of gravitation in the spacetime medium in the viewpoint of general relativity, one sees that in order to properly define the context of pre-metric electromagnetism one must really discuss not only electricity and magnetism, but also gravity, as well.

Hence, before we embark upon the detailed discussion of the mathematical and physical bases for the theory of pre-metric electromagnetism, we shall briefly recall the conceptual evolution of the three relevant physical phenomena of electricity, magnetism, and gravity.

1. The unification of electricity and magnetism [1]. In ancient times, it is unlikely that man ever suspected that the natural phenomena associated with electricity, such as lightning and static electricity on animal furs, could possibly be associated with magnetic phenomena, which were discovered somewhat later in history, and most likely in the Iron Age in the context of lodestones, which are magnetite deposits that have become magnetized from long-term exposure to the Earth's magnetic field.

However, when Europe emerged from the Dark Ages into the Renaissance the science of electricity gradually gave way to the development of batteries [2], wires, and currents, on the one hand, with the development of compasses for marine navigation on the other. It was only a matter of time before the early experimenters, primarily Hans Christian Oersted (1777-1851) and Michael Faraday (1791-1867), noticed that electrical currents could produce measurable magnetic fields around conductors, while time-varying magnetic fields could conversely induce electrical currents in current loops. Actually, this is not a precise converse, since a time-constant electrical current can induce a time-constant magnetic field, while a time-constant magnetic field will *not* induce an electrical current of any sort. This reciprocal set of phenomena was called *electromagnetic induction*.

One of Faraday's other innovations in the name of electromagnetism was the introduction of the concept of invisible force fields distributed in space that accounted to the forces of attraction or repulsion that "test" electric charges and magnetic dipoles experienced when placed at each point. Although nowadays the concept of vector fields

[1] For a comprehensive historical timeline of the theories of electricity and the ether, one should confer the tomely two-volume treatise of E. T. Whittaker [**1**].

[2] Apparently, the concept of a battery had been developed in a rudimentary form by ancient cultures, as pottery that seemed to involve weak acids and metal electrodes has been unearthed by archeologists.

seems rather commonplace and above dispute, nevertheless, in its day the ideas of invisible lines of force apparently seemed rather mystical and dubious. It is important to note that the key to defining such fields is the association of a purely dynamical notion – namely, force – with more esoteric ones, in the form of electrical and magnetic fields.

In addition to this key development, one must also observe the evolution of the theory of electrostatic forces from the early measurements by Charles Augustin de Coulomb (1736-1806) and the formulation of the empirical law that bears his name to its formulation in terms of potential theory by Laplace, Poisson, and the host of contributors to the theory of boundary-value problems in the Laplace or Poisson equation, such as Green, Cauchy, Dirichlet, Neumann, Robin, and many more. It was also found that although the static magnetic field was not apparently due to a magnetic charge monopole, but a magnetic dipole, nevertheless, the Laplace equation could still be used in the context of a magnetic vector potential.

Part of the shift in emphasis from Coulomb's law to the Poisson equation involves the introduction of electrical flux and the use of Gauss's law, which is due to the German mathematician Karl Friedrich Gauss (1777-1855). One can also introduce a corresponding notion of magnetic flux and obtain an analogous law that relates the electrical current in a loop that bounds a surface with the total magnetic flux through the surface; this law was due to the French physicist André Ampère (1775-1835).

The capstone of the unification of electricity and magnetism into a single unified field theory of electromagnetism was laid by James Clerk Maxwell (1831-1879) [**2**], when he postulated that the sort of electromagnetic induction that was discovered by Faraday might also work in reverse. That is, a time-varying electric flux through a surface with boundary might induce a *magnetomotive* force, or *displacement current*, in the boundary loop; effectively, this amounts to saying that it induces a magnetic field.

However, there was a significant difference between the two types of electromagnetic induction, namely, the one was 180^{o} out of phase with the other one. This took the form of a relative minus sign in the resulting field equations for the curl of the electric field strength vector field **E** and the curl of the magnetic field strength vector field **H**.

The resulting set of four first-order partial differential equations for the vector fields **E**, **B**, **D**, **H** [3]:

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \qquad \nabla\cdot\mathbf{E} = 4\pi\rho, \qquad \nabla\times\mathbf{H} = +\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{J}, \qquad \nabla\cdot\mathbf{B} = 0,$$

which constitute Maxwell's equations in one of their simplest forms, represent the culmination of Faraday's law, Coulomb's law, Ampère's law, combined with Maxwell's displacement current, and the non-existence of magnetic monopoles, respectively. The vector field **J** represents an electric current density, which can be associated with the electric charge density ρ by way of its motion with flow velocity vector field **v** in the form of $\rho\mathbf{v}$, or it can also be an electric source current that is independent of ρ.

Actually, the nature of the vector field **J** is not entirely arbitrary, since taking the divergence of the third equation gives the conservation of charge in the form:

[3] We present these equations in the form that they are given in the standard reference work by J. D. Jackson [**3**], but, as we shall discuss in due course, many other forms exist.

$$0 = \frac{\partial(\nabla \cdot \mathbf{D})}{\partial t} + 4\pi \nabla \cdot \mathbf{J}\,.$$

One can also regard this as an integrability condition for the over-determined system of linear first-order partial differential equations for **H** and **D** that the third set of Maxwell equations, in the form above, represents.

The constant *c* is the speed of propagation of electromagnetic waves *in vacuo*. However, its introduction at this point seems unrelated to general considerations, since the system of equations is valid in media other than the vacuum, and at this point the constant *c* serves as more of a units conversion constant than a fundamental property of an electromagnetic medium of a particular sort. We shall discuss these issues in more detail in the next section.

Moreover, in order to state Maxwell's equations in their traditional form, we have introduced two further vector fields **D** and **B**, which were classically referred to as the electric displacement and magnetic flux density vector fields. Hence, the system of equations as it was stated above is underdetermined; viz., we have defined nine component equations for the fifteen components of the vector fields **E**, **B**, **D**, **H**, **J**. The six remaining equations that we require take the form of the *electromagnetic constitutive law*:

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{B}), \qquad \mathbf{H} = \mathbf{H}(\mathbf{E}, \mathbf{B}),$$

for the medium in question that relates the fields **E** and **B** to the fields **D** and **H** in a manner that is closely analogous to the way that mechanical constitutive laws couple the stress tensor for a material medium to its strain tensor. We shall discuss this aspect of Maxwell's equations in more detail in the next section of this introduction, as well, since it defines the key to understanding why pre-metric electromagnetism is based in established experimental physics, as well as purely mathematical refinements.

2. The evolution of geometrical optics into wave optics. Perhaps the most profound consequence of Maxwell's hypothesis was that it led to the existence of solutions of his equations that took the form of electromagnetic waves. This opened up a new realm of possibilities for explaining the optical phenomena that seemed beyond the reach of the earlier methods of geometrical optics.

The study of optics seems as old as man's awareness of the presence of light in the world. One can find writings on the subject as far back as Euclid, who also wrote about geometry more generally. Indeed, one sees that the concept of "light rays," which is at the root of geometrical optics, seems to be the first way of modeling the behavior of light.

One of the first enduring results of geometrical optics was due to Willebrod Snell (1581-1626), who experimentally derived the largely empirical result that the ratio of the sines of the angles of refraction and incidence at an optical interface could be expressed in terms of the ratio of two numbers that were characteristic of the materials. Although we now understand that these numbers are the indices of refraction of the materials and they are inversely proportional to the speeds of propagation of electromagnetic waves in them, at the time Snell had no such insight into the nature of the numbers. Interestingly,

after Snell's contribution the next significant advance in optics did not come about until fifty-two years after his death.

Similarly, although Sir Isaac Newton (1642-1727) published his celebrated treatise [**4**] in 1704, in which he suggested that light was due to the motion of infinitesimal "corpuscles," nonetheless, the Dutch physicist Christiaan Huygens (1629-95) had published a treatise [**5**] in 1678 in which he was introducing the notion that the behavior of light had more to do with envelopes of "elementary waves."

One also notes that after Newton's treatise the next major advance to optics did not seem to come until 104 years after its publication, which would be the work of Etienne-Louis Malus (1775 –1812), who did many experiments on the polarization of light and derived a law for the resulting intensity of a light beam that passes through a polarizer, as well as observing that reflected light is always polarized. His work [**6**] also made considerable use of what later become *line geometry*, which was first developed by the German geometer Julius Plücker (1801 – 1868).

As we will see, one of the foundations of pre-metric electromagnetism is due to the work of Augustin-Jean Fresnel (1788 – 1827), whose research in optics [**7**] further reinforced the wavelike conception that Huygens had previously introduced. The concept of the Fresnel wave surface or ray surface also reinforces the fundamental role of line geometry, as it admits such a formulation and interpretation quite naturally.

One the most far-reaching conceptual advances for geometrical optics was due to the contributions of Sir William Rowan Hamilton (1805–1865) to the Royal Irish Society [**8**] over a period of time from 1828 to1837. He essentially established that the same general mathematical techniques that allowed one to describe the trajectories of point particles in the motion of matter could suffice to determine the curves – viz., light rays – that were followed by light corpuscles. Perhaps some inkling of how long it took for physics to completely digest new ideas in those days was in the fact that when the German geometer Felix Klein (1849 – 1925), a former student of Plücker's who also advanced the cause of line geometry, commented upon the work of Hamilton some fifty-five years after his last supplement on the theory of rays had been published the title of his paper [**9**] (in translation) was "On *recent* English work in mechanics."

In roughly the same year as Klein's article, Pierre de Fermat (1601 or 1607/8 – 1665) began doing his own work on geometrical optics, which further reinforced the applicability of Hamilton-Jacobi theory by showing that one could derive the equations of geometrical optics by starting with a least-action principle, just as Hamilton's equations are derived from Hamilton's least action principle. The trick is to define an appropriate action functional for curves between points of space if one is to make the curves one has in mind – viz., light rays, in this case – take the form of the paths along which the action function has a minimum. The action that Fermat introduced for geometrical optics was the functional that associated each spatial curve segment with the time it took for light to go from one end to the other.

In 1895, H. Bruns published a lengthy treatise [**10**] in the papers of the mathematical physics class of the Saxon Academy of Sciences in which he introduced the notion of the *eikonal*, a function that embodied the essential information for the propagation of light rays. Klein also responded to this contribution in 1901 with a paper [**11**] in Zeitschrift für Mathematik und Physik in which he showed that the eikonal of Bruns was virtually the same thing as the *characteristic function* of Hamilton-Jacobi theory in mechanics.

Another tributary of research that was emerging in the later Nineteenth Century that eventually met up with both Hamiltonian mechanics and Hamiltonian optics was the method of contact transformations and contact geometry. They had been introduced by Marius Sophus Lie (1842-1899) in his monumental 1888 treatise [**12**] on transformation groups in the form of "Berührungstransformationen." Not only did one-parameter families of such transformations describe the motion of massive particles in Hamilton mechanics, but, as Ernest Vessiot (1865-1952) observed in 1906 [**13**] they also serve to describe the motion of light corpuscles in geometrical optics.

To return to the wave conception of light, around the time of Maxwell the general drift in thinking was towards a mechanical conception of light waves as behaving somewhat like elastic waves in a medium they referred to as the "ether." However, it eventually emerged that the optical wave constructions of Huygens, which predated the work of Maxwell by a considerable margin, were nonetheless of sufficient generality as to continue to apply to the electromagnetic waves that Maxwell's equations implied.

Consequently, the impact of Maxwell's equations on physics was not only in their complete unification of the theories of electricity and magnetism, but also the fact that they gave a precise physical basis for the notion of *wave optics*; i.e., the idea that the behavior of light in optical media was best explained by the propagation of electromagnetic waves, rather than light corpuscles.

However, since geometrical optics, namely, the optical constructions that were based in the Newtonian conception of corpuscular light, was not only adequate in its experimental accuracy in some contexts, such as reflection and refraction, but also more convenient, it was important to also establish the way that geometrical optics might emerge from wave optics, at least as an approximate class of solutions. One simply accepted that there were well-established optical phenomena, such as dispersion, diffraction, and polarization that seemed largely foreign to the methods of geometrical optics.

In order to account for diffraction, one must return to the wave optics of Huygens and regard each point of any spatial isophase surface (momentary wave front) as a potential source of elementary waves, such as expanding spherical waves. The form of these elementary waves is intimately related to the form of the Fresnel surfaces, and it is generally only isotropic optical media in which the elementary wave fronts are expanding sphere. One finds that if waves, in general, are characterized by a phase function, which defines the shape of the momentary wave fronts, and an amplitude function, which defines how the basic physical quantity is carried along by the wave fronts, then Huygens's principle by itself only allows one to propagate the momentary isophase surfaces.

In order to propagate the amplitude, as well, one usually must look more specifically at the nature of the wave equations that one is concerned with. When that equation is the linear wave equation, which again pertains to optically isotropic media, one often resorts to the methods of integral operators and fundamental solutions, such as Green functions, in order to propagate the amplitude. When one reduces from time-varying wave solutions to stationary solutions, the linear wave equation, after separating the time variable becomes the *Helmholtz equation*, after the German physicist and physician Hermann Ludwig Ferdinand von Helmholtz (1821 – 1894), who published his treatment of the problem of propagating amplitudes in 1859 [**14**].

As is often the case with integrals, general solutions are not possible, and one often must resort to approximations. The case of diffraction is no different and the approximate evaluation of the Helmholtz integral for the treatment of diffraction was first achieved by Sir George Gabriel Stokes (1819–1903) in his 1849 memoir [**15**] on the "Dynamical Theory of Diffraction" and refined by Gustav Robert Kirchhoff (1824–1887) in 1882 [**16**].

What eventually emerged was that even Kirchhoff's approximate integral was too complicated to admit general solutions, and the method of asymptotic approximations was introduced into the theory of diffraction. This method originated in previous work [**17**] of Henri Poincaré (1854–1912) on the three-body problem of celestial mechanics and involved the introduction of series expansions for solutions that did not have to converge, except in some asymptotic limit. One then obtains the solution for the propagation of amplitude in terms of successive contributions to the "classical" solution that one obtains from the geometrical optics approximation, which one then regards as the "diffracted fields" of each order.

It is mostly in the study of optical phenomena that one sees the full scope of the previous remark on the necessity of introducing electromagnetic constitutive laws for the medium in which one propagates electromagnetic waves. Due to the vast and increasing variety of optical media that have been studied up to this point in time, one sees that even though the constitutive laws are a system of algebraic equations, not a system of differential equations, nonetheless, one must leave open a considerable degree of generality concerning the properties that go into them depending upon the medium in question.

In particular, the first issue that seems to arise is that of linearity, namely, the possibility that the system in question is a system of linear equations. As usual, one finds that linearity is a simplifying approximation that one introduces only in order to make tangible progress in analyzing the equations, not a deep assumption about the laws of Nature. Indeed, if Nature has any fundamental law in that regard it would have to be: Linearity is always a simplifying approximation for something more nonlinear and complicated. In fact, nowadays, the field of nonlinear optics – i.e., the optics of nonlinear media – has grown to quite vast proportions in its own right in theory, experiment, and practical applications.

Other material properties that one must consider in the name of constitutive laws are homogeneity and isotropy. That is, the optical properties of a medium can vary from point to point, although often the change is a discontinuity at the interface between two different media, and the propagation of electromagnetic waves might even have a strong correlation with the direction of propagation. Furthermore, optical properties often depend upon the "state of polarization" of the wave, as well; by this term, we are referring to the fact that electromagnetic waves are generally polarized linearly, circularly, or elliptically.

Hence, one must realize that the model that one traditionally uses for the classical electromagnetic vacuum state is characterized by a constitutive law that is linear, isotropic, and homogeneous. Thus, one can concisely define it by two constants ε_0 and μ_0, which represent the electric permittivity (or dielectric constant) and magnetic permeability of the vacuum. It is important to note that speed of propagation c, which

was introduced above in a somewhat *ad hoc* way, then becomes a *derived* constant, by way of:

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}},$$

not merely a convenient basis for a unit conversion.

As we shall see later on, this situation is generic to optics, except that one does not always derive a single constant, such as *c*, but more generally a set of three functions of position.

3. Geometrization of gravity. Although the primary focus of this book is, of course, pre-metric *electromagnetism*, nonetheless, in order to fully account for the appearance of a spacetime metric, as well as to discuss its role in the equations of electromagnetism, one must unavoidable touch upon the fact that the spacetime metric plays the fundamental role in Einstein's theory of *gravitation*.

If the natural phenomena that pertained to magnetism seemed remote in their manifestation from those of electricity, it is just as much the case they are both remote from the phenomena that pertained to gravitation. Indeed, one encounters gravitational phenomena, such as falling, in primitive settings more frequently than one encounters electrical or magnetic ones.

However, as humanity learned more about nature, it eventually emerged that there was a close analogy between the law of gravitational attraction, as it was described by Newton, and the law of electrostatic attraction and repulsion, as it was described by Coulomb. Of course, there were some perplexing difficulties associated with making that analogy precise.

For one thing, the fact that electrostatic forces could be either attractive or repulsive, while gravitational forces were only observed to be attractive, suggested that there were two types of charges – viz., positive and negative ones – but only one type of gravitational mass. Indeed, if one postulates the existence of negative masses and examines their effect on Newton's equations of motion when the force is due to gravitation, one obtains contradictory conclusions. In particular, look at the one-dimensional picture, which is governed by:

$$G\frac{Mm}{r^2} = ma_m = -Ma_M,$$

in which M is one mass, m is the other, a_m and a_M are their respective accelerations, r is the distance between their center of mass, and G is Newton's gravitational constant.

Now, assume that the sign of M is positive, while that of m is negative. Although this implies the intriguing "anti-gravitational" possibility that the gravitational force between them is repulsive, nevertheless, if one assumes that Galileo's experiment is equally valid for negative masses as it is for positive ones – i.e., gravitational mass equals inertial mass, including their signs – then one must accept that the acceleration of a negative mass is in the *opposite* direction to the applied force, while the acceleration of the positive mass is in the *same* direction. Hence, the effect is that the negative mass seems

to accelerate in the same direction as the positive mass and they "chase each other" off to infinity, rather than flying apart as electric charges of the same sign would.

Of course, one way around this is to weaken Galileo's equality of gravitational and inertial mass to simply apply it to their *absolute values*. However, in the absence of experimental confirmation for the gravitational and inertial behavior of negative masses, such speculations are rather moot.

Another fascinating aspect of the analogy between electrostatics and gravitostatics is that historically no one pursued the possibility that it might extend to an analogy between electrodynamics and gravitodynamics, as well, until rather recent – i.e., post-Einsteinian – history. To be fair, this is because the gravitostatic force is quite small in its magnitude to begin with, except in astronomical contexts, and one usually expects the magnetic forces that are produced by moving charges to be considerably weaker than the electrostatic ones, as well. Hence, the only laboratory in which any "gravitomagnetic" forces produced by moving masses, in addition to the gravitostatic ones, might possibly be measurable would have to be astrophysical. It is only with the advances of high-precision measuring technology and orbiting space vehicles that science has managed to discover that there are expansions in the scope of Newtonian gravitation that logically precede the refinements of general relativity, which represents essentially a "strong-field" theory of gravitation, while Newton's theory represents its "weak-field" form.

Although nowadays general relativity is usually thought of as Einstein's theory of *gravitation*, one must understand the crucial and unavoidable fact that he did not originally set out to address gravity. Indeed, Einstein's earliest work on relativity grew out of his studies of *electromagnetism*, namely, "The Electrodynamics of Moving Bodies" (cf., [**18**]). The main issue at that point in time was the way that the equations of electromagnetism transformed from one measurer/observer to another, when one assumed that the medium itself was in its own state of "motion."

Now, when the electromagnetic medium is a material one, such as glass, it is easier to justify the assumption that it is in a state of motion relative to the measurer/observer. Indeed, there are experimentally established phenomena associated with such a relative motion; for instance, the Fresnel-Fizeau effect, which is due to the French physicists Fresnel and Armand Hippolyte Louis Fizeau (1819-1896). However, when one is considering an immaterial – i.e., massless – electromagnetic medium, such as the vacuum, it is harder to make the concept of its relative motion logically rigorous. Indeed, the negative result of Michelson and Morley concerning the possibility of measuring a difference between the speed of propagation of light in the direction of motion of the Earth, relative to the vacuum of space, and transverse to it showed that there were severe limitations to the mechanical analogy for electromagnetic wave propagation that the ether model had suggested.

As is well-known by now, the resolution of the paradox was effected by assuming that one had to add the time dimension to the spatial manifold to produce a four-dimensional spacetime manifold, and that the positive-definite, or "spherical," Euclidian metric had to be replaced with an indefinite hyperbolic one. This had the effect of replacing the origin as the sole "isotropic" point of Euclidian space with the light-cone of Minkowski space, which was named for Hermann Minkowski (1864-1909), who had observed that geometrical aspect of Einstein's theory of special relativity in his address to the 80th Assembly of German Natural Scientists and Physicians at Cologne in 1908 [**19**].

Largely at the suggestion of Marcel Grossman, Einstein then used the extension of Euclidian geometry to Minkowski geometry as the basis for a further extension to the differential geometry of curved spaces [**20**], which was primarily due to Georg Bernhard Friedrich Riemann (1826-1866) at that point in time, although the geometry of Riemann was actually still positive-definite in its signature. Nowadays, one refers to the spacetime manifold as a *Lorentzian manifold*, in honor of the Dutch physicist Hendrik Antoon Lorentz (1853-1928. cf., e.g., [**21**]), who actually did not define such manifolds, although his research in the relativity of electromagnetism led him to introduce transformations, namely, boosts, that augmented the Euclidian spatial rotations to give a six-dimensional group that is now called the *Lorentz group*. Hence, putting his name on the group or the manifolds is simply a gesture of respect for his fundamental role in relativity, like many of the names that get associated with units of measurement.

Eventually, Einstein showed that when one extended Minkowski geometry to Lorentzian geometry, the resulting curvature of spacetime, when coupled to the energy, momentum, and stress in the distribution of matter in spacetime, produced a set of field equations for gravitation. This curvature manifested itself primarily in the deviation of "geodesics" from straight lines into curves, and the solutions to these field equations then represented the Lorentzian metric tensor field, whose components took on the interpretation of gravitational potentials. Indeed, Einstein showed that one could recover the Newtonian law of gravitation in the weak-field limit in the form of the Poisson equation for the gravitational potential functions.

As the theoretical physics community became sufficiently comfortable with the new picture for gravitation as a manifestation of the geometry of the spacetime manifold, they returned to the problem of the relativistic formulation of electromagnetism, first, in Minkowski space and then in a Lorentzian manifold. Eventually, it was found that the most elegant way of representing Maxwell's equations was to first absorb the **E** and **B** field into a single second-rank antisymmetric tensor field F:

$$F = \tfrac{1}{2} F_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu}, \qquad F_{0i} = E_i\,, \qquad F_{ij} = \varepsilon_{ijk} B^{k}.$$

In these expressions, the Greek induces range from 0 to 3, with 0 representing the time dimension of the spacetime manifold, at least for the particular choice of coordinate system, and the Latin indices range over the spatial values of 1, 2, 3. The Levi-Civita symbol ε_{ijk} is completely anti-symmetric and normalized such that $\varepsilon_{123} = 1$. The wedge symbol in the definition of F represents an anti-symmetrized tensor product of the 1-forms dx^{μ} and dx^{ν}; we shall discuss the definitions of all these terms in due course, but, for now, we refer to them casually.

Notice that although we are treating the components of **E** as if they are fundamentally those of a "spatial" covector field, nevertheless, we seem to treating the components of **B** as if they are fundamentally those of a spatial *vector* field that is associated with the components of a 2-form by way of the spatial "duality" that one obtains from the Levi-Civita symbol. As we shall see, this duality plays a key role in the geometry of electromagnetism.

One can then represent four of Maxwell's equations, namely, the three equations for the curl of **E** and the one for the divergence of **B**, quite concisely as:

$$dF = 0,$$

in which the d operator is the exterior derivative operator that acts on completely anti-symmetric covariant tensor fields, which are referred to as *exterior differential forms.*

In order to obtain the remaining four Maxwell equations, one similarly forms the 2-form:

$$G = \tfrac{1}{2} G_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu}, \qquad G_{0i} = D_i\,, \qquad G_{ij} = \varepsilon_{ijk} H^k.$$

One then introduces the Hodge * isomorphism, which is an invertible linear map from 2-forms to 2-form that takes G to:

$$*G = \tfrac{1}{2} *G_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu}, \qquad *G_{0i} = -H_i\,, \quad *G_{ij} = \varepsilon_{ijk} D^k.$$

Here, we have implicitly used the spatial Euclidian scalar product to raise and lower indices.

The last four of Maxwell's equations then take the form:

$$d*G = 4\pi *J \qquad (J = \rho dt + J_i\, dx^i),$$

or, if one defines the codifferential operator $\delta = *^{-1} d *$:

$$\delta G = 4\pi J.$$

One can then append the electromagnetic constitutive law in the form:

$$G = \kappa(F).$$

Before we return to criticize the necessity or desirability of introducing a Lorentzian metric in order to obtain the Maxwell equations in the present form, we return to our historical perspective in order to give further physical reasons for seeking a pre-metric form for them.

4. Attempts at unifying electromagnetism with gravitation. By the time that Einstein's theory of gravitation was becoming widely known to theoretical physicists, in combination with Maxwell's theory of electromagnetism, there was also an increasing suspicion – which mostly originated with Einstein himself – that perhaps the theories of electromagnetism and gravitation could be effectively unified, just as the theories of electricity and magnetism had been.

Many attempts at arriving at such a unification were subsequently made throughout the balance of the Twentieth Century [4], although after the 1930's the emphasis shifted away from the "classical" (more properly, neo-classical) problem of unifying gravitation

[4] An excellent historical reference on the various attempts that were made can be found in Vizgin [**22**]. The geometric aspects of Kaluza-Klein and Einstein-Schrödinger are covered in detail in Part II of Lichnerowicz [**23**].

and electromagnetism to the "quantum" problems that came about with the emergence of quantum field theory as the preferred approach to the theory of elementary particles and their interactions.

What all of the neo-classical models had in common was that they started with the assumption that one still had to consider the geometry of spacetime, and then enlarged the scope of the geometry in one manner or another. Generally, they were also of the Einstein-Maxwell type, in the sense that they attempted to account for the Einstein equations for gravitation, along with the Maxwell equations for electromagnetism. However, we shall treat the solutions to the Einstein field equations that involved finding the metric tensor for a four-dimensional Lorentzian spacetime when one includes the energy, momentum and stress of an electromagnetic field as a source for the curvature as being distinct from this class of unified field theories; such attempts were made by Rainich [**24**] and Wheeler [**25**].

One of the early set of attempts were made by the German mathematical physicist Hermann Weyl [**26**], the French mathematician Élie Cartan [**27**], who was also largely responsible for first emphasizing the utility of exterior differential forms in the expression of differential geometry, and the English astronomer and physicist Sir Arthur Stanley Eddington [**28**]. The basic expansion of scope in the geometry was in weakening the requirement that the spacetime connection, which allows one to define locally define the parallel translation of tangent vectors and frames, did not have to be a metric connection; that is, parallel translation did not have to preserve lengths and angles. This had the effect of introducing four additional components for the connection 1-form, which would then eventually be associated with the components of the electromagnetic potential 1-form $A = A_\mu \, dx^\mu$. Although the actual attempt at unification eventually failed – for instance, Einstein was concerned that the length of a tangent vector under parallel translation might depend upon the choice of curve used for the translation – nevertheless the mathematical apparatus that was introduced still drew considerable attention to the concerns of conformal geometry, as well as defining the tributary of research that eventually became gauge field theory.

Another attempt at unification that showed much promise, and is still being discussed in various forms, was made by Theodore Kaluza [**29**] and Oskar Klein [**30**]. The expansion of scope took the form of the addition of yet another dimension to the spacetime manifold and the extension of the Lorentzian metric $g^{\mu\nu}$, which locally takes on the form of the hyperbolic normal metric of Minkowski space η = diag[+1, –1, –1, –1] in an orthonormal frame field, to a Lorentzian metric G^{AB}, $A, B = 0, \ldots, 4$ with the signature type [+1, –1, –1, –1, –1]. This had the effect of introducing five more components to the metric tensor field, four of which, namely, $G^{\mu 0}$, indirectly accounted for the electromagnetic potentials.

An intriguing consequence of the use of this metric was that when one formed the d'Alembertian operator $\Box_5 = G^{AB} \, \partial^2 \, / \, \partial x^A \partial x^B$ that allows one to define the five-dimensional linear wave equation $\Box_5 \Psi = 0$, one found that separating out the fifth coordinate in the wave function by introducing a separation constant of the form k^2 produced a four-dimensional equation that took the form $\Box\psi + k^2 \psi = 0$ of the Klein-

Gordon equation [5]. In effect, the appearance of the mass in four-dimensions was a by-product of separating out the fifth coordinate when one started with a massless wave equation in five dimensions.

To say that the Kaluza-Klein program failed is somewhat unfair, since it did obtain both the Einstein equations for gravitation and the Maxwell equations of electromagnetism; hence, the theory did not fail in the sense of implying an unacceptable contradiction. The main issues that emerged as controversial were in the interpretation of the fifth coordinate x^4, as well as the remaining extra component of the metric, namely G^{00}, and the fact that no new couplings between electromagnetism and gravitation seemed to emerge. That is, there seemed to be no "induction" terms to examine experimentally that would say that electromagnetic fields might affect gravitational ones or vice versa. Hence, the unification of the field theories was really more of a concatenation.

Actually, there is good reason to assume that the fifth coordinate takes the form of the proper time parameter τ. This has the effect of making the fifth component of velocity vectors represent the speed of propagation of light, so instead of restricting velocity vectors to lie on the unit proper time hyperboloid in a four-dimensional Minkowski space, one requires them to lie on a light cone in a five-dimensional one, along with lightlike ones, which are associated with $\tau = 0$. This way of extending spacetime has an immediate interpretation in terms of manifolds of jets of differentiable curves, although we shall not return to that aspect of jet manifolds when we discuss their geometry later on.

In the process of accounting for fifth coordinate and the G^{00} component of the Kaluza-Klein metric, various researchers, notably Einstein and Mayer [**32**], Cartan [**33**], Oswald Veblen [**34**], and the Dutch mathematicians Jan Schouten and David van Dantzig [**35**] pursued the possibility that introducing the extra coordinate really represented the transition from inhomogeneous coordinates to homogeneous coordinates that one uses in projective geometry. Furthermore, this was consistent with the way that the Kaluza-Klein picture related to the Klein-Gordon wave equation. Of course, since it was still basically the same Kaluza-Klein model, the same criticisms applied, except that one could account for fifth coordinate and the extra metric component. However, to this day, projective relativity still manages to attract serious attention (cf., Schmutzer [**36**]).

Another attempt at Einstein-Maxwell unification that continues to attract modern attention was the one that Einstein, and later Mayer, made [**37**] under the name of *teleparallelism*. It was based the fact that a linear 4-coframe field $\{h_\mu, \mu = 0, \ldots, 3\}$ on a four-dimensional differentiable manifold has sixteen independent components $h_{\mu\nu}$, in general, which is also the total number of independent components that one expects from the metric tensor field $g_{\mu\nu}$ and the electromagnetic potential 1-form A_μ . Now, Roland Weitzenböck [**38**] had already discussed the geometry of parallelizable manifolds – which are sometimes called Weitzenböck spaces, for that reason – which are differentiable manifolds on which there exists a global frame field. For instance, given a choice of global frame field one can define a canonical volume element, Lorentzian

[5] Not only was this the same Klein that came up with the Kaluza-Klein theory, but one finds that his father Felix had also done some work on the subject of how arbitrary geodesic equations could be represented as null geodesic equation by introducing the extra coordinate as one would in projective geometry; this is discussed at length in Rumer [**31**].

metric, and linear connection. However, unlike the Levi-Civita connection that one obtains from a metric tensor, which has zero torsion and generally non-vanishing curvature, the Weitzenböck connection, which is defined to make the given frame field parallel, has just the opposite property. Hence, all of the geometry is in the torsion tensor field. One finds that not only are Lie groups important examples of parallelizable manifolds, but general parallelizable manifolds are, in a sense, "almost Lie groups." In particular, one replaces left-translation with parallel translation and the Maurer-Cartan connection with the Weitzenböck connection.

The failure of teleparallelism took the form of unacceptable contradictions, namely, it did not admit the Schwarzschild solution for the case of a spherically symmetric stationary purely gravitational field and it did admit a stable configuration of gravitating masses with no other forces acting to stop their mutual attraction. However, it is interesting that Einstein and Mayer made no mention of the topological obstructions to the existence of global frame fields, which are sufficiently severe that even such homogeneous spaces as most spheres (except in dimension 0, 1, 3, and 7) do not admit them. Indeed, the first definitive work on the subject of topological obstructions to parallelizability was made by Ethan Stiefel [**39**] in 1936, several years after Einstein and Mayer went on to other things. Stiefel constructed characteristic $\mathbb{Z}_2$-homology classes, whose dual $\mathbb{Z}_2$-cohomology classes are now referred to as *Stiefel-Whitney classes*, since Hassler Whitney preceded the work of Stiefel with his own derivation of the characteristic classes in 1935 [**40**]; however, the paper of Whitney does not explicitly address the problem of parallelizability.

The idea for such characteristic classes came to Stiefel as a generalization of the theorem of his advisor Heinz Hopf on obstructions to non-zero vector fields [**41**], which was also discussed by Henri Poincaré, in which the characteristic class one considers is the Euler class, whose integral, when it is represented by a differential form, over the manifold equals the Euler-Poincaré characteristic. Since the work of Stiefel and Whitney did not seem to find a path to Einstein and Mayer in that era, there was no attempt to extend the theory to the case of a non-parallelizable manifold; i.e., a singular frame field whose singularities might very well generate non-vanishing curvature of topological origin.

It is worth pointing out that in the eyes of pure mathematics the topological issue of whether the spacetime manifold is parallelizable or not is an unavoidable fundamental question to address. Hence, regardless of whether teleparallelism failed to unify gravitation and electromagnetism, the possible physical manifestation of these topological singularities will continue to be a fundamental problem.

Another way of weakening the hypotheses made on the spacetime connection that general relativity introduces to account for gravitation, besides dropping the requirement that it be a metric connection, is to weaken the assumption that it has vanishing torsion. Perhaps as a result of Einstein's prior exposure to non-zero torsion by way of teleparallelism, he [**42**], and later Erwin Schrödinger [**43**], pursued that extension of scope in the geometry of spacetime. Although the Einstein-Schrödinger unification program did not ultimately succeed, nevertheless, it did introduce the importance of considering spacetimes with non-vanishing torsion and the general role it plays, and interest in that topic has not vanished completely even to the present day. Perhaps to

some extent this is due to the seminal papers of Cartan [**44**] on the subject of the geometry of spaces with torsion and their application to the spacetime of general relativity, and partly to the fact that the geometry of connections with non-vanishing torsion and curvature had become quite established in the literature of the theory of plastic materials with continuous distributions of defects that are referred to as *dislocations* and *disclinations*, respectively [6]. Hence, the applicability of the mathematical concept is more general than spacetime structure. The reference by Lichnerowicz also contains a thorough discussion of the Einstein-Schrödinger models.

Yet another extension of Lorentzian spacetime geometry was made by John L. Synge [**46**] and Vranceanu [**47**]. To them, a promising way of interpreting the fifth coordinate in the Kaluza-Klein model was to imagine that actually the four-dimensional spacetime manifold was an *anholonomic hypersurface* in a five-dimensional manifold. This term is really a misnomer, in the same way that the phrase "non-inertial coordinate system" is not rigorously justified, and for the same reason. What one is really dealing with, as in the case of dynamical systems with anholonomic constraints, is a sub-bundle of the tangent bundle to a differentiable manifold that has corank one and is not assumed to be integrable as a differentiable system on the manifold. That is, one is associating a hyperplane in each tangent space in such a manner that there is no foliation of the manifold into leaves of codimension one that might represent spacetime manifolds. In effect, one sees them locally in the tangent spaces, even though they do not exist globally.

Although the method of anholonomic submanifolds attracted only passing attention, again the fact that it addresses fundamental issues, such as integrability and anholonomic constraints, suggests that it is still worthwhile to understand the possible implications of such considerations.

5. Rise of quantum electrodynamics. Eventually, Einstein himself developed some suspicions about the Einstein-Maxwell unification problem that were probably quite definitive. For one thing, the mainstream of physics was increasingly focusing on the problems of the emerging field of quantum physics, which was pursuing directions of approach to its problems and interpretations that were largely inconsistent with the very spirit of Einstein-Maxwell physics. Because the nature of electromagnetism at the atomic-to-subatomic level seemed to be seriously at odds with Maxwellian electromagnetism, Einstein increasingly suspected that the unification of electromagnetism with gravitation might be more meaningful when one introduced the quantum considerations.

One way of seeing that this is probably unavoidable is to note that the modern acceptance of gravito-electromagnetism as a valid extension of Newtonian gravitostatics to gravitodynamics suggests that really the Maxwell equations of electromagnetism – or gravito-electromagnetism, for that matter – are manifestly weak field equations, while the Einstein field equations of gravitation are mostly significant in the realm of large gravitational field strengths, such as in the neighborhood of compact astronomical bodies (neutron stars, black holes, and the like). Hence, one might expect that the Einstein

[6] This train of thought has roots in the early days of relativity theory and has generated a considerable body of literature. For a modern reference that contains citations to classical references one might confer Kleinert [**45**].

equations should be unified with some corresponding strong-field equations of electromagnetism that might pertain to the field strengths that one encounters in the close proximity to elementary charge distributions, such as electrons and nuclei.

Here, one encounters the most fundamental obstacle that separates Einstein-Maxwell field theories from quantum field theories, such as quantum electrodynamics, in particular. Because the nature of sub-atomic physics includes the idea that one will ever directly observe the inner structure of atoms, nuclei, and nucleons, the very methodology of quantum physics quickly took on an *ad hoc* sort of character that is now simply referred to as *phenomenology*. This philosophy is essentially a variation on the basic tenet of solipcism that says "to be is to be measured." Hence, one always treats the system that one is investigating as something of a "black box," such that all one can consider is the way that it responds to inputs. Eventually, one hopes that a sufficiently complete set of input-output relationships will allow one to construct a model for the states of the system inside the box and then speculate on the nature of the system itself. For instance, this is how geophysics constructs models for the Earth's interior out of seismic data and geomagnetic information.

One of the first leaps of faith that quantum electrodynamics made was to stop trying to model the fields of elementary particles as actual fields in the Einstein-Maxwell or continuum-mechanical sense and replace the emphasis on "fields of force" with a new emphasis on the "exchange particle" concept, which was largely due to Werner Heisenberg. Hence, it was only the *interactions* of particle fields that was important, not the fields themselves. A secondary effect of this was to increasingly focus on the scattering of particles as the primary source of information about their structure. At no point did anyone attempt to pose systems of partial differential equations for the fields and solve boundary-value problems for the static case or Cauchy problems for the time evolution of the fields.

Rather, one immediately turned to the problem of constructing the momentum-space kernel for the scattering operator, and eventually a vast set of largely algorithmic procedures for doing this emerged, including algorithms for dealing with the unphysical infinities that appeared as a consequence of the initial steps. Interestingly, what one is implicitly constructing in momentum space is not ultimately the kernel for a nonlinear differential operator in configuration space – indeed the method of Green functions and linear integral operators is not applicable to nonlinear differential operators – but the kernel for a *linear pseudo-differential operator* in configuration space. Hence, if one imagines that the most natural expansion of scope of Maxwell's equations is from linear to nonlinear differential equations as a consequence of going from linear to nonlinear constitutive laws then clearly the momentum-space constructions that follow from the method of Feynman diagrams are not giving one such an extension, at least directly.

Nevertheless, the phenomenological methods of quantum electrodynamics eventually reconnect with classical electromagnetic field theory by the use of *effective* models, such as the ones that were defined by Heisenberg, in conjunction with Hans Euler [**48**], as well Max Born and Leopold Infeld [**49**]. That is, although one usually starts by assuming that the action associated with the fields in question has a classical sort of form, after one has "quantized" it and renormalized it to something that is physical again, one generally finds that supplementary terms have appeared in the action that one identifies as "quantum corrections." For instance, if one quantizes the field theory with the method of functional

integrals, which evolved from the Feynman path integrals of quantum mechanics, then one often expands the Green function in a "loop expansion," which is a perturbation series expansion indexed by the number of loops in the Feynman diagram; the expansion parameter in a loop expansion is then Planck's constant $\hbar$. For instance, zero loops defines the "tree" level of expansion, and corresponds to the original classical theory, while one loop is analogous to the WKB approximation of quantum mechanics. Actually, the methodology of loop expansions is also based in the asymptotic expansions that Poincaré introduced, so geometrical optics represents essentially a "tree-level" approximation to wave optics, while diffraction introduces quantum corrections.

One sees that the effective actions, which usually suggest effective potentials, give one strongly-worded hints as to how one might expand the scope of one's "classical" model to include the quantum corrections. Of course, the ultimate challenge is not to simply refine the numerical accuracy, but to find a better model at the fundamental level. One is reminded of how long astronomy labored in its Ptolemaic phase of adding epicycles to cycles and then epi-epicycles in order to account for the motion of planets with increasing numerical accuracy when the elegant solution was to make the Copernican hypothesis at the outset. Perhaps one day the historians of physics will regard the Feynman diagrams as a sort of Ptolemaic algorithm for obtaining better numerical accuracy in one's agreement with experiments when the elegant solution was something of a Copernican revolution at the fundamental level.

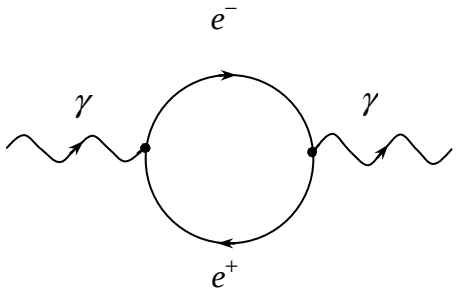
The ultimate challenge to quantum electrodynamics is then to respect the largely empirical nature of its greatest successes while using them as qualitative insights into how one might construct a more fundamental model of the physics at the atomic-to-subatomic level. One can see that the problem defined by the modeling of quantum – i.e., atomic-to-subatomic – phenomena is closely analogous to that of modeling the Earth's interior or that of the Sun in that one cannot expect direct information, so one must always propose effective models, and the measure of effectiveness for any effective model is in its conciseness, as well as its relationship to more general models of physics. For instance, concepts such as spontaneous symmetry breaking are just as important, as well as more directly observable, in the context of condensed matter physics as they are in theoretical particle physics.

In the name of qualitative lessons that one can learn from the empirical successes of quantum electrodynamics, one should start with the fact that every charged fermion, such as an electron or a proton, has an antiparticle that shares its properties except for having an opposite charge, while the photon, being a boson, has no anti-particle. Furthermore, the fact that electrons and positrons are anti-particles manifests itself in the phase transition that takes the form of pair annihilation, which makes anti-particle pairs combine to produce photons, which are generally in the gamma range of energies. The opposite process by which a photon of sufficient energy can split into an anti-particle pair is not as spontaneous, but still just as experimentally established. In particular, one usually needs an external electric or magnetic field in order to convert a photon into an anti-particle pair.

At photon energies slightly above 1 MeV (the rest energy of an electron and a positron), the particles will be electrons and positrons with the excess photon energy being converted into the kinetic energies of the resulting particles. When the photon energy exceeds the rest energy of a muon and its anti-muon (about 211 MeV), there is the

possibility that the particles created are a muon and an anti-muon. Once one reaches the energy level of the rest energy of a pion and an anti-pion (about 280 MeV), the fact that one is now involved with strongly-interacting particles implies that quantum electrodynamics is insufficient and one must recast the problem in the context of quantum chromodynamics. The converse process of pair annihilation is even more involved since an electron-positron collision at a sufficiently high total kinetic energy can create pion-anti-pion pairs, although their lifetimes will be brief, and ultimately one is left with various combinations of lower-energy electron-positron pairs and gamma rays as the stable particles.

These complementary processes of pair creation and annihilation then give rise to the possibility of *vacuum polarization*, in which the presence of photon as an intermediate stage in a particle interaction, such as an electromagnetic exchange particle, can be associated with the spontaneous creation and annihilation of "virtual" anti-particle pairs; i.e., intermediate steps in the process that are not observed directly. Since this sort of process can be represented graphically in the form:



one sees how the concept of loop expansions emerges naturally from elementary principles. In particular, there is nothing to say that the gamma photons in the diagram might not give rise to further virtual pairs with an increasing amount of loops.

The fact that vacuum polarization actually takes place in various quantum scattering and bound-state problems is well-established by numerous experiments by now. One of the most celebrated experimental verifications is the Lamb shift of atomic electron energy levels due to the presence of an anomalous magnetic moment to the electron, in addition to the one that is associated with its spin, since the origin of this anomalous magnetic moment is presumed to be vacuum polarization. Vacuum polarization can also affect the interaction of photons with charges, as well as other photons. The (Delbrück) scattering of photons by nuclear electric fields, which does not exist in Maxwellian electromagnetism, was established by the experiment, and although the scattering of photons by other photons, which is similarly non-existent in Maxwell's theory, was proposed theoretically in the early days of quantum electrodynamics. Interestingly, although the experimental physics community has been promising theoreticians that high-intensity laser technology has been "ten years" away from the critical field strengths for light pulses for quite some decades now, nevertheless, the actual experimental verification of photon-photon scattering has yet to emerge.

Besides the ubiquitous appearance of vacuum polarization in the quantum corrections to classical electromagnetic scattering and bound state problems, another fundamental concept that keeps asserting itself is that of a non-trivial electromagnetic "vacuum state."

Indeed, the very definition of that notion is a fundamental problem. One way of looking at the complexity of the problem is to recall that an electromagnetic wave can be regarded as a continuous distribution of coupled simple harmonic oscillators and then regard the quantum version of that wave – at least heuristically – as a continuous distribution of coupled *quantum harmonic oscillators.* Of course, one key difference between the simple and the quantum harmonic oscillator is the fact that the quantum harmonic oscillator has a small, but non-zero, ground state energy of $1/2\hbar\omega_n$, where ω_n is the natural frequency of the oscillation. Hence, one would expect this to imply that the lowest-energy quantum electromagnetic wave – i.e., photon – would have to also be of non-vanishing total energy, hence, of non-vanishing electric and magnetic field strengths. However, even though one can shift any scale of potential energy for a single oscillator to make the ground state energy precisely zero, nevertheless, when one extends this process to an infinitude of oscillators the necessary shift is not unique, and one finds that there is likely to be some – generally non-unique – *zero-point-field.* One finds that the existence of such a field, which presumably is not due to any specific charge or current source, has been experimentally observed in the form of the *Casimir effect* [**50, 51**]. This effect amounts to the statement that an uncharged perfectly conducting parallel plate capacitor will experience a measurable force of attraction between the plates. However, it is important to observe that the region between the plates is a compact manifold with boundary, so often the effect is attributed to that topological situation, as well as quantum electrodynamics.

The form that quantum electrodynamics eventually settled on (e.g., [**52**]) made it clear that one was dealing with a gauge field theory. That is, the fundamental field, namely, the photon, was best represented by a potential 1-form, which could also be regarded as a connection 1-form on a $U(1)$-principal bundle that one referred to as the *gauge structure* of the field theory; one also refers to the connection form as a *gauge field* or *gauge boson*, since the fact that one is also dealing with a covector field means that the spin of the representation of the Lorentz group that governs its frame transformations is one, and particles that are described by fields of integer spins must obey Bose-Einstein statistics in their energy levels. By contrast, the charged particles, such as electrons and positrons, are *fermions*, since they are represented by Dirac bi-spinor wave-functions, which are due to half-odd-integer spin representation of the Lorentz group and obey Fermi-Dirac statistics. This association of spin and statistics is widely regarded as one of the foundations upon which quantum field theories all rest (see, for instance, Streater and Wightman [**53**]).

The theoretical successes that followed, in the form of the further unification of the quantum theory of electromagnetism with that of the weak interaction, further established the methodology of gauge field theories as a powerful tool in the formulation of quantum field theories [**54, 55**]. In particular, the concept of spontaneous symmetry breaking in the vacuum state played a key role in this unification. Once again, the key qualitative notion is that of a non-unique vacuum ground state for the gauge field in question, although when the non-uniqueness follows from the existence of an unbroken vacuum symmetry group the set of vacuum states will have the geometrical and topological character of a "homogeneous space," such as circles, spheres, and torii.

6. Spacetime topology and electromagnetism. In addition to the increasing emphasis that was placed on the problems of quantum electrodynamics, another tributary of research in electromagnetism that grew more out of general relativistic consideration was the increasing interest in the role that spacetime topology played in electromagnetism.

Part of this increasing interest was due to the possible role of spacetime topology in the theory of gravitation, such as the spacetime singularities that one expected to find in the vicinity of black holes and the Big Bang singularity. Indeed, when one models spacetime as a differentiable manifold, it becomes unavoidable that one must deal with the global nature of its topology, and not just the local nature, which is still Euclidian.

Another contribution to the increasing interest in the role of topology in electromagnetism was due to the concerns of gauge field theory. In particular, when one is dealing with connections on principal fiber bundles from the outset one must similarly consider the global nature of the situation, such as whether the principal bundle is trivial or not; this is equivalent to the physical question of whether a global choice of gauge is possible. One finds that the issues associated with the triviality of G-principal bundles $P \to M$ over the spacetime manifold M, where G is the gauge group of the field theory, depend upon both the topology of M, as well as the topology of G, in a deep and complicated way.

Finally, the ambition to examine the role of topology in electromagnetism can arise at a much more elementary level that precedes the considerations of either general relativity or quantum electrodynamics, namely, the very nature of the constructions that one makes in classical electromagnetism itself.

For instance, the elementary concepts of charge, flux, and current can be most effectively defined in the language of topology, or, more to the point, *homology*. One finds that the sources of electromagnetic fields are usually represented by elementary finite chain complexes with real coefficients, such as finite sets of points or networks of current loops, while the concepts of flux and potential difference take the form elementary cochains with real coefficients. The introduction of field strength vector fields and covector fields with such cochains then follows naturally from the considerations of de Rham's theorem. Furthermore, electrostatics and potential theory can be elegantly formulated in the language of Hodge theory.

A key construction in all of this is that of an *orientation* on the spacetime manifold M (or really, its tangent bundle $T(M)$). This is because one must introduce a volume element on a manifold in order to define the integration of differential forms that allows one to speak of flux and the bilinear pairing of k-chains with k-cochains, and before one can introduce a volume element one must assume the orientability of the vector bundle $T(M)$. Since not every differentiable manifold is orientable, one must accept that there are topological obstructions to the existence of global orientations. However, every simply connected manifold is orientable and every manifold has a simply connected orientable covering manifold.

The existence of a volume element on an orientable manifold allows one to define a set of linear isomorphisms between k-vector fields and $n–k$-forms that one can call *Poincaré duality*. Although its role in homology theory is well-established by now, one should realize that it started off as a more projective-geometric sort of concept than a topological one. In particular, any k-dimensional subspace of an n-dimensional vector

space can either be spanned directly by k linearly independent vectors, whose exterior product is then a non-zero k-vector, or annihilated by $n–k$ linearly independent covectors, whose exterior product is then a non-zero n-k-form.

Of course, there is nothing unique about the choice of spanning vectors or annihilating covectors, but any other choice would affect the resulting k-vector or $n–k$-form only by a non-zero scalar multiplication. One then confronts the *Plücker-Klein* representation of k-dimensional vector subspaces in an n-dimensional vector space as lines through the origin in either the vector space of k-vectors over it or the vector space of $n–k$-forms over it; that is, one considers points in the projective spaces that these vector spaces define.

Hence, one sees that the same concept of orientation that is so fundamental to the basic concepts of electromagnetism has both a projective-geometric and a topological aspect to it. Mostly, we shall be concerned with the projective-geometric aspect in what follows. Indeed, a recurring theme is that just as metric differential geometry seems to be the natural language for the description of gravitation, projective differential geometry seems to be the natural language for the description of electromagnetism.

7. Pre-metric electromagnetism. As one sees, the mainstream of theoretical physics in the Twentieth Century largely split into the two warring camps of gravitational theorists, whose main concern was the geometry, and sometimes the topology of the spacetime manifold, and the quantum field theorists, whose main concern was to obtain the best possible agreement between the relativistic particle scattering amplitudes that were obtained by the method of Feynman diagrams and the actual scattering data that was being obtained by high-energy particle scattering experiments.

It is therefore not surprising that one of the lesser-discussed topics in electromagnetism that largely fell through the cracks between these two communities was based in the early observations of the German physicist Friedrich Kottler [**56**] that actually the metric tensor field that was so fundamental to Einstein's theory of gravitation seemed to play only an incidental role in formulating the Maxwell equations of electromagnetism. Indeed, one could formulate them without the necessity of introducing a metric.

Other physicists and mathematicians, such as Hargeaves [**57**], Cartan [**44**], and Bateman [**58**] made similar observations in the course of their own discussion of electromagnetism, and eventually van Dantzig [**59**] published a series of papers that expanded upon the formulation of Maxwell's equations in the absence of a metric. The topic of pre-metric electromagnetism did not die away completely, though, and occasionally resurfaced in the work of Murnaghan [**60**], Post [**61**], Truesdell and Toupin [**62**], Hehl, Obukhov, and Rubilar [**63**, **64**], and was masterfully presented in the modern language of exterior differential forms by Hehl and Obuhkov [**65**]. The present work comes about as the author's attempt to summarize the topics that he himself pursued in response to the work of Hehl and Obukhov, and is intended to only enlarge the scope of the subject of pre-metric electromagnetism with other specialized techniques and issues, and not to replace their definitive treatise.

The picture of electromagnetism that emerges is based in the idea that the Lorentzian structure g on the spacetime manifold enters into Maxwell equations only by way of the

Hodge * isomorphism, as it acts on 2-forms, in particular. Hence, one must take a closer look at this isomorphism from both a mathematical and a physical perspective.

One finds that it is best to factor the isomorphism *: $\Lambda^2 M \to \Lambda^2 M$ into a product of two isomorphisms, where $\Lambda^2 M \to M$ is the bundle of 2-forms over M. The first one $i_g \wedge i_g$: $\Lambda^2 M \to \Lambda_2 M$ is the one that "raises both indices" on 2-forms, where $\Lambda_2 M \to M$ is the bundle of bivectors on M. The second one #: $\Lambda_2 M \to \Lambda_2 M$ is the aforementioned Poincaré duality that one obtains from the assumption that M has a volume element on it.

Now, if one were to be more explicit about the introduction of an electromagnetic constitutive law into Maxwell's equations then one would find that the role of the isomorphism $i_g \wedge i_g$ could be absorbed into that constitutive law, as long as one represented it as a vector bundle isomorphism κ: $\Lambda^2 M \to \Lambda_2 M$, at least for linear media.

One then finds that in order to account for the introduction of a Lorentzian structure, one can first consider the dispersion law for electromagnetic waves that follows from the Maxwell equations, when formulated in this "pre-metric" fashion. What one obtains is generally a quartic hypersurface in the cotangent bundle $T^*M \to M$, rather than the usual quadratic one that derives from the light cones of a Lorentzian metric g. A further reduction of the homogeneous quartic polynomial that one obtains from the dispersion law into a square of a quadratic one of Lorentzian type is possible only if the constitutive law has the proper sort of symmetries, such as spatial isotropy.

When viewed in this light, one sees that in a real sense the gravitational structure of spacetime, as defined by the Lorentzian structure on its cotangent bundle, is a specialized *consequence* of assumptions that one makes about the electromagnetic constitutive laws of the spacetime manifold. Indeed, if one goes back to the historical progression that led Einstein from electromagnetism to gravitation then one sees that the connecting link between them related to the structure of the symbol of the wave operator that governed the propagation of electromagnetic waves in the spacetime manifold. Hence, one sees that gravitation relates to *light* cones and not purely gravitational ones that are unrelated to the propagation of electromagnetic waves.

What emerges from the foregoing picture is nothing short of a major paradigm shift in the consideration of spacetime geometry from the metric geometry of tangent vectors that pertains to gravitational geodesics to the projective geometry of 2-forms and bivectors (i.e., 2-planes) that pertains to the propagation of electromagnetic waves. This also suggests that one is redirecting one's focus from the geometry of Riemann to the geometry of Klein, who once asserted that "projective geometry is all geometry." Hence, there is something mathematically satisfying about the fact that one is simply moving to a higher plane of generality in one's discussion of geometry.

Another subtlety that appears is the fact that the Hodge * isomorphism has the property that if the metric g is Lorentzian then $*^2 = -I$ when * is applied to 2-forms. This means the linear operator * defines an "almost-complex structure" on the real vector bundle $\Lambda^2 M$. However, as observed by the author [**66**], if one starts with an electromagnetic constitutive law and defines $* = \# \cdot \kappa$ then one must note that not all of the physically meaningful laws give * isomorphisms with such a property. In particular, anisotropic optical media will not define almost-complex structure in this way. When one is dealing with such a class of media, though, one can also introduce concepts of

complex projective geometry accordingly. For instance, the complex projective space $\mathbb{C}P^2$ and its dual play an important role in the geometry of electromagnetic waves.

Since the geometry of metrics and geodesics seems to follow only after one passes from the pre-metric Maxwell equations to the characteristic equation that they define in the geometrical optics approximation, one sees that the transition from wave optics to geometrical optics implies a corresponding transition from wave geometry to geodesic geometry, just as one goes from wave mechanics to geometrical mechanics in quantum theory. The question then arises of how one might define wave geometry in general, and one finds that the solution of this problem is well-known: the geometry of waves is the contact geometry that was previously mentioned in the context of geometrical optics.

One begins to gain some inkling of how Einstein's suspicions about the unification of electromagnetism and gravitation were probably quite accurate: It would probably have to involve some simultaneous incorporation of quantum considerations and expansion of scope in spacetime geometry. Of course, the problem of resolving the growing gap between the very formalism, if not the natural philosophy, of relativity and gravitation with that of quantum field theory has always been regarded as every bit as perplexing as the Einstein-Maxwell unification problem. One might suspect that a better understanding of wave geometry might lead to a resolution of both the problem of reconciling general relativity with quantum physics and unifying the theory of electromagnetism with that of gravitation.

One also begins to see that gravity appears as an "emergent" phenomenon when one starts with the electromagnetic structure of spacetime; i.e., the dispersion law that follows from its electromagnetic constitutive law. Hence, one suspects that the best way to formulate the mathematics of gravity is in terms of 2-forms to begin with. Of course, this approach has been around since the 1960's in the form of "complex relativity," except, as we shall point out later, it becomes redundant to complexify the vector bundle $\Lambda^2 M$ when one has already introduced an almost-complex structure.

Another possible expansion of scope that is associated with pre-metric electromagnetism is the fact that the very significance of Lorentz invariance in physics is based on the assumption that one is concerned with a quadratic dispersion law for electromagnetism. One sees that the more general case of a quartic dispersion law implies that one might be required to alter one's conception of the invariance of the laws of Nature accordingly. This gives one a more tangible origin for the possible violation of Lorentz invariance.

So far, the emphasis on electromagnetic constitutive laws that result in quartic dispersion laws has mostly been based in macroscopic optical sorts of considerations. However, one finds that the effective models for quantum electrodynamics that take the form of the Heisenberg-Euler effective Lagrangian and the Born-Infeld both produce constitutive laws of the "bi-isotropic" type, in the language of Lindell [**67**], as well as dispersion laws that are of the "birefringent" type. Hence, one sees that pre-metric electromagnetism might give some new insight into the problem of the modeling of the electromagnetic vacuum state when one takes into account the considerations of quantum physics.

8. Summary of contents. Since the application of the calculus of exterior differential forms to the formulation of problems in electromagnetism as a substitute for vector calculus is better known to theoretical physicists who are usually only interested in Maxwellian electromagnetism as a springboard to more general gauge field theories, we begin our discussion of pre-metric electromagnetism with a chapter on that more recent [7] calculus. Furthermore, since most of the theoretical discussions are immediately concerned with topological matters – e.g., triviality of principal bundles – it is never emphasized that that exterior calculus can be applied at a much more elementary level in electromagnetism than the topological level, namely, at the level of a typical undergraduate course in electromagnetism. Hence, we shall attempt to exhibit the formulation of as many of the traditional non-theoretical topics in electromagnetism as possible in terms of differential forms to show that they are more than just a topological necessity.

Of course, the topological aspects of differential forms are still essential in any discussion of the fundamental concepts in electromagnetism, so Chapter II will summarize the usual issues in the topology of differential manifolds to the extent that is necessary for the discussion of those fundamental concepts. Although some attempt has been made at making the chapter self-contained and based only upon a knowledge of linear algebra and multi-variable calculus, nonetheless, it is suggested that readers with no prior exposure to differentiable manifolds will probably find the chapter a bit too concise.

In Chapter III, we discuss the topological nature of charge, flux, current, and field strengths in both the static electric and magnetic contexts. We also discuss the distinction between the field strength and the associated excitations of the medium that they produce (also called "inductions," by many researchers), as well as the introduction of potential functions for electric field strength 1-forms and potential 1-forms for magnetic field strength 2-forms, which then represent the classical "vector potentials".

Then, in Chapter IV, we show how one goes from electrostatics and magnetostatics to electrodynamics by the introduction of the two types of electromagnetic induction, namely, the ones that are described by Faraday's law and Maxwell's law. One can also assemble electric charge density and the electric current into a four-dimensional vector field, the vanishing of whose divergence is equivalent to the conservation of charge. We finally assemble all of the accumulated laws for electric and magnetic fields into the pre-metric Maxwell equations, which we express in both their three-dimensional time+space form, as well as the more general four-dimensional form. We also discuss the four-dimensional introduction of electromagnetic potential 1-forms for the field strength 2-form, which is at the basis of all gauge approaches to electromagnetism, whether classical or quantum.

In Chapter V, we discuss the largely phenomenological nature of electromagnetic constitutive laws, except that we continue to use the language of differential forms and multivector fields to the greatest extent possible, while most treatments of the subjects are oriented towards experimental physicists, whose preference is for the methods of vector

[7] Apparently, the transfer of wisdom from mainstream mathematics to mainstream physics is getting sufficiently sluggish that "new mathematics" generally encompasses most of the Twentieth Century. In particular, the concept of differential forms was discussed by pure mathematicians such as Darboux and Goursat in the late Nineteenth Century.

calculus. Among the examples of specific constitutive laws, we give not only ones familiar to experimental physicists, such as optical media and plasmas, but also more theoretical ones, such as Lorentzian and almost-complex media. The example of bi-isotropic media is shown to include both the constitutive laws of the Heisenberg-Euler effective model for the propagation of electromagnetic waves in the presence of background electromagnetic fields when one includes one-loop quantum corrections to the Maxwellian Lagrangian, and the Born-Infeld model, which is closely related to the Heisenberg-Euler model, but based in other considerations.

Because the pre-metric Maxwell equations give partial differential equations when one expresses them locally, Chapter VI is concerned with the formulation of the basic notions from the theory of partial of differential equations that relate to classical electromagnetism when one formulates those equations on differentiable manifolds. Since there is no universal agreement amongst mathematicians as to the "best" way to formulate partial differential equations on manifolds, we discuss three of the most popular approaches: differential operators on vector bundles, hypersurfaces in jet manifolds, and exterior differential systems, as well as how they relate to each other. We then attempt to formulate the traditional boundary-value problems of electrostatic or magnetostatic potential theory, along with the Cauchy problem of electrodynamics, in that language. Although the methods of Green functions and Fourier transforms are fundamentally limited to the treatment of linear differential equations, as well as being rather difficult to deal with specifically when the manifold in question does have a high degree of homogeneity (affine spaces, spheres, etc.), nevertheless, we attempt to at least define the nature of the mathematical problems involved.

In Chapter VII, we return to the physics of electromagnetism and discuss the issues that arise when one looks at all of the ways that electromagnetic fields and source distributions can interact with each other. However, since the interaction of sources and fields that takes the form of electromagnetic radiation is quite involved in its own right, we content ourselves with only a cursory discussion of some of the issues and defer a more detailed analysis of pre-metric radiation theory – if there even is such a thing – for a later monograph.

Chapter VIII represents the culmination of the pre-metric theory of electromagnetism in which one sees that the Lorentzian structure of spacetime can emerge from the dispersion laws for the medium that one obtains from the field equations when one specifies a constitutive law. However, the quadratic dispersion law that gives a Lorentzian metric is a degenerate case of the more general quartic polynomial law, and is usually associated with isotropic media.

In Chapter IX, one then finds that the geometrical optics approximation also represents an approximation to the geometry of waves by null geodesics in the same way that it represents an approximation to the physics of electromagnetic waves by light rays. Interestingly, this was also the driving force behind the early formulation of quantum wave mechanics, namely, to make its relationship to classical mechanics be analogous to the relationship of wave optics to geometrical optics. In optics, one usually attempts to go beyond the geometrical optics approximation by the introduction of diffraction effects and asymptotic expansions, so we discuss what that might mean in the context of pre-metric electromagnetism.

In Chapter X, we discuss the way that energy and action are associated with electromagnetic fields in both the static and dynamics contexts. We then show how one defines pre-metric Lagrangians for the static and dynamic field equations, as well as the equations of motion for an electric charge in the presence of the Lorentz force. We finally show that the usual statement of Fermat's principle for the variational formulation of the spatial light rays (spatial projections of null geodesics) is fraught with subtleties when one attempts to generalize from metric geometry to pre-metric geometry since even the nature of the projection of spacetime onto space needs to be considered in the language of projective geometry, along with a rethinking of the basic action functional that gives the elapsed-time along a curve.

The last three chapters essentially summarize some of the author's previous research into the extension of some traditional results in classical electromagnetism and general relativity. The expansion of scope that comes about when one attempts to examine the symmetries of the pre-metric field equations themselves is discussed in Chapter XI, when it is known that in the Lorentzian case, one must go from Lorentzian invariance to conformal Lorentzian invariance, among other extensions. Since projective geometry has been persistently suggesting itself as an unavoidable expansion of the scope of metric geometry, the topic is discussed at a more elementary level in Chapter XII and applied to elementary physical mechanics, as well as electromagnetism. A key construction is the Plücker-Klein association of linear planes in $\mathbb{R}^4$ with lines through the origin in either the vector space of bivectors or 2-forms over $\mathbb{R}^4$ by way of decomposable bivectors and 2-forms. Finally, Chapter XIII discusses the fact that the usual formulation of "complex relativity" in terms of 2-forms is naturally embedded in the present context, and indeed, is probably an essential part of incorporating "gravitational" considerations into pre-metric electromagnetism, in the form of more general spacetime connections than the Levi-Civita connection that is defined by its Lorentzian structure.

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Chapter I

Calculus of exterior differential forms

Although at the present time it is traditional for mathematicians – and even some physicists – to first introduce the calculus of exterior differential forms in the context of differentiable manifolds, nevertheless, since the purpose of this study is to begin the discussion of electromagnetism in the same place that conventional physics does, it seems more appropriate to illustrate that there are really three advantages to the use of differential forms over vector calculus: computational conciseness, dimensional generality, and their sensitivity to the topology of the space in question. Hence, the sole purpose of this first chapter will be to show how the calculus of exterior differential forms on vector spaces neatly subsumes the main results of vector calculus. Chapter II will then address the topological aspects of differential forms as a separate issue.

Most of the material in this chapter can be found, in one form or another, in the introductory chapters of most books on differential manifolds [**1-3**], the later chapters of some books on advanced calculus [**4-6**], and various books on differential forms themselves [**7**, **8**]. The list of references at the end of the chapter is only a sample of these possibilities.

1. Multilinear algebra. The topic of *exterior algebra* is actually a special case of the more general topic of *tensor algebra*, or, as it is sometimes called, *multilinear algebra* [**9**]. Hence, we shall begin by summarizing a few generalities from the latter branch of mathematics.

Let V represent a vector space of dimension n whose scalars come from the field $\mathbb{R}$ of real numbers; later, we shall return to some of the same notions when it becomes necessary to discuss complex vector spaces. We also let V^* represent the dual vector space to V – viz., the vector space of all linear functionals on V. Hence, if $\phi \in V^*$, $\mathbf{v}, \mathbf{w} \in V$, and $\alpha, \beta \in \mathbb{R}$ then we will always have:

$$\phi(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha\,\phi(\mathbf{v}) + \beta\,\phi(\mathbf{w}). \tag{I.1}$$

a. Bilinear functionals. A function T: $V \times V \to \mathbb{R}$ is said to be *bilinear* if it is linear in each variable separately:

$$T(\alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w}) = \alpha\,T(\mathbf{u}, \mathbf{w}) + \beta\,T(\mathbf{v}, \mathbf{w}), \tag{I.2a}$$
$$T(\mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w}) = \alpha\,T(\mathbf{u}, \mathbf{v}) + \beta\,T(\mathbf{u}, \mathbf{w}), \tag{I.2b}$$

If $\{\mathbf{e}_i, i = 1, \ldots, n\}$ is a basis for V, so $\mathbf{u} = u^i\,\mathbf{e}_i$, $\mathbf{v} = v^j\,\mathbf{e}_j$ (the summation convention is always in effect unless specified to the contrary) then from bilinearity one will have:

$$T(\mathbf{u}, \mathbf{v}) = u^i v^j \, T(\mathbf{e}_i, \mathbf{e}_j). \tag{I.3}$$

If we introduce the notation $T_{ij} = T(\mathbf{e}_i, \mathbf{e}_j)$, which one refers to as the *components of T with respect to the basis* $\mathbf{e}_i$, then we can write (I.3) in the component form:

$$T(\mathbf{u}, \mathbf{v}) = T_{ij} \, u^i v^j \, . \tag{I.4}$$

Suppose we perform a change of basis $\mathbf{e}_i \rightarrow \overline{\mathbf{e}}_i$, which defines an invertible linear transformation of V with a matrix A_i^j relative to the basis $\mathbf{e}_i$ that is defined by the basic equations:

$$\overline{\mathbf{e}}_i = A_i^j \mathbf{e}_j \, . \tag{I.5}$$

The components $\overline{u}^i$ of $\mathbf{u}$ with respect to the transformed basis $\overline{\mathbf{e}}_i$ are then obtained from the assumption that both of the linear combinations $u^i \mathbf{e}_i$ and $\overline{u}^i \overline{\mathbf{e}}_i = \overline{u}^i A_i^j \mathbf{e}_j$ must define the same vector, namely, $\mathbf{u}$. Hence, one must have:

$$\overline{u}^i = \tilde{A}_j^i u^j \, , \tag{I.6}$$

in which we are using a tilde to denote the matrix inverse to A_i^j. That is, the transformation of components for vectors is inverse to the transformation of bases, a situation that one sometimes refers to as *contragredience*, while the vector $\mathbf{u}$ is said to be *contravariant*. Since the purely component-oriented formulation of tensor algebra usually performs the transformation of components first, modern mathematics adapted by usually assuming that bases – or *frames* – are the objects that transform by the inverse.

The effect of this change of basis on the components T_{ij} is obtained by inserting $\mathbf{u} = \overline{u}^i \overline{\mathbf{e}}_i$ and $\mathbf{v} = \overline{v}^j \overline{\mathbf{e}}_j$ into $T(\mathbf{u}, \mathbf{v})$ in (I.4), which gives:

$$\overline{T}_{ij} \overline{u}^i \overline{v}^j = T_{ij} \, u^i v^j \, . \tag{I.7}$$

By means of (I.6), and a similar equation for the v^j, this gives:

$$\overline{T}_{ij} = A_i^k A_j^l T_{kl} \, . \tag{I.8}$$

Since the effect of the frame change in V shows up in the components of T directly, not inversely, one says that the bilinear functional T on V is *covariant.* Often, one hears T referred to as a *(doubly-) covariant second rank tensor* on V.

When one considers bilinear functionals on V^* one obtains formulas that are analogous to those of the covariant case, and which then represent contravariant second rank tensors on V. In particular, a *coframe* for V^* is a basis $\{\theta^i, \, i = 1, \ldots, n\}$, so any covector $\alpha \in V^*$ can be expressed in the form $\alpha = \alpha_i \, \theta^i$. Relative to this coframe, the value of a contravariant bilinear functional $\mathbf{T}(\alpha, \beta)$ when evaluated on two covectors α and β is:

$$\mathbf{T}(\alpha, \beta) = T^{ij}\, \alpha_i\, \beta_j\,. \tag{I.9}$$

When one chooses a frame $\mathbf{e}_i$ on V there is a unique coframe θ^i that it defines on V^* by the requirement that one must have:

$$\theta^i(\mathbf{e}_j) = \delta^i_j\,; \tag{I.10}$$

one calls this coframe the *reciprocal (or inverse) coframe to* $\mathbf{e}_i$. Since the reciprocal coframe $\overline{\theta}^i$ to any other frame $\overline{\mathbf{e}}_i$ must satisfy the analogue of (I.10), as well, one sees that under the transformation from $\mathbf{e}_i$ to $\overline{\mathbf{e}}_i = A^j_i \mathbf{e}_j$, the reciprocal coframe θ^i must go to:

$$\overline{\theta}^i = \tilde{A}^i_j \theta^j\,. \tag{I.11}$$

Hence, coframes transform contragrediently to frames. The effect of the transformation of $\mathbf{e}_i$ on the components of covectors must then be:

$$\overline{\alpha}_i = A^j_i \alpha_j\,. \tag{I.12}$$

This then has the effect of making the transformation of the components of $\mathbf{T}$ a contravariant one:

$$\overline{T}^{ij} = \tilde{A}^i_k \tilde{A}^j_l T^{kl}\,. \tag{I.13}$$

b. Multilinear functionals. In order to go from bilinearity to multilinearity, all that one has to do is to require that a multilinear functional $V \times \ldots \times V \to \mathbb{R}$, with k factors of V in each the Cartesian product, be linear in of its factors individually.

If we choose a basis $\mathbf{e}_i$ for V then the components:

$$T_{ij\ldots k} = T(\mathbf{e}_i\,, \mathbf{e}_j\,, \ldots, \mathbf{e}_k) \tag{I.14}$$

of T for this basis will have k indices, so:

$$T(\mathbf{u}, \mathbf{v}, \ldots, \mathbf{w}) = T_{ij\ldots k}\, u^i\, v^j \ldots w^k\,. \tag{I.15}$$

Under a change of basis on V, the components will transform covariantly:

$$\overline{T}_{ij\cdots k} = A^l_i A^m_j \cdots A^n_k\, T_{lm\cdots n}\,. \tag{I.16}$$

Analogously, a multilinear functional $\mathbf{T}$: $V^* \times \ldots \times V^* \to \mathbb{R}$ will have components with respect to the reciprocal coframe θ^i to the frame $\mathbf{e}_i$ that are defined by:

$$T^{ij\ldots k} = \mathbf{T}(\theta^i\,, \theta^j\,, \ldots, \theta^k) \tag{I.17}$$

and transform contravariantly:

$$\overline{T}^{ij\cdots k} = \tilde{A}^i_l \tilde{A}^j_m \cdots \tilde{A}^k_n T^{lm\cdots n}. \tag{I.18}$$

The set of all multilinear functionals on V of rank k can be made into a vector space in its own right, since $\mathbb{R}$ is a vector space. One defines a scalar combination $\alpha T_1 + \ldots + \beta T_m$ of m multilinear functionals by letting each T_a, $a = 1, \ldots, m$ act on the m-tuple of vectors $(\mathbf{u}, \ldots, \mathbf{w})$ to produce m numbers $T_a(\mathbf{u}, \ldots, \mathbf{w})$ and then forming the corresponding scalar combination of real numbers:

$$(\alpha T_1 + \ldots + \beta T_m)(\mathbf{u}, \ldots, \mathbf{w}) = \alpha T_1(\mathbf{u}, \ldots, \mathbf{w}) + \ldots + \beta T_m(\mathbf{u}, \ldots, \mathbf{w}). \tag{I.19}$$

Since the vector space of linear functionals on V is denoted by V^*, we denote the vector space of k-linear functionals by $V^*\otimes\ldots\otimes V^*$, for consistency, and refer to it as the *tensor product* of k copies of V^*. Hence, the tensor product of a finite number of vector spaces is another vector space[8]. Its dimension is n^k, which can be seen by defining a basis $\{\theta^i \otimes\ldots\otimes \theta^j\}$ for $V^*\otimes\ldots\otimes V^*$ by forming all *tensor products* of the basis vectors θ^i for V*. Although it is possible to give a mathematically rigorous definition of the tensor product of vectors or covectors (see [**9**]), for our purposes it is only necessary to know that it is bilinear, so the tensor product of k vectors is k-linear, and the tensor product of vectors belongs to a higher-dimensional vector space.

One can then express a k-linear functional T in component form relative to this tensor product basis as:

$$T = T_{i\ldots j}\, \theta^i \otimes\ldots\otimes \theta^j. \tag{I.20}$$

Similarly, one can form the tensor product $V\otimes\ldots\otimes V$ to represent the vector space of all k-linear functionals on V^*. It is also n^k-dimensional and has a basis $\{\mathbf{e}_i \otimes\ldots\otimes \mathbf{e}_j\}$ that is defined by all tensor products of the basis elements $\mathbf{e}_i$. In fact, the basis $\{\theta^i \otimes\ldots\otimes \theta^j\}$ on $V^*\otimes\ldots\otimes V^*$ is easily seen to be reciprocal to the basis $\{\mathbf{e}_i \otimes\ldots\otimes \mathbf{e}_j\}$ on $V\otimes\ldots\otimes V$ when θ^i is reciprocal to $\mathbf{e}_i$.

A general k-linear functional $\mathbf{T}$ on V* can be represented in component form relative to this basis as:

$$\mathbf{T} = T^{i\ldots j}\, \mathbf{e}_i \otimes\ldots\otimes \mathbf{e}_j\,. \tag{I.21}$$

2. Exterior algebra. Whenever a multilinear functional acts on a Cartesian product of copies of the same vector space – such as V or V^* – with itself, one can consider the issue of whether the functional is symmetric under permutations of the vectors that it acts upon.

a. Antisymmetric multilinear functionals. For instance, a bilinear functional T on V is *symmetric* iff $T(\mathbf{u}, \mathbf{v}) = T(\mathbf{u}, \mathbf{v})$ and *antisymmetric (or skew-symmetric)* iff $T(\mathbf{u}, \mathbf{v}) = -$

[8] Some members of the physics community find this ambivalence between the use of the words "vector" and "tensor" confusing, since tensors have more indices than vectors, to them. Hopefully, if one can understand that vector spaces are the general concept and tensor products of vector spaces are a specialization of the concept of a vector space then this confusion will pass in time.

$T(\mathbf{u}, \mathbf{v})$. This has the effect of making its components with respect to any frame $\mathbf{e}_i$ on V be symmetric or antisymmetric under permutation, as well:

$$T_{ij} = \pm\, T_{ji}\ . \tag{I.22}$$

One finds that the symmetric bilinear functionals on V form a vector subspace in $V^*\otimes V^*$, as do the antisymmetric ones. This follows from the fact that scalar combinations of (anti-)symmetric bilinear functionals are also (anti-)symmetric. In fact, one can polarize any bilinear functional into a symmetric part T_+ and an antisymmetric part T_- :

$$T = T_+ + T_-, \quad T_\pm(\mathbf{u}, \mathbf{v}) \equiv \tfrac{1}{2}[T(\mathbf{u}, \mathbf{v}) \pm T(\mathbf{v}, \mathbf{u})]. \tag{I.23}$$

Correspondingly, the components of T polarize in the form:

$$T_{ij} = T_{(ij)} + T_{[ij]}, \qquad T_{(ij)} \equiv \tfrac{1}{2}[T_{ij} + T_{ji}], \qquad T_{[ij]} \equiv \tfrac{1}{2}[T_{ij} - T_{ji}]\ . \tag{I.24}$$

One can then think of the association of T with T_+ and T_- as defining linear projections of $V^*\otimes V^*$ onto subspaces that we denote by $V^* \odot V^*$ and $V^* \wedge V^*$, respectively. This means that we can express $V^* \otimes V^*$ as the direct sum:

$$V^*\otimes V^* = (V^* \odot V^*) \oplus (V^* \wedge V^*). \tag{I.25}$$

We think of the vector space $V^* \odot V^*$ as the *symmetric* tensor product of V^* with itself and the vector space $V^* \wedge V^*$as the *exterior product* of V^* with itself.

We shall be primarily concerned with the antisymmetric case in what follows.

If $\alpha, \beta \in V^*$ then one defines their *exterior product* by antisymmetrizing their tensor product:

$$\alpha \wedge \beta = \tfrac{1}{2}(\alpha\otimes\beta - \beta\otimes\alpha). \tag{I.26}$$

Relative to the coframe θ^i, this takes the component form:

$$\alpha \wedge \beta = \tfrac{1}{2}(\alpha_i \beta_j - \alpha_j \beta_i)\ \theta^i \wedge \theta^j. \tag{I.27}$$

One notices that the antisymmetry of the "wedge" product makes:

$$\alpha \wedge \alpha = 0 \tag{I.28}$$

in any case. Another of way of looking at this is to observe that the tensor product $\alpha\otimes\alpha$ is always symmetric, so its projection into the space of antisymmetric tensors must be zero.

The vector space $V^* \wedge V^*$is seen to have dimension $n(n - 1)/2$ if one considers the number of linearly independent antisymmetric combinations of basis covectors $\theta^i \wedge \theta^j$. For instance, if V^* is of dimension 1, 2, 3, 4, resp. then $V^* \wedge V^*$ is of dimension 0, 1, 3, 6,

resp. By contrast, the (complementary) dimension of $V^* \odot V^*$ is $n(n + 1)/2$, which is zero only when $n = 0$.

In terms of bilinear functionals, the action of $\alpha \wedge \beta$ on any two vectors $\mathbf{u}, \mathbf{v} \in V$ gives:

$$(\alpha \wedge \beta)(\mathbf{u}, \mathbf{v}) = \tfrac{1}{2}(\alpha \otimes \beta - \beta \otimes \alpha)(\mathbf{u}, \mathbf{v}) = \tfrac{1}{2}[\alpha(\mathbf{u})\, \beta(\mathbf{v}) - \beta(\mathbf{u})\alpha(\mathbf{v})], \tag{I.29}$$

which has the component form:

$$(\alpha \wedge \beta)(\mathbf{u}, \mathbf{v}) = \tfrac{1}{2}(\alpha_i\, \beta_j - \beta_i\, \alpha_j)\, u^i\, v^j; \tag{I.30}$$

one could also obtain this from (I.27).

When one goes beyond the symmetry of second-rank covariant tensors, one sees that polarization no longer gives a simple dichotomy of the higher-rank tensor product spaces; i.e., there are more than two types of symmetry since there is more than one way to permute pairs of vectors. (Indeed, this is at the root of the decomposition of tensor product representations of groups into irreducible representations.) In order to define the exterior algebra over a vector space, one need only focus on the completely antisymmetric subspaces of the higher-rank tensor products. That is, any transposition of a pair of vectors that the multilinear functional acts on will change the sign of its value.

More generally, if π: {1, 2, …, k} → {1, 2, …, k} is a permutation (i.e., a bijection) then if T is a k-linear functional on V one says that T is *completely antisymmetric* iff:

$$T(\mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(k)}) = \mathrm{sgn}(\pi)\, T(\mathbf{u}_1, \ldots, \mathbf{u}_k) \tag{I.31}$$

where sgn(π) is + when the number of transpositions in π is even and – when it is odd.

One finds that the set of all completely antisymmetric k-linear functionals forms a vector under the scalar combinations that are defined by the obvious extension of (I.19). We shall denote this vector space by $\mathrm{A}^k(V)$ and refer to its elements as *algebraic k-forms* on V. It has a dimension that vanishes when $k > n$ and is given by the binomial coefficient $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ for $k \leq n$. This can be seen from the fact that $\mathrm{A}^k(V)$ has a basis that is given by the linearly independent k-fold exterior products $\theta^{i_1} \wedge \ldots \wedge \theta^{i_k}$ of the θ^i, and from the fact that any permutation of the indices $[i_1 \ldots i_k]$ will change the functional thus defined by at most a sign, while there are $k!$ such permutations. One can express the arbitrary k-form α in terms of this redundant basis as:

$$\alpha = \frac{1}{k!} \alpha_{i \cdots j} \theta^i \wedge \cdots \wedge \theta^j \tag{I.32}$$

or in terms of a non-redundant basis by using only those k-fold products for which the indices are in ascending order:

$$\alpha = \sum_{i_1 < \cdots < i_k} \alpha_{i_1 \cdots i_k} \theta^{i_1} \wedge \cdots \wedge \theta^{i_k} \ . \tag{I.33}$$

Although the term "redundant" sounds pejorative, there are actually times when calculations are more conveniently carried out in the redundant basis.

Note that in order for the right-hand side of (I.32) to make sense the indices of $\alpha_{i\ldots j}$ need to have the same permutation symmetry as the k-fold wedge product of the θ^i:

$$\alpha_{i_{\pi(1)}\cdots i_{\pi(k)}} = \mathrm{sgn}(\pi)\,\alpha_{i_1\cdots i_k} \,. \tag{I.34}$$

Indeed, even if one had formed the linear combination in question when starting with components $\alpha_{i\ldots j}$ of unspecified symmetry, the complete antisymmetry of the exterior product of basis elements would select out only the completely antisymmetric part of $\alpha_{i\ldots j}$; for instance, $\alpha_{ij}\,\theta^i \wedge \theta^j = 0$ when $\alpha_{ij} = \alpha_{ji}$.

Due to the symmetry $\left\{ {n \atop k} \right\} = \left\{ {n \atop n-k} \right\}$, one sees that the dimension of $\mathrm{A}^k(V)$ is the same as the dimension of $\mathrm{A}^{n-k}(V)$. Hence, they are linearly isomorphic, but not canonically so; a typical way of defining the isomorphism would be to choose bases for both spaces. For instance, when $n = 3$, one has $\mathrm{A}^0(V) \cong \mathrm{A}^3(V) \cong \mathbb{R}$, $\mathrm{A}^1(V) \cong \mathrm{A}^2(V) \cong \mathbb{R}^3$.

b. Volume elements. One notes that in the extreme case $k = n$, one always has $\mathrm{A}^n(V) \cong \mathbb{R}$. A basis for this one-dimensional vector space is simply a non-zero n-form $\mathcal{V}$. In terms of a basis θ^i for V^*, it can be written as either:

$$\mathcal{V} = \theta^1 \wedge \ldots \wedge \theta^n \tag{I.35}$$

or:

$$\mathcal{V} = \frac{1}{n!}\varepsilon_{i_1\cdots i_n}\,\theta^{i_1} \wedge \cdots \wedge \theta^{i_n} \,. \tag{I.36}$$

The ε symbol represents the usual Levi-Civita symbol with n indices, which equals +1 when $i_1 \ldots i_n$ is an even permutation of $12\ldots n$, equals –1 when it is an odd permutation, and is zero otherwise.

One generally refers to a choice of $\mathcal{V}$ as a *volume element* on V since the effect of applying the n-linear function $\mathcal{V}$ to an n-tuple of vectors in V is:

$$\mathcal{V}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \frac{1}{k!}\varepsilon_{i_1\cdots i_n} v_1^{i_1} \cdots v_n^{i_n} = \det(v_j^i) \,, \tag{I.37}$$

where v_j^i is the $n\times n$ matrix whose columns are the components of $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Hence, the value of $\mathcal{V}$ when applied to an n-tuple of vectors in V is to give the volume of the parallelepiped that they span. This volume will vanish unless they are all linearly independent.

Under a change of coframe from θ^i to $\bar{\theta}^i = A^i_j \theta^j$, the redundant form (I.36) shows that $\mathcal{V}$ goes to:

$$\bar{\mathcal{V}} = \det(A)\, \mathcal{V}. \tag{I.38}$$

This situation is often described by physicists as implying that *n*-forms are "pseudo-scalars."

c. Exterior products of general k-forms. So far, we have really only discussed the exterior products of covectors as a means of defining completely antisymmetric multilinear functionals on vector spaces. Now, we shall extend the exterior product to include the product of a *k*-form α and an *l*-form β. If we demand that the result be a *k*+*l*-form then we see that although a simple tensor product of α and β will produce a tensor of the required rank, it will only have the desired anti-symmetry if one completely anti-symmetrizes the resulting tensor product accordingly. Naively, this involves (*k*+*l*)! permutations of the factors, but since α and β are antisymmetric to begin with *k*! of those permutations will affect only the sign of α and *l*! will affect only the sign of β; that is, one only needs to anti-symmetrize the transpositions that are not already antisymmetric. The result is:

$$(\alpha \wedge \beta)(\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) = \sum_\pi \frac{(k+l)!}{k!\,l!} \operatorname{sgn}(\pi)\, \alpha(\mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(k)}) \beta(\mathbf{v}_{\pi(k+1)}, \ldots, \mathbf{v}_{\pi(k+l)}). \tag{I.39}$$

The effect on components is similar:

$$(\alpha \wedge \beta)_{1\ldots k,\, k+1\ldots k+l} = \sum_\pi \frac{(k+l)!}{k!\,l!}\, \operatorname{sgn}(\pi)\, \alpha_{\pi(1)} \ldots_{\pi(k)} \beta_{\pi(k+1)} \ldots_{\pi(k+l)}. \tag{I.40}$$

For instance if α is a 1-form and β is a 2-form then the components of $\alpha \wedge \beta$ will take the form:

$$(\alpha \wedge \beta)_{ijk} = (l+1) \sum_\pi \operatorname{sgn}(\pi) a_{\pi(i)} b_{\pi(j)\pi(k)}\,. \tag{I.41}$$

Hence, if we define the direct sum $\mathrm{A}^*(V) = \mathrm{A}^0 \oplus \mathrm{A}^1 \oplus \ldots \oplus A^n$ of all the vector spaces $\mathrm{A}^k(V)$, which we abbreviate by A^k when *V* is unambiguous, then we obtain a vector space of dimension 2^n whose elements then represent finite sums of multilinear functionals on *V*. When all of the functionals in a sum have the same rank *k*, one calls the linear combination *homogeneous*, and it will define an element of A^k; the other elements are called *mixed* forms.

The exterior product, as we have extended it, defines a bilinear map $\mathrm{A}^* \times \mathrm{A}^* \to \mathrm{A}^*$, $(\alpha, \beta) \mapsto \alpha \wedge \beta$, which then makes A^*, with this bilinear product, into an *algebra* over the vector space A^*. It is associative:

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma), \qquad (\text{all } \alpha, \beta, \gamma) \tag{I.42}$$

and has a unity element – namely, the real number $1 \in A^0$ – but it is not commutative, since if α is a k-form and β is an l-form, one has:

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha. \tag{I.43}$$

Hence, although we have obtained the exterior product by a process of complete anti-symmetrization, the exterior product itself can be either antisymmetric or symmetric. In particular, as long as either k or l is even, the product will be symmetric.

Since the product of exterior algebraic forms always has a rank that is greater than or equal to the individual ranks, and the unity element 1 has rank 0, one sees that the only possible *units* in the algebra – i.e., elements that have multiplicative inverses under the wedge product – will be non-zero real numbers. Similarly, from (I.43), one sees that the only way that a k-form α can commute with all l-forms β, regardless of l, is if $kl = 0$ for all l; hence, $k = 0$. This says that the *center* of the algebra A^* – viz. the set of all elements in A^* that commute with all other elements – is defined by the elements of A^0.

The fact that the exterior product takes $A^k \times A^l$ to A^{k+l} means that, by definition, the algebra over A^* that is defined by the exterior product, which one calls the *exterior algebra over* V^*, is a *graded* algebra.

d. The algebra of multivectors. One can also define an exterior algebra over the vector space V itself by an analogous process of the complete anti-symmetrization of tensor products. For instance, the exterior product of vectors $\mathbf{v}$ and $\mathbf{w}$ in V is the *bivector:*

$$\mathbf{v} \wedge \mathbf{w} = \tfrac{1}{2}(\mathbf{v} \otimes \mathbf{w} - \mathbf{w} \otimes \mathbf{v}). \tag{I.44}$$

If $\mathbf{e}_i$ is a basis for V then all $\mathbf{e}_i \wedge \mathbf{e}_j$ will define a redundant basis for $A_2(V)$, and in order to eliminate the redundancy, one must use only exterior products with $i < j$. The components of $\mathbf{v} \wedge \mathbf{w}$ are then:

$$(\mathbf{v} \wedge \mathbf{w})^{ij} = v^i w^j - v^j w^i , \tag{I.45}$$

and:

$$\mathbf{v} \wedge \mathbf{w} = \tfrac{1}{2}(v^i w^j - v^j w^i)\, \mathbf{e}_i \wedge \mathbf{e}_j . \tag{I.46}$$

One could regard a bivector as an antisymmetric linear functional on A^2, since the expression:

$$(\mathbf{v} \wedge \mathbf{w})(\alpha \wedge \beta) = (\alpha \wedge \beta)(\mathbf{v} \wedge \mathbf{w}) = (\alpha \wedge \beta)(\mathbf{v}, \mathbf{w}) = \tfrac{1}{2}[\alpha(\mathbf{v})\beta(\mathbf{w}) - \alpha(\mathbf{w})\beta(\mathbf{v})], \tag{I.47}$$

can be extended by linearity to all finite linear combinations of 2-forms, but it is generally preferable to regard a non-zero bivector as more like a pair of linearly independent vectors in V, although this is true only in the *simple* case, where the bivector is of the form $\mathbf{v} \wedge \mathbf{w}$ for some pair of vectors $\mathbf{v}$ and $\mathbf{w}$, which is not, however, unique.

One defines the vector spaces $A_k(V)$ in an analogous manner to the way that one defined the $A^k(V)$, only in terms of vectors in V, not covectors in V^*. The elements of $A_k(V)$ – or just A_k , for short – are called *k-vectors* or *multivectors*, in general. One similarly defines the exterior product of k-vectors and l-vectors, and ultimately the

exterior algebra A_* becomes a graded algebra that is isomorphic to A^* as an algebra by means of any choice of basis for V. More specifically, each vector space A_k is linearly isomorphic to the corresponding A^k. However, none of these isomorphisms are canonical – i.e., defined uniquely in the absence of further assumptions.

e. Interior products. Just as (I.47) can be generalized to give a bilinear pairing $A^k \times A_k \to \mathbb{R}$, $(\alpha, \mathbf{A}) \mapsto \alpha(\mathbf{A})$, which represents the evaluation of a k-form on a k-vector, one can generalize this construction, as well, to give a bilinear pairing $A_l \times A^k \to A^{k-l}$ when $k > l$ and $A_l \times A^k \to A_{k-l}$ when $k < l$.

The construction begins by looking at how it works for the bilinear pairing of a vector $\mathbf{v}$ and a simple 2-form $\alpha \wedge \beta$. We define:

$$i_{\mathbf{v}}(\alpha \wedge \beta) = (i_{\mathbf{v}}\alpha)\,\beta - \alpha\,(i_{\mathbf{v}}\beta) = \alpha(\mathbf{v})\beta - \beta(\mathbf{v})\alpha. \tag{I.48}$$

and extend to a simple k-form similarly:

$$i_{\mathbf{v}}(\alpha_1 \wedge \ldots \wedge \alpha_k) = \sum_{m=1}^{k} (-1)^{m+1} \alpha_m(\mathbf{v})(\alpha_1 \wedge \cdots \wedge \hat{\alpha}_m \wedge \cdots \wedge \alpha_k), \tag{I.49}$$

in which the caret over the α_m means that the indicated 1-form is omitted from the exterior product.

Since any k-form can be expressed as a finite linear combination of simple k-forms – e.g., the basis elements – the action of $\mathbf{v}$ on simple k-forms can be extended by linearity to all of A^k, which then gives a bilinear pairing $A_1 \times A^k \to A^{k-1}$ that one refers to as the *interior product* of a k-form with a vector.

One can further extend this interior product from vectors to simple l-vectors by iteration:

$$i_{\mathbf{v} \wedge \ldots \wedge \mathbf{w}}\alpha = (-1)^{\tau}\, i_{\mathbf{v}}(\ldots i_{\mathbf{w}}\alpha), \tag{I.50}$$

and to all l-vectors by linearity. The sign comes from the number τ of transpositions that it takes to permute the sequence $\mathbf{v} \ldots \mathbf{w}$ into its reverse sequence $\mathbf{w} \ldots \mathbf{v}$. For instance, **vw** goes to **wv** by one transposition, as does **uvw** to **wvu**, while it takes two transpositions to take **tuvw** to **wvut**. Therefore, in the cases that will be of recurring interest to us:

$$i_{\mathbf{v} \wedge \mathbf{w}}\alpha = -\, i_{\mathbf{v}}(i_{\mathbf{w}}\alpha), \qquad i_{\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}}\alpha = -\, i_{\mathbf{u}}(i_{\mathbf{v}}\,(i_{\mathbf{w}}\alpha), \qquad i_{\mathbf{t} \wedge \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}}\alpha = i_{\mathbf{t}}(i_{\mathbf{u}}(i_{\mathbf{v}}(i_{\mathbf{w}}\alpha), \tag{I.51}$$

Hence, we now have our bilinear pairing of $A_l \times A^k \to A^{k-l}$, when $k > l$.

In order to define the bilinear pairing when $k < l$, we start by defining the interior product of a simple k-vector by a 1-form, analogously to (I.49):

$$i_{\alpha}(\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k) = \sum_{m=1}^{k} (-1)^{m+1} \alpha(\mathbf{v}_m)(\mathbf{v}_1 \wedge \cdots \wedge \hat{\mathbf{v}}_m \wedge \cdots \wedge \mathbf{v}_k), \tag{I.52}$$

and then extend this by linearity to all k-vectors. This gives a bilinear pairing $\mathrm{A}_k \times \mathrm{A}^1 \to \mathrm{A}_{k-1}$. We then extend to a bilinear pairing $\mathrm{A}_k \times \mathrm{A}^l \to \mathrm{A}_{l-k}$ for $k < l$ analogously to (I.50):

$$i_{\alpha \wedge \ldots \wedge \beta}\mathbf{A} = (-1)^{\tau}\, i_{\alpha}(\ldots\, i_{\beta}\mathbf{A}). \tag{I.53}$$

f. Poincaré duality. An important consequence of the bilinear pairings that we discussed in the last section is that whenever one chooses a volume element $\mathcal{V} \in \mathrm{A}^n$ for V one then defines a linear map #: $\mathrm{A}_k \to \mathrm{A}^{n-k}$, $\mathbf{A} \mapsto i_{\mathbf{A}}\mathcal{V}$ for each $0 \le k \le n$. These linear maps are seen to be, in fact, linear isomorphisms.

Conversely, if one chooses a volume element $\boldsymbol{\mathcal{V}} \in \mathrm{A}_n$ for V^* then one can define linear maps in the opposite directions #′: $\mathrm{A}^k \to \mathrm{A}_{n-k}$, $\alpha \mapsto i_{\alpha}\boldsymbol{\mathcal{V}}$, which are also linear isomorphisms. In fact, as long as one chooses $\boldsymbol{\mathcal{V}}$ to be the n-vector that makes $\mathcal{V}(\boldsymbol{\mathcal{V}}) = 1$, for consistency, the isomorphism #′ will be the inverse to #.

The complete set of linear isomorphisms that are defined by a choice of volume element on V is then referred to as *Poincaré duality*. Its geometric origin is in projective geometry, as we shall discuss later, and amounts to the idea that a k-dimensional subspace of V can either be spanned by k linearly independent vectors in V or annihilated by $n - k$ linearly independent covectors in V^*. It plays an essential role in the foundations of electromagnetism, as we shall see.

We illustrate the nature of Poincaré duality by showing how it works in the low dimensional vector spaces in terms of frame and coframes.

In two dimensions, if $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis and $\{\theta^1, \theta^2\}$ is its reciprocal basis then we can define $\boldsymbol{\mathcal{V}} = \mathbf{e}_1 \wedge \mathbf{e}_2$ and $\mathcal{V} = \theta^1 \wedge \theta^2$. The dual of a 0-vector $\lambda \in \mathbb{R}$ is the form $\lambda\mathcal{V}$, and conversely. As for the duals of the basis elements, by direct calculation, one verifies that:

$$\#\mathbf{e}_1 = i_{\mathbf{e}_1}(\theta^1 \wedge \theta^2) = \theta^2, \qquad \#\mathbf{e}_2 = -\,\theta^1. \tag{I.54}$$

More generally, one can say:

$$\#\mathbf{e}_i = \varepsilon_{ij}\,\theta^j, \qquad \varepsilon_{ij} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{I.55}$$

Although this all looks somewhat trivial at this point, actually when one goes from real to complex scalars, as we shall do later on, one finds that these isomorphisms are of great help in understanding the representation of Lorentz transformations in terms of $SL(2, \mathbb{C})$, which acts naturally on $\mathbb{C}^2$.

The three-dimensional case pertains directly to conventional vector algebra, as we shall discuss in more detail shortly. The non-trivial isomorphism is now #: $\mathrm{A}^1 \to \mathrm{A}^2$, which one can think of as taking "polar" vectors to "axial" vectors in the traditional physics terminology. If $\boldsymbol{\mathcal{V}} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ and $\mathcal{V} = \theta^1 \wedge \theta^2 \wedge \theta^3$ now, then we have:

$$\#\mathbf{e}_1 = \theta^2 \wedge \theta^3, \qquad \#\mathbf{e}_2 = \theta^3 \wedge \theta^1, \qquad \#\mathbf{e}_3 = \theta^1 \wedge \theta^2 , \tag{I.56}$$

or:

$$\#\mathbf{e}_i = \tfrac{1}{2}\, \varepsilon_{ijk}\, \theta^j \wedge \theta^k. \tag{I.57}$$

In four dimensions, both the isomorphisms $A_1 \to A^3$ and $A_2 \to A^2$ are non-trivial:

$$\begin{aligned} \#\mathbf{e}_1 &= \theta^2 \wedge \theta^3 \wedge \theta^4, \qquad \#\mathbf{e}_2 = \theta^3 \wedge \theta^4 \wedge \theta^1, \\ \#\mathbf{e}_3 &= \theta^4 \wedge \theta^1 \wedge \theta^3, \qquad \#\mathbf{e}_4 = \theta^1 \wedge \theta^2 \wedge \theta^3 \end{aligned} \tag{I.58a}$$

$$\begin{aligned} &\#(\mathbf{e}_1 \wedge \mathbf{e}_2) = \theta^3 \wedge \theta^4, \quad \#(\mathbf{e}_1 \wedge \mathbf{e}_3) = \theta^4 \wedge \theta^2, \quad \#(\mathbf{e}_1 \wedge \mathbf{e}_4) = \theta^3 \wedge \theta^2, \\ &\#(\mathbf{e}_2 \wedge \mathbf{e}_3) = \theta^1 \wedge \theta^4, \quad \#(\mathbf{e}_2 \wedge \mathbf{e}_4) = \theta^3 \wedge \theta^1, \quad \#(\mathbf{e}_3 \wedge \mathbf{e}_4) = \theta^1 \wedge \theta^2, \end{aligned} \tag{I.58b}$$

or, more concisely:

$$\#\mathbf{e}_i = \varepsilon_{ijkl}\, \theta^j \wedge \theta^k \wedge \theta^l, \qquad \#(\mathbf{e}_i \wedge \mathbf{e}_j) = \varepsilon_{ijkl}\, \theta^k \wedge \theta^l. \tag{I.59}$$

These will be the isomorphisms that are most useful for electrodynamics.

g. Algebraic operators defined on exterior algebras. The concepts of exterior product and interior product allow one to define various linear operators on A_* and A^* that prove useful in treating more elaborate topics.

Since the exterior product takes $A^k \times A^l$ to A^{k+l}, one sees that by fixing a k-form α, one can define a linear map e_α: $A^l \to A^{k+l}$, $\beta \mapsto \alpha \wedge \beta$, which one might think of as the *adjoint map* defined by α in the algebra, or simply, *left multiplication* by α.

Naively, the dimension of the image of A^l – i.e., the rank of e_α – under this map is less than or equal to $\min\{\dim(A^l), \dim(A^{k+l})\}$. In order for the dimension to be $\dim(A^l)$ the map would have to be an injection, which would imply that its kernel would have to vanish. This, in turn, would have to imply that $\alpha \wedge \beta \neq 0$ as long as $\beta \neq 0$. However, this is not always the case. For instance, in the elementary case of a 1-form α acting on other 1-forms, one sees that all β of the form $\lambda\alpha$ for a real scalar λ will give $\alpha \wedge \beta = 0$. Hence, the kernel of e_α: $A^1 \to A^2$ is 1-dimensional, so its image is n – 1-dimensional; its restriction to any hyperplane in $A^1 = V^*$ that is transverse to α will then be injective.

As we have seen, the interior products define a bilinear pairing $A_k \times A^l \to A^{l-k}$, when $k < l$ and $A_k \times A^l \to A_{k-l}$ when $k > l$. Hence, when one fixes a k-vector $\mathbf{A}$ one defines a linear map $i_{\mathbf{A}}$: $A^l \to A^{l-k}$, for $k < l$, and when one fixes an l-form α one defines a linear map i_α: $A_k \to A_{k-l}$ when $k > l$.

A crucial property of the interior product operator is that when α is a k-form and β is an l-form one has for any vector $\mathbf{v} \in V$:

$$i_{\mathbf{v}}(\alpha \wedge \beta) = i_{\mathbf{v}}\alpha \wedge \beta + (-1)^k\, \alpha \wedge i_{\mathbf{v}}\beta\,. \tag{I.60}$$

Suppose $\mathbf{e}_i$ is a basis for V and θ^i is the reciprocal basis for V^*. Any vector $\mathbf{v} \in V$ can be expressed as:

$$\mathbf{v} = v^i \mathbf{e}_i = \theta^i(\mathbf{v})\mathbf{e}_i = (e_{\mathbf{e}_i} \circ i_{\theta^i})(\mathbf{v})\,. \tag{I.61}$$

Hence, $e_{\mathbf{e}_i} \circ i_{\theta^i} = I$ when applied to vectors in V. In fact, each individual $e_{\mathbf{e}_i} \circ i_{\theta^i}$ (i not summed) acts as a projection of $\mathbf{v}$ onto the line in the direction $\mathbf{e}_i$. Therefore, the sum of terms over i represents a sort of spectral decomposition of the identity operator into projections onto the basis elements.

Similarly, $e_{\theta^i} \circ i_{\mathbf{e}_i} = I$ when applied to covectors in V^*.

Now, let us examine the opposite operator $i_{\theta^i} \circ e_{\mathbf{e}_i}$. When applied to $\mathbf{v}$ it gives:

$$(i_{\theta^i} \circ e_{\mathbf{e}_i})(\mathbf{v}) = i_{\theta^i}(\mathbf{e}_i \wedge \mathbf{v}) = \theta^i(\mathbf{e}_i)\mathbf{v} - \mathbf{e}_i\, \theta^i(\mathbf{v}) = n\mathbf{v} - \mathbf{v} = (n-1)\mathbf{v}. \tag{I.62}$$

This means that when we sum the operators thus defined, we get:

$$e_{\mathbf{e}_i} \circ i_{\theta^i} + i_{\theta^i} \circ e_{\mathbf{e}_i} = nI. \tag{I.63}$$

More generally, we would also find that for any non-zero vector $\mathbf{v}$ and covector α such that $\alpha(\mathbf{v}) = 1$, one would have:

$$I = e_{\mathbf{v}} \cdot i_\alpha + i_\alpha \cdot e_{\mathbf{v}}\,. \tag{I.64}$$

The first term represents a projection of a vector in V onto the direction [$\mathbf{v}$] spanned by $\mathbf{v}$, so the second term represents a projection of that vector onto the hyperplane Ann(α) in V that is annihilated by α, which is transverse to the line through $\mathbf{v}$. Hence, the pair ($\mathbf{v}$, α) defines a direct sum decomposition $V = [\mathbf{v}] \oplus \mathrm{Ann}(\alpha)$, and (I.64) corresponds to the fact that one will have a unique decomposition $\mathbf{w} = \mathbf{w}_\parallel + \mathbf{w}_\perp$ of $\mathbf{w} \in V$ into projections parallel to $\mathbf{v}$ and transverse to it.

One finds that the operator $e_{\mathbf{v}} \cdot i_\alpha + i_\alpha \cdot e_{\mathbf{v}}$ also acts as the identity operator on all of the other spaces A_k. Analogous statements to the preceding ones apply to the action of the operator $I = e_\alpha \cdot i_{\mathbf{v}} + i_{\mathbf{v}} \cdot e_\alpha$ on A^k. These operators will prove essential later when treating the effect of defining a timelike observer on the spacetime manifold; i.e., a space + time decomposition of the tangent bundle to the spacetime manifold.

3. Exterior derivative. From multi-variable calculus [**4**, **5**], one presumably knows how to define the differential df of a differentiable function f on a vector space V with real values and the fact that it represents a linear functional on V.

A choice of basis $\mathbf{e}_i$ for V defines a linear isomorphism $V \to \mathbb{R}^n$, $\mathbf{x} \mapsto (x^1(\mathbf{x}), \ldots, x^n(\mathbf{x}))$ which can also be treated as a global coordinate system on V. Each of the coordinate functions x^i on V is presumed to be differentiable and defines a differential dx^i. This makes:

$$df = \frac{\partial f}{\partial x^i}\, dx^i = f_{,i}\, dx^i, \tag{I.66}$$

in which we have introduced a common notation for partial derivatives as being indicated by a comma.

If one regards smooth functions on V as *(differential) 0-forms* and their differentials as *(differential) 1-forms* then one sees that the effect of differentiation is to take 0-forms to 1-forms. However, since functions take their values in $\mathbb{R}$ without actually being elements of $\mathbb{R}$, and the components of differential 1-forms are functions, not constants, in general, we see that we are not really dealing with A^0 and A^1 directly, but rather with smooth functions on V that take their values in these vector spaces.

In general, we will then denote the vector space of all smooth functions on V with values in A^k by $\Lambda^k(V)$, or Λ^k, for short. For instance, elements of Λ^0 will be smooth functions on V, elements of Λ^1 will be smooth maps from V to V^*, and elements of Λ^2 will be of the form:

$$F = \tfrac{1}{2} F_{ij}(x)\, dx^i \wedge dx^j \,. \tag{I.67}$$

The exterior derivative operator is an extension of the differential operator $d\colon \Lambda^0 \to \Lambda^1$ to a more general linear operator $d\colon \Lambda^k \to \Lambda^{k+1}$. As it turns out (see [**1**]), besides requiring linearity and agreement with the differential on 0-forms, this operator is uniquely defined by further requiring only that d be an anti-derivation with respect to the exterior product and that its square be zero, in any case. That is, if α is a k-form and β is an l-form then:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k\, \alpha \wedge d\beta \,. \tag{I.68a}$$

$$d^2 = 0. \tag{I.68b}$$

For instance, let us see what this produces for the exterior derivatives of 1-forms. If $\alpha = \alpha_i(x)\, dx^i$ then a direct application of the rules gives:

$$d\alpha = d\alpha_i \wedge dx^i + \alpha_i\, d^2x^i = (\alpha_{i,j}\, dx^j) \wedge dx^i = -\tfrac{1}{2}(a_{i,j} - a_{j,i})\, dx^i \wedge dx^j \,. \tag{I.69}$$

If $F = \tfrac{1}{2} F_{ij}\, dx^i \wedge dx^j$ is a 2-form then:

$$\begin{aligned} dF &= \tfrac{1}{2}\, dF_{ij} \wedge dx^i \wedge dx^j = \tfrac{1}{2}(F_{ij,k}\, dx^k) \wedge dx^i \wedge dx^j \\ &= \tfrac{1}{3}(F_{ij,k} + F_{jk,i} + F_{k,ij})\, dx^i \wedge dx^j \wedge dx^k. \end{aligned} \tag{I.70}$$

The fact that we are making d^2 vanish in any case is really a generalization of the fact that the second derivative of f defines a symmetric bilinear functional on V at each point of V; i.e., the mixed partial derivatives $\partial^2 f / \partial x^i\, \partial x^j$ are symmetric in i and j, since we are assuming that f has continuous second derivatives. Hence, the antisymmetric part of the bilinear functional must vanish. To illustrate how the symmetry of mixed partial derivatives implies the vanishing of d^2, apply it to a smooth function f:

$$d(df) = d(f_{,i}\, dx^i) = -\tfrac{1}{2}(f_{,i,j} - f_{j,i})\, dx^i \wedge dx^j = 0. \tag{I.71}$$

4. Divergence operator. When one has chosen a volume element $\mathcal{V}$ for a vector space V, one can use Poincaré duality to define a differential operator on multivector fields that is "adjoint" to d. For our immediate purposes, a k-vector field on V will be a smooth function from V to A_k. Hence, if $\{\partial_i, i = 1, \ldots, n\}$is a basis for V then a k-vector field $\mathbf{A}$ on V has the local form [9]:

$$\mathbf{A} = \frac{1}{k!} A^{ij\cdots m}(x)\, \partial_i \wedge \partial_j \wedge \cdots \wedge \partial_m . \tag{I.72}$$

We use Poincaré duality and the exterior derivative operator to define a divergence operator δ: $\Lambda^k \to \Lambda^{k-1}$ by way of:

$$\delta = \#^{-1} \cdot d \cdot \#. \tag{I.73}$$

This can be represented schematically by a "commutative diagram:"

$$\begin{array}{ccc} \Lambda^{n-k} & \xrightarrow{\ d\ } & \Lambda^{n-k+1} \\ \#\uparrow & & \#\uparrow \\ \Lambda_k & \xrightarrow{\ \delta\ } & \Lambda_{k-1} \end{array}$$

The properties of δ are not as convenient as those of d, although some of them are derived from properties of d, such as linearity, and the fact that:

$$\delta^2 = \#^{-1} \cdot d^2 \cdot \# = 0. \tag{I.74}$$

An immediate consequence of (I.73) is the useful relation:

$$d\# = \delta\#. \tag{I.75}$$

We also have:

$$\delta\partial_i = \#^{-1} \cdot d \cdot \#\partial_i = \varepsilon_{ij\ldots m}\, \#^{-1} \cdot d\,(dx^j \wedge \ldots \wedge dx^m) = 0. \tag{I.76}$$

Hence, as long as the coordinate basis ∂_i for V is reciprocal to the differential basis dx^i for Λ^1, the vanishing of the divergences of these basis vectors follows from $d^2 = 0$.

The relation (I.75) suggests that, in some sense, the linear operator δ is "adjoint" to the operator d. In order to clarify the sense in which this is meaningful, we introduce the bilinear pairing $\langle .,. \rangle$: $\mathrm{A}^k \times \mathrm{A}_k \to \mathrm{A}^n$, that takes $(\alpha, \mathbf{A})$ to:

$$\langle \alpha, \mathbf{A} \rangle = \alpha(\mathbf{A})\, \mathcal{V} = \alpha \wedge \#\mathbf{A} . \tag{I.77}$$

Now, let $\alpha \in \mathrm{A}^{k-1}$, $\mathbf{A} \in \mathrm{A}_k$ and compare $\langle d\alpha, \mathbf{A} \rangle$ to $\langle \alpha, \delta\mathbf{A} \rangle$:

[9] Although we shall eventually wish to regard the symbols ∂_i as relating to directional derivative operators, for the present, we shall only regard them as vectors in V.

$$<d\alpha, \mathbf{A}> = d\alpha \wedge \#\mathbf{A} = d(\alpha \wedge \#\mathbf{A}) - (-1)^k\, \alpha \wedge d\#\mathbf{A}$$
$$= d(\alpha \wedge \#\mathbf{A}) - (-1)^k\, \alpha \wedge \#\delta\mathbf{A}, \tag{I.78}$$

which leads to:

$$<d\alpha, \mathbf{A}> + (-1)^k <\alpha, \delta\mathbf{A}> = d(\alpha \wedge \#\mathbf{A}). \tag{I.79}$$

Once we have discussed the integration of differential forms and Stokes's theorem, we can show this leads to a "graded" form of adjointness between d and δ, at least for compact, orientable manifolds without boundary.

What complicates the use of δ in computations is that since # is a linear isomorphism, but not an algebra isomorphism – viz., it does not take $\alpha \wedge \beta$ to $\#\alpha \wedge \#\beta$ – the divergence operator is not an anti-derivation. One does, however, have the following useful result when f is a smooth function and $\mathbf{A}$ is a k-vector field:

$$\delta(f\mathbf{A}) = \#^{-1}(df \wedge \#\mathbf{A}) + f\, \delta\mathbf{A} \tag{I.80}$$

For $k = 1$, δ produces the usual divergence of a vector field:

$$\delta\mathbf{v} = \delta(v^i\, \partial_i) = \#^{-1}(dv^i \wedge \#\partial_i) + v^i\, \delta\partial_i = v^i{}_{,j}\, (dx^j \wedge \#\partial_i)(\mathbf{V}) = v^i{}_{,i}\,. \tag{I.81}$$

In the last step, we made use of what will later be described as an "orthogonality" condition:

$$dx^j \wedge \#\partial_i = \delta^j_i\, \mathcal{V}. \tag{I.82}$$

As we will also discuss later, when one is dealing with a Riemannian – or even Lorentzian – manifold, which has a metric tensor field $g_{ij}\, dx^i\, dx^j$ defined on it, the local form of the volume element $\mathcal{V}$ will also include a factor of $\sqrt{g}$, where $g \equiv |\det g_{ij}|$; hence, the volume element $\boldsymbol{\mathcal{V}}$ will have a factor of $1/\sqrt{g}$. One then sees that (I.81) takes on the customary form:

$$\delta\mathbf{v} = \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g}\, v^i)}{\partial x^i}. \tag{I.83}$$

Although the discussion of such matters at this point seems somewhat premature, the reason for mentioning it is to point out that in its most fundamental form the divergence operator is related solely to volume elements, not metrics, and this fact is essential to understanding pre-metric electromagnetism. In the next section, we shall see that vector fields with vanishing divergence generate flows of volume-preserving diffeomorphisms.

5. Lie derivative. Now that we have defined the exterior derivative and the interior product, we have enough tools to construct the Lie derivative operator. However, before we do so, in order to interpret that operator we first mention a few elementary notions from the theory of systems of ordinary differential equations (see [**10**] for all references to matters concerned with that theory).

a. Systems of first order ordinary differential equations. A vector field $\mathbf{v}$: $V \to V$, $x \mapsto \mathbf{v}(x) = v^i(x)\, \partial_i$ on an n-dimensional vector space V defines a system of n first order ordinary differential equations by starting with the assumption that $\mathbf{v}(x)$ represents the velocity vector field of a differentiable curve γ: $\mathbb{R} \to V$, $\tau \mapsto \gamma(\tau) = x^i(\tau)\, \partial_i$ at each point. The system of equations then takes the form:

$$\frac{dx^i}{d\tau} = v^i(x^j(\tau)) \tag{I.84}$$

relative to this choice of coordinate system on V.

Actually, this system is more general than it looks on first glance. For one thing, any ordinary differential equation of order k can be converted into a system of k first order equations, so the order of the system is no restriction. Furthermore, although the fact that $\mathbf{v}$ is only indirectly a function of τ implies that the system is autonomous – i.e., time-invariant – one can extend $\mathbf{v}$ to a vector field on $\mathbb{R} \times V$ that gives a system that is autonomous on that extended space.

From the theorem of existence and uniqueness of such systems of ordinary differential equations, as long as one assumes that $\mathbf{v}$ is continuously differentiable one always has the existence of *local flows* about each point $x \in V$; that is, there is always a sufficiently small neighborhood $(-\varepsilon, +\varepsilon)$ of $0 \in \mathbb{R}$ and a neighborhood U of x such that:

a. There is a one-parameter family of differentiable maps Φ: $(-\varepsilon, +\varepsilon) \times U \to V$, $(\tau, x) \mapsto \Phi_\tau(x)$, such that for each τ the map Φ_τ: $U \to V$ is a diffeomorphism onto its image. That is, it is invertible onto its image, and both it and its inverse are continuously differentiable.

b. The differentiable curves that are defined through each $x_0 \in U$ by way of $\gamma_0(\tau) = \Phi_\tau(x)$ are solutions to the equation $d\gamma / d\tau = \mathbf{v}$.

Because of the uniqueness of solutions in U these curves cannot intersect within U, and one finds that U is "foliated" by a *congruence* of integral curves; i.e., U is partitioned into distinct curves that are each a local solution of the system of ordinary differential equations that is defined by $\mathbf{v}$.

Two obvious questions to ask about the local flows that we have defined are those of whether $(-\varepsilon, +\varepsilon)$ can be extended to all of $\mathbb{R}$ for a given U and whether U can be extended to all of V, for a given ε. In the former case, the local flow would be called *complete*, and in the latter case, it would be a *global* flow. Neither possibility is obtained in all cases, but the analytical details are beyond the scope of the present discussion, so we refer the reader to the reference cited above.

b. Differentiation along a flow. To return to the calculus of exterior differential forms, the concept of a Lie derivative is based in the notion of looking at the time rate of change of something as it flows along a given integral curve for a system of first order ordinary differential equations. When our "something" takes the form of a smooth

function F: $\mathbb{R} \times V \to W$, $(\tau, x) \mapsto F(\tau, x)$, where W is a vector space of dimension N, this derivative with respect to τ along a curve $x(\tau)$ becomes:

$$\left.\frac{dF}{d\tau}\right|_{(\tau_0, x(\tau))} = \lim_{\tau \to 0} \frac{1}{\tau}[F(\tau_0 + \tau, x(\tau_0 + \tau)) - F(\tau_0, x(\tau_0))] = \left(\frac{\partial F^A}{\partial \tau} + \frac{dx^i}{d\tau} \frac{\partial F^A}{\partial x^i} \right) \partial_A , \qquad \text{(I.85)}$$

in which $\{\partial_A , A = 1, \ldots, N\}$ is the basis for W defined by a choice of coordinates.

Not surprisingly, this sort of derivative plays a key role in continuum mechanics, where it called variously the "material derivative" or the "substantial derivative." If we set $v^i = dx^i/d\tau$ then we shall also regard this derivative as the *Lie derivative of F with respect to* **v**, and denote it by $L_{\mathbf{v}}F$. We shall state without proof some useful facts about Lie derivatives, and refer the curious to other references for the proofs (e.g., [**1-3**])

The Lie derivative of a vector field $\mathbf{X} = X^j \partial_j$ on V with respect to another vector field $\mathbf{v} = v^i \partial_i$ takes the form:

$$L_{\mathbf{v}}\mathbf{X} = [\mathbf{v}, \mathbf{X}] = \left(v^j \frac{\partial X^i}{\partial x^j} - X^j \frac{\partial v^i}{\partial x^j} \right) \partial_i . \qquad \text{(I.86)}$$

The square brackets that we introduced are referred to as the *Lie bracket* of the vector fields in question. Because they are bilinear, anti-symmetric:

$$[\mathbf{X}, \mathbf{Y}] = -[\mathbf{Y}, \mathbf{X}], \qquad \text{(I.87)}$$

and satisfy the *Jacobi identity:*

$$[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] = 0, \qquad \text{(I.88)}$$

the product [.,.]: $\mathfrak{X}(V) \times \mathfrak{X}(V) \to \mathfrak{X}(V)$ defines, by definition, a *Lie algebra* on the (infinite-dimensional) vector space $\mathfrak{X}(V)$ of all vector fields on V.

In fact, the usual vector (or cross) product on $\mathbb{R}^3$ satisfies these requirements and thus defines a Lie algebra that is isomorphic to the Lie algebra of all infinitesimal Euclidian rotations in space. As we shall see, all that one needs to do to make the cross product useful in relativistic physics is to define it on $\mathbb{C}^3$, instead. The Lie algebra thus defined is then isomorphic to that of the infinitesimal Lorentz transformations.

The Lie derivative of more general multivector field on V is obtained from (I.86) by adding the assumption that the Lie derivative with respect to **v** acts on $\Lambda^*(V)$ as a *derivation* with respect to the exterior product:

$$L_{\mathbf{v}}(\mathbf{A} \wedge \mathbf{B}) = L_{\mathbf{v}}\mathbf{A} \wedge \mathbf{B} + \mathbf{A} \wedge L_{\mathbf{v}}\mathbf{B} . \qquad \text{(I.89)}$$

For instance, when one takes the Lie derivative of a simple bivector field $\mathbf{X} \wedge \mathbf{Y}$, one gets:

$$L_{\mathbf{v}}(\mathbf{X} \wedge \mathbf{Y}) = [\mathbf{v}, \mathbf{X}] \wedge \mathbf{Y} + \mathbf{X} \wedge [\mathbf{v}, \mathbf{Y}]. \qquad \text{(I.90)}$$

The Lie derivative of a covector field $\alpha = \alpha_i\, dx^i$ with respect to **v** is given by:

$$\mathrm{L}_{\mathbf{v}}\alpha = (i_{\mathbf{v}}d + di_{\mathbf{v}})\alpha = i_{\mathbf{v}}d\alpha + d\alpha(\mathbf{v}) = \left(v^j \frac{\partial \alpha_i}{\partial x^j} + \frac{\partial (v^j \alpha_j)}{\partial x^i} \right) dx^i , \qquad \text{(I.91)}$$

and can be similarly extended to all *k*-forms by assuming that $\mathrm{L}_{\mathbf{v}}$ also acts as a derivation with respect to the exterior product on differential forms:

$$\mathrm{L}_{\mathbf{v}}(\alpha \wedge \beta) = \mathrm{L}_{\mathbf{v}}\alpha \wedge \beta + \alpha \wedge \mathrm{L}_{\mathbf{v}}\beta . \qquad \text{(I.92)}$$

However, one finds that the expression in parentheses in the first equality of (I.90) is general to all differential forms. One obtains what is sometimes referred to as *Cartan's Magic Formula:*

$$\mathrm{L}_{\mathbf{v}} = i_{\mathbf{v}}d + di_{\mathbf{v}} . \qquad \text{(I.93)}$$

Of particular interest is the Lie derivative of a volume element $\mathcal{V}$ – or any other *n*-form – along the integral curves of a vector field **v**:

$$\mathrm{L}_{\mathbf{v}}\mathcal{V} = i_{\mathbf{v}}d\mathcal{V} + di_{\mathbf{v}}\mathcal{V} = d\#\mathbf{v} = \#\delta\mathbf{v} = (\delta\mathbf{v})\mathcal{V}. \qquad \text{(I.94)}$$

(In the second step, we implicitly used the fact that $d\mathcal{V}$ is an n+1-form on an n-dimensional vector space, hence, zero.) This gives, as a consequence the fact that the flow of **v** preserves the volume element iff **v** has vanishing divergence.

6. Integration of differential forms. Since a differential *n*-form $\phi = f(x^i)\, dx^1 \wedge \ldots \wedge dx^n$ looks suspiciously reminiscent of the integrand $f(x^i)\, dx^1 \ldots dx^n$ for the integration of a function f: $V \to \mathbb{R}$ over some *n*-dimensional region *B* in a vector space *V* of dimension at least *n*, it should come as no surprise that the integral of a differential *n*-form over such an *n*-dimensional region can be defined in much the conventional way that is defined in multivariable calculus by replacing the expression $f(x^i)\, dx^1 \wedge \ldots \wedge dx^n$ with the expression $f(x^i)\, dx^1 \ldots dx^n$. The possible change in sign that comes about by changing the order of the factors in $dx^1 \wedge \ldots \wedge dx^n$ then simply represents the possibility that ultimately there will be two orientations for *B*, and opposite orientations give opposite signs for the integral.

There is another possibility, namely, that *B* is not orientable, in the first place. In that case, *B* does not admit an *n*-form that is non-zero everywhere. We shall discuss the subject of orientation in more detail in the next chapter when we can give it its proper topological context.

We denote the integral of an *n*-form *f* over an *n*-dimensional region *B* by:

$$\int_B \phi = \int_B f(x^1, \cdots, x^n)\, dx^1 \cdots dx^n . \qquad \text{(I.95)}$$

It is important to recognize the integral of an n-form is defined only over an n-dimensional region. This is because one expects the integral of anything over B to be independent of the parameterization of B, and the only n for which n-forms transform properly is $n = \dim(B)$; in effect, this is the only dimension in which the differential map of a change in parameterization contributes only a determinant to the transformation of the n-form components. That is, if $y^i = y^i(x^j)$ represents a change in parameterization – i.e., a diffeomorphism – for the region B then the n-form $f(y^i)\, dy^1 \wedge \ldots \wedge dy^n$ goes to $f(x^i)\, J(x)\, dx^1 \wedge \ldots \wedge dx^n$, in which:

$$J(x) = \det\left(\frac{\partial y^i}{\partial x^j}\right) \tag{I.96}$$

is the *Jacobian* of the diffeomorphism.

Note that since the differential map to a diffeomorphism must be invertible (this follows from the chain rule for differential maps), one must have that $J(x) \neq 0$ everywhere. If $J(x) > 0$ everywhere then the change in parameterization is *orientation-preseving*; if it is < 0 everywhere then it is *orientation-reversing*.

Ultimately, one can only take line integrals of 1-forms, surface integrals of 2-forms, and volume integrals of 3-forms, etc. Although this may sound trivial, actually, it is common practice in physics and engineering to integrate components of vector fields and tensor fields whose rank is largely unrelated to the dimension of the region, such as when one defines the total linear momentum or total angular momentum of a three-dimensional extended object by integrating the components of the corresponding densities. One is cautioned that the resulting integrals are not invariant under changes of frames that are not constant at all points, such as the transition from a holonomic to an anholonomic frame. Hence, just as differentiation can produce fictitious – i.e., frame-dependent – velocities and accelerations, integration can produce "fictitious moments."

If α is a k-form then $d\alpha$ is a k+1-form. Hence, if B is an orientable k+1-dimensional region with a k-dimensional boundary ∂B then it is possible to define the integral of α over ∂B and $d\alpha$ over B invariantly. If $\mathcal{V}$ is the k+1-dimensional volume element on B and $\mathbf{n}$ is the unit normal to the k-dimensional boundary (assuming that V is equipped with a Euclidian metric) then the k-form $i_{\mathbf{n}}\mathcal{V}$, when restricted to the points of ∂B gives it a k-dimensional volume element. In an "adapted" coordinate system for B, if $\mathcal{V} = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^{k+1}$ and x^1 is adapted to the normal direction then $i_{\mathbf{n}}\mathcal{V} = dx^2 \wedge \ldots \wedge dx^{k+1}$.

The two integrals are related in a simple, but far-reaching, way by the fact that:

$$\int_{\partial B} \alpha = \int_B d\alpha\,. \tag{I.97}$$

Depending upon the dimension of B this equality can be interpreted as the fundamental theorem of calculus (dim = 1), Green's theorem (dim = 2), or Stokes's theorem (dim = 3). We shall discuss this more in the next section.

In fact, Gauss's theorem (i.e., the divergence theorem) also relates to the three-dimensional form, by way of Poincaré duality. We shall have more to say about this in Chapter III after we have defined charge and flux, but, for now, suppose that $\alpha = \#\mathbf{a}$. Then (I.97) becomes:

$$\int_{\partial B} \#\mathbf{a} = \int_B (\delta\mathbf{a})\mathcal{V} . \tag{I.98}$$

Proof:

$$\int_{\partial B} \#\mathbf{a} = \int_B d\,\#\mathbf{a} = \int_B \#\,\delta\mathbf{a} = \int_B (\delta\mathbf{a})\mathcal{V}$$

In the standard treatments on differential forms, the generic term for this theorem is *Stokes's theorem* [10]. Since the topological significance of this relationship is quite subtle, but important, we shall have more to say about this theorem in Chapter II.

One can further note that, from (I.97), whenever a k-form α is *closed* – viz., $d\alpha = 0$ – the integral of α over the boundary of a k+1-dimensional region must always vanish.

We can now return to the discussion of adjointness between d and δ that we suspended in a previous section. We simply redefine our bilinear pairing to be $<.,.> : A^k \times A_k \to \mathbb{R}$, which takes $(\alpha, \mathbf{A})$ to:

$$<\alpha, \mathbf{A}> = \int_B \alpha \wedge \#\mathbf{A} , \tag{I.99}$$

when B is a compact, orientable n-dimensional manifold with boundary.

By integration and an application of Stokes's theorem, (I.79) takes the form:

$$<d\alpha, \mathbf{A}> + (-1)^k <\alpha, \delta\mathbf{A}> = \int_{\partial B} \alpha \wedge \#\mathbf{A} . \tag{I.100}$$

Hence, when B has no boundary, one can say that:

$$<d\alpha, \mathbf{A}> = -(-1)^k <\alpha, \delta\mathbf{A}>. \tag{I.101}$$

In particular, in even dimensions d and δ are skew-adjoint, while in odd dimensions they are self-adjoint.

7. Relationship to vector calculus. The first thing that one notices about the relationship between exterior differential forms and vector analysis is that the exterior product subsumes the vector cross product, as it defined in $V = \mathbb{R}^3$.

To see this, give $A_1 = \mathbb{R}^3$ the canonical basis $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$, which one depicts as column vectors, so the reciprocal basis θ^i has the same triples of numbers as row vectors. One can give A_2 the basis $\mathbf{e}_1 \wedge \mathbf{e}_2$, $\mathbf{e}_2 \wedge \mathbf{e}_3$, $\mathbf{e}_3 \wedge \mathbf{e}_1$, which means:

$$\mathbf{e}_i \wedge \mathbf{e}_j = \varepsilon_{ijk}\, \mathbf{e}_k . \tag{I.102}$$

[10] Vladimir Arnol'd may have made a valid point in [**11**] when he suggested that if one wished to be consistent with the modern tendency to hyphenate all of the names that contributed to the general form of a modern theorem then one might wish to call it the *Newton-Leibniz-Gauss-Green-Ostragradskii-Stokes-Poincaré theorem!* (…and perhaps abbreviate this to the acronym NLGGOSP!)

Hence, one can think of the symbol ε_{ijk} as essentially the matrix of the isomorphism of A_1 with A_2.

Now, strictly speaking, this isomorphism only comes about because we chose a basis for $\mathbb{R}^3$. In point of fact, the natural frame-invariant isomorphism is the Poincaré duality between A_1 and A^2:

$$\theta^i \wedge \theta^i = \varepsilon^{ijk}\, \mathbf{e}_k \,. \tag{I.103}$$

The volume element on V that defines this is given by the 3-form $\theta^1 \wedge \theta^2 \wedge \theta^3$.

In order to make this a frame-invariant isomorphism of A_1 with A_2, one must define either an isomorphism of A_1 with A^1 or A_2 with A^2, such as one usually gets from the Euclidian metric. However, the spirit of pre-metric electromagnetism is to treat the spacetime metric as something that shows up at a later stage than the most fundamental assumptions about the spacetime manifold. Indeed, we shall find that both the electric permittivity and the magnetic permeability by themselves can also define such isomorphisms.

For now, we notice that the components of $\mathbf{a} \wedge \mathbf{b}$ with respect to the stated basis are:

$$(\mathbf{a} \wedge \mathbf{b})_{ij} = a_i\, b_j - a_j\, b_i \,, \tag{I.104}$$

and those of $\mathbf{a} \times \mathbf{b}$ are:

$$(\mathbf{a} \times \mathbf{b})_i = \tfrac{1}{2}\, \varepsilon_{ijk}\, (\mathbf{a} \wedge \mathbf{b})_{jk} \,. \tag{I.105}$$

Hence, one is basically associating the two sets of components by way of the isomorphism of vectors and bivectors that we have chosen.

Of course, the advantage of exterior forms then becomes the generality of their application in the eyes of dimension, while the cross product can only be defined in dimension three.

The triple product $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$, which also involves the Euclidian scalar product, is related to an even more elementary exterior product:

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = (\mathbf{a} \cdot \mathbf{b} \times \mathbf{c})\boldsymbol{\mathcal{V}} = \det[\mathbf{a} \mid \mathbf{b} \mid \mathbf{c}]\boldsymbol{\mathcal{V}}. \tag{I.106}$$

One finds that the exterior derivative operator immediately subsumes both the gradient operator, as applied to smooth functions, and the curl operator, as applied to vector fields, and we have already showed that the divergence operator, as we have defined it, agrees with the classical one on vector fields:

$$(df)_i = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (\nabla f)_i \,, \tag{I.107a}$$

$$(dv)_{ij} = v_{i,j} - v_{j,i} = \varepsilon_{ijk} (\nabla \times \mathbf{v})_k \,, \tag{I.107b}$$

$$\delta \mathbf{v} = \nabla \cdot \mathbf{v} \,. \tag{I.107c}$$

However, as is usually the case in three-dimensional computations, one is always implicitly defining the Euclidian scalar product on $\mathbb{R}^3$ and using it to associate vectors

with covectors. For instance, the gradient of a function is usually thought of as a vector field, not a covector field. Consequently, in the component formulation of vector and tensor analysis, there is no attention paid to whether one is implicitly raising or lowering indices as it suits the calculations; indeed, all indices are usually written as subscripts, for simplicity. This is really a bad habit to get into if one is also going to be concerned with spaces of other dimensions and metrics of other signature types, since it often involves remembering when such tacit constructions were being carried out, if one is to replace them with the more general constructions.

The fact that $d^2 = 0$, in any case, subsumes the vanishing of $\nabla\times\nabla f$ for any f and $\nabla\cdot(\nabla\times\mathbf{v})$ for any $\mathbf{v}$; that is, the curl of a gradient and the divergence of a curl vanish identically. Ordinarily, one assumes that the vanishing of $\nabla \times \mathbf{v}$ implies that $\mathbf{v} = \nabla f$ for some (non-unique) function f, but, as we shall see in the next chapter, this really goes back to the fact that we are looking at vector fields and k-forms on a vector space, which is, among other things, simply connected. That is, whether $d\alpha = 0$ implies that there is a ϕ such that $\alpha = d\phi$ depends upon deeper topological considerations, which we will discuss in Chap. III.

In order to relate the general notation for the integration of a k-form α over a k-dimensional region B in an n-dimensional vector space V to the various notations for the integral of a function or vector field over regions of $\mathbb{R}^3$ that are one, two, and three-dimensional, one mostly has to show how the free index of a vector field $\mathbf{A}(x) = (A^1(x), A^2(x), A^3(x))$ gets absorbed to produce a k-form that looks like $f\mathcal{V}_k$.

For line integrals, it is sufficient to turn the vector field $\mathbf{A}$ into a 1-form $A = A_i\, dx^i$ by means of the Euclidian scalar product ($A_i = \delta_{ij}A^i$) and show that this is equal to:

$$A = \mathbf{A}\cdot d\mathbf{s}\,, \tag{I.108}$$

in which:

$$d\mathbf{s} = \mathbf{v}\, d\tau = (dx^1, dx^2, dx^3) \tag{I.109}$$

is the differential element of arc length; however, the result is immediate by this definition. Hence, we can unambiguously write:

$$\int_C A = \oint_C \mathbf{A}\cdot d\mathbf{s}\,. \tag{I.110}$$

In two dimensions, the elimination of the free vector index is accomplished by means of taking the scalar product of $\mathbf{A}$ with the normal vector field $\mathbf{n} = \#^{-1}dS$ to the surface S over which the integration is performed; here dS refers to the surface element on S (i.e., a non-vanishing 2-form in $\Lambda^2(S)$). We can also associate the vector field $\mathbf{A}$ with the 2-form:

$$\#\mathbf{A} = i_{\mathbf{A}}\mathcal{V} = A_n\, i_{\mathbf{n}}\mathcal{V} = A_n\, dS. \tag{I.111}$$

Indeed, the only 2-forms that can be integrated over S have to be tangent to S, and therefore of the form $A_n\, dS$.

If one uses an adapted coordinate system in the neighborhood of S that makes:

$$\mathcal{V} = n \wedge dS \qquad (n_i = \delta_{ij}\, n^j) \tag{I.112}$$

then one has:

$$\#\mathbf{A} = i_{\mathbf{A}}\mathcal{V} = n(\mathbf{A})\, dS - n \wedge i_{\mathbf{A}}\, dS\,. \tag{I.113}$$

and the part of this that is tangential to S is:

$$n(\mathbf{A})\, dS = (\mathbf{A} \cdot \mathbf{n})\, dS\,. \tag{I.114}$$

Thus, in order to make sense of the equality:

$$\int_S A = \oiint_S (\mathbf{A} \cdot \mathbf{n})\, dS \tag{I.115}$$

the 2-form A and the vector field $\mathbf{A}$ must be related by:

$$A = P_S(\#\mathbf{A}), \tag{I.116}$$

in which P_S refers to the projection onto the tangential 2-forms. This implies that, although the map # is a bijection, nonetheless, a sizable subspace of vector fields will produce the same tangential 2-form, namely the ones for which $i_{\mathbf{A}}dS$ is non-vanishing; i.e., ones that differ by a vector that is tangent to S.

As for a three-dimensional B, the only things that can be integrated invariantly are pseudo-scalars of the form $f\mathcal{V}_3$. A non-algebraic way of turning $\mathbf{A}$ into a scalar is to take its divergence, which gives Gauss's theorem (I.99) the form:

$$\oiint_{\partial B} (\mathbf{A} \cdot \mathbf{n})\, dS = \oiiint_B (\nabla \cdot \mathbf{A})\, \mathcal{V}\,. \tag{I.117}$$

Something that one begins to notice after working with the calculus of exterior differential forms long enough is that even though it is possible to find differential form equivalents for many of the tabulated formulas of vector calculus beyond the elementary ones that were discussed above, nevertheless, part of the beauty of differential forms is in the way that many of these vector calculus identities become largely unnecessary, since one generally only needs them as lemmas for the proofs of more fundamental results that can be proved more directly with differential forms.

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Chapter II

Topology of differentiable manifolds

From the standpoint of physics, there are two main reasons for extending the calculus of exterior differential forms from the formulation that was presented in the previous chapter for vector spaces to a formulation that pertains to more general differentiable manifolds:

1. The introduction of coordinate systems is inevitable in physics and differentiable manifolds are the mathematical structures that have evolved to address that situation.

2. The physical foundations of electromagnetism, such as charge and flux, are manifestly topological in character.

Consequently, the purpose of this chapter is to carry out that extension so that the aforementioned foundations of electromagnetism can be presented in their topological form in the next chapter. We shall present only the elements of point-set topology to begin with, and then we shall introduce some of the more advanced topological notions in a form that is adapted to the nature of the problem at hand.

1. Differentiable manifolds. In the previous chapter, we made use of coordinate systems on vector spaces, which often came about as a consequence of defining a basis for the vector space. Actually, such a construction will produce only "rectilinear" or "Cartesian" coordinate systems. If one wishes to discuss "curvilinear" coordinate systems then one will have to deal with the fact that they usually come about as an adaptation to the demands of dealing with nonlinear spaces such as cylinders, spheres, and ellipsoids.

First, we need to recall some useful generalities at the level of point-set topology.

a. Topological spaces [**1, 2**]. In mathematics, one of the most useful nonlinear generalizations of a vector space is that of *topological space*. Such a space is a set S that is given a *topology* – viz., a collection of *open subsets* that satisfy certain axioms regarding unions and intersections. In particular, the union of any family of open subsets is another open subset and the intersection of any *finite* family is another open subset [11]. One can then define a *closed* subset as the set complement of an open set and obtain a collection of closed subsets that satisfy the complementary axioms (finite unions, any intersections). Conversely, one can start with closed subsets and define open subsets by complementation. Note that a given subset of S can be open, closed, both, or neither.

Some elementary elementary examples of topological spaces that will be of interest in what follows are the real line $\mathbb{R}$ and the Cartesian product $\mathbb{R}^n$ of ordered n-tuples of real numbers. The topology that is commonly used for $\mathbb{R}$ is defined by means of its total

[11] Often, one encounters the axioms that S and the empty set are open subsets, although to some authors, these axioms follow logically from the two that were stated.

ordering <, namely, the interval topology, whose open subsets are unions of open intervals of the form $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$. The closed subsets are then intersections of finite unions of closed intervals $[a, b] = (a, b) \cup \{a, b\}$. One can also define a topology for $\mathbb{R}$ by means of the metric $d(x, y) = | x - y |$, which allows one to define open "ε-balls" $B_x(\varepsilon)$ about each point by way of $B_x(\varepsilon) = \{y \in \mathbb{R} \mid | x - y | < \varepsilon\}$. Since these will also define open intervals of the form $(x - e, x + e)$, one sees that the two topologies just defined amount to the same open subsets; hence, they are topologically equivalent. As for $\mathbb{R}^n$, one can extend the interval topology on $\mathbb{R}$ by means of the product topology on $\mathbb{R}^n$, which makes every open subset the union of finite intersections of products $(a_1, b_1) \times \ldots \times (a_n, b_n)$ of open intervals in $\mathbb{R}$. An equivalent topology is given by extending the concept of ε-balls to n dimensions by extending the norm in the Euclidian manner $\| \mathbf{v} \| = (v_1^2 + \cdots + v_n^2)^{1/2}$. Of course, one needs to show that every ε-ball is the union of n-cubes or vice versa, but, as it turns out, in order to show the topological equivalence of the two topologies, it is only necessary to show that there is a continuous, invertible map from one to the other that also has a continuous inverse, as we shall discuss shortly. Such a map is given by radially projecting the points of one onto the other.

The effect of defining a topology is to define an abstract way of characterizing "closeness" without introducing an actual numerical way of defining the concept, such as a metric. A *neighborhood* of a point $x \in S$ is any subset $U \subset S$ that contains an open subset that includes x. Hence, any point of S defines a localization of the topology on S to a system of neighborhoods of x. The partial ordering of subset inclusion allows one to the think of the relative "closeness" of a point y to x, as compared to another point z as being related to whether there is a neighborhood of x that contains one point, but not the other.

One can use the partial ordering of subset inclusion to define a partial ordering on the set of topologies over a given set. A topology τ on S is *finer* than another topology τ' iff every open subset in τ' is also an open subset in τ. Hence, τ potentially has "more open subsets" than τ'. One also says that the topology τ' is *coarser* than the topology τ.

Another issue that is related to the "fineness" of the topology is the question of whether there is always an open neighborhood of x that excludes any other given point of S. This gets one into the *separation* axioms, the most common of which is the Haussdorff axiom: A topological space is *Haussdorff* iff given any two distinct points there are disjoint open neighborhoods of each point. Another axiom that shows up in the study of manifolds is normality: A topological space is *normal* iff any two disjoint closed subsets have disjoint open neighborhoods. One sees that as long as subsets that consist of only individual points are always closed normal spaces must be Haussdorff.

So far, the nature of topological spaces seems rather set-theoretic and prosaic. The real power of the topological structure on a set emerges when one uses it to define the continuity of maps. A map $f: A \to B$ between topological spaces A and B is said to be *continuous* iff the inverse image $f^{-1}(V) = \{x \in S \mid f(x) \in V\}$ of any open subset of B is an

open subset of A. All that one needs to do to reconcile this with the "ε–δ" definition that one encounters in calculus is to consider the topologies on normed vector spaces V, W that are given by considering open balls $B_\delta(x)$ in V and $B_\delta(y)$ in W. Hence, one of the subtle sources of the power of topology is that it simultaneously generalizes elements of both geometry and analysis.

An even stronger condition on the map f than continuity is that it should be invertible and have a continuous inverse. Such a map between topological spaces is called a *homeomorphism*. They are fundamental because they represent topological equivalences, since one has not only a one-to-one correspondence between the points of the spaces one also has a one-to-one correspondence between the open subsets of the topologies.

For instance, the map f might be the identity map on a set S that has been given two different topologies. Although the map is clearly invertible, whether it is continuous in one direction or another is another way of characterizing the fineness of one topology relative to the other one. For instance, the previous example of $\mathbb{R}^n$, when given both the product topology and the norm topology, has the property that the identity map is continuous, along with its inverse, since every open subset of one topology is an open subset of the other one; hence, we can say that the two topologies are homeomorphic.

Having defined topological structures and maps that relate to them, one also wishes to know what sort of properties of topological spaces are preserved by such maps. Since homeomorphic spaces have equivalent topologies, it is essentially a matter of definition to say that a topological property of a space is one that is equally true for homeomorphic spaces. The properties that are preserved by merely continuous maps, at least on their images, are somewhat more definitive.

One such property of continuous maps is *compactness:* a topological space S is called *compact* iff it is Haussdorff and every open covering of it can be reduced to a finite sub-covering; note that this does not say that any space S that can be covered by a finite number of open subsets is compact, since S itself covers any topological space, in any case. One finds that any closed subset of a compact space is compact, and any compact subset of a Haussdorff space is closed. By the *Heine-Borel theorem*, the compact subsets of $\mathbb{R}^n$, when given the Euclidian norm topology, are the closed and bounded subsets. The fact that the image of a compact topological space by a continuous map is compact also explains why any continuous real-valued function on a compact space, such as a sphere, must have a maximum value, along with a minimum value, at some points of the space.

Another property that is preserved by continuous maps, which also gets generalized to more sophisticated methods in topology, such as algebraic topology, is connectedness. A topological space is *connected* iff it is not the union of two disjoint open subsets; equivalently, the only subsets that are both open and closed are the empty set and the entire set. The fact that continuous real-valued function from a connected space S onto the interval (a, b) in the real line must attain each intermediate value between a and b is due to the fact that the connected subsets of $\mathbb{R}$ are the intervals.

Conversely, a continuous map $\gamma\colon [a, b] \to S$ must have a connected image. Such a map is described as a *path* in S from $\gamma(a)$ to $\gamma(b)$. If any two points of S admit at least one path that connects them then S is called *path-connected.* The image of a path-connected

space under a continuous map is clearly path-connected. One also finds that any path-connected space is connected, although the converse is not necessarily true.

A particular class of paths in a space that is of fundamental interest is the class of all loops: A *loop* in S is a path $\gamma:[a, b] \to S$ such that $\gamma(a) = \gamma(b)$. Note that this includes the possibility that the image of γ is just one point.

b. Homotopy theory [**1-3**]. A fundamental question about a topological space is whether every loop can be continuously deformed to a point. That is, two continuous maps $f, g: A \to B$ between topological spaces are called *homotopic* iff there is a continuous map $F: A \times [0, 1] \to B$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. Hence, if f is homotopic to g then one can continuously deform f into g by means of a one-parameter family of maps in between. In effect, this is also like defining a path in the "topological space" of continuous maps from A to B, although defining the topology on the space that would make this statement well-defined proves to be the less practical way of looking at homotopies. One finds that homotopy is an equivalence relation between continuous maps, since every map is homotopic to itself, if f is homotopic to g then g is homotopic to f, and if f is homotopic to g while g is homotopic to h the f is homotopic to h. Hence, one can partition the set of all continuous maps from A to B into equivalences classes, namely, *homotopy classes*, which we denote by $[f]$.

In the case of homotopies of loops in a topological space S, one finds that the set $\pi_1(S)$ of homotopy classes $[\gamma]$ of loops can often have a very elementary character to it. When there is only one such class – i.e., all loops are homotopic to constant loops – one calls S *simply-connected*, and denotes this by $\pi_1(S) = 0$. For instance, any vector space is simply-connected. The notion of simple-connectedness is neither stronger nor weaker than path-connectedness, since a plane minus a point – i.e., a *punctured plane* – is path-connected, but not simply-connected, while a pair of disjoint discs in the plane is simply-connected, but not path-connected. The homotopy classes of loops in a circle – i.e., the homotopy classes $[f]$ of continuous maps $f: S^1 \to S^1$ – are in one-to-one correspondence with the integers $\mathbb{Z}$, by way of the "winding number" of the map f.

The set $\pi_1(S)$ can also be given a group structure by composing subsequent loops, and with that group structure one calls $\pi_1(S)$ the *fundamental group* of S. The subscript "1" on $\pi_1(S)$ is suggestive of a sequence of other such groups. Indeed, one can define *higher homotopy groups* for a topological space S by defining the k^{th} homotopy group $\pi_k(S)$ to consist of homotopy classes of continuous maps of the k-dimensional sphere S^k into S. The group structure is somewhat more difficult to describe (cf. [**3**]), although, as it turns out, homotopy groups are always Abelian in dimensions higher than one. Interestingly, although clearly $\pi_n(S^n) = \mathbb{Z}$, by way of a higher-dimensional winding number argument, and $\pi_1(S^n) = 0$ whenever $n > 1$, nonetheless, finding the higher homotopy groups of spheres in general is more difficult than it sounds.

For the sake of completeness, one usually denotes the set of all path connected components of S by $\pi_0(S)$, although it is not generally given a group structure.

Since the composition of continuous maps is continuous, one finds that the image of a loop γ in A by a continuous map $f: A \to B$ is a loop $f \cdot \gamma$ in B. Since homotopic loops in A will have homotopic images in B under f – in particular, homotopic loops – a continuous

map will take $\pi_1(A)$ to a subgroup of $\pi_1(B)$. Indeed, this situation extends to the higher dimensions, and when two topological spaces are homeomorphic they will have isomorphic homotopy groups in all dimensions. The converse, however, is not always true, although it has finally been proven for spheres, which was the generalized Poincaré conjecture. (The original conjecture was for three-dimensional spheres, which was also the last dimension in which the conjecture was proved.)

The most extreme form of a homotopy is a *contraction:* A topological space is called *contractible* iff the identity map is homotopic to a constant map; i.e., the whole space is homotopic to one of its points. For example, any vector space is contractible, as is a space that consists of only one point, to begin with. The homotopy groups of a contractible space vanish in every dimension, since this is the case for a point. It is a deep result of homotopy theory that the converse statement is also true: Any topological space whose homotopy groups vanish in every dimension is contractible.

c. Differential structures [**4-9**]. The next step beyond considering topological spaces that are homeomorphic to vector spaces, which are only so topologically interesting, is considering topological spaces that are locally homeomorphic to vector spaces. That is, a *topological manifold* is a topological space M such that every point $x \in M$ has a neighborhood U that is homeomorphic to $\mathbb{R}^n$. One can think of the homeomorphism ϕ: $U \to \mathbb{R}^n$, $p \mapsto (x^1(p), \ldots, x^n(p))$ as defining a *coordinate system* on U, and we call the pair (U, ϕ) a *coordinate chart* on M about x.

Examples of such spaces are spheres, tori, polygons, polyhedra, and many algebraic sets. Counter-examples would be such things as intersecting curves, a pair of tangent spheres, and a disk with a line segment attached to its perimeter at one endpoint. In effect, since $\mathbb{R}^n$ is homeomorphic to any open n-ball (regardless of radius), to say that a point of M has a neighborhood that looks like $\mathbb{R}^n$ topologically is to say that it has a neighborhood that looks like a (sufficiently small) open n-ball.

If one has another neighborhood V of x and another homeomorphism ψ: V: $\to \mathbb{R}^n$, $p \mapsto (y^1(p), \ldots, y^n(p))$ then on the overlap $U \cap V$ one can invert ϕ and compose it with ψ to obtain a homeomorphism $\psi \cdot \phi^{-1}$: $\mathbb{R}^n \to \mathbb{R}^n$, $(x^1, \ldots, x^n) \mapsto (y^1, \ldots, y^n)$ which then represents a *coordinate transformation*.

From the examples given above, it is clear that topological equivalence is really quite vague in character, since it ignores many of the geometric properties of spaces. For instance, every point of a circle has a unique tangent, but the vertices of a triangle, which is homeomorphic to a circle, have double tangents. Hence, if one wishes to refine the equivalence classes of topological equivalence, a next step might be to include some consideration for how tangent spaces are associated with the points of the space. Since the main point of differentiation is to locally linearize nonlinear functions by associating them with linear ones, to some degree of approximation, this brings one into the realm of differential topology.

In order to refine the structure of a topological manifold, since the differentiation of maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ can be defined (when they are differentiable) we restrict the class of allowable coordinate transformations, as defined above, to the ones that are differentiable and have a differentiable inverse; i.e., to *diffeomorphisms* of $\mathbb{R}^n$. Hence, since such a coordinate transformation $\psi \cdot \phi^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ will have a differential map $D(\psi \cdot \phi^{-1})|_y : \mathbb{R}^n \to \mathbb{R}^n$ that is defined at each $y \in \mathbb{R}^n$, and since this is an invertible linear map it defines an element of the group $GL(n)$ of all invertible $n \times n$ real matrices (by means of the standard basis on $\mathbb{R}^n$) we can define a *transition function* g_{UV}: $U \cap V \to GL(n)$, $x \mapsto D(\psi \cdot \phi^{-1})|_{\phi(x)}$. If the coordinates that are defined by ϕ are x^i and those defined by ψ are y^i then the coordinate transformation takes the form $y^i = y^i(x^j)$ and the transition function takes the form:

$$\left[g_{U\cap V}(x)\right]^i_j = \left.\frac{\partial y^i}{\partial x^j}\right|_x . \tag{II.1}$$

Notice that is it the coordinate *transformations* that we are restricting, not the coordinate *charts*. However, having fewer allowable transformations will lead to having fewer charts.

A *differential structure* on a topological manifold M is a collection $\{(U_\alpha, \phi_\alpha), \alpha \in \Gamma\}$ of charts – which one calls an *atlas* – such that:

1. The atlas covers M, i.e., every point of M is contained in at least one chart.
2. All of the coordinate transformations are diffeomorphisms of $\mathbb{R}^n$.
3. The atlas is maximal, in the sense that any chart whose intersections with the existing charts gives coordinate transformations that are diffeomorphisms must already be a chart of the atlas.

A topological manifold that has been given a differential structure is called a *differentiable manifold.* If $f: M \to N$ is a map from a differentiable manifold M to another one N then for every chart (U, ϕ) about a point $x \in M$ and every chart (V, ψ) about the point $f(x) \in N$ one can define the map $\psi \cdot f \cdot \phi^{-1}: \mathbb{R}^m \to \mathbb{R}^n$ by inversion and composition.

If this map is differentiable for every possible x, (U, ϕ), and (V, ψ) then one calls the map f itself *differentiable.* Furthermore, if f is invertible, differentiable, and has a differentiable inverse then one calls f a *diffeomorphism* of M and N. This is a stronger statement than homeomorphism, since every differentiable map is continuous, but not always the converse.

When M is given two differential structures, one can consider the differentiability of the identity map. If it is a diffeomorphism then the two differential structures are equivalent. Although the definition of differential structure sounds sufficiently general as to encompass all of the possibilities for a given M, that is not the case. Even in the simplest example of differential structures on $\mathbb{R}^n$ itself, it is now known (thanks to a theorem of Simon Donaldson in 1983) that all of the spaces $\mathbb{R}^n$ have a unique differential

structure except for $\mathbb{R}^4$, which has an infinitude of inequivalent differential structures that are often called "fake $\mathbb{R}^4$'s." It had been known for a long time, from the work of Milnor, that the seven-dimensional sphere admits two inequivalent differential structures: the conventional one that it inherits from $\mathbb{R}^8$ and an "exotic" one.

More specifically, when an n-sphere is defined as a point set in Euclidian $\mathbb{R}^{n+1}$ in the usual way, one cannot cover it with one chart since S^n is compact, but $\mathbb{R}^n$ is not. At the very least, one can cover it with two charts, one that is homeomorphic to an open n-disc centered at the North pole and covers everything except the South pole and another one with the poles reversed. Their intersection is everything except for the two poles, which is an open subset that is homeomorphic to the "cylinder" $\mathbb{R} \times S^{n-1}$, with S^{n-1} playing the role of "equator."

A closely related compact differentiable manifold to S^n is the n-dimensional real projective space $\mathbb{R}\mathrm{P}^n$, which we shall return to later in this work. It consists of all lines through the origin in $\mathbb{R}^{n+1}$, and since every such line intersects S^n (when centered at the origin) in two antipodal points, one can also think of $\mathbb{R}\mathrm{P}^n$ as consisting of all pairs of antipodal points on S^n. At the very minimum, one can cover $\mathbb{R}\mathrm{P}^n$ with n+1 charts, which are defined by the fact that the projection of *homogeneous coordinates* $(x^0, \ldots, x^n) \in \mathbb{R}^{n+1}$ for a point of $\mathbb{R}\mathrm{P}^n$ onto its *inhomogeneous coordinates* $(X^1, \ldots, X^n)$ can be accomplished in n+1 ways by choosing a non-zero coordinate x^k and defining $X^i = x^i/x^k$, all $i \neq k$.

d. Tangency, tangent spaces [**4-9**]. In multivariable calculus, one learns that differentiation allows one to associate a straight line with each point of a differentiable curve and, more generally, a k-dimensional vector space with each point of a differentiable k-dimensional hypersurface by way of the concept of tangency. The notion of a differential structure on a topological manifold also allows one to associate an n-dimensional vector space T_xM with each point x of an n-dimensional differentiable manifold M by a more general notion of tangency.

The key to this construction is to note that when the images of two curves in M in $\mathbb{R}^n$ for some coordinate chart (U, ϕ) have a common tangent line in at a particular point $x \in U \subset M$ the same will be true in any other coordinate chart; i.e., tangency is a coordinate-invariant concept. One can then define an equivalence class $[\gamma]_x$ of all curves through x M that have a common tangent in $\mathbb{R}^n$ at x for some, and therefore all, coordinate charts about x. One calls this equivalence class a *tangent vector* to M at x, since it is associated with a vector in $\mathbb{R}^n$. The set of all tangent vectors to M at x is called the *tangent space* to M at x and is usually denoted by T_xM.

One can give T_xM the structure of an n-dimensional vector space by way of the fact that $\mathbb{R}^n$ has that structure, but here is where differential topology takes a convenient, but non-obvious, turn and associates a tangent vector $\mathbf{v}_x \in T_xM$, not merely with an n-tuple of real numbers, such as $(v^1, \ldots, v^n)$, that is defined by a choice of coordinate chart (U, x^i) about $x \in M$, but with the *directional derivative operator* that these components define:

$$\mathbf{v}_x = v^i \frac{\partial}{\partial x^i} . \tag{II.2}$$

The key to making this association work is to see that this directional derivative also has a coordinate invariant character, as well, from the chain rule for differentiation. In another coordinate system (V, y^i) about x, one has:

$$\mathbf{v}_x = \overline{v}^i \frac{\partial}{\partial y^i} = \overline{v}^i \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j} , \tag{II.3}$$

which is consistent with (II.2) as long as:

$$v^j = \left. \frac{\partial x^j}{\partial y^j} \right|_x \overline{v}^j = [g^{-1}_{U \cap V}]^i_j \, \overline{v}^j . \tag{II.4}$$

Note that the components of $\mathbf{v}$ transform by means of the inverse of the transition function that is defined by the coordinate transformation. Hence, one can think of transition functions as transformations that act on tangent objects to $\mathbb{R}^n$, while coordinate transformations act on points of $\mathbb{R}^n$.

Now that we have defined a notion of differential maps and functions on M and tangent vectors as directional derivatives, we can give an invariant sense to the action of the tangent vector $\mathbf{v}_x \in T_xM$ on a differentiable function f: $M \to \mathbb{R}$, which actually only needs to be defined in some neighborhood of x:

$$\mathbf{v}_x f = v^i \left. \frac{\partial f}{\partial x^i} \right|_x . \tag{II.5}$$

There is a slight generalization of the n operators $\partial_i = \partial/\partial x^i$ that one uses as a generalized basis for T_xM, at least relative to a choice of chart, that has far-reaching applications and fundamental significance to physics, in general. That is the concept of a tangent *frame* in T_xM , namely, a set $\{\mathbf{e}_i, i = 1, \ldots, n\}$ of n linearly independent tangent vectors in T_xM. Hence, one can represent any tangent vector $\mathbf{v}_x \in T_xM$ uniquely in the form:

$$\mathbf{v}_x = v^i \, \mathbf{e}_i \tag{II.6}$$

relative to this frame.

Since the members of the frame $\mathbf{e}_i$ are themselves tangent vectors, when one chooses a coordinate chart (U, x^i) about x one can express them as:

$$\mathbf{e}_i = A_i^j \partial_j \tag{II.7}$$

for a unique invertible matrix of components A_i^j. Hence, the n operators ∂_i can just as well be regarded as defining a tangent frame in T_xM. Since they are only defined by a choice of coordinate chart, one refers to the frame $\{\partial_i\}$ as the *natural frame* at x that is defined by (U, x^i).

e. Vector fields [**4-9**]. So far, the components of a tangent vector, such as $\mathbf{v}_x$, relative to some chosen tangent frame are just constant scalars. Since a choice of coordinate chart (U, x^i) allows us to define a tangent vector at every $x \in U$, one can extend the concept of a tangent vector at x to a *tangent vector field* on U – or *vector field,* for short – whose components relative to the natural frame ∂_i on $\mathbb{R}^n$ will be functions on U*:*

$$\mathbf{v}(x) = v^i(x)\, \partial_i \,. \tag{II.8}$$

In order to extend the concept of a vector field on U to a vector field on all of M, we need to introduce one more general construction, in the form of the *tangent bundle* to M. It is a differentiable manifold $T(M)$ of dimension $2n$ that one obtains by taking the disjoint union of all T_xM as x ranges over all of the points of M. Hence, as a disjoint union, when x and y are distinct points of M the vector spaces T_xM and T_xM will be distinct from each other, as well, no matter how "close" the two points are. There is a natural projection $T(M) \to M$, $\mathbf{v}_x \mapsto x$, and the set of all $\mathbf{v}_x$ that project to a given x, which is, of course T_xM, is called the *fiber* of the projection over x. One can also restrict the projection to $T(U) \to U$ for any $U \subset M$.

The coordinate systems about each tangent vector $\mathbf{v}_x \in T(M)$ take the form of homeomorphisms $T(U) \to U \times \mathbb{R}^n$ for some, but not all, open neighborhoods U about each $x \in M$, such that the projection of $T(U)$ onto U corresponds to the projection of $U \times \mathbb{R}^n$ onto its first factor. Such a coordinate chart is called a *local trivialization* of $T(M)$ over U, since the most elementary sort of *fibration* $A \to B$ of one topological space A over another B (i.e., a projection that is local trivial) is the projection $M \times N \to M$ of a product space onto one of its factors; one calls such a fibration *trivial*. In that sense, fiber bundles do not become topologically interesting until one considers the ones for which the local trivializations cannot be extended beyond some limit to global trivializations. For instance, even on a manifold as elementary as the two-dimensional sphere S^2 the tangent bundle $T(S^2)$ is not globally trivializable, although one can define a local trivialization over an open subset $U \subset S^2$ that omits only a single point. Indeed, the only dimensions for which spheres are trivializable are dimensions 0, 1, 3, and 7. (This is actually closely related to the fact that the only real "division algebras," i.e., algebras with inverses, are defined over $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^4$, and $\mathbb{R}^8$.)

Once one has the concept of the fibration $T(M) \to M$ to work with, one can define the opposite concept of a (global) *section* of this fibration, namely, a differentiable map $\mathbf{v}$: $M \to T(M)$, $x \mapsto \mathbf{v}(x)$ such that when one projects $T(M)$ back to M each $\mathbf{v}(x)$ goes back to x; this is equivalent to the statement that $\mathbf{v}(x) \in T_xM$ for each $x \in M$. One can also define local sections over any $U \subset M$ by the same means. Since the local sections correspond to what we called local vector fields above, we see that a global section of the tangent bundle fibration is a reasonable candidate for a global vector field on M. Now, more general fiber bundles, whose fibers are not always vector spaces, do not have admit global sections, while $T(M) \to M$ always has at least one global section, namely the *zero section* Z: $M \to T(M)$ that takes each $x \in M$ to the origin of T_xM. However, as the example of S^2 shows, one cannot always find a global *non-zero* section of the tangent bundle fibration, since every vector field on S^2 must have at least one zero, a result that is usually described rather colorfully as the "Hairy Ball Theorem," when one thinks of the tangent vectors as hairs.

One can now extend the concept of a tangent frame in T_xM to a *local frame field* on $U \subset M$, namely, a set of n vector fields $\mathbf{e}_i : U \to T(U)$, $x \mapsto \mathbf{e}_i(x)$ that are linearly independent at each x. A local vector field $\mathbf{v} : U \to T(U)$ can then be expressed in terms of this local frame field as:

$$\mathbf{v}(x) = v^i(x)\, \mathbf{e}_i(x), \tag{II.9}$$

in which it is important to see that we are allowing the frame members themselves to vary over the points of U. In fact, if U admits a coordinate system x^i the local frame field can be expressed in the form:

$$\mathbf{e}_i(x) = A_i^j(x)\, \partial_j\,, \tag{II.10}$$

in which the component matrices $A_j^i(x)$ are now differentiable functions on U, and are invertible at each $x \in U$. Among other things, this means that every coordinate chart defines a local frame field by way of ∂_i that one calls the *natural frame field* defined by this chart. However, not every local frame field is the natural frame field for some coordinate chart. This gets one into the difference between holonomic and anholonomic local frame fields, which we shall discuss later in this book.

Although global vector fields always exist on a general manifold, global frame fields do not, in general. In fact, the existence of a global frame field on M is equivalent to the possibility that $T(M)$ admits a global trivialization. When this is possible, one calls M *parallelizable*, because one then has unique linear isomorphisms between each pair of tangent spaces in $T(M)$ that allow one to clarify the meaning of parallelism between tangent vectors at finitely-separated points.

In the case of the 2-sphere, it is clear that any manifold that does not admit a non-zero vector field cannot admit a global frame field, but it is not true that the existence of a non-zero vector field implies the existence of a global frame field. Hence, the existence

of global frame fields is a much stronger condition than the existence of non-zero vector fields [12].

By defining vector fields on M in terms of directional derivative operators, we introduce the possibility of composing the action of two vector fields **X** and **Y** on a smooth function f on M, such as **XY**f, which has the local form:

$$\mathbf{XY}f = X^i \frac{\partial}{\partial x^i}\left(Y^j \frac{\partial f}{\partial x^j} \right) = X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} + X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j}. \tag{II.11}$$

However, this construction does not have the coordinate-invariant (or really, *frame*-invariant) character that we desire for objects that are defined on manifolds. Since it is the second term in the final expression that is undesirable, we find that in order to obtain a frame-invariant object, we need only to subtract **YX**f and obtain:

$$[\mathbf{X}, \mathbf{Y}]\, f = \mathbf{XY}f - \mathbf{YX}f = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right)\frac{\partial f}{\partial x^j}, \tag{II.12}$$

so we can define the vector field:

$$[\mathbf{X}, \mathbf{Y}] = \mathbf{XY} - \mathbf{YX} = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right)\frac{\partial}{\partial x^j}, \tag{II.13}$$

which can also be defined globally.

This means that not only is the set $\mathfrak{X}(M)$ of all vector fields on M an infinite-dimensional vector space under pointwise finite linear combinations, but with this bracket, it also becomes an infinite-dimensional Lie algebra. Note that one also has a local Lie algebra $\mathfrak{X}(U)$ for each $U \subset M$. A deep problem in differential topology is that of determining how much detail concerning the differential structure on M can be expressed in terms of the algebraic structure of $\mathfrak{X}(M)$. For instance, not all n-dimensional Lie algebras on $\mathbb{R}^n$ can be represented in $\mathfrak{X}(M)$, or even $\mathfrak{X}(U)$. In particular, the Abelian Lie algebra on $\mathbb{R}^n$ is often hard to find globally.

A basic property of natural frame fields is that they are *holonomic*:

$$[\partial_i, \partial_j] = 0, \tag{II.14}$$

which follows from the equality of the mixed partial second derivatives. (Simply set $X^i = Y^i = \text{const.}$ in (II.13).)

[12] One can even introduce an intermediate notion of *k-parallelizability*, which means that there are k globally linearly independent vector fields on M, but not k+1. One calls k the *degree of parallelizability* of M. Not surprisingly, the topological nature of this gets quite involved.

e. Covector fields [**4-9**]. There are dual constructions to those of the preceding subsection that have just as much, if not more, application to the problems of physics. One starts by defining a *covector* α_x at $x \in M$ to be a linear functional on T_xM. Since T_xM is presumed to be a vector space, in its own right, it has a dual vector space consisting of all linear functionals on T_xM, and we denote it by T_x^*M ; one calls this vector space the *cotangent space* to M at x. Hence, we can say $\alpha_x \in T_x^*M$, and if $\mathbf{v}_x \in T_xM$ then we write the evaluation of α_x on $\mathbf{v}_x$ as $\alpha_x(\mathbf{v}_x)$.

We can basically jump ahead to define the *cotangent bundle on M* as the disjoint union of all the cotangent spaces, which then has a projection $T^*M \to M$, and is locally trivial by way of local trivializations $T^*U \to U \times \mathbb{R}^{n*}$ over all of the same open subsets that locally trivialize $T(M)$, if one chooses a linear isomorphism of $\mathbb{R}^n$ with its dual; for instance, one could map the standard frame on $\mathbb{R}^n$ to its reciprocal coframe. Similarly, this means that the bundle $T(M)$ is isomorphic to the bundle T^*M, but not canonically so[13]. One also has restrictions $T^*U \to U$ over any open subset $U \subset M$.

Going in the opposite direction, a *local covector field* on U is a section $\alpha: U \to T^*U$, $x \mapsto \alpha_x$ and a *global covector field* on M is a section $\alpha: M \to T^*M$. If U carries a coordinate system x^i then the coordinate differentials dx^i define a *coframe* on $\mathbb{R}^{n*}$ that is reciprocal to the frame defined by the ∂_i :

$$dx^i(\partial_j) = \delta^i_j \, . \tag{II.15}$$

Any local covector field $\alpha: U \to T^*U$ can then be expressed in local form as:

$$\alpha(x) = \alpha_i(x)\, dx^i \, . \tag{II.16}$$

in which the components $\alpha_i(x)$ are presumed to be smooth functions on U.

Global covector fields are defined predictably, and the same caveat that applies to the existence of global non-zero vector fields applies to existence of global non-zero covector fields. A local coframe field $\theta^i : U \to T^*U$ will be reciprocal to a unique local frame field $\mathbf{e}_i$ on U, and if U carries a local coordinate system x^i then one can express θ^i in the form:

$$\theta^i(x) = \tilde{A}^i_j(x)\, dx^j \, , \tag{II.17}$$

where $\tilde{A}^i_j(x)$ is the matrix inverse to $A^i_j(x)$ at each x, and one also has:

$$\mathbf{e}_i(x) = A^i_j(x)\, \partial_i \, . \tag{II.18}$$

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[13] By "vector bundle isomorphism," we simply mean that the manifolds $T(M)$ and T^*M are diffeomorphic by a diffeomorphism that takes each fiber $T_x(M)$ to the fiber T^*_x linearly.

One might think that the isomorphism of $T(M)$ and T^*M would imply that anything that is true of one bundle is true of the other. However, this shows one the danger of extending equivalence relations beyond their natural limits. For instance, the infinite-dimensional vector space of sections $\alpha: M \to T^*M$, i.e., global covector fields, does not admit a natural Lie algebra structure, as does $\mathfrak{X}(M)$. Conversely, as we shall see, one can define the exterior derivative of a covector field, but not a vector field. Interestingly, these last two statements are weakly related.

f. Vector bundles in general [**4-9**]. Since the fibers of $T(M)$ and T^*M over any point $x \in M$ are both vector spaces, it is useful to generalize to the concept of (differentiable) *vector bundle*, which is a differentiable manifold E that projects onto M in such a manner that the fiber E_x over each $x \in M$ is a vector space whose dimension n – which is assumed constant over M – is called the *rank* of the vector bundle [14]. Furthermore, one assumes that E is locally trivial, which means that each $x \in M$ has some open neighborhood U such that $E(U)$ is diffeomorphic to $U \times \mathbb{R}^n$ in such a way that the projection of $E(U)$ onto U corresponds to the projection of $U \times \mathbb{R}^n$ onto its first factor. These local trivializations also define the atlas of coordinate charts for the manifold E.

Although the mathematics of vector bundles can go off into abstruse, esoteric realms that often seem hopelessly divorced from the problems of physics, the same can be said of group theory; so the challenge to the theoretical physicist is to isolate from the potentially vast set of irrelevant sidetracks that one "critical path" that is most immediately relevant to the problem at hand, which is the spirit of Occam's Razor. For our purposes, the main objective of generalizing to vector bundles is merely to define the bundle of exterior differential forms on M and the bundle of multivector fields on M. Hence, we shall only point out some of the elementary constructions on vector bundles that will give us that much.

Some of the same constructions that one carries out with vector spaces can be applied to vector bundles, as well. In particular, just as one can form direct sums $V \oplus W$ and tensor products $V \otimes W$ of vector spaces, one can form direct – or *Whitney* – sums $E \oplus F$ and tensor products $E \otimes F$ of vector bundles that are both fibered over the same manifold M. All that is basically necessary is to do these operations on each fiber individually; that is, the fiber of $E \oplus F$ over $x \in M$ is $E_x \oplus F_x$, and the fiber of $E \otimes F$ is $E_x \otimes F_x$. Indeed, one can extend these constructions to direct sums of families of vector bundles and tensor products of finite families.

Both of these constructions can be applied to the vector spaces $\Gamma(E)$ and $\Gamma(F)$ of sections of the vector bundles E and F. The result is that $\Gamma(E) \oplus \Gamma(F)$ represents the vector space of sections of $E \oplus F \to M$, and $\Gamma(E) \otimes \Gamma(F)$ represents the vector space of sections of the vector bundle $E \otimes F \to M$.

One can also define the dual E^* to a vector bundle E by looking at all linear functionals on the vectors of E.

[14] A non-constant rank tends to suggest a non-connected base manifold M, which we shall mostly exclude from our considerations.

Some of things that one expects of tangent and cotangent bundles that no longer have any meaning for more general vector bundles are: the interpretation of elements in the fibers as directional derivatives, the existence of a natural Lie algebra on the vector space of sections, and the existence of a naturally-defined exterior derivative operator.

2. Differential forms on manifolds [4-10]. When the basic constructions that we discussed above are applied to the vector bundles $T(M)$ and T^*M, one finds that it is straightforward to define the bundles $\Lambda_k M \to M$ and $\Lambda^k M \to M$, which represent the k^{th} exterior products of the vector bundles $T(M)$ and T^*M, respectively. Hence, the fibers of these vector bundles are the vector spaces that one obtains by taking the k^{th} exterior power of the vector spaces T_xM and T_x^*M at each $x \in M$.

A section of the bundle $\Lambda_k M \to M$ is called a *k-vector field* on M and a section of its dual $\Lambda^k M \to M$ is called a *differential k-form*. At each point $x \in M$, the value of such a field will be a k-vector in $\Lambda_{k,x}$ or an algebraic k-form in Λ_x^k, respectively. Hence, one can think of a non-zero k-vector field on M as associating k linearly independent vectors in the tangent space to each point (at least in the simple case), and a differential k-form associates a completely anti-symmetric k-linear functional on the tangent vectors at x.

The local forms of k-vector fields $\mathbf{A}$ and k-forms α over $U \subset M$ are:

$$\mathbf{A}(x) = \frac{1}{k!} A^{ij\cdots m}(x)\, \partial_i \wedge \partial_j \wedge \cdots \partial_m, \qquad \alpha(x) = \frac{1}{k!} \alpha_{ij\cdots m}(x)\, dx^i \wedge dx^j \wedge \cdots \wedge dx^m \tag{II.19}$$

when U carries a coordinate system x^i and, more generally:

$$\mathbf{A}(x) = \frac{1}{k!} A^{ij\ldots m}(x)\, \mathbf{e}_i \wedge \mathbf{e}_j \wedge \ldots \wedge \mathbf{e}_m, \quad \alpha(x) = \frac{1}{k!} \alpha_{ij\ldots m}(x)\, \theta^i \wedge \theta^j \wedge \ldots \wedge \theta^m \tag{II.20}$$

when U carries a local frame field $\mathbf{e}_i$ and its reciprocal coframe field θ^i. (Of course, the component functions will not generally be the same in these two cases.)

One can define bilinear pairings of k-forms and l-vector fields for the cases of $k < l$, $k = l$ and $k > l$. However, one must be careful in the case of $k = l$ to notice that although the evaluation of an algebraic k-form on a k-vector is a *number*, nonetheless, when this is going on at each point of a manifold, the evaluation of a differential k-form on a k-vector field gives a *smooth function* on M.

The basic rules for interior products that were discussed in Chapter I still apply with only a new interpretation. For instance:

$$i_{\mathbf{v}}(\alpha_1 \wedge \ldots \wedge \alpha_k) = \sum_{i=1}^{k} (-1)^{i+1} \alpha_i(\mathbf{v})\, \alpha_1 \wedge \cdots \wedge \hat{\alpha}_i \wedge \cdots \wedge \alpha_k, \tag{II.21a}$$

$$i_{\alpha}(\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k) = \sum_{i=1}^{k} (-1)^{i+1} \alpha(\mathbf{v}_i)\, \mathbf{v}_1 \wedge \cdots \wedge \hat{\mathbf{v}}_i \wedge \cdots \wedge \mathbf{v}_k. \tag{II.21b}$$

The exterior derivative operator that we previously defined in Chapter 1 can be extended to a linear anti-derivation d: $\Lambda^k \to \Lambda^{k+1}$ that not only agrees with df on smooth functions, but also agrees with the exterior derivative operator that we defined in Chapter I for local k-forms over open subsets $U \subset M$ that carry coordinate systems or local coframe fields. Hence, if $\alpha \in \Lambda^k$ and $\beta \in \Lambda^l$ then:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta . \tag{II.22}$$

Sometimes it useful to have a way of evaluating exterior derivatives without having to introduce local coframe fields and components. For that purpose, one has the intrinsic formula for the exterior derivative. Although it can be generalized (see [**1**]), we shall present it only for 1-forms. Basically, one must evaluate the 2-form $d\alpha$ on an arbitrary pair of vector fields **X**, **Y** and obtain:

$$d\alpha(\mathbf{X}, \mathbf{Y}) = \mathbf{X}(\alpha(\mathbf{Y})) - \mathbf{Y}(\alpha(\mathbf{X}) - \alpha([\mathbf{X}, \mathbf{Y}]). \tag{II.23}$$

Here, we see our first inkling of the relationship between Lie brackets of vector fields and exterior derivatives of 1-forms.

The Lie derivative of a k-vector field **A** or a k-form α with respect to a vector field **v** still works as it did in Chapter I, as well. One has, for instance:

$$\mathrm{L}_{\mathbf{v}}(\mathbf{X} \wedge \mathbf{Y}) = \mathrm{L}_{\mathbf{v}}\mathbf{X} \wedge \mathbf{Y} + \mathbf{X} \wedge \mathrm{L}_{\mathbf{v}}\mathbf{Y} = [\mathbf{v}, \mathbf{X}] \wedge \mathbf{Y} + \mathbf{X} \wedge [\mathbf{v}, \mathbf{Y}], \tag{II.24a}$$

$$\mathrm{L}_{\mathbf{v}}\alpha = (i_{\mathbf{v}}d + di_{\mathbf{v}})\, \alpha . \tag{II.24b}$$

3. Differentiable singular cubic chains [11-13]. In order to define the integration of differential forms on differentiable manifolds, one basically has to show how the integrals on manifolds can be converted – viz., "pulled back" – to conventional multiple integrals over regions in $\mathbb{R}^n$ in a coordinate-invariant manner. Hence, it helps to be mapping regions of $\mathbb{R}^n$ into the manifold in question in a differentiable manner that permits such a pull-back.

A particularly useful class of differentiable mappings of regions in $\mathbb{R}^n$ into a manifold M that is adapted to this purpose is that of *differentiable singular cubic chains*. These objects also have bonus that they define elementary building blocks for the topology of the manifold, as well.

To begin with, a *k-cube* $I^k \subset \mathbb{R}^k$ is simply any region of $\mathbb{R}^k$ that is homeomorphic to the k-fold Cartesian product $[0, 1] \times \ldots \times [0, 1]$. Note that this means that a k-cube could just as well be a closed k-ball or a k-simplex, which is the k-dimensional generalization of a triangle or a tetrahedron. Of course, since we are going to be dealing with objects in the differentiable category, it is important to recall that the corners and edges of a cube prevent it from being *diffeomorphic* to a ball.

A *singular cubic k-simplex* in a topological space M is a continuous map σ_k: $I^k \to M$. Note that since we did not require that it be a homeomorphism onto – i.e., a topological

embedding of a k-cube – there is nothing to say that the image of I^k is still k-dimensional; indeed, it could very well be merely a point. This is why one calls such simplexes [15] "singular." As it turns out the effect of such degenerate cases eventually disappears when one goes on to homology. A *differentiable singular cubic k-simplex* is then a singular cubic k-simplex for which the defining map is differentiable. Since the k-cube is not really a differentiable manifold, in the first place, the way that one addresses this detail is to extend σ_k to a differentiable map on *any* open neighborhood of I^k in $\mathbb{R}^k$. Because differentiation is a purely local process, the differentials of all of the extensions must agree everywhere on I^k, as well as the functions themselves.

Once again, differentiable k-cubes, which is how we will abbreviate the more cumbersome expression, do not have to be diffeomorphisms onto. Also, one finds that in the eyes of homotopy theory, differentiability is no severe restriction, since a "smoothing" argument shows that any homotopy class $[\sigma_k]$ of continuous maps of I^k into M contains a differentiable representative.

A *differentiable singular cubic k-chain*, or *differentiable k-chain*, for short, is a formal linear combination $\sum_{m=1}^{N} \alpha_i \sigma_i$ of a finite number of k-cubes σ_i with real coefficients α_i . We should point out that singular homology (see, [**11, 12**]) usually starts with coefficients in more general rings than $\mathbb{R}$, such $\mathbb{Z}$, and the resulting homology has more detail than what we will be considering, but we are using real coefficients to be consistent with the de Rham cohomology that we shall discuss in a later section. If one is justifiably suspicious of all "formal" constructions as having no rigorous basis, then be assured that these formal linear combinations can be easily made rigorous. All one needs to do is define the "free $\mathbb{R}$-vector space" $C_k(M; \mathbb{R})$ that has the set of all k-cubes in M for its basis. Suffice it to say the main consequence of this construction is that one can regard the set of all k-cubes in M as the basis for an uncountably-infinite dimensional vector space. The miracle of homology is that one can usually reduce this uncountable infinitude to a finite set of equivalence classes that pertains to the topology of M.

Since non-zero real numbers have signs, one can think of a k-cube as being oriented by the sign of its coefficient. The actual numerical value of the coefficient should be regarded as simply a scalar that one associates with the k-cube, like a "charge."

The nature of "free" constructions in mathematics, such as free groups, rings, modules, etc., is that the only role that the set of generators plays is due to solely to its cardinality. That is, the free $\mathbb{R}$-vector space over a set of three apples is no different from the one over a set of three oranges; they are both linearly isomorphic to $\mathbb{R}^3$. In particular,

[15] Although in the early days of homology everyone was content to form the plural of "simplex" just as they formed the plural of "complex" – with an "es" – sometime in the 1960's mathematicians apparently had second thoughts because "vertex" forms its plural with "ices," and took to saying "simplices;" note that the spelling checker on most word processors does not approve of that construction. One could say it is really just a matter of personal taste, but that is also the sort of attitude towards the English language that gave us "Canterbury Tales."

the topology of M does not affect the free $\mathbb{R}$-vector space of all k-chains in M, at this point. The way that topology enters the picture is by way of defining a *boundary map* for each k-chain in $C_k(M; \mathbb{R})$. This means that one is regarding the k-cubes in a given k-chain as being "glued" to each other on their mutual boundaries in some way.

In general, the boundary map is a linear map ∂: $C_k(M; \mathbb{R}) \to C_{k-1}(M; \mathbb{R})$ with the property that $\partial^2 = 0$. Since one assumes linearity, it is sufficient to define the effect of the boundary operator on k-cubes. By definition, one regards any 0-cube – i.e., any point in M – as having boundary zero. The boundary of a 1-cube – i.e., an oriented path AB in M – is $\partial(AB) = B - A$. If one represents a 2-cube in M by the images of its vertices as $ABCD$ then one sees that its boundary is the 1-chain:

$$\partial(ABCD) = AB + BC + CD + DA. \tag{II.25}$$

A second application of ∂ to this 1-chain gives:

$$\partial^2(ABCD) = (B - A) + (C - B) + (D - C) + (A - D) = 0, \tag{II.26}$$

as expected. We illustrate this situation in Fig. 1.

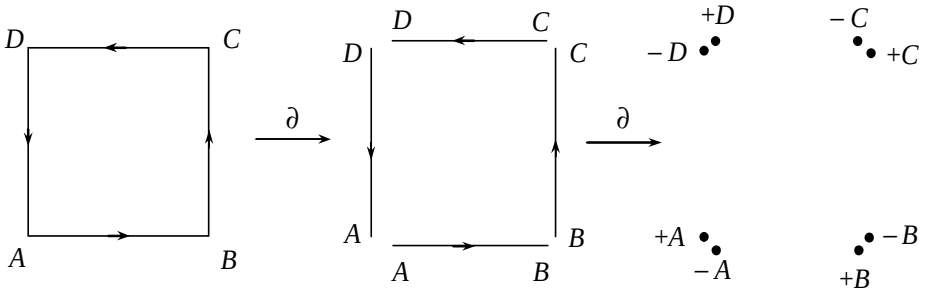

Figure 1. The boundary of a boundary of a 2-cube.

An elementary example that shows how the boundary operator attaches k-cubes along their boundaries is that of a circle, which we think of as the 1-chain $AB + BA$, whose boundary is then $B - A + A - B = 0$.

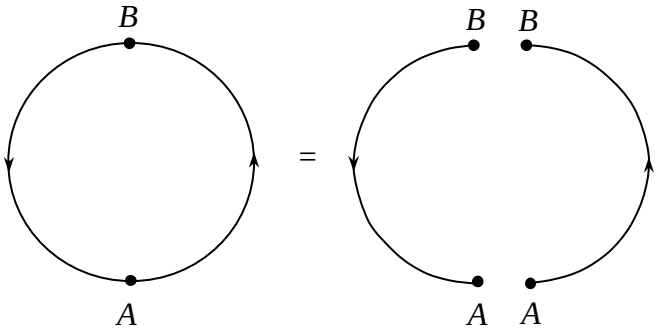

Figure 2. The representation of a circle as a 1-chain with boundary zero.

However, this means that we must regard the 1-cube BA as basically distinct from $-AB$, since otherwise the 1-chain itself would vanish, as well. We illustrate this situation in Fig. 2.

Many two-dimensional examples can be represented as squares with sides and corners identified by means of the boundary operator. We depict some of them in Fig. 3.

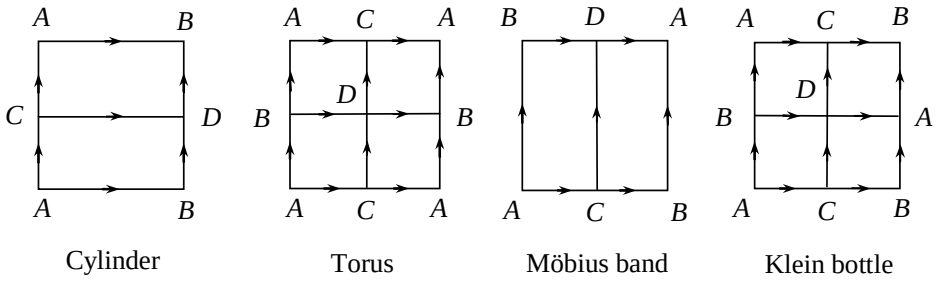

Figure 3. Two-dimensional spaces that are described by 2-chains.

We can represent these four examples by 2-chains c_2 and boundary operators as follows:

Cylinder: $$\begin{aligned}\partial c_2 = {} & CA + AB - BD - CD \\ & + AC - AB - DB + CD \\ = {} & AC + CA - (BD + DB),\end{aligned}$$

which then represents the formal difference of two circles.

Torus: $$\begin{aligned}\partial c_2 = {} & BA + AC - CD - BD \\ & - AB + CA + DC - DB \\ & + AB - AC - DC + BD \\ & - BA - AC + CD + DB \\ = {} & 0,\end{aligned}$$

which is what one would expect for a torus. This time, one sees that the 1-cycles $AC + CA$ and $AB + BA$ do not bound any 2-chains, nor does their difference.

Möbius band: $$\begin{aligned}\partial c_2 = {} & AB - BD - CD - CA \\ & - AB + CD + DA - BC \\ = {} & BD + DA - (CA + BC).\end{aligned}$$

If one replaces $BD + DA$ with the homologous arc BA, while replacing $CA + BC$ with $-AB$ then one sees that the boundary of a Möbius band is a circle, which is completed twice.

Klein bottle: $$\begin{aligned}\partial c_2 = {} & BA + AC - CD - BD \\ & - AB + DC + CB - DA \\ & + AB - AC - DC + BD \\ & - BA + CD - CB + DA \\ = {} & 0\end{aligned}$$

One can think of a Klein bottle as a twisted cylinder with its ends identified in the same way that a Möbius band is a twisted rectangle with its ends identified.

It is important to understand that, in some sense, the boundary operator must be defined "by hand" to reflect the topological nature of the manifold M and that it is not, by any means, canonically defined by the bases for the vector spaces $C_k(M; \mathbb{R})$.

We have now introduced the fact that there are three different types of k-chains c_k as far as the boundary operator is concerned: the ones for which ∂c_k is non-vanishing, the ones for which ∂c_k is vanishing, and the ones for which $c_k = \partial c_{k+1}$ for some k+1-chain c_{k+1}. Since $\partial^2 = 0$, the last possibility implies the second-to-last one. The first type of k-chain has no special name, but those of the second type are called *k-cycles*, and those of the third type are called *k-boundaries*. Since ∂: $C_x(M; \mathbb{R}) \to C_{k+1}(M; \mathbb{R})$ is linear, its image is a vector subspace of $C_{k+1}(M; \mathbb{R})$ that represents all k+1-boundaries, and we denote it by $B_{k+1}(M; \mathbb{R})$. Similarly, its kernel is a vector subspace of $C_x(M; \mathbb{R})$ that consists of all k-cycles, and we denote it by $Z_k(M; \mathbb{R})$.

The power of homology theory to define elementary building blocks for the topology of M is that even though both $Z_k(M; \mathbb{R})$ and $B_k(M; \mathbb{R})$ are generally infinite-dimensional vector spaces, nonetheless, their quotient vector space [16] $H_k(M; \mathbb{R}) = Z_k(M; \mathbb{R}) / B_k(M; \mathbb{R})$ is often finite-dimensional; indeed, it is often 0-dimensional. One calls the vector space $H_k(M; \mathbb{R})$ the *k^{th} homology space of M*. Ordinarily, one uses coefficients in a more general ring and obtains homology modules, but, once again, we shall be mostly concerned with the field of real coefficients, for the sake of de Rham cohomology. In order for a homology space to vanish in dimension k all of the k-cycles must bound k+1-cycles. A convenient way of regarding the basis elements for the homology space $H_k(M; \mathbb{R})$ is to think of them as essentially "k-dimensional holes" in M, such as circles that do not bound disks and spheres that do not bound balls.

The equivalence relation of homology is actually more tangible than it might seem. One sees that two k-cycles z_k and z_k' are *homologous* iff their difference is the boundary of a k-chain c_{k+1}: $z_k - z_k' = \partial c_{k+1}$. As we saw above, the boundary of a cylinder is a pair of circles. Hence, since circles are 1-cycles, one can say that the two boundary components of a cylinder are homologous, and the 2-chain that effects this homology is the cylinder itself. When the boundary of a k-chain has only one component, such as the boundary of a disk, one says that this component is *homologous to zero*. Often, physics uses a more general homology to describe this sort of situation that one calls *cobordism*. The main difference at the elementary level is that one does not assume a triangulation of the differentiable manifold whose boundary components are being connected together, but

[16] Recall that the quotient A/B of two vector spaces $B \subset A$ is the set of all translates of the subspace B by the vectors in A. One can also say that its elements are equivalence classes $[a]$ of vectors $a_1, a_2 \in A$ whose difference $a_1 - a_2$ is some vector $b \in B$.

the effect in the long run is that generally much less is known about the structure of the cobordism ring of compact submanifolds of a given manifold than is known about its singular homology. Generally, the power of cobordism is more necessary when one is dealing with purely mathematical aspects of differentiable manifolds, such as defining equivalence classes of them and classifying them.

One of the advantages of using cubes for ones basic topological objects is that a homotopy of a singular k-cube is itself a singular k+1-cube. This follows from the fact that, by definition, if σ_k: $I^k \to M$ is a singular k-cube in a topological space M then a homotopy of σ_k is a continuous map F: $I^k \times I \to M$, that is, a continuous map F: $I^{k+1} \to M$, which is then simply a singular k+1-cube in M. The maps $F(x, 0)$ and $F(x, 1)$ then represent two opposite k-faces of the k+1-cube F.

As we saw above, a 1-cycle can take the form of a loop, up to homeomorphism. If a loop bounds a 2-chain, which is then homeomorphic to a two-dimensional disk, then the boundary is homotopic to any point of the disk. Conversely, if no such disk exists then the boundary loop cannot be contracted to any point in M. Hence, one sees that there is clearly a close relationship between homology and homotopy, at least in dimension one. In fact, they are related in all dimensions, but not as conveniently as one might wish. The closest that one can come, at present, is Hurwitz's theorem, which says, for our purposes, that the first non-zero homotopy group – say $\pi_k(M)$ – has as many generators as the dimension of $H_k(M; \mathbb{R})$; for $k = 1$, one must specify that one is looking at generators of the "Abelianization" of $\pi_1(M)$, if it is not already Abelian. For all dimensions above that dimension, one only has a possibly many-to-one map of generators for $\pi_m(M)$ to basis vectors for $H_m(M; \mathbb{R})$.

For instance, as Fig. 2 shows, the homology of the circle S^1 is $H_0(M; \mathbb{R}) = \mathbb{R}$, $H_1(M; \mathbb{R}) = \mathbb{R}$, $H_k(M; \mathbb{R}) = 0$, for $k > 1$. For any contractible topological space – whose homotopy groups also vanish – all homology vector spaces vanish, as well. Such topological spaces are sometimes called *acyclic*.

It is always true that if the dimension of a manifold M is n then $H_m(M; \mathbb{R}) = 0$ for all $m > n$.

As an example of one of the limitations to doing homology with coefficients in a field such as $\mathbb{R}$, consider the difference between the 2-sphere S^2 and the two-dimensional projective space $\mathbb{R}\mathrm{P}^2$. Whereas S^2 is simply-connected, $\mathbb{R}\mathrm{P}^2$ has a fundamental group that is isomorphic to $\mathbb{Z}_2$, which comes from the two-to-one projection of S^2 onto $\mathbb{R}\mathrm{P}^2$. Had we considered homology with integer coefficients, as one usually does in singular homology, we would have found that $H_1(S^2) = 0$, but $H_1(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}_2$. However, the use of coefficients in a field does not permit the appearance of "finite cyclic summands" in the homology modules, and we then find that $H_1(S^2; \mathbb{R}) = H_1(\mathbb{R}\mathrm{P}^2; \mathbb{R}) = 0$. Therefore, one sees that there is something topologically "incomplete" about looking at homology with real coefficients.

4. Integration of differential forms [4-10]. Before going on to a discussion of the duals of the homology spaces, we first return to the more elementary business of integrating differential forms over manifolds.

In effect, we shall be restricting ourselves to manifolds that can be "triangulated," i.e., ones that are homologically equivalent to finite chain complexes with suitably-defined boundary operators. In fact, this is not much of a restriction, since it can be shown (cf., [**14**]) that any compact differentiable manifold can be triangulated.

One first notes that the only things that can be integrated over an n-dimensional compact manifold M are n-forms, which must, moreover, take the form $f\mathcal{V}_n$, where $\mathcal{V}_n$ is a globally non-zero volume element on M. Of course, this presupposes that such a volume element actually exists, which means that first we must address the issue of orientability for differentiable manifolds.

The fibers of $\Lambda^n(M) \to M$ are each one-dimensional, so if one takes away the zero section from $\Lambda^n(M)$ then what will be left looks like the disjoint union $(-\infty, 0) \cup (0, +\infty)$ at each point of M. Up to homotopy, one can contract each such union to a pair of disjoint points, which we denote by $\{-, +\}$. This defines a fibration $\mathcal{O}(M) \to M$ whose fibers all look like $\{-, +\}$, but it does not have to be trivial. In fact, it is trivial iff it admits a global section, which we will then call an *orientation* for M. (Really, it is an orientation for $T(M)$.)

Not all manifolds are orientable. For instance, the aforementioned Klein bottle and Möbius band, as well as all of the projective spaces, are all non-orientable. Since all of them have $\mathbb{Z}_2$ for a fundamental group, one suspects that this is relevant; it is, but we shall not go into the matter of "Stiefel-Whitney classes" at the moment. All simply-connected manifolds are orientable, but not all orientable manifolds are simply-connected (counterexample: torus). In any event, $\mathcal{O}(M)$ itself is always an orientable manifold, and one refers to it as the *orientable covering manifold* of M.

Actually, although orientability of a manifold is necessary and sufficient for the existence of a volume element, nonetheless, the choice of volume element is still open to one's discretion. Hence, one has to specify that a manifold M be not only orientable, but also oriented, and a choice of volume element has made before one can define integrals on it.

Having made such restrictions on a compact manifold M, as well as a choice of triangulation, $M = c_n = \sum \alpha_i \sigma_i$ one then defines the integral of an n-form $f\mathcal{V}_n$ over M by assuming that integration is linear with respect to linear combinations of n-chains:

$$\int_M f\mathcal{V}_n = \sum \alpha_i \int_{\sigma_i} f\mathcal{V}_n \,. \tag{II.27}$$

One then defines the integral over any basis simplex σ_i: $I^n \to M$ by pulling back the integral over its image in M to an integral over I^n:

$$\int_{\sigma_i} f\mathcal{V}_n = \int_0^1 \cdots \int_0^1 f(\sigma_i) J(\sigma_i)\, dx^1 \cdots dx^n \,, \tag{II.28}$$

in which $J(\sigma_i)$ is the Jacobian of the map σ_i. Hence, in order for this to be non-vanishing, one must consider only the *non-singular* n-simplexes; i.e., the ones that are diffeomorphisms onto their images, which makes them *embeddings*.

Stokes's theorem can be generalized to the integration of n-forms over n-chains in a manner that has deep topological significance, as we shall see in the next section. If $\alpha \in \Lambda^n(M)$ and $M = \partial c_{n+1}$ is an n-chain then Stokes's theorem says that:

$$\int_{\partial c_{n+1}} \alpha = \int_{c_{n+1}} d\alpha \ . \tag{II.29}$$

We introduce the following bracket notation for the integration of an n-form α on an n-chain c_n :

$$<\alpha, c_n> = \int_{c_n} \alpha . \tag{II.30}$$

With this notation, Stokes's theorem takes the form:

$$<\alpha, \partial c_{n+1}> = <d\alpha, c_{n+1}>. \tag{II.31}$$

In this form, Stokes's theorem appears to be asserting that in some sense the exterior derivative operator on differential forms is "adjoint" to the boundary operator on chains. In the next section, we shall clarify the extent to which this is indeed the case.

An important consequence of Stokes's theorem is that if z_k and z_k' are homologous k-cycles then the integral of a closed k-form α is the same over both z_k and z_k'. Abstractly, this only says that the value of the integral $<\alpha, z_k>$ can be expressed as $<\alpha, [z_k]>$.

At the more practical level of physics applications, this invariant pairing of closed k-forms and k-dimensional homology classes by way of integrals serves as the basis for certain *integral invariants* when the homology of two k-cycles comes about in less abstract ways. For instance, the interpolating k+1-chain c_{k+1} – that is, one that makes $z_k - z_k' = \partial c_{k+1}$ – might take the form of a differentiable homotopy of z_k to z_k', or even a one-parameter family of diffeomorphisms that represent the flow of a vector field on c_{k+1}.

5. De Rham's theorem [5, 10]. From (II.30), we see that for a fixed $\alpha \in \Lambda^k$ the bracket $<\alpha, c_k>$ defines a linear functional on the vector space $C_k(M; \mathbb{R})$. If we denote the dual vector space to $C_k(M; \mathbb{R})$ by $C^k(M; \mathbb{R})$ then we can say that $<\alpha, \ . \ > \in C^k(M; \mathbb{R})$. Since the elements of $C_k(M; \mathbb{R})$ are called k-chains, we call the elements of $C^k(M; \mathbb{R})$ *cochains.*

We can define a *coboundary* operator on cochains that is adjoint to the boundary operator under the bilinear pairing $(c^k, c_k) = c^k(c_k)$ of k-cochains with k-chains that amounts to the evaluation of a linear functional on a vector:

$$(\delta c^k, c_{k+1}) \equiv (c^k, \partial c_{k+1}). \tag{II.32}$$

Note the resemblance to (II.31).

Hence, from the way that we defined things the coboundary operator δ: $C^k(M; \mathbb{R}) \rightarrow C^{k+1}(M; \mathbb{R})$ is a linear operator with the property that $\delta^2 = 0$. This means that most of what we said in the context of chains applies just as well to cochains. For instance, there are three types of cochains in the eyes of δ: cochains for which $\delta c^k \neq 0$, those for which $\delta c^k = 0$, and those for which there is a k–1-cochain c^{k-1} such that $c^k = \delta c^{k-1}$. The first type has no special name, but the second type is called a *cocycle*, while the last type is called a *coboundary*. We denote the vector space of all k-cocycles by $Z^k(M; \mathbb{R})$ and the vector space of all k-coboundaries by $B^k(M; \mathbb{R})$; the former space is the kernel of δ: $C^k(M; \mathbb{R}) \rightarrow C^{k+1}(M; \mathbb{R})$, while the latter is the image of δ: $C^{k-1}(M; \mathbb{R}) \rightarrow C^k(M; \mathbb{R})$.

Just as we defined the real singular homology vector spaces by quotients of spaces of cycles by spaces of boundaries, we now do the same thing with cocycles and coboundaries. We call the vector space $H^k(M; \mathbb{R}) = Z^k(M; \mathbb{R}) / B^k(M; \mathbb{R})$ the *real singular cohomology vector space* in dimension k. One of the advantages of using real coefficients that partially compensates for the fact that we cannot consider the "torsion" summands in our homology or cohomology modules – i.e., finite cyclic Abelian groups – is the fact that one does indeed have that the vector space $H^k(M; \mathbb{R})$ is the dual to the vector space $H_k(M; \mathbb{R})$; in particular, they have the same dimension, so if one of them vanishes then the other one does.

Now, since the exterior derivative operator d: $\Lambda^k \rightarrow \Lambda^{k+1}$ works in a manner that is clearly analogous to the coboundary operator, we see that we can define vector spaces $Z^k_{dR}(M)$, $B^k_{dR}(M)$, and $H^k_{dR}(M) = Z^k_{dR}(M) / B^k_{dR}(M)$ by means of the kernel, image, and quotient constructions. One calls the vector space $Z^k_{dR}(M)$, the space of closed k-forms, $B^k_{dR}(M)$, the space of exact k-forms, and $H^k_{dR}(M)$, the *de Rham cohomology vector space* in dimension k. A de Rham cohomology class can then be represented by a closed k-form, and two closed k-forms α, α' are *cohomologous* if they differ by an exact k-form: $\alpha - \alpha' = d\beta$, for some k–1-form β. One must note that β is not unique, since one can add any closed k–1-form γ to it and still produce the same exterior derivative: $d(\beta + \gamma) = d\beta$.

Since a de Rham cohomology class $[\alpha] \in H^k_{dR}(M)$ can be represented by a closed k-form α and a k-form defines a singular cochain by way of $<\alpha, .>$, we see that we have a linear map from $Z^k_{dR}(M)$ to $Z^k(M; \mathbb{R})$. One finds that this map also "descends to cohomology" since it commutes with the action of the exterior derivative and coboundary operators. Hence, there is a corresponding linear map of $H^k_{dR}(M)$ to $H^k(M; \mathbb{R})$. It was the profound and far-reaching contribution of Georges de Rham that this map could be shown to be an isomorphism. Basically, it meant that the otherwise analytic issue of whether a closed k-form was also an exact k-form, which amounts to a question of

integrability for the operator d, was rooted in purely topological matters, such as whether the manifold M has "holes" in dimension k.

In dimension one this usually takes the form of multiple connectedness. Hence, one finds many classical references to the difference between "single-valued" potentials and "many-valued" potentials in the context of vector calculus, such as in electromagnetism or hydrodynamics. One should be advised that since $d^2 = 0$ follows from the equality of mixed partial derivatives of functions that are twice continuously differentiable, the introduction of many-valued potentials is one way of getting around that identity by dropping the assumption that the second derivative exists everywhere. Rather than deal with topology by way of closed forms that are not exact, one introduces topology in the form of the singular subset of the many-valued potential ϕ for which $d^2\phi \neq 0$, which would be the points at which $d^2\phi$ has jump discontinuities.

If M is a contractible space, such as any vector space, then any closed k-form will be exact, and this will be true in each dimension k. Since every point of any manifold has a neighborhood U that is homeomorphic to $\mathbb{R}^n$ – hence, contractible – one then sees that $H^k(U; \mathbb{R}) \cong H^k_{dR}(U)$ vanishes in every dimension, and one can say that every closed k-form on M is *locally exact;* this is referred to as the *Poincaré lemma.*

One way in which cohomology differs from homology, despite the linear isomorphism of the spaces, is in the fact that there is a natural ring structure on cohomology that is does not appear in homology, in general. It is easiest to account for the ring structure in terms of de Rham cohomology because the "cup" product $[\alpha] \cup [\beta]$ of a cohomology class $[\alpha]$ in dimension k with another one $[\beta]$ in dimension l is simply the $k+l$-dimensional cohomology class $[\alpha \wedge \beta]$. In order to show that this definition does not depend upon the choice of closed k-form α and closed l-form β, one actually only needs to show that $\alpha \wedge \beta$ is also closed; of course, this is an elementary calculation.

Now, suppose that M is orientable and given a choice of volume element $\mathcal{V} \in \Lambda^n$. We have already seen that this implies isomorphisms of the vector spaces A_k and A^{n-k}, which define the fibers of the vector bundles in question. Hence, there is an isomorphism of the vector bundles themselves #: $\Lambda_k \to \Lambda^{n-k}$, $\mathbf{A} \mapsto i_{\mathbf{A}}\mathcal{V}$ that one calls *Poincaré duality*, as well.

In the singular homology of orientable manifolds (see [**11, 12**]), the phrase "Poincaré duality" generally refers to a set of isomorphisms of the homology modules $H_k(M; R)$ with the cohomology modules $H^{n-k}(M; R)$, where R is a more general coefficient ring. We could use de Rham's theorem to replace $H^{n-k}(M; R)$ with $H^{n-k}_{dR}(M)$ for the case where $R = \mathbb{R}$, but we find that on orientable manifolds it is also possible to define homology vector spaces that are directly dual to the de Rham cohomology spaces, and which make Poincaré duality even more straightforward.

We simply define our *de Rham k-chains* [17] to be k-vector fields and our boundary operator to be the divergence operator δ: $\Lambda_k \to \Lambda_{k-1}$, which still takes the same form $\delta =$

[17] Strictly speaking, de Rham did not define the homology that was dual to his cohomology in terms of multivector fields and the divergence operator, but in terms of *currents*, which amount to continuous linear functionals on k-forms – i.e., k-form distributions. However, since the structure of the homology that we

$\#^{-1} \cdot d \cdot \#$, as it did in the previous chapter. (From now on, we drop the use of δ to mean the coboundary operator in singular cohomology.) The kernel of δ, which we denote by $Z_k^{dR}(M)$, consists of divergenceless k-vector fields, which play the role of de *Rham k-cycles*. The image of δ: $\Lambda_{k-1} \to \Lambda_k$, which we denote by $B_k^{dR}(M)$, then plays the role of *de Rham k-boundaries.* The quotient vector space $H_k^{dR}(M)$ is then the *de Rham homology* vector space in dimension k, and is isomorphic to $H_{dR}^k(M)$, but not canonically, and is therefore also isomorphic to the real singular homology vector space $H_k(M; \mathbb{R})$.

One sees that Poincaré duality $\#: H_k^{dR}(M) \to H_{dR}^{n-k}(M)$ follows naturally from the way that we defined δ, since it implies that:

$$d\# = \#\delta. \tag{II.33}$$

Hence, # takes de Rham k-cycles to n–k-cocycles and k-boundaries to n–k-boundaries, which allows one conclude that the isomorphism $\#: C_k^{dR}(M) \to C_{dR}^{n-k}(M)$ "descends to homology."

One finds that there is another intriguing difference between homology and cohomology in the fact that there is a Lie algebra structure defined over vector fields that is not defined over covector fields. In fact, the Lie bracket of divergenceless vector fields is also divergenceless, so one can define the Lie bracket of de Rham homology classes in dimension one. It is even possible to extend the Lie bracket to include k-vector fields for $k > 1$, but since the divergence operator is not an anti-derivation on k-vector fields, it is more difficult to establish whether the Lie bracket of divergenceless k-vector fields is still divergenceless. Furthermore, the possible role that topology might play in a Lie algebra structure on the homology spaces would have to be explored.

6. Hodge theory [4, 15]. In the study of Riemannian manifolds, one has another isomorphism of k-vector fields and k-forms at one's disposal. It is rooted in the fact that a Riemannian metric on a differentiable manifold M, which is a positive-definite symmetric non-degenerate second-rank covariant tensor field g on M, defines an isomorphism $i_g : T(M) \to T^*M$, $\mathbf{v} \mapsto i_g\mathbf{v}$, of the tangent bundle with the cotangent bundle. In this definition, we intend that the covector field $i_g\mathbf{v}$ is defined by:

$$(i_g\mathbf{v})(\mathbf{w}) = g(\mathbf{v}, \mathbf{w}) \tag{II.34}$$

for any tangent vector $\mathbf{w}$ on M.

This isomorphism of bundles defines a corresponding isomorphism of the vector space $\mathfrak{X}(M) = \Lambda_1(M)$ of vector fields with the space $\Lambda^1(M)$ of covector fields (i.e., 1-forms). By tensor product, and consequently, exterior product, one can extend this to an isomorphism $i_g \wedge \ldots \wedge i_g : \Lambda_k \to \Lambda^k$ of k-vector fields and k-forms.

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are defining is clearly derived from corresponding concepts in de Rham cohomology, we shall make a minor abuse of terminology.

If $g = g_{\mu\nu}\, dx^\mu\, dx^\nu$ in some local coordinate system (U, x^μ) then:

$$i_g \mathbf{v} = (g_{\mu\nu} v^\nu)\, dx^\mu = v_\mu\, dx^\mu. \tag{II.35}$$

Hence, the isomorphism i_g amounts to what is usually called "lowering the index" in this case, as well as in its extension to k-vector fields, which then amounts to lowering all indices. In particular, for bivector fields, we have:

$$(i_g \wedge i_g)\mathbf{A} = \tfrac{1}{2} A_{\mu\nu}\, dx^\mu \wedge dx^\nu = \tfrac{1}{2}(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha})\, A^{\alpha\beta}\, dx^\mu \wedge dx^\nu . \tag{II.36}$$

Therefore, we can think of the linear map $i_g \wedge i_g$ as having the matrix:

$$[i_g \wedge i_g]_{\mu\nu\alpha\beta} = g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}, \tag{II.37}$$

relative to this choice of local coframe field.

Understanding this isomorphism is crucial to all of what follows, since the whole point of pre-metric electromagnetism is to replace this isomorphism with one that is defined not by the spacetime metric, but by the electromagnetic constitutive laws of the medium.

When one composes the inverse $[i_g \wedge \ldots \wedge i_g]^{-1}$: $\Lambda^k \to \Lambda_k$ of the metric isomorphism with the Poincaré isomorphism #: $\Lambda_k \to \Lambda^{n-k}$ one gets a linear isomorphism *: $\Lambda^k \to \Lambda^{n-k}$:

$$* = \# \cdot [i_g \wedge \ldots \wedge i_g]^{-1} \tag{II.38}$$

that one refers to as *Hodge duality*.

Since * is defined for every k from 0 to n one can take its square and obtain:

$$*^2 = (-1)^{k(n-k)}\, I. \tag{II.39}$$

Of particular interest is the middle dimension k in an even-dimensional space, for which $*^2 = I$, so the eigenvalues of * in this case are +1 and −1. This allows one to decompose Λ^k into a direct sum of eigen-bundles that one calls the bundles of *self-dual* and *anti-self-dual* k-forms, respectively. However, we shall be generalizing from the case of a *Lorentzian g*, which is not positive-definite, and whose eigenvalues in the case of 2-forms on a four-dimensional manifold are imaginary, not real. A decomposition of Λ^2 into essentially "real" and "imaginary" sub-bundles is not canonical, and must be specified, much as one chooses a frame for a vector space.

The metric isomorphism allows one to map the divergence operator on k-vector fields over to a *codifferential* operator δ: $\Lambda^k \to \Lambda^{k-1}$ on k-forms:

$$\delta = i_g \cdot \delta \cdot i_g^{-1} = *^{-1} d * = (-1)^{n(k+1)} * d * \tag{II.40}$$

Clearly, one has $\delta^2 = 0$, as a consequence of the fact that $d^2 = 0$. A k-form α for which $\delta\alpha$ vanishes is called *co-closed* and one for which there is a k+1-form β such that $\alpha = \delta\beta$ is called *co-exact*.

Using the operators d and δ, one can form a second-order differential operator Δ: $\Lambda^k \to \Lambda^k$: on k-forms:

$$\Delta = d\delta + \delta d \tag{II.41}$$

that one calls the *Laplacian operator* for the Riemannian manifold (M, g).

A k-form α for which $\Delta\alpha = 0$ is called *harmonic*. Clearly, it is sufficient that a harmonic k-form be both closed and co-closed:

$$d\alpha = 0, \qquad \delta\alpha = 0. \tag{II.42}$$

In fact, if M does not have a boundary then this is also necessary.

This follows from the fact that if $\alpha, \beta \in \Lambda^k$ then:

$$\alpha \wedge *\beta = \beta \wedge *\alpha \tag{II.43}$$

is always an n-form, and if we assume that M is compact then we can define the following symmetric, bilinear functional on k-forms:

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta\,, \tag{II.44}$$

which is also positive-definite.

From (II.39) and (II.43), * becomes a (graded) *self-adjoint* operator under this inner product.

$$(*\alpha, \beta) = (-1)^{k(n-k)}(\alpha, *\beta). \tag{II.45}$$

Suppose $\alpha \in \Lambda^k$ and $\beta \in \Lambda^{k+1}$, so that $(d\alpha, \beta)$ and $(\alpha, \delta\beta)$ make sense. Now:

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^k\, \alpha \wedge d*\beta = d\alpha \wedge *\beta - \alpha \wedge *\delta\beta\,, \tag{II.46}$$

so, by Stokes's theorem, one has:

$$(d\alpha, \beta) - (\alpha, \delta\beta) = \int_{\partial M} \alpha \wedge *\beta\,. \tag{II.47}$$

Hence, when M is closed (= compact, without boundary) d and δ are adjoint to each other.

This makes:

$$\begin{aligned}(\Delta\alpha, \beta) &= (d\delta\alpha, \beta) + (\delta d\alpha, \beta) \\ &= (-1)^k[(\delta\alpha, \delta\beta) + (d\alpha, d\beta)] + \int_{\partial M} (\alpha \wedge *d\beta + \delta\alpha \wedge *\beta)\,.\end{aligned} \tag{II.48}$$

Therefore, if $\Delta\alpha = 0$, which is a generalization of the Laplace equation, and one specializes this last formula by setting $\beta = \alpha$ then since the inner product is positive-definite, we see that when M has no boundary one must have the vanishing of $d\alpha$ and $\delta\alpha$ as a consequence of the vanishing of $\Delta\alpha$.

Another formula that is useful in the solution of boundary-value problems for the Laplace equation is *Green's formula* (one of many, actually; cf., [**16, 17**]):

$$(\Delta\alpha, \beta) - (\alpha, \Delta\beta) = \int_{\partial M} [\alpha \wedge *d\beta - \beta \wedge *d\alpha + \delta\alpha \wedge *\beta - \delta\beta \wedge *\alpha]. \quad \text{(II.49)}$$

Once again, we see that whether the Laplacian operator itself is self-adjoint depends upon the vanishing of the boundary of M.

The *Hodge decomposition theorem* says that on a compact, orientable manifold M *without boundary* the vector spaces Λ^k can all be expressed as direct sums $Z^k \oplus Y^k \oplus \mathcal{H}^k$, in which Y^k represents the space of all co-closed k-forms and $\mathcal{H}^k$ represents the space of all harmonic k-forms. In other words, any k-form can be expressed as the sum of a closed form, a co-closed form, and a harmonic form.

A well-known consequence of this decomposition is the fact that if M is a compact, orientable manifold without boundary then each de Rham cohomology class $[\alpha] \in H^k_{dR}(M)$ contains a unique harmonic representative. In particular, if M is a vector space then there should be no harmonic forms in any dimension. This has, as a consequence, the well-known result of vector calculus that is called *Helmholtz's theorem*, which says that any vector field (i.e., 1-form α) can be uniquely expressed as the sum of an irrotational (i.e., closed) vector field and a solenoidal (i.e., co-closed) one. Actually, the way that this changes in the case of non-vanishing cohomology is that one simply has to specify the generators γ_a, $a = 1, \ldots, b_k$ of $H^k_{dR}(M)$, or equivalently, the values of the $<\alpha, \gamma_a>$, in addition to the irrotational and solenoidal parts; b_k refers to the *Betti number* in dimension k, which is the dimension of $H^k_{dR}(M)$. This decomposition was discovered by Lord Kelvin in the case of multiply-connected spaces ($k = 1$, $b_1 > 0$).

Of course, if one has been exposed to the rich variety of harmonic functions that arise by way of solving boundary-value problems in the Laplace equation – also known as potential theory – then one will find it disappointing that Hodge theory gives such a result. The resolution of the discrepancy is in the fact that the Hodge decomposition theorem is purely related to manifolds *without boundary*, which obviously eliminates the possibility of posing boundary-value problems.

There are, however, some results for the case of manifolds with boundary. They mostly involve going to what one calls *relative homology*, in which a chain in M is a *relative cycle* modulo ∂M iff its boundary is a chain in ∂M. Since this is going beyond the scope of the present study, we simply refer the curious to the work of Duff and Spencer [**16, 17**].

Another physical context in which Hodge theory does not apply is when one generalizes the construction of the Laplacian operator on k-forms to metrics of more general signature type, such as Lorentzian manifolds. One finds that the proof of the Hodge decomposition theorem breaks down due to the fact that the kernel of the d'Alembertian operator, which is hyperbolic, is infinite-dimensional, while the kernel of the Laplacian, which is elliptic for Riemannian manifolds, is finite-dimensional.

Basically, Hodge theory will apply to the physics of static or stationary fields, in which the four-dimensional hyperbolic picture reduces to a three-dimensional Euclidian one. However, this reduction must be treated with care, as it bears upon fundamental

issues in relativistic physics, such as simultaneity, and fundamental issues in mathematics, such as integrability.

7. Space-time splittings of the spacetime manifold. Although the subtle geometrical, topological, and physical issues that are involved with space-time splittings (see [**18**] for more discussion and references), since our treatment of the foundations of electromagnetism begins in the traditional realm of static fields, it is unavoidable that we make at least some perfunctory remarks about the subject.

If the spacetime manifold M takes the form of a four-dimensional vector space V, as it does in special relativity, then the main geometrical issue associated with decomposing spacetime into a three-dimensional spatial manifold Σ and a one-dimensional time line L is simply one of decomposing V into a direct sum $L \oplus \Sigma$.

This implies that any vector $\mathbf{v} \in V$ can be uniquely expressed as a sum $\mathbf{v}_t + \mathbf{v}_s$ of a *temporal* vector $\mathbf{v}_t \in L$ and a *spatial* vector $\mathbf{v}_t \in \Sigma$. One also has canonically-defined projections P_t: $V \to L$, $\mathbf{v} \mapsto \mathbf{v}_t$ and P_s: $V \to \Sigma$, $\mathbf{v} \mapsto \mathbf{v}_s$.

Usually, it is the line L that is defined first, in the form of a line that is tangent to the motion of the measurer/observer that defines a rest space. In other words, one is "modding out" the motion of the measure/observer by defining a comoving frame field along its world line. The complementary spatial vector space Σ then becomes essentially the normal space V/L to L in V. However, this normal space is really a subspace of V^*, not a subspace of V, so if one is to define a complement to L in V, one must either choose it arbitrarily or introduce – say – a scalar product, such as the Minkowski scalar product, and define Σ to be the orthogonal complement to L. Of course, that is not in the spirit of pre-metric physics, but fortunately there is much that can still be said about space-time splittings at a pre-metric level, as we shall see.

Now, when one goes from a vector space V to a more general manifold M there are actually two ways that one can speak of space-time separability. The most stringent form is to demand that the manifold M take the form of a product manifold $L \times \Sigma$, where L is a one-dimensional temporal manifold and Σ is a three-dimensional spatial manifold. Hence, the time manifold might be either $\mathbb{R}$, as in the vector space, or possibly the circle S^1, which would be more appropriate to situations in which periodicity featured prominently.

An immediate consequence of assuming that M has a product structure is the fact that the tangent bundle $T(M)$ can be given a direct sum – i.e., *Whitney* – decomposition $L(M) \oplus \Sigma(M)$, which then means that the tangent space T_x at any point x M admits a direct sum decomposition $L_x \oplus \Sigma_x$ as a vector space, with the previous consequences applying locally. Similarly, one has a Whitney sum decomposition of the cotangent bundle T^*M into $L^*M \oplus \Sigma^*M$.

One then has direct sum decompositions of the various tensor products of tangent and cotangent bundles. For instance:

$$T(M) \otimes T(M) = [T(L) \otimes T(L)] \oplus [T(L) \otimes T(\Sigma)] \oplus [T(\Sigma) \otimes T(L)] \oplus [T(\Sigma) \otimes T(\Sigma)],$$

and similarly for $T^*M \otimes T^*M$.

One often hears the projections T_{tt}, T_{ts}, T_{st}, T_{ss} of a second-rank tensor field T when it is decomposed in this way as its *time-time*, *time-space*, *space-time*, and *space-space* projections. (See, for instance, Cattaneo [**19**]).

Under symmetrization or anti-symmetrization, one finds that the sub-bundle $[T(L) \otimes T(\Sigma)] \oplus [T(\Sigma) \otimes T(L)]$ becomes either $T(L) \odot T(\Sigma)$ or $T(L) \wedge T(\Sigma)$, respectively. In particular, the exterior algebra $\Lambda^*(M)$ of T^*M becomes:

$$\begin{aligned}
&\Lambda^0 M = \Lambda^0 L \otimes \Lambda^0 \Sigma, \\
&\Lambda^1 M = \Lambda^1 L \oplus \Lambda^1 \Sigma, \\
&\Lambda^2 M = [\Lambda^1 L \wedge \Lambda^1 \Sigma] \oplus \Lambda^2 \Sigma, \\
&\Lambda^3 M = [\Lambda^1 L \wedge \Lambda^2 \Sigma] \oplus \Lambda^3 \Sigma, \\
&\Lambda^4 M = \Lambda^1 L \wedge \Lambda^3 \Sigma.
\end{aligned}$$

Note that since L is one-dimensional, $\Lambda^k L = 0$ for all $k > 1$, and similarly, $\Lambda^k \Sigma = 0$ for all $k > 3$. It is particularly convenient that each of the bundles $\Lambda^k L$ for $k = 1, 2, 3$ admits a decomposition into only a temporal sub-bundle and a spatial one, where "temporal" in this case means that it has $\Lambda^1 L$ as an exterior factor.

One can then extend the projection operators P_t and P_s to $\Lambda^k M$, $k = 1, 2, 3$ such that P_t: $\Lambda^k M = \Lambda^1 L \wedge \Lambda^{k-1} \Sigma$, P_s: $\Lambda^k M = \Lambda^k \Sigma$.

The specific forms that k-forms in each dimension then take are:

$$f(t, x) = T(t)S(x), \tag{II.49a}$$
$$\phi = \phi_0\, dt + \phi_s, \tag{II.49b}$$
$$F = dt \wedge E_s + F_s, \tag{II.49c}$$
$$G = dt \wedge F_s + G_s, \tag{II.49d}$$
$$\rho = \rho_0\, dt \wedge \rho_s, \tag{II.49e}$$

in which the subscript s denotes k-forms in $\Lambda^*\Sigma$ in each case.

A subtlety that is easy to overlook is the fact that since the component functions for k-forms on M are functions on M the component functions for the temporal and spatial parts of any k-form will still be functions on M, in general, not purely temporal or spatial functions. For instance, one will have:

$$\phi = \phi_0(t, x)\, dt + \phi_i(t, x)\, dx^i . \tag{II.50}$$

This is especially important to keep in mind when differentiating.

Of course, there is an analogous decomposition for the exterior algebra $\Lambda_* M$ of multivector fields on M that amounts to lowering all of the superscripts in the decompositions above.

One finds that the exterior derivative operator d: $\Lambda^k(M) \to \Lambda^{k+1}(M)$, $k = 0, 1, 2, 3$ admits a decomposition into a temporal part d_t: $\Lambda^k(M) \to \Lambda^1 L \wedge \Lambda^k \Sigma$, and a spatial one d_s: $\Lambda^k(M) \to \Lambda^{k+1} \Sigma$. One simply composes the operator d with the projections P_t and P_s, respectively. For instance:

$$d\phi = d_t\phi + d_s\phi = -\phi_{0,i}\, dt \wedge dx^i - \tfrac{1}{2}[\phi_{i,j} - \phi_{j,i}]\, dx^i \wedge dx^j. \tag{II.51}$$

By means of such compositions of operators with projections, one can similarly decompose the divergence operator $\delta\colon \Lambda_k(M) \to \Lambda_{k-1}(M)$, $k = 1, 2, 3, 4$, as well as the various exterior multiplication and interior multiplication operators on k-forms or k-vector fields.

As we mentioned above, there are two ways of imposing a space-time decomposition, the most demanding of which is a product structure $L \times \Sigma$ on the manifold M. One finds that quite often since most of the physical constructions are local and define differential equations, the reason that one needs a space-time decomposition first asserts itself in the local context. That is, one does not always require that M to decompose as a manifold, but only that $T(M)$ decompose as a vector bundle.

When $T(M)$ is given a Whitney sum decomposition $T(L) \oplus T(\Sigma)$, with no other restrictions on M, one says that M has been given an *almost-product* structure. Whether such an almost-product structure implies a product structure on M is a deep and subtle matter of integrability. For further discussion, one might confer the author's work in [**20**], which includes references to other approaches to the problem.

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Chapter III

Static electric and magnetic fields

In addressing the laws of nature in their most fundamental form, however enigmatic, one must first address the issue of what constitute the most directly observable phenomena of nature. In physics, these phenomena tend to be the ones that appeal to one's vision, hearing, and sense of touch. (Taste and smell seem to be more chemical in character.) Not surprisingly, much of physics is concerned with light, sound, and motion, and, more generally, wave phenomena.

One can immediately distinguish classes of physical phenomena, such as passive and reactive phenomena, or static and dynamic ones. One sees that, in effect, these two ways of distinguishing phenomena are really the same. Passive or static phenomena are the ones that are observable in the absence of input from other parts of the system, including the observer, such as light from the distant stars or the equilibrium state of a mechanical structure. The reactive or dynamic phenomena are then the ones that pertain to the response of the state of a system to an input from elsewhere in the system, which then includes the measurements performed by an observer.

At the root of the physics of electricity and magnetism, one must realize that such a fundamental concept as charge does not represent a directly observable phenomenon, but only an indirectly observable one. The directly observable phenomenon that pertains to charge is the acceleration of some – but not all – matter as a result of the presence of some – but not all – other matter, in a manner that cannot be accounted for by the other known forces, such as gravitation. One only *postulates* the existence of an invisible, intangible quality, such as charge, to account for the dynamics of that interaction, just as thermodynamics once postulated the concept of "phlogiston" to account for the fact that some materials were flammable and others were not.

Of course, nowadays one must add the atomic hypothesis to the list of basic axioms, namely, that not only does charge come in three basic types +, –, and 0, but furthermore it does not really exist on an entire continuum of possible values. Rather, one always considers integer multiples of a nearly infinitesimal basic charge unit that is due to the presumed "indivisibility" of elementary matter. Nonetheless, just as the study of ideal gases is not best approached by looking at an ensemble of individual molecules whose cardinality is Avogadro big ($> 10^{23}$), but by replacing it with suitable totals and density functions defined on the volume in question, similarly, macroscopic electromagnetism is usually defined in terms of total and average quantities that are traceable to microscopic origins.

An unexpected consequence of the history of Twentieth Century physics is that it seems that the success of the atomic hypothesis reached its zenith during the early years of quantum physics, when it successfully accounted for the structure of periodic table. After that, it became clear that atoms were not indivisible, and neither were nuclei, or even nucleons. Hence, in the present era, one must quibble over whether the most elementary unit of electric charge is due to the electron, which is observable in a free state, or the quarks whose charge has an absolute value of one-third that of the electron, but are, apparently, observable only in bound states. Thus, one can only think of quarks

as having a somewhat more tentative character than electrons, positrons, protons, and the like. Furthermore, if one regards "elementary" particles as synonymous with their fields, and fields belong to an infinite-dimensional space, then one must realize that matters are not getting simpler as one goes deeper into the realm of elementary matter, but more complicated! Hence, we shall try to be objective about the existence of a fundamental duality between discreteness and continuity in the manifestation of matter, and use one or the other as it seems convenient.

Another issue that affects the way that one perceives physics at its most fundamental level – viz., the observation and measurement of physical phenomena in nature – is that there seems to be an unavoidable "duality" between statics and dynamics. That is, should one think of dynamics as the response of statics to "external" (i.e., external to the subsystem in question) perturbations, or should one think of statics as the limiting state of dynamics in the absence of external perturbations? This question actually bears upon the difference in spirit between quantum physics and relativistic (but non-quantum) physics. In the former approach to physics, one is essentially looking at the response of static – or, at least, *stationary* – systems to the perturbations – viz., measurements – of an external "measurer." Indeed, most of one's early exposure to quantum mechanics is exclusively concerned with the structure of the stationary states. In the latter approach, one must regard the four-dimensional world of dynamics as more fundamental in character, and the static world only comes about as the rest space that is defined by a choice of "observer."

Perhaps, the best way to reconcile these two viewpoints is to keep both of them in mind and consider the observation of Max Born that all measurements are made in the rest space of the measuring devices. Hence, there is something unavoidable, indeed *fundamental*, about the role of measurer/observer in the description of physical phenomena.

1. Electric charge [1-4]. We shall start with the concept of electric charge – with all of the aforementioned caveats – as the new logical primitive that we add to the laws of kinematics and dynamics as a starting point for our theory of electrostatics. The axioms are then that *total electric charge* is a real number $Q[V]$ that one associates with a volume V of space Σ (which is a differentiable manifold of dimension one, two, or three) like an *extensive* variable, in the language of thermodynamics. That is, when one combines disjoint volumes, one adds the total electric charges that they contain:

$$Q\left[\sum_{i=1}^{N} V_i\right] = \sum_{i=1}^{N} Q[V_i]\,. \tag{III.1}$$

This amounts to saying that total electric charge defines a *measure* on space, in the language of measure theory, as long as the volume V is a measurable subset.

The value $Q[V]$ of charge is an integer multiple of an elementary charge, which we call e. Interestingly, although e is very small in units of Coulombs, that fact is really a criticism of the applicability of the latter unit, since one Coulomb of static charge is harder to configure than it sounds. (It is, nevertheless, quite dynamically practicable. Consider one Ampere of current flowing through the cross-section of a wire, which amounts to one Coulomb every second.)

Although a region of space that contains no elementary charge sources will be associated with zero charge, the converse is not true. In fact, most macroscopic matter, such as stars, seems to have close to zero total charge, despite the fact that it contains an enormous number of elementary charges. That is why gravitational forces seem to be dominant at the astronomical level, even though the force of gravitation is almost infinitesimal by comparison to that of electrostatic attraction/repulsion.

The intensive variable that is associated with total electric charge is *electric charge density*, which one first obtains as an average charge density in a given volume by dividing the total charge in a volume by its volume:

$$\bar{\rho}[V] = \frac{1}{V} Q[V] \tag{III.2}$$

and then passing to the limit of zero volume while regarding each V as a neighborhood of each of its points:

$$\rho(x) = \lim_{V_x \to 0} \bar{\rho}(V_x) \,. \tag{III.3}$$

Of course, we are then assuming that such a limit actually exists. This is where the concept of a point charge introduces a predictable infinity in the form of the electric charge density at the point where the charge is localized. If one takes the position that infinity never manifests itself in physical reality (i.e., in measurements) then one must regard the point charge as merely a convenient approximation to something more difficult to fathom, namely, a charge distribution that is concentrated in a volume that is too small to probe directly. Furthermore, the fact that quantum mechanics routinely regards electrons as wavelike in character suggests that they have finite spatial extent.

Here, we introduce the methods of the last chapter by restricting ourselves to volumes that take the form of differentiable singular cubic n-chains, where $n = \dim(\Sigma)$. Hence, the additivity that we postulated in (III.1) suggests that we might wish to regard total electric charge as an n-cochain; i.e., a linear functional on n-chains. All that we need to add is the requirement that it also be homogeneous of degree one with respect to scalar multiplication by a real number λ:

$$Q[\lambda V] = \lambda Q[V]. \tag{III.4}$$

Admittedly, the concept of multiplying a volume by a real scalar is less physically intuitive than that of adding disjoint volumes, but we shall see that this is a small price to pay for the utility of homology in describing charge and flux.

Since we are clearly assuming that Σ is orientable, oriented, and endowed with a volume element $\mathcal{V}$, we already have one linear functional on n-chains that is defined by the volume functional. On any n-cube σ_n it takes the form:

$$V[\sigma_n] = \int_{\sigma_n} \mathcal{V} \,. \tag{III.5}$$

One then extends this definition to more general n-chains by the assumption of linearity.

From (III.2), one can then define the average electric charge density in the region described by σ_n as the ratio:

$$\overline{\rho}[\sigma_n] = \frac{Q[\sigma_n]}{V[\sigma_n]}. \tag{III.6}$$

It is in passing to the limit that one must deal with the existence of elementary charges, since that fact suggests that ultimately a volume must contain either one elementary charge (with its sign) or none at all. If an elementary charge is distributed over a finite volume then, in a sense, it is not really elementary, since it is presumed to be further reducible. If it is pointlike then one must accept the non-existence of a finite limiting charge density for the neighborhoods that it contains.

One can consider some sort of "confinement" hypothesis for charges less than e, similar to the hypotheses that make free quarks unobservable. However, in the next section, we shall resolve the issue by a simple topological device, namely, by removing the point to which the volumes are converging from space itself. This then has the effect of saying that the concept of electric charge density is useful only as a macroscopic approximation.

Hence, we accept that the concept of electric charge density is only so fundamental at the elementary level and assume the electric charge density $\rho(x)$ at a point $x \in \Sigma$ actually exists as a function. One can then think of it as the "distribution kernel" of the electric charge functional:

$$Q[\sigma_n] = \int_{\sigma_n} \rho \mathcal{V}. \tag{III.7}$$

The non-existence of a limit to the electric charge density for a point charge is closely related to the concept of the Dirac delta function, which is not really a function, at all, but a fictitious kernel for a much-better-defined distribution called the "evaluation" functional, which we briefly describe.

If S is a set and $F(S; \mathbb{R})$ is the vector space of real-valued functions on S then the *evaluation functional* that is associated with each $x \in S$ is the linear functional E_x: $F(S; \mathbb{R}) \to \mathbb{R}$, $f \mapsto E_x[f] = f(x)$, which simply evaluates the function f: $S \to \mathbb{R}$ at the point x. We can extend this to a linear functional on n-forms E_x: $\Lambda^n \to \mathbb{R}$ by assuming that all elements of Λ^n take the form $f\mathcal{V}$, so $E_x[f\mathcal{V}] = f(x)\mathcal{V}_x$. Despite the fact that this distribution does not have a kernel, we follow tradition and formally write:

$$E_x[f\mathcal{V}] = \int_\Sigma \delta(x, y) f(y) \mathcal{V}_y = f(x)\mathcal{V}_x, \tag{III.8}$$

in which $\delta(x, y)$ is the *Dirac delta function.* Here, we are using the notation $\mathcal{V}_y$ to suggest that the integration is over the y variable, while the x is held fixed.

If we integrate over a subset $V \subset \Sigma$, instead of Σ itself, then we can define the *selection functional* for $x \in V$ using this delta function:

$$\psi_x[V] = \int_V \delta(x,y)\mathcal{V}_y = \begin{cases} 1 & x \in V \\ 0 & \text{otherwise.} \end{cases} \tag{III.9}$$

Hence, the selection functional for x identifies all subsets that have x as an element; i.e., all neighborhoods of x. Of course, this makes perfect sense as an n-cochain, as long as we restrict ourselves to subsets V that can be parameterized by k-chains.

Usually, when Σ is a vector space, one defines $\delta(x)$ to give evaluation at the origin and then replaces x with $x - y$ to give evaluation at $x = y$, but one can see that this definition breaks down when Σ does not have an affine structure that would allow us to make sense of the expression $x - y$ for two points $x, y \in \Sigma$. This is a recurring theme for the problems that are associated with generalizing linear constructions to nonlinear manifolds.

By means of this fictitious kernel, one can define electric charge densities for finite sets of point charges of charge Q_i at points $x_i \in \Sigma$ as linear combinations of delta functions:

$$\rho(y) = \sum_{i=1}^{N} Q_i \delta(x_i, y)\,. \tag{III.10}$$

Although our sum over points suggests that we are defining a 0-cochain, from (III.9), we see that this sum makes better sense as an n-cochain, which only means multiplying both sides by $\mathcal{V}_y$ and integrating over an n-chain V:

$$Q[V] = \int_V \rho\mathcal{V} = \sum_{i=1}^{N} Q_i \int_V \delta(x_i, y)\mathcal{V}_y = \text{total charge in } V. \tag{III.11}$$

2. Electric field strength and electric flux [1-4]. If a point charge Q at a point $x \in \Sigma$ experiences a force $F(x; Q) \in T_x^*M$, which we presume to be of electrostatic origin, then we define the *electric field strength 1-form for this charge* to be the covector field E: $\Sigma \rightarrow T^*(\Sigma)$:

$$E(x; Q) = \frac{1}{Q} F(x; Q). \tag{III.12}$$

However, this definition has two unsavory aspects to it:

1. We are defining it for point charges, and we have already expressed doubts about the limits of that approximation.

2. We are defining E in terms of a particular charge Q, which suggests that a different charge Q' might define a different E.

The traditional way of getting around the second objection is to introduce the *test particle hypothesis:* For every real number Q, no matter how small, there exists a charged particle in nature whose charge is Q. Clearly, this is totally inconsistent with the assumption that there is a non-zero minimum charge.

If one accepts this hypothesis then one can factor out the effect of Q by passing to the limit – if there is one – and defining:

$$E(x) = \lim_{Q\to 0} \frac{1}{Q} E(x; Q). \tag{III.13}$$

If one does not wish to introduce test particles then one can define $E(x)$ to be the force on a *unit charge*.

$$E(x) \equiv F(x; 1). \tag{III.14}$$

Now, we can deal with an even more subtle issue concerning the nature of the electrostatic force: Is $E(x; Q)$ going to be this same 1-form for $Q \neq 1$, and if not, how does it change? At first, this suggests the question of the linearity of the field equations for E in disguise; i.e., the principle of superposition. However, in order for the nature of the electrostatic force to be consistent with the independence of $E(x)$ from Q, it is necessary and sufficient to assume that $F(x, Q)$ is homogeneous of degree one in Q:

$$F(x; \lambda Q) = \lambda F(x; Q), \tag{III.15}$$

which then makes:

$$F(x; Q) = QF(x; 1) = QE(x). \tag{III.16}$$

We shall simply work with the definition of E that we get from (III.14) and keep in mind what that might entail.

If we represent a path in Σ by a 1-chain c_1 then since the work done by a force F along that path is the 1-cochain:

$$\Delta U[c_1] = \int_{c_1} F\,, \tag{III.17}$$

we define the *change in potential energy* along c_1 due to the influence of $F(x; Q)$ on a charge Q to be:

$$\Delta U[c_1; Q] = \int_{c_1} F(x;Q)\,, \tag{III.18}$$

and the *change in electric potential* along c_1 due to the influence of $F(x; Q)$ to be the change in potential energy experienced by a unit charge:

$$\mathcal{E}[c_1] = \Delta U[c_1; 1] = \int_{c_1} F(x;1) = \int_{c_1} E\,. \tag{III.19}$$

We are using the notation $\mathcal{E}$ to be consistent with the classical term "electromotive force," but that term, despite it continued popularity, is a misnomer, since the quantity being described has units of *work* per unit charge, not force per unit charge.

Usually, in elementary physics, one regards the issue of whether this work done is independent of the path between two points as being equivalent to the question of whether it vanishes around any loop (i.e., any 1-cycle). Of course, that is because one is

usually concerned with life in $\Sigma = \mathbb{R}^3$, which is contractible, and therefore simply connected, *a fortiori*.

Since we are focusing more on topological issues in this study, we shall clearly distinguish between the two conditions. Hence, one must treat the cases of loops that do or do not bound 2-chains separately. If z_1 is a bounding 1-cycle $z_1 = \partial c_2$ then, by Stokes's theorem:

$$\mathcal{E}[\partial c_2] = \int_{\partial c_2} E = \int_{c_2} dE \,. \tag{III.20}$$

Hence, if the work done on a unit charge by an electrostatic force F vanishes around any loop in Σ that bounds a 2-chain then one must have:

$$dE = 0. \tag{III.21}$$

This amounts to the statement that the electric field strength 1-form E is closed; one can also say that it is *irrotational*. Hence, E defines a de Rham cohomology class in dimension 1: $[E] \in H^1_{dR}(M)$.

Of course, if Σ is simply connected then every 1-cycle bounds a 2-chain and (III.21) is equivalent to the condition that E be exact; we shall return to this concept in a later section.

Another way of looking at (III.21) is that it represents an integrability condition for the exterior differential system that is defined by $E = 0$. That is, since E is a 1-form, at each point of Σ there is a hyperplane $\mathrm{Ann}(E)_x$ in $T_x\Sigma$ that consists of the tangent vectors $\mathbf{v}$ at x that are annihilated by E: $E(\mathbf{v}) = 0$. This set of hyperplanes defines a sub-bundle of $T(M)$ of codimension one that calls a *differential system* on M. It is completely integrable iff every point $x \in \Sigma$ has a hypersurface that passes through it, and its tangent hyperplane agrees with $\mathrm{Ann}(E)_x$. Such a hypersurface is called an *integral hypersurface* or *integral submanifold* and Σ is said to be *foliated* by these integral submanifolds.

The most general necessary and sufficient condition for the complete integrability of $\mathrm{Ann}(E)$ is given by *Frobenius's theorem*, which says that $\mathrm{Ann}(E)$ is completely integrable iff it is *involutory*. By definition, this means that the vector space of vector fields on Σ that take their values in the fibers of $\mathrm{Ann}(E)$ is closed under the Lie bracket; i.e., it is a Lie subalgebra of $\mathfrak{X}(\Sigma)$. An equivalent condition is that:

$$E \wedge dE = 0, \tag{III.22}$$

which is equivalent to the condition that there be a 1-form η such that:

$$dE = \eta \wedge E \,. \tag{III.23}$$

Hence, (III.21) is a stronger condition than Frobenius requires.

An even stronger condition is that E be exact, which gets us into the realm of electric potential functions and shows us that the nature of the integral hypersurfaces is simply that of equipotentials. We shall return to this topic shortly.

3. Electric excitation (displacement) [1-4]. When an electric field E is present in a macroscopic medium that is composed of atomic ions and electrons that are bound into a crystal lattice or molecules that are bound into a more amorphous solid or fluid, the charges in the medium will react to E to a greater or lesser degree. In conductors, for which the electron mobility is high, the response of the medium will generally take the form of an electric current – i.e., the translational motion of the electrons. In an insulator, for which the electron mobility is low, the effect will often be the alignment of its electric dipoles, which are spatially separated charge pairs of the form $\{+Q, -Q\}$. If the spatial separation is described by the displacement vector **d** that points from $+Q$ to $-Q$ then the *electric dipole moment* of the pair is $Q\mathbf{d}$.

When this is going on macroscopically, so we can consider an electric charge density ρ in place of Q, we can replace $Q\mathbf{d}$ with a vector field **D** that represents an electric dipole density. We shall call this vector field the *electric excitation* of the medium; it is also called the *electric displacement*, following Maxwell's terminology.

Actually, one usually associates such a vector field with E even in the otherwise uninteresting case of the electromagnetic vacuum. However, the association, which we write in the form:

$$\mathbf{D} = \varepsilon_0\, i_g E \qquad (D^i = \varepsilon_0\, g^{ij} E_j) \tag{III.24}$$

has some suspicious features that get passed over when one is only doing topology, geometry, and physics in $\mathbb{R}^3$:

1. To characterize the response of the vacuum state to E by the constant ε_0, which one calls the *electric permittivity* or *dielectric constant* of the vacuum, is to assume that the vacuum is a linear, homogeneous, isotropic dielectric medium, which ignores a lot of lessons from quantum electrodynamics, such as vacuum polarization. (We shall discuss this in more detail in a later chapter.)

2. We are using the spatial Euclidian metric g on Σ as if it were given independently of electrostatic considerations. One of the main assumptions of pre-metric electromagnetism is that the four-dimensional Lorentzian structure on spacetime is a consequence of the electromagnetic constitutive laws as they relate to the propagation of electromagnetic waves, so we wonder if this also extends to the electrostatic level, as well.

For now, we simply assume that there is a diffeomorphism ε_x: $\Lambda^1_x \to \Lambda_{1,x}$ that takes each fiber of Λ^1 at $x \in \Sigma$ to the corresponding fiber of Λ_1 at x. When this diffeomorphism is a linear isomorphism, the map ε: $\Lambda^1 \to \Lambda_1$ is a vector bundle isomorphism. Of course, this would leave out the possibility of nonlinear electrostatics, such as one encounters in the effective field theories of quantum electrodynamics [**2**].

Since we are assuming that Σ is orientable and given a volume element $\mathcal{V}$, the vector field **D** can be associated with an n–1-form:

$$\#\mathbf{D} = i_{\mathbf{D}}\mathcal{V} = \frac{1}{3!}\,\varepsilon_{ijk}\, D^k\, dx^i \wedge dx^j \tag{III.25}$$

by Poincaré duality.

We call this n–1-form the *electric flux density*. The corresponding integral of #**D** over an n–1-chain:

$$\Phi_{\mathbf{D}}[c_{n-1}] = \int_{c_{n-1}} \#\mathbf{D} \tag{III.26}$$

is then the *total electric flux though* c_{n-1}. This makes it clear that the total electric flux defines an n–1-chain on Σ.

If $c_{n-1} = \partial c_n$ then from Stokes's (viz., Gauss's, in this case) theorem

$$\Phi_{\mathbf{D}}[c_{n-1}] = \int_{c_n} d\,\#\mathbf{D} = \int_{c_n} (\delta\mathbf{D})\mathcal{V} = Q[c_n], \tag{III.27}$$

in which we need only set our charge density ρ equal to:

$$\delta\mathbf{D} = \rho \tag{III.28}$$

in order to make (III.26) valid for every bounding n-chain.

Hence, one can say that the total electric flux through the boundary of an n-cycle is equal to the total electric charge that is contained in the n-cycle itself. (We are omitting a multiplicative constant, such as 4π, which can be absorbed into the definition of $\mathcal{V}$.)

One can also include the constitutive law that connects **D** with E and express (III.28) as first order partial differential equation in E:

$$\delta \cdot \varepsilon(E) = \rho. \tag{III.29}$$

If ε is expressed by local component functions of the form $\varepsilon^{ij}(x, E)$ then (III.29) can be expressed in local form as:

$$\tilde{\varepsilon}^{ij}\frac{\partial E_j}{\partial x^i} + \frac{\partial \varepsilon^{ij}}{\partial x^i}E_j = \rho, \tag{III.30}$$

in which we have introduced the notation:

$$\tilde{\varepsilon}^{ij} = \varepsilon^{ij} + \frac{\partial \varepsilon^{ik}}{\partial E_j}E_k\,. \tag{III.31}$$

Therefore, we conclude that the inhomogeneity in ε affects E directly, while the nonlinearity affects the spatial derivatives of E.

We see from (III.28) that outside the support of ρ (supp(ρ) = closure of the set of points at which ρ is non-vanishing) the vector field **D** is divergenceless. Hence, if one thinks of ρ as the generator of a one-parameter family of local diffeomorphisms of Σ then outside of supp(ρ) these diffeomorphisms preserve the volume element $\mathcal{V}$. The integral curves of the vector field **D** are what one commonly refers to as *electric field lines*, or, more precisely, *electric flux lines.* In the case of $\mathbf{D} = \varepsilon\, i_g E$, where ε is a smooth function, **D** will also represent the direction of the force and acceleration that acts on a charge Q of

non-zero mass at each given point. It is not, in general, the direction of the velocity vector for the subsequent motion.

Since any $n-1$-chain that bounds will be an $n-1$-cycle, the question arises of what happens to the total electric flux through a non-bounding $n-1$-cycle. Of course, in order for this to be possible $H_{n-1}(\Sigma; \mathbb{R})$ must not vanish. From Poincaré duality, this is equivalent to the non-vanishing of $H^1(\Sigma; \mathbb{R}) \cong H_1(\Sigma; \mathbb{R})$, so it is sufficient that Σ be non-simply connected.

Another possibility for making $H_{n-1}(\Sigma; \mathbb{R})$ non-vanishing is that Σ is simply connected, but the first non-vanishing homotopy group is in dimension $n-1$, which is the case with $\mathbb{R}^n - \{0\}$. A generator for $H_{n-1}(\Sigma; \mathbb{R}) \cong \mathbb{R}$ in that case is then defined by any $n-1$-sphere (i.e., any $n-1$-cycle) that includes the origin in the ball that it bounds.

When z_{n-1} is not a bounding cycle, Stokes's theorem no longer applies, so $\Phi_D[z_{n-1}]$ can be non-vanishing even when the total charge contained in the region "bounded" by z_{n-1} can no longer be defined, since there is no such region. This allows one to give a purely topological origin to charge, which is probably more appropriate at the elementary level than the macroscopic one that involves a more statistically-defined charge density: Elementary charge is due to the presence of non-bounding $n-1$-cycles in Σ; i.e., the non-vanishing of its homology in dimension $n-1$.

Furthermore, if $H_{n-1}(\Sigma; \mathbb{R})$ is non-vanishing then so is $H_{dR}^{n-1}(\Sigma)$, and there are closed $n-1$-forms that are not exact. Hence, it is possible to find $\#\mathbf{D}$ such that $d\#\mathbf{D} = 0$, but $\Phi_{\mathbf{D}}[z_{n-1}] \neq 0$; for such a $\mathbf{D}$, one clearly has $\delta\mathbf{D} = 0$. This is essentially the spirit of what Wheeler and Misner [**5**] called "charge without charge;" i.e., non-vanishing flux with vanishing divergence.

For example, the Coulomb $\mathbf{D}$-field:

$$\mathbf{D}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}, \tag{III.32}$$

has this property.

4. Electrostatic field equations [2]. If we combine the equations that we proposed for E and $\mathbf{D}$ so far then we have:

$$dE = 0, \qquad \delta\mathbf{D} = \rho, \qquad \mathbf{D} = \varepsilon(E). \tag{III.33}$$

As we have seen, the presence of ρ tends to be more of interest in macroscopic problems, in which the concept of a charge density makes sense as an approximation to a large ensemble of elementary charges. When dealing with elementary charges, it is probably preferable to consider the homogeneous form of these equations, which is also valid outside $\text{supp}(\rho)$:

$$dE = 0, \qquad \delta\mathbf{D} = 0, \qquad \mathbf{D} = \varepsilon(E). \tag{III.34}$$

In this form, one can see the manifestly topological nature of the first two equations, since the first one says that E defines a de Rham cohomology class for Σ in dimension one and the second one says that $\mathbf{D}$ defines a de Rham homology class in dimension one; equivalently, $\#\mathbf{D}$ defines a de Rham cohomology class in dimension $n - 1$. This also introduces a topological origin for elementary charges by the possibility that either of – hence, both of – $H^1_{dR}(\Sigma)$ and $H^{n-1}_{dR}(\Sigma)$ are non-trivial vector spaces. For instance, Σ might be multiply connected or it could be simply connected and have its first non-vanishing homotopy group in dimension $n - 1$, like $\mathbb{R}^n - \{0\}$, among other possibilities.

Although one often encounters the equations of electrostatics as differential equations, one can also express them in integral form:

$$\int_{\partial c_2} E = 0, \qquad \int_{\partial c_3} \#\mathbf{D} = \int_{c_3} \rho\,\mathcal{V}, \qquad \mathbf{D} = \varepsilon(E). \tag{III.35}$$

One can think of the first two equations as expressing conservation of energy and conservation of charge, respectively.

The integral form of differential equations is equivalent to regarding the differential equations as having distributions for solutions, which is necessary when one is dealing with solutions that have singularities, such as jump discontinuities or poles.

5. Electrostatic potential functions [2]. Previously, we only required that the work done by an electrostatic force around a loop that bounded a 2-cycle vanished, and showed that if this were true for all 1-boundaries then one could conclude that E was closed. A closely-related issue associated with work and potential energy, independently of the nature of F as we discussed above, is the issue of whether the work done along a path connecting two points A and B is path-dependent or not. However, one must keep in mind that if c_1 is a path from A to B, we must have $\partial c_1 = B - A$. This suggests that we might define ΔV as a 0-cochain; i.e.:

$$\Delta V[A, B] = \int_{\partial c_1} V = \int_{c_1} dV = \int_{c_1} E\,. \tag{III.36}$$

If this is true for all 1-chains then we can conclude that E is exact:

$$E = dV. \tag{III.37}$$

Of course, the function V is not defined uniquely, but only up to an additive constant. This is because it is only the difference:

$$\Delta V[A, B] = V(B) - V(A) \tag{III.38}$$

that is uniquely defined.

If one chooses a point, such as A, and a value $V(A)$ to associate with A then one can convert the two-point function $\Delta V[A, B]$ on $\Sigma \times \Sigma$ to a one-point function on Σ, at least, under the assumption that it is path-connected:

$$V(x) = V(A) + \Delta V[A, x]. \tag{III.39}$$

One calls such a function V an (electrostatic) *potential function* for E.

Note that (III.38) implies that the work done around any 1-cycle – bounding or not – vanishes. One simply puts two points A and B on the loop and treats the two arcs AB and BA as two paths from A to B, one which has a negative orientation with respect to the other.

As we pointed out above, exactness is the strongest integrability requirement that one can put on the exterior differential system $E = 0$. Basically, it makes the integral submanifolds take the form of the level hypersurfaces $V(x) = \text{const.}$ of the function V; a different choice of $V(A)$ will not change the nature of the hypersurfaces, only the value of V on each of them.

If E admits a potential function then we can combine the three field equations (III.33) into a single one:

$$\Delta_\varepsilon V = \rho, \tag{III.40}$$

in which:

$$\Delta_\varepsilon = \delta \cdot \varepsilon \cdot d \tag{III.41}$$

is the generalized Laplacian operator on 0-forms that is defined by ε. If ε is a nonlinear map then Δ_ε will be a nonlinear second-order elliptic differential operator.

From (III.30), all that we have to do to get the local form of (III.40) is substitute $E_i = V_{,i}$:

$$\tilde{\varepsilon}^{ij} \frac{\partial^2 V}{\partial x^i \partial x^j} + \frac{\partial \varepsilon^{ij}}{\partial x^i} \frac{\partial V}{\partial x^j} = \rho. \tag{III.42}$$

6. Magnetic charge and flux [1, 4]. If we continue our argument that the most fundamental level of any force of nature involves the observation/measurement of changes in the state of a natural systems that are consistently correlated with some other natural phenomenon then we admit that magnetism first manifests itself as forces on some, but not all, materials (viz., magnetic materials) as a result of being exposed to other influences that we simply identify as magnetic forces. Of course, this all sounds very tautological, unless we get more specific about the origin of magnetic forces.

Here, the atomic hypothesis is only so helpful. If we break a bar magnet into a geometric progression of smaller pieces then once again the series will terminate with electrons, but not in the same way as for electrostatic forces. If a bar magnet is a macroscopic ensemble of elementary magnetic dipoles then one sees that the most elementary manifestation of a magnetic field is due to a *dipole*, not a pair of monopoles. Namely, the ultimate source of magnetic fields seems to be the *relative motion* of elementary electric charges. In the case of elementary dipoles, this usually takes the form of electron or nucleon spin.

Of course, at this point in the history of physics, the possible existence of magnetic monopoles is a hotly-contested subject. They were first postulated by Dirac in the 1930's as a possible consequence of the formulation of electromagnetism as a $U(1)$ gauge theory. In particular, if the $U(1)$-principal bundle on spacetime that represented the gauge structure for electromagnetism happened to be non-trivial – so no global choice of $U(1)$ gauge was possible – then this would suggest topological obstructions (viz., the first Chern class of the bundle) that might bring about magnetic monopoles. Because these topological considerations seem to be so fundamental and unavoidable, the fact that, to date, no experimental evidence for the existence of magnetic monopoles has emerged has not deterred theoretical physics from continuing to publish thousands of books and papers on the subject. Nowadays, one can always use spontaneous symmetry breaking as a convenient excuse for the non-existence of theoretically-predicted phenomena in experiments: One simply postulates that the energy level of spontaneous symmetry breaking for the phenomenon in question is many orders of magnitude beyond any reasonable experimental bounds (e.g., 10^{15} or 10^{19} GeV vs. 10^{3} GeV for terrestrial particle colliders, 10^{6} GeV for ultra-high energy cosmic rays). This usually means that everything unobservable on Earth probably existed in the first nanosecond of the Big Bang when the universe was that energetic.

We shall err on the side of caution in this presentation and take the experimentally established position that the source of all magnetic fields is the relative motion of electric charges, whether that takes the form of the rectilinear motion of charges in currents or the rotational motion of spinning charge distributions. Even then, we must caution the reader that, as Jackson [**6**] pointed out, there does not always exist a rest frame for an observer that makes a given magnetic field disappear. There is certainly such a frame for a single elementary charge moving along a curve in spacetime; one simply chooses a co-moving frame, in which the only field is the electrostatic field. However, as the example of a bar magnet shows, when one is concerned with macroscopic magnetic fields, trying to find a co-moving frame gets hopeless. In fact, there is always the question of whether a measurer/observer that is orbiting around a rotating charge distribution with the same orbital period as the angular period of the rotation will similarly see no magnetic field, since there are fictitious forces that come about due to the orbital motion.

One sees that if one truly believes that the origin of all magnetic forces is the relative motion of electric charges then the existence of magnetic monopoles seems to be a patent absurdity. We shall express this situation topologically by saying that elementary charges represent non-trivial generators of $\pi_2(\Sigma)$ – i.e., monopoles, in the language of topological defects [**7, 8**] – and elementary magnetic sources represent non-trivial generators of $\pi_1(\Sigma)$ – i.e., vortex defects [18]. That is, the spatial sources of magnetic fields will, for us, exist as curves in Σ, not individual points.

One immediately sees that the apparent non-existence of magnetic monopoles makes it harder to define a magnetic field in Σ than it was to define the electric field E. We cannot speak of the magnetic force on a unit magnetic charge if there is no such thing as a magnetic charge. Since magnetic fields exert torques on magnetic dipoles, one might

[18] Although these are also called *string defects*, since the use of the term "string" in theoretical physics is so well-established in the literature of Big-Bang cosmology and life beyond the Planck scale of space, time, and energy, we shall avoid its usage in the context of tangible physics that is easily observed in terrestrial experiments.

define the magnetic field strength $B(x)$ at a point $x \in \Sigma$ to be the torque on a unit dipole. This has the advantage of making B a 2-form on Σ, by its nature. However, a magnetic dipole moment $\boldsymbol{\mu}$ has to have a direction in space, as well as a magnitude, since it is a vector, not a scalar.

If one looks at the way that the torque $\boldsymbol{\tau}$ is coupled to $\mathbf{B}$ and $\boldsymbol{\mu}$, namely:

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B} = *(\boldsymbol{\mu} \wedge \mathbf{B}) \tag{III.43}$$

then one sees that although one cannot simply divide $\boldsymbol{\tau}$ by $\boldsymbol{\mu}$ to get $\mathbf{B}$, nevertheless, in principle, a complete knowledge of the $\boldsymbol{\tau}$ as a function of $\boldsymbol{\mu}$ will give a unique vector $\mathbf{B}$.

This is three-step process for a given "test dipole" $\boldsymbol{\mu}$:

1. Obtain the magnitude B of $\mathbf{B}$ from the maximum value of τ/μ as the direction of $\boldsymbol{\mu}$ varies over all possibilities.
2. Obtain the direction of $\mathbf{B}$ – up to orientation – from the direction of $\boldsymbol{\mu}$ that gives a null to $\boldsymbol{\tau}$, since $\boldsymbol{\mu} \times \mathbf{B} = 0$ iff $\boldsymbol{\mu}$ is parallel to $\mathbf{B}$.
3. Since $\boldsymbol{\tau}$ cannot be parallel to a non-zero $\boldsymbol{\mu}$, the plane of $\boldsymbol{\tau}$–$\boldsymbol{\mu}$ then divides $\mathbb{R}^3$ into two disjoint halves. By choosing one of them to contain the positive normal to that plane, one can define that to also contain the direction of $\mathbf{B}$.

In order for this construction to produce a vector field $\mathbf{B}(x)$ on Σ, one simply understands that the aforementioned process is concerned with tangent vectors at each point of Σ.

We point out that it is generally more traditional to define $\mathbf{B}$ in terms of the Lorentz linear force density $\mathbf{F} = \mathbf{I} \times \mathbf{B}$ that acts on a "test" current vector $\mathbf{I}$, but currents do not seem to suggest the same kind of minimal unit as the Bohr magneton defines for the magnetic dipole moment of the electron.

The total magnetic flux through a 2-chain is then:

$$\Phi_{\mathbf{B}}[c_2] = \int_{c_2} \#\mathbf{B}\,. \tag{III.44}$$

When c_2 is a bounding 2-cycle ∂c_3 Stokes's theorem gives:

$$\Phi_{\mathbf{B}}[\partial c_3] = \int_{c_3} d\,\#\mathbf{B} = \int_{c_3} (\delta\mathbf{B})\,\mathcal{V} = Q_M[c_3]. \tag{III.45}$$

Since this is supposed to equal the total "magnetic charge" that is contained in the region of Σ defined by c_3, to say that there are no magnetic monopoles is to say that $\Phi_{\mathbf{B}}[\partial c_3]$ must vanish for all bounding 2-cycles, which leads to the differential equation:

$$\delta\mathbf{B} = 0. \tag{III.46}$$

Of course, if there are non-bounding 2-cycles z_2, which implies that $H_2(\Sigma; \mathbb{R})$ is non-vanishing, it is entirely possible that $\Phi_{\mathbf{B}}[z_2]$ is non-vanishing even though one still has (III.46). Hence, this is another way of accounting for "magnetic charge without magnetic

charge" by a purely topological device. Furthermore, in the gauge formulation of electromagnetism, a necessary, but not sufficient, condition for the non-triviality of the $U(1)$-principal bundle that defines the gauge structure is that $H_2(\Sigma; \mathbb{R}) \cong H^2_{dR}(M)$ be non-vanishing, since one is concerned with the first Chern class of the bundle, and this will be a closed 2-form; i.e., a de Rham cohomology class in dimension two.

7. Magnetic excitation (induction). When a magnetic material is exposed to a magnetic field, there is a tendency for the elementary magnetic dipoles to align themselves to a greater degree. As opposed to the situation described in the context of electric fields there is no corresponding issue of the translational motion of the elementary magnetic monopoles, only the rotational motion of the dipoles. There is a resulting magnetic dipole moment density associated with the medium, which we refer to as the *magnetic excitation* of the medium, and describe by a 1-form H on Σ. Although the term "magnetic induction" is also used, since that tends to lead to confusion in the context of electromagnetic induction, we shall use the more precise term of "excitation."

The response H of a given medium to an applied **B** field will then be described by a *magnetic constitutive law:*

$$H = \mu^{-1}(\mathbf{B}), \tag{III.47}$$

in which μ: $\Lambda^1 \to \Lambda_1$ is a diffeomorphism of each cotangent space with its corresponding tangent space. Here, we see that classical electromagnetism seemed to be treating the vector field **B** and the 1-form H as having the opposite roles to what later seemed to work better in the eyes of relativistic electromagnetism. Hence, we shall have to use the inverse of the map μ, instead of the map itself.

In the case of the classical electromagnetic vacuum, which is linear, isotropic, and homogeneous with respect to its magnetic response, this relation takes the form:

$$H = \frac{1}{\mu_0}\, i_g^{-1}\mathbf{B}, \qquad (H_i = \frac{1}{\mu_0}\, g_{ij} B^j). \tag{III.48}$$

The constant μ_0 is referred to as the *magnetic permeability* of the vacuum. This, too, is subject to vacuum polarization in the eyes of quantum electrodynamics.

When one integrates the 1-form H along a 1-chain c_1 the resulting number:

$$\mathcal{M}[c_1] = \int_{c_1} H \tag{III.49}$$

is called the *magnetomotive force*, which is also an unfortunate term, since the units of the quantity are work done per unit magnetic charge, not force.

If the 1-chain is a bounding 1-cycle ∂c_2 then Stokes's (or rather, Green's) theorem gives:

$$\mathcal{M}[\partial c_2] = \int_{c_2} dH = \int_{c_2} \#\mathbf{i} = I[c_2], \tag{III.50}$$

in which **i** is the *electric current density* – or electric charge flux density – in that region of Σ so $I[c_2]$ then represents the *total electric current* through c_2. In magnetostatics, this law is referred to as *Ampère's Law.*

If this relationship (III.50) is valid for all bounding 1-cycles then one can deduce the differential equation:

$$dH = \#\mathbf{i}\ . \tag{III.51}$$

If one expresses $H = \#\mathbf{H}$, which would make **H** a bivector field, then this latter equation can be put into the form:

$$\delta\mathbf{H} = \mathbf{i}\ . \tag{III.52}$$

In the macroscopic case, the vector field **i**, which represents the source of the **H** field, commonly takes the form:

$$\mathbf{i} = \rho\,\mathbf{v}, \tag{III.53}$$

in which ρ is the electric charge density and **v** is the velocity of the charge cloud. Both ρ and **v** then have the same support.

The vector field **i**, or equivalently **v**, defines a congruence of integral curves. Outside the support of **i**, the field equations (III.52) become:

$$\delta\mathbf{H} = 0. \tag{III.54}$$

In local form, with $\mathbf{B} = B^i\,\partial_i$, $H = H_i\,dx^i$, (III.51) becomes:

$$H_i = \tilde{\mu}_{ij}\,B^j, \qquad \tfrac{1}{2}(H_{i,j} - H_{j,i}) = -\,\varepsilon_{ijk}\,i^k, \tag{III.55}$$

and (III.52) becomes:

$$H^{ij} = \varepsilon^{ijk}\,H_k, \qquad H^{ki}{}_{,i} = i^k. \tag{III.56}$$

If one thinks of Σ as being the disjoint union of supp(**i**) and its complement $\Sigma' = \Sigma -$ supp(**i**) then the restriction of **H** to Σ′ is a de Rham homology class in dimension two. However, since the integral of #**H** around a 1-cycle in not in Σ′, which bounds in Σ, but Σ′, is non-zero, we must conclude that **H** is divergenceless, but not the divergence of a 3-vector field. This is analogous to the way that Coulomb **D** field (III.32), which is not defined at the origin, has zero divergence, even though its integral over any sphere centered at the origin is Q/ε_0 . In either case, when one removes the source points from Σ, the same 2-cycles that bound in Σ might no longer bound in Σ′, and Stokes's theorem no longer applies.

In the extreme case of a point charge moving along a single differentiable curve, Σ′ will be missing that curve, which might have the effect of introducing a non-trivial generator for the fundamental group of Σ′, as well as $H_1(\Sigma'; \mathbb{R})$. Hence, the source singularity can be regarded as a topological defect in Σ′ of vortex type.

Perhaps the fact that the elementary sources of electric fields are of monopole type and the elementary sources of magnetic fields are of vortex type might shed some light

on the reason for the continued absence of magnetic monopoles from the world of experimental physics.

8. Magnetostatic field equations. We can summarize the field equations of magnetostatics that we have obtained up to this point as differential equations:

$$\delta\mathbf{B} = 0, \qquad dH = \#\mathbf{i}, \qquad H = \mu^{-1}(\mathbf{B}). \tag{III.57}$$

However, for the sake of introducing a vector potential, which we will do in the next section, it is also convenient to give them the form:

$$dB = 0, \qquad \delta\mathbf{H} = \mathbf{i}, \qquad \mathbf{H} = \mu^{-1}(B), \tag{III.58}$$

in which $B = \#\mathbf{B}$ is now a 2-form and $\mathbf{H} = \#^{-1}H$ is a bivector field. The constitutive map $\mu^{-1}\colon \Lambda^2 \to \Lambda_2$, $B \mapsto \mathbf{H}$, is usually represented by the corresponding map of vector fields to 1-forms $\# \cdot \mu^{-1} \cdot \# : \Lambda_1 \to \Lambda^1$, $\mathbf{B} \mapsto H$, which one locally denotes by:

$$H_i = \mu_{ij}B^j. \tag{III.59}$$

Hence, we can represent the local components of the map on 2-forms in the form:

$$\mu^{ijkl} = \varepsilon^{ijm}\varepsilon^{kln}\mu_{mn}\,. \tag{III.60}$$

One can also express the differential equations in integral form:

$$\int_{\partial c_2} H = \int_{c_2} \#\mathbf{i}\,, \qquad \int_{\partial c_3} \#\mathbf{B} = 0\,. \tag{III.61}$$

The second of these admits the topological interpretation that the (singular) 2-cochain $\mathbf{\Phi_B}[.]$, which associates the total magnetic flux through a 2-chain, is a 2-cocycle. In terms of de Rham cohomology, this is equivalent to the statement that the 2-form $B = \#\mathbf{B}$ is closed, which is the first equation in (III.58). When $H_2(\Sigma; \mathbb{R})$ does not vanish, neither does $H^2(\Sigma; \mathbb{R})$ or $H^2_{dR}(\Sigma)$, and it is possible for there to be closed – but non-bounding – surfaces through which the total magnetic flux is non-zero. This is equivalent to the possibility that there are closed 2-forms B that are not exact. We shall return to this in the next section.

The first of equations (III.61) takes on a similar topological interpretation outside the support of $\mathbf{i}$, where the 1-cochain $\mathcal{M}[.]$ vanishes on every bounding 1-cycle. Hence, it too becomes a 1-cocycle, and the 1-form H becomes a closed 1-form; i.e., a de Rham cohomology class in dimension one. However, since we have changed the topology of Σ by deleting supp($\mathbf{i}$), we see that $H \in H^1_{dR}(\Sigma')$, and if the deletion of supp($\mathbf{i}$) renders this cohomology vector space non-trivial then the fact that H is closed but not exact implies

that $\mathcal{M}[z_1]$ is non-vanishing. Although Stokes's theorem does not apply when z_1 is not a boundary, one can treat the non-vanishing of $\mathcal{M}[z_1]$ as a topological source for the field H that surrogates for the missing current **i**.

9. Magnetostatic potential 1-forms. Now that we have drawn attention to the possible non-exactness of the closed 2-form B, let us consider the case in which B is exact. Recall that from the Poincaré lemma this is always possible locally about any point of Σ. Hence, we are assuming that there is a 1-form A such that:

$$B = \#\mathbf{B} = dA. \tag{III.62}$$

Such a 1-form is called a *potential 1-form* for B, or more imprecisely, a *vector potential*. It is clearly not uniquely defined since the addition of any closed 1-form α to A will produce the same B. This defines an equivalence relation on 1-forms, which one calls *gauge equivalence*, by the requirement that gauge-equivalent 1-forms A and A' must differ by a closed 1-form:

$$A' - A = \alpha \in Z^1(\Sigma). \tag{III.63}$$

If one is dealing with α locally, or if Σ is simply connected, then α is also exact – say, $\alpha = d\lambda$ – and one obtains the conventional form of a gauge transformation (of the second kind):

$$A \mapsto A + d\lambda\,. \tag{III.64}$$

From Stokes's theorem, the integral of A around a bounding 1-cycle ∂c_2 is:

$$\int_{\partial c_2} A = \int_{c_2} dA = \int_{c_2} B = \Phi_{\mathbf{B}}[c_2]; \tag{III.65}$$

i.e., it gives the total magnetic flux through the 2-chain.

When B is expressed in terms of a potential 1-form A, one can consolidate the field equations (III.58) into a single second-order equation:

$$(\delta \cdot \mu^{-1} \cdot d)A = \mathbf{i}\,. \tag{III.66}$$

If the action of μ^{-1} on 2-forms is locally expressed by the action of a matrix of functions of the form $\mu^{ijkl}(x, B)$ then (III.66) can be given the local form:

$$\tilde{\mu}^{ijkl}\frac{\partial^2 A_k}{\partial x^i \partial x^j} + \frac{\partial \mu^{ijkl}}{\partial x^i}\frac{\partial A_k}{\partial x^j} = i^l, \tag{III.67}$$

in which we have introduced:

$$\tilde{\mu}^{ijkl} = \mu^{ijkl} + \frac{\partial \mu^{ijkl}}{\partial B_{rs}} A_{r,s}\,. \tag{III.68}$$

When the medium is linear, homogeneous, and isotropic, such as the classical magnetic vacuum, one can express μ_{mn} as $1/\mu_0\ \delta_{mn}$, and (III.60) becomes:

$$\mu^{ijkl} = \frac{1}{\mu_0}\,\varepsilon^{ijm}\varepsilon^{klm} = \frac{1}{2\mu_0}\,(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}), \tag{III.69}$$

while $\tilde{\mu}^{ijkl} = \mu^{ijkl}$, $\mu^{ijkl}{}_{,i} = 0$, which makes (III.66) take the form:

$$\delta dA = \mu_0\,\mathbf{i}\,. \tag{III.70}$$

and (III.67) takes the form:

$$\delta^{ij}\,\frac{\partial^2 A^k}{\partial x^i \partial x^j} = \mu_0\, i^k. \tag{III.71}$$

We have implicitly raised the index of A_k by means of the Euclidian spatial metric that is naturally associated with any linear, isotropic (but not necessarily homogeneous) magnetic constitutive law.

When magnetostatics has the luxury of a spatial metric at its disposal – or, at least, a Hodge * operator that is defined in all dimensions of Λ^k – one can choose the gauge potential A to be one that has vanishing codifferential:

$$\delta A = 0, \tag{III.72}$$

which allows one to write (III.70) in the form of Poisson's equation for A:

$$\Delta A = \mu_0\, i\ ; \tag{III.73}$$

now, we have lowered the index on the source current vector field **i** in order to make it a 1-form.

Of course, in pre-metric magnetostatics, one must simply deal with (III.70) directly.

10. Field-source duality [3]. There has been a recurring theme in the foregoing presentation: As a general rule, the most elementary fields are not defined at their sources, and, as a result, one must deal with the topology of the space that is complementary to the region in which the source is defined. Frequently, this deletion will render the applicability of Gauss's theorem invalid, and one will be dealing with non-zero fluxes over k-cycles that are associated with vector fields that nonetheless have zero divergence.

For instance, Coulomb's law of electrostatics breaks down at the origin – if that is where the point charge is located. However, one notices that there is a further equivalence between the field that is exterior to a spherically-symmetric extended charge distribution and that of a point charge. Since a ball is contractible to a point, one sees that in the eyes of homotopy it is only the absence of a single point from the space in which the field exists that dictates the appearance of the field.

One can interpret the deletion of a point from a closed n-ball $\bar{B}(x;r)$ in a topological space Σ from either the standpoint of homotopy or homology. In the context of homotopy, the absence of a point from $\bar{B}(x;r)$ means that its boundary n–1-sphere is (probably) no longer homotopic to a point. Hence, there is at least one non-trivial generator to $\pi_{n-1}(\Sigma)$, depending upon what is happening with the topology of Σ, more globally. In the context of homology, it means that the boundary n–1-cycle is (probably) not the boundary of an n-chain, which implies that there is a non-trivial generator for $H_{n-1}(\Sigma; \mathbb{R})$.

Actually, when one goes to the case of one-dimensional field sources, such as line charges and currents in (non-closed) infinite wires, one finds that, up to homotopy, deleting a straight line from $\mathbb{R}^n$ is the same thing as deleting a point from $\mathbb{R}^{n-1}$. That is because the line being deleted is contractible to a point by a contraction that takes the rest of $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$ minus the point that the line turns into. For instance, $\mathbb{R}^3$ minus the z-axis contracts to $\mathbb{R}^2$ minus the origin, and $\mathbb{R}^2$ minus the y-axis contracts to two non-zero points on the x axis.

Of course, $\mathbb{R}^n$ minus a circle is an entirely different matter. When $n = 2$, one sees that a plane minus a circle is homeomorphic to the disjoint union of an open disc and a plane minus a closed disc, which is then homotopically equivalent, by contraction, to the disjoint union of a point and a circle. When $n = 3$, one already sees that the resulting space is rather homotopically complicated. In addition to contractible loops, there are also non-contractible ones that encircle the missing circle, but one cannot retract the space to the circle without deleting – say – the z-axis, which must then be added as a "line at infinity."

When one deletes $\mathbb{R}^2$ from $\mathbb{R}^n$ the resulting space is homotopically equivalent to $\mathbb{R}^{n-2}$ minus a point. For instance, $\mathbb{R}^3$ minus a plane is homotopically equivalent to two points. This is useful in approaching the electric field of a charge distribution on an infinite plane.

Similarly to the situation in the second-to-last paragraph, the deletion of a 2-sphere – i.e., a basic 2-cycle – from $\mathbb{R}^n$ is not generally equivalent to the deletion of a plane. For instance, in the case of $\mathbb{R}^3$, the resulting space is contractible to the disjoint union of a point and a 2-sphere, which is reminiscent of the deletion of a circle from a plane, only with the next higher dimension of sphere. More generally, deleting an n-sphere from $\mathbb{R}^{n+1}$ will produce a space that is homotopically equivalent to a point and an n-sphere.

Since a point is a 0-cycle, we begin to see that the sources of electric and magnetic fields seem to take the form of a finite set $\{z_i, i = 1, \ldots, N\}$ of singular cycles of varying dimensions. In the most elementary case, this is a finite set of points. Furthermore, we can associate a number Q_i with each cycle that represents an electric charge in the case of

an electrostatic field and an electric current in the case of a magnetostatic field. Hence, one can form the linear combination:

$$Q = \sum_{i=1}^{N} Q_i z_i \,, \tag{III.74}$$

which is then a cycle of mixed dimension, in general, and we call this cycle the *source complex*. For instance, when one has a set of N charged points it will be a 0-cycle.

Hence, we shall always think of the field – say, **D** – that is produced by any source complex as being defined in the space $\Sigma - Q$ that is complementary to (the carrier of) Q in Σ. When we take the total flux $\Phi_{\mathbf{D}}[z_{n-1}]$ of **D** over an n–1-cycle z_{n-1} that bounds in Σ, but not in $\Sigma - Q$, we shall *define* the value of $\Phi_{\mathbf{D}}[z_{n-1}]$ to be Q, even though Gauss's theorem is in applicable in $\Sigma - Q$.

In fact, that is exactly what happens in the case of Coulomb's law: the field **D** is not defined at the point charge Q, but the total flux over any sphere that includes it in its interior equals Q, even though **D** has zero divergence. Since the total flux functional is a singular cocycle, in the event that **D** has vanishing divergence we shall call this coupling of the homological information in the source complex to the cohomological information in the field space *field-source duality.*

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Chapter IV

Dynamic electromagnetic fields

Much of the modern fascination, if not obsession, with the unification of the theories of the fundamental forces of nature is probably traceable to the stellar success of the unification of the theories of electricity and magnetism into a single theory of electromagnetism, which was not only capable of explaining both theories as they existed independently up to that point in history, but was also capable of explaining natural phenomena that had not been recognized as being explainable in electromagnetic terms, namely, optical phenomena.

Here, we see a perfect example of the difference between a unification of theories and a mere "concatenation" of them. Generally, the unification of theories that pertain to phenomena that do not seem to be otherwise related will produce unpredicted consequences due to unexplored interactions between the individual theories. In general, one calls these interactions *inductions,* because the state of one field is inducing a response in the state of the other field.

1. Electromagnetic induction. One can distinguish static inductions from dynamic ones by the constraint that with a static induction the field itself induces a response, whereas with a dynamic response, it is one of the derivatives of the field that is provoking the response. For instance, a time-varying magnetic field induces a time-varying electric field. However, although it is true that a time-constant electric current will induce a time-constant magnetic field, nevertheless the converse statement is not true.

a. Faraday's law of induction [**1**]. Historically, the first dynamical coupling of electric and magnetic fields to be discovered was the coupling of time-varying magnetic fields to time-varying electric fields. More precisely, the first manifestation of that phenomenon was the coupling of time-varying magnetic fields to electric *currents* that was observed by Michael Faraday. However, if one regards the current in a conductor as being a response to the imposition of an electric field on the medium then one sees that it is more intrinsic to look at the coupling of time-varying magnetic fields and electric fields.

One first encounters Faraday's law of induction in the form of the coupling of time-varying magnetic *flux* to an induced *electromotive force:*

$$\mathcal{E}[\partial c_2] = -\frac{d\Phi_{\mathbf{B}}[c_2]}{d\tau}. \tag{IV.1}$$

In words, this says: When a loop ∂c_2 bounds a 2-chain c_2, the time-variation of the magnetic flux that links the 2-chain will induce an electromotive force in the loop that is 180 degrees out of phase with the time-variation in the magnetic flux.

When expressed in integral form, (IV.1) becomes:

$$\int_{\partial c_2} E = -\frac{d}{d\tau}\int_{c_2} \#\mathbf{B}\,, \tag{IV.2}$$

and an application of Stokes's theorem makes this take the form:

$$\int_{c_2} dE = -\frac{d}{d\tau}\int_{c_2} \#\mathbf{B}\,. \tag{IV.3}$$

Of course, when phrased in the manifestly topological form (IV.1), one notices certain features that are never discussed in a first exposure to Faraday's law:

1. The loop has to be a bounding 1-cycle. This leaves open the possibility that when the spatial manifold Σ is not simply connected there might non-zero emf's in loops that do not bound.

2. In some sense, we are defining a "differentiable curve" in the infinite-dimensional vector space $Z^2(\Sigma; \mathbb{R})$ that the 2-cocyle $\Phi_{\mathbf{B}}[.]$ lives in, although we have not said anything up till now about putting a topology or differential structure on it.

3. We now have a good example of the non-conservation of energy in an electromagnetic system that has nothing to do with dissipation in the thermodynamic sense. Basically, the reason that E cannot be exact anymore is because it is not closed and the non-vanishing of dE is due to the time-variation of $\Phi_{\mathbf{B}}[.]$. Hence, energy is being added to the loop in such a manner that the work done on a unit charge as it goes around the loop does not vanish.

4. If E were the covelocity 1-form for a fluid flow then one would say that the time-variation of $\Phi_{\mathbf{B}}[.]$ is acting like a torque that induces vorticity in the flow around the loop ∂c_2.

We see that what (IV.1) accomplishes is to effectively generalize our previous law of electrostatic fields, which becomes the limiting form of Faraday's law when the magnetic flux is constant in time.

Of course, one always learns that the minus sign in Faraday's law is a law unto itself, namely, *Lenz's law.* One can think of the induced emf as a sort of "damping" term that shows up to oppose the changes that create it, just as viscous damping in a fluid medium acts to oppose the change in position of an object immersed in the medium.

Even though Faraday's law is usually applied to sinusoidally changing magnetic fluxes, it works just fine for linearly and exponentially changing fluxes, as well. In particular, a linearly growing/decaying magnetic flux induces a constant emf, while an exponentially growing/decaying flux induces an exponentially growing/decaying emf.

Due to the derivative coupling, the actual magnitude of the change in flux does not have to be large to produce potentially enormous emf's. Indeed, a serious issue in most practical situations is what happens as the change in flux approaches a jump discontinuity, such as a switching transient, for which the magnitude of the derivative can grow quite large, even though the current itself is small.

Now suppose that the induced emf in ∂c_2 induces a current $I[\partial c_2]$, which is coupled to an induced magnetic flux $\Phi_{\mathbf{B}}[c_2]$ in the 2-chain that is bounded by the current loop. If the coupling of current in the loop to the resulting magnetic flux in the 2-chain it bounds is simply:

$$\Phi_{\mathbf{B}}[c_2] = L\, I[\partial c_2] \tag{IV.4}$$

in which the constant L is the *self-inductance* of the loop, then we can write Faraday's law as:

$$\mathcal{E}[\partial c_2] = -\, L\, \frac{d}{d\tau} I[\partial c_2]\,. \tag{IV.5}$$

Notice that now the coupling takes place solely on the bounding 2-cycle.

Part of the subtle profundity in Faraday's law is based in the fact the emf that is induced in a loop does not depend upon it being composed of a conducting medium, and, in fact, emf's can be induced in a vacuum, just the same. Of course, in that event one usually thinks in terms of induced E fields directly. Hence, we need to examine the process of going from the integral form of Faraday's law to its differential form.

Since the magnetic flux integral can be regarded as the evaluation $<\Phi_{\mathbf{B}}, c_2>$ of a linear functional on a vector – or rather, the evaluation $<\#\mathbf{B}, c_2>$ of a 2-form on a 2-chain – if one wishes to make the resulting number time-varying then one can make both $\mathbf{B}$ and c_2 time-varying, in their own right. However, since allowing c_2 to vary in time is more interesting to engineering applications, such as motors and generators, we shall simply consider the possibility that only $\mathbf{B} = \mathbf{B}(\tau, x^i)$ is a (differentiable) function of time.

This makes the magnetic flux derivative take the form:

$$\frac{d\Phi_{\mathbf{B}}[c_2]}{d\tau} = \int_{c_2} \frac{\partial(\#\mathbf{B})}{\partial\tau}\,. \tag{IV.6}$$

We can now write Faraday's law (IV.3) in the form:

$$\int_{c_2} [dE + \partial_\tau(\#\mathbf{B})] = 0, \tag{IV.7}$$

and if this is true for any possible 2-chain c_2 then one obtains the differential equation:

$$dE + \partial_\tau(\#\mathbf{B}) = 0. \tag{IV.8}$$

Of course, this means that we are regarding both E and $\mathbf{B}$ as time-varying fields, now.

b. Maxwell's law of induction [**1**]. It was Maxwell who put the capstone on the classical theory of electromagnetism by the intuitive leap of assuming that there is a partial converse to Faraday's law of induction that takes the form of saying that a time-varying electric flux through a 2-chain c_2 will induce a magnetomotive force in its boundary loop. However, this time the induction is in phase:

$$\mathcal{M}[\partial c_2] = +\frac{d\Phi_{\mathbf{D}}[c_2]}{d\tau}\,. \tag{IV.9}$$

In integral form, this is:

$$\int_{\partial c_2} H = +\frac{d}{d\tau}\int_{c_2} \#\mathbf{D}\,, \tag{IV.10}$$

and by an application of Stokes's theorem, this becomes:

$$\int_{c_2} dH = +\frac{d}{d\tau}\int_{c_2} \#\mathbf{D}\,. \tag{IV.11}$$

If (IV.10) is to serve as a generalization of Ampère's law (III.49) then we must add a contribution to the right-hand side that accounts for the current that is also producing the field:

$$\mathcal{M}[\partial c_2] = +\frac{d\Phi_{\mathbf{D}}[c_2]}{d\tau} + I[c_2], \tag{IV.12}$$

in which:

$$I[c_2] = \int_{c_2} \#\mathbf{i} \tag{IV.13}$$

is the total electric current through c_2 . Keep in mind that the actual support of the electric current density **i** will generally be different from c_2 , so the integral will involve only the intersection of those two sets.

Once again, if we consider only the possibility that **D** is time varying, but not c_2 itself, this leads to the differential equation:

$$dH - \partial_\tau \#\mathbf{D} = \#\mathbf{i}\,. \tag{IV.14}$$

It was this reciprocity in the coupling between time-varying electric fields and time-varying magnetic fields that eventually led to the possibility of wavelike solutions to the electromagnetic field equations, and the wave theory of optics.

2. Conservation of charge [**1**]**.** If we wish to go beyond the static perspective on electric and magnetic fields, we need to also go beyond the static perspective on their sources. Of course, we have actually made a first step in this direction by pointing out that electric currents already represent electric charges in a state of relative motion, but in order to produce static magnetic fields it was necessary to consider only time-invariant currents.

Now we shall consider the situation in its full generality, and assume that we have a time-varying electric charge density $\rho = \rho(\tau, x)$ that is defined on the spatial manifold Σ, or really, on $\mathbb{R} \times \Sigma$. On the support $\mathrm{supp}(\rho)$ of this density, we also have a time-varying vector field $\mathbf{v} = \mathbf{v}(\tau, x)$ that is defined, and which represents the flow velocity vector field of the charge distribution.

Together, these two data define another time-varying vector field:

$$\mathbf{i} = \rho\mathbf{v}, \tag{IV.15}$$

whose support is clearly the same as that of ρ and **i**. We call this vector field the *electric current density* for the distribution ρ.

The sense in which we think of the charge as being carried along by the motion defined by the vector field **v** – i.e., its congruence of integral curves – is defined by looking at the time rate of change for the total charge $Q[c_3]$ that is contained in a spatial 3-chain c_3. (By "spatial," we mean it is a 3-chain in $\tau_0 \times \Sigma$ for some fixed value τ_0 of τ.):

$$\frac{dQ[c_3]}{d\tau} = \frac{d}{d\tau}\int_{c_3} \#\rho = \int_{c_3}\left(\frac{\partial\rho}{\partial\tau}\right)\mathcal{V}\,. \tag{IV.16}$$

We couple this to **i** by the assumption that the physical meaning of the total flux of **i** through the boundary of c_3:

$$\Phi_{\mathbf{i}}[c_3] = \int_{\partial c_3} \#\mathbf{i} \tag{IV.17}$$

is the time rate at which charge is flowing out of the region described by c_3:

$$\frac{dQ[c_3]}{d\tau} = -\,\Phi_{\mathbf{i}}[\partial c_3]. \tag{IV.18}$$

This coupling of total charge with total charge current through the boundary is the most fundamental form of the *law of charge conservation*.

In integral form, this is:

$$\int_{c_3}\left(\frac{\partial\rho}{\partial\tau}\right)\mathcal{V} = -\int_{\partial c_3}\#\mathbf{i}\,, \tag{IV.19}$$

and an application of Stokes's theorem makes this become:

$$\int_{c_3}\left(\frac{\partial\rho}{\partial\tau} + \delta\mathbf{i}\right)\mathcal{V} = 0, \tag{IV.20}$$

after a few self-evident manipulations.

If (IV.20) is to be true for all possible spatial 3-chains then one must have the validity of the following differential equation:

$$\frac{\partial\rho}{\partial\tau} + \delta\mathbf{i} = 0. \tag{IV.21}$$

Hence, the differential form of (IV.18) amounts to the statement that the divergence of the vector field **i**, when restricted to the boundary of a 3-chain, is minus the time rate of change of the electric charge density at each point.

In local form, with $\mathbf{i} = \rho v^i \partial_i$, one has:

$$\frac{\partial\rho}{\partial\tau} + \frac{\partial(\rho v^i)}{\partial x^i} = 0. \tag{IV.22}$$

3. Pre-metric Maxwell equations. Let us summarize the equations that we have accumulated for the fields E, $\mathbf{D}$, H, and $\mathbf{B}$. In integral form, they are:

Gauss's law for $\mathbf{D}$:	$\Phi_{\mathbf{D}}[\partial c_3] = Q[c_3]$,	(IV.23a)
Gauss's law for $\mathbf{B}$:	$\Phi_{\mathbf{B}}[\partial c_3] = 0$,	(IV.23b)
Faraday's law of induction:	$\mathcal{E}[\partial c_2] = -\partial_\tau \Phi_{\mathbf{B}}[c_2]$,	(IV.23c)
Maxwell's law of induction:	$\mathcal{M}[\partial c_2] = +\partial_\tau \Phi_{\mathbf{D}}[c_2] + I[c_2]$,	(IV.23d)

and in differential form, they are:

Gauss's law for $\mathbf{D}$:	$\delta\mathbf{D} = \rho$,	(IV.24a)
Gauss's law for $\mathbf{B}$:	$\delta\mathbf{B} = 0$,	(IV.24b)
Faraday's law of induction:	$dE + \partial_\tau \# \mathbf{B} = 0$,	(IV.24c)
Maxwell's law of induction:	$dH - \partial_\tau \# \mathbf{D} = \# \mathbf{i}$.	(IV.24d)

Furthermore, we must add the constitutive laws that we have been using, so far, namely:

$$\mathbf{D} = \varepsilon(E), \qquad H = \mu(\mathbf{B}). \tag{IV.25}$$

Collectively, (IV.24a-d) represent the formulation of Maxwell's equations in terms of differential forms, at least in the eyes of non-relativistic physics. The first two equations basically express the divergences of $\mathbf{D}$ and $\mathbf{B}$, while the last two equations express their curls, if one takes into account the constitutive laws (IV.25) that couple them.

One of the most important advances to these classical equations came from the work of Lorentz, Poincaré, Minkowski, and others to give these non-relativistic equations a relativistic form (as one might find in [**2-8**]). Basically, one must replace the four-dimensional time + space manifold $\mathbb{R} \times \Sigma$, in which the time dimension really represents the proper time parameter τ that would be measured by a particular measurer/observer, with a more general four-dimensional spacetime manifold M. Now, the time dimension only shows up as one of the coordinates x^μ in a choice of coordinate chart (U, x^μ) about each point of M.

It was eventually established that the most intuitive and computationally useful formulation of Maxwell's equations on M came about by consolidating E and $\mathbf{B}$ together into a single 2-form on M:

$$F = dt \wedge E - \# \mathbf{B}, \tag{IV.26}$$

while consolidating D and $\mathbf{H}$ into the bivector field:

$$\mathfrak{H} = \partial_t \wedge \mathbf{H} + \#^{-1} D \, . \tag{IV.27}$$

We relate $\mathfrak{H}$ to F by a more general constitutive law than (IV.25):

$$\mathfrak{H} = \kappa(F) \, . \tag{IV.28}$$

We shall discuss the nature of four-dimensional constitutive laws in the next chapter, but, for now, we just treat it as a diffeomorphism of each fiber of $\Lambda^2(M)$ at each $x \in M$ with the corresponding fiber of $\Lambda_2(M)$ at the same point.

Furthermore, we consolidate the sources ρ and **i** into a single four-dimensional current vector:

$$\mathbf{J} = \rho\, \partial_t + \mathbf{i}\,. \tag{IV.29}$$

With these consolidations, the Maxwell equations can be expressed in the form:

$$dF = 0, \qquad \delta\mathfrak{h} = \mathbf{J}, \qquad \mathfrak{h} = \kappa(F). \tag{IV.30}$$

One sees that conservation of charge follows as an unavoidable consequence:

$$\delta\mathbf{J} = 0. \tag{IV.31}$$

One can also treat this equation as an integrability – or compatibility – condition for a given electric current four-vector field **J** to be the source of a bivector field **H**.

If one expresses the 2-form F, the bivector field $\mathfrak{h}$, and the electric current **J** in local form as:

$$F = \tfrac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu, \qquad \mathfrak{h} = \tfrac{1}{2} H^{\mu\nu}\, \partial_\mu \wedge \partial_\nu, \qquad \mathbf{J} = J^\mu\, \partial_\mu \tag{IV.32}$$

then Maxwell's equations take on their familiar local form:

$$F_{\mu\nu,\kappa} + F_{\nu\kappa,\mu} + F_{\kappa\mu,\nu} = 0, \qquad H^{\mu\nu}{}_{,\nu} = J^\mu, \qquad H^{\mu\nu} = \kappa(F_{\mu\nu}). \tag{IV.33}$$

When the constitutive map k is linear, the last equation can be expressed in local form as:

$$H^{\mu\nu} = \kappa^{\mu\nu\alpha\beta} F_{\alpha\beta}\,. \tag{IV.34}$$

We point out that at no point has it been necessary to introduce a metric of any signature in order to define the system of equations (IV.30), only a volume element. The usual role of the Lorentzian metric g in the formulation of Maxwell's equations in terms of differential forms is purely confined to the definition of the Hodge duality isomorphism *, and only for 2-forms, moreover.

In order to see that we have essentially replaced the necessity for g by means of the constitutive map κ; we first rephrase the middle equation in (IV.30) as:

$$d\#\kappa(F) = \#\mathbf{J}\,. \tag{IV.35}$$

If we consider the case of a linear map κ and define *: $\Lambda^2 M \to \Lambda^2 M$, $F \mapsto \#\kappa(F)$ then we see that * is an isomorphism of 2-forms with 2-forms. We can then put (IV.35) into the form:

$$d{*}F = \#\mathbf{J}. \tag{IV.36}$$

This is consistent with the metric form of Maxwell's equations that gets presented in most texts on relativistic electrodynamics, as long as our isomorphism * has one of the key properties of the Hodge isomorphism, at least, as it acts on 2-forms on a four-dimensional Lorentzian manifold, namely:

$$*^2 = -I, \tag{IV.37}$$

which actually means that the * isomorphism defines an "almost-complex structure" on the vector bundle $\Lambda^2 M$. We shall have more to say about this later, but, for now, we point out that generally all that one can assume is that:

$$*^2 = \#\kappa\#\kappa. \tag{IV.38}$$

The integral form of Maxwell's equations is then:

$$\Phi_F[\partial c_3] = 0, \quad \Phi_{\mathfrak{h}}[\partial c_3] = \Phi_{\mathbf{J}}[c_3], \qquad \mathfrak{h} = \kappa(F). \tag{IV.39}$$

Once again, we see that we can give topological interpretations to the field equations of electromagnetism (IV.30), just as we did for the static field equations of electric and magnetic fields individually. Now, we see that F must be a de Rham cocycle in dimension 2, while $\mathfrak{h}$, outside the support of the source current $\mathbf{J}$, is a de Rham cycle in dimension two.

4. Electromagnetic potential 1-forms. The first of Maxwell's equations – viz., $dF = 0$ – takes the form of saying that the 2-form F is closed. Depending upon whether $H^2_{dR}(M)$ is trivial or not, F might be globally exact or locally exact, resp. That is, there will be a 1-form A such that:

$$F = dA. \tag{IV.40}$$

One calls this 1-form an *electromagnetic potential 1-form*, or a choice of *gauge* for the electromagnetic field F.

a. Gauge equivalence. As before, A is not unique because any other 1-form A' that differs from A by a closed 1-form z^1 will give the same 2-form F under exterior differentiation. The relation $A \sim A'$ iff $dA = dA'$ iff $A - A' \in Z^1_{dR}(M)$ is an equivalence relation that is usually called *gauge equivalence*. We denote the set of all gauge equivalent 1-forms (mod F) by $\mathcal{A}(M; F)$.

Note that two cohomologous closed 2-forms F and F' will generally differ by an exact 2-form $d\alpha$; hence, if $F = dA$ and $F' = dA'$ then:

$$F - F' = d(A - A') = d\alpha \neq 0. \tag{IV.41}$$

Since α is not closed A and A' cannot be gauge equivalent. That is, cohomologous 2-forms will not generally have gauge-equivalent potential 1-forms.

It is not true that if $A \sim A'$ then any linear combination $\alpha A + \beta A'$ will be an element of the same gauge equivalence class as A and A', since:

$$d(\alpha A + \beta A') = (\alpha + \beta)\, F, \tag{IV.42}$$

which is not generally equal to F. Hence, $\mathcal{A}(M; F)$ is not a vector space. However, the fact that the expression $A - A' \in Z^1_{dR}(M)$ is well-defined implies that it is an *affine* space that is modeled on the vector space $Z^1_{dR}(M)$. The fact that it is infinite-dimensional follows from the fact that at long as M has finite-dimensional de Rham cohomology in dimension one the dimension of $Z^1_{dR}(M)$ is the same as the dimension of $B^1_{dR}(M)$, and since each of its elements take the form $d\lambda$ for some smooth function λ, the dimension of $B^1_{dR}(M)$ is the same as the dimension of $C^\infty(M; \mathbb{R})$, which is denumerably or non-denumerably infinite depending upon whether M is compact or not, respectively.

Since gauge equivalence takes the local form $A - A' = d\lambda$, one can think of this as defining a *gauge transformation (of the second kind)* by the replacement $A \to A + d\lambda$. In order to define the gauge transformations of the first kind, one must express λ in the form $\lambda = \ln g$, which makes:

$$d\lambda = g^{-1}\, dg\,. \tag{IV.43}$$

Since multiplication by real numbers is commutative, one can express the gauge transformation of the second kind as the replacement of A with:

$$A' = g^{-1}Ag + g^{-1}\, dg. \tag{IV.44}$$

although the excess complexity in the expression is, of course, unnecessary for the Abelian case at hand. It does, however, become non-trivial for non-Abelian gauge theories.

c. Field equations in terms of a potential. If $F = dA$ then the first of equations (IV.30) becomes an identity. The constitutive law makes $\mathfrak{h} = \kappa(dA)$, and this puts the non-trivial differential equation of the set into the form:

$$\Box_\kappa A = (\delta \cdot \kappa \cdot d)A = \mathbf{J}. \tag{IV.45}$$

This means that the operator $\Box_\kappa$: $\Lambda^1 \to \Lambda_1$ plays the role of a "pre-metric d'Alembertian." Ordinarily – i.e., in the metric theory, which provides a Hodge * isomorphism in all dimensions, not just dimension two – we could say that this operator *equals* the d'Alembertian operator by the Lorentz choice of gauge for A, namely $\delta A = 0$. However, in the absence of a metric, we cannot generally define the codifferential operator for δA to make any sense.

If we represent A in a local coordinate chart as $A_\mu\, dx^\mu$, $\mathbf{J}$ as $J^\mu\, \partial_\mu$, and κ by means of functions $\kappa^{\mu\nu\alpha\beta}(x, F)$ then the field equation (IV.45) takes the local form:

$$\frac{\partial}{\partial x^\mu}\left(\kappa^{\mu\nu\alpha\beta}\, \frac{\partial A_\alpha}{\partial x^\beta} \right) = J^\nu, \tag{IV.46}$$

and, upon carrying out the partial derivatives, it becomes:

$$\tilde{\kappa}^{\mu\nu\alpha\beta}\, \frac{\partial^2 A_\alpha}{\partial x^\mu \partial x^\beta} + \frac{\partial \kappa^{\mu\nu\alpha\beta}}{\partial x^\mu}\, \frac{\partial A_\alpha}{\partial x^\beta} = J^\nu, \tag{IV.47}$$

with:

$$\tilde{\kappa}^{\mu\nu\alpha\beta} \equiv \kappa^{\mu\nu\alpha\beta} + \frac{\partial \kappa^{\mu\nu\alpha\beta}}{\partial F_{\kappa\lambda}}\, \frac{\partial A_\kappa}{\partial x^\lambda}\,. \tag{IV.48}$$

c. U(1) gauge structures [**9**, **10**]. Because the expression (IV.44) looks so formally similar to the way that the local representative of a connection 1-form A on a $U(1)$-principal bundle $P \to M$ would transform from one local trivialization over $U \subset M$ to another one over $V \subset M$ that overlaps the first, we briefly summarize the physical roots of the rest of that construction.

An element of the Abelian Lie group $U(1)$ can be represented by a complex number of the form $e^{i\theta}$. Hence, an element of its Abelian Lie algebra will take the form $i\theta$. Our group of gauge transformations of the second kind can then be represented by the group $C^\infty(U, U(1))$ of smooth maps from U to $U(1)$.

A $U(1)$-principal bundle $P \to M$ has a fiber over each point of M that looks like $U(1)$, which is topologically a circle. However, as usual, the diffeomorphism of each fiber with $U(1)$ is not canonical, and one finds that there are generally inequivalent $U(1)$-principal bundles over a given M. In particular, they are not all generally trivial – i.e., equivalent to the projection $M \times U(1) \to M$ – since that is equivalent to saying that there is a global section of the bundle.

The physical meaning of a local section $\phi\colon U \to P$ is that it represents a local choice of $U(1)$ gauge for an electromagnetic field F, which we now assume to be a 2-form on P. By means of ϕ, one can then pull F down to a 2-form $\phi^* F$ on U. If $F = dA$ then if one interprets the 1-form A on P as a $\mathfrak{u}(1)$-connection 1-form then this would make F the curvature 2-form that it defines (since the Lie algebra u(1) is Abelian). The 1-form A also pulls down to a 1-form $\phi^* A$ on U.

If $V \subset M$ is an open subset that overlaps U and supports a local section $\psi\colon V \to P$ then on the overlap $U \cap V$ there is a transition function $g\colon U \cap V \to U(1)$ that allows one to transform from one choice of local gauge to the other by way of:

$$\psi(x) = g(x)\phi(x). \tag{IV.49}$$

This is what one commonly calls a *gauge transformation of the first kind.*

One finds that the pull-down of A transforms to $A' = \psi^* A$ by way of:

$$A' = g^{-1}Ag + g^{-1}\,dg = A + d\lambda, \tag{IV.50}$$

and the curvature 2-form F transforms to:

$$F' = g^{-1}Fg = F. \tag{IV.51}$$

In the eyes of modern differential geometry [**11**], this implies that it is possible to interpret electromagnetism as involving basic constructions that are consistent with the geometry of a $U(1)$-principle bundle over spacetime that has been given a choice of connection. In the vocabulary of modern theoretical physics, one calls such a theory a $U(1)$ *gauge theory* of the interaction in question.

We pointed out that the triviality of a G-principal bundle $P \to M$ is equivalent to the existence of a global section of the fibration. The branch of topology called obstruction theory [**12**, **13**] gives the obstruction to the extension of a local section to a global one in terms of the non-vanishing of a certain cocycle called the *primary obstruction cocycle* whose dimension is one more than the dimension of the first non-vanishing homotopy group $\pi_k(G)$ of the fiber – namely, G – and whose coefficient group is $\pi_k(G)$. In the case of $G = U(1)$ the first, and only, non-vanishing homotopy group is $\pi_1(U(1)) = \pi_1(S^1) = \mathbb{Z}$, so the primary obstruction cocycle for a $U(1)$-principal bundle over a manifold M is a cocycle $c_1 \in H^2(M; \mathbb{Z})$ that one calls the *first Chern class* for P. The gist of the *Chern-Weil homomorphism* is that this integer cocycle can be represented in de Rham cohomology by the 2-form:

$$c_1 = \frac{1}{2\pi} F\,. \tag{IV.52}$$

As it turns out, the choice of $\mathfrak{u}(1)$-connection 1-form A is irrelevant, because any other possible choice A' would give a curvature F' that is cohomologous to F.

In the case of $M = S^2$, the issue of triviality leads to the consideration of Dirac monopoles, although not in the form that Dirac described. Suppose one has a a $U(1)$ principal bundle $P \to S^2$. Basically, one cannot cover the 2-sphere with a single local trivialization of P, but must cover it with two overlapping hemispheres U_N and U_S centered on the North and South pole. One assumes, moreover, that the overlap $U_N \cap U_S$ is homotopic to a circle.

We will call two local sections of the fibration over the two hemispheres ϕ_N and ϕ_S. There must be a transition function $g_{NS}: U_N \cap U_S \to U(1)$ that relates them be way of:

$$\phi_N = g_{NS}\,\phi_S\,. \tag{IV.53}$$

Since $U_N \cap U_S$ is homotopic to S^1, as is $U(1)$, the homotopy classes $[g_{NS}]$ are all elements of $\pi_1(S^1) = \mathbb{Z}$. Hence, there is a denumerable sequence of homotopically inequivalent transition functions, which also implies a corresponding sequence of homotopically inequivalent $U(1)$-principal bundles over S^2. In fact, the integers that index this sequence of bundles can be obtained by a simple integration of c_1 over S^2:

$$c_1[P] = \frac{1}{2\pi}\int_{S^2} F \,. \qquad \text{(IV.54)}$$

This integer $c_1[P]$ is called the *first Chern number* for P, and since it happens to equal to the *Euler-Poincaré characteristic* [19] of S^2 in this case, equation (IV.54) is another form of the *Gauss-Bonnet theorem*, which is what Chern was originally attempting to generalize when he was eventually led to define the characteristic classes that bear his name. For a trivial $U(1)$-principal bundle over S^2 the first Chern class will vanish, so a non-vanishing first Chern number is necessary and sufficient for the non-triviality of such a bundle.

The way that this relates to magnetic monopoles is that the right-hand side of (IV.54) is also proportional to the total magnetic flux though S^2 when one thinks of F as an electromagnetic field strength 2-form, hence, the total magnetic charge contained in it, assuming that S^2 bounds a closed 3-ball.

An interesting point to ponder, since magnetic monopoles have not been experimentally demonstrated, and even seem to contradict one's basic intuition about the nature of magnetic field sources, is whether it is actually necessary that one assume that the gauge transformation $A \mapsto A + d\lambda$ actually has to imply that one dealing with $U(1)$ indeed. Basically, it is only a statement about Lie *algebras*, and there is another Lie *group* that has a Lie algebra that is isomorphic to $\mathbb{R}$, namely, the additive group $(\mathbb{R}, +)$ of real numbers. The usual argument for going the route of $U(1)$ is purely reasoning by analogy with wave mechanics, on the assumption that gauge transformations of the first kind work the same way for potential 1-forms as they do for quantum wave functions. However, if this analogy were weak, and the actual gauge group was $(\mathbb{R}, +)$, not $U(1)$, that would account for the non-existence of magnetic monopoles, since $\mathbb{R}$ is contractible, so all of its homotopy groups vanish, and there is no obstruction to a global section of any $\mathbb{R}$-principal bundle; i.e., they are always trivial. Then again, the experimentally-verified validity of the Bohm-Aharonov effect suggests that $U(1)$ might be the correct choice, after all.

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[19] The Euler-Poincaré characteristic $\chi[M]$ of a topological space is the alternating sum $\sum (-1)^k b_k$ of its Betti numbers b_k, which, in our case, will be the dimensions of the de Rham cohomology vector spaces in each dimension $k = 0, 1, \ldots, \dim(M)$. For a 2-sphere, the only non-vanishing Betti numbers are $b_0 = b_2 = 1$, so $\chi[S^2] = 2$. The fact that $\chi[S^2]$ is non-zero is also the reason that there are no everywhere non-zero vector fields on S^2, which is a special case of the *Poincaré-Hopf* theorem that a differentiable manifold M admits a global non-zero vector field iff $\chi[M] = 0$.

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Chapter V

Electromagnetic constitutive laws

Although the subject of electromagnetic constitutive laws seems to be best established for the response of macroscopic matter to electromagnetic fields, it seems to be gradually emerging from quantum electrodynamics that there is much more internal structure to the electromagnetic vacuum state than can be conveniently described by the constants ε_0 and μ_0 . Since the nature of that state is apparently beyond the limits of direct observation, one must hope that it is permeable to intuitive probes that are based on analogies with more established macroscopic phenomena. Therefore, we shall first discuss some of the issues that pertain to macroscopic matter and then summarize the most established aspects of the quantum electrodynamical vacuum state that suggest analogies with nonlinear optics.

1. Electromagnetic fields in macroscopic matter [1-4]. As pointed out previously, the application of an electromagnetic field F to a macroscopic medium will imply the application of forces and torques to the elementary charges and dipoles – both electric and magnetic – in the medium. The main issue for a given medium is the degree of freedom that those elementary charges and dipoles have in the medium. This, in turn, reverts to the question of how they interact.

Any model for macroscopic matter, such as one finds in condensed matter physics [**5**], is going to be based in some empirical approximation to the interaction of the elementary constituents of the medium. For instance, the ideal gas model assumes that the gas molecules all have zero volume and interact only by elastic collisions. Hence, the limits of the model for the medium are usually defined by the limits of the interaction model for its constituents. One often finds that the nature of phase transitions in such media is closely related to the transition from one interaction model to another. For instance, the ideal gas model begins to break down when the molecules get close enough for inter-atomic forces to take over, which suggests a high gas density, as one would expect during condensation from the gaseous phase to the liquid phase. Similarly, when a gas goes to the plasma state the outer electrons of the gas molecules are no longer bound to the molecules, so the interaction model for the electrons and gaseous ions must change accordingly.

The elementary charges in a macroscopic electromagnetic medium come in two basic forms: electrons and atomic ions. The electrons are free to move in response to applied electromagnetic fields to a greater or lesser degree that mostly determines whether the medium is regarded as a conductor, insulator, or something more exotic, such as a semiconductor. That is, the translational response of atomic electrons to applied electric and magnetic forces will take the form of an electric current that flows in the medium.

Atomic ions in a solid medium are often bound to each other into a crystal lattice, so they do not generally translate very far from their equilibrium positions, but they are still capable of exhibiting collective vibrational modes of response, as well as electronic

energy level transitions. Atomic ions can appear in non-solid media such as electrolytic solutions and in the plasma state.

Elementary electric dipoles are most commonly associated with rather complex molecules, such as one finds in plastic dielectrics, but also in the silicon dioxide that mostly constitutes many optical media, such as quartz and the various glasses. It is no coincidence that optically transparent media are not generally conductors, although we shall not go into the details here.

Elementary magnetic dipoles are mostly associated with the angular momenta, both orbital and intrinsic, of elementary charges. However, in the case of both nuclear and electronic magnetic moments, it is the intrinsic angular momentum – i.e., *spin* – that tends to dominate in macroscopic phenomena over the orbital angular momentum. In particular, the response of most magnetic media to applied magnetic fields is generally due to the degree of freedom that the electron spins have in the medium.

As mentioned above, the electromagnetic properties of macroscopic media tend to be partitioned according to the phase of the medium. This, in turn, is often related to the magnitudes of the field strengths involved. One simply has to consider the possible phase transitions that might come about when field strengths exceed critical limits. For instance, in most optical media, the propagation of waves involves the responses of the atomic electrons to the impinging photons. One can see that there will be various types of photons in that regard: photons that bring about level transitions and result in absorption or stimulated emission of photons, photons of sufficient energy to bring about the ionization of electrons, photons that get Compton scattered by the electrons, etc. Hence, it is clear the optical character of the medium will change dramatically when the energy of the photon – i.e., the field strengths of its electric and magnetic fields – exceeds the ionization limit or has the proper frequency to initiate stimulated emission, as in lasers, or perhaps when it equals a level transition that would cause absorption.

In this chapter, which is more phenomenological than the previous ones, we shall start with the linear electromagnetic media and then go on to the nonlinear ones. From that study, we shall comment upon some of the possible analogies with the quantum electrodynamical vacuum state. In all of the discussion, the primary objective will be to obtain possible forms for the electromagnetic constitutive law that shows up in the pre-metric Maxwell equations.

2. Linear constitutive laws [1-4]. Even though linearity is invariably an approximation in physics, nonetheless, it can also be such a useful approximation to a wide class of natural phenomena – which usually amounts to the realm of weak field strengths in the case of electromagnetism – that we still need to discuss the linear case. Indeed, one finds that the extension to nonlinearity is so fraught with complexity that it is only by starting with the firm intuitive foundation that one gains from the linear phenomena that one can even hope to choose a fruitful direction of mathematical generalization into the domain of nonlinear phenomena.

We shall first deal with the most general case of non-local linear constitutive laws, so that we can give the local laws their proper context. We will then discuss some of the classes of natural phenomena that pertain to the structure of local linear constitutive laws.

We shall also be considering only non-conducting media, so the applied electromagnetic field will not be assumed to produce an electric current in the medium.

a. Non-local linear laws. In its most general form, when an electromagnetic medium exhibits a linear response to an applied electromagnetic field strength $F \in \Lambda^2 M$ the resulting bivector field $\mathfrak{h} \in \Lambda_2 M$ is related to F by an linear operator κ: $\Lambda^2 M \rightarrow \Lambda_2 M$ that is not necessarily local. That is, it does not induce linear maps on the individual fibers of these vector bundles at each point of M. Rather, it can generally be represented as an integral operator:

$$\mathfrak{h}(x) = \int_M K(x, y) \wedge F(y), \tag{V.1}$$

in which K: $M \times M - \Delta \rightarrow \Lambda_2 M \otimes \Lambda^{n-2}\mathrm{M}$, $(x, y) \mapsto K(x, y) \in \Lambda_{2,x} \otimes \Lambda^{n-2}_y$ is a kernel function for the operator K that is not, however, defined on the diagonal Δ of $M \times M$; viz., the set of all (x, x). One way of characterizing the kernel function is to think of it as the *impulse-response* function for the medium. That is, use $F(y) = \delta(y)$ and note that the resulting $\mathfrak{h}(x)$ will equal $K(x, 0)$.

This non-locality takes two basic forms: temporal and spatial.

In the temporal case, one must consider the fact that a real-world medium will not respond to an applied field instantaneously, nor will its response cease immediately after the applied field itself vanishes. There will generally be a certain time delay before the medium begins to respond, as well as a certain decay envelope after the impulse ceases. One must also add causality constraints on the kernel to insure that $\mathfrak{h}$ is not responding to the future state of the field F.

Nonlocality in the spatial case is based on the notion that generally neighboring dipoles, whether electric or magnetic, are not completely independent of each other, but generally have some sort of interaction. Often, this interaction takes the form of essentially a "nearest-neighbor" interaction, as one finds in the Heisenberg ferromagnet. Spatial nonlocality does not involve a causality constraint in the same way as temporal nonlocality, but one might imagine that the state of a given dipole could not depend upon the state of dipoles that are not sufficiently close that electromagnetic waves could pass between them within a lightlike time interval.

If M is a compact region in $\mathbb{R}^4$ and we assume that K is translation-invariant, so it can be expressed in the form $K(x - y)$, then we can take the Fourier transform of (V.1) to get an algebraic equation in frequency-wave number space, which we express in component form for a choice of coordinates on $\mathbb{R}^4$:

$$\mathfrak{h}^{\mu\nu}(\omega, k_i) = \tfrac{1}{2} K^{\mu\nu\alpha\beta}(\omega, k_i)\, F_{ab}(\omega, k_i). \tag{V.2}$$

Hence, we are now dealing with a complex 2-form $F \in \Lambda^2(\mathbb{R}^{4*})$ on frequency-wave number space $\mathbb{R}^{4*}$, a complex bivector field $\mathfrak{h} \in \Lambda_2(\mathbb{R}^{4*})$, and a complex linear map K from one to the other, which is now local. In general, temporal nonlocality will manifest

itself as a frequency dependence of the components of K, and spatial nonlocality will imply wave-number dependence. The appearance of non-zero real parts to the components of K is related to the possibility that the medium might tend to absorb electromagnetic energy.

b. Local linear laws. From the preceding discussion, it should be clear that there is a fundamental approximation associated with passing from nonlocal linear constitutive laws to local linear ones, just as there is a fundamental approximation that is associated with the restriction to linear laws, in the first place. Basically, the approximation is that the temporal impulse response must involve a short time delay and a quick decay afterwards, while the spatial response must be confined to only a sufficiently small neighborhood of each elementary dipole.

In dealing with the general form of the component matrix for a local linear constitutive map κ, it is generally more convenient to ignore the specific nature of the elements of the vector spaces $\Lambda^2_x M$ and $\Lambda_{2,x} M$ as exterior products of other vector spaces and treat them as simply six-dimensional real vector spaces, at least when M is four-dimensional. Furthermore, since most of what we shall be discussing is local and linear in character, we shall consider only the vector spaces $A^2(\mathbb{R}^4)$ and $A_2(\mathbb{R}^4)$, which serve as typical fibers of the vector bundles $\Lambda^2 M$ and $\Lambda_2 M$.

Hence, a basis for A_2 will consist of six linearly independent vectors $\{\mathbf{b}_I\,,\ I = 1, \ldots, 6\}$ (which are really *bi*vectors) and a basis for A^2 will consist of six linearly independent covectors$\{b^I\,,\ I = 1, \ldots, 6\}$ (which are really algebraic 2-forms). As in any other vector space, they will be reciprocal to each other iff $b^I(\mathbf{b}_J) = \delta^I_J$.

If $\{\mathbf{e}_\mu,\ \mu = 0, \ldots 3\}$ represents a basis for $\mathbb{R}^4$ and $\{\theta^\mu,\ \mu = 0, \ldots, 3\}$ its reciprocal basis for $\mathbb{R}^{4*}$ then one can define a basis for A_2 by means of:

$$\mathbf{b}_i = \mathbf{e}_0 \wedge \mathbf{e}_i\,, \qquad \mathbf{b}_{i+3} = \tfrac{1}{2}\,\varepsilon_{ijk}\,\mathbf{e}_j \wedge \mathbf{e}_k, \qquad i, j, k = 1, 2, 3 \tag{V.3}$$

and one for A^2 by means of:

$$b^i = \theta^0 \wedge \theta^i\,, \qquad b^{i+3} = \tfrac{1}{2}\,\varepsilon^{ijk}\,\theta^j \wedge \theta^k. \tag{V.4}$$

It is easy to see that these bases are themselves reciprocal.

For instance, in this basis we can represent the electromagnetic field strength F as:

$$F = E_i\, b^i + B_i\, b^{i+3}\,. \tag{V.5}$$

Because the bases seem to split evenly into three members that have a common exterior factor of $\mathbf{e}_0$ (θ^0, resp.) and three members that do not involve it, we shall find it convenient, as well as mathematically and physically significant, to represent A_2 as the direct sum $A_2^{\mathrm{Re}} \oplus A_2^{\mathrm{Im}}$ and A^2 as the direct sum $A^2_{\mathrm{Re}} \oplus A^2_{\mathrm{Im}}$ of two three-dimensional real vector spaces. Our notations "Re" and "Im" are suggestive of "real" and "imaginary,"

respectively, although we have more to say about why that is actually appropriate in Chapter XII.

When a basis has been chosen for both A_2 and A^2, any linear transformation L: $A^2 \to A_2$, $\alpha \mapsto L(\alpha)$ can be represented by a matrix L^{IJ} with respect to these bases that is defined by:

$$L(b^I) = L^{IJ}\, \mathbf{b}_J \,. \tag{V.6}$$

In block-matrix form, the matrix of a local linear constitutive map κ looks like:

$$\kappa^{IJ} = \left[\begin{array}{c|c} \alpha^{ij} & \beta^{ij} \\ \hline \gamma^{ij} & \eta^{ij} \end{array}\right]. \tag{V.7}$$

So far, we have said nothing concerning the symmetry of the indices I and J. Indeed, one can first regard them as having no specified symmetry. However, one can polarize κ^{IJ} into a sum $\kappa^{IJ}_+ + \kappa^{IJ}_-$ of a symmetric part κ^{IJ}_+ and an anti-symmetric part κ^{IJ}_- , where:

$$\kappa^{IJ}_+ = \tfrac{1}{2}(\kappa^{IJ} + \kappa^{JI}), \qquad\qquad \kappa^{IJ}_- = \tfrac{1}{2}(\kappa^{IJ} + \kappa^{JI}) \,. \tag{V.8}$$

However, κ^{IJ}_+ can be decomposed further, since the inverse $\#^{-1}$ of the Poincaré duality isomorphism also has a matrix that is symmetric:

$$\tilde{\#}^{IJ} = \left[\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right]. \tag{V.9}$$

In the terminology of Hehl and Obukhov [**3**], the *principal part* of κ^{IJ} is then defined to be the part of κ^{IJ}_+ that does not include this contribution:

$${}^{(1)}\kappa^{IJ} = \kappa^{IJ}_+ - \tfrac{1}{6}\mathrm{Tr}(\kappa^I_J)\,\tilde{\#}^{IJ} = \left[\begin{array}{c|c} -\varepsilon^{ij} & \gamma^{ij} \\ \hline \overline{\gamma}^{ji} & \tilde{\mu}^{ij} \end{array}\right]. \tag{V.10}$$

In the second term, we have used $\kappa^I_J \equiv \#_{JK}\, \kappa^{KI}$.

The sub-matrix ε^{ij}, which is Hermitian, represents the *electric permittivities* of the medium, which generalize the dielectric constant; the reason for the minus sign will become clear shortly. The matrix $\tilde{\mu}^{ij}$, which is also Hermitian, represents the inverse of the matrix of *magnetic permeabilities* of the medium. The remaining off-diagonal matrices γ^{ij} and its Hermitian transpose $\overline{\gamma}^{ji}$ represent the contribution of the magnetic field B_i to the electric excitation D^i and that of the electric field E_i to the magnetic excitation H^i. We shall discuss possible origins for these contributions in a bit.

The anti-symmetric part of κ – namely, ${}^{(2)}\kappa^{IJ} = \kappa^{IJ}_{-}$ – is called the *skewon* part of κ and the remaining "trace-class" contribution ${}^{(3)}\kappa^{IJ} = \frac{1}{6}\mathrm{Tr}(\kappa^{I}_{J})\tilde{\#}^{IJ}$ is called its *axion* part, which represents a contribution from the volume element on $\mathbb{R}^4$.

3. **Examples of local linear media**. It will prove essential in what follows to have some specific examples of constitutive laws to consider, so we shall show how one accounts for some of the most commonly used electromagnetic media in the aforementioned formalism. One thing that should be self-evident is that when one is referring to the constancy of matrix components this unavoidably begs the question of "in what frame?" In most cases the answer is simply: the rest frame of the medium itself. Of course, in the case of massless media, such as the electromagnetic vacuum – whether classical or quantum – it is meaningless to speak of a rest frame for the medium, which was, of course, the problem with the former concept of the "ether," and one must then accept the more ambiguous answer: the frame of some measurer/observer.

a. Linear isotropic media. The most elementary electromagnetic media of all are the *linear isotropic media*, which include the classical vacuum itself. The term "isotropic" refers to invariance under spatial rotations in the chosen frame, such as the rest frame of the medium, when it is not massless. At the elementary level, this generally means that the electric and magnetic dipoles of the medium must be capable of aligning themselves to an imposed field just the same way at any point and in any direction.

Such media are characterized by the fact that κ^{IJ} agrees with its principal part, so it is symmetric, and its constituent submatrices are of the form:

$$\varepsilon^{ij} = \varepsilon\, \delta^{ij}, \qquad \tilde{\mu}^{ij} = (1/\mu)\, \delta^{ij}, \qquad \gamma^{ij} = 0. \tag{V.11}$$

The function ε is referred to as the *electric permittivity* of the medium. The function μ is its *magnetic permeability*. When these functions are constant in the chosen frame one calls the medium *homogeneous*, as well. The constant ε is usually referred to as the *dielectric constant* of the medium, in that case.

The classical electromagnetic vacuum is assumed to be linear, isotropic, and homogeneous; one denotes its dielectric constant by ε_0 and its magnetic permeability by μ_0 . We shall denote its constitutive map by κ_0, and it will play a recurring role in what follows as essentially an asymptotic state that one finds in the absence of electromagnetic fields.

b. Linear optical media. The next level of generality is defined by the *linear optical media*, which are basically like (V.11), except that one allows ε^{ij} to possibly be inhomogeneous or anistropic. In regular optical practice, the most common inhomogeneity that one treats involves matrix components that are piecewise constant and undergo jump discontinuities on the interfaces between dissimilar media. However, an important example of continuous inhomogeneity is defined by the variation of the optical properties of some transparent media under stress. Since the magnetic

permeability of an optical medium generally plays no role, one customarily thinks of them as dielectric media; i.e. $\tilde{\mu}^{ij} = (1/\mu_0)\ \delta^{ij}$, with μ_0 a constant that is usually set to unity.

When ε^{ij} is symmetric, but anisotropic, it is customary to define the *principal frame*, in which it becomes the diagonal matrix diag[ε_x , ε_y , ε_z]. The diagonal elements, which are the *principal permittivities*, are the eigenvalues of the matrix $\varepsilon^i_j = \delta_{jk}\ \varepsilon^{ik}$, and the vectors of the principal frame (in $\mathbb{R}^3$) are its normalized eigenvectors. Because ε^{ij} is symmetric, these eigenvalues exist and are real, and the eigenvectors associated with distinct eigenvalues must be orthogonal. Note that one implicitly introduces the Euclidian metric on the spatial manifold in order to speak of eigenvectors and eigenvalues.

There are three basic ways that the eigenvalues of ε^{ij} can equal or differ from each other:

1. When they are all equal, one calls the medium *isotropic*.
2. When two are equal, and the third one differs, the medium is called *uniaxial*. (The axis in question belongs to the distinct eigenvalue.)
3. When all three are unequal, the medium is called *biaxial*.

Generally, the symmetry type of a dielectric is traceable to the symmetry type of the crystal lattice of the material that it is composed of.

The complementary class to that of optical media is defined by *magnetic media*, for which $\varepsilon^{ij} = \varepsilon_0\ \delta^{ij}$, where ε_0 is regarded as constant, but $\tilde{\mu}^{ij}$ is possibly inhomogeneous or anisotropic. For magnetic media, one has an analogous notion of a principal frame for the symmetric matrix $\tilde{\mu}^{ij}$, whose eigenvalues are $1/\mu_x$, $1/\mu_y$, $1/\mu_z$, are then *principal permeabilities.* Clearly, the principal frame for $\tilde{\mu}^{ij}$ will be the same as the principal frame for its inverse matrix μ_{ij} , and the eigenvalues of the inverse matrix will be μ_x , μ_y , μ_z .

It is important to understand that the physical nature of electric and magnetic polarization in a medium suggests that there is no reason to believe that in the general case where ε^{ij} and $\tilde{\mu}^{ij}$ are both anisotropic they will have a common principal frame. Basically, this is equivalent to the condition that the matrices commute under multiplication, which is always the case when one of them is a scalar multiple of δ^{ij}, but not true in general. When both ε^{ij} and $\tilde{\mu}^{ij}$ have a common principal frame, we shall call such a medium *electromagnetically diagonalizable*. In such a frame, one has:

$$\kappa^{IJ} = \text{diag}[-\varepsilon_x , -\varepsilon_y , -\varepsilon_z, 1/\mu_x , 1/\mu_y , 1/\mu_z]. \qquad \text{(V.12)}$$

In all of the cases for which $\alpha^{ij} = 0$ one can summarize the constitutive relations by the conventional three-dimensional component equations:

$$D^i = -\varepsilon^{ij} E_j , \qquad B_i = \mu_{ij} H^j . \qquad \text{(V.13)}$$

c. Bi-isotropic media. An elementary example of a medium in which the off-diagonal matrices can be non-vanishing is that of bi-isotropic media (see Lindell [**4**]), which are like isotropic media in their diagonal matrices, but have:

$$\gamma^{ij} = \overline{\gamma}^{ji} = \gamma\,\delta^{ij}\,. \tag{V.14}$$

In this case, the off-diagonal matrices are symmetric, and the part of κ^{IJ} is $\gamma\tilde{\#}^{IJ}$.

As we shall see, these media include the case of greatest interest in quantum electrodynamics, namely, the Heisenberg-Euler media.

d. Lorentzian media. Recall that the crucial step in pre-metric electromagnetism is the replacement of the purely geometric tensor field $\imath_g \wedge \imath_g$ that represents the isomorphism of the vector space of 2-forms with the vector space of bivectors at each point of the spacetime manifold with a purely physical – indeed, as we have seen, largely phenomenological – tensor field that represents the response of the medium to the presence of electric and magnetic fields. It is then heuristically useful to examine the form that the isomorphism $\imath_g \wedge \imath_g$ takes when expressed in terms of the *I, J* indices in order to compare the effect that it has to that of an electromagnetic constitutive law.

One simply has to start with the expression for the raising of two indices:

$$\mathfrak{h}^{\mu\nu} = g^{\mu\alpha}\,g^{\nu\beta}\,F_{\alpha\beta} = \tfrac{1}{2}(g^{\mu\alpha}\,g^{\nu\beta} - g^{\mu\beta}\,g^{\nu\alpha})F_{\alpha\beta} \tag{V.15}$$

and expand:

$$\mathfrak{h}^{0i} = (g^{00}\,g^{ij} - g^{0i}\,g^{0j})\,F_{0j} + \tfrac{1}{2}(g^{0j}\,g^{ik} - g^{0k}\,g^{ij})\,F_{jk}\,, \tag{V.16a}$$

$$\mathfrak{h}^{ij} = \tfrac{1}{2}(g^{i0}\,g^{jk} - g^{0j}\,g^{ik})\,F_{0k} + \tfrac{1}{2}(g^{ik}\,g^{jl} - g^{il}\,g^{jk})\,F_{kl}\,. \tag{V.16b}$$

If we set:

$$F_{0i} = E_i\,, \qquad F_{ij} = \varepsilon_{ijk}B^i, \qquad \mathfrak{h}^{0i} = D^i, \qquad \mathfrak{h}^{ij} = \varepsilon^{ijk}H_k \tag{V.17}$$

then (V.16a, b) take the form:

$$D^i = -\varepsilon^{ij}\,E_j + \gamma^i_j B^j, \qquad H_i = \gamma^j_i\,E_j + \tilde{\mu}_{ij}\,B^j, \tag{V.18}$$

as long as we set:

$$\varepsilon^{ij} = g^{0i}\,g^{0j} - g^{00}\,g^{ij}, \qquad \gamma^i_j = \varepsilon_{klj}\,g^{0k}\,g^{il}, \qquad \tilde{\mu}_{ij} = \varepsilon_{nmi}\,\varepsilon_{klj}\,g^{nk}\,g^{ml}. \tag{V.19}$$

Of particular interest is the case of the Minkowski space metric:

$$g^{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}[+1, -1, -1, -1], \tag{V.20}$$

which makes:

$$\varepsilon^{ij} = \delta^{ij}, \qquad \gamma^i_j = 0, \qquad \tilde{\mu}_{ij} = \delta_{ij}\,. \tag{V.21}$$

Hence, we can put κ^{IJ} into the form:

$$\kappa^{IJ} = \left[\begin{array}{c|c} -\delta^{ij} & 0 \\ \hline 0 & \delta_{ij} \end{array}\right] \tag{V.22}$$

in this case; we now see the reason for the inclusion of the minus sign in the general form (V.10) for ${}^{(1)}\kappa^{IJ}$ above.

Note that one cannot generally solve (V.19) for the $g^{\mu\nu}$ when given the ε^{ij}, γ^{i}_{j}, $\tilde{\mu}_{ij}$ since the manifold Lor(4) of all $g^{\mu\nu}$ is 10-dimensional, whereas the manifold Cons(4) of all ε, μ, and γ, in which the first two are symmetric and the last one has no specified symmetry, is 6 + 6 + 9 = 21-dimensional. Therefore, not all possible combinations (ε, μ, γ) will be consistent with the mapping Lor(4) $\rightarrow$ Cons(4), $g \mapsto (\varepsilon, \mu, \gamma)$ that that is defined by (V.19), and one must specify 11 compatibility conditions, in order to define the image of that map as a submanifold of Cons(4).

Actually, there is an increasing body of literature, both mathematical and physical (see, e.g., [**6**]), that is based on the fact that the principal part of a linear electromagnetic constitutive law defines a fourth-rank totally covariant tensor field κ on the spacetime manifold of a symmetry type that is sometimes called an *algebraic curvature tensor* [**7**], which also defines an *area metric* – i.e., a metric on the vector bundle $\Lambda^2 M$. We shall return to discuss this latter aspect of k in Chapters XII and XIII, but for now, we simply point out that a key result is the *Gilkey decomposition* [**7**]:

$$\kappa_{\kappa\lambda\mu\nu} = \sum_{(i)=1}^{N} Z_{(i)} \left[{}^{(i)}g_{\kappa\mu}\, {}^{(i)}g_{\lambda\nu} - {}^{(i)}g_{\kappa\nu}\, {}^{(i)}g_{\lambda\mu} \right], \tag{V.23}$$

in which the $Z(i)$, $i = 1, \ldots, N$ are frame-invariant scalar functions and the ${}^{(i)}g_{\kappa\lambda}$ are metric tensor fields. This decomposition is not, however, unique, since, at the very least one can choose other scalar factors.

We shall refer to an electromagnetic medium whose constitutive law takes the form of (V.19) for some Lorentzian metric as a *Lorentzian* medium.

e. Almost-complex media [**8**]. If one left-multiplies κ^{IJ} in (V.22) by the matrix $\#_{IJ}$ that is inverse to (V.9) then one gets a matrix:

$$*^{I}_{J} = \#_{JK}\, \kappa^{KI} = \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array} \right] \tag{V.24}$$

that represents a linear isomorphism $*: \Lambda^2 \rightarrow \Lambda^2$ with the property that:

$$*^2 = -I; \tag{V.25}$$

In particular, it is the Hodge duality isomorphism that is associated with the Lorentzian metric, as it acts on 2-forms.

Now, a linear isomorphism of an even-dimensional real vector space with itself that has the property (V.25) defines a *complex structure* on that vector space, because multiplication by the imaginary i has that same effect on the vectors in a complex vector space. We shall have much more to say about the role of complex structures in pre-

metric electromagnetism in Chapter XII, but, for now, we confine ourselves to the aspects that pertain to constitutive laws.

Since an electromagnetic constitutive law κ is supposed to replace the * isomorphism that one derives from a Lorentzian structure, it is natural to *postulate* that it have the same property (V.25). However, in the simplest case of an isotropic medium (V.11), since:

$$\tilde{\kappa}^I_J = \left[\begin{array}{c|c} 0 & (1/\mu)I \\ \hline -\varepsilon I & 0 \end{array}\right], \qquad (\tilde{\kappa} \equiv \# \cdot \kappa) \tag{V.26}$$

one must have:

$$\tilde{\kappa}^2 = -\frac{\varepsilon}{\mu} I\,. \tag{V.27}$$

Hence, it probably more prudent to postulate that there is some function λ on the spacetime manifold M such that:

$$\tilde{\kappa}^2 = -\,\lambda^2\, I\,. \tag{V.28}$$

One can then define * by normalization:

$$* = (1/\lambda)\,\tilde{\kappa}\,. \tag{V.29}$$

The function λ can be shown to have the dimension of an admittance (i.e., 1/impedance), and for the vacuum it has the numerical value of 1/377 mhos.

An immediate consequence of the condition (V.28) is that it implies further restricting conditions on the submatrices that represent $\tilde{\kappa}$. If κ agrees with its principal part, so its matrix has the form (V.10), then:

$$\tilde{\kappa} = \left[\begin{array}{c|c} \gamma^T & \tilde{\mu} \\ \hline -\varepsilon & \gamma \end{array}\right]. \tag{V.30}$$

From the condition (V.28), a direct evaluation of $\tilde{\kappa}^2$ gives a set of four matrix equations that reduce to just two:

$$\varepsilon = \lambda^2 \mu + \mu\,\gamma^2, \qquad \mu\gamma^T = \gamma\mu\,. \tag{V.31}$$

Here, we see that this constraint is actually more restrictive than it sounds like it would be. If we assume that $\gamma = 0$ then we are left with the constraint that the matrix ε must be proportional to the matrix μ. Hence, the medium must have the same symmetry under spatial rotations for both its electric and magnetic properties; i.e., both must be isotropic, uniaxial, or biaxial, resp. Whereas this is certainly the case for the classical vacuum, and any other isotropic space, it is not generally true of most anisotropic optical media, since one assumes that they are magnetically isotropic and homogeneous.

For a bi-isotropic medium, the second equation in (V.31) is trivial, and the first says that one must have:

$$\varepsilon = (\lambda^2 + \gamma^2)\,\mu \tag{V.32}$$

for some λ. Once again, ε must be proportional to μ, but the factor of proportionality depends upon γ.

Nonetheless, we shall see that media for which κ satisfies (V.33), which we call *almost-complex media*, provide a considerable wealth of applications for the methods of complex projective geometry, which is also closely suggested by the use of 2-forms and bivector fields.

f. Purely axionic media. A *purely axionic* electromagnetic medium will have a constitutive tensor field of the form:

$$\kappa = \alpha \left[\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right], \tag{V.33}$$

which makes:

$$* = \alpha I. \tag{V.34}$$

This sort of relationship between F and $H = *F$ is strongly analogous to a variation on Ohm's law for coupling current to voltage in a two-port electrical network that was obtained by Tellegen [**9**, **10**]. For the network that he devised, which he called a "gyrator" instead of the usual $V = IR$ rule, he obtained:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = s \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}, \tag{V.34}$$

in which i_a, v_a, $a = 1, 2$ are the currents and voltages at the two ports, respectively, while s has the dimensions of a resistance.

Physical examples of electromagnetic media that exhibit such laws also exist. For instance, Lindell and Sihvola [**10**] have described such media as *perfect electromagnetic conductors*, and the emerging class of *metamaterials* [**11**] appears to offer the promise of implementing such a law in a tangible medium.

g. Magneto-electric materials [**12**]. As a contribution to an electromagnetic constitutive law, and not a law unto itself, an axion part represents one type of magneto-electric coupling, in addition to the contributions of skewon type. In the next section, we shall discuss how magneto-electric couplings can come about as a result of frame transformations, and some of the experimentally established physical that exhibit them, but, for now, we cite as an example of a *magneto-electric material* the much-studied anti-ferromagnetic material chromium sesquioxide (Cr_2O_3).

It gives a * isomorphism whose matrix has the block matrix form:

$$[*] = \left[\begin{array}{c|c} \varepsilon & 0 \\ \hline 0 & \tilde{\mu} \end{array}\right] + \alpha \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right], \tag{V.35}$$

in which:

$$\varepsilon = \varepsilon_0 \begin{bmatrix} \varepsilon_\perp - \alpha_\perp^2/\mu_\perp & 0 & 0 \\ 0 & \varepsilon_\perp - \alpha_\perp^2/\mu_\perp & 0 \\ 0 & 0 & \varepsilon_\parallel - \alpha_\parallel^2/\mu_\parallel \end{bmatrix}, \qquad \tilde{\mu} = \frac{1}{\mu_0} \begin{bmatrix} 1/\mu_\perp & 0 & 0 \\ 0 & 1/\mu_\perp & 0 \\ 0 & 0 & 1/\mu_\parallel \end{bmatrix}, \tag{V.36}$$

$$\alpha = \frac{1}{3}\left(2\frac{\alpha_\perp}{\mu_\perp} + \frac{\alpha_\parallel}{\mu_\parallel} \right) \sqrt{\frac{\varepsilon_0}{\mu_0}}\,. \tag{V.37}$$

Hence, such a medium has no skewon contribution, and its magneto-electric coupling is solely due to the axionic part.

Experiments with Cr_2O_3 [**12**] have verified that $\alpha \neq 0$.

h. Fresnel-Fizeau effect [**1**, **2**]. It has long been established that magneto-electric couplings in an electromagnetic medium can appear in as a result of the propagation of electromagnetic waves in a massive electromagnetic medium that is in a state of motion relative to the measurer/observer. This effect of relative motion on such a medium was first studied by Fresnel and later by Fizeau; it was also discussed by Einstein in his early work on special relativity.

Basically, the effect of a relative velocity **v** on the medium is to couple magnetic fields to electric ones and vice versa in a manner that is analogous to the Lorentz force, at least up to first order. All that one needs to do is subject the **E** and **B** field to a Lorentz transformation that corresponds to a transformation to a frame that has a relative velocity of **v**. One finds (see Jackson [**13**], Landau and Lifschitz [**14**]) that the transformation of fields is, to first order:

$$\mathbf{E}' = \gamma\mathbf{E} + \gamma c\, \mathbf{v} \times \mathbf{B}, \qquad \mathbf{B}' = \gamma\mathbf{B} - \gamma c\, \mathbf{v} \times \mathbf{E}\,. \tag{V.38}$$

Hence, the effect of the relative velocity is to make:

$$\gamma^{ij} = 1/c\ \varepsilon^{ijk} v_k = \frac{1}{c} \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}. \tag{V.39}$$

Note that although κ^{IJ} remains symmetric the individual off-diagonal matrices γ^{ij} and $\overline{\gamma}^{ji}$ are anti-symmetric.

i. Plasmas [**15-18**]. One of the most important examples of an electromagnetic medium is defined by the plasma state. This state, which comes about from the ionization of gases as a result of high temperatures or applied electric fields, is characterized by an ensemble of (usually) two oppositely charged fluids that are collectively neutral. The positively-charged fluid, which is composed of the atomic or molecular ions, has a low mobility due to the higher masses of the ions, while the negatively-charged fluid is composed of free electrons, which will then have a high degree of mobility. Examples of plasmas are found at all levels of scale, including

possibly the scale of strongly-interacting particles. Commonly, one encounters them in the form of flames, glowing gases in discharge tubes, such as neon lights, the Earth's ionosphere, as well as the solar atmosphere, and even interstellar space seems to be composed of a very dilute plasma whose ions are mostly protons – i.e., hydrogen ions. Since the scope of plasma physics is so vast as to touch upon most of the established branches of physics, we shall confine our remarks to only those aspects of plasmas that affect the electromagnetic constitutive properties and later, the propagation of electromagnetic waves.

The key aspect of plasmas that affects their electromagnetic constitutive properties is the fact that high electron mobility implies high conductivity. Hence, in the usual Maxwell equations for electromagnetic fields in plasmas, one must add an electric current $\mathbf{i}_\sigma = \mathbf{i}(E)$ that comes about in response to the applied electric field. Customarily, one assumes that the response is linear but not necessarily isotropic or homogeneous, so the induced current obeys Ohm's law in the form:

$$i^i_\sigma = \sigma^{ij}(t, x)E_i \,, \tag{V.40}$$

in which the matrix:

$$\sigma^{ij} = \begin{bmatrix} \sigma_x & \sigma_{xy} & \sigma_{xz} \\ -\sigma_{xy} & \sigma_y & \sigma_{yz} \\ \sigma_{xz} & -\sigma_{yz} & \sigma_z \end{bmatrix} \tag{V.41}$$

is the electrical conductivity of the plasma and is generally complex.

For as *lossless* plasma one will have that σ is anti-Hermitian (i.e., $\sigma^\dagger = -\sigma$). In this case, the elements σ_{xy} and σ_{yz} will be real, while the others are imaginary.

Generally, the origin of the anisotropy in the conductivity of a plasma is the presence of a background magnetic field, which then introduces a preferred direction and an orbital motion to the charged particles, although the electrons will be dynamically affected by the magnetic field more than the ions.

The electric current due to the applied E can be absorbed into the dielectric constant of the plasma to give an effective dielectric constant:

$$\varepsilon^{ij} = \varepsilon_0\, \delta^{ij} + \frac{1}{i\omega}\,\sigma^{ij} \,. \tag{V.42}$$

Hence, ε^{ij} will have the same symmetry as σ^{ij}, except that for lossless plasmas, one will have that ε^{ij} is Hermitian (i.e., $\varepsilon^\dagger = \varepsilon$).

In the case of an isotropic plasma, such as when no background magnetic field is present, one has that $\sigma^{ij} = \sigma\delta^{ij}$, with:

$$\sigma = -\, i\frac{ne^2}{m_e} \,, \tag{V.43}$$

in which m_e is the electron mass and n is the number density of the electrons.

This makes the effective dielectric constant take the form:

$$\varepsilon(\omega) = \left(1 - \frac{ne^2}{m_e \omega}\right)\varepsilon_0 . \qquad \text{(V.44)}$$

One immediately sees that the dependence of the dielectric constant on the frequency of the electric field that is present is such that it will vanish when that frequency equals:

$$\omega_p(n) = \left(\frac{4\pi ne^2}{m_e}\right)^{1/2} = 2p(8980)\sqrt{n}\,, \qquad \text{(V.45)}$$

and is referred to as the *plasma frequency*.

The process of electric polarization in a medium with considerable charge mobility implies the formation of a counter-electric field that tends to oppose the applied field. For instance, in the close proximity of a charged electrode in a plasma one can expect that there will be a shielding layer with a characteristic thickness that one calls the *Debye screening length:*

$$\lambda_D = \left(\frac{\varepsilon_0 KT_e}{ne^2}\right)^{1/2} . \qquad \text{(V.46)}$$

The fact that this length depends upon the temperature T_e of the electrons, in addition to their number density n, derives from the fact that when the electrons have sufficiently high temperature they can escape the electrostatic well that is created by the screening effect.

Note that such a statistical argument is applicable only when the total number N of electrons in the shielding layer is appreciable. This also defines a characteristic number, in the form of the total number of free electrons in a "Debye sphere":

$$N_D = n\left(\frac{4}{3}\pi\lambda_D^3\right) = 1.38\times 106\left(\frac{T^3}{n}\right)^{1/2} \qquad (T \text{ in } {}^{o}\text{K}). \qquad \text{(V.47)}$$

Actually, since the number density of the free electrons is not constant, but is assumed to vary in response to the ambient electric and magnetic fields, one must add a further equation to the Maxwell equations that is based in the Boltzmann equation of physical kinetics [**18**] and expresses the balance law for the number density in terms of these ambient fields. This equation is often called the *Vlasov equation.*

4. Some physical phenomena due to magneto-electric couplings. The presence of magneto-electric couplings in an electromagnetic medium can give rise to various phenomena that are well-documented in the world of experimental physics and sometimes the most powerful tools that physics – especially astrophysics – has in determining the nature of the medium that electromagnetic waves are propagating through. Although we shall discuss the propagation of electromagnetic waves in a later

chapter, for now we shall mostly concentrate on the way that these effects manifest themselves in changes to the constitutive properties of the medium.

a. Faraday effect [**1**, **2**]. The way that electromagnetic waves propagate through an electromagnetic medium can be influenced by the presence of "external" electric and magnetic fields in the medium, in addition to those of the wave itself. For instance, in astrophysics, one generally has to deal with the presence of galactic magnetic fields that might affect the propagation of photons from the stars of the galaxy in question.

In particular, the presence of a magnetic field can bring about a rotation of the polarization planes (which is generated by the **E** and **k** vectors) within the tangent spaces. This is because an imposed magnetic field, which we assume to point in the positive *z*-direction, will affect both the dielectric and magnetic properties of the medium, and either effect is called the *Faraday effect.*

The dielectric Faraday effect changes the ε^{ij} matrix of an isotropic dielectric with permittivity ε to one of the form:

$$\varepsilon^{ij} = \begin{bmatrix} \varepsilon' & 0 & 0 \\ 0 & \varepsilon & i\varepsilon_{yz} \\ 0 & -i\varepsilon_{yz} & \varepsilon \end{bmatrix}. \tag{V.48}$$

Note, in particular, that the contribution is imaginary and the medium is no longer isotropic, but uniaxial.

Similarly, the magnetic Faraday effect changes the $\tilde{\mu}^{ij}$ matrix for an isotropic magnetic medium with a permeability of μ to one of the form:

$$\tilde{\mu}^{ij} = \begin{bmatrix} 1/\mu' & 0 & 0 \\ 0 & 1/\mu & i\mu_{yz} \\ 0 & -i\mu_{yz} & 1/\mu \end{bmatrix}. \tag{V.49}$$

b. Natural optical activity [20]. In Landau, Lifschitz, and Pitaevski [**1**], the phenomenon of natural activity is described as something that occurs in optical media with no center of symmetry and takes the form of allowing ε^{ij} to be a function of both ω and **k**.

To first order, this takes the form:

$$\varepsilon^{ij}(\omega, \mathbf{k}) = \varepsilon_0^{ij}(\omega) + i\gamma^{ijk} k_k . \tag{V.50}$$

Hence, the effect of natural optical activity is equivalent to the dielectric Faraday effect for a background magnetic field in an optically inactive medium.

However, in Post [**2**] the effect of natural optical activity is to make:

[20] Since the mathematical representations of this phenomenon that were given in Landau, et al [**1**] and Post [**2**] are inconsistent, we shall defer to the former reference.

$$\gamma^{ij} = -\overline{\gamma}^{ji} = i\gamma\delta^{ij} . \qquad \text{(V.51)}$$

That is, it affects the off-diagonal matrices in κ^{IJ}, not the off-diagonal terms in ε^{ij} itself.

5. Nonlinear constitutive laws [19-21]. Since the appearance of linearity in physics is invariably based in some simplifying empirical approximation, such Hooke's law, Ohm's law, or others, the consideration of nonlinear phenomena is also invariably the most promising mathematical horizon for further exploration in physics.

Generally, the transition from the linear regime to the nonlinear regime is indexed by some magnitude, such as displacement, temperature, or field strength. Quite often, a complicating factor that can drastically change the nature of a system's response from linear to nonlinear is the possibility of a phase change. For instance, in the case of Ohm's law (viz., $I = \Delta V/R$) one finds that current I flowing through a resistor R in response to an applied voltage difference ΔV will cause it to heat up, which increases the resistance. That, in its own right, would make the current through the resistor related to the voltage drop across the resistor in a nonlinear manner, but ultimately the heat will bring about complete vaporization of the resistor, as in fuses. This then represents the sort of catastrophic nonlinearity that one associates with phase transitions.

In the case of electromagnetic constitutive laws, since the elementary electric and magnetic dipoles are associated with more complicated systems, such as atoms and crystal lattices, it is not surprising that linear constitutive laws are just as much of an approximation as in any other linear law of nature. For instance, one can imagine that an intense laser beam in a transparent plastic fiber will have a similar effect to a high current in a resistor, and at a threshold level of intensity the fiber will melt or vaporize.

As with linear effects, one can distinguish between non-local nonlinear effects and local nonlinear effects. An example of a non-local nonlinear effect that is quite common in electromagnetics is *magnetic hysteresis*. The idea is that as one increases the magnitude B of $\mathbf{B}$ in most magnetic media, the resulting H field will involve a time lag in its response that depends upon the magnitude B in such a way that if one decreases the value of B back to its original value then the medium will respond to the same values of B differently. This situation is illustrated in Fig. 4.

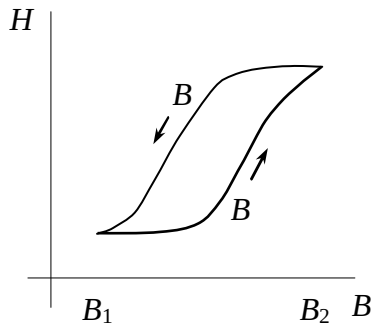

Figure 4. Magnetic hysteresis.

The same sort of situation occurs when an incandescent filament heats up at one rate as the current in it increases and then cools down at a different rate when the current decreases, so instead of the straight-line plot of Ohm's law one has something more like Fig. 4 for I vs. ΔV.

a. Electromagnetic susceptibilities. There is a regime between linearity and the onset of some phase transition that is sometimes referred to as "weak nonlinearity." It is characterized by the possibility of representing the diffeomorphism κ: $A^2 \to A_2$, $F \mapsto \mathfrak{h} = \kappa(F)$ by the first terms of a Taylor series in F:

$$\mathfrak{h}(F) = \mathfrak{h}_0 + d\kappa|_0(F) + \tfrac{1}{2} d^2\kappa|_0(F, F) + \ldots \tag{V.52}$$

which can be expressed in coordinates as:

$$\mathfrak{h}^{\mu\nu}(F_{\alpha\beta}) = \mathfrak{h}_0^{\mu\nu} + \left.\frac{\partial \mathfrak{h}^{\mu\nu}}{\partial F_{\alpha\beta}}\right|_{F=0} F_{\alpha\beta} + \frac{1}{2}\left.\frac{\partial^2 \mathfrak{h}^{\mu\nu}}{\partial F_{\alpha\beta}\partial F_{\gamma\delta}}\right|_{F=0} F_{\alpha\beta}F_{\gamma\delta} + \ldots \tag{V.52}$$

Actually, the leading term in this expansion, which we shall suggestively call the *zero-point field*, represents a possible source of nonlinearity, in the sense that it makes the linear relationship that is defined by the second term into an affine relationship. In the case of magnetism the presence of a non-zero constant term in **H** is usually attributed to *ferromagnetism*; in the case of electric fields, it is called *ferroelectricity*.

For the purposes of nonlinear optics, it is usually preferable to use a Taylor expansion in the electric polarization vector field **P**, which relates to **E** by way of:

$$\mathbf{D} = \varepsilon_0\mathbf{E} + 4\pi\mathbf{P}. \tag{V.53}$$

Similarly, in magnetic media one introduces the *magnetization* vector field **M** by way of:

$$\mu_0\mathbf{H} = \mathbf{B} + 4\pi\mathbf{M}. \tag{V.54}$$

In order to combine these into something more useful to pre-metric electromagnetism – i.e., a four-dimensional construction – we must use the classical vacuum isomorphism κ_0 to define a sort of reference medium, in order to define the subtraction above.

We call the bivector field:

$$\mathbf{Q}(F) = \frac{1}{4\pi}(\kappa - \kappa_0)(F) = P_i\, b^i + M_i\, b^{i+3}, \tag{V.55}$$

the *electromagnetic polarization* bivector field.

Now, we can expand $4\pi Q(F)$ in a Taylor series:

$$4\pi\mathbf{Q}(F) = \chi^{(0)} + \chi^{(1)}(F) + \chi^{(2)}(F, F) + \ldots \tag{V.56}$$

The successive terms $\chi^{(i)}$, $i = 0, 1, \ldots$, are the *electromagnetic susceptibilities*. To the zeroth and first order, they relate to κ by way of:

$$\chi^{(0)} = 4\pi\eta_0, \qquad \chi^{(1)} = 4\pi\, d(\kappa - \kappa_0)|_{F=0}, \quad \ldots, \chi^{(i)} = 4\pi\, d^{(i)}(\kappa - \kappa_0)|_{F=0}, \ldots, \qquad \text{(V.57)}$$

in which the $d^{(i)}$ refers to the i^{th} differential map. Therefore, the essential difference between the susceptibilities and the corresponding terms in the Taylor series for the constitutive law is in the linear terms.

b. Nonlinear optical effects. Although the field of nonlinear optics has expanded by now into a vast volume of literature (for some elementary treatments, however, cf. Mills [**20**] or Butcher and Cotter [**21**], as well as the synopsis in Delphenich [**22**]), most of which is considerably more applied and empirical than is relevant to our present theoretical discussion, there are some general phenomena that occur as a result of the nonlinearity itself that we shall find intuitively illuminating to summarize. Basically, the nonlinear effects are classified according to the level of susceptibility that first introduces the effect.

At second order, one first encounters a departure from the law of superposition that is definitive of linear fields or waves. In particular, when two interacting electromagnetic waves have frequencies ω_1 and ω_2, one can find that waves with their difference and sum frequencies appear in addition to the original frequencies. When they both have same frequency, the resulting waves can include a DC component and one with the double frequency. The former effect is called *optical rectification*, while the latter one is called *second-harmonic generation*.

In some cases, such as when one F_0 of two fields F_0 and F can be treated as a "background" field, in some sense, instead of dealing with – say – second-order effects in the combined field $F_0 + F$, one can absorb the background field into the second-order susceptibility:

$$\chi^{(2)}(F_0 + F, F_0 + F) = \chi^{(2)}(F_0, F_0) + 2\chi^{(2)}(F_0, F) + \chi^{(2)}(F, F) \qquad \text{(V. 58)}$$

to produce a first-order effect, in which the background field has "modulated" the first order susceptibility. For instance, one might consider, among other terms:

$$4\pi\mathbf{Q} = \chi^{(1)}(F) + \chi^{(2)}(F_0, F) = [\chi^{(1)} + 2\chi^{(2)}(F_0, .)](F). \qquad \text{(V.59)}$$

Hence, the background field has effectively altered the first order susceptibility. In the context of electric fields, which is of interest to optics, this is referred to as the *linear electro-optic* – or *Pockels – effect*. Similarly, the cubic term $3\chi^{(2)}(F_0, F_0, .)$ can also contribute to the linear term, which is referred to as the *quadratic electro-optic* – or *Kerr – effect* in the electric case.

Another cubic effect that gets considerable attention is due to the fact that in addition to the original frequencies and their sums and differences, one can also produce waves of other combination frequencies, such as $2\omega_1 - \omega_2$. This case is called *four-wave mixing*. However, one must always impose the constraint on the incoming and outgoing frequencies and wave numbers that they sum to zero, which is essentially a form of the conservation of energy-momentum.

When the background field is a magnetic field, there are magnetic phenomena that correspond to the electrical ones. For instance, there are linear and quadratic magneto-optical effects due to the modulation of the *electrical* susceptibilities by a background magnetic field. The conversion of circularly polarized waves into elliptically polarized ones under reflection is called the *magneto-optic Kerr effect*. The presence of non-zero magnetization in a medium can make an electrically isotropic medium behave like a uniaxial dielectric, with the associated birefringence [21], and this is referred to as the *Couton-Mouton effect*.

Many of the optical phenomena that nonlinear optics is concerned with are based in the idea that macroscopic media will generally have resonant frequencies, due to either the discrete nature of electron level transitions in atoms or the normal modes of mechanical vibrations in the crystal lattice. When the incoming frequency mixes with a resonant frequency to produce sum and difference frequencies, one gets *Raman scattering*; the difference frequency is the *Stokes* component of the scattered wave and the sum frequency is the *anti-Stokes* component. In *Brillouin scattering*, the acoustic modes of the lattice create an effective diffraction grating, producing dispersion.

The nonlinear wave equations of solitons have found considerable application in nonlinear optics, as well. In particular, the nonlinear Schrödinger equation relates to the phenomenon of *self-focusing* in cylindrical beams, while the sine-Gordon equation describes *self-transparency* in a nonlinear medium. In the former case, the index of refraction can vary with the beam intensity, which varies with radial distance from the center of the beam, and in the latter case, the frequency of the incoming wave couples to a resonant frequency that originates in an electronic level transition.

c. Nonlinear plasma phenomena [**15**]. The range over which the plasma state of matter manifests itself – indeed, over ninety percent of the matter in the universe is in that state – is sufficiently vast in scope that the breakdown of linear approximations in its mathematical modeling can also be quite broad-ranging.

Most of the nonlinear phenomena first manifest themselves in the form of nonlinear wave phenomena, which we will defer to a later discussion. However, as far as the actual constitutive properties of plasma are concerned, we can mention here that the electrical resistivity of plasmas can take on anomalous contributions due to the fact that the diffusion equation that one uses is fundamentally nonlinear, while treating it as a linear partial differential equation is only a limiting approximation.

6. Effective quantum constitutive laws. Beyond a doubt, the most challenging task for pre-metric electromagnetism is to see if some sort of formal extension of the scope of classical – i.e., linear – Maxwellian electromagnetism makes it possible to absorb some of the phenomenological successes of quantum electrodynamics into the domain of things that can be explained by the theoretical methodology that is sometimes called “classical.”

[21] Birefringence – or “double refraction” – means that the index of refraction depends upon the direction of the wave vector for the wave. As we shall see in Chapter VIII, it plays a fundamental role in pre-metric electromagnetic dispersion laws.

At this point, it is necessary to clarify that the use of the word "classical" to refer to physics or mathematics that was not actually being discussed by the classical physicists – say, the Copenhagen school of quantum physics – is incorrect, if not pejorative. The essential difference between what is currently dismissed as "classical" formalism and what is considered to be the more modern quantum formalism is essentially the difference between field theories and scattering theories.

That is, modern quantum electrodynamics (cf., for instance, Berestetskii, Lifschitz, and Pitaevski [**23**] or Jauch and Rohrlich [**24**]) does not even attempt to pose boundary-value problems in electrostatics or Cauchy problems in electrodynamics because it has long since been agreed that the details of physical processes at that level of resolution are beyond the limits of experimental measurement or observation, and therefore it would be pointless for theoretical physics to speculate on them. (Of course, this argument does not seem to apply to the nature of the early Big Bang, or physics at the Planck energy scale, or at the scale of supersymmetry breaking, all of which is many orders of magnitude beyond the limits of experiment, but nonetheless quite dominant in the topics that modern theoretical physics is concerned with!)

Rather, the methodology of quantum physics is based in the statistical interpretation of wave mechanics, combined with the notion that if the only things that experimental physics will ever "know" about the nature of physics at the atomic-to-subatomic scale must manifest themselves in the results of particle scattering experiments then there is no loss in theoretical scope associated with treating the scattering theory as if it were identical with the field theory. In fact, scattering theories follow from dynamical field theories as an asymptotic approximation whereby one does not address the time evolution of a set of fields in interaction during that interaction itself, but only the relationship of the asymptotic incoming fields at a time long before the interaction took place to the asymptotic fields at a time long afterwards (cf., e.g., Lax and Phillips [**25**]). Hence, in a sense, a scattering theory follows from a field theory in a manner that is similar to the way that static fields relate to dynamic fields.

Nevertheless, one must respect the empirical successes of quantum electrodynamics, even if the theoretical formalism seems to be manifestly limiting in its scope. Here, it helps to remind oneself that the problems of the qualitative and quantitative description of natural phenomena are quite distinct from that of the mathematical modeling of those phenomena. Hence, one can accept that charged particles have certain properties that are entirely consistent with the formalism of quantum electrodynamics without taking the position that the established formalism is the only possible solution to the problem of mathematical modeling. For instance, one must address the unavoidable facts that – at least as far as the electromagnetic interaction is concerned – there is a minimum total electric charge e that forms the basis for all larger total charges, that charged particles all have anti-particles, that the most elementary charges have non-vanishing internal angular momentum or spin, and other established truths of electromagnetism at the elementary level.

One of the recurring themes of quantum electrodynamics is that the fundamental process by which photons of more than a critical energy – viz., $2m_ec^2$, or about 1 MeV – can split into an electron-positron pair plays a fundamental role in all of the other processes. Conversely, an electron-positron pair can annihilate each other to form a photon. One calls this process and its inverse *pair creation and annihilation*,

respectively, and it can lead to what one calls *vacuum polarization*, a phenomenon that accounts for some of the most definitive experimental confirmations of quantum electrodynamics, such as the anomalous magnetic moment of the electron and the associated Lamb shift of atomic electron energy levels.

Since we have been treating the electromagnetic polarization of a medium as a purely macroscopic process that eventually resolves to the alignment of elementary atomic dipoles there is always the question of whether one might take the phrase "vacuum polarization" literally and propose to regard it as the basis for replacing the elementary constants ε_0 and μ_0 – and thus also $c_0 = 1/\sqrt{\varepsilon_0\mu_0}$ – with more elaborate functions, most likely, functions of the electric and magnetic field strengths. There are various arguments for and against taking this step.

On the one hand, one might argue that is it not really the vacuum itself that is polarizing, but the photon. That is, electron-positron pairs originate in the splitting of photons, not from regions of space in which there is not even so much as a photon present. However, this argument is weakened by other considerations.

For one thing, one regularly uses vacuum polarization as a means of accounting for the difference between the "bare" charge and the "dressed," or renormalized, charge of a particle, namely, one assumes that the dressed charge comes about as a result of the polarization of the surrounding vacuum in the presence of the high electric field strength that one finds close to elementary charges. In that event, there is no photon present, only the static electric field of the charge distribution and the static magnetic field due to its spin.

Perhaps the most compelling argument for treating vacuum polarization more generally is based in the idea that nowadays the electromagnetic vacuum is not regarded as a region of space in which no fields are present, and which is associated with various empirical constants, such as ε_0 and μ_0. Rather, the vacuum is regarded as a state in a large state space of electromagnetic fields, which is not only infinite-dimensional to begin with, but, more to the point, one must clearly distinguish the vacuum state, in the sense of energetic ground state, from the "zero" state, if indeed the space in question is a linear space, and not some more general manifold. This opens the possibility that the vacuum state might even be non-unique, as with the phenomenon of spontaneous symmetry breaking, which gives ground states non-zero expectation values for their energies.

In the case of the electromagnetic field, one must intuitively replace the classical conception of that field as being a spatial distribution of simple harmonic oscillators with its conception as a spatial distribution of *quantum* harmonic oscillators. A consequence of this is that since quantum harmonic oscillators have a non-zero ground state energy the quantum electromagnetic field must also have a non-zero ground state configuration, which one calls the *zero-point field*. Note that this field is entirely distinct from the 2.7K cosmic microwave background radiation that is also ubiquitous to space.

The existence of a zero-point electromagnetic field has actually been confirmed directly by experiments in the form of the *Casimir effect* [**26**]. This effect amounts to the fact that an ideal parallel-plate capacitor in a vacuum will experience a slight, but measurable, force of attraction between its plates.

One sees that an immediate problem with the conception of the zero-point field as a spatial distribution of quantum harmonic oscillators is that if one adheres to the constraint

that *all* of them must be in their non-zero ground state then no matter how small that energy is, the fact that there are an infinitude of points in space will make the total energy of the vacuum state infinite, as well. Hence, one must accept that "most" of that infinitude of points must be in the *zero* energy state and that only the *total* energy of the field must exist in some ground state. Of course, this suggests that the vacuum ground state is anything but unique, since once generally imagines an infinite set of possible "vacuum fluctuations" of a largely stochastic nature.

One of the compelling experimental verifications of vacuum polarization in the neighborhood of elementary charges takes the form of *Delbrück scattering*. In that process, a photon can be scattered by the electrostatic field of an atomic nucleus, which is a classically non-existent possibility, due to the fact that the combined field of the photon and nucleus might exceed the threshold for electron-positron pair production, and the electron-positron pair then couples to the nuclear field electromagnetically.

Another non-classical possibility that follows from vacuum polarization is *photon-photon scattering*. This would imply that the superposition of the photon fields is nonlinear, since the only linear effect of combining electromagnetic waves is possible interference where they intersect each other, but no lasting effects on their subsequent propagation. Interestingly, this process has yet to be experimentally verified, although the minimum energy level for observing it seems only incrementally beyond the limits of laser technology.

Although it seems unavoidable that the proper context for the mathematical modeling of the electromagnetic vacuum must involve infinite-dimensional spaces, nevertheless, since experimental physics always involves a finite set of measurements, which can only span a finite-dimensional space, one must eventually come back to *effective models* for the vacuum state. In our case, these will be effective quantum constitutive laws that are derived from effective field theories for quantum electrodynamics, so we briefly discuss that notion.

a. Effective actions [**27**-**31**]. Although we shall have more to say about the variational formulation of electromagnetism later, at this point, since we are mostly concerned with simply stating electromagnetic constitutive laws for nonlinear field theories that have emerged from quantum electrodynamics, we shall assume a minimal familiarity with conventional Lagrangian field theory.

Without going into the details here, we simply say that the difference between a full quantum field theory and an effective quantum field theory goes back to some of the early problems in quantum electrodynamics [22] that all originated in the concept of the "Dirac Sea" as a model for the electromagnetic vacuum state. In that model, electrons were positive energy states and positrons were negative energy states, with an energy gap of $2m_e c^2$ between them that represented the difference between the non-zero rest energies of the electron and the positron. The problem was that in the eyes of classical physics there is no mirror symmetry between positive energy, which represents free particle states, and negative energy, which represents bound particle states. Furthermore, an energetic state is stable iff it is minimal; i.e., iff there are no lower energy states to decay

[22] An engrossing discussion of the early years of quantum electrodynamics can be found in Miller [**32**], along with translations of many of the key papers.

into. Hence, because the energy states of the Dirac Sea went down to negative infinity they would all be unstable.

The "solution" to this dilemma was to assume that all of the negative energy states were occupied, so that the Pauli exclusion principle would prevent the runaway decay of particles to negative infinity. However, this implies that the vacuum state would have to have infinite rest mass – hence, infinite rest energy – and infinite charge, which was clearly absurd.

The approach that Heisenberg took was a combination of the "exchange-particle concept" and the so-called "subtraction of infinities". The exchange-particle concept effectively replaced all consideration of fields and forces at the fundamental level with the consideration of particles that were exchanged during interactions. Basically, subtraction of infinities amounted to treating the unphysical infinities in the vacuum state as being passive to the process of interaction. That is, the only energy states that were actually affected by the process of interaction would be a finite number of low-energy states. The subtraction of infinities eventually turned into the more modern techniques for the regularization and renormalization of unphysical infinities.

A similar transition occurs between a complete quantum formulation of a field theory and an effective theory of that field. In the complete theory one starts with a classical action for the fields in question and "quantizes" it, either by regarding some of the fields as taking their values in an operator algebra or by forming an integral over an infinite-dimensional space of fields (or, at least, gauge equivalence classes of them) that gives the transition probability for the incoming scattering state to turn into the outgoing one. This then leads to unphysical infinities, such as mass and charge, which are then corrected by the processes of regularization of the integral and renormalization of the action.

Finally, one can often define an effective action for the process in question to be a correction to the original classical action that includes the effects of renormalization. In the language of the functional integral approach, one is performing a "loop expansion" of the scattering amplitude (really, the Green function). This is an asymptotic series expansion in powers of $\hbar$, in which the "tree level," which has no loops, represents the classical action. At the one-loop level, one must renormalize the first-order radiative corrections that come from the possibility of the creation and subsequent annihilation of one electron-positron pair from a photon. Similarly, succeeding levels of approximation will involve increasing numbers of loops on external and internal lines of the Feynman diagrams for the interaction.

One sometimes hears it said that what effective field theories amount to are low-energy theories that result from integrating out the higher-energy states. This is, of course, quite reminiscent of the Heisenberg approach to the subtraction of infinities from the Dirac Sea. They can be an invaluable tool in probing the quantum domain because what they provide is a strong sense of direction when desires to go beyond the classical theories into an infinite set of possible directions.

b. Heisenberg-Euler action. What Heisenberg and Euler were addressing in their seminal paper [**27**] was essentially the question of what happens to the classical action for an electromagnetic field F in vacuo, which is based on a Lagrangian of the form:

$$\mathcal{L}_{\text{em}} = \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} = \tfrac{1}{2} \hat{\kappa}(F, F) \tag{V.60}$$

when one includes the possibility that the vacuum might also polarize. That is, the electromagnetic field might produce any number of virtual electron-positron pairs. Although $\hat{\kappa}$ is basically distinct from the ultimate nonlinear constitutive law κ; it still defines a scalar product on 2-forms, as well as an isomorphism of 2-forms and bivectors.

Hence, the full quantum treatment of this situation must include contributions to the action that represent the fields of the fermions and their interactions with the given electromagnetic field. This must then be renormalized beyond the tree level, and one finds that the one-loop correction to $\mathcal{L}_{\text{em}}$ is the effective Lagrangian:

$$\mathcal{L}_{\text{HE}} = \frac{\alpha}{2\pi}\int_0^\infty d\eta \frac{e^{-\eta}}{\eta^3}\left\{(E_c^2 - \tfrac{1}{3}\eta^2\mathcal{F}) - i\eta^2\mathcal{G}\frac{\cos\left(\frac{\eta}{E_c}\sqrt{\mathcal{F}-i\mathcal{G}}\right)+c.c}{\cos\left(\frac{\eta}{E_c}\sqrt{\mathcal{F}-i\mathcal{G}}\right)-c.c}\right\}. \qquad \text{(V.61)}$$

In this expression, $\alpha = e^2/\hbar c$ = 1/137 is the fine structure constant; the fact that it appears to the first power is associated with the fact that this is a one-loop correction. The symbol $E_c = m_e^2c^3/e\hbar$ refers to the critical field strength for pair production; it equals either 1.3×10^{16} V/cm or 4.4×10^{13} G. The notations $\mathcal{F}$ and $\mathcal{G}$ refer to the fundamental Lorentz-invariant expressions that F defines:

$$\mathcal{F} = \tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} = \hat{\kappa}(F, F), \qquad \mathcal{G} = \tfrac{1}{2}F_{\mu\nu}{}^*F^{\mu\nu} = \mathcal{V}(F, F). \qquad \text{(V.62)}$$

We have, as a consequence, that:

$$\mathcal{L}_{\text{em}} = \tfrac{1}{2}\mathcal{F}. \qquad \text{(V.63)}$$

The Lagrangian (V.61) is, of course, quite complicated to work with directly, and one most often encounters it in applications in its weak-field ($E < E_c$) Taylor series expansion:

$$\mathcal{L}_{\text{HE}} = \frac{\alpha}{360\pi E_c^2}(\mathcal{F}^2 + \frac{7}{4}\mathcal{G}^2). \qquad \text{(V.64)}$$

As Barcelo, Liberati, and Visser [**33**] point out, since this is a one-loop correction, it is absurd to carry the expansion to higher-order terms. One also needs to note that the numerical value of the leading scalar factor in c.g.s units is 10^{-42} cm^2/V^2 if one is to get some sense of how the correction term compares to the classical – i.e., tree-level – term (V.63).

In order to obtain an electromagnetic constitutive law from the combination $\mathcal{L} = \mathcal{L}_{\text{em}} + \mathcal{L}_{\text{HE}}$, one need only know at this point that in Lagrangian electromagnetics one has that the electromagnetic excitation 2-form $\mathfrak{H}$ is related to the field strength 2-form F by:

$$\mathfrak{H} = 2\frac{\partial \mathcal{L}}{\partial F} = 2\frac{\partial \mathcal{L}}{\partial \mathcal{F}}\frac{\partial \mathcal{F}}{\partial F} + 2\frac{\partial \mathcal{L}}{\partial \mathcal{G}}\frac{\partial \mathcal{G}}{\partial F} = \frac{\partial \mathcal{L}}{\partial \mathcal{F}}\hat{\kappa}(F) + \frac{\partial \mathcal{L}}{\partial \mathcal{G}}\#(F)$$
$$= \left(\frac{\partial \mathcal{L}}{\partial \mathcal{F}}\hat{\kappa} + \frac{\partial \mathcal{L}}{\partial \mathcal{G}}\#\right)(F). \qquad \text{(V.65)}$$

Hence, we can make the general expression for the constitutive isomorphism:

$$\kappa = \frac{\partial \mathcal{L}}{\partial \mathcal{F}}\hat{\kappa} + \frac{\partial \mathcal{L}}{\partial \mathcal{G}}\#. \qquad \text{(V.66)}$$

One sees from this that if $\hat{\kappa}$ does not have an axion part to begin with then the origin of an axion contribution to κ will be in the functional dependency of the field Lagrangian on $\mathcal{G}$.

For the classical electromagnetic Lagrangian $\mathcal{L}_{\text{em}}$, one finds that:

$$\frac{\partial \mathcal{L}}{\partial \mathcal{F}} = 1, \qquad \frac{\partial \mathcal{L}}{\partial \mathcal{G}} = 0, \qquad \text{(V.67)}$$

and for the Lagrangian in question – viz., $\mathcal{L} = \mathcal{L}_{\text{em}} + \mathcal{L}_{\text{HE}}$ – one has:

$$\frac{\partial \mathcal{L}}{\partial \mathcal{F}} = 1 + \frac{\alpha}{360\pi}\frac{\mathcal{F}}{E_c^2}, \qquad \frac{\partial \mathcal{L}}{\partial \mathcal{G}} = \frac{7\alpha}{360\pi}\frac{\mathcal{G}}{E_c^2}. \qquad \text{(V.68)}$$

As E_c grows arbitrarily large, the Lagrangian $\mathcal{L}$ converges to the classical vacuum expression.

When the matrix $\hat{\kappa}^{IJ}$ is associated with the classical vacuum, we find that the constitutive matrix κ^{IJ} is of the bi-isotropic form:

$$\kappa^{IJ} = \left[\begin{array}{c|c} -\varepsilon\delta^{ij} & \gamma\delta^{ij} \\ \hline \gamma\delta^{i}_{j} & (1/\mu)\delta^{ij} \end{array}\right], \qquad \text{(V.69)}$$

in which:

$$\varepsilon = \left(1 + \frac{\alpha}{360\pi}\frac{\mathcal{F}}{E_c^2}\right)\varepsilon_0, \qquad \gamma = \frac{7\alpha}{360\pi}\frac{\mathcal{G}}{E_c^2}, \qquad 1/\mu = \left(1 + \frac{\alpha}{360\pi}\frac{\mathcal{F}}{E_c^2}\right)(1/\mu_0). \qquad \text{(V.70)}$$

We see that κ^{IJ} is symmetric, so its skewon part vanishes, and it has an axion part whose proportionality factor is γ.

As we shall see later, when we discuss complex geometry, the representation of κ by a 6×6 real matrix that is given by (V.69) is equivalent (by a rescaling of the field strengths to make $\varepsilon = 1/\mu$) to its representation by the 3×3 complex matrix $(\gamma + i\varepsilon)I$.

Hence, its effect on 2-forms, when regarded as elements of a complex three-dimensional vector space, is that of complex scalar multiplication. This also shows that a constitutive law of Heisenberg-Euler type is almost-complex iff γ vanishes, hence $\mathcal{G} = 0$.

c. Born-Infeld action. Born and Infeld [**29, 30**] were motivated by the fact that Coulomb's law leads to not only an infinite field strength at the origin, but also an infinite total self-energy for an electron. They postulated that if there were a maximum allowable field strength E_c then the infinite field strength predicted by Coulomb's law for a pointlike particle would be unphysical. They deduced a largely heuristic electromagnetic Lagrangian, which is also based in the Lorentz invariants $\mathcal{F}$ and $\mathcal{G}$, as before, and which has the property that the field strength of a pointlike charge origin will be E_c at the origin.

The Born-Infeld Lagrangian is:

$$\mathcal{L}_{\mathrm{BE}} = -2E_c^2\left(1-\frac{\alpha}{180\pi}\frac{\mathcal{F}}{E_c^2}-\frac{7\alpha}{720\pi}\frac{\mathcal{G}^2}{E_c^4}\right)^{1/2}. \tag{V.71}$$

In order to compute κ from this, we only need to find:

$$\frac{\partial\mathcal{L}}{\partial\mathcal{F}}=\left(1-\frac{\alpha}{180\pi}\frac{\mathcal{F}}{E_c^2}-\frac{7\alpha}{720\pi}\frac{\mathcal{G}^2}{E_c^4}\right)^{-1/2}, \tag{V.72a}$$

$$\frac{\partial\mathcal{L}}{\partial\mathcal{G}}=\left(1-\frac{\alpha}{180\pi}\frac{\mathcal{F}}{E_c^2}-\frac{7\alpha}{720\pi}\frac{\mathcal{G}^2}{E_c^4}\right)^{-1/2}\frac{7\alpha}{360}\frac{\mathcal{G}}{E_c^2}. \tag{V.72b}$$

Thus, in the limit as E_c grows arbitrarily large these expressions also converge to the classical vacuum expressions. If we expand the expression in parenthesis by means of the binomial theorem then we get:

$$\left(1-\frac{\alpha}{180\pi}\frac{\mathcal{F}}{E_c^2}-\frac{7\alpha}{720\pi}\frac{\mathcal{G}^2}{E_c^4}\right)^{-1/2}\approx 1+\frac{\alpha}{360\pi}\frac{\mathcal{F}}{E_c^2}, \tag{V.73}$$

and we see that the partial derivatives in (V72a,b) are approximately equal to the corresponding ones (V.68) for the Heisenberg-Euler Lagrangian.

Similarly, by direct inspection, one can see that the Born-Infeld vacuum is also of the bi-isotropic variety, as was the Heisenberg-Euler vacuum, and we see that the expressions in (V.70) now take the form:

$$\varepsilon=\left(1-\frac{\alpha}{180\pi}\frac{\mathcal{F}}{E_c^2}-\frac{7\alpha}{720\pi}\frac{\mathcal{G}^2}{E_c^4}\right)^{-1/2}\varepsilon_0, \tag{V.74a}$$

$$\gamma=\left(1-\frac{\alpha}{180\pi}\frac{\mathcal{F}}{E_c^2}-\frac{7\alpha}{720\pi}\frac{\mathcal{G}^2}{E_c^4}\right)^{-1/2}\frac{7\alpha}{360}\frac{\mathcal{G}}{E_c^2}, \tag{V.74b}$$

$$1/\mu = \left(1 - \frac{\alpha}{180\pi}\frac{\mathcal{F}}{E_c^2} - \frac{7\alpha}{720\pi}\frac{\mathcal{G}^2}{E_c^4}\right)^{-1/2} (1/\mu_0). \qquad \text{(V.74c)}$$

Therefore, the essential difference between the Heisenberg-Euler constitutive law and the Born-Infeld one amounts to the differences between precise values of the corresponding scalar multipliers.

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Chapter VI

Partial differential equations on manifolds

Since the most fundamental statements of electromagnetism, in any of its representations, involve systems of partial differential equations, and we are trying to maintain a consistent use of the topological and geometric benefits of differentiable manifolds to the greatest extent possible, it is inevitable that we must say something about the representation of such systems of partial differential equations on manifolds.

We hasten to point out that such a task can be quite formidable in the modern era, since, in effect, the first problems regarding partial differential equations on manifolds to be formally addressed by the mathematics community were not concerned with redefining the elementary methods that one applies to the equations of physics, but with more general problems, such as the nature of the eigenvalues of the Laplacian operator on Riemannian manifolds, the topological aspects of the Yang-Mills field equations, or the formal integrability of overdetermined systems of partial differential equations.

Certainly, these problems are not insignificant, even insofar as physics applications are concerned. However, many of the methods of mathematical physics that get used most widely by the physics community are more in the nature of applied mathematical techniques for solving specific problems, not general techniques for proving purely mathematical theorems. Hence, the purpose of this chapter will be to show how many of these elementary concepts can be represented in terms of things that pertain to differentiable manifolds and the elementary vector bundles that relate to them. In particular, we need to eventually address the linear differential operators on the vector bundles of k-forms and k-vector fields over the spatial manifold Σ or the spacetime manifold M that take the form of d and δ.

There are three basic ways of representing a system of partial differential equations on a manifold M: one can represent it by a differential operator $L: E \to F$ between vector bundles over M, by a hypersurface in a manifold $J^k(M, N)$ of k-jets of C^k maps from M to another manifold N, or by an exterior differential system on M. The choice of method is usually determined by the class of problems that one is ultimately addressing, so we shall begin by discussing each method at an elementary level.

Next, we discuss the nature of the most common types of problems that one encounters in the partial differential equations: boundary-value problems and initial-value problems. In the case of a linear system of partial differential equations, these problems can be reformulated in terms of a linear integral equation involving an integral operator whose kernel takes the form of a Green function, so we then discuss the extent to which the usual definitions can or cannot be carried over to corresponding construction on manifolds.

Finally, we briefly discuss the much more involved problem of what becomes of the Fourier transform on a differential manifold that is not necessarily an affine space, or even a homogeneous space. Indeed, this problem has been a predictable obstacle to the formulation of the foundations of quantum wave mechanics in the language of differentiable manifolds.

1. Differential operators on vector bundles [1, 2]. As far as field theories are concerned, the most convenient way of representing most of the fields of interest to physics is by means of sections ϕ: $M \to E$ of vector bundles E over the manifold M in which the fields are assumed to reside. Since the vector space V that represents the typical fiber of E – up to linear isomorphism – can be the tensor product of other vector spaces, one sees that any tensor or spinor field on M can be represented in such a manner. In particular, electromagnetism is often concerned with those vector bundles over the space manifold Σ or the spacetime manifold M whose sections are differential forms and multivector fields.

As pointed out above, the set $\Gamma(E)$ of sections of a vector bundle $E \to M$ can be given the structure of an infinite-dimensional vector space. Basically, one defines scalar combinations of sections pointwise. That is, if ϕ, $\psi \in \Gamma(E)$ are sections and $\alpha, \beta \in \mathbb{R}$ are scalars then the scalar combination $\alpha\phi + \beta\psi$ is defined to take each $x \in M$ to the vector:

$$(\alpha\phi + \beta\psi)(x) = \alpha\phi(x) + \beta\psi(x) \tag{VI.1}$$

in the vector space E_x .

If $E \to M$ and $F \to M$ are vector bundles over M then it is simple enough to define a linear operator L: $\Gamma(E) \to \Gamma(F)$ from the vector space of sections of the former bundle to the vector space of sections of the latter. Such an operator takes any linear combination $\alpha\phi + \beta\psi \in \Gamma(E)$ to the section:

$$L(\alpha\phi + \beta\psi) = \alpha L(\phi) + \beta L(\psi) \tag{VI.2}$$

in $\Gamma(F)$.

In order to give an operator $\mathcal{O}$: $\Gamma(E) \to \Gamma(F)$, whether linear or nonlinear, a local expression as a system of equations, one must choose a local trivialization $U \times E$ of E over a subset $U \subset M$. The best way to do this is to define the frame fields $\mathbf{e}_a$: $U \to E$, $x \mapsto \mathbf{e}_a(x)$, $a = 1, \ldots, N = \text{rank}(E)$ and $\mathbf{f}_b$: $U \to F$, $x \mapsto \mathbf{f}_b(x)$, $b = 1, \ldots, N' = \text{rank}(F)$; both fields consist of N (N', resp.) sections over U that are linearly independent at each point. Hence:

$$\mathcal{O}(\phi) = \mathcal{O}(\phi^a \mathbf{e}_a) = \mathcal{O}^b(\phi^a \mathbf{e}_a)\, \mathbf{f}_b \,, \tag{VI.3}$$

in which the ϕ^a and $\mathcal{O}^b(\phi^a \mathbf{e}_a)$ are smooth functions on U that represent the components of ϕ and $\mathcal{O}(\phi)$ with respect to the chosen frames. This gives rise to the following local system of N' equations in the N unknown functions ϕ^a:

$$\mathcal{O}^b = \mathcal{O}^b(\phi^a). \tag{VI.4}$$

When $\mathcal{O}$ is linear, one can replace the right-hand side of the latter expression with the column matrix $\mathcal{O}^b_a \phi^a$ of functions on U.

An operator $\mathcal{O}$: $\Gamma(E) \to \Gamma(F)$ is called *algebraic* iff it induces a corresponding map $\mathcal{O}_x$: $E_x \to F_x$ on each fiber of E. In that case, (VI.4) also works for the individual values:

$$\mathcal{O}^b(x) = \mathcal{O}^b(\phi^a(x)). \tag{VI.5}$$

If $\mathcal{O}$ is a linear algebraic operator then $\mathcal{O}_x$ will be a linear map at each $x \in M$. Some examples of linear algebraic operators that we have previously encountered include scalar multiplication, the identity operator, left-multiplication of k-forms by a 1-form, the interior product operator on either differential forms or multivector fields, and Poincaré duality.

Of the non-algebraic types of operators on sections the two that will be of interest to us will be *differential* and *integral* operators. We shall first discuss differential operators and then return to the discussion of integral operators.

Ordinarily – i.e., when M is a vector space – one tends to think of the action of a differential operator $\mathcal{D}$ of degree k on a smooth function $f \in C^\infty(M, \mathbb{R})$ as essentially a "functional" expression:

$$\mathcal{D}f(x) = F(x, f(x), df|_x, \ldots, d^k f|_x) \tag{VI.6}$$

in the points of M, the values of f at each point, and the values of its differentials $d^m f$ up to order k.

Actually, this way of looking at partial differential equations leads directly into the methods of jet manifolds, which we will discuss in the next section. However, for now, we point out that when M is not a manifold and the function f takes its values in a vector bundle E, instead of $\mathbb{R}$, one runs into the problem of whether the differential df of the section f: $U \to E$ transforms properly under changes of local frames in E.

In general, the only way to get around this is to introduce a connection on E and replace the differential d with the covariant differential ∇ that goes with the chosen connection. However, one advantage of using differential forms, i.e., sections of $\Lambda^k(M) \to M$ is the fact that the exterior derivative operator d transforms properly without the necessity of introducing a connection.

Hence, since we will be mostly concerned with the field equations of pre-metric electromagnetism, the differential operators of order k that will be of interest to us will consist of scalar combinations of compositions:

$$\mathcal{D} = A_1 \cdot d \cdot A_2 \cdot \ldots \cdot A_k \cdot d \cdot A_{k+1} \tag{VI.7}$$

of k+1 algebraic operators and the exterior derivative operator d. Note that if any of the algebraic operators besides A_1 and A_{k+1} are a scalar multiple of the identity operator then the resulting operator $\mathcal{D} = 0$.

This restricted class of differential operators on differential forms and multivector fields nevertheless includes the exterior derivative operator, the divergence operator $\delta = \#^{-1} \cdot d \cdot \#$, and the Lie derivative operator $\mathrm{L}_\mathbf{v} = i_\mathbf{v} d + d i_\mathbf{v}$, which are all first order, as well as the field operator $\Box_\kappa = \delta \cdot \kappa \cdot d$, which is second order.

a. Integrability. Suppose $\mathcal{D}$: $\Gamma(E) \to \Gamma(F)$ is a differential operator from sections of a vector bundle $E \to M$ to sections of a vector bundle $F \to M$. A typical problem for differential equations that $\mathcal{D}$ might define could take the form of the inhomogeneous problem: Given a section $\rho \in \Gamma(F)$, find a section $\phi \in \Gamma(E)$ such that:

$$\mathcal{D}\phi = \rho\,. \tag{VI.8}$$

The homogeneous problem is defined by choosing $\rho = 0$.

In general, the inhomogeneous problem is not going to have a solution, since it will usually be *overdetermined*; that is, the map $\mathcal{D}$ might not be a surjection. In such an event the image $\mathrm{im}(\mathcal{D}) = \mathcal{D}(\Gamma(E)) \subset \Gamma(F)$ is a proper subset; when $\mathcal{D}$ is linear, it is a proper linear subspace.

When $\rho \in \mathrm{im}(\mathcal{D})$ one says that the system of equations defined by (VI.8) is *integrable*. A set of *integrability conditions* – or *compatibility conditions* – usually takes the form of a set of equations that define the subset $\mathrm{im}(\mathcal{D})$. For instance, one might have another differential operator $\mathcal{D}'$: $\Gamma(F) \to \Gamma(G)$, and then characterize $\mathrm{im}(\mathcal{D})$ as the kernel of $\mathcal{D}'$. That is, $\rho \in \mathrm{im}(\mathcal{D})$ iff:

$$\mathcal{D}'\rho = 0. \tag{VI.9}$$

Of course, in the case of the exterior derivative operator this sort of condition introduces a subtlety in the form of the fact that whether or not $\ker(d) = \mathrm{im}(d)$ at some step in the sequence $\ldots \xrightarrow{d} \Lambda^k \xrightarrow{d} \Lambda^{k+1} \xrightarrow{d} \ldots$ depends upon the vanishing of de Rham cohomology in that dimension, which then defines a topological constraint on the manifold M; analogous remarks apply to the divergence operator δ.

b. Symbol of a differential operator. If $\mathcal{D}$: $\Gamma(E) \to \Gamma(F)$ is a first-order differential operator from sections of a vector bundle $E \to M$ to sections of a vector bundle $F \to M$ then its *symbol* is a bundle map $\sigma[\mathcal{D}]$: $T^*(M) \otimes \Gamma(E) \to \Gamma(F)$ that takes any $df \otimes \phi$ to $\mathcal{D}(f\phi) - f\mathcal{D}(\phi)$. Hence, it is a linear algebraic map between the fibers $T^*_x \otimes E_x$ and F_x at each point $x \in M$. When one fixes a covector field k the map $\sigma[\mathcal{D}, k]$ takes sections of E to sections of F linearly; hence, $\sigma[\mathcal{D}, k]$: $\Gamma(E) \to \Gamma(F)$.

In the case of the ordinary differential operator, which we denote by D to avoid confusion with the exterior derivative d, since $D(f\phi) = Df \otimes \phi + fD\phi$, the symbol of D is tensor multiplication by a covector field k:

$$\sigma[D, k](\phi) = k \otimes \phi. \tag{VII.10}$$

From this, by polarization, we find that:

$$\sigma[D_s, k](\phi) = k \odot \phi = \tfrac{1}{2}(k \otimes \phi + \phi \otimes k), \qquad \text{(VII.11a)}$$

$$\sigma[d, k](\phi) = k \wedge \phi = \tfrac{1}{2}(k \otimes \phi - \phi \otimes k). \qquad \text{(VII.11b)}$$

Here, the operator D_s represents symmetrized differentiation and d represents anti-symmetrized differentiation; i.e., the exterior derivative operator.

Hence, the linear algebraic operator on sections of E that corresponds to symmetrized or anti-symmetrized differentiation is symmetrized or anti-symmetrized tensor multiplication by k, respectively. By abuse of notation, we shall denote any of the three maps $\sigma[D, k]$, $\sigma[D_s, k]$, $\sigma[d, k]$ by e_k: $E \to F$ and let the context of its usage dictate the precise meaning implied.

Had we gone the route of replacing D with a covariant differential, we would have found that the symbol of the operator would not change, since the difference between ordinary and covariant differentiation, in any case, is an algebraic operator, whose symbol is then zero. Similarly, if a differential operator is quasi-linear – i.e., linear in its highest-order derivatives, but possibly nonlinear in its lower-level ones – then the symbol (really, the *principal* symbol) of that operator is the same as that of the linear operator that is defined by its highest-order term.

The symbol of the divergence operator δ is:

$$\sigma[\delta, k](\mathbf{\Phi}) = k(\mathbf{\Phi}) = i_k\mathbf{\Phi}, \qquad \text{(VII.12)}$$

i.e., interior multiplication by k.

In order to extend this conception of the symbol of a differential operator to operators of order higher than one, we confine ourselves to linear operators of the form (VII.7) and assume that:

i. The symbol of a composition of linear differential and algebraic operators is the composition of the symbols.

ii. The symbol of a linear algebraic operator (in such a sequence) is itself.

Hence, the symbol of an operator defined of the form (VII.7) is:

$$\sigma[\mathcal{D}, k] = A_1 \cdot e_k \cdot A_2 \cdot \ldots \cdot A_k \cdot e_k \cdot A_{k+1}\,. \qquad \text{(VI.13)}$$

In the case where any of the sequences $A \cdot e_k \cdot A'$ take the form of $i_k = \#^{-1} \cdot e_k \cdot \#$, which is the symbol of δ, it is generally simpler to replace them with i_k. For instance, the symbol of the field operator $\Box_\kappa = \delta \cdot \kappa \cdot d$, when κ is linear is most simply expressed as:

$$\sigma[\Box_\kappa, k] = i_k \cdot \kappa \cdot e_k = \kappa^{\mu\nu\alpha\beta} k_\nu k_\beta\,. \qquad \text{(VI.14)}$$

c. *Characteristic variety*. Since $\sigma[\mathcal{D}, k]$: $\Gamma(E) \to \Gamma(F)$ is a linear algebraic map, there is always the question of its invertibility to be addressed. As long as E and F both have finite rank, this amounts to the question of the integrability of the linear map $\sigma[\mathcal{D}, k]_x : E_x \to F_x$ at each $x \in M$. If the dimensions of both fibers are unequal then the map can never be invertible. When they are equal, invertibility can be characterized in a

frame-invariant way by considering the determinant of $\sigma[\mathcal{D}, k]_x$. This will depend not only upon x, but also upon the choice of $k \in T_x^*$, and one calls the set of all k such that:

$$\det\{\sigma[\mathcal{D}, k]_x\} = 0 \qquad \text{(VII.15)}$$

the *characteristic variety* of $\mathcal{D}$ at x; it will then be a hypersurface in T_x^*M. This has the immediate consequence that for differential operators of the type that we are considering, from (VI.13), we obtain:

$$\det\{\sigma[\mathcal{D}, k]\} = \det(A_1)\det(e_k)\det(A_2)\ \ldots\ \det(A_k)\det(e_k)\det(A_{k+1}). \qquad \text{(VI.16)}$$

However, one immediately sees that there is a problem, since all of the linear maps in the sequence (VI.13) must be invertible in order for the product to be non-vanishing, in general, and the map e_k is not invertible in any case. It is not injective, since its kernel in dimension m will be spanned by all simple m-forms that contain at least one exterior factor of k. It is not generally surjective, except for n–1 forms, since its image will also be spanned by simple m+1-forms that contain k as an exterior factor. Therefore, in order to arrive at a non-trivial characteristic variety, one must always restrict the domain of e_k to the quotient $\Lambda^m/\ker(e_k)$ and the range to $\mathrm{im}(e_k) = k \wedge \Lambda^m$ at each $x \in M$. For instance, in the case of e_k: $\Lambda^1 \to \Lambda^2$ for an n-dimensional manifold one must restrict e_k to any n–1-dimensional subspace of Λ^1 that is transverse to the line generated by k and the range to the linear subspace of Λ^2 that is spanned by all 2-forms of the form $k \wedge \alpha$ for some 1-form α.

Now, the determinant function on an $n\times n$ matrix represents a homogeneous polynomial of degree n in the components of the matrix. The number of times that the components of k appear in the terms of the *characteristic polynomial* that is defined by $\det\{\sigma[\mathcal{D}, k]\}$, with suitable restriction on the domains and ranges, will be equal to the order of the differential operator. Hence, in the case of a m^{th}-order differential operator on sections of a vector bundle of rank n, after restriction, the characteristic polynomial will be a homogeneous polynomial in k of degree mn.

In the cases of interest to us, when the relevant constitutive laws are linear:

i. The second-order field operator Δ_ε: $\Lambda^0 \to \Lambda_0$, with $\Delta_\varepsilon = \delta \cdot \varepsilon \cdot d$, has the symbol:

$$\sigma[\Delta_\varepsilon, k] = i_k \cdot \varepsilon \cdot e_k = \varepsilon(k, k) = \varepsilon^{ij} k_i k_j, \qquad \text{(VI.17)}$$

which is also the characteristic quadratic polynomial, since the matrix is 1×1.

ii. For three-dimensional space Σ, the second-order field operator Δ_μ: $\Lambda^1 \to \Lambda_1$, with $\Delta_\mu = \delta \cdot \tilde{\mu} \cdot d$, has the symbol:

$$\sigma[\Delta_\mu, k] = i_k \cdot \mu^{-1} \cdot e_k. \qquad \text{(VI.18)}$$

However, in order to derive the characteristic polynomial we must first note that since e_k is not invertible, for each choice of $k \in \Lambda^1$ we must restrict the map $\sigma[\Delta_\mu, k]$ to a two-dimensional subspace at each point of Σ that is transverse to k, and similarly, one must

restrict oneself to the two-dimensional subspaces of Λ_1 that define its image. If θ^a, θ^3 is an adapted coframe (a = 1, 2, $k = \kappa\theta^3$) and $\mathbf{e}_a$, $\mathbf{e}_3$ is its adapted reciprocal frame then the 2×2 matrix $\sigma[\Delta_\mu, k]^{ab}$ takes the form:

$$\sigma[\Delta_\mu, k]^{ab} = -\kappa^2 \varepsilon^a_c \tilde{\mu}^{cd} \varepsilon^d_b , \qquad (\varepsilon^a_b = \varepsilon_{ab}). \tag{VI.19}$$

The characteristic polynomial is actually a degenerate quartic in κ:

$$P[\Delta_\mu, k] = \kappa^4/\mu^2, \qquad (\mu = 1/\det[\tilde{\mu}]). \tag{VI.20}$$

iii. The second-order field operator $\Box_\kappa$: $\Lambda^1 \to \Lambda_1$, with $\Box_\kappa = \delta \cdot \kappa \cdot d$, has a symbol that is given by (VI.14). We shall devote a section of chapter VIII to the discussion of this case, since it is fundamental to geometrical optics, as well as the manner by which a Lorentzian structure "emerges" from the laws of pre-metric electromagnetism.

2. Jet manifolds [2-4]. If we return to the expression (VI.6) then we see that if we were to define a manifold that represented the space that is locally described by the components (x^μ, y^a, $y^a{}_\mu$, … , $y^a{}_{\mu_1 \cdots \mu_k}$) then we could regard a system of N partial differential equations of order k in the unknown functions y^a on the n-dimensional manifold M whose local coordinates are described by the functions x^μ as a level hypersurface of a function F on this manifold:

$$F(x^\mu, y^a, y^a{}_{,\mu}, \ldots, y^a{}_{,\mu_1,\cdots,\mu_k}) = \text{const.} \tag{VI.21}$$

For instance, the linear wave equation $\Box\psi = g^{\mu\nu}(x)\psi_{,\mu,\nu} = 0$ can be defined by the 0-hypersurface of the quadratic function:

$$F(x^\mu, \psi, \psi_\mu, \psi_{\mu\nu}) = g^{\mu\nu}(x)\psi_{\mu\nu} . \tag{VI.22}$$

The function F is often restricted by the requirement that one be able to solve (VI.21) for the highest-order derivatives:

$$y^a{}_{,\mu_1,\cdots,\mu_k} = F'(x^\mu, y^a, y^a{}_{,\mu}, \ldots, y^a{}_{,\mu_1,\cdots,\mu_{k-1}}). \tag{VI.23}$$

Courant and Hilbert [**5**] call this the *normal form* for a system of partial differential equations. This also makes the system *quasilinear*, since the only possible nonlinearity in the system (VI.23) must be in the derivatives of less than maximal order.

From the implicit function theorem, the condition on F that makes this possible is that one must have:

$$\frac{\partial F}{\partial y^a{}_{,\mu_1,\cdots,\mu_k}} \neq 0 \qquad \text{(all } a, \mu_1, \ldots, \mu_k). \tag{VI.24}$$

Otherwise, one must deal with a singular system of equations whose order is reduced at some points. For instance, in the case of systems of first-order ordinary differential equations, which can be represented by vector fields on manifolds, the points at which the derivative vanishes will also be zeroes of the vector field, and therefore fixed points of its local flow.

i. Jets of functions and sections. The manifold that F is defined on in (VI.20) is called the manifold $J^k(M; N)$ of *k-jets* of C^k functions $f: M \to N$. The k-jet $j^k_x f$ of such a C^k function f at $x \in M$ is defined to be the equivalence class of all C^k functions that are defined in some neighborhood of x, which may vary with the function, and have the same values at x as functions, along with the same values of their first k derivatives. Hence, it is easy to see how this gives rise to local coordinate charts on $J^k(M; N)$ of the form $(x^\mu, y^a, y^a{}_\mu, \ldots, y^a{}_{\mu_1 \cdots \mu_k})$; note that we do not include commas in the subscripts, which will become significant shortly.

One immediately has three manifold projections for any k that are defined by:

$$\begin{aligned} &J^k(M; N) \to M, && j^k_x y \mapsto x, \\ &J^k(M; N) \to N, && j^k_x y \mapsto y, \\ &J^k(M; N) \to M \times N, && j^k_x y \mapsto (x, y). \end{aligned}$$

The first two are referred to as the *source* and *target* projections.

In general, these projections do not define fiber bundles, but only give $J^k(M; N)$ the structure of a *fibered manifold*, since they are surjective submersions; i.e., the projections have differential maps that have maximal rank at each point of $J^k(M; N)$.

Of the three possible types of sections for the projections above, the ones that are most fundamental for us are the sections of the first projection, which then take the form of differentiable maps s: $M \to J^k(M; N)$ that give the identity when composed with the projection. This basically means that the value $s(x)$ of s at each $x \in M$ is an element of the fiber $J^k_x(M; N)$. Locally, the values of $s(x)$ have the coordinates

$$s(x) = (x^\mu, y(x), y_\mu(x), \ldots, y_{\mu_1 \cdots \mu_k}(x)). \tag{VI.25}$$

Of particular interest are the *integrable* sections, for which these coordinates also locally satisfy:

$$y_\mu(x) = y_{,\mu}(x), \ldots, \; y_{\mu_1 \cdots \mu_k}(x) = y_{,\mu_1, \cdots, \mu_k}(x). \tag{VI.26}$$

The global way of characterizing integrable sections is that they represent *k-jet prolongations* of differentiable functions on M, which can be thought of as differentiable sections of the third projection above. One then notates the k-jet prolongation of a function $f(x)$ by $j^k f(x)$. It is basically defined by the function f and its first k derivatives. One is cautioned that not all sections of the projection $J^k(M; N) \to M$ are integrable.

ii. Contact form. Although the contact form can be defined for manifolds of k-jets when $k > 1$, we shall confine our attention to the case of $k = 1$ for the moment.

The integrable sections of $J^1(M; N) \to M$ have the property that the pull-backs $s^*\theta^a$ of a certain set of N one-forms θ^a on $J^1(M; N)$ by a section s vanish iff the section is integrable. The 1-forms θ^a are collectively called the *contact form* on $J^1(M; N)$ and can be locally represented in the form:

$$\theta^a = dy^a - y^a{}_\mu \, dx^\mu \, . \tag{VI.27}$$

One is cautioned that the coordinates $y^a{}_\mu$ are functions on an open subset of $J^1(M; N)$, not an open subset of M. However, pulling θ^a down to M by means of s turns dy^a into $y^a{}_{,\mu}\, dx^\mu$ and θ^a into:

$$s^*\theta^a \; = (y^a{}_{,\mu} \; - y^a{}_\mu) \, dx^\mu \qquad \text{(all } a\text{)}, \tag{VI.28}$$

in which the $y^a{}_\mu$ are now functions on M.

Hence, locally, s is integrable iff:

$$y^a{}_\mu = y^a{}_{,\mu} \qquad \text{(all } \mu, a\text{)}. \tag{VI.29}$$

iii. Differential equations. The notion of higher jet prolongations gives a particular concise way of explaining the usual process by which a differential equation (either ordinary or partial) of order k in a C^k function y on M can be converted into an equivalent system of k first-order differential equations. All that one is doing is introducing the higher jet coordinates of $J^k(M; \mathbb{R})$, such as y_μ, $y_{\mu\nu}$, …, and coupling them to the derivatives of y by means of first order differential equations $y_\mu = y_{,\mu}$, etc.

For instance, suppose that we have an nth-order ordinary differential equation in normal form:

$$\frac{d^n y}{d\tau^n} = F(\tau, y, \dot{y}, \cdots, \overset{(n-1)}{y}) \, . \tag{VI.30}$$

If one introduces supplementary variables $v = \dot{y}$, $\overset{(1)}{v} = \ddot{y}$, …, $\overset{(n-2)}{v} = \overset{(n-1)}{y}$ for the successive derivatives then (VI.30) can be converted into the equivalent system of n first-order equations:

$$\begin{cases} \dfrac{dy}{d\tau} = v, \\ \quad \vdots \\ \dfrac{d\overset{(n-2)}{v}}{d\tau} = \overset{(n-2)}{v} \, , \\ \dfrac{d\overset{(n-1)}{v}}{d\tau} = F(\tau, y, v, \cdots, \overset{(n-2)}{v}) \end{cases} \tag{VI.31}$$

Since the supplementary variables are clearly the coordinates of $J^{n-1}(\mathbb{R}, \mathbb{R})$ beyond τ and y, and the first n–1 equations in (VI.27) amounts to the vanishing of the contact form, one sees that the n^{th}-order differential equation (VI.26) is equivalent to a first-order system on $J^{n-1}(\mathbb{R}, \mathbb{R})$, as well as a hypersurface in $J^n(\mathbb{R}, \mathbb{R})$. One can thus conclude that any non-singular system of n^{th}-order differential equations (ordinary or partial, linear or nonlinear) in normal form is equivalent to a first-order system of quasilinear equations.

Of particular interest to us are k-jets of C^k sections of vector bundles $E \to M$. In principle, the definitions of the k-jet $j^k_x\phi$ of a C^k section ϕ: $M \to E$ at $x \in M$ and its k-jet prolongation $j^k\phi$ are analogous to those for more general functions between these manifolds. The difference is in the fact that the fiber structure on E translates into a fiber structure on the target projection $J^1(E) \to E$; in fact, it is a vector bundle. One can actually regard the case of general C^k functions f: $M \to N$ as a special case of C^k sections by regarding each f as a section of the trivial fiber bundle $M \times N \to M$.

The way that the k-jet formulation of systems of partial differential equations relates to their formulation in terms of differential operators is quite simple to explain if one considers the case of k-jets of sections of vector bundles. If one wishes to represent a k^{th} order differential operator (linear or not) $\mathcal{D}$: $E \to F$, by which we really mean $\mathcal{D}$ acts on sections, as a hypersurface in a manifold of jets, one need only define a C^k fiber-preserving map $\mathcal{O}$: $J^kE \to F$ such that if s: $M \to E$ is a section then $\mathcal{D}s$ takes the form $\mathcal{D}s = \mathcal{O} \cdot j^k s$. Hence, $\mathcal{O}$ plays the same role in this case that F did in the more elementary case that we first considered.

3. Exterior differential systems [6-9]. A third way of representing systems of partial differential equations that can be of advantage to some classes of problems is the method of exterior differential equations. Much of that methodology goes back to the seminal works of Cartan [**6**] and Kähler [**7**].

In general, a *differential system of rank k* on a manifold M is a vector sub-bundle $D(M) \subset T(M)$ of constant rank k. That is, one associates a k-plane in T_xM with each $x \in M$. For instance, a line field on M is a differential system of rank 1 and a field of hyperplanes is a differential system of *corank* 1.

An *integral submanifold* of a differential system of rank k on M is a submanifold σ: $N \to M$ such that tangent space to the submanifold – i.e., $d\sigma|_x(T_xN)$ – is contained $D_{\sigma(x)}M$ for each $x \in N$. When the dimension of N equals k, one calls such an integral submanifold *maximal*. The differential system $D(M)$ is called *integrable* iff there is an integral submanifold through each of its points and *completely integrable* iff there is a maximal integral submanifold through point. In the latter case, the images of the integral submanifolds partition M into what one calls a *foliation* of dimension k (or codimension $n - k$, n being the dimension of M); the integral submanifolds are then called *leaves*.

The necessary and sufficient condition for complete integrability is given by *Frobenius's theorem*, which can be stated in various forms. The one that pertains to the present definition of a differential system is that $D(M)$ is completely integrable iff the

vector space $\mathfrak{X}(D)$ of all sections of $D(M) \to M$ is closed under the Lie bracket of vector fields. This also means that $\mathfrak{X}(D)$ is a Lie subalgebra of $\mathfrak{X}(M)$; one also calls $D(M)$ *involutory* when this is true.

The way that an exterior differential system differs from a more general one is in the representation of the sub-bundle $D(M)$. Any k-form $\alpha \in \Lambda^k M$ defines an n-k-dimensional linear subspace of T_xM at each $x \in M$ by way of the set D_xM of all tangent vectors with the property that for any k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in D_xM one must always have:

$$a(\mathbf{v}_1, \ldots, \mathbf{v}_k) = 0. \tag{VI.32}$$

More concisely, one says that D_xM is the solution to the exterior algebraic equation:

$$\alpha = 0. \tag{VI.33}$$

An exterior differential system on M is then a differential system on M that is the simultaneous solution of a set of exterior algebraic equations:

$$0 = \alpha_i , \qquad i = 1, \ldots, p. \tag{VI.34}$$

in which the α_i are p differential forms of varying degree.

Of particular interest are the *Pfaffian systems*, for which all of the differential forms in the systems are 1-forms. Since 1-forms are obstructed in the same way as vector fields from being globally non-zero, one sees that in the general case – at least when M is compact – one will be dealing with differential systems with singularities.

The complete integrability of an exterior differential system of the form (VI.34) is given by a variant form of Frobenius: The exterior differential system (VI.34) is completely integrable iff:

$$0 = \alpha_i \wedge d\alpha_i \qquad \text{(all } i\text{)} \tag{VI.35}$$

which, in turn, is equivalent to the condition that there exist 1-forms η^i_j such that [23]:

$$d\alpha_j = \eta^i_j \wedge \alpha_i . \tag{VI.36}$$

The actual representation of a system of partial differential equations by an equivalent exterior differential system is not generally uniquely defined. For one thing, it is often more convenient to put higher-order partial differential equations into the form of systems of first-order equations. Furthermore, the manifold on which the exterior differential system is ultimately defined will generally have to be adapted to the nature of the problem.

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[23] One can also say that the "ideal" in the exterior algebra $\Lambda^* M$ that is generated by the set $\{\alpha_1 , \ldots, \alpha_p\}$, namely, the vector space spanned by all finite linear combinations of expressions of the form $\beta \wedge \alpha_i$ where β is arbitrary, is closed under the exterior derivative. This is yet another way of stating Frobenius that is favored by the school of Chern, et al. [**9**].

In the case of the pre-metric vacuum Maxwell equations in the form $dF = 0$, $d{*}F = 0$ on a manifold M, and with $* = \# \cdot \kappa$; one might think on first glance that they already represent an exterior differential system. However, their solutions are 2-forms, not submanifolds of M, so if one desires to define an exterior differential system whose integral submanifolds are 2-forms then it is better to realize that since a 2-form is a smooth map F: $M \to \Lambda^2 M$ it is also a four-dimensional submanifold of the ten-dimensional manifold $\Lambda^2 M$. Hence, if a solution is to be an integral submanifold, one should define the exterior differential system on $\Lambda^2 M$, not on M; in effect, one must "lift" the system from M to $\Lambda^2 M$.

Since a local coordinate system on $\Lambda^2 M$ – i.e., a local trivialization – has the form $(x^\mu, F_{\mu\nu})$, by differentiation of the local expressions $F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$, $*F = \frac{1}{2} *F_{\mu\nu}\, dx^\mu \wedge dx^\nu$, one sees that the exterior differential system on $\Lambda^2 M$ that represents the vacuum Maxwell equations takes the form:

$$\Theta^1 = \Theta^2 = 0, \tag{VI.37}$$

where [24]:

$$\Theta^1 = dF_{\mu\nu} \wedge dx^\mu \wedge dx^\nu, \qquad \Theta^2 = d{*}F_{\mu\nu} \wedge dx^\mu \wedge dx^\nu. \tag{VI.38}$$

Be careful to note that in these expressions the $F_{\mu\nu}$ and $*F_{\mu\nu}$ represent functions on $\Lambda^2 M$, not functions on M. However, if F is a section of $\Lambda^2 M \to M$ then the pull-back of Θ^1 by F becomes dF itself, and similarly one pulls Θ^2 back to $d{*}F$ by the section $*F$.

Often, one defines exterior differential systems on jet manifolds. For instance, the methods of prolongations of exterior differential systems to integrable ones that were first suggested by Cartan and later proved by Kuranishi [**10**] essentially amount to higher jet prolongations in which the supplementary variables that one is extending to are the higher jet manifold coordinates. We have already encountered one exterior differential system on a jet manifold in the form of the integrability condition for a section of $J^1(M, M) \to M$, as it is expressed in terms of the contact form θ^a, namely, the exterior differential system that is defined by the vanishing of those 1-forms. The integral submanifolds are then the integrable sections.

4. Boundary-value problems. [**11-13**]. When one is confronted with a linear differential operator L: $C^\infty(M) \to C^\infty(M)$ on smooth functions on a manifold, a natural problem to pose is that of solving the overdetermined linear differential equation:

$$Lf = \rho. \tag{VI.39}$$

As we pointed out above, the first question to address is that of integrability, which simply amounts to the statement that unless ρ is in the image $\mathrm{im}(L) = L(C^\infty(M))$ of the map L there can be no solution to begin with.

The second question to address, once one has identified the subspace of $C^\infty(M)$ in which the solutions exist, is whether a solution is unique. Of course, in the case of

[24] Actually, one can generally expand $d{*}F_{\mu\nu}$ into an expression in dx^κ and $dF_{\alpha\beta}$, but, for now, we pass over that fact for the sake of brevity.

differential operators this is generally impossible since even for the most elementary case of the operator d/dx acting on $C^\infty(\mathbb{R})$, smooth functions that differ by a constant function will map to the same derivative. Hence, one can hope to find, at best, a section of a surjection.

The issue then becomes one of how to characterize the nature of a particular section, and this is, of course, where one introduces initial/boundary conditions on the function f that would single it out from an infinite class of possibilities. Hence, we shall assume that M has boundary ∂M, so the boundary conditions on f – and possibly its derivatives – will be defined on ∂M. Therefore, the initial/boundary-value problem that is defined by (VI.39) for an integrable ρ is: Find that unique function $f \in C^\infty(M)$ that satisfies equation (VI.39) and has specified functions $\phi, \phi', \ldots \in C^\infty(\partial M)$ for the restriction to ∂M of f and its derivatives up to some order.

As long as L is linear, one will expect a solution of (VI.39) for a well-posed initial/boundary value problem to take the form of a linear operator L^{-1}: $\mathrm{im}(L) \to C^\infty(M)$, where our notation is suggestive only of the operator being a *right inverse* to L, so $LL^{-1} = I$; i.e.:

$$LL^{-1}\rho = \rho \tag{VI.40}$$

when the boundary conditions have been imposed on $L^{-1}\rho$.

One can also represent this situation in the form:

$$f = L^{-1}\rho \qquad (\mathrm{mod}\ C^\infty(\partial M)). \tag{VI.41}$$

The actual distinction between an initial-value problem and a boundary-values problem does not become clear until one looks at the specific nature of L – e.g., elliptic, hyperbolic, parabolic – and how that affects the nature of a "well-posed" problem; i.e., one that has a unique solution that depends continuously upon the given data. For instance, elliptic problems, such as might be defined by Poisson's equation, are fairly restrictive as far as what sort of boundary data one can specify.

The *Dirichlet problem* is defined by specifying only the boundary values of the function f, while the *Neumann problem* is defined by specifying only the boundary values of its normal derivative – viz., its derivative in the normal direction $\mathbf{n}$:

$$f_n = \mathbf{n}f = n^i \frac{\partial f}{\partial x^i} \tag{VI.42}$$

(Of course, we have to assume that ∂M is orientable in order to define $\mathbf{n} = \#^{-1}\mathcal{V}_\Sigma$, where $\mathcal{V}_\Sigma$ is the volume element on ∂M.)

One can also envision *mixed* or *Robin* boundary-value problems, at least when ∂M is composed of more than one connected component. In such problems, one might define f on some components and its normal derivative on others.

5. Initial-value problems [4, 5, 11]. Often, the problems that one poses in the case of dynamical fields take the form of initial-value – or *Cauchy* – problems. If one were dealing with a system of n^{th}-order ordinary differential equations then an initial-value problem would take the form of finding that particular integral curve that has a given kinematical state, in the sense of the position, and subsequent time derivatives up to order $n - 1$ at the initial time t_0 . In the case of partial differential equations, the initial point becomes an initial hypersurface, the initial position becomes the initial values of the solution function or field on the initial hypersurface, and the various subsequent time derivatives become corresponding normal derivatives of the solution.

One usually finds that only the normal derivative can be specified independently of the initial function, at least when it has derivatives of all orders involved, since that would determine the derivatives in all of the directions are tangent to the initial hypersurface.

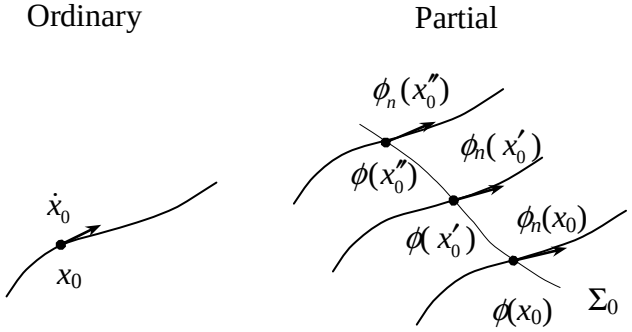

Figure 5. Initial-value problems for ordinary and partial differential equations.

a. Existence and uniqueness. It was Cauchy who first made significant progress towards the solution of the problem that bears his name, and later the proof was extended by Sonja Kowalevski [25] (see Courant and Hilbert [**5**] for the references). In essence, the Cauchy-Kowalevski theorem says that if a system of partial differential equations, in normal form, is given by an analytic function F on $J^{n-1}(M, \mathbb{R}^m)$ and the Cauchy (i.e., initial) data is analytic on some initial hypersurface Σ_0 then a unique solution to the Cauchy problem exists in a neighborhood of each point of Σ_0 for a sufficiently small time interval into the future.

One must be careful to distinguish positive time evolution from negative time evolution in the context of partial differential equations since in many cases the two are quite distinct. One can say that the "flows" of partial differential equations are then due to one-parameter *semigroups* of transformations, not one-parameter groups, as is the case for ordinary differential equations; this is especially true of diffusion equations.

Although the Cauchy-Kowalevski theorem is sufficiently general as to make no mention of the type of system that one dealing with – e.g., elliptic, parabolic, or hyperbolic – one finds that generally the Cauchy problem is not well-posed for elliptic

[25] Since there are a number of spellings for her name that are used in the literature, we follow the argument of John [**6**] in using this spelling, on the grounds that she herself used that spelling in her papers.

systems [26]. As it turns out, the very assumption that one can put the partial differential equation into normal form includes the constraint that initial hypersurface is not "characteristic," in a sense that follows from the signature type of the principal symbol. Furthermore, the restriction to analytic initial data is more severe than it sounds, since an important class of initial-value problems in the theory of waves concerns initial data that have discontinuities in their derivatives at some order, such as shock waves and acceleration waves. We shall discuss the Hadamard theory of the Cauchy problem for acceleration waves in chapter VIII.

The Cauchy-Kowalevski theorem was extended into the context of exterior differential systems by Cartan [**7**] and Kähler [**8**], although its very statement requires a considerable amount of preliminary definitions and constructions. We shall not elaborate here, but only refer the curious to Choquet-Bruhat [**9**] or Bryant, Chern, et al. [**10**]

b. Method of characteristics: first-order case. The simplest Cauchy problem is posed for a first-order partial differential equation in a real-valued function $y(x)$ on a manifold M. In local coordinates, the partial differential equation can be expressed in the form:

$$F(x^\mu, y, y_\mu) = 0, \tag{VI.43}$$

and for a function y that is a solution to this equation, one will have:

$$y_\mu = \frac{\partial y}{\partial x^\mu} \tag{VI.44}$$

in addition to (VI.43).

The initial-value problem for this class of partial differential equations is completely solvable by Cauchy's method of characteristics. This method reduces the solution of the initial-value problem for the partial differential equation (VI.43) to the solution of an initial-value problem for a system of first-order ordinary differential equations on the manifold $J^1(M; \mathbb{R})$, whose local coordinates take the form (x^μ, y, y_μ).

In order to define the characteristic equations for a first-order partial differential equation of the form (VI.43) we first regard the function F as a differentiable function on $J^1(M; \mathbb{R})$, so the partial differential equation in question represents a hypersurface in $J^1(M; \mathbb{R})$. Hence, F defines a 1-form on $J^1(M; \mathbb{R})$, namely:

$$dF = \frac{\partial F}{\partial x^\mu} dx^\mu + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial y_\mu} dy_\mu . \tag{VI.45}$$

For an integrable section s: $M \to J^1(M; \mathbb{R})$ this pulls down to:

[26] One can, however, think of equipotential hypersurfaces as "evolving" along the field lines.

$$d(s^*F) = s^*dF = \left(\frac{\partial F}{\partial x^\mu} + y_{,\mu} \frac{\partial F}{\partial y} \right) dx^\mu + \frac{\partial F}{\partial y_\mu} dy_\mu . \qquad \text{(VI.46)}$$

In order to turn (VI.45) into a vector field on $J^1(M; \mathbb{R})$, we first look at the 2-form that the contact form $\theta = dy - y_\mu\, dx^\mu$ defines, namely:

$$d\theta = dx^\mu \wedge dy_\mu . \qquad \text{(VI.47)}$$

If one has a vector field:

$$X = X^\mu \frac{\partial}{\partial x^\mu} + X^y \frac{\partial}{\partial y} + X_\mu \frac{\partial}{\partial y_\mu} \qquad \text{(VI.48)}$$

on $J^1(M; \mathbb{R})$ then one can define a 1-form using X and $d\theta$, namely:

$$i_X d\theta = X^\mu\, dy_\mu - X_\mu\, dx^\mu . \qquad \text{(VI.49)}$$

If one requires, moreover, that θ itself annihilate X, so:

$$X^y = y_{,\mu} X^\mu , \qquad \text{(VI.50)}$$

then the combined algebraic equations:

$$i_X d\theta = dF, \qquad \theta(X) = 0, \qquad s^*\theta = 0 \qquad \text{(VI.51)}$$

define X uniquely, and its local components are then:

$$X^\mu = \frac{\partial F}{\partial y_\mu} , \qquad X^y = y_{,\mu} X^\mu , \qquad X_\mu = -\left(\frac{\partial F}{\partial x^\mu} + y_{,\mu} \frac{\partial F}{\partial y} \right) . \qquad \text{(VI.52)}$$

This vector field on $J^1(M; \mathbb{R})$ is then the *characteristic vector field* for the partial differential equation that F defines. The system of $2n + 1$ ordinary differential equations that X defines is then the system of *characteristic equations*:

$$\frac{dx^\mu}{d\tau} = \frac{\partial F}{\partial y_\mu} , \qquad \frac{dy}{d\tau} = y_{,\mu} X^\mu , \qquad \frac{dy_\mu}{d\tau} = -\left(\frac{\partial F}{\partial x^\mu} + y_{,\mu} \frac{\partial F}{\partial y} \right) . \qquad \text{(VI.53)}$$

To solve the initial-value problem that is defined by giving the values $y(x_0)$ of the function y on some initial hypersurface Σ_0 in M and requiring that y be a solution to the partial differential equation (VI.43) that is defined on all of M, one first converts it into an initial-value problem for the system of ordinary differential equations (VI.53) by means of the 1-jet prolongation of y on Σ_0. Hence, each pair $(x_0, y(x_0))$ turns into a 1-jet $(x_0, y(x_0), y_{,\mu}(x_0))$ and if one projects the unique integral curve $(x^\mu(\tau, x_0), y(\tau, x_0), y_{,\mu}(\tau, x_0))$ to

X that passes through this point back down to M then one obtains a curve $x^\mu(\tau) = x^\mu(\tau, x_0)$ in M that is transverse to the initial hypersurface Σ_0 at x_0 for each choice of x_0.

The solution $y(x)$ is then obtained by the requirement that y be constant on each of the projected integral curves. Hence, in a coordinate system (τ, x_0^i) that is adapted to Σ_0 one will have:

$$y(\tau, x_0^i) = y(x_0^i) \tag{VI.54}$$

for all τ in the range of values for which the integral curve is defined.

One must observe that for the case of an F that does not depend upon y, numerous reductions of scope follow. The manifold $J^1(M; \mathbb{R})$ reduces to T^*M, $F = F(x^i, y_i)$ becomes a function on T^*M, θ becomes its canonical 1-form, so $d\theta = -\, y_i\, dx^i$ is the canonical symplectic form, and the characteristic equations of F become:

$$\frac{dx^i}{d\tau} = \frac{\partial F}{\partial y_i}, \qquad \frac{dy_i}{d\tau} = -\frac{\partial F}{\partial x^i}, \tag{VI.55}$$

which are Hamilton's equations is one regards F as a Hamiltonian function.

Note that a section s: $M \to T^*M$ can be regarded as either a covector field on M or a 1-form. It is integrable relative to d iff:

$$s = dS \tag{VI.56}$$

for some differentiable function S on M; i.e., iff it is an exact 1-form.

The partial differential equation:

$$F(x^i, y_{,i}) = 0 \tag{VI.57}$$

that one obtains for any section s: $M \to T^*M$ is then the (stationary) *Hamilton-Jacobi equation* that F defines.

Although it may seem like a restriction in scope to eliminate the functional dependency of F on y, actually, it is not. One simply adds y, which we now call τ, as an extra dimension to M – i.e., one goes to $M\times\mathbb{R}$ – so $T^*(M\times\mathbb{R})$ takes on a fiber dimension y_0 that refers to partial differentiation with respect to τ.

In order to obtain the usual time-varying Hamilton-Jacobi equation, one must replace $F(x^i, y_i)$ with $y_0 + F(x^i, y_i)$. For an integrable section, $y_0 = \partial S/\partial\tau$ and $y_i = \partial S/\partial x^i$, so the Hamilton-Jacobi equation takes the time-varying form:

$$\frac{\partial S}{\partial \tau} + F\left(x^i, \frac{\partial S}{\partial x^i}\right) = 0. \tag{VI.58}$$

One must be aware that although the method of characteristics gives a complete solution to the Cauchy problem for a first-order partial differential equation in a real-valued function, it does not extend to the case of a system of first-order equations. This is because, as we mentioned above, any n^{th}-order partial differential equation can be

converted into an equivalent system of first-order equations by introducing the successive derivatives up to order $n - 1$ as auxiliary variables.

c. Method of characteristics: second-order case. A second-order partial differential equation in a C^2 function y on M takes the form:

$$F(x^\mu, y, y_{,\mu}, y_{,\mu,\nu}) = 0. \tag{VI.59}$$

Hence, F is now a differentiable function on the manifold $J^2(M; \mathbb{R})$ of 2-jets of C^2 functions on M. A 2-jet is simply the next level of differentiation from a 1-jet, namely, an equivalence class of functions that are defined on some neighborhood of each point x M that have common values as functions at x, along with their first and second derivatives. A local coordinate system about a point $j^2_x y \in J^2(M; \mathbb{R})$ then has the form $(x^\mu, y, y_\mu, y_{\mu\nu})$, in which $y_{\mu\nu} = y_{\nu\mu}$, since they must behave like second partial derivatives.

When one considers jet prolongations beyond the first one, one must be very careful concerning a certain subtlety: the first prolongation of a section s of $J^1(M; \mathbb{R}) \to M$ does not have to be a section of $J^2(M; \mathbb{R}) \to M$; i.e., a 1-jet does not always prolong to a 2-jet. This is because if s takes the local form $s(x) = (x^\mu, y(x), y_\mu(x))$ then its first prolongation will take the form $j^1 s(x) = (x^\mu, y(x), y_\mu(x), y_{\mu,\nu}(x))$. However, unless s is an integral section to begin with, so $y_\mu(x) = y_{,\mu}(x)$ the resulting functions $y_{\mu,\nu}(x))$ will not generally be symmetric in their lower indices. Hence, the manifold $J^1(J^1(M; \mathbb{R}))$ is higher-dimensional than $J^2(M; \mathbb{R})$, which can be embedded as a submanifold in the latter manifold.

The Cauchy problem for a second-order partial differential equation of the form (VI.59) for an initial (orientable) hypersurface Σ_0 in M then amounts to defining not only the values of y on Σ_0, but also its normal derivative y_n .

Although it would be nice if there were a characteristic vector field on $J^2(M; \mathbb{R})$ that would allow one to convert the initial-value problem for the partial differential equation (VI.55) into an initial-value problem for a system of ordinary differential equations, alas, such is not the case. However, as we shall see in the next chapter, there is a process that is almost as convenient: One first converts the second-order partial differential equation into a – generally, nonlinear – first-order partial differential equation, which one also calls the "characteristic equation," and which usually captures the essentials of the second-order equation to a lesser extent. One then solves this first-order partial differential equation by the method of characteristics, and, to avoid confusion, one calls the characteristic equations for the first-order equation "bicharacteristic" equations for the second-order one.

6. Distributions on differential forms [15-17]. In order to lead into a proper treatment of solving boundary-value problems for systems of linear differential equations – whether ordinary or partial – one must generalize from differential operators that act on

functions or differential forms to differential operators that act on distributions, which are also sometimes called "generalized functions." This is largely due to the fact that in order to define the Green function for an integral operator, which we shall do in the next section, one must unavoidably introduce the Dirac delta function, which is not a function, at all, but a distribution. Consequently, the Green function itself becomes a distribution, as well, and the defining equation makes sense only in the distributional sense.

a. Continuous linear functionals. The infinite-dimensional vector spaces $\Lambda^k M$ and $\Lambda^k_C M$, which consists of smooth *k*-forms with compact support, can be given topologies, although we shall not elaborate here [27], except to say that one can make sense of the notion of a *null sequence* of *k*-forms; viz., a sequence α_i, $i = 1, 2, \ldots$ of *k*-forms that converges uniformly to 0. One refers to the elements of $\Lambda^k_C M$ as *test k-forms.*

A linear functional *T* on either vector space is called *continuous* if the sequence of real numbers $T[\alpha_i]$ converges to 0 for every null sequence α_i. A *distribution* on either space $\Lambda^k M$ and $\Lambda^k_C M$ is a continuous, linear functional on that space. Hence, a distribution is an element of the dual spaces to $\Lambda^k M$ and $\Lambda^k_C M$, as topological vector spaces, which we denote by $(\Lambda^k)'$ and $(\Lambda^k_C)'$, resp. In the case of distributions on test *k*-forms, the term that de Rham used was "currents," and they defined the foundations for his formulation of his famous theorem.

This latter fact gives us our first example of a distribution on *k*-forms, regardless of whether their support is compact or not, namely, any *k*-chain $c_k \in C_k(M; \mathbb{R})$ defines a distribution by way of:

$$c_k[\alpha] = \int_{c_k} \alpha \,. \tag{VI.60}$$

Hence, one has a linear map [.]: $C_k(M; \mathbb{R}) \rightarrow (\Lambda^k M)'$ that takes the *k*-chain c_k to the distribution c_k[.]. Since we have not given $C_k(M; \mathbb{R})$ a topology, it is meaningless to speak of continuity for this map, but we shall not need that condition, anyway. However, we can say something about injectivity, which amounts to the issue of whether the kernel of the map [.] is trivial or not. An element of the kernel is a *k*-chain such that any *k*-form will integrate to zero over it. However, that can only be the 0-chain, since we are not allowing degenerate *k*-chains of dimension less than *k*, which might have "measure zero." Hence, the map in question is injective. It is not surjective, but the image of $C_k(M; \mathbb{R})$ is dense in $(\Lambda^k M)'$ (see de Rham [**15**]).

Any $(n - k)$-form α defines a distribution on test *k*-forms by way of:

$$\alpha[\beta] = \int_M \alpha \wedge \beta \,. \tag{VI.61}$$

[27] The details can be found in de Rham [**15**] or Hörmander [**16**]. For a general discussion of distributions, one might confer Friedlander [**17**].

Now, we have a linear map $[.]: \Lambda^{n-k} \to (\Lambda^k_C)'$, $\alpha \mapsto \alpha[.]$. It is continuous and injective since $\alpha \wedge \beta = 0$ for all β iff $\alpha = 0$.

Finally, since the evaluation $\alpha(\mathbf{A})$ of a k-form α on a k-vector field $\mathbf{A}$ produces a function, when α has compact support, one can use this to define yet another distribution on $\Lambda^k_C M$:

$$\mathbf{A}[\alpha] = \int_M \alpha(\mathbf{A})\, \mathcal{V} . \tag{VI.62}$$

b. Operations on distributions. Many of the operations that that one performs on differential forms can be performed on distributions, as well. For instance, one can clearly form scalar combinations of distributions.

One can form the exterior product of a distribution T that acts on $k+l$-forms with an l-form α to form a distribution $T \wedge \alpha$ that acts on k-forms:

$$(T \wedge \alpha)[\beta] = T[\alpha \wedge \beta]. \tag{VI.63}$$

We can thus define left-multiplication of distributions by α as a linear map $l_\alpha : (\Lambda^{k+l})' \to (\Lambda^k)'$, which then works more like the operator of taking the interior product by α does on $(k+l)$-vector fields.

The tensor product $T_1 \otimes T_2$, of two distributions can be defined as a distribution on $\Lambda^k \otimes \Lambda^l$:

$$(T_1 \otimes T_2)[\alpha \otimes \beta] = T_1(\alpha)\, T_2(\beta). \tag{VI.64}$$

Similarly, the exterior product of two distributions $T_1 \in (\Lambda^k)'$ and $T_2 \in (\Lambda^l)'$ can be defined as a distribution $(T_1 \wedge T_2)$ on Λ^{k+l}, but the anti-symmetrization of (VI.63) is not always immediate. Thus, we can define the exterior algebra $(\Lambda^*)'$ of distributions on differential forms as being generated by $(\Lambda^1)'$.

The interior product of a distribution $T \in (\Lambda^k)'$ with a vector field $\mathbf{v}$ can be defined predictably:

$$(i_\mathbf{v} T)[\alpha] = T[i_\mathbf{v} \alpha]. \tag{VI.65}$$

Hence, it is a linear map $i_\mathbf{v} : (\Lambda^{k-1})' \to (\Lambda^k)'$, so it behaves more like $e_\mathbf{v}$ does on Λ_{k-1} .

The divergence of a distribution T can be defined to be adjoint to the exterior derivative of the differential forms that it acts on:

$$\delta T[\alpha] = -(-1)^{n-k}\, T[d\alpha]. \tag{VI.66}$$

The choice of sign comes from the product rule for the exterior derivative if one uses $T = \beta[.]$, where β is an $(n-k)$-form:

$$d(\beta \wedge \alpha) = d\beta \wedge \alpha + (-1)^{n-k}\, \beta \wedge d\alpha \tag{VI.67}$$

which vanishes since $\beta \wedge \alpha$ is an n-form on an n-dimensional manifold.

We can also define distributions on the vector spaces Λ_k or $\Lambda_{k,C}$ by similar means to the foregoing. An $(n-k)$-chain c_{n-k} defines a distribution on Λ_k by way of:

$$c_{n-k}[\mathbf{A}] = \int_{c_{n-k}} \#\,\mathbf{A}\,. \tag{VI.68}$$

A k-form α defines a distribution on $\Lambda_{k,C}$ by way of:

$$\alpha[\mathbf{A}] = \int_M \alpha(\mathbf{A})\,\mathcal{V} = \int_M \alpha \wedge \#\,\mathbf{A}\,. \tag{VI.69}$$

We denote the topological vector spaces of distributions on Λ_k and $\Lambda_{k,C}$ by $(\Lambda_k)'$ and $(\Lambda_{k,C})'$, predictably.

The Poincaré duality isomorphism #: $\Lambda_k \to \Lambda^{n-k}$ can be transposed to a corresponding isomorphism #′: $(\Lambda^{n-k})' \to (\Lambda_k)'$ in the obvious way:

$$\#'T[\mathbf{A}] = T[\#\mathbf{A}]. \tag{VI.70}$$

The exterior derivative operator d: $(\Lambda_k)' \to (\Lambda_{k+1})'$ is basically the transpose of the divergence operator on Λ_{k+1}:

$$d\tau[\mathbf{A}] = -(-1)^{n-k}\tau[\delta\mathbf{A}]. \tag{VI.71}$$

In order to verify this, it is simplest to use $\tau[\mathbf{A}]$ in the form $\int_M \alpha \wedge \#\,\mathbf{A}$ and apply the product rule for exterior derivatives, as before.

c. Vector-valued distributions. It will prove necessary for us to extend our notion of a real-valued distribution on differential forms to that of a vector-valued distribution on differential forms. We define a *vector-valued distribution* on differential forms to be a continuous linear functional on Λ^k or Λ^k_C that takes its values in a topological vector space V, such as $\mathbb{R}^m$, a specified fiber E_x of a vector bundle $E \to M$, or the vector space $\Gamma(E)$ of sections of that vector bundle.

One example of a vector-valued distribution on k-forms is the *evaluation functional* $\delta_x[]$ for $x \in M$, which takes any k-form α to its value at x:

$$\delta_x[\alpha] = \alpha_x\,. \tag{VI.72}$$

Hence, the vector space V is the fiber Λ^k_x of the vector bundle $\Lambda^k M \to M$.

Another elementary example of a vector-valued distribution on k-forms is the *identity map*:

$$I[\alpha] = \alpha\,. \tag{VI.73}$$

The vector space V in this case is Λ^k itself.

A broad class of vector-valued distributions on differential forms is defined by *integral operators*, which take the form K: $\Lambda^k_C \to \Lambda^l_C$:

$$K[\alpha(y)] = \beta(x) = \int_M K(x,y) \wedge \alpha(y)\,. \tag{VI.74}$$

in which the integration is over all $y \in M$. (The fact that the notation for the independent variable changes is irrelevant to the fact that both x and y range over all of M.)

The *kernel* $K(x, y)$ of the operator then takes its values in the vector space $\Lambda_x^l \otimes \Lambda_y^{n-k}$ for each pair $(x, y) \in M \times M$; the integral operator K can then be regarded as a *two-point distribution.* In local coordinate terms, the kernel will be expressible in the form:

$$K(x, y) = \frac{1}{l!(n-k)!} K_{i_1 \cdots i_l, j_1 \cdots j_{n-k}}(x, y) dx^{i_1} \wedge \cdots \wedge dx^{i_l} \otimes dy^{j_1} \wedge \cdots \wedge dy^{j_{n-k}} . \qquad \text{(VI.75)}$$

The kernel will be called *decomposable* iff it takes the form $K(x, y) = \alpha(x) \otimes \beta(y)$, which implies that the local component functions take the form:

$$K_{i_1 \cdots i_l, j_1 \cdots j_{n-k}}(x, y) = K'_{i_1 \cdots i_k}(x) K''_{j_1 \cdots j_l}(y) . \qquad \text{(VI.76)}$$

It is commonplace to represent both the evaluation functional $\delta_x[.]$ and the identity operator $I[.]$ in terms of integral operators, even though both operators are purely algebraic – hence, local – and not integral operators, which are global in character:

$$\delta_x[\alpha] = \int_M \delta(x, y) \wedge \alpha(y) = \alpha(x), \qquad \text{(VI.77a)}$$

$$I[\alpha(x)] = \int_M I(x, y) \wedge \alpha(y) = \alpha(x). \qquad \text{(VI.77b)}$$

The fictitious kernel for the former distribution is the *Dirac delta function* $\delta(x, y)$; the equally fictitious kernel $I(x, y)$ is referred to as the *reproducing kernel.*

d. Fredholm theory. What Fredholm was originally concerned with was the solution of three basic classes of integral equations. When expressed in terms of an integral operator K that is the right inverse to a differential operator D they take the forms:

$$K\rho = f, \qquad \text{(VI.78a)}$$

$$(K - \lambda I)\rho = f, \qquad \text{(VI.78b)}$$

$$(K - \lambda I)\rho = 0. \qquad \text{(VI.78c)}$$

One then looks for solutions in the form of ρ.

Hence, we see that what he was concerned with were the solutions to the general inhomogeneous differential equation $Df = \rho$ and two equations that grew out of the eigenvalue equation for K:

$$K\rho = \lambda\rho . \qquad \text{(VI.79)}$$

The non-vanishing eigenvalues λ of K are easily seen to be inverses to the eigenvalues of D since one must have $DK = I$.

As for the zero eigenvalues, the corresponding eigenfunctions ρ belong to the kernel of the operator K. Hence, since a linear operator is injective iff it has a vanishing kernel,

one deduces the *Fredholm alternative:* Either (VI.78c) has a non-vanishing solution for $\lambda = 0$ or (VI.78a) has a unique solution for any f in the image of K.

7. Fundamental solutions [12-14, 18-20]. As we observed above, solving systems of linear differential equations, whether ordinary or partial, amounts to finding a right inverse to the linear differential operator that satisfies some boundary/initial-value problem that makes the solution exist uniquely. Such a right inverse will then represent a linear integral operator on the vector space of functions or sections that the differential operator acts on. In this section, we shall discuss the nature of this construction when one is concerned with functions and sections on more general manifolds than vector spaces.

a. General definitions. When the integral operator in question is the right-inverse operator D^{-1} that is associated with the differential operator D, the two-point function $\gamma(x, y)$ that represents the kernel of the integral operator is called a *fundamental solution*. Since the operator equation is $DD^{-1} = I$, the distributional equation that is associated with it is:

$$D\gamma_y(x, y) = -\delta(x, y). \tag{VI.80}$$

The subscript y indicates the independent variables that the differentiation affects.

Let us illustrate the representation of the solution of a boundary-value problem for a differential equation by fundamental solutions in the simplest possible case of solving:

$$\frac{df(x)}{dx} = \rho(x), \qquad f(0) = f_0 . \tag{VI.81}$$

A simple quadrature gives the solution as:

$$f(x) = f_0 + \int_0^x \rho(y)dy . \tag{VI.82}$$

Note that the only definitive restrictions on f and ρ are that they be differentiable and integrable, respectively.

At this point, we can introduce the fundamental solution:

$$G(x, y) = \begin{cases} 1 & x \le y, \\ 0 & \text{otherwise}, \end{cases} \tag{VI.83}$$

whose graph looks like:

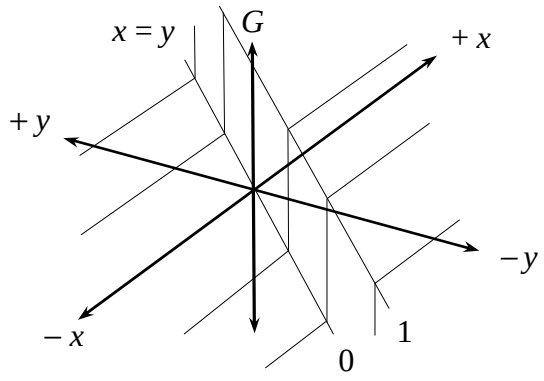

Figure 6. The fundamental solution for *d/dx*.

Note that both of the partial derivatives of $G(x, y)$ are zero everywhere except for the diagonal subset $\Delta = \{(x, y) \mid x = y\}$, where they are undefined and essentially infinite:

$$\frac{\partial G}{\partial x} = \frac{\partial G}{\partial y} = \begin{cases} 0 & x \neq y \\ \text{"}\infty\text{"} & x = y. \end{cases} \tag{VI.84}$$

(Our quotation marks around ∞ are there to suggest that it is more proper to say that the functions are simply undefined on the diagonal.)

Let us apply $\partial G/\partial y$ to $f(y)$ as a distribution:

$$\begin{aligned} \partial G/\partial y\,[f] &= \int_0^1 \frac{\partial G}{\partial y} f(y)\,dy = \int_0^1 \frac{\partial (Gf)}{\partial y}\,dy - \int_0^1 G(x, y)\frac{df}{dy}\,dy \\ &= \int_0^1 d(Gf) - \int_0^1 G(x, y)\rho(y)dy\,. \end{aligned} \tag{VI.85}$$

The first integral vanishes since it is equal to $G(x, 1)f(1) - G(x, 0)f(0)$ and the second integral equals:

$$-f(x) = -\int_0^1 \delta(x, y)\, f(y)dy\,. \tag{VI.86}$$

Hence, as a distributional equation, we have:

$$\frac{\partial G(x, y)}{\partial y} = -\,\delta(x, y). \tag{VI.87}$$

One finds that generally fundamental solutions are undefined on the diagonal, and not merely discontinuous, as in the present case.

The fundamental solution $\gamma(x, y)$ can also be used to construct solutions to boundary-value problems for (VI.39). One starts with *Green's formula* as the definition of a *self-adjoint* linear operator L: $C^\infty(M) \to C^\infty(M)$:

$$\int_M (uLv - vLu)\mathcal{V} = \int_{\partial M} \left(uv_n - vu_n\right)\mathcal{V}_{\partial M} \,. \tag{VI.88}$$

Then, one applies this to the particular case in which the function v is the fundamental solution for L and u is a solution to the inhomogeneous equation $Lu = \rho$. (VI.88) then takes the form:

$$u(x) = \int_M \gamma(x,y)\rho(y)\mathcal{V}_y + \int_{\partial M} \frac{\partial \gamma(x,y)}{\partial n_y} u(y)\mathcal{V}_{\partial M} - \int_{\partial M} \gamma(x,y)\frac{\partial u(y)}{\partial n_y}\mathcal{V}_{\partial M} \,. \tag{VI.89}$$

The first integral on the right-hand side is called the *volume potential*, and it represents the contribution to u that is due to the presence of a non-zero source ρ. The second one is called the *single-layer surface potential*, and one sees that it is driven by the boundary values of u, while it is the normal derivative of γ that serves as the integral kernel. The final term is referred to as the *double-layer surface potential*, and one sees that it is driven by the normal derivatives of u on ∂M.

In the homogeneous case, where $\rho = 0$, one can think of the boundary-value functions f and f_n as the "source" of the field f inside of the boundary ∂M.

In the case where L is the Laplacian operator Δ, if one poses the Dirichlet problem for u then this implies the boundary condition on γ:

$$\gamma|_{\partial M} = 0, \qquad \frac{\partial \gamma(x,y)}{\partial n_y} = G(x,y), \tag{VI.90}$$

in which $\partial/\partial n_y$ refers to the normal derivative in terms of the y variables. The function $G(x, y)$ is then referred to as the *Green function* [28] for this problem.

If one poses the Neumann problem then the corresponding boundary conditions for γ are:

$$\left.\frac{\partial \gamma(x,y)}{\partial n_y}\right|_{\partial M} = 0, \qquad \gamma(x,y) = N(x,y), \tag{VI.91}$$

and one refers to the function $N(x, y)$ as the *Neumann function* for the problem.

The reader should be advised that it quite commonplace for the term "Green function" to be used as a generic reference to all integral operator kernels. In particular, the usage of that term in quantum field theory is not specific to the Dirichlet problem, and indeed it is entirely possible for a Green function to represent something that is not even a differential operator, such as a pseudo-differential operator.

The method of Green functions can still be used in the case of hyperbolic linear second-order partial differential equations. In particular, the construction of a solution to the Cauchy problem for a linear forced wave equation $\Box u = \rho$ by means of (VI.89) is still

[28] As pointed out by Jackson [**18**], Rohrlich [**19**], and others, the popular phrases "the Green's function" and "a Green's function" are ungrammatical, since one should not mix articles with possessive adjectives. After all, one does not say "the Laplace's equation" or "a Bessel's function."

valid. The main difference is that one must use "fractional hyperbolic potentials" in order to define the Green function, and this has the consequence that the character of the resulting wave solution depends upon whether the dimension of M is even or odd. One can also compute the Green function for $\Box$ by means of the Fourier transform, which we shall discuss below.

b. Topological integral operators [29]. Since we have already seen that the linear differential operators d and δ have topological significance, by way of de Rham cohomology and homology, respectively, it should come as no surprise that their operators also have topological significance.

A *chain map* f: $C_*(M) \to C_*(N)$ from one chain complex $C_*(M)$ to another one $C_*(N)$ is a linear map that commutes with the boundary operator ∂. That is, if c_k is a k-chain in M then the boundary of $f(c_k)$ is the image of c_k under f; in other words, a chain map takes boundaries to boundaries.

A *chain homotopy* from one chain map f to another chain map g is a linear operator H: $C_*(M) \to C_{*+1}(N)$, such that:

$$\partial H + H\partial = f - g. \tag{VI.92}$$

When this is applied to a k-cycle z_k the result is:

$$\partial H z_k = f(z_k) - g(z_k). \tag{VI.93}$$

Since f and g are chain maps the images $f(z_k)$ and $g(z_k)$ will be cycles in some dimension and (VI.92) then says that they will also be homologous. Hence, f and g induce the same map in homology since this means that $f[z_k] = g[z_k]$.

In particular, one might consider *chain contractions*, which are chain homotopies from the identity map to the zero map, which makes:

$$\partial H + H\partial = I. \tag{VI.94}$$

When applied to a specific k-chain c_k this says:

$$\partial H c_k + H\partial c_k = c_k\,. \tag{VI.95}$$

In particular, when applied to a k-cycle z_k, the result is:

$$\partial H z_k = z_k\,. \tag{VI.96}$$

However, here we see the limitation on whether such an H can even exist, since (VI.96) says that the k-cycle z_k must also be a k-boundary, namely, the boundary of the k+1-chain Hz_k ; that is, z_k must be homologous to zero. If H is to be defined on all k-cycles then one must have that they are all k-boundaries; i.e., $H_k(\Sigma)$ must be trivial.

[29] For the basic definitions in homology, see Vick, Rotman, or Greenberg, as cited in Chap. II. The representation in terms of differential forms is given in de Rham[**15**] or Bott and Tu, also cited in Chap. II.

We can represent this situation in de Rham homology, which pertains directly to static electric and magnetic fields. Now, we represent a chain contraction by a map H: $\Lambda_k \rightarrow \Lambda_{k+1}$ such that:

$$\delta H + H\delta = I. \tag{VI.97}$$

When applied to a k-cycle $\mathbf{A}$ this becomes:

$$\delta H\mathbf{A} = \mathbf{A}, \tag{VI.98}$$

which is to say that H is a right-inverse for the linear differential operator δ.

A local representation for the operator H can be defined about any point $x_0 \in M$ that is associated with a coordinate chart (U, x^μ), which is associated with a *radius vector field* $\mathbf{r}$:

$$\mathbf{r}(x) = x^\mu(x)\frac{\partial}{\partial x^\mu}. \tag{VI.99}$$

This vector field vanishes at the point x_0, which corresponds to the origin in $\mathbb{R}^n$ under the coordinate map.

By means of this vector field, one can define a differentiable curve $x(\lambda)$ from x_0 to any other $x \in U$ by means of $\lambda\mathbf{r}(x)$, where $\lambda \in [0, 1]$, and we abbreviate the expression $\lambda\mathbf{r}(x)$ to simply λx. If we are given an arbitrary k-vector field $\mathbf{A} \in \Lambda_k$ then we can define a k+1-vector field along this curve by way of $e_\mathbf{r}\mathbf{A}(\lambda\mathbf{r}) = \mathbf{r} \wedge \mathbf{A}(\lambda\mathbf{r})$ and a k+1-vector-valued 1-form $\lambda^{k-1}\, \mathbf{r} \wedge \mathbf{A}(\lambda\mathbf{r})\, d\lambda$. By integration along the curve, this gives a k+1-vector at x:

$$H\mathbf{A}(x) = \int_0^1 \lambda^{k-1}\, \mathbf{r} \wedge \mathbf{A}(\lambda x)\, d\lambda. \tag{VI.100}$$

As for the exterior derivative operator d: $\Lambda^k \rightarrow \Lambda^{k+1}$, a cochain contraction operator for it will take the form H': $\Lambda^{k+1} \rightarrow \Lambda^k$ such that:

$$dH' + H'd = I. \tag{VI.101}$$

Since the operator d is the transpose to the operator δ, the operator H' is the transpose to the operator H above. All that is actually necessary for the construction of H is to replace the operator $e_\mathbf{r}$ with its transpose $i_\mathbf{r}$. Hence, if $\alpha \in \Lambda^{k+1}(U)$ then we can define a k-form by means of $i_\mathbf{r}\alpha$, and for every curve of the form $x(l)$ we can then define a 1-form with values in $\Lambda^k(U)$ out of $\lambda^k\, i_\mathbf{r}\alpha_{\lambda x}\, d\lambda$. We then obtain our desired k-form $H'\alpha$ by integration:

$$H'\alpha_x = \int_0^1 \lambda^k i_\mathbf{r}\alpha_{\lambda x}\, d\lambda. \tag{VI.102}$$

Although we have constructed integral operators that represent right-inverses to the divergence and exterior derivative operators, nevertheless, we can see that they are not presented in a form that would suggest a Green function for a kernel. For one thing, we

are not integrating over *M* itself. In order to define such Green functions, one generally needs to look at specific problems, such as radially-symmetric solutions, which we shall do in the next chapter, since we are getting back into the realm of traditional electromagnetism.

8. Fourier transforms [14, 16, 17, 21, 22]. Obviously, since entire books have been written on just the subject of Fourier analysis, not to mention its applications to physics, we cannot hope to make a very far-reaching discussion of it in a single section of a chapter. However, since the ultimate objective of this book is to examine the aspects of electromagnetism that have a pre-metric character to them, we cannot avoid some discussion of Fourier analysis as it relates to such considerations.

a. Linear theory. Although sometimes Fourier transforms are defined in terms of scalar products, nevertheless, upon closer inspection one finds that as long as one regards a wave vector *k* as a *covector* the real issue is not one of scalar products but *bilinear pairings* of vectors and covectors. The scalar product then makes its first appearance in the context of the inner product on the function space in question.

Just to make the discussion more specific, let us first examine the usual Fourier transform for a complex function *f* on an *n*-dimensional vector space *V*:

$$\hat{f}(k) = \mathcal{F}f[k] = \int_V e^{-ik(x)} f(x)\, \mathcal{V}\,. \tag{VI.103}$$

In this definition, $k \in V^*$, so $\hat{f}$ – or $\mathcal{F}f$ – is a complex function on V^*, and $\mathcal{V}$ is the volume element for *V*.

Following Duistermaat [**21**], we restrict ourselves to complex functions in the *Schwartz spaces* $\mathcal{S}$ and $\mathcal{S}'$, which consist of complex functions on *V* and V^*, respectively, that satisfy the constraint that the local functions $x^\alpha(\partial^\beta f / \partial x^\beta)$ [$k_\alpha(\partial^\beta \hat{f} / \partial k_\beta)$, resp.] are bounded for all multi-indices α, β. (A *multi-index* is a notation for abbreviating elaborate algebraic expressions, as one often encounters in multivariable analysis. In the former expression, x^α means a product $x^{\alpha_1}x^{\alpha_2}\cdots x^{\alpha_k}$ while $\partial^\beta f / \partial x^\beta$ refers to the mixed partial derivative of $f(x^1, \ldots, x^n)$ with respect to the multi-indexed variables, and analogously for the expressions in *k* and $\hat{f}$.) The motivation for this constraint is to be found in the extension of Fourier transforms to distributions, namely, that it is not enough to be dealing with smooth functions of compact support, but one must restrict oneself to "tempered" distributions.

One then finds that the Fourier transform can be regarded as a linear isomorphism $\mathcal{F}\colon \mathcal{S} \to \mathcal{S}', f \mapsto \hat{f}$, whose inverse transform is:

$$\mathcal{F}^{-1}\hat{f}[x] = \frac{1}{(2\pi)^n}\int_{V^*} e^{ik(x)}\hat{f}(k)\,\mathcal{V}_k\,; \tag{VI.104}$$

this time $\mathcal{V}_k$ is the volume element on V^*.

There are various ways of interpreting the preceding constructions in terms of other mathematical concepts. For instance, one can think of the set $B = \{e^{-ik(x)} \mid k \in V^*\}$ as an uncountable basis for the vector space $\mathcal{S}$, and similarly, the set $B' = \{e^{ik(x)} \mid x \in V\}$ defines an uncountable basis for $\mathcal{S}'$. If one restricts the function f to some compact subset of V then the former basis can be reduced to a countable one.

The integrals:

$$<f, g> = \int_V f(x)\overline{g(x)}\mathcal{V}, \qquad < \hat{f}, \hat{g} > = \frac{1}{(2\pi)^n}\int_{V^*} \hat{f}(k)\overline{\hat{g}(k)}\mathcal{V}_k \qquad \text{(VI.105)}$$

then define inner products on $\mathcal{S}$ and $\mathcal{S}'$ that make them into Hilbert spaces. One of the forms that *Parseval's identity* takes is to say that the map $\mathcal{F}$ is also an isometry; i.e.:

$$<f, g> = < \hat{f}, \hat{g} > . \qquad \text{(VI.106)}$$

The integrands in the right-hand sides of (VI.103) and (VI.104) then represent the orthogonal projections of f ($\hat{f}$, resp.) onto the orthonormal bases that are defined by the sets B and B', while the integrals themselves represent the orthogonal decompositions of f ($\hat{f}$, resp.) with respect to the bases.

In this way of regarding Fourier transforms, the function $\hat{f}$ is considered to be the *spectral density* of the function f. In effect, it gives the probability density function (when suitably normalized) for the spectrum of wave numbers that contribute to the orthogonal decomposition of f.

One can also regard $\mathcal{F}$ as an integral operator whose kernel $K(k, x)$ is the complex function K on $V^* \times V$:

$$K(k, x) = e^{-ik(x)} . \qquad \text{(VI.107)}$$

The kernel of $\mathcal{F}^{-1}$ is then simply $\overline{K(k, x)}$.

One finds that the Fourier transform of $\partial f / \partial x^i$ is $ik_i \hat{f}$. Hence, one can think of the linear operator on $\mathcal{S}'$ that is defined by multiplication by ik_i as the Fourier transform of the operator of partial differentiation by x^i on $\mathcal{S}$.

This can be extended to linear differential operators with constant coefficients. Suppose the operator in question is $D = a^\alpha \partial^\alpha / \partial x^\alpha$ and it has order m. It Fourier transform – or *symbol* – is then the generally inhomogeneous polynomial of degree m in k:

$$\mathcal{F}[D](k) = a^\alpha (ik_\alpha). \qquad \text{(VI.108)}$$

The homogeneous polynomial $\sigma[D, k]$ of degree m in k that represents the highest-degree terms in the symbol is called the *principal symbol.* It agrees with the more abstract definition that we cited above, except for the factor of i before k.

One can find the Green function for the linear differential operator D when it has constant coefficients by means of the Fourier transform. However, one must take into account that the Fourier transform of the distributional equation $DG = -\delta$ is $\mathcal{F}[D](k)\hat{G}(k) = 1$. Hence, one can solve for the Fourier transform of the desired Green function:

$$\hat{G}(k) = \frac{1}{\mathcal{F}[D](k)}. \tag{VI.109}$$

If one then takes the inverse Fourier transform, one obtains the Green function in the form:

$$G(x, y) = G(\mathbf{x} - \mathbf{y}) = \frac{1}{(2\pi)^n}\int_{V^*} \frac{e^{ik(\mathbf{x}-\mathbf{y})}}{F[D](k)}\mathcal{V}_k. \tag{VI.110}$$

One sees that the assumption that the coefficients of D are constant in space, which already implies some sort of affine structure in order to define homogeneity, manifests itself in the fact that the function G must also be translation invariant. This is yet another fundamental limitation on the use of the Fourier transform in the eyes of the extension from vector spaces to manifolds.

Clearly, there will also be analytical problems with the convergence of the integral when the symbol of D has real roots k. Customarily, physics deals with such poles in the integrand of (VI.110) by analytically continuing the integrand to a complex function and replacing the real integrations by complex ones along contours that avoid the poles. However, this generally alters the character of the resulting Green function accordingly.

Since the linear differential operator with constant coefficients D is associated with a polynomial $F[D](k)$ of finite degree, one can envision a generalization of the integral operator G that involves replacing $F[D](k)$ with a more general function $\sigma[P, k]$ on V^*; at the very least, one might extend to non-constant coefficients. One might also assume that $\sigma[P, k]$ is assumed to be analytic in k, which is like generalizing from finite degree polynomials to infinite degree ones, and if it is smooth, then one is generalizing to formal power series expansions in k.

The resulting integral operator P: $\mathcal{S} \to \mathcal{S}$ that is associated with the symbol $\sigma[P, k]$ and kernel (VI.110) is then called a *pseudo-differential operator*. It then takes the general form:

$$Pf(x) = \frac{1}{(2\pi)^n}\int_{V\times V^*} \frac{e^{ik(x-y)}}{\sigma[P,k]} f(y)\mathcal{V}_y\mathcal{V}_k. \tag{VI.111}$$

Actually, the concept of a Fourier integral operator can be generalized in such a way that a pseudo-differential operator is a special case of it (cf., Duistermaat [**21**]).

b. Nonlinear extensions. The first point at which one might wish to generalize the foregoing discussion is to adapt it to the demands of spatial or spacetime manifolds that are not vector spaces. However, one finds that it is not enough to simply replace V with a more general n-dimensional differentiable manifold M.

For one thing, although it is conceivable that one could generalize $\mathcal{S}$ to a Schwartz space of complex functions on M, nonetheless, when M is not a vector space it is absurd to speak of its dual space. However, one does have vector spaces at each point of M in the form of tangent and cotangent spaces. This means that wave covector k must be associated with some specified point $x \in M$, or perhaps a covector field on M, as opposed to its definition in the linear case, which is independent of $x \in V$.

When one considers the expression $k(x) = k_i x^i$ one sees that when x does not belong to a linear space one must consider a more general *phase function* $\theta: M \to \mathbb{R}$. We assume that θ is analytic, or at least locally approximated by a finite-degree Taylor polynomial about each point of M:

$$\theta(x^i) = \theta_0 + d\theta|_0(x^i) + \mathcal{O}^2(x^i). \tag{VI.112}$$

Hence, since the initial phase θ_0 is essentially arbitrary, and can be neglected, we see that our previous phase function $k(x)$ represents the leading term of this series; i.e.:

$$k = d\theta. \tag{VI.113}$$

If we substitute $e^{-i\theta(x)}$ for the exponential in the Fourier transform then we obtain:

$$\mathcal{F}'f[k] = \int_M e^{-ik(x)} \left[e^{-i\mathcal{O}^2(x)} f(x) \right] \mathcal{V} = \mathcal{F}f'[k], \tag{VI.114}$$

where:

$$f'(x) = e^{-i\mathcal{O}^2(x)} f(x). \tag{VI.115}$$

That is, we have deformed the function that is being transformed with the higher-degree terms of the phase function.

Another reasonable assumption concerning the generalization of Fourier transforms from vector spaces to manifolds is that the space $V \times V^*$ will undoubtedly be replaced by the cotangent bundle T^*M, which looks like the latter vector space in any local trivialization.

Considerable progress has been made towards using the geometry of the cotangent bundle, especial its symplectic structure, to facilitate the generalization of *Fourier integral operators*, which take the form:

$$Af(x) = \int_{V \times V^*} e^{i\phi(x,y,k)} a(x,y,k) f(y) \mathcal{V}_y \mathcal{V}_k \tag{VI.116}$$

in the case where M is a vector space V. The function ϕ is a generalized phase function, which is assumed to be homogeneous of degree one in k, while the function a is a generalized amplitude function.

Of particular interest is the way that such expressions relate to asymptotic expansions, such as when a is a sum of terms a_j, $j = 0, 1, \ldots$ that are homogeneous of positive degree $\mu - j$, where μ is then called the *order* of the operator A, and go to zero asymptotically as some parameter, such as wave number or frequency, goes arbitrarily large. Such

asymptotic expansions play a fundamental role in not only the extension of geometrical optics into diffraction phenomena, but quantum wave mechanics, as well.

Of course, we are rapidly going beyond the scope of the present discussion, so we simply refer the curious reader to the literature cited at the beginning of this section.

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Chapter VII

The interaction of fields and currents

One can argue that there is always something solipsistic about any discussion of free field solutions to field equations, since the only way that one knows about the very existence of a field in a region of spacetime is by its interaction with other physical phenomena, often in the form of the motion of matter. However, this illustrates one of the subtle distinctions between linear field theories and nonlinear ones, since in a linear field theory as long as both fields that are present are solutions to the same field equations their combined field, which is obtained by summation, is also a solution, so the space of solutions is sufficiently rich to include combinations of fields. By contrast, if the field equations are nonlinear then there is no guarantee that the sum of two fields will still be a solution. However, this is not merely a technical nuisance, but also a statement about the nature of field interactions, since in the context of electromagnetic wave solutions linear superposition gives interference in the region of overlap without lasting changes to the fields, while nonlinear superposition can bring about lasting alterations to the form of the interacting waves, such as photon splitting.

Therefore, the purpose of this chapter will be to briefly explore the nature of the interactions between electromagnetic fields and electric currents, which can take the form of either the sources of electromagnetic fields or currents external to other fields. One sees that there are basically four possible pairings to consider: current $\rightarrow$ field, field $\rightarrow$ current, field $\rightarrow$ field, current $\rightarrow$ current, where the arrow suggests a cause-and-effect relationship.

In the first case, one must consider the way that source charges and currents generate electromagnetic fields. One sees that in the rest frame of the measurer/observer there are three distinct types of fields that couple to the successive terms in the kinematical state of a charge distribution that is moving relative to the measurement devices: Electrostatic fields couple to the relative position, magnetic fields couple to the relative velocity, and radiation fields couple to the relative acceleration.

Conversely, electromagnetic fields affect currents mostly by exerting forces on the currents. The simplest force to consider, beyond the electrostatic force, is the Lorentz force, which couples to the velocity. However, it is in attempting to account for the effects of radiation that one finds that the theory itself becomes questionable. The problem seems to be in the fact that the way that the acceleration of a moving charge in an electromagnetic field couples to that field is by way of its radiation reaction – or radiation damping – which actually produces a third-order system of ordinary differential equations for the motion of a point charge in an electromagnetic field, namely, the Lorentz-Dirac equation. This system admits various unphysical solutions that make one suspicious about whether the equations are more heuristic than definitive.

As mentioned above, the interaction of electromagnetic fields themselves has a very different character depending upon whether one considers linear constitutive laws or nonlinear ones. It is in the phenomena of nonlinear optics and quantum electrodynamics

that one can gain some intuition as to the best way to proceed into the nonlinear realm of electromagnetism.

Finally, one can consider that two currents can interact in various ways. The simplest way, beyond the electrostatic interaction of their charge distributions, is due to the fact that one current produces a magnetic field that exerts a force on the other current, and vice versa. The result is a mutual magnetostatic force of attraction and repulsion between currents that basically has the opposite sign to the electrostatic one; i.e., like currents attract and unlike ones repel. This sort of static "action-at-a-distance" is, of course, a non-relativistic oversimplification of a more involved interaction picture in which the currents can only propagate electromagnetic waves that effect the interaction in a more causal way.

1. Construction of fields from sources. The most elementary, if not the most fundamental, interaction between fields and sources is the one that amounts to the statement that a field source must generate a field, in some sense. Here again, one can distinguish the problem of the generation of static fields from the generation of dynamic fields since it will affect the type of boundary/initial-value problems that one can pose.

Generally, the association of a field ϕ with its source ρ follows from the solution of the inhomogeneous system of partial differential equations $D\phi = \rho$, where D is a differential operator. One is basically looking for a left inverse operator D^{-1} for D that makes $\phi = D^{-1}\rho$. Since a left inverse is not generally unique, one must then narrow down the set of possibilities by imposing boundary/initial-value conditions. Furthermore, the source ρ must satisfy the integrability condition that it lie in the image of D.

Hence, one can immediately distinguish the problem of generating fields in linear media from generating ones in nonlinear media.

In the case of linear partial differential equations, such as $\delta\mathfrak{h} = \mathbf{J}$, the association of a field with a source by the method of Green functions is preferred. When the manifold in which the fields live is an affine space, or possibly a more general homogeneous space, such as a sphere, the method of the Fourier transform is also very powerful.

a. Static fields. The linear differential operators that are most fundamental in electrostatics and magnetostatics are basically the exterior derivative operator d and the divergence operator δ. One can absorb the constitutive law into the divergence operator so that one has two first-order differential equations in a single field – viz., E or H. If one assumes the existence of a potential – whether local or global – then one can further reduce the system to a single second-order differential equation in the potential.

In the electrostatic case, the basic problem to be solved is the inhomogeneous partial differential equation $\delta\mathbf{D} = \rho$, for $\mathbf{D} = \varepsilon(E)$, along with the constraint on E that $dE = 0$. One can define a pair of equations for E:

$$dE = 0, \qquad \delta_\varepsilon E = \delta \cdot \varepsilon(E) = \rho. \tag{VII.1}$$

Hence, the solution E to this pair of equations will involve an operator G_ε: $\Lambda_0 \to \Lambda^1$ such that:

$$E = G_\varepsilon \rho = (\varepsilon^{-1} \cdot G_\delta)\rho, \tag{VII.2}$$

where G_δ is a right-inverse to d.

When ε is linear, so is δ_ε, as well as G_ε, and as an integral operator it will have a Green function $G_\varepsilon(x, y)$ for its kernel that satisfies the distributional equations:

$$dG_\varepsilon(x, y) = 0, \qquad \delta_\varepsilon G_\varepsilon(x, y) = -\delta(x, y). \tag{VII.3}$$

However, actually solving this equation for $G_\varepsilon(x, y)$ is simply a special case of solving the inhomogeneous problem by assuming that ρ represents essentially a point source of unit charge.

If the constitutive law ε is not only linear, but homogeneous, which naturally assumes that the spatial manifold Σ is an affine space, then we can use the Fourier transform to solve for $G_\varepsilon(x, y) = G_\varepsilon(\mathbf{x} - \mathbf{y})$. The Fourier-transformed equations that one obtains from (VII.3) are:

$$k \wedge \hat{G}_\varepsilon(k) = 0, \qquad i_k[\varepsilon(\hat{G}_\varepsilon(k))] = -i\,. \tag{VII.4}$$

The first one can be solved, up to multiplication by a scalar function $\alpha(k)$, in the form:

$$\hat{G}_\varepsilon(k) = \alpha(k)k. \tag{VII.5}$$

Substituting this in the second one gives:

$$\alpha(k)\,\varepsilon(k, k) = -i\,, \tag{VII.6}$$

which allows us to solve for $\alpha(k)$; in this expression $\varepsilon(k, k) = \varepsilon^{ij} k_i k_j$ represents the quadratic form that e defines on k. We then find our fundamental solution in k-space to be:

$$\hat{G}_\varepsilon(k) = \frac{-ik}{\varepsilon(k, k)}. \tag{VII.7}$$

The inverse Fourier transform of this is:

$$G_\varepsilon(x) = \frac{-i}{(2\pi)^3}\int_{\mathbb{R}^{3*}} \frac{k}{\varepsilon(k, k)} e^{ik(x)}\, \mathcal{V}_k\,. \tag{VII.8}$$

When ε is the Euclidian scalar product, we know that taking the inverse Fourier transform will give the usual Coulomb law expression for the Green function. However, since ε is symmetric, non-degenerate, and positive-definite, one can just as well use ε in place of the Euclidian scalar product as long as one regards the expression $\varepsilon^{ij} k_i k_j$ as simply the form that it takes in a non-orthonormal frame.

As far as physics is concerned, there is another problem with using ε as a metric: it still has the units of electric permittivity. Hence, we normalize it to a dimensionless metric:

$$g_\varepsilon = (\det \varepsilon)^{-1/3}\, \varepsilon, \tag{VII.9}$$

so we can set:

$$\varepsilon(k,k) = (\det \varepsilon)^{1/3}\, g_\varepsilon(k,k), \qquad k(\mathbf{x}) = g_\varepsilon(k, x), \qquad x \equiv \tilde{g}_\varepsilon(\mathbf{x}). \tag{VII.10}$$

This puts the *k*-space Green function (VII.7) into the form:

$$\hat{G}_\varepsilon(k) = \frac{1}{(\det \varepsilon)^{1/3}} \frac{-ik}{g_\varepsilon(k,k)}, \tag{VII.11}$$

and the integral (VII.8) into the form:

$$G_\varepsilon(x) = \frac{-i}{(2\pi)^3 (\det \varepsilon)^{1/3}} \int_{\mathbb{R}^{3*}} \frac{k}{g_\varepsilon(k,k)} e^{ig_\varepsilon(k,x)}\, \mathcal{V}_k . \tag{VII.12}$$

The integrand now takes the standard Euclidian form that makes the spatial Green function:

$$G_{\varepsilon i}(\mathbf{x}) = \frac{1}{4\pi(\det \varepsilon)^{1/3}} \frac{\tilde{g}_\varepsilon(\mathbf{x})}{[\tilde{g}_\varepsilon(\mathbf{x},\mathbf{x})]^{3/2}} = \frac{1}{4\pi} \frac{\tilde{\varepsilon}(\mathbf{x})}{[\tilde{g}_\varepsilon(\mathbf{x},\mathbf{x})]^{3/2}}; \tag{VII.13}$$

the tilde on the g_ε signifies that we are dealing the inverse metric on $T(\Sigma)$ to the one that g_ε defines on $T^*\Sigma$.

One immediately verifies that when the medium is isotropic, as well as linear and homogeneous, so one has:

$$\varepsilon^{ij} = \varepsilon\, \delta^{ij}, \qquad \varepsilon_{ij} = 1/\varepsilon\, \delta_{ij}, \qquad \det \varepsilon = \varepsilon^{3/2}, \tag{VII.14}$$

the form that the Green function takes is:

$$G_{\varepsilon i}(x) = \frac{1}{4\pi\varepsilon} \frac{x_i}{\| x \|^{3/2}}, \tag{VII.15}$$

in which one uses the Euclidian components δ^{ij} for the scalar product.

Given $G_\varepsilon(x)$, one obtains E by convolving G_ε with ρ:

$$\begin{aligned} E(\mathbf{x}) &= \int_{\operatorname{supp}\rho} G_\varepsilon(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})\, \mathcal{V}_y \\ &= \varepsilon^{-1} \left[\frac{1}{4\pi} \int_{\operatorname{supp}\rho} \frac{\rho(\mathbf{y})}{[\tilde{g}_\varepsilon(\mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{y})]^{3/2}} (\mathbf{x}-\mathbf{y})\, \mathcal{V}_y \right]. \end{aligned} \tag{VII.16}$$

Note that the expression inside the brackets is nothing but the vector field $\mathbf{D}(\mathbf{x})$.

In the case of a linear, isotropic, homogeneous electrostatic medium with a point charge Q at the origin for a source one then arrives at *Coulomb's law*:

$$\mathbf{D}(\mathbf{x}) = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \qquad E(\mathbf{x}) = \frac{1}{4\pi\varepsilon}\frac{Q}{r^2}\hat{r}. \tag{VII.17}$$

with:

$$r^2 = x^2 + y^2 + z^2, \qquad \hat{\mathbf{r}} = \frac{x^i}{r}\partial_i. \tag{VII.18}$$

The basic potential equation for electrostatics is $\Delta_\varepsilon\phi = \rho$, with $\Delta_\varepsilon = \delta \cdot \varepsilon \cdot d$: $\Lambda^0 \to \Lambda_0$. When ε is linear, its symbol is the function:

$$\sigma[\Delta_\varepsilon; k] = i_k \cdot \varepsilon \cdot e_k = \varepsilon(k, k). \tag{VII.19}$$

Since ε is assumed to be positive definite, this function is always non-zero when k is non-zero.

When the medium is electrically homogeneous, the Fourier transform for the Green function for Δ_ε is then:

$$\hat{G}_{\Delta\varepsilon}(k) = \frac{-i}{\varepsilon(k,k)} = \frac{-i}{(\det\varepsilon)^{1/3}}\frac{1}{g_\varepsilon(k,k)}. \tag{VII.20}$$

Taking the inverse Fourier transform gives:

$$G_{\Delta\varepsilon}(\mathbf{x}) = \frac{1}{4\pi(\det\varepsilon)^{1/3}[\tilde{g}_\varepsilon(\mathbf{x},\mathbf{x})]^{1/2}}, \tag{VII.21}$$

and in a linear, homogeneous medium that is isotropic, as well, this takes the Coulomb form:

$$G_{\Delta\varepsilon}(\mathbf{x}) = \frac{1}{4\pi\varepsilon\,\|\mathbf{x}\|}. \tag{VII.22}$$

The solution to the inhomogeneous potential problem for a source charge density ρ in an electrostatically linear and homogeneous medium is then obtained by convolution:

$$\phi(\mathbf{x}) = \frac{1}{4\pi(\det\varepsilon)^{1/3}}\int_{\mathrm{supp}(\rho)}\frac{\rho(\mathbf{y})}{[\tilde{g}_\varepsilon(\mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{y})]^{1/2}}\mathcal{V}_y. \tag{VII.23}$$

In magnetostatics, the corresponding equations for the magnetic field B are:

$$dB = 0, \qquad \delta\mathbf{H} = \mathbf{I}, \qquad \mathbf{H} = \tilde{\mu}(B), \tag{VII.24}$$

which can be consolidated into:

$$dB = 0, \qquad \delta\cdot\tilde{\mu}(B) = \mathbf{I}. \tag{VII.25}$$

The integrability condition for the source current **I** is then:

$$\delta \mathbf{I} = 0, \tag{VII.26}$$

which also represents conservation of charge.

Now, let us examine the Fourier-transforms of equations (VII.25) in the linear homogeneous case:

$$k \wedge \hat{B}(k) = 0, \qquad i_k \cdot \tilde{\mu}(\hat{B}) = -i\hat{\mathbf{I}}(k). \tag{VII.27}$$

One can solve the first one by means of:

$$\hat{B}(k) = k \wedge \hat{A}(k), \tag{VII.28}$$

in which $\hat{A}(k) \in \Lambda^1(\mathbb{R}^{3*})$ is some 1-form on $\mathbb{R}^{3*}$, which we shall see in a bit represents the Fourier transform of a covector potential for B. Hence, $\hat{B}(k)$ can only lie in a 2-dimensional subspace of $\Lambda^2_k(\mathbb{R}^{3*})$ at each $k \in \mathbb{R}^{3*}$, namely, the image of $\Lambda^1_k(\mathbb{R}^{3*})$ under e_k. This subspace gets mapped isomorphically to a two-dimensional subspace of $\Lambda_{2,k}(\mathbb{R}^{3*})$ under $\tilde{\mu}$.

When we attempt to solve the second equation in (VII.27) for $\hat{B}(k)$ we are immediately obstructed by the fact that the map i_k is not invertible. However, it does behave somewhat like a projection of the fibers of $\Lambda_2(\mathbb{R}^{3*})$ onto two-dimensional subspaces of the fibers of $\Lambda_1(\mathbb{R}^{3*})$ at each $k \in \mathbb{R}^{3*}$. Hence, if we restrict $\hat{\mathbf{I}}(k)$ to these subspaces we can define a left-inverse to i_k in the form of $e_{\mathbf{k}}$, where $\mathbf{k}$ is a vector field such that $k(\mathbf{k}) = 1$. This defines two problems: how to characterize the subspaces in $\Lambda_1(\mathbb{R}^{3*})$ and how to define the vector field $\mathbf{k}$.

The solution to the first problem follows immediately from the fact that $i_k \cdot i_k = 0$, namely, $\hat{\mathbf{I}}(k)$ must lie in the kernel of i_k. However, this follows naturally from the conservation of charge constraint on $\mathbf{I}$, namely, (VII.26), which leads to the Fourier-transformed constraint on $\hat{\mathbf{I}}(k)$ that it must lie in the hyperplane defined by:

$$0 = i_k \hat{\mathbf{I}} = k_i \hat{I}^i. \tag{VII.29}$$

In order to solve the second problem, we must confront another subtle distinction between the magnetostatic case and the previous electrostatic one that is also related to the fact that we are dealing with bivector fields and 2-forms. It is the fact that the magnetic constitutive law μ: $\Lambda_2 \rightarrow \Lambda^2$ not only "goes backwards," but it also most naturally defines a scalar product on either Λ_2 or Λ^2:

$$\mu(\mathbf{A}, \mathbf{B}) = \mu(\mathbf{A})(\mathbf{B}), \qquad \tilde{\mu}(A, B) \equiv \tilde{\mu}(A)(B). \tag{VII.30}$$

What we ultimately need, though, is a scalar product on Λ_1 or Λ^1. The solution to this problem is simply to use Poincaré duality to define the isomorphism:

$$\# \cdot \mu \cdot \#: \Lambda_1 \to \Lambda^1,$$

which then defines scalar products on the tangent and cotangent bundles, as desired. At the risk of confusion, we denote them by the same letters that we used for the scalar products that μ defines directly.

Furthermore, we need to normalize the resulting metric to be dimensionless:

$$g_\mu = (\det \mu)^{-1/3} \mu . \tag{VII.31}$$

In order to make $k(\mathbf{k}) = 1$, we then divide $\mathbf{k}$ by $g_\mu(k, k)$.

We find that the vector field $\mathbf{k}$ can be obtained from the covector field k by mapping k to a vector field using the normalized form of μ:

$$\mathbf{k} = g_\mu(k), \tag{VII.32}$$

which then makes:

$$k(\mathbf{k}) = g_\mu(k, k). \tag{VII.33}$$

We can then solve the second equation in (VII.27) for $\hat{B}(k)$:

$$\hat{B}(k) = -i\,\mu \cdot \frac{e_{\mathbf{k}}}{g_\mu(k,k)} \cdot \hat{\mathbf{I}}(k) = -i\mu\left[\frac{1}{g_\mu(k,k)}\mathbf{k} \wedge \hat{\mathbf{I}}(k)\right]. \tag{VII.34}$$

We then obtain the k-space Green function:

$$\hat{G}_\mu(k) = \frac{-i}{g_\mu(k,k)}\mu(\mathbf{k}), \tag{VII.35}$$

which is analogous to (VII.11).

Its inverse Fourier transform is:

$$G_\mu(\mathbf{x}) = \frac{1}{4\pi}\frac{1}{[\tilde{g}_\mu(\mathbf{x},\mathbf{x})]^{3/2}}\mu(\mathbf{x}), \tag{VII.36}$$

and the form that it takes in a magnetically isotropic medium is:

$$G_{\mu,i}(\mathbf{x}) = \frac{\mu}{4\pi}\frac{x_i}{\|\mathbf{x}\|^{3/2}}. \tag{VII.37}$$

If we express the integral operator whose kernel is $G_\mu(\mathbf{r})$ in the form:

:

$$B(\mathbf{x}) = \int_{\mathrm{supp}\,\mathbf{I}} G_\mu(\mathbf{x}-\mathbf{y}) \wedge \mathbf{I}(\mathbf{y})\, \mathcal{V}_y \tag{VII.38}$$

then the solution for B that corresponds to (VII.16) is:

$$B(\mathbf{x}) = \mu(\mathbf{H}) = \mu\left[\frac{1}{4\pi} \int_{\mathrm{supp}\,\mathbf{I}} \frac{1}{[\tilde{g}_\mu(\mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y})]^{3/2}} (\mathbf{x}-\mathbf{y}) \wedge \mathbf{I}(\mathbf{y})\, \mathcal{V}_y \right]. \tag{VII.39}$$

In the case of a magnetically isotropic medium, this solution takes the form of the *Biot-Savart law*:

$$B(\mathbf{x}) = \frac{\mu}{4\pi} \int_{\mathrm{supp}\,\mathbf{I}} \frac{1}{\| \mathbf{x}-\mathbf{y} \|^3} (x-y) \wedge I(\mathbf{y})\, \mathcal{V}_y\,, \tag{VII.40}$$

in which x, y, and I denote the covector fields that correspond to the vector fields **x**, **y**, and **I** under the isomorphism that is defined by the Euclidian scalar product.

As for the magnetostatic potential problem $\Delta_\mu A = \mathbf{I}$, the analogy with electrostatics is also imprecise, since the magnetic Laplacian $\Delta_\mu = \delta \cdot \tilde{\mu} \cdot d$: $\Lambda^1 \to \Lambda_1$ has a symbol:

$$\sigma[\Delta_\mu, k] = i_k \cdot \tilde{\mu} \cdot e_k\,, \tag{VII.41}$$

that is not invertible. Not only does i_k: $\Lambda_2(\mathbb{R}^{3*}) \to \Lambda_1(\mathbb{R}^{3*})$ define a projection onto a two-dimensional subspace of $\Lambda_1(\mathbb{R}^{3*})$, but the kernel of the map e_k: $\Lambda^1(\mathbb{R}^{3*}) \to \Lambda^2(\mathbb{R}^{3*})$ is one-dimensional, namely, the line $[k]$ in $\Lambda^1(\mathbb{R}^{3*})$ that is generated by k.

This is where a choice of gauge for the potential 1-form A is useful. If one imposes a generalized Coulomb condition on A*:*

$$\delta \cdot \mu(A) = 0 \tag{VII.42}$$

then the corresponding constraint on the Fourier transform $\hat{A}(k)$ of A is:

$$0 = i_k \mu(\hat{A}) = \mu(k, \hat{A}). \tag{VII.43}$$

That is, $\hat{A}$ is confined to the orthogonal complement to $[k]$. When $\sigma[\Delta_\mu, k]$ is restricted to this subspace of the domain space $\Lambda^1(\mathbb{R}^{3*})$ and its image in the range space $\Lambda_1(\mathbb{R}^{3*})$ one finds that it is invertible. Of course, this also means that the Fourier transform $\hat{I}$ of the current can only lie in the image, but that is a natural consequence of the conservation of charge, as before.

One finds that the inverse of $\sigma[\Delta_\mu, k]$ then takes the form:

$$\sigma[\Delta_\mu, k]^{-1} = -\hat{G}_\mu(k) = i_{\mathbf{k}} \cdot \mu \cdot e_{\mathbf{k}} = \frac{1}{g_\mu(k,k)}\mu\,; \tag{VII.44}$$

hence:

$$\hat{G}_\mu(k) = -\frac{1}{g_\mu(k,k)}\mu\,. \tag{VII.45}$$

In a magnetically homogeneous medium this takes the component form:

$$\hat{G}_{\Delta\varepsilon}(k) = \frac{\mu}{k^2}\delta_{ij}\,. \tag{VII.46}$$

The inverse Fourier transform of (VII.45) then becomes:

$$G_{\Delta\varepsilon}(\mathbf{x}) = \frac{1}{4\pi}\frac{1}{[\tilde{g}_\mu(\mathbf{x},\mathbf{x})]^{1/2}}\mu\,. \tag{VII.47}$$

In a magnetically isotropic medium, we then have:

$$G_{\Delta\varepsilon}(\mathbf{x}) = \frac{\mu}{4\pi\,\|\,\mathbf{x}\,\|}\delta_{ij}\,. \tag{VII.48}$$

The covector potential $A(\mathbf{x})$ that is produced by a source current density **I** is then:

$$A(\mathbf{x}) = \mu\left[\frac{1}{4\pi}\int_{\mathrm{supp}(\rho)}\frac{\mathbf{I}(\mathbf{y})}{[\tilde{g}_\mu(\mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{y})]^{1/2}}\mathcal{V}_y\right]. \tag{VII.49}$$

One should compare the expressions that we have derived here with the corresponding ones in – say – Jackson [**1**] to see how much vector calculus takes for granted as a consequence of dealing with a three-dimensional Euclidian space. For instance, it regards vectors, covectors, bivectors, and 2-forms as essentially the same, since one has isomorphisms of all four spaces. As we saw, just dropping the assumption that the Euclidian structure is purely geometric in origin makes it necessary to reevaluate all of these isomorphisms.

b. Dynamic fields. In the case of dynamic fields, we must use the spacetime manifold M in place of the spatial manifold Σ and a source current density **J** on M in place of ρ or **I**.

However, the inhomogeneous problem that one poses for a given source current **J** is still essentially the same: one must solve the partial differential equations:

$$dF = 0, \qquad \delta_\kappa F = \delta \cdot \kappa(F) = \mathbf{J} \tag{VII.50}$$

for F, with the integrability condition for **J** that:

$$\delta \mathbf{J} = 0, \tag{VII.51}$$

which still represents conservation of charge.

On the surface of things, this problem is no different from the one that we defined above for magnetostatics. However, we shall see that the previous logic breaks down at a crucial point.

The Fourier-transformed equations that one obtains when κ is linear and homogeneous are:

$$k \wedge \hat{F} = 0, \qquad i_k \cdot \kappa(\hat{F}) = -i\hat{\mathbf{J}}. \tag{VII.52}$$

The solution to the first equation still takes the form:

$$\hat{F} = k \wedge \hat{A} \tag{VII.53}$$

for a suitable 1-form $\hat{A}$. Hence, $\hat{F}$ must lie in a three-dimensional subspace of the fiber of $\Lambda^2(\mathbb{R}^{4*})$ at each point, and it gets mapped to a three-dimensional subspace in $\Lambda_2(\mathbb{R}^{4*})$.

Once again, i_k not invertible unless we restrict ourselves to a hyperspace in $\Lambda_1(\mathbb{R}^{4*})$ that that takes the form of the kernel of i_k: $\Lambda_1(\mathbb{R}^{4*}) \rightarrow \Lambda_0(\mathbb{R}^{4*})$. This again comes from the integrability condition on **J** that charge be conserved:

$$0 = i_k \hat{\mathbf{J}}(k). \tag{VII.54}$$

One can then invert i_k when restricted to this subspace by means of $e_{\mathbf{k}}$ when **k** is a vector field with $k(\mathbf{k}) = 1$.

However, this is where things break down compared to the three-dimensional magnetostatic case. Basically, Poincaré duality no longer takes vector fields to 2-forms, but to 3-forms; similarly, it now takes bivector fields to 2-forms, not 1-forms. Hence, we can no longer use the # isomorphism to associate a scalar product on $\Lambda^1(\mathbb{R}^{4*})$ with the scalar product on $\Lambda^2(\mathbb{R}^{4*})$ that is defined by κ; and we will have to find some other way of obtaining **k**.

A further complication arises when one considers the inhomogeneous potential problem for a dynamic field, which is defined by substituting $F = dA$ in the field equation for F to give $\Box_\kappa A = \mathbf{J}$, where $\Box_\kappa = \delta \cdot \kappa \cdot d$: $\Lambda^1 \rightarrow \Lambda_1$ is the field operator for κ:

In the linear case, its symbol is:

$$\sigma[\Box_\kappa, k] = i_k \cdot \kappa \cdot e_k = Q(k, k). \tag{VII.55}$$

One finds that, in addition to the aforementioned restrictions, one must contend with the fact that if $Q(k, k)$ is to behave like a generalization of the Lorentzian dispersion law then it will vanish not just at the origin, but on an algebraic hypersurface in $\Lambda^1(\mathbb{R}^{4*})$.

Since this is the starting point for any discussion of electromagnetic wave motion and the ultimate appearance of a Lorentzian structure as a consequence of the pre-metric electromagnetic structure of spacetime – i.e., its constitutive law and field equations – we shall simply return to the topic in the next chapter of this book.

c. Electromagnetic radiation. We pause to observe that each of the successive terms in the kinematical state of a relatively moving electric charge distribution $\rho(t, x^i)$ seems to be associated with its own special field. The relative position of the points of the distribution generates the electrostatic field and the relative velocity generates a magnetic field; if the relative velocity is "constant in time" then this magnetic field is static, as well.

Since historically the rigorous geometrical treatment of the concept of acceleration for motion in manifolds that have no natural affine structure opened up the subtleties of general relativity, and the consequent representation of gravity as a sort of "fictitious force," it is not surprising that one of the most debatable extensions of Maxwellian electromagnetism regards the manner by which the relative acceleration of an electric charge distribution is associated with an electromagnetic field.

It is generally agreed that the electromagnetic field that is generated by a relative acceleration is a *radiation field*, which implies, among other things, that it will be a wavelike solution of the field equations. It is not, however, generally agreed that such a radiation field will be observed by a comoving measure-observer; e.g., a static electron observed in a freely-falling laboratory. Basically, the issue is that of the relativity of acceleration, which leads into the study of conformal Lorentzian geometry, since the transformations between frames that differ by a constant relative acceleration is related to a composition of translations and inversions. Although the former class of transformations is contained in Poincaré group, the latter class is not. Inversions are more projective-geometric in character, and relate to the conformal Lorentz group.

Although we shall make a more complete discussion of electromagnetic waves in the next chapter, including the manner by which a Lorentzian structure might emerge from the electromagnetic field equations, for now, we provisionally revert to the metric formulation of the Maxwell equations, namely:

$$dF = 0, \qquad \delta F = J, \tag{VII.56}$$

in which δ is the codifferential operator that is associated with the Hodge dual isomorphism *, which is defined using a Lorentzian metric g on the $T(M)$, and the 1-form J is associated with the source current density **J** by way of g; i.e.:

$$J = i_{\mathbf{J}} g = (g_{\mu\nu} J^{\nu})\, dx^{\mu}. \tag{VII.57}$$

If we take the codifferential of both sides of the first equation in (VII.56) and the exterior derivative of the second one then the result is:

$$\Box F = dJ, \tag{VII.58}$$

in which $\Box = \delta d + d\delta$ is the d'Alembertian operator for g.

Equation (VII.58) represents a forced linear wave equation for the 2-form F with a source term that consists of the 2-form dJ. Its solution will then have a different character from an electromagnetic field that has J for its source, and we refer to a solution F_{rad} to (VII.58) as a *radiation field*.

In order to get a better physical intuition for the nature of the forcing term, we assume that J takes the form ρu, where ρ is an electric charge density function and $u = u_\mu \, dx^\mu$ is its associated covelocity 1-form; both ρ and u are assumed to have the same spatially compact support. One sees that:

$$dJ = d\rho \wedge u + \rho \, du. \tag{VII.59}$$

Hence, there are two essentially distinct sources of radiation fields: the non-constancy of ρ over its support and the "kinematical vorticity" du of its flow. In time + space form, du looks like:

$$du = -dt \wedge a - \omega = -dt \wedge a - \#_s \, \boldsymbol{\omega}. \tag{VII.60}$$

in which the 1-form a and the 2-form ω have the local component forms:

$$a = (u_{i,0} - u_{0,i}) \, dx^i, \qquad \omega = \tfrac{1}{2}(u_{i,j} - u_{j,i}) \, dx^i \wedge dx^j . \tag{VII.61}$$

The 1-form a amounts to the spatial acceleration of the distribution ρ, with a correction for the spatial gradient of the Fitzgerald-Lorentz factor, which reverts to the spatial gradient of the spatial speed $v = \| u_s \|$. The 2-form ω defines the spatial vorticity of the covelocity u, which is essentially the curl of the spatial velocity vector field $\mathbf{v}$.

If one has a Green function for the operator $\Box$, which we now represent as $G(x, y) \in \Lambda^2 \otimes \Lambda^2$, then the volume potential term in (VI.88) can be given the form:

$$F_{\text{rad}}(x) = \int_M G(x, y) \wedge dJ(y) = \int_{\text{supp}(J)} G(x, y) \wedge dJ(y). \tag{VII.62}$$

We can then substitute from (VII.59) and (VII.60) to express F_{rad} as the sum of three contributions:

$$\begin{aligned} F_{\text{rad}}(x) = & \int_{\text{supp}(J)} G(x, y) \wedge (d\rho \wedge u)(y) \\ & - \int_{\text{supp}(J)} G(x, y) \wedge (dt \wedge a)(y) - \int_{\text{supp}(J)} G(x, y) \wedge \omega(y). \end{aligned} \tag{VII.63}$$

It is important to point out that when one gets into the realm of interactions that must be moderated by the exchange of dynamic fields, such as photons, that move with finite speeds one must restrict the Green function accordingly. That is, the support of G must be *causal*, which usually means that it can only be a subset of the past light cone at each point. We shall not elaborate at the moment, as the theory of radiation, especially when one includes the quantum consideration, is a deep problem that requires further analysis.

Of the three contributions to F_{rad} in (VII.63), the ones that seem to get the most attention in practice are the first two.

The first one usually gets applied in the context of antenna theory, where it is not the spatial variation of the charge density ρ that is important, but the time variation. One sees that since G defines a linear operator from 2-forms to 2-forms, it is meaningful and useful to speak of eigenvalues and eigenforms. For instance, sinusoidal variations in the charge will produce sinusoidal variations in the radiation field.

The second contribution says that accelerating (or, of course, decelerating) charges generate radiation fields. For instance, *brehmstrahlung*, or "braking radiation," is an example of this situation. *Čerenkov radiation* is the form of braking radiation that is produced when the speed of a charge is greater than the speed of light in the medium.

It is the fact that accelerating charges, such as charges confined to circular orbits, must radiate energy, and therefore lose energy, that defined one of the first unavoidable flaws in the Bohr planetary model for atomic electrons, since this classical radiation model would suggest that a planetary electron would only lose energy continuously and spiral down into the nucleus itself. There were two problems with this picture in the eyes of experimental physics:

1. A continuous decay of energy would produce a continuous spectrum of frequencies for the radiated field, which contradicts the discrete nature of atomic spectra.

2. There is a non-zero ground state energy for atomic electrons that they reach before they are ever absorbed into the nucleus.

Although it is traditional to say that all discussion of the radiation of atomic electrons must necessarily fall within the purview of quantum electrodynamics, one must keep in mind that there is still a sizable gap between the field-theoretic formalism of Maxwellian electromagnetism and the formalism of the scattering approximation of quantum electrodynamics. Hence, there is still some validity to posing the problem of extending the field-theoretic formalism into the realm that is usually treated in the scattering approximation. Therefore, many of the generalizations that we shall consider in the name of pre-metric electromagnetism are guided by the hope that they will extend the field-theoretic formalism further into the quantum – i.e., atomic-to-subatomic – domain.

2. Lorentz force. In this section, we address the converse of the problem of the previous section: That is, we now consider the effect of an electromagnetic field on a current that is not its source. Of course, there is something unavoidably linear about assuming that an electromagnetic field can be "external" to an electric current, because the current will, of course, generate an electromagnetic field of its own. It is only linear superposition that allows one to treat the combined effect as a simple addition of 2-forms.

a. Lorentz force on one current. Suppose we have an electric current $\mathbf{J} = \rho\mathbf{u} = \rho\partial_t + \rho\mathbf{v}$ in an external electromagnetic field $F = dt \wedge E - \#_s\mathbf{B}$. We have already discussed the force density that the electric field E exerts on the charge density ρ, namely:

$$\rho E = \rho\, i_{\partial t}(dt \wedge E). \tag{VII.64}$$

We now address the force density that $\mathbf{B}$ exerts on $\rho\mathbf{v}$. In the conventional formulation of electromagnetism in terms of vector analysis, it takes the form of the *Lorentz force (density):*

$$\mathbf{f} = \rho\mathbf{v} \times \mathbf{B} = \mathbf{I} \times \mathbf{B}. \tag{VII.65}$$

in which we have introduced $\mathbf{I} = \rho\mathbf{v}$ as the spatial current density.

If we wish to put this into the language of exterior algebra then we need only represent f by a spatial 1-form, and we can rewrite (VII.65) in the form:

$$f = \rho\, \#_s(\mathbf{v} \wedge \mathbf{B}) = \#_s(\mathbf{I} \wedge \mathbf{B}). \tag{VII.66}$$

But:

$$\#_s(\mathbf{v} \wedge \mathbf{B}) = i_{\mathbf{v} \wedge \mathbf{B}}\mathcal{V} = -\, i_{\mathbf{v}}(\#_s\mathbf{B}) = -\, i_{\mathbf{u}}(\#_s\mathbf{B}). \tag{VII.67}$$

We find that the combined Coulomb force on the charge density and the Lorentz force on the spatial current density, namely:

$$f = \rho[E + \#_s(\mathbf{v} \wedge \mathbf{B})] = \rho[i_{\partial t}(dt \wedge E) - i_{\mathbf{v}}(\#_s\mathbf{B})], \tag{VII.68}$$

can be combined into the four-dimensional expression:

$$f = i_{\mathbf{J}}F = J^{\mu}F_{\mu\nu}\,dx^{\nu}. \tag{VII.69}$$

However, the f of (VII.69) describes only the spatial force density, so there is one extra piece to $i_{\mathbf{J}}F$ that is missing from (VII.68), namely, the temporal piece $-\,\rho\, E(\mathbf{v})\, dt$. Hence, we correct (VII.68) to:

$$f = \rho\,[-\, E(\mathbf{v})\, dt + E + \#_s(\mathbf{v} \wedge \mathbf{B})]. \tag{VII.70}$$

From a consideration of its basic units, the temporal contribution represents a power density.

Now, let us define the energy-momentum density associated with the vector field $\mathbf{v}$ to be:

$$p = \mu u, \tag{VII.71}$$

in which μ is the rest mass density, whose support is the same as for ρ and $\mathbf{v}$, while the 1-form u is a covelocity 1-form that relates to $\mathbf{v}$ by means of:

$$u(\mathbf{v}) = \text{const.} \tag{VII.72}$$

We define the proper-time derivative of something to be its Lie derivative along the flow of $\mathbf{v}$:

$$\frac{d}{d\tau} = \mathrm{L}_{\mathbf{v}} = i_{\mathbf{v}}d + di_{\mathbf{v}}, \tag{VII.73}$$

so:

$$\frac{dp}{d\tau} = \frac{d\mu}{d\tau}u + \mu\frac{du}{d\tau}. \tag{VII.74}$$

The first term vanishes iff mass is conserved along the flow of $\mathbf{v}$.

If we express **v** and u in time + space form as:

$$\mathbf{v} = v^0\partial_t + \mathbf{v}_s\,, \qquad u = u_0\, dt + u_s \tag{VII.75}$$

then:

$$\frac{du}{d\tau} = i_{\mathbf{v}}du + di_{\mathbf{v}}u = i_{\mathbf{v}}du = -\, a_s(\mathbf{v}_s) + [v^0 a_s + \#_s(\mathbf{v}_s \wedge \boldsymbol{\omega}_s)], \tag{VII.76}$$

where the term $di_{\mathbf{v}}u$ vanishes on account of (VII.72). The spatial vector field $\boldsymbol{\omega}_s$ and 1-form a_s that we introduced are defined by (VII.59) and (VII.60).

Hence, we can regard a_s as the acceleration, in the sense of the convected derivative of the covelocity, and $\boldsymbol{\omega}_s$ is the (kinematical) vorticity vector field.

Thus, if we consider the four-dimensional form of Newton's second law of motion – i.e., conservation of energy-momentum:

$$f = \frac{dp}{d\tau}\,, \tag{VII.77}$$

then we arrive at the following pair of equations:

$$\mu a_s(\mathbf{v}_s) = \rho E(\mathbf{v}_s), \tag{VII.78a}$$
$$\mu[v^0 a_s + \#_s(\mathbf{v}_s \wedge \boldsymbol{\omega}_s)] = \rho[v^0 E + \#_s(\mathbf{v}_s \wedge \mathbf{B})]. \tag{VII.78b}$$

The first one is a statement about power, i.e., the rate at which energy is being added or subtracted from the motion of the charge cloud. The second one is a generalization of the usual equation of motion defined by the combined Coulomb and Lorentz force that one encounters for pointlike charges, for which the vorticity $\boldsymbol{\omega}_s$ vanishes. It also vanishes for irrotational flows, such as ones for which $\mathbf{v}_s$ is spatially uniform.

When $E = 0$ one finds that the power equation says simply that:

$$a_s(\mathbf{v}_s) = 0. \tag{VII.79}$$

In the metric case, for which:

$$a_s(\mathbf{v}_s) = g(\mathbf{a}_s, \mathbf{v}_s) = \tfrac{1}{2}\,\frac{dv_s^2}{d\tau}\,, \tag{VII.80}$$

this says that the motion of a charge cloud in a magnetic field is uniform circular.

b. Radiation reaction. When an electromagnetic field exerts a force on a current – i.e., a moving charge distribution – that charge distribution will accelerate or decelerate. Consequently, since accelerating charges emit radiation fields, which carry energy and momentum, the charge distribution will also decelerate due to the fact that it is losing energy and momentum. This loss of energy-momentum due to radiation, or rather, its proper time derivative, is referred to as the *radiation reaction* or *radiation damping*.

A full accounting of the equations of motion for a charge distribution in an external electromagnetic field must then include not only the Lorentz force but the radiation reaction, as well. However, since a full accounting of the radiation reaction involves a

more detailed discussion of the problem of electromagnetic radiation, we shall defer that study to future research.

3. Interaction of fields. The main issue that concerns the interaction of more than one field in a region of space is linearity versus nonlinearity. In effect, the very assumption that there is interaction to begin with has a distinctly nonlinear sort of character, but there are still enough linear phenomena in nature to make a brief discussion of linear field interactions worthwhile. Furthermore, what starts off as a linear combination in the field space might very well project to a nonlinear interaction in the measurement space.

Generally, there are three basic regimes in the name of linearity, which are parameterized by some appropriate magnitude: the linear regime, which is characterized by weak magnitudes, strong nonlinearity, which is usually characterized by the onset of some phase transition at a critical magnitude, and the regime of weak nonlinearity, which is an intermediate sort of realm between linearity and the onset of a phase transition.

For instance, in the response of elastic materials to applied stresses, one starts out with Hooke's law for small strains, which is the linear regime. Between linear response and the onset of plastic deformation, one is in the regime of nonlinear elasticity, which is what we are calling weak nonlinearity. Once plastic deformation begins, a phase transition has taken place, and the concept of elastic deformation is no longer applicable.

Since the actual differential equations that we consider in pre-metric electromagnetism, namely, $dF = 0$, $\delta\mathfrak{H} = \mathbf{J}$, are both linear, the only possible source of nonlinearity is in the constitutive law $\mathfrak{H} = \kappa(F)$ that couples the excitation $\mathfrak{H}$ of a medium to the presence of an electromagnetic field F. We then see that for small field strengths the response of the medium is approximately linear, whereas for some high enough field strength, phase transitions can change the very nature of the medium. For instance, one can get melting of solids, ionization of gases, the onset of ferromagnetism, and the formation of particle-anti-particle pairs.

a. Linear constitutive laws. When one is concerned with weak enough field strengths to remain well within the linear response regime for a medium, one is dealing with a vast variety of possible phenomena. In particular, electronics, optics, and communications are mostly concerned with linear phenomena. Indeed, most practical applications are greatly complicated by the onset of nonlinearity. For instance, in the design of capacitors, the onset of a phase transition is clearly undesirable, since the phase transition that one has to contend with is the breakdown of the dielectric at high field strengths.

The equivalent statement to saying that one is in the linear response regime for a medium is the principle of superposition, which says, in effect, that if the constitutive map $\kappa\colon \Lambda^2 \to \Lambda_2$ is linear on each fiber – hence, linear on sections – then the combined operator $\delta\cdot\kappa\colon \Lambda^2 \to \Lambda_1$, $F \mapsto \delta\kappa(F)$ is linear, as well. Its kernel $\ker(\delta\kappa) = \{F \in \Lambda^2 \mid \delta\kappa(F) = 0\}$ is therefore a linear subspace of Λ^2, and since an element of $\ker(\delta\kappa)$ is a solution to the first-order partial differential equation $\delta\kappa(F) = 0$ this implies that linear combinations of solution fields will be solution fields.

One of the most far-reaching consequences of dealing with the linear regime of phenomena of any sort is the applicability of the methods of Fourier analysis when one is also dealing with a homogeneous medium. Indeed, if one has a nonlinear operator from a Hilbert space of "input" fields to another Hilbert space of "output" fields, there is nothing to stop one from performing Fourier transforms of the inputs and outputs. However, it is only in the case of a homogeneous linear operator that the relationship between the Fourier transformed fields is linear, as well. One can see how this is immensely useful in the problem of modulating and demodulating information into carrier signals. It is also the basis for most of the constructions of quantum field theory, in which one is chiefly concerned with constructing the Fourier transform of the integral kernel for a unitary map from a Hilbert space of "incoming" scattering states to a Hilbert space of "outgoing" ones that one thinks of as the scattering operator or *S* matrix for the interaction in question.

There is a subtle relationship between linear supposition and the optical phenomenon of interference, which leads to diffraction, which defines the most definitive difference between classical mechanics and wave mechanics. Basically, one is dealing with a linear superposition of *complex* wave functions whose effect in the eyes of the measurer/observer is what we might call "pseudo-nonlinear." This takes the form of a nonlinear projection of a linear combination. In the case of wave interference, the linear combination is formed in the complex vector space $\mathbb{C}$, while the nonlinear projection maps $\mathbb{C} - \{0\}$ onto the non-negative real axis $\mathbb{R}^+$ by way of the transformation from Cartesian to polar coordinates $\mathbb{C} \to \mathbb{R}^+$, $x + iy \mapsto \sqrt{x^2 + y^2} = \| z \|$.

Although this map is homogeneous of degree one with respect to real scalar multiplication, since $\lambda(x + iy)$ goes to $\lambda\sqrt{x^2 + y^2}$, nevertheless, the sum of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ goes to $\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} = \| z_1 + z_2 \|$, which differs from $\sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} = \| z_1 \| + \| z_2 \|$. Indeed, since we are dealing with a norm on $\mathbb{C}$ the sum satisfies the triangle inequality $\| z_1 + z_2 \| \leq \| z_1 \| + \| z_2 \|$. Hence, the projection is nonlinear whenever strict integrability is obtained.

Another example of nonlinear superposition is given by the addition of velocities in special relativity, which originates in the fact that the projection of four-dimensional spacetime events into the rest space of a measure/observer is not a simple Cartesian projection of a four-tuple (v^0, v^1, v^2, v^3) of components onto a triple (v^1, v^2, v^3) of components, but more like the projection of homogeneous coordinates (v^0, v^1, v^2, v^3) for $\mathbb{R}\mathrm{P}^3$ onto inhomogeneous coordinates (V^1, V^2, V^3), which is a nonlinear projection, since $V^i = v^i/v^0$. Since we shall have much more to say about the role of projective geometry in special relativity and electromagnetism later in Chapter XI, we shall suspend our discussion of the subject for now.

b. Nonlinear constitutive laws. When κ is a nonlinear map on the fibers, $\ker(\delta \cdot \kappa)$ is not generally a linear subspace, but a more elaborate nonlinear subspace of the vector space Λ^2 that will generally be infinite-dimensional, as well. Hence, if F_1 and F_2 are both solutions of $\delta\kappa(F) = 0$ then their sum is not another solution, in general. One concludes

that the effect of superposing the two fields in a nonlinear medium will be to produce a field $\Sigma(F_1, F_2)$ that is generally distinct from $F_1 + F_2$, and satisfies $\delta\kappa(\Sigma(F_1, F_2)) = 0$.

Of course, since we dealing with a vector space we can always define a 2-form $\Delta(F_1, F_2)$ such that:

$$\Sigma(F_1, F_2) = F_1 + F_2 + \Delta(F_1, F_2). \tag{VII.81}$$

However, unless there is some convenient way of characterizing the interaction term $\Delta(F_1, F_2)$ one must realize that its utility is mostly in the weakly nonlinear regime.

Indeed, if we assume that $\Sigma(F_1, F_2)$ can be approximated by a Taylor series:

$$\Sigma(F_1, F_2) \approx F_1 + F_2 + d\Sigma|_0(F_1 + F_2) + \tfrac{1}{2} d^2\Sigma|_0(F_1 + F_2, F_1 + F_2) + \ldots \tag{VII.82}$$

then we see that:

$$\Delta(F_1, F_2) \approx d\Sigma|_0(F_1 + F_2) + \tfrac{1}{2} d^2\Sigma|_0(F_1 + F_2, F_1 + F_2) + \ldots \tag{VII.83}$$

Since $d\Sigma|_0$ is linear and $d^2\Sigma|_0$ is bilinear, we can expand this into:

$$\begin{aligned}\Delta(F_1, F_2) \approx\ & d\Sigma|_0(F_1) + d\Sigma|_0(F_2) \\ & + \tfrac{1}{2} d^2\Sigma|_0(F_1, F_1) + d^2\Sigma|_0(F_1, F_2) + \tfrac{1}{2} d^2\Sigma|_0(F_2, F_2) + \ldots\end{aligned} \tag{VII.84}$$

Of course, the problem now reverts to that of characterizing the nature of the successive derivatives in this expression in terms of something that pertains to the nonlinearity in κ. One way of doing this is to note that since the operator $\delta \cdot \kappa$ annihilates the left-hand side of (VII.81), one has:

$$\delta\kappa[F_1 + F_2 + \Delta(F_1, F_2)] = 0. \tag{VII.85}$$

If one expands κ in terms of nonlinear susceptibilities then presumably one might arrive at successive levels of perturbation to the law of linear superposition, but that also appears to be a major undertaking in terms of computational complexity. Of course, there might be useful parallels to the more established perturbational methods of quantum field theory that could prove heuristically probative.

4. Interaction of currents. Since all physical fields presumably have sources one can consider the interaction of fields as being, in a sense, dual to the problem of the interaction of the source currents themselves. From that viewpoint, one regards the fields of the sources as being essentially intermediaries for the interaction of the sources themselves.

We see that there are two basic levels of complexity associated with the interaction of sources, which correspond to the non-relativistic and relativistic approximations, respectively. Namely, when one is dealing with slowly-varying fields produced by closely-spaced sources, one can use the non-relativistic approximation of action-at-a-distance. By contrast, when the currents are rapidly varying or the sources are

sufficiently distant from each other, one must respect causality and regard the interaction of sources as only taking place by the intermediary of electromagnetic waves.

a. Force between two currents (linear case). In the case of static – or at least, slowly-varying – fields with neighboring sources $\mathbf{J}_1$ and $\mathbf{J}_2$, one can combine the coupling of – say – $\mathbf{J}_1$ to the field F_1 that it generates with the Lorentz force density $f_{12} = i_{\mathbf{J}_2} F_1$ that the field F_1 exerts on $\mathbf{J}_2$. Since there is nothing, at this point, to distinguish $\mathbf{J}_1$ from $\mathbf{J}_2$, one could just as well consider the force density $f_{21} = i_{\mathbf{J}_1} F_2$ that the field F_2, which is generated by $\mathbf{J}_2$, exerts on $\mathbf{J}_1$. By Newton's third law of motion, one expects that $f_{12} = -f_{21}$.

When the medium in which $\mathbf{J}_1$ and $\mathbf{J}_2$ are defined is linear in its response, one can use the method of Green functions to establish the map from each source to its corresponding field; i.e., one can obtain a linear function $F_i = F_i(\mathbf{J}_i)$, $i = 1, 2$. As a consequence, one can define f_{12} as an anti-symmetric bilinear functional:

$$f_{12}(\mathbf{J}_1, \mathbf{J}_2) = -f_{12}(\mathbf{J}_2, \mathbf{J}_1) = -f_{21}(\mathbf{J}_1, \mathbf{J}_2). \tag{VII.86}$$

For instance, in the electrostatic case, Coulomb's law (VII.17) associates an electrostatic field:

$$E_1(\mathbf{r}) = \frac{1}{4\pi\varepsilon} \frac{Q_1}{\| \mathbf{r} - \mathbf{r}_1 \|^2} d \| \mathbf{r} - \mathbf{r}_1 \| \tag{VII.87}$$

with a point charge Q_1 that is located at $\mathbf{r}_1 \in \mathbb{R}^3$.

The force that Q_1 exerts on a point charge Q_2 that is located at $\mathbf{r}_2$ is:

$$f_{12}(Q_1, Q_2) = Q_2 E_1(\mathbf{r}_2) = \frac{1}{4\pi\varepsilon} \frac{Q_1 Q_2}{\| \mathbf{r}_2 - \mathbf{r}_1 \|^2} d \| \mathbf{r}_2 - \mathbf{r}_1 \| = -f_{12}(Q_2, Q_1). \tag{VII.88}$$

(The sign change originates in the 1-form $d\| \mathbf{r}_2 - \mathbf{r}_1 \|$.)

Actually, the anti-symmetry in $f_{12}(Q_1, Q_2)$ is closely related to the fact that the charge distributions associated with Q_1 and Q_2 are pointlike. If one replaced them with more general charge distributions ρ_1 and ρ_2 then $f_{12}(\rho_1, \rho_2)$ would become a force density that was distributed over the support of ρ_2, which would take the form:

$$f_{12}(\rho_1, \rho_2)(y) = \rho_2(y) \int_{\text{supp}\, \rho_1} G_\varepsilon(x, y) \rho_1(x) \mathcal{V}_x = \int_{\text{supp}\, \rho_1} \rho_1(x) G_\varepsilon(x, y) \rho_2(y) \mathcal{V}_x, \tag{VII.89}$$

while $f_{12}(\rho_2, \rho_1)$ would be a force density that was distributed over the support of ρ_1 that would take the form:

$$f_{12}(\rho_2, \rho_1)(x) = \int_{\text{supp}\, \rho_2} \rho_1(x) G_\varepsilon(x, y) \rho_2(y) \mathcal{V}_y \tag{VII.90}$$

Even with an anti-symmetric kernel $G_\varepsilon(x, y)$, it makes no sense to speak of the symmetry properties of $f_{12}(\rho_1, \rho_2)(x)$ with respect to $f_{12}(\rho_2, \rho_1)(y)$ since they have their supports on two different regions of space. However, if one integrates each of them over

their respective supports then one obtains a total force of interaction between the two distributions:

$$f_{12}(\rho_1, \rho_2) = \int_{\mathrm{supp}\,\rho_1} \mathcal{V}_x \int_{\mathrm{supp}\,\rho_2} \rho_1(x) G_\varepsilon(x, y) \rho_2(y)\, \mathcal{V}_y\,, \qquad \text{(VII.91)}$$

which is anti-symmetric when $G_\varepsilon(x, y)$ is.

Of course, this process of reducing extended charge distributions to point distribution by integration is not entirely mathematically rigorous, since one obtains a 1-form that is not associated with any specific point of Σ in the general case. Furthermore, the integral does not transform properly under frame changes that involve transition functions that are not constant. However, one does see how the anti-symmetry of the integral kernel implies the anti-symmetry of the force between point charges as approximations to extended charges.

The situation with the magnetic forces of interaction between currents is similar, but complicated by various factors, mostly relating to the non-existence of point currents, except in the form of the transversal intersection of curves with surfaces, and the fact that the Green function for the interaction acts on a current density by the exterior product, not the scalar product.

The Lorentz force density f_{12} that a current $\mathbf{I}_1$ exerts at a point y of $\mathbf{I}_2$ is of the form:

$$\begin{aligned} f_{12}(\mathbf{I}_1, \mathbf{I}_2)(y) &= \#_s(\mathbf{I}_2(y) \wedge \mathbf{B}(y; \mathbf{I}_1) \\ &= \#_s\left[\mathbf{I}_2(y) \wedge \int_{\mathrm{supp}\,\mathbf{I}_1} G_\mu(x, y) \wedge \mathbf{I}_1(x)\, \mathcal{V}_x \right] \\ &= \#_s\left[\int_{\mathrm{supp}\,\mathbf{I}_1} \mathbf{I}_2(y) \wedge G_\mu(x, y) \wedge \mathbf{I}_1(x)\, \mathcal{V}_x \right], \end{aligned} \qquad \text{(VII.92)}$$

and conversely, the force density that $\mathbf{I}_2$ exerts on a point x of $\mathbf{I}_1$ is:

$$f_{12}(\mathbf{I}_2, \mathbf{I}_1)(x) = \#_s\left[\int_{\mathrm{supp}\,\mathbf{I}_2} \mathbf{I}_1(x) \wedge G_\mu(x, y) \wedge \mathbf{I}_2(y)\, \mathcal{V}_y \right]. \qquad \text{(VII.93)}$$

If one then integrates over y or x, resp., then one obtains a total force of the form:

$$f_{12}(\mathbf{I}_1, \mathbf{I}_2) = \#_s\left[\int_{\mathrm{supp}\,\mathbf{I}_1} \mathcal{V}_x \int_{\mathrm{supp}\,\mathbf{I}_2} \mathbf{I}_1(x) \wedge G_\mu(x, y) \wedge \mathbf{I}_2(y)\, \mathcal{V}_y \right] = -\, f_{12}(\mathbf{I}_2, \mathbf{I}_1) \qquad \text{(VII.94)}$$

that is anti-symmetric.

Since the only way of reducing currents to points is by reducing three-dimensions to two dimensions, one can then think of this expression as something like the interaction of magnetic charges, except that there is a significant difference: In the case of currents in parallel infinite wires, one knows, from elementary physics that like – i.e., parallel – currents *attract*, while unlike ones *repel*. Hence, the magnetic interaction of currents behaves more like the Newtonian interaction of masses than the electrostatic interaction of charges.

b. Causal interactions between currents. When the time variation of a current reaches the point that its second derivative is no longer negligible, or the spatial separation between two currents is appreciable, the approximation of action-at-a-distance breaks down and one sees that interactions between currents must be mediated by radiation fields and take into account the time lag that it takes for a photon to go from one source to the other.

Something else that must break down in the process is the anti-symmetry of the interaction, since one is essentially dealing with "signals" that travel from one place to another in a finite amount of time, so there will be a noticeable difference between a signal that is sent from point A at time t_A to a point B at t_B and one that is sent from B at time t_A and arrives at A at time t_B.

It is also reasonable to ask whether the response of the medium itself will depend upon the direction of travel, since there are such things as one-way mirrors. If the constitutive law is invariant under such a replacement of signal source with receiver then one calls the medium *reciprocal*.

Once again, in order to do justice to the causal interactions of currents, one must discuss the theory of electromagnetic radiation in greater detail. Hence, we content ourselves for the moment with only these cursory remarks.

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Chapter VIII

Electromagnetic waves

In this chapter, we finally get to the most fundamental aspect of pre-metric electromagnetism, which is to account for the "emergence" of the Lorentzian structure on the spacetime manifold as a *possible consequence* of a more general set of possibilities. These possibilities follow from examining the dispersion law for the propagation of electromagnetic waves that one defines from the field equations and the constitutive law of the medium. In particular, the quadratic nature of the characteristic hypersurfaces for the Maxwell field equations, as they are usually presented for isotropic media, is a degenerate case of the quartic dispersion law that appears in anisotropic media.

It is essential to understand that the process of reduction by which we obtain the dispersion law implicitly involves not only a linearization of a potentially nonlinear constitutive law, as well as the fact that dispersion laws are generally subordinate to the basic assumption that one makes about the form of the wavelike solutions to the field equations. However, many of the common forms that wavelike solutions take produce the same dispersion equation. This is closely related to the fact that when one passes from a differential operator to its symbol one loses a considerable amount of information, since the (principal) symbol really only tells one about the highest order of derivatives, and a quasi-linear differential operator will have the same principal symbol as a linear one.

Nevertheless, one thing that all of the common forms for wavelike solution have in common is the fact that they involve some sort of amplitude function, which defines the spatial shape of the wave envelope, and a phase function, which defines the spatial shape of the momentary wave fronts themselves. Although the concept of a plane wave, for which the amplitude function is constant and the phase function is linear, is of limited utility in differentiable manifolds that are not affine spaces, one finds that the concepts of amplitude and phase function can be generalized to nonlinear manifolds.

The geometrical optics approximation that one makes into order to go from the propagation of electromagnetic waves to the behavior of null geodesics, which become light rays when one looks at their spatial projections, amounts to ignoring the contribution of the amplitude function to the wave and concentrating on the phase function as root of the essential information in the wave. As we shall see, this is equivalent to considering only the dispersion law that follows from passing to the symbol of the second-order differential operator that defines the electromagnetic field equations.

Eventually, we intend to show that the geometry of wave motion, at least for characteristic waves, is best addressed in the framework of projective geometry. In particular, both of the concepts of wave normal covector and group velocity vector are naturally defined in projective terms by means of inhomogeneous coordinates, as well as the relationship between them and the Fresnel surfaces for both. Although we shall return to a more general discussion of the role of projective geometry in electromagnetism in Chapter XII, we mention these aspects of it in this chapter for the sake of motivating that more general study.

1. Electromagnetic waves. For the sake of completeness, we rewrite the pre-metric Maxwell equations here:

$$dF = 0, \qquad \delta\mathfrak{h} = \mathbf{J}, \qquad \mathfrak{h} = \kappa(F). \tag{VIII.1}$$

In gauge form, they are:

$$F = dA, \qquad \delta\mathfrak{h} = \mathbf{J}, \qquad \mathfrak{h} = \kappa(F), \tag{VIII.2}$$

which can be combined into a single second-order equation:

$$\Box_\kappa A = \mathbf{J}, \tag{VIII.3}$$

in which:

$$\Box_\kappa = \delta \cdot \kappa \cdot d\colon \Lambda^1 \to \Lambda_1 \tag{VIII.4}$$

is the *electromagnetic field* operator.

In a local coordinate chart (U, x^μ) on M, when the constitutive law is both inhomogeneous and nonlinear, so κ has local components $\kappa^{\mu\nu\alpha\beta}(x, F)$, the operator $\Box_\kappa$ takes the local form:

$$\Box_\kappa^{\nu\alpha} = \tilde{\kappa}^{\mu\nu\alpha\beta}\frac{\partial^2}{\partial x^\mu \partial x^\beta} + \frac{\partial \kappa^{\mu\nu\alpha\beta}}{\partial x^\mu}\frac{\partial}{\partial x^\beta} \tag{VIII.5}$$

in which:

$$\tilde{\kappa}^{\mu\nu\alpha\beta} = \kappa^{\mu\nu\alpha\beta} + \frac{\partial \kappa^{\mu\nu\alpha\beta}}{\partial F_{\kappa\lambda}}\frac{\partial A_\kappa}{\partial x^\lambda}. \tag{VIII.6}$$

Although, as we shall see later when we discuss the dispersion law that follows from this operator, the operator $\Box_\kappa$ is something like a "pre-metric d'Alembertian," actually that is misleading. In particular, the field equation (VIII.3) admits solutions that represent static electric and magnetic fields, as well as time-varying solutions that are not wavelike in character. For instance, one can have fields that vary linearly or exponentially in time.

Hence, since the field equations (VIII.3) are more general than the usual wave equations of mathematics – whether linear or nonlinear – we must treat wavelike solutions as essentially subspaces of the (not necessarily linear) space of solutions A in Λ^1. Some of the common classes of solutions treated in conventional books on electromagnetic waves (e.g., [**1-6**]) are time-periodic electromagnetic fields, fields that are describable by the geometrical optics approximation, and waves as propagating discontinuities, so we show how these topics can be treated in the present formalism.

a. Time-periodic electromagnetic fields. Suppose the spacetime manifold M is space-time separable; i.e., expressible as a product $\mathbb{R} \times \Sigma$, with $\mathbb{R}$ playing the role of the time manifold and Σ, a three-dimensional manifold that plays the role of a spatial manifold. From the previous discussion of space-time splittings of $T(M)$ and its tensor

algebra, we then see that, in particular, all of the bundles of exterior differential forms on M will be decomposed into direct sums of temporal forms and spatial forms, and that the Poincaré duality isomorphism # permutes the temporal and spatial sub-bundles.

However, it is important to notice that the components of the spatial and temporal tensor fields are still functions on M, in general. Hence, when M itself can be decomposed one can also distinguish temporal and spatial functions, as well; that is, a *temporal function* is a function on $\mathbb{R}$ and a *spatial function* is a function on Σ. As a result, we can also distinguish *purely temporal* differential forms on M, which are forms on M that have components in $C^\infty(\mathbb{R})$ – for some adapted coordinate system on M – from *purely spatial* forms on M, which only have components in $C^\infty(\Sigma)$. However, one must clearly distinguish the purely spatial k-forms on M, which can locally have terms involving dt, from the k-forms on Σ, which cannot. We shall use the notation $\Lambda^k_{s'}(M)$ for the bundle of purely spatial k-forms on M, which must be clearly distinguished from the bundle $\Lambda^k_s(M)$ of spatial k-forms on M that is defined by the splitting $T(M) = T(\mathbb{R}) \oplus T(\Sigma)$. Locally, these k-forms look like:

purely temporal: $1/(k-1)!\ \alpha_{0\mu\ldots\nu}(t)\ dt \wedge dx^\mu \wedge \ldots \wedge dx^\nu$,

purely spatial ($a \in \Lambda^k_{s'}(M)$): $1/(k-1)!\ \alpha_{0\mu\ldots\nu}(x^1, \ldots, x^n)\ dt \wedge dx^\mu \wedge \ldots \wedge dx^\nu$,

$\alpha \in \Lambda^k\Sigma$: $1/k!\ \alpha_{\mu\ldots\nu}(x^1, \ldots, x^n)\ dx^\mu \wedge \ldots \wedge dx^\nu$.

In this section, we shall be concerned with a special class of differential forms on M that we call *stationary*. A stationary k-form $\alpha \in \Lambda^k(M)$ takes the form of $\alpha(t, x) = T(t)\alpha'(x)$ where T is a function on $\mathbb{R}$ that is the same for all such forms and α' is a purely spatial k-form on M.

For instance, if an electromagnetic field $F \in \Lambda^2(M)$ on a space-time separable manifold $M = \mathbb{R} \times \Sigma$ is stationary then it can be expressed in the form:

$$F(t, x) = T(t)f(x), \tag{VIII.7}$$

in which $T(t)$ is a smooth function on $\mathbb{R}$ and $f(x) \in \Lambda^k\Sigma$. In particular, the components f_{ij} of f in any coordinate system (t, x^i) on an open subset of M that is adapted to the product structure must be functions of only the x^i:

$$f(x) = \tfrac{1}{2} f_{ij}(x)\ dx^i \wedge dx^j. \tag{VIII.8}$$

From now on, we shall omit the explicit reference to x in the symbol f.

The exterior derivative of F then becomes:

$$dF = dT \wedge f + T\ df = T[d(\ln T) \wedge f + df] \equiv T[\varpi\, dt \wedge f + df], \tag{VIII.9}$$

in which we have introduced the function:

$$\varpi(t) = d(\ln T). \tag{VIII.10}$$

This integrates immediately to:

$$T(t) = \exp\left[\int_0^t \varpi(\tau)d\tau\right], \tag{VIII.11}$$

in which we have suppressed the integration constant by setting $t_0 = 0$.

Of course, in the usual case that is treated by geometrical optics, ϖ is a negative imaginary constant $-i\omega$ so the function $T(t)$ takes the form $e^{-i\omega t}$; from now on, we shall make that assumption. When T takes that form, a stationary field on M is then called *time-periodic*.

In order to make sense of the multiplication by the imaginary i, we could assume that the constitutive law defines an almost complex structure * on $\Lambda^2(M)$, but, as it turns out, in order get agreement with the stationary form of the Maxwell equations, we must not use that structure to define multiplication by i. For one thing, we shall need to multiply 1-forms and 3-forms by i, as well. Rather, we understand that if α is a k-form and $X_1, \ldots, X_k$ are vector fields on M then $i\alpha(X_1, \ldots, X_k)$ is the imaginary number that is obtained by multiplying the real number $\alpha(X_1, \ldots, X_k)$ by i. Hence, in this case we are passing to the complexification of $\Lambda^*(M)$.

For simplicity, we set $* = \# \cdot \kappa$; so the second Maxwell equation takes the form $d*F = 0$ in the absence of sources. Furthermore, we assume that $*^2 = -I$, so it does, in fact, define an almost-complex structure.

Since $*F$ will take the form $T(t)*f(x)$, we see that a straightforward replacement of f with $*f$ in (VIII.9), combined with aforementioned replacement of ϖ with the constant $-i\omega$, will give:

$$dF = T[-i\omega\, dt \wedge f + df]. \tag{VIII.12a}$$
$$d*F = T[-i\omega\, dt \wedge *f + d*f]. \tag{VIII.12b}$$

Maxwell's sourceless equations then imply that:

$$df = i\omega\, dt \wedge f, \qquad d*f = i\omega\, dt \wedge *f. \tag{VIII.13}$$

To see that these indeed give the customary E-B form, substitute:

$$F = dt \wedge E - *(dt \wedge B) = T[dt \wedge u - *(dt \wedge v)], \tag{VIII.14}$$

in which we have assumed that the 1-forms E and B on M take the form:

$$E(t, x) = T(t)u(x), \qquad B(t, x) = T(t)v(x), \tag{VIII.15}$$

so $u(x)$ and $v(x)$ are 1-forms on Σ. Hence, they will have the component form:

$$u(x) = u_i(x)\, dx^i, \qquad v(x) = v_i(x)\, dx^i, \tag{VIII.16}$$

in an adapted coordinate system.

From (VIII.14), we see that:

$$*F = T[dt \wedge v + *(dt \wedge u)], \tag{VIII.17}$$

Hence, between (VIII.14) and (VIII.17), we see that:

$$f = dt \wedge u + *_s v, \qquad *f = dt \wedge v - *_s u. \tag{VIII.18}$$

We have also replaced the expressions $*(dt \wedge u)$ and $*(dt \wedge v)$ with $-*_s u$ and $-*_s v$, respectively.

It is physically illustrative to see what sort of matrix the map $*_s : \Lambda^1\Sigma \to \Lambda^2\Sigma$ will have. We use a basis $\{b^i, i = 1, 2, 3\}$ for a fiber of $\Lambda^1\Sigma$ that is adapted to a basis for $\Lambda^1 M = \Lambda^1 L \oplus \Lambda^1\Sigma$, and the corresponding basis $*b^i$ for $\Lambda^2\Sigma$ that is a subset of the basis $\{dt \wedge b^i, *b^i\}$ for $\Lambda^2 M$. The linear map $*_s = -*e_{dt}$ then has a 3×3 matrix relative to these bases that is the composition of the matrices $[e_{dt}][*][P_{\mathrm{Im}}]$, since the components of $u = u_i\, b^i$ are row matrices. Here, the linear map P_{Im}: $\Lambda^2 M = \Lambda^2_{\mathrm{Re}} M \oplus \Lambda^2_{\mathrm{Im}} M$ is the canonical projection of any 2-form onto its imaginary or "magnetic" part. If one recalls the discussion of almost-complex media from Chapter V, and represents the 6×6 matrix [*] in the form of (V.34) then the product of matrices in question is:

$$[*_s] = -\begin{bmatrix} I & 0 \end{bmatrix} \left[\begin{array}{c|c} \gamma & -\varepsilon \\ \hline \tilde{\mu} & \gamma^T \end{array}\right] \begin{bmatrix} 0 \\ \hline I \end{bmatrix} = [\varepsilon]. \tag{VIII.19}$$

This somewhat surprising result, that spatial duality is due solely to the electric permittivity of the medium, is actually subordinate to the assumption that we are considering an almost-complex medium, since (VIII.35) suggests that in order for an electromagnetic constitutive law to define such a structure, one must satisfy rather severe restrictions on the form of the submatrices of κ. In particular, in the absence of electromagnetic couplings ($\gamma = 0$), one must have that ε is proportional to μ.

Exterior differentiation of the resulting expressions in (VIII.18) gives:

$$df = -dt \wedge du + d*_s v = i\omega dt \wedge f = i\omega dt \wedge *_s v, \tag{VIII.20a}$$
$$d*f = -dt \wedge dv - d*_s u = i\omega dt \wedge *f = -i\omega dt \wedge *_s u, \tag{VIII.20b}$$

which gives:

$$-dt \wedge du + d*_s v = dt \wedge (i\omega *_s v), \tag{VIII.21a}$$
$$dt \wedge dv + d*_s u = dt \wedge (i\omega *_s u). \tag{VIII.21b}$$

From (VIII.13), (VIII.18), and (VIII.20), and considering the temporal or spatial nature of the 3-forms in question, we deduce:

$$du = -i\omega *_s v, \qquad d*_s u = 0, \tag{VIII.22a}$$
$$dv = +i\omega *_s u, \qquad d*_s v = 0. \tag{VIII.22b}$$

If we apply the $*_s$ operator to both sides of each equation then we see that they take the vector calculus form:

$$\nabla\times\mathbf{u} = +\, i\omega\mathbf{v}, \qquad \nabla\cdot\mathbf{u} = 0, \tag{VIII.23a}$$

$$\nabla\times\mathbf{v} = -\, i\omega\mathbf{u}, \qquad \nabla\cdot\mathbf{v} = 0, \tag{VIII.23b}$$

that is more customary in geometrical optics (see Luneberg [**6**]).

b. Stationary potential 1-forms. Since Maxwell's first equation suggests that the 2-form F can be locally represented by dA for some non-unique potential 1-form A, we next look at what happens when this 1-form is also stationary:

$$A(t, x) = T(t)a(x). \tag{VIII.24}$$

A change of gauge $A \mapsto A + d\lambda$ must also contain T as a common factor in both terms. In particular, $\lambda(t, x)$ must take the form $T(t)l(x)$. As a consequence, a change of gauge for A implies a change of the form $a \mapsto a + dl - i\omega l\, dt$ for a.

One must be careful to note that this time, although we are assuming that the components of $a(x)$ are functions on Σ, nevertheless, we are allowing it to take the form:

$$a = \phi(x)\, dt + a_i(x)dx^i \equiv \phi\, dt + a_s\,; \tag{VIII.25}$$

hence, it is what we have been calling a purely spatial 1-form on M.

We then have:

$$da = -\, dt \wedge d\phi + da_s\,. \tag{VIII.26}$$

However, in optical problems it is customary to assume that the electrostatic field $d\phi$ is null.

This makes:

$$F = dA = T(-\, i\omega dt \wedge a + da) = T[-\, dt \wedge (d\phi + i\omega a_s) + da_s]. \tag{VIII.27}$$

Hence:

$$f = -\, dt \wedge (d\phi + i\omega a_s) + da_s\,. \tag{VIII.28}$$

By matching up temporal and spatial terms in (VIII.28) and the first of (VIII.18), we get:

$$u = -\, d\phi - i\omega a_s, \qquad *_s v = -\, da_s\,. \tag{VIII.29}$$

Since:

$$*f = -\, *[dt \wedge (d\phi + i\omega a_s)] + *da_s = *_s\, d\phi + i\omega *_s\, a_s + *da_s\,, \tag{VIII.30a}$$

$$d*f = d*_s\, d\phi + i\omega d*_s\, a_s + d*da_s\,, \tag{VIII.30b}$$

the remaining Maxwell equation for $*F$ gets converted into the form:

$$\begin{aligned} 0 &= d*f - i\omega dt \wedge *f \\ &= [d*da_s + \omega^2\, dt \wedge *_s\, a_s] + [d*_s\, d\phi - i\omega dt \wedge *_s\, d\phi\,] + i\omega d*_s\, a_s\,. \end{aligned} \tag{VIII.31}$$

With the choice of gauge:

$$d\phi = 0, \qquad d^*_s a_s = 0, \tag{VIII.32}$$

which is essentially the Coulomb gauge, equation (VIII.30) takes the form:

$$0 = d^* d a_s + \omega^2\, dt \wedge {}^*_s a_s. \tag{VIII.33}$$

In order to put this into the customary Helmholtz form, first apply the * operator to both sides:

$$0 = {}^* d^* d a_s - \omega^2\, a_s\,. \tag{VIII.34}$$

By following the sequence of operators d, $*$, d, $*$ acting on the 1-form $a_s \in \Lambda^1(\Sigma)$, one sees that $*d*d$ has the same effect as:

$$-\,{}^*_s d^*_s d = -\,\Delta + d^*_s d^*_s\,. \tag{VIII.35}$$

However, with the choice of gauge (VIII.32) the second term on the right-hand side will vanish, and we finally obtain:

$$0 = \Delta a_s + \omega^2 a_s\,. \tag{VIII.36}$$

c. Geometrical optics approximation. The form that electromagnetic fields take in geometrical optics is a generalization of the aforementioned time-periodic form. In particular, rather than factoring $F(t, x)$ into a purely temporal part and a purely spatial part by way of the time function $e^{-i\omega t}$, we now assume that M still takes the form $\mathbb{R} \times \Sigma$, but we factor F as:

$$F(t, x) = e^{-i\phi} f(x), \tag{VIII.37}$$

in which $\phi(t, x)$ is a smooth function on M that one calls the *phase function*. Its level hypersurfaces in M are then referred to as *isophases.*

The most common form that ϕ takes is the *plane-wave* form:

$$\phi(t, x) = k_\mu x^\mu = \omega t \pm k_i x^i\,, \tag{VIII.38}$$

which amounts to assuming that $M = \mathbb{R}^4$ and ϕ is linear in the coordinates x^μ. The isophases are then affine hyperplanes. By means of other choices of coordinate system on M one can similarly define cylindrical and spherical isophases.

One sees that time-periodic electromagnetic fields are the special case of (VIII.38) for which $k_i = 0$ for all $i = 1, 2, 3$. However, one then refers to the isophases as *isochrones*, or *simultaneity hypersurfaces*. Clearly, this notion demands a more relativistic treatment, but we shall only point the curious to the author's work [**7**], and the references that were cited therein. Furthermore, the spirit of pre-metric electromagnetism is that causality is a *consequence* of the manner by which electromagnetic waves propagate through spacetime, not a prerequisite for it.

The physical significance of an isophase is that it represents the time evolution of a *momentary wave surface* in the space Σ. That is, the projection of an isophase in *M* onto Σ will be, by definition, a momentary wave surface. Hence, if $f(x)$ represents the overall "shape" of the wave motion in Σ then each value $f(x)$ will propagate in *M* on a fixed isophase. In effect, an isophase is a higher-dimensional analogue of the world line of a point particle.

When *F* takes the form (VIII.1), one derives:

$$dF = e^{-i\phi}(-ik \wedge f + df), \tag{VIII.39}$$

in which we have set:

$$k = d\phi. \tag{VIII.40}$$

Similarly, since $*F = e^{-i\phi}*f$, one has:

$$d*F = e^{-i\phi}(-ik \wedge *f + d*f). \tag{VIII.41}$$

Maxwell's sourceless equations then give:

$$df = ik \wedge f, \qquad d*f = ik \wedge *f. \tag{VIII.42}$$

If we compare these equations with (VIII.13) then we see that we have essentially replaced $\omega\, dt$ with $k = \omega\, dt + k_i\, dx^i$.

Note that, from Frobenius, the existence of such a *k* as in (VIII.42) implies that when *f* – hence, $*f$ – is a decomposable (i.e., rank two) 2-form the exterior differential systems $f = 0$ and $*f = 0$ must be completely integrable.

Now, this is always the case for electromagnetic waves, due to the basic properties of wavelike solutions to the Maxwell equations:

$$0 = f \wedge f = f \wedge *f. \tag{VIII.43}$$

Hence, *M* will be doubly foliated by the two-dimensional leaves to each of these foliations. At each point of *M* one leaf will be tangent to the characteristic variety of the field operator and the other one will be normal to it, and their intersection will be tangent to a generator of the characteristic variety; we shall discuss this situation in more detail shortly. When projected into Σ, the leaf that is tangent will produce a momentary wave surface, while the leaf that is normal will produce the normal trajectory that is followed by a point on that surface in time.

This is the point at which geometrical optics makes an approximation. One assumes that the spatial variation of the amplitude 2-form *f* is sufficiently slow compared to the variation of the phase function θ in time, which is referred to as either the *high-frequency* limit or the *small-wavelength* limit. For wavelengths on the order of visible light and smaller, this approximation is generally more than sufficient, although for the purposes of radio waves, which can have wavelengths in meters, it is generally inadequate.

In order to make the statement of this approximation somewhat more precise, we introduce the following notation:

$$k = \omega\hat{k}, \qquad \hat{k} \equiv dt - n_i\, dx^i, \qquad n_i \equiv k_i / \omega. \tag{VIII.44}$$

The components n_i then amount to indices of refraction in the various spatial directions, as they are reciprocal to the phase velocities ω/k_i, which we shall discuss below.

This allows us to rewrite (VIII.41) in the form:

$$1/\omega\, df = i\hat{k} \wedge f, \qquad 1/\omega\, d{*}f = i\hat{k} \wedge {*}f. \tag{VIII.45}$$

If one takes the high-frequency limit of these expressions then the Maxwell equations reduce to the algebraic equations:

$$0 = \hat{k} \wedge f = \hat{k} \wedge {*}f, \tag{VIII.46}$$

along with the partial differential equation (VIII.40). One can interpret equations (VIII.46) as saying that $\hat{k}$ – or k, for that matter – is incident on both the annihilating plane of f and the annihilating plane of $*f$, which then implies that $\hat{k}$ generates their line of intersection.

Now, let us examine the 1+3 form of equations (VIII.46). Thus, we let $f = dt \wedge u + *_s v$ and $*f = dt \wedge v - *_s u$, as before. We further let $\hat{k} = dt - n$, which makes:

$$\hat{k} \wedge f = + dt \wedge (*_s v + n \wedge u) - n \wedge *_s v, \tag{VIII.47a}$$
$$\hat{k} \wedge {*}f = - dt \wedge (*_s u - n \wedge v) + n \wedge *_s u. \tag{VIII.47b}$$

By considering the vanishing of the temporal and spatial parts independently, (VIII.47) gives:

$$*_s v = - n \wedge u, \qquad n \wedge *_s v = 0, \tag{VIII.48a}$$
$$*_s u = + n \wedge v, \qquad n \wedge *_s u = 0. \tag{VIII.48b}$$

One sees that these equations are similar to (VIII.22a,b), except that now there is no differentiation involved, and we have absorbed the factor of ω into n.

Note that now the second equation in each set follows unavoidably from the first.

The effect of these equations can be better understood by relating them to the conventional forms that they take. By operating on both sides of the first equations with $*_s$, and taking advantage of the fact that $*_s(\mathbf{a} \wedge \mathbf{b}) = \mathbf{a} \times \mathbf{b}$ and $a \wedge *_s b = (\mathbf{a} \cdot \mathbf{b})V_s$, we can put (VIII.48a, b) into the vector form:

$$\mathbf{v} = + \mathbf{n} \times \mathbf{u}, \qquad \mathbf{n} \cdot \mathbf{v} = 0, \tag{VIII.49a}$$
$$\mathbf{u} = - \mathbf{n} \times \mathbf{v}, \qquad \mathbf{n} \cdot \mathbf{u} = 0. \tag{VIII.49b}$$

Hence, given **u** one can deduce **v**, and vice versa. Similarly, if one given **u** and **v**, which are basically the Cauchy data for the initial-value problem associated with the field equations, when expressed in space-time form, one can derive **n** by using the first equation in either (VIII.48a) or (VIII.48b). One simply takes the interior product of both sides of (VIII.48a) with **u**, taking into account that $u(\mathbf{u}) = u^2$, $n(\mathbf{u}) = 0$:

$$n = u^{-2} *_s(u \wedge v), \tag{VIII.50}$$

Since the vector form of this is:

$$\mathbf{n} = \| \mathbf{u} \|^{-2} \mathbf{u} \times \mathbf{v}, \tag{VIII.51}$$

we see that the 2-form $u \wedge v$ is essentially the Poincaré dual of the *Poynting vector* for the wave.

Furthermore, when one includes the space-time form of (VIII.43), namely:

$$u \wedge *_s v = u \wedge *_s u - v \wedge *_s v = 0, \tag{VIII.52}$$

which has the vector form:

$$\mathbf{u} \cdot \mathbf{v} = u^2 - v^2 = 0, \tag{VIII.53}$$

one sees that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{k}_s\}$ represents a set of orthogonal, but not orthonormal, vectors at each point of Σ, relative to the spatial metric that is defined by $\langle u, v \rangle = \mathcal{V}_s(u \wedge *_s v)$.

We can also substitute the values of $*_s u$ and $*_s v$ that we obtained from (VIII.47a, b) into the definitions of f and $*f$ to deduce that:

$$f = \hat{k} \wedge u, \qquad\qquad *f = \hat{k} \wedge v. \tag{VIII.54}$$

This makes it clearer just what the nature of the 2-planes that are associated with f and $*f$ amounts to. The plane of f in a cotangent space is spanned by the orthogonal covectors k and u, while that of $*f$ is spanned by k and v. It is traditional to call the plane that is spanned by f the *polarization plane*.

d. Propagating discontinuities. Although the Fourier conception of waves as being linear superpositions of elementary plane waves is quite mathematically powerful and pervasive in its practical applications, there are nonetheless several disadvantages to this approach as far as physics is concerned.

For one thing, the very concept of a plane wave in $\mathbb{R}^3$ is unphysical, since such a wave ends up having infinite total momentum and infinite total kinetic energy, due to the non-compactness of its support. Of course, one almost always restricts one's consideration to plane waves in a compact region of $\mathbb{R}^3$, but, strictly speaking, such a spatial truncation of the wave must necessarily introduce higher-wave-number contributions to the spectral density of the wave that imply that it is not really a plane wave. Another disadvantage of planes waves is that they can only be defined globally in affine spaces, so their use in more general differentiable manifolds has a purely local character.

In the Hadamard approach to wave motion [**8-12**], a wave is defined to be a disturbance in a region of a medium that propagates through that medium. In particular, the region of the disturbance is assumed to be bounded by a surface $\mathcal{F}$ that one can call a *singular hypersurface* or the *wave front*. There might also be a bounding surface at the trailing edge of the wave, although generally one expects the shape of the wave envelope to decay smoothly from dissipative forces in the medium. Indeed, waves in even-

dimensional spaces (i.e., odd-dimensional spacetimes) will always be associated with such a decaying "tail."

Across this surface, one expects that there is a finite jump discontinuity in the field under consideration or one of its normal derivatives at some order. For instance, acoustic shock waves represent jump discontinuities in the covelocity 1-form u, which represents the exterior derivative of the velocity potential function. One also thinks of the sources of mechanical waves as being due to jump discontinuities in the acceleration vector field that originate in the fact that the driving force that produces them is approximately time-impulsive. That is, as a function of time, it looks like a Dirac delta function about some initial time point.

For electromagnetic waves, a time-impulsive source current $\mathbf{J}$ will produce a jump discontinuity in the bivector field $\mathfrak{h}$, and we assume that the set of all points at which $\mathfrak{h}$ is discontinuous is the wave front $\mathcal{F}$. Whether the jump discontinuity in $\mathfrak{h}$ also produces a jump discontinuity in F depends upon the linearity of κ: For the sake of progress, we assume for the rest of this section that F also has a jump discontinuity across $\mathcal{F}$. We denote the discontinuities in the fields in question by $[\mathbf{J}]$, $[F]$ and $[\mathfrak{h}]$, respectively. They then represent a smooth vector field, a smooth 2-form, and a smooth bivector field on $\mathcal{F}$, respectively.

In order to relate these discontinuities to the pre-metric Maxwell equations, one needs to first recognize that the most rigorous formalism in which to treat jump discontinuities analytically is in the language of distributions. We must then reformulate the pre-metric Maxwell equations in terms of distributions on differential forms and multivector fields.

Since we described several ways of defining continuous linear functionals on Λ^* and Λ_*, we need to first specify which definition that we are using. Because we are using the differential operators d and δ, the definition that is most convenient to our immediate purposes is the one that makes these operators adjoint to each other, namely:

$$\langle F, \mathbf{A}\rangle = \int_M F \wedge \#\mathbf{A} = \int_M F(\mathbf{A})\,\mathcal{V}, \tag{VIII.55a}$$

$$\langle \mathfrak{h}, \alpha\rangle = \int_M \#\mathfrak{h} \wedge \alpha = \int_M \alpha(\mathfrak{h})\,\mathcal{V}, \tag{VIII.55b}$$

$$\langle \mathbf{J}, \beta\rangle = \int_M \#\mathbf{J} \wedge \beta = \int_M \beta(\mathbf{J})\,\mathcal{V}. \tag{VIII.55c}$$

In these expressions, $\mathbf{A}$ is an arbitrary smooth bivector field of compact support, α is a smooth 2-form of compact support, and β is a smooth 1-form of compact support.

The Green formula for d and δ is obtained by integrating the product rule for $d(\alpha \wedge \#\mathbf{A})$:

$$\langle d\alpha, \mathbf{A}\rangle + (-1)^k \langle \alpha, \delta\mathbf{A}\rangle = \int_{\partial M} \alpha \wedge \#\mathbf{A}, \tag{VIII.56}$$

which vanishes when M has no boundary or a boundary that is disjoint from the support of $\mathbf{A}$. In that event, the operators d and δ are adjoint with respect to this bilinear pairing.

Hence, we can define the exterior product of the distribution F and the divergence of the distribution $\mathfrak{h}$ by way of:

$$\langle dF, \mathbf{A}\rangle = -\langle F, \delta\mathbf{A}\rangle, \qquad \langle \delta\mathfrak{h}, \alpha\rangle = -\langle \mathfrak{h}, d\alpha\rangle. \qquad \text{(VIII.57)}$$

Therefore, these definitions allow one to define the differentiation of fields with jump discontinuities by differentiating the test fields on which the distributions act.

Thus, in order to make sense of the pre-metric Maxwell equations as equations involving distributions, one then says that what they really mean is that for every smooth trivector field **A** of compact support and every smooth 3-form α of compact support, one has:

$$\langle dF, \mathbf{A}\rangle = 0, \qquad \langle \delta\mathfrak{h}, \alpha\rangle = \langle \mathbf{J}, \alpha\rangle, \qquad \mathfrak{h} = C(F), \qquad \text{(VIII.58)}$$

or:

$$\langle F, \delta\mathbf{A}\rangle = 0, \qquad \langle \mathfrak{h}, d\alpha\rangle = -\langle \mathbf{J}, \alpha\rangle, \qquad \mathfrak{h} = C(F), \qquad \text{(VIII.59)}$$

One can then consolidate these equations into equations for F alone or $\mathfrak{h}$ alone by using the third equation. In the sourceless case, this gives:

$$\langle F, \delta\mathbf{A}\rangle = 0, \qquad \langle C(F), d\alpha\rangle = 0, \qquad \text{(VIII.60)}$$

or:

$$\langle C^{-1}(\mathfrak{h}), \delta\mathbf{A}\rangle = 0, \qquad \langle \mathfrak{h}, d\alpha\rangle = 0, \qquad \text{(VIII.61)}$$

respectively.

If F has a jump discontinuity across a hypersurface $\phi(x) = 0$ then we let F_- represent the restriction of F to the half-space $\phi(x) < 0$ and F_+ represents its restriction to $\phi(x) > 0$. By the notation $[F] = F_+ - F_-$, we intend to denote a smooth 2-form on the hypersurface itself that represents the actual jump itself.

e. Polarization. Upon closer inspection of most treatments of the polarization of electromagnetic fields [**1**-**6**], one sees that the fact that one is using complex electric and magnetic field vectors is largely irrelevant to the nature of the discussion. In particular, one does not make use of either the electromagnetic constitutive laws of the medium or the Maxwell equations. Indeed, the essence of the construction amounts to the description of how the orbit of $U(1)$ on Euclidian $\mathbb{C}^3$, by way of scalar multiplication by the factor $e^{-i\omega t}$, projects onto the real $\mathbb{R}^3$ subspace.

Clearly, since $U(1)$ is a circle as a real manifold and its action on $\mathbb{C}^3$ is faithful, the image of $U(1)$ is diffeomorphic to a circle. If $\mathbf{a} = \mathbf{a}_R + i\mathbf{a}_I \in \mathbb{C}^3$ is an arbitrary non-zero vector then the orbit of $\mathbf{a}$ under the action of $e^{-i\omega t}$ takes the form:

$$\begin{aligned} e^{-i\omega t}\mathbf{a} &= \cos(t\omega)\,\mathbf{a} - i\sin(\omega t)\,\mathbf{a} \\ &= [\cos(t\omega)\,\mathbf{a}_R + \sin(t\omega)\,\mathbf{a}_I] - i\,[\sin(t\omega)\,\mathbf{a}_R - \cos(t\omega)\,\mathbf{a}_I] \end{aligned} \qquad \text{(VIII.62)}$$

The projection of $e^{-i\omega t}\mathbf{a}$ onto the real $\mathbb{R}^3$ subspace is then:

$$\mathrm{Re}[e^{-i\omega t}\mathbf{a}] = \cos(t\omega)\,\mathbf{a}_R + \sin(t\omega)\,\mathbf{a}_I\,. \qquad \text{(VIII.63)}$$

Although this is a loop in $\mathbb{R}^3$, one can perform a projection of $\mathbb{R}^3$ onto $\mathbb{R}^2$ (and possibly just $\mathbb{R}$) by taking the vectors $\mathbf{a}_R$ and $\mathbf{a}_I$ to $(a_R, 0)$ and $(0, a_I)$, respectively, unless $\mathbf{a}_R$ and $\mathbf{a}_I$ are collinear as real vectors, in which case, they go to $(a_R, 0)$ and $(a_I, 0)$, respectively. Hence, the image of $\mathrm{Re}[e^{-i\omega t}\mathbf{a}]$ in $\mathbb{R}^2$ takes the form $(a_R \cos(t\omega), a_I \sin(t\omega))$, in the non-degenerate case, and $(a \cos \omega t, 0)$, in the degenerate case, with $a = \max\{a_R, a_I\}$. In general, this curve takes the form of an ellipse with the given vectors as semi-major and semi-minor axes, depending upon the relative magnitudes of the vectors. As we have defined things, the vectors $(a_R, 0)$ and $(0, a_I)$ will be orthogonal even when $\mathbf{a}_R$ and $\mathbf{a}_I$ are not, which differs from the standard treatment of polarization. However, one can still distinguish the same three polarization classes, as usual.

One can then classify the type of ellipse in terms of the relationship between the two real 3-vectors $\mathbf{a}_R$ and $\mathbf{a}_I$. In the generic case, they are non-parallel and unequal in length; this is referred to as *elliptical* polarization. When they are equal in length, this is called *circular* polarization. The degenerate case, in which the ellipse flattens into a line segment, is called *linear* polarization.

Since a circle has two orientations – viz., clockwise and counter-clockwise – and the action of $U(1)$ preserves orientation, we can also speak of the sense in which the orbit of $\mathbf{a} \in \mathbb{C}^3$ is traversed. Rather than using the word "orientation" to describe this situation, one uses the word *helicity*. That quantity will have the value +1 or –1, depending upon the convention that one chooses for the orientation of circles in $\mathbb{R}^3$.

2. Characteristics. For the purposes of this section, in order to avoid confusion we shall refer to the constitutive map by the notation C, instead of κ:

a. Characteristic polynomial. In order to find the symbol of the second order differential operator $\Box_C$ we need to replace C with its linearization DC if it is nonlinear to begin with; from now on, we simply assume that C is linear. We then find that the symbol of the field operator $\Box_C$ is:

$$\sigma[\Box_C, k] = i_k \cdot C \cdot e_k\,. \qquad \text{(VIII.64).}$$

From (VIII.5), we see that the components of the linear operator $\sigma[\Box_C, k]$ are:

$$\sigma^{\mu\nu}[\Box_C, k] = -\,C^{\mu\kappa\lambda\nu}\,k_\kappa k_\lambda\,. \qquad \text{(VIII.65)}$$

It is illuminating to compute the 4×4 matrix $\sigma^{\mu\nu}[\Box_C, k]$ explicitly by regarding (VIII.64) as the product of the 4×6 matrix $\left[i_k\right]_I^\mu$, the 6×6 matrix C^{IJ}, and the 6×4 matrix

$[e_k]^\nu_J$, respectively, when we choose the coframe θ^μ for $\Lambda^1 M$, the coframe$\{b^i, \#\mathbf{b}_i\}$ for $\Lambda^2 M$, and the reciprocal frames $\{\mathbf{b}_i, \#b^i\}$ and $\mathbf{e}_\mu$ for $\Lambda_1 M$ and $\Lambda_2 M$, respectively. As usual, we set $b^i = \theta^0 \wedge \theta^i$ and $\mathbf{b}_i = \mathbf{e}_0 \wedge \mathbf{e}_i$. By direct computation, one finds that:

$$[i_k]^\mu_I = \left[\begin{array}{c|cc} -k_i & 0 & 0 \\ \hline \omega\delta^i_j & \multicolumn{2}{c}{-ad(k)^i_j} \end{array}\right], \qquad ad(k)^i_j \equiv \varepsilon^{ijk} k_k = \begin{bmatrix} 0 & -k_2 & k_3 \\ k_2 & 0 & -k_1 \\ -k_3 & k_1 & 0 \end{bmatrix}. \quad \text{(VIII.66)}$$

The matrix $[e_k]^\nu_J$ is simply the transpose of the matrix $[i_k]^\mu_I$ since the maps are adjoint to each other. If we give C^{IJ} the usual form that we discussed in Chapter V then the multiplication of the matrices gives:

$$\sigma^{\mu\nu}[\Box_C, k] = \left[\begin{array}{c|c} \varepsilon(k,k) & -\omega\varepsilon^{im}k_m - k_m\gamma^m_n ad(k)^{in} \\ \hline -\omega\varepsilon^{jm}k_m + k_m\hat{\gamma}^m_n ad(k)^{jn} & \omega^2\varepsilon^{ij} + \omega[\gamma\cdot ad(k) - ad(k)\cdot\hat{\gamma}]^{ij} - ad(k)^{im}\tilde{\mu}_{mn}ad(k)^{nj} \end{array}\right] \quad \text{(VIII.67)}$$

In general, the bundle map $\sigma[\Box_C, k]$: $T^*(M) \to T(M)$ does not have to be invertible, and this depends upon the choice of k. As we discussed in the chapter on partial differential equations, the definition:

$$P[k] \equiv \det \sigma[\Box_C, k] \quad \text{(VIII.68)}$$

then defines a polynomial in k that one calls the *characteristic polynomial* of the differential operator $\Box_C$. It vanishes iff $\sigma[\Box_C, k]$ is not invertible, and the zero locus of $P[k]$ is called the *characteristic hypersurface (or variety)* for $\Box_C$; we shall then call a k that lies in this hypersurface *characteristic*. The algebraic equation in k that is thus defined:

$$P[k] = 0 \quad \text{(VIII.69)}$$

is then what we call the *dispersion law* for the wave medium in question.

From (VIII.68), one can see that since k appears twice in $\sigma[\Box_C, k]$ this implies that the polynomial $P[k]$ will have a degree that is equal to $2n$. It is, moreover, homogeneous in k of degree $2n$. However, there are some traditional reductions that get applied to the degree.

First, one generally deals with the case of time-invariant constitutive laws on space-time separable manifolds, so M takes the form $\mathbb{R} \times \Sigma$, where $\mathbb{R}$ plays the role of the time manifold and Σ is the spatial manifold. This reduces our map $\sigma[\Box_C, k]$ to a map from $T^*\Sigma$ to $T(\Sigma)$, and the degree of $P[k]$ to $2(n-1)$; basically, one considers only the lower

right-hand sub-matrix in (VIII.67). For instance, when $n = 4$ the polynomial in k that one considers is homogeneous and sextic.

Second, as we are dealing with the case of electromagnetic waves, the map e_k: $\Lambda^1(\Sigma) \rightarrow \Lambda^2(\Sigma)$, $\phi \mapsto k \wedge \phi$ is not invertible for any k, since its kernel is one-dimensional, namely, all ϕ that take the form λk for some scalar $\lambda \in \mathbb{R}$. This is related to the fact that electromagnetic waves have no longitudinal modes of vibration, but are confined to the *Poynting plane,* which is 2-plane in T^*M that is spanned by E and H. Hence, the characteristic polynomial reduces to a homogeneous quartic polynomial in k in the electromagnetic case:

$$P[k] = P^{\kappa\lambda\mu\nu}\, k_\lambda\, k_\kappa\, k_\mu\, k_\nu\,. \tag{VIII.70}$$

By the fact that any polynomial of degree d in the coordinates k_μ of $\mathbb{R}^{n*}$ is associated with a completely symmetric covariant tensor of rank d, one can define a fourth-rank completely symmetric covariant tensor field on M that is associated with the polynomial $P[k]$. Locally, it looks like:

$$P = P^{\kappa\lambda\mu\nu}\, \partial_\kappa \partial_\lambda \partial_\mu \partial_\nu\,. \tag{VIII.71}$$

In the book by Hehl and Obukhov [**13**], this tensor field is referred to as the *Tamm-Rubilar tensor,* since it represents an enlargement of the scope of a tensor that was first defined by Tamm [**14**] in the early Twentieth Century that was defined by Rubilar [**15**]. In particular, the latter author derived an expression for the components $P^{\kappa\lambda\mu\nu}$ in terms of the components $\kappa^{\kappa\lambda\mu\nu}$ of the constitutive law:

$$P^{\kappa\lambda\mu\nu} = \frac{1}{4!}\, \varepsilon_{\alpha\beta\gamma\delta}\, \varepsilon_{\rho\sigma\tau\eta}\, \kappa^{\alpha\beta\rho(\kappa}\, \kappa^{\lambda\,|\gamma\delta|\mu}\, \kappa^{\nu)\sigma\tau\eta}\,. \tag{VIII.72}$$

One sees that, like the polynomial that spawned it, the tensor field is also homogeneous of degree four.

b. Propagation of discontinuities. One of the many good reasons for representing waves as propagating discontinuities, besides their generality and independence of linear space structures, is the fact that if the Cauchy problem for a second-order partial differential equation is well-posed then a second-order jump discontinuity in the solution – i.e., an acceleration wave – can only be defined across a characteristic hypersurface. Hence, the second-order jump discontinuities can only represent wavelike solutions.

This result was something that Hadamard showed in his seminal work [**8**] on the basis of compatibility relations for the wave function ψ that followed from the assumption that the function $[\psi] = \psi_+ - \psi_-$, which is defined on S, is continuously differentiable to at least second order. The only potentially unresolved aspect of the behavior of ψ across S is then in its normal – i.e., time – derivatives. Now, the first time derivative on S is assumed to be continuous, as part of the Cauchy problem, so the issue is whether one is free to specify the second time derivative of ψ on S or does it follow automatically by

specifying the Cauchy data and demanding that ψ satisfy the wave equation in question, viz., $\Box_C \psi = 0$, everywhere.

It is in the process of resolving the latter question that one finds yet another way of passing to essentially the symbol of $\Box_C$ and obtaining the characteristic equation. We exhibit this process in the form:

Theorem:

If F is a weak solution of the sourceless pre-metric Maxwell equations that has a jump discontinuity $[F]$ *across the hypersurface* $\phi(x) = 0$ *then that hypersurface is characteristic. In particular:*

$$d\phi \wedge [F] = 0, \qquad i_{d\phi}[\mathfrak{h}] = 0, \qquad [\mathfrak{h}] = C([F]), \tag{VIII.73}$$

so:

$$F[d\phi] = 0. \tag{VIII.74}$$

Proof:

Suppose $c = c_- + c_+$ is a 4-cycle that intersects $\phi(x) = 0$ such that c_- lies in the half-space $\phi(x) < 0$, while c_+ lies in the half-space $\phi(x) > 0$; hence, one also has $\partial c_- = -\partial c_+$. This situation is depicted in Fig. 7.

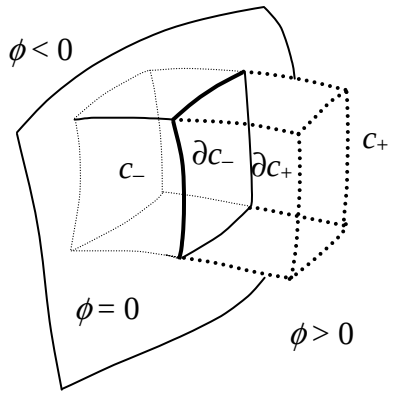

Figure 7. A typical 4-cycle that intersects the hypersurface $\phi(x) = 0$.

On c_- and c_+ the adjointness relations take the form:

$$<dF_\pm, \mathbf{G}> - <F_\pm, \delta\mathbf{G}> = \int_{\partial c_\pm} F \wedge \#\mathbf{G},$$
$$<\delta\mathfrak{h}_\pm, \alpha> - <\mathfrak{h}_\pm, d\alpha> = \int_{\partial c_\pm} \#\mathfrak{h} \wedge \alpha.$$

Since $F_\pm$ and $\mathfrak{h}_\pm$ are smooth solutions of the sourceless pre-metric Maxwell equations, the left-hand sides vanish. As for the right-hand sides, by addition, we see that for every

smooth bivector field **G** of compact support and every smooth 1-form α of compact support one must have:

$$0 = \int_{\partial c_-} F_- \wedge \#\mathbf{G} + \int_{\partial c_+} F_+ \wedge \#\mathbf{G} = \int_{\partial c_+} [F] \wedge \#\mathbf{G},$$
$$0 = \int_{\partial c_-} \#\mathfrak{h}_- \wedge \alpha - \int_{\partial c_+} \#\mathfrak{h}_- \wedge \alpha = \int_{\partial c_+} \#[\mathfrak{h}] \wedge \alpha .$$

In particular, they are true when $\#\mathbf{G} = \alpha = d\phi$.

Since they are also true for any choice of c, one has:

$$d\phi \wedge [F] = 0, \qquad [\#\mathfrak{h}] \wedge d\phi = 0, \qquad [\mathfrak{h}] = C([F]),$$

which is equivalent to (VIII.73).

By solving the first equation as $[F] = d\phi \wedge [A]$, where $[A]$ is a 1-form on the hypersurface, one can combine the equations into:

$$(i_{d\phi} \cdot C \cdot e_{d\phi})[A] = 0,$$

which is the same as:

$$\sigma[\Box_C, d\phi][A] = 0.$$

The condition for this to admit a non-trivial solution $[A]$ is the vanishing of $\det(\sigma[\Box_C, d\phi])$, which gives (VIII.74).

3. Examples of dispersion laws. In order to derive specific dispersion laws for specific constitutive laws, one finds that it is often easiest to start with the block matrix (VIII.67) and evaluate the determinant when one restricts the submatrices of the constitutive law accordingly.

a. Isotropic media. In the case of an *isotropic* medium, for which $\varepsilon_{ij} = \varepsilon(x)\delta_{ij}$, $\mu_{ij} = \mu(x)\delta_{ij}$, $\gamma^i_j = \hat{\gamma}^j_i = 0$, one finds that (VIII.67) takes the form:

$$\sigma^{\mu\nu}[\Box_C, k] = \left[\begin{array}{c|c} \varepsilon k^2 & -\varepsilon\omega k^i \\ \hline -\varepsilon\omega k^j & \varepsilon\omega^2\delta^{ij} - (1/\mu)\delta_{mn}ad(k)^{im}ad(k)^{nj} \end{array}\right]$$
$$= \varepsilon\left[\begin{array}{c|c} \kappa^2 & -\omega k^i \\ \hline -\omega k^j & \omega^2\delta^{ij} - (1/\varepsilon\mu)k^i k^j \end{array}\right], \qquad \text{(VIII.75)}$$

in which $\kappa^2 = \delta^{ij} k_i k_j$ takes the form of the square of the Euclidian spatial norm of the spatial wave number covector $k_i\, dx^i$.

Taking the determinant gives:

$$\det(\sigma^{\mu\nu}[\Box_C, k]) = \varepsilon^4 (\omega^2 - c^2\kappa^2)^2, \qquad \text{(VIII.76)}$$

in which $c^2 = 1/\varepsilon\mu$ then becomes the speed of propagation.

The vanishing of the determinant then reduces to the vanishing of a quadratic polynomial – viz.:

$$g^{\mu\nu} k_\mu k_\nu = \omega^2 - c^2 \delta^{ij} k_i k_j , \tag{VIII.86}$$

which involves the constitutive properties of the spacetime only by way of c.

Since the resulting quadratic form on k that this equation defines has a normal hyperbolic signature type (+1, –1, …, –1), one sees that this is where the light cones finally originate, as well as the unit proper time hyperboloids and mass shells.

b. Anisotropic optical media - Fresnel analysis. In traditional geometrical optics [**3-6**], one deals with electromagnetic constitutive laws κ of a very particular form, as we discussed in Chapter V. Mostly, one assumes that the γ and $\hat{\gamma}$ matrices vanish, while the magnetic properties of the medium are linear, homogeneous, and isotropic, so $\mu_{ij} = \mu\delta_{ij}$, and the dielectric tensor ε_{ij} is symmetric. One can then reduce the matrix (VIII.67) to:

$$\sigma^{\mu\nu}[\Box_C, k] = \left[\begin{array}{c|c} \varepsilon(k,k) & -\omega\varepsilon^{im}k_m \\ \hline -\omega\varepsilon^{jm}k_m & \omega^2\varepsilon^{ij} - ad(k)^i_m\, ad(k)^{mj} \end{array}\right] \tag{VIII.87}$$

Next, one factors out the ω^2, while keeping in mind that $n_i = k_i/\omega$, and obtains:

$$\sigma^{\mu\nu}[\Box_C, k] = \left[\begin{array}{c|c} \varepsilon(n,n) & -\varepsilon^{im}n_m \\ \hline -\varepsilon^{jm}n_m & \varepsilon^{ij} - ad(n)^i_m\, ad(n)^{mj} \end{array}\right] \tag{VIII.88}$$

In the usual formulation, the matrix that one obtains from the algebraic form of the Maxwell equations – viz., $\mathbf{v} = c\mathbf{n} \times \mathbf{u}$, $\mathcal{E}(\mathbf{u}) = -c\mathbf{n} \times \mathbf{v}$ – is:

$$\sigma^{ij} = \varepsilon^{ij} - (n^2\delta^{ij} - n^i n^j) \tag{VIII.89}$$

However, we see that this matrix is precisely the spatial sub-matrix in (VIII.87).

We can then think of the polynomial in the components of the covector:

$$\Phi[n] = \det \sigma^{ij} \tag{VIII.90}$$

as a polynomial in the inhomogeneous coordinates of $\mathbb{R}\mathrm{P}^{3*}$, which we call the *Fresnel polynomial*.

If one further considers ε^{ij} in its principal frame, which then gives ε^{ij} the form diag[ε_x, ε_y, ε_z], then when one takes the determinant of σ^{ij} one obtains a characteristic equation in the form:

$$\begin{aligned} 0 = \Phi[n] = {} & n^2[\varepsilon^x n_x^2 + \varepsilon^y n_y^2 + \varepsilon^y n_y^2] \\ & -[n_x^2\varepsilon^x(\varepsilon^y + \varepsilon^z) + n_y^2\varepsilon^y(\varepsilon^x + \varepsilon^z) + n_z^2\varepsilon^z(\varepsilon^x + \varepsilon^y)] + \varepsilon^x\varepsilon^y\varepsilon^z , \end{aligned} \tag{VIII.91}$$

which can be put into the more elegant forms:

$$\frac{n_x^2}{n^2-\varepsilon^x}+\frac{n_y^2}{n^2-\varepsilon^y}+\frac{n_z^2}{n^2-\varepsilon^z}=\frac{1}{n^2} \tag{VIII.92}$$

or:

$$\frac{n_x^2}{\frac{1}{v_p^2}-\frac{1}{v_x^2}}+\frac{n_y^2}{\frac{1}{v_p^2}-\frac{1}{v_y^2}}+\frac{n_z^2}{\frac{1}{v_p^2}-\frac{1}{v_z^2}}, \tag{VIII.93}$$

in which $v_p = \omega/\kappa = 1/n$ is the phase velocity of the wave whose normal is n_i and we have introduced the principal velocities $v^i = 1/\sqrt{\varepsilon^i}$ of the medium.

However, the form (VIII.91) makes it more explicit that one is dealing with a quartic polynomial in the inhomogeneous coordinates n_i of the wave normal n.

The quartic hypersurface in $\mathbb{R}P^{3*}$ that is defined by either (VIII.91), (VIII.92), or (VIII.93) is called the *Fresnel normal hypersurface.*

The general case of ε^x, ε^y, ε^z all distinct corresponds to the case of a *biaxial* medium. If we assume, without loss of generality that $\varepsilon^x < \varepsilon^y < \varepsilon^z$ then one octant of the Fresnel normal hypersurface can be depicted as in Fig. 8, along with its intersections with the coordinate planes.

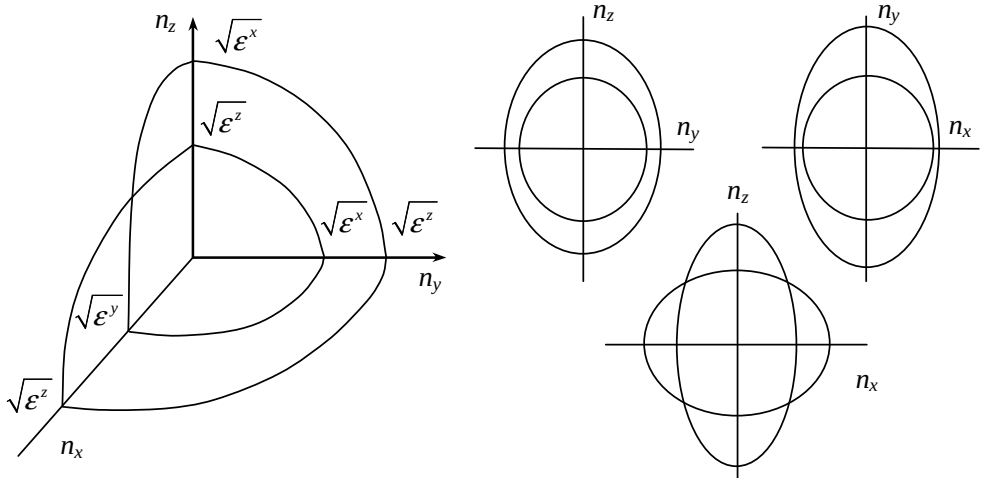

Figure 8. The general Fresnel quartic (biaxial media).

As one can see from the figure, it is not unheard of for the Fresnel quartic to be self-intersecting and the existence of such a singularity corresponds to the possibility of *conical refraction*. That is, one incoming ray can produce a cone of outgoing ones.

A possible property of $P[k]$ that is of considerable interest in electromagnetism is *birefringence*. In optics, this is associated with a certain type of anisotropy that is found in "uniaxial" media (see Landau, et al. [**3**]). For such media, two of the principal values of ε^{ij} – say, ε^x and ε^y – are equal. One refers to the two equal values as the *ordinary* values, while the third one is the *extraordinary* value, and one denotes them by ε^o and ε^e, respectively.

Birefringence refers to the fact that when $P[k_\mu]$ is quartic, if one fixes the spatial components k_i of k then the remaining polynomial $P[\omega, k_i]$ is quadratic in ω^2, and its roots can be shown to be real. This then implies that for any spatial direction of propagation there will generally be two distinct positive values of ω, and therefore two distinct values of the phase velocity ω/κ, where $\kappa^2 = \delta^{ij} k_i\, k_j$. This leads to double refraction of a given light ray. It is customary to refer to the resulting waves of the pair as the *ordinary* and *extraordinary* waves. This phenomenon can be observed by placing a slab of calcite over a page of print, which produces double images of the letters. The fainter image is then due to the extraordinary waves.

In terms of the Fresnel quartic, the effect of a uniaxial dielectic is to cause the quartic polynomial to factor into a product of quadratic ones:

$$(n^2 - \varepsilon^o)[\varepsilon^e n_z^2 + \varepsilon^0 (n_x^2 + n_y^2) - \varepsilon^0 \varepsilon^e] = 0. \qquad \text{(VIII.94)}$$

The factorization of the polynomial then implies that the quartic consists of the union of a sphere and an ellipsoid that intersect at their North and South poles. The sphere is indicative of an isotropic medium, so one sees why the corresponding principal value of ε^{ij} would be referred to as ordinary.

There are two possibilities for the extraordinary ellipsoid corresponding to whether it is outside or inside the ordinary sphere. In the former case, one says that the medium is *positive*, while in the latter case one refers to it as *negative*. We depict these two possibilities in Fig. 9 by means of the x–z sections of the surfaces.

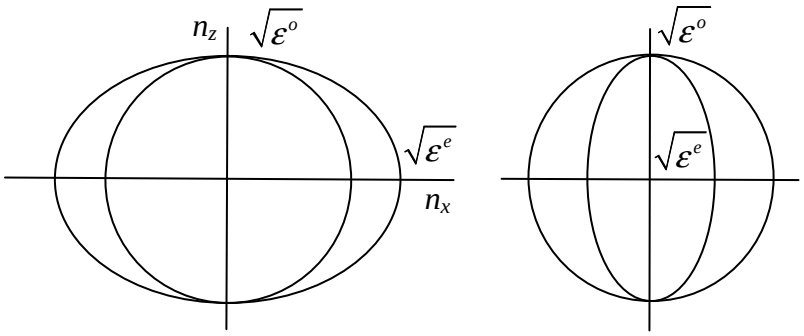

Figure 9. Fresnel quartic (uniaxial media).

b. Skewon contributions [**16**]. Although a purely skewonic medium would give a vanishing dispersion polynomial, which would imply the absence of wave modes, nonetheless, as a contribution to the principal part of a constitutive tensor field, it can have an effect.

According to the analysis that was given to the subject in [**16**, loc. cit.], one must distinguish between real and imaginary skewon contributions. Furthermore, one can classify skewons in terms of a basic decomposition as having "electric Faraday,"

"magnetic Faraday," and "magneto-electric optical activity" type. (see, loc. cit. for all definitions involved).

A real skewon contribution amounts to a source of resistivity and energy absorption for waves. For a skewon of "electric Faraday" type, the Fresnel wave surface can take on the character of a torus. That is, it has changed its topology, as well as its shape. When a real skewon has the "magneto-electric optical activity" type, the Fresnel surface can become two intersecting torii.

In the case of imaginary skewon contributions, one has to distinguish not only types of skewons, but also the ones that are "small," "medium," and "large" in magnitude when compared to the principal part. Generally, they are associated with natural optical activity and the Faraday effects. For the large ones of "magneto-electric optical activity type," the Fresnel surface will become merely an oblate spheroid, while for small ones it will be two concentric spheroids. For those of "magnetic Faraday" type, the large ones give intersecting hyperboloids, the small ones give intersecting spheroids, and in between one finds intersecting paraboloids.

c. Axion contributions. A direct verification using (VIII.67) shows that a axion part to a constitutive law does not contribute to the dispersion polynomial. That is, it does not affect the propagation of waves *in the geometrical optics approximation.* Perhaps that is why sometimes the vanishing of the axion part is a basic axion of electromagnetic constitutive laws that is referred to as the "Post constraint," since it was Post [**17**] who advocated that axiom.

However, as Itin [**18**] observes, if one goes beyond that approximation and considers waves for which the amplitude function is not effectively constant then the first derivatives of that amplitude can couple to the axionic part in a non-trivial way. In particular, one finds that dispersion polynomial is no longer a homogeneous polynomial in the wave covector, but an inhomogeneous one.

d. Bi-metric media. There often exists a factorization of the quartic polynomial $P[k]$ into a product of quadratic polynomials (which is called *bi-metricity* by Barcello, Liberati, and Visser [**19**]):

$$P[k] = g(k, k)\,\bar{g}(k, k)\,. \tag{VIII.95}$$

Hence, the Tamm-Rubilar tensor field that is associated with $P[k]$ takes the form of symmetrized tensor product of two Lorentzian metrics on $T^*(M)$:

$$P = g \odot \tilde{g}\,. \tag{VIII.96}$$

The components of P are then obtained from those of g and $\tilde{g}$ by way of:

$$P^{\kappa\lambda\mu\nu} = \tfrac{1}{6}(g^{\kappa\lambda}\tilde{g}^{\mu\nu} + g^{\mu\lambda}\tilde{g}^{\kappa\nu} + g^{\nu\lambda}\tilde{g}^{\mu\kappa} + g^{\kappa\mu}\tilde{g}^{\lambda\nu} + g^{\kappa\nu}\tilde{g}^{\mu\lambda} + g^{\mu\nu}\tilde{g}^{\kappa\lambda})\,. \tag{VIII.97}$$

Although fourth-degree polynomials in more than one real variable do not always have to factorize into products of quadratics, nonetheless, in the case of electromagnetic waves, it is widely known that as long as the Lagrangian for the electromagnetic field F

depends only upon the Lorentz invariants $F \wedge F$ and $F \wedge {}^*F$, the characteristic polynomial *will* factorize, even in the nonlinear case. (See [**15**].)

e. One-loop effective vacua. From the analysis that was given in Barcello, Liberati, and Visser [**19**], one sees that the dispersion laws for both the Heisenberg-Euler and Born-Infeld media take the general bimetric form:

$$(\eta_{\mu\nu} + \varepsilon_1 T_{\mu\nu})(\eta_{\mu\nu} + \varepsilon_2 T_{\mu\nu}) = 0. \qquad \text{(VIII.98)}$$

In this equation, the components $T_{\mu\nu}$ refer to the Faraday stress-energy-momentum tensor field for the background electromagnetic field, which we will discuss more thoroughly in Chapter X, while the scalar factors ε_1 and ε_2 depend upon the electromagnetic field strengths and would generally have units of 1/energy density. Hence, in this case the geometry of spacetime is perturbed quite directly by the presence of a background electromagnetic field.

f. Plasmas. In plasmas, not only the constitutive laws can vary, but so can the field equations themselves, depending upon what degree of interaction (usually collisions) one is assuming. Furthermore, plasmas are capable of propagating not only electromagnetic waves, but also mechanical – i.e., acoustic – ones. Rather than derive the dispersion laws, which would take us quite far afield, we simply summarize some of them that one deals with in the linear case (cf., [**20**]).

In general, waves in plasmas can be either acoustic or electromagnetic, and can be concerned with the basic oscillations of either the electrons or the ions. Furthermore, as we already saw in Chapter V, the presence or absence of a background magnetic field $\mathbf{B}_0$ can affect whether the dielectric – hence, optical – properties of the plasma are isotropic or anisotropic. They can also exhibit linear or nonlinear wave behavior, depending upon their amplitudes, even though often the actual number densities or temperatures involved can be relatively casual.

Electron waves can be electrostatic or electromagnetic and the dispersion laws for electrostatic electron waves are:

$$\omega^2 - 3/2 v_{th}^2 k^2 = \omega_p^2, \qquad (\mathbf{B}_0 = 0 \text{ or } \mathbf{k} \parallel \mathbf{B}_0) \qquad \text{(VIII.99a)}$$

$$\omega^2 = \omega_h^2 \equiv \omega_p^2 + \omega_c^2 \qquad (\mathbf{k} \perp \mathbf{B}_0) \qquad \text{(VIII.99b)}$$

respectively. In these expressions, $v_{th} = (2KT_e/m)^{1/2}$ is the thermal velocity of the electrons, while $\omega_p = (n_0 e^2/\varepsilon_0 m)^{1/2}$ is the *plasma frequency* of the electrons when their equilibrium number density is n_0, and ω_h is called the *upper hybrid frequency.* The plasma oscillations come about due to the fact that the equilibrium configuration of the electrons is stable, so any perturbation of an electron from equilibrium will produce a counter-electric field that tends to restore the equilibrium, bur produces a characteristic oscillation about the equilibrium configuration.

Note that the second relation (VIII.99b) does not give an actual dispersion law, so no traveling wave actually propagates, only a stationary oscillation.

For electromagnetic electron waves, one has:

$$\omega^2 - c^2 k^2 = \omega_p^2, \qquad (\mathbf{B}_0 = 0) \qquad \text{(VIII.100a)}$$

$$= \omega_p^2, \qquad (\mathbf{k} \perp \mathbf{B}_0, \mathbf{E}_1 \parallel \mathbf{B}_0) \qquad \text{(VIII.100b)}$$

$$= \omega_p^2 \left(\frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2} \right), \quad (\mathbf{k} \perp \mathbf{B}_0, \mathbf{E}_1 \perp \mathbf{B}_0) \qquad \text{(VIII.100c)}$$

$$= \omega_p^2 \left(\frac{\omega}{\omega - \omega_c} \right), \qquad (\mathbf{k} \parallel \mathbf{B}_0) \qquad \text{(VIII.100d)}$$

$$= \omega_p^2 \left(\frac{\omega}{\omega + \omega_c} \right), \qquad (\mathbf{k} \parallel \mathbf{B}_0), \qquad \text{(VIII.100e)}$$

in which $\omega_c = eB_0/m$ is the cyclotron frequency of the electron in the magnetic field.

One sees that in the absence of a background magnetic field, the dispersion law has the same "Klein-Gordon" form as in the electrostatic case, but with a different speed of propagation, and in which the plasma frequency plays a role that is analogous to the Compton frequency of a massive wave in relativistic quantum wave mechanics. In particular, the electromagnetic waves are not obtained from the vanishing of the characteristic polynomial – i.e., its zero locus – but from a non-zero locus. Whether this analogy between plasma waves and matter waves proves to be a useful tool in understanding quantum physics clearly deserves more attention.

The presence of a background field, the dielectric tensor becomes anisotropic, which leads to a bi-metric situation. When the background magnetic field is perpendicular to the wave vector, a wave that obeys (VIII.100b) is referred to as an *ordinary* wave, while (VIII.100c) describes the *extraordinary* wave, although the convention is reversed from the usual optical terminology. When $\mathbf{B}_0$ is parallel to the wave vector, one has two types of waves: (VIII.100d) describes the *whistler* mode and (VIII.100e) describes *L* waves.

Ion waves also come in the two types according to electrostatic and electromagnetic, and these can be further classified according to the relationship between the wave vector **k** and the background magnetic field.

For the electrostatic waves, one has:

$$\omega^2 - v_s^2 k^2 = 0, \qquad (\mathbf{B}_0 = 0 \text{ or } \mathbf{k} \parallel \mathbf{B}_0) \qquad \text{(VIII.101a)}$$

$$\omega^2 - v_s^2 k^2 = \Omega_c^2, \qquad (\mathbf{k} \perp \mathbf{B}_0) \qquad \text{(VIII.101b)}$$

$$\omega^2 = \omega_l^2. \qquad \text{(VIII.101c)}$$

Now, the speed $v_s = [(\gamma_e KT_e + \gamma_i KT_i)/M]^{1/2}$ is the speed of sound in the plasma, when the number fraction of the electrons is γ_e and that of the ions is γ_i, since the first type of wave is acoustic in character. $\Omega_c = eB_0/M$ is the cyclotron frequency of the ions in the magnetic field, and one calls the waves described by (VIII.101b) *electrostatic ion cyclotron waves.* The last law (VIII.101c) does not describe a wave, but only a state of oscillation at the *lower hybrid frequency* $\omega_l = (\Omega_c \omega_c)^{1/2}$.

As for electromagnetic ion waves, one first finds that there are no such things in the absence of $\mathbf{B}_0$, but when it is non-vanishing, there are two types:

$$\omega^2 - v_A^2 k^2 = 0, \qquad (\mathbf{k} \parallel \mathbf{B}_0) \qquad \text{(VIII.102a)}$$

$$\omega^2 - c^2 \left(\frac{v_s^2 + v_A^2}{c^2 + v_A^2} \right) k^2 = 0. \qquad (\mathbf{k} \perp \mathbf{B}_0) \qquad \text{(VIII.102b)}$$

The first ones are called *Alfvén waves*, which are hydromagnetic in character and $v_A = B_0 / \sqrt{\mu_0 n_0 M}$ is then their speed of propagation, while the second ones are called *magnetosonic waves*, and one sees that their speed of propagation is more involved.

One notes the following recurring aspects of these dispersion laws:

1. The ubiquitous role that seems to be played by laws of the Klein-Gordon type. $\omega^2 - v^2 k^2 = \omega_0^2$ for suitable choices of the parameters v and ω_0.

2. The fact that longitudinal electromagnetic wave modes are possible in plasmas, as well as transverse ones.

These dispersion laws define certain characteristic frequencies namely *cutoffs* and *resonances*. A cutoff frequency is defined by a frequency at which k (and therefore the index of refraction $n = k/\omega$, as we shall see)) vanishes. This is equivalent to saying that the phase velocity v_p goes infinite. Conversely, a resonance is a frequency at which either k or n goes infinite (i.e., v_p goes to zero).

In the case of the extraordinary wave (VIII.100c), one has a resonance at the hybrid frequency and the cutoff frequencies are the roots of:

$$\omega^2 \mp \omega_c \omega - \omega_p^2 = 0, \qquad \text{(VIII.103)}$$

namely:

$$\omega_R = \tfrac{1}{2}\left[\omega_c + \sqrt{\omega_c^2 + 4\omega_p^2} \right], \qquad \omega_L = \tfrac{1}{2}\left[-\omega_c + \sqrt{\omega_c^2 + 4\omega_p^2} \right]. \qquad \text{(VIII.104)}$$

The propagation of transverse electromagnetic waves in the Earth's ionosphere is limited by a cutoff frequency on the order of 10 MHz. Hence, radio waves of lower frequency will not penetrate the ionosphere, but only reflect back. Although this makes them useless as a means of communicating with spacecraft outside the ionosphere, nonetheless, it makes it possible for shortwave radio transmissions of low power to communicate with stations over the Earth's horizon from the transmitting antenna by means of successive reflections.

An important aspect of plasma oscillations is that they are damped by what amounts to the interaction of the plasma particles with passing waves. In effect, there is a resonance in this coupling when the particles have a velocity equal to the phase velocity $v_\phi = \omega/k$ of the wave. When the velocities v are far from v_ϕ, there is essentially no energy exchanged between the wave and the particle. As v approaches v_ϕ from below, a particle gains energy from the wave and as it approaches v_ϕ from above it loses energy to the wave. When the velocities agree there is no energy lost or gained and the particle is pushed along by the wave like a surfboard.

If one assumes that the number density function $f(x, v)$ for the velocities of the particles is Maxwellian then this tends to favor slow particles, which implies a damping of the wave itself due to the energy that it is losing to the particles and this damping,

which is not associated with any actual collisions between particles, is called *Landau damping*.

If one linearizes the Vlasov equation for *f*, which is the form that the Boltzmann equation takes for plasma dynamics, around a Maxwellian equilibrium distribution $f_0(x, v)$ then the resulting dispersion law for plasma oscillations is:

$$\omega(k) = \omega_p \left(1 + i \frac{\pi}{2} \frac{\omega_p^2}{k^2} \left[\frac{\partial f_0}{\partial v} \right]_{v=v_\phi} \right). \qquad \text{(VIII.105)}$$

The presence of damping in this expression derives from the fact that the imaginary part of $\omega(k)$ is negative.

Nonlinear wave phenomena are almost too numerous to mention. We briefly mention some of them just to give an impression of how wide-ranging they are.

1. One finds that "drift waves," in plasmas, whose amplitudes should grow exponentially in time, actually seem to reach a saturation limit, which is a symptom of nonlinearity in the dynamics of the wave.

2. Another common symptom of nonlinearity in the dynamics of waves is the changing of the shape of waves over time, although sometimes that can be accounted for by dispersion in a linear model. For instance, the breaking of surface wave on the ocean as it approaches the shore is due to the fact that the speed of propagation depends upon the depth of the water, which is different for the leading and trailing edges.

3. Waves in fluid media of sufficient amplitude often give rise to turbulence, and plasma waves are no exception.

4. The Landau damping that first appears in the linear approximation for the diffusion equation (viz., the Vlasov equation) eventually takes on a nonlinear form when one gives the Vlasov equation its full nonlinear treatment.

5. One has interactions between the plasma waves and the particles of the plasmas, such as particle trapping and plasma echoes, as well as interactions between the waves, which are somewhat analogous to photon-photon scattering in quantum electrodynamics.

6. One finds that the same nonlinear Schrödinger equation that played a role in nonlinear optics also plays a role in nonlinear plasma waves, while the Koortweg-deVries (KdV) equations, which originally described one-dimensional waves in shallow water, also appears.

4. Speed of wave propagation. Although the speed *c* of electromagnetic wave propagation in vacuo is treated as a "fundamental constant" in the eyes of special relativity, as well as quantum electrodynamics, nonetheless, the very fact that vacuum polarization seems to be fundamental to almost all quantum electrodynamical effects suggests that it is better to regard *c* as a *derived* constant, namely:

$$c = 1/\sqrt{\varepsilon_0 \mu_0}\ , \qquad \text{(VIII.106)}$$

and to recognize that the constancy of *c* would only be a consequence of the constancy of ε_0 and μ_0, or at least their product.

When one takes vacuum polarization into account, one must consider the possibility that both of these vacuum parameters are functions of the electromagnetic field strengths that are present in the region of space under consideration. Hence, one might recover the classical definition of c as an asymptotic limit in the absence of fields:

$$c = \lim_{F\to 0} \frac{1}{\sqrt{\varepsilon_0(F)\mu_0(F)}}. \qquad \text{(VIII.107)}$$

In full generality, however, since the parameters ε_0 and μ_0 are part of the constitutive law of the medium in question we should like to regard the speed of propagation of waves in a given medium as being a property of the medium that is derived from more fundamental assumptions about that constitutive law. In fact, it is only in the case of isotropic media that one can give any meaning to the notion of a unique speed of propagation at each point, and only with the further restriction of homogeneity that one can speak of constancy. In the general inhomogeneous, anisotropic medium the speed of propagation depends upon both position and direction, as well as possibly time and non-local considerations.

In order to derive the speed of propagation of electromagnetic waves in a medium from the constitutive law, one first has to recognize that since there is more than one way of defining a "wave" in the first place there is also more than one way of defining its speed of propagation [1]. The two that we shall consider are the phase velocity and the group velocity.

First, assume that the tangent and cotangent bundle of the spacetime manifold have been given a specific choice of space-time splitting. The *phase velocity* of propagation of a wave is a spatial vector field that is associated with the wave covector $k = \omega dt - k_i\, dx^i$, namely:

$$\mathbf{v}_p = (v_p^1, v_p^2, v_p^3), \qquad v_p^i = \frac{\omega}{k_i}, \qquad i = 1, 2, 3. \qquad \text{(VIII.108)}$$

Hence, $\mathbf{v}_p$ depends only upon k and the choice of space-time splitting, and not on the constitutive law. This basically accounts for the choice of the term "phase" in the definition, since it effectively amounts to the velocity that is associated with the isophase foliation defined by k itself when one chooses a space-time splitting.

One must note that there is something geometrically unnatural about taking the inverses of the components of vectors, since the operation is certainly not invariant under changes of frames. However, if we form the triple of components:

$$n = (n_1, n_2, n_3), \qquad n_i = -\frac{k_i}{\omega} \qquad \text{(VIII.109)}$$

then we see that the map from $(\omega, -k_i)$ to (n_1, n_2, n_3) makes perfect *projective* geometrical sense as the projection of the homogeneous coordinates for a chart on the

[1] For a thorough treatment of the other definitions that we do not use, see the book by Brillouin [**21**].

projective space $\mathbb{R}P^3$ onto the inhomogeneous coordinates, as we pointed out in chapter II. Hence, the *normal covector n* at each point $x \in M$ that is so defined is really the line [*k*] through the origin in T_x^*M that is generated by the covector *k*. It is important to note that this line is actually defined independently of any space-time splitting. Hence, the three-dimensional "rest space" that is most appropriate to geometrical optics is really a projective space, not an affine one. One then considers the notion of the *projectivized cotangent bundle PT*M*, whose fibers are the projective spaces PT_x^*M obtained from the cotangent spaces T_x^*M .

In order to define the *group velocity*, one must also consider the dispersion law *P*[*k*] = const. that follows from the constitutive law and field equations. Although in elementary physics, one usually assumes that the dispersion law has been solved for ω as a function of k_i , so one can define:

$$v_g^i = \frac{\partial \omega}{\partial k_i}, \tag{VIII.110}$$

since this implies that $\partial P/\partial \omega$ is non-vanishing, one can also define:

$$v_g^i = -\frac{\partial P / \partial k_i}{\partial P / \partial \omega}. \tag{VIII.111}$$

This definition has an interpretation in the language of projective geometry that is analogous to the previous interpretation of the normal covector. As we shall see below, the partial derivatives in the quotient are all components of a velocity vector field **v** on T^*M that represents part of the characteristic vector field X_P that is associated with *P*[*k*]:

$$v^\mu = \frac{1}{4}\frac{\partial P}{\partial k_\mu}, \tag{VIII.112}$$

which makes:

$$v_g^i = -\frac{v^i}{v^0}. \tag{VIII.113}$$

Hence, we see that the triple of group velocity components (v_g^1, v_g^2, v_g^3) is better regarded as the inhomogeneous coordinates of a line [**v**] in the *projectivized tangent bundle PT*(*M*), whose fibers are then the projective spaces PT_xM associated with all lines through the origins of the tangent spaces, than as a section of the tangent bundle itself. Since we shall have much more to say about the role of projective geometry in spacetime structure in chapter XI, we suspend our discussion on that point.

We shall point out that there is a duality defined by *P*[*k*] that links *k* with **v**, such that the dispersion law *P*[*k*] = 0 becomes *k*(**v**) = 0, while the image of this in the projective spaces is $n(\mathbf{v}_g) = 1$. Indeed, this sort of duality depends only upon the assumption of the homogeneity of the function *P*[*k*] and the invertibility of its Hessian, and not the assumption that it is a quadratic polynomial, as well.

Naturally, it is illuminating to see how the foregoing constructions work in the familiar case of a quadratic – i.e., Lorentzian – dispersion law:

$$P[k] = \tfrac{1}{2}(\omega^2 - c^2 k^2) = \tfrac{1}{2}\omega_0^2, \qquad k^2 = g^{ij} k_i k_j, \tag{VIII.114}$$

in which ω_0 is a constant that may or may not be zero and c has its usual meaning, although it is mostly being used to convert the spatial dimension to the time dimension.

This makes:

$$v^0 = \omega, \qquad v^i = -c^2 g^{ij} k_j, \tag{VIII.115}$$

so:

$$v_g^i = \frac{c^2 k^i}{\omega}, \tag{VIII.116}$$

and in the Euclidian case, in which $k_i = k^i$, one has:

$$v_p^i \, v_g^i = c^2. \tag{VIII.117}$$

One thus sees that generally the phase velocity and the group velocity are quite distinct from each other.

Let us compare the two velocities that one obtains in the lightlike case of $\omega_0 = 0$ with the one that one obtains in the timelike case in which $\omega_0 > 0$. In the former case, one can say that the dispersion law is simply:

$$\omega(k) = ck. \tag{VIII.118}$$

This makes:

$$v_p^i = \frac{ck}{k_i}, \qquad v_g^i = \frac{ck^i}{k}. \tag{VIII.119}$$

When one computes the spatial norms of these vectors relative to g_{ij}, which is inverse to g^{ij}, one gets:

$$v_p = v_g = c. \tag{VIII.120}$$

Thus, in this case either the phase velocity or the group velocity represents the most commonly-used way of referring to the speed of propagation of electromagnetic waves.

One also verifies that:

$$n_i v_g^i = \frac{k_i}{\omega}\frac{c^2 k^i}{\omega} = \left(\frac{ck}{\omega}\right)^2 = 1. \tag{VIII.121}$$

When $\omega_0 > 0$, one can rewrite the dispersion law as:

$$\omega = ck\sqrt{1 + \left(\frac{\omega_0}{ck}\right)^2}. \tag{VIII.122}$$

This dispersion law approaches the lightlike one as *k* grows indefinitely large.

This time, we get:

$$v_p^i = \frac{ck}{k_i}\sqrt{1+\left(\frac{\omega_0}{ck}\right)^2}\,, \quad v_g^i = \frac{ck^i}{k\sqrt{1+(\omega_0/ck)^2}}\,, \tag{VIII.123}$$

so:

$$v_p = c\sqrt{1+\left(\frac{\omega_0}{ck}\right)^2} = \frac{\omega}{k}\,, \qquad v_g = \frac{c}{\sqrt{1+(\omega_0/ck)^2}}\,. \tag{VIII.124}$$

From the facts that v_g goes to zero as *k* goes to zero and that it goes to *c* as *k* grows indefinitely large, we then see that the group velocity behaves more like the conventional velocity of a massive particle than the phase velocity, which grows indefinitely large as *k* goes to zero.

An interesting consequence of the first of these equations is that:

$$\sqrt{1-\frac{v_g^2}{c^2}} = \frac{\omega_0}{\omega}\,, \tag{VIII.125}$$

which shows that the Fitzgerald-Lorentz factor in special relativity can just as well be regarded in terms of the wave covector as in terms of the phase velocity vector.

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CHAPTER IX

Geometrical optics

When one has obtained a dispersion polynomial $P[k]$, which is a differentiable function on the cotangent bundle T^*M to the spacetime manifold M, one can take advantage of its natural symplectic structure to obtain a characteristic vector field on T^*M and a bicharacteristic flow whose integral curves project to generalized null geodesics in M. One obtains supplementary contributions to the geodesic equations, in addition to the generalized Levi-Civita contribution that one obtains for quadratic dispersion polynomials, which are based in the non-quadratic nature of the dispersion law. Hence, if the origin of the non-quadratic nature of the dispersion law is vacuum polarization due to high field strengths then one is justified in thinking of the departure of the null geodesics from the curves that one obtains in the quadratic case as a form of "quantum fluctuations about classical extremals."

Furthermore, since the derivation of a dispersion law follows only from the geometrical optics approximation, the deduction of null geodesics as the bicharacteristics of the field equations, which is so fundamental to the geometry of spacetime in the eyes of general relativity, is nevertheless a high-frequency (short-wavelength) approximation to a more involved geometrical picture that pertains to wave mechanics. Hence, one must also regard the diffraction effects that geometrical optics overlooks as being another source of quantum fluctuations.

Since the role of projective geometry in pre-metric electromagnetism is so fundamental and pervasive, we shall return to address it more thoroughly in Chapter XII. However, since the topic appears naturally in the context of geometrical optics, we shall nonetheless mention it briefly before then. At the root of the introduction of projective geometry is the fact that light rays have no preferred parameterization as curves so one must think in terms of tangent lines to the null geodesics, not tangent vectors. Similar considerations also apply to the wave covector, namely, it defines the same tangent hyperplane as any other covector that differs by a non-zero scalar multiple. Hence, one must pass from the tangent and cotangent bundles to their projectivizations in order to obtain the three-dimensional "rest spaces" in which geometrical optics takes place, rather than the usual space-time decompositions that conventional relativity considers.

We must also address the geometric nature of Huygens's principle in the context of pre-metric electromagnetism. As it turns out, the basic shift in emphasis that is required is from the arc-length functional that one obtains from a metric to the elapsed-time functional that is defined by the wave covector and the dispersion law. One finds that the resulting propagation of phase– i.e., the envelope of a family of elementary hypersurfaces that all represent the same elapsed time along a null geodesic – has a natural interpretation in terms of contact geometry, while the usual propagation of amplitude by Green function techniques is only appropriate to linear wave equations. However, one can consider more general "transport equations" for the propagation of amplitude.

1. The generalized eikonal equation. So far, by means of the geometrical optics approximation, we have reduced the pre-metric Maxwell equations from a set of first-order partial differential equations in the time-varying electric and magnetic fields to an algebraic equation involving the wave covector $k = \omega\, dt - k_i\, dx^i$ and a first-order linear partial differential equation for the phase function ϕ.

$$P[k] = 0, \qquad k = d\phi. \tag{IX.1}$$

Hence, one can combine them to obtain the generalized eikonal equation in the form:

$$P[d\phi] = 0, \tag{IX.2}$$

which will be nonlinear first-order partial differential equation for ϕ.

In order to obtain the spatial form of the eikonal equation one must first observe that, from what we said above, the three-dimensional "space" in question is really the *projective* space $\mathbb{R}\mathrm{P}^{3*}$, in the form of the projective cotangent spaces, not the *vector* space $\mathbb{R}^3$, as represented by hyperplanes in the cotangent spaces. Hence, we must first look at the projection of the dispersion law $P[k] = 0$ onto PT^*M.

We accomplish this by factoring k into $\omega(dt - n)$ with $n = n_i\, dx^i$, $n_i = k_i / \omega$. Since $P[k]$ is a homogeneous quartic polynomial in k, this gives:

$$0 = \omega^4 P[n], \tag{IX.3}$$

with:

$$P[n] = P_0 + P_1[n] + P_2[n] + P_3[n] + P_4[n], \tag{IX.4}$$

in which $P_k[n]$ refers to a homogeneous polynomial in n of degree k.

However, since P is also quadratic in ω^2 one expects that the odd-degree polynomials will vanish, which leaves only:

$$0 = P_0 + P_2[n] + P_4[n], \tag{IX.5}$$

as a spatial dispersion law.

We must also replace the requirement that the spacetime 1-form k be exact with the requirement that spatial 1-form $n = n_i\, dx^i$ be exact:

$$n = d\theta, \tag{IX.6}$$

where $\theta(x^i)$ is then a spatial phase function on $\mathbb{R}\mathrm{P}^3$.

By combining the dispersion law $\Phi[n] = 0$ with this differential expression for the wave normal n one obtains:

$$0 = P_0 + P_2[d\theta] + P_4[d\theta] \tag{IX.7}$$

for the spatial form of the generalized eikonal equation.

This equation represents a nonlinear first order partial differential equation for the spatial phase function θ, which takes the form of the time-invariant Hamilton-Jacobi equation when one regards the dispersion polynomial $P[k]$ as a Hamiltonian function on the cotangent bundle of Σ.

It takes the component form:

$$0 = P^{0000} + 6P^{00ij}\frac{\partial\theta}{\partial x^i}\frac{\partial\theta}{\partial x^j} + P^{ijkl}\frac{\partial\theta}{\partial x^i}\frac{\partial\theta}{\partial x^j}\frac{\partial\theta}{\partial x^k}\frac{\partial\theta}{\partial x^l} \ . \qquad \text{(IX.8)}$$

At this point, it helps to look at the form that equation (IX.8) takes when P is the quadratic form η that defines Minkowski space:

$$0 = \eta(dt, dt) + 2\eta(dt, d\theta) + \eta(d\theta, d\theta), \qquad \text{(IX.9)}$$

which then reduces to:

$$0 = 1 - c^2\delta^{ij}\,\theta_{,i}\,\theta_{,j} \qquad \text{(IX.10)}$$

by orthogonality.

Finally, this gives the partial differential equation for θ:

$$\delta^{ij}\frac{\partial\theta}{\partial x^i}\frac{\partial\theta}{\partial x^j} = n^2. \qquad \text{(IX.11)}$$

Of course, (IX.8) is still considerably more complicated than (IX.11), so we consider the fact that for a broad class of electromagnetic constitutive laws, including most of the popular nonlinear ones, the dispersion relation factorizes into a product of quadratic relations, as in (VIII.72). We see that it is not necessary to formulate a single quartic eikonal equation because it is sufficient to note that a polynomial $P[k]$ of the form (VIII.72) vanishes iff either $g(k, k)$ or $\bar{g}(k,k)$ vanishes, independently of each other. Hence, one simply obtains the union of the solution sets for quadratic eikonal equations of the form (IX.7), when δ^{ij} is replaced with either $-\,g^{ij}$ or $-\bar{g}^{ij}$, and there will be different functions for n^2.

A subtle point that one must consider in the foregoing is that since $n = 1/\omega k_s$, with $k_s = \phi_{,i}\,dx^i$, unless one assumes that ω is spatially homogeneous, one has no guarantee that the resulting 1-form is exact, even when k_s is. A necessary integrability condition for this is that n must be closed, which gives:

$$dk_s = d\omega \wedge n = d\ln\omega \wedge k_s \ . \qquad \text{(IX.12)}$$

From Frobenius, this is also the condition for the exterior differential system $k_s = 0$ to be completely integrable.

2. Bicharacteristics – null geodesics. Although the generalized eikonal equation (IX.2) is usually nonlinear, due to the polynomial character of the dispersion law, nevertheless, it is still a first-order partial differential equation in a single function ϕ.

Hence, one can solve it by Cauchy's method of characteristics. However, since we are already using the word "characteristic" in the context of the second-order system of partial differential equations in the potential 1-form A as a way of reducing it to a first-order equation, it is traditional to refer to the characteristic curves of the eikonal equation as the *bicharacteristics* of the original second-order system.

Consider the case of electromagnetic waves whose dispersion law is (VIII.69), which we rewrite in the form:

$$P(x)[k] = 0. \tag{IX.13}$$

The bicharacteristic equations are simply the Hamilton equations that one obtains by treating $F = 1/4\ P$ as a Hamiltonian:

$$\frac{dx^\mu}{d\tau} = \frac{\partial F}{\partial k^\mu} = \gamma^{\mu\nu}(x,\, k)\, k_\nu, \tag{IX.14a}$$

$$\frac{dk_\mu}{d\tau} = -\frac{\partial F}{\partial x^\mu} = -\frac{1}{4}\frac{\partial \gamma^{\kappa\lambda}}{\partial x^\mu} k_\kappa k_\lambda, \tag{IX.14b}$$

in which we have defined:

$$\gamma^{\mu\nu}(x,\, k) = P^{\mu\nu\kappa\lambda}(x)\, k_\kappa k_\lambda\,. \tag{IX.15}$$

Although the tensor field γ appears to define a second-rank covariant tensor field on M, actually, the fact that the local components are functions on T^*M shows that we are really "lifting" T^*M to the *vertical* sub-bundle $V(T^*M)$ of the tangent bundle $T(T^*M))$ to T^*M. That is, under the differential of the projection $T^*M \to M$ the tangent vectors in each $V_{(x,k)}(T^*M)$ project to 0 in each T_xM.

As usual, there is no natural complementary "horizontal" bundle to $V(T^*M)$ in $T(T^*M)$. However, from (IX.14a), we *do* have a natural algebraic correspondence between covectors on M and tangent vectors on M.

b. Dimension-codimension duality. If $k_x \in T_x^*M$ is a covector on M then one can associate k with the characteristic vector $X_P(k_x)$ on T_x^*M that is defined by P and obtain a tangent vector in $T_k T_x^* M$. Under the projection π: $T^*M \to M$ the vertical part of $X_P(k_x)$ will vanish and one obtains a tangent vector $d\pi|_{(x,\ k)}\ X_P(k_x)$ in T_xM. Hence, by composition, we obtain a map $i_P : T^*M \to T(M)$, $k_x \mapsto \mathbf{v}(k_x) = d\pi|_{(x,\ k)}\ X_P(k_x)$ In local form, it simply looks like:

$$v^\nu(x,\, k) = P^{\mu\nu\kappa\lambda}(x)\, k_\kappa k_\lambda\, k_\mu\ = \gamma^{\nu\mu}(x,\, k)\, k_\mu\,. \tag{IX.16}$$

If this system of homogeneous cubic equations is to replace the system of linear equations $v^\nu = g^{\nu\mu}\, v_\mu$ that one customarily derives from a Lorentzian structure g then just as one usually *postulates* the invertibility of the linear system, we must also postulate that the cubic system is invertible, as well. A necessary, but only locally sufficient, condition for this invertibility is given by the inverse function theorem, namely, the invertibility of the matrix:

$$\frac{\partial v^{\nu}}{\partial k_{\mu}} = \frac{\partial^2 P}{\partial k_{\mu} \partial k_{\nu}} = 3\gamma^{\mu\nu}(x, k) \tag{IX.17}$$

for every $(x, k) \in T^*M$. Hence, if we only assume that the algebraic correspondence $\mathbf{v} = \mathbf{v}(k_x)$ satisfies (IX.16) then the correspondence might possibly be neither one-to-one nor onto; for example, one might have folds and cusps.

Another complicating factor associated with the cubic case is the fact that now, since the polynomial $P^{\mu\nu\kappa\lambda}(x)\ k_{\kappa}\ k_{\lambda}\ k_{\mu}$ is homogeneous of degree three in k, the inverse algebraic relationship:

$$k_{\mu} = k_{\mu}(x, \mathbf{v}) \tag{IX.18}$$

is homogeneous of degree 1/3 in $\mathbf{v}$. Hence, whereas in the case of a quadratic dispersion law the map i_g and its inverse were both homogeneous of degree one, we now have a situation where the inverse map to i_P is not even a polynomial map anymore.

Moreover, in the Lorentzian case of a quadratic P the corresponding function $\bar{P}[\mathbf{v}] = P[k(\mathbf{v})]$ on $T(M)$ would also be quadratic. In other words, the light cones in the cotangent spaces would become light cones in the tangent spaces. However, that situation no longer obtains in the present case, since the function $\bar{P}$ does not have the same degree of homogeneity as the function P; in particular, it will have degree of homogeneity 4/3. One might consider $\bar{P}[\mathbf{v}] = (P[k(\mathbf{v})])^3$, which will have homogeneity degree four, but one is cautioned that this does not imply that $\bar{P}$ represents a homogeneous quartic *polynomial*, since there are many non-polynomial functions, such as rational functions, which are quotients of polynomials, that can still be homogeneous of degree four.

The fact that the function $\bar{P}[\mathbf{v}]$ on $T(M)$ is still homogeneous, but not generally a polynomial, implies that the hypersurface $\bar{P}[\mathbf{v}] = 0$ projects to a surface in $PT(M)$, which we will call the *Fresnel ray surface*, to be consistent with the literature on geometrical optics.

Due to the homogeneity of P, one finds that the normal surface in PT^*M and the ray surface in $PT(M)$ are related by a simple relationship that is true for any degree of homogeneity, even though it is usually established for only the quadratic case (cf., Kommerel [**1**]). One starts with the fact that Euler's formula for homogeneous functions of degree r, when combined with the dispersion law, gives us that:

$$0 = rP[k] = k_{\mu} \frac{\partial P}{\partial k_{\mu}} = k_{\mu} v^{\mu} \tag{IX.19}$$

for any r.

When one expresses k as $\omega dt - k_i\, dx^i$, this becomes $\omega v^0 = k_i\, v^i$ or:

$$n_i V^i = 1. \tag{IX.20}$$

In the case of optical media, one obtains an equations that are quite analogous to (VIII.102), (VIII.103), or (VIII.104) for the inhomogeneous coordinates (V^1, V^2, V^3) of the ray. These coordinates are obtained from the components v^{μ} of any velocity vector by

the usual process of regarding them as homogeneous coordinates for the ray and setting $V^i = v^i/v^0$. One also finds that the principal values ε^i of ε^{ij} get replaced by their reciprocals, since it is the inverse matrix ε_{ij} that now figures. The optical equation that corresponds to (VIII.104) is then:

$$\frac{(V^x)^2}{\frac{1}{v_r^2}-\frac{1}{v_x^2}}+\frac{(V^y)^2}{\frac{1}{v_r^2}-\frac{1}{v_y^2}}+\frac{(V^z)^2}{\frac{1}{v_r^2}-\frac{1}{v_z^2}}=0. \qquad \text{(IX.21)}$$

In this equation, we are now using the *ray velocity* v_r, which relates to the flow of energy in the direction of the Poynting vector, as opposed to the phase velocity v_p, which only relates to the normal to the wave front. The two are traditionally related by the relation:

$$\frac{v_p}{v_r}=\hat{n}(\hat{\mathbf{V}}), \qquad \text{(IX.22)}$$

in which $n = (1/v_p)\,\hat{n}$ and $\mathbf{V} = v_r\,\hat{\mathbf{V}}$, so $\hat{n}$ and $\hat{\mathbf{V}}$ are the unit covector and vector in those directions, relative to the Euclidian metric, which then gives $\hat{n}(\hat{\mathbf{V}})$ the interpretation of the cosine of the angle between the vectors **n** and **V**. In particular, this means that the wave normal does not have to be collinear with the ray velocity. Of course, in the pre-metric formulation of electromagnetism the very introduction of a metric onto the fibers of PT^*M and $PT(M)$ must be a consequence of the dispersion law for waves in the medium in question.

One can think of the "classical" – i.e., correspondence principle – limit of the quartic theory as being the degenerate case in which the metric $\gamma^{\mu\nu}(x, k)$ degenerates to $g^{\mu\nu}(x)$, so the cubic map i_P goes to the linear map i_g, because the dispersion polynomial $P[k]$ degenerates to the square of a quadratic. Hence, in general, the components of the metric γ we are using will depend on the covector k that we start with, as well as the point of M. This situation is sometimes referred to as a "rainbow metric" (cf., e.g., [**2**]).

It is important to understand that the linear map $d\pi \cdot X_P$ is only "generically" injective, since there is always the possibility that it might have projective singularities – i.e., points of T^*M at which X_P goes vertical, – which then depends upon the nature of P. One then has a second source of singularity in the association that did not appear in the linear isomorphism i_g: $T^*M \to T(M)$ that one obtained from a Lorentzian structure g.

As we have seen, one can use the polynomial $P[k]$ to define a corresponding map i_P : $\Lambda^1 M \to \Lambda_1 M$, $k(x) \mapsto \mathbf{v}(k(x))$. In the invertible case, this association between covector fields and vector fields defines what one might call *dimension-codimension duality*, since the differential system that is defined by a covector field has a dimension that is complementary to the dimension of the differential system that is defined by a vector field. One can eventually see that this is quite similar to the spirit of wave-particle duality if one understands that the isophase hypersurfaces that are defined by an integrable k represent the motion of wave surfaces and the integral curves of **v** represents the paths of the points of the wave surfaces when regarded as pointlike particles.

This situation also points out that there is an essential physical difference between the map i_P and the map i_g in terms of the units and interpretation of the covectors k_μ and v_μ that result from applying the inverses of these two maps to the same velocity vector v^μ. One sees that i_g^{-1} does not change the units of velocity in mapping it to covelocity, while i_P^{-1} essentially inverts the units of velocity (times time) into the corresponding units of frequency and wave number. The best way to get around this is to express both velocity and frequency-wave number in dimensionless units by rescaling in terms of some characteristic value of each.

c. Null geodesics. If we "solve" (IX.14a) for $k_\mu = \gamma_{\mu\nu} v^\nu$, where $\gamma_{\mu\nu}$ is the inverse matrix of $\gamma^{\mu\nu}$, and substitute in (IX.14b) then, after various straightforward manipulations of the expressions, we ultimately obtain a first-order system of differential equations for v^μ whenever $k(x)$ has been chosen:

$$\frac{dv^\mu}{d\tau} + \Gamma^\mu_{\kappa\lambda}(k) v^\kappa v^\lambda = --B^\mu_{\kappa\lambda}(k) v^\kappa v^\lambda + C^{\mu\kappa}_\lambda(k) k_\kappa v^\lambda , \tag{IX.23}$$

in which we have defined:

$$\Gamma^\mu_{\kappa\lambda}(k) = \tfrac{1}{2}\gamma^{\mu\nu}(\gamma_{\nu\lambda,\kappa} + \gamma_{\nu\kappa,\lambda} - \gamma_{\kappa\lambda,\nu}) \tag{IX.24a}$$

$$B^\mu_{\kappa\lambda}(k) = \tfrac{1}{4}\gamma^{\mu\nu}\frac{\partial\gamma_{\kappa\lambda}}{\partial x^\nu}, \tag{IX.24b}$$

$$C^{\mu\nu}_\lambda(k) = \gamma^{\mu\nu}\frac{\partial\gamma_{\nu\lambda}}{\partial k_\kappa}. \tag{IX.24c}$$

We recognize that for each choice of k the expressions $\Gamma^\mu_{\kappa\lambda}(k)$ take the form of the components of the Levi-Civita connection for the metric $\gamma_{\mu\nu}$. The differential equations for **v** then amount to perturbations of the conventional geodesic equations for the metric $\gamma_{\mu\nu}$ by the contributions on the right-hand side of (IX.23). Furthermore, the fact that k is characteristic suggests that we are justified in calling the geodesics thus obtained *null geodesics*. However, we must emphasize that this means that the "spectrum" of rainbow metrics $\gamma_{\mu\nu}(x, k)$ that we obtain as k varies over the characteristic quartic is associated with a spectrum of connections and a spectrum of null geodesics, as well.

The expressions $B^\mu_{\kappa\lambda}(k)$ and $C^{\mu\nu}_\lambda(k)$ essentially embody the perturbations to geodesic motion, in the Levi-Civita sense, that originates in the possibility that the quartic form $P[k]$ is not the square of a quadratic form of Lorentzian type. In order to see this, when $P[k] = (Q[k])^2$, with $Q[k] = g^{\mu\nu} k_\mu k_\nu$ one must first replace the dispersion law with its square root:

$$g^{\mu\nu}(x) k_\mu k_\nu = 0. \tag{IX.25}$$

One then finds that the bicharacteristic equations that result from using $Q[k]$ in place of $P[k]$ are the conventional geodesic equations for the Levi-Civita connection that is

defined by the metric $g^{\mu\nu}(x)$. Hence, $\Gamma^{\mu}_{\kappa\lambda}$ would no longer depend upon k, while $B^{\mu}_{\kappa\lambda}(k)$ and $C^{\mu\nu}_{\lambda}(k)$ would vanish.

It is tempting to identify the spectrum of geodesics that originate in the quartic nature of $P[k]$ as being, in some sense, "quantum fluctuations about the classical extremals;" for instance, one might have zitterbewegung in mind. However, one must remember that the quartic nature of $P[k]$ has more to do with the symmetry of constitutive laws of the medium, while quantum fluctuations are more commonly associated with going beyond the geometrical optics approximation that is implicit in the present discussion. We shall return to this issue at the end of this chapter in our discussion of diffraction, but for now we simply point out that quartic dispersion laws can also constitute quantum corrections.

As an example of the foregoing, let us look at what happens to the geodesic equations in the bi-metric case where:

$$P[k] = \tfrac{1}{4} g(k, k)\, \bar{g}(k, k)\,. \tag{IX.26}$$

The bicharacteristic equations are then:

$$\frac{dx^{\mu}}{d\tau} = \tfrac{1}{2}(\bar{\kappa}^2 g^{\mu\nu} + \kappa^2 \bar{g}^{\mu\nu})k_{\nu}\,, \quad \frac{dk_{\mu}}{d\tau} = -\tfrac{1}{4}(\bar{\kappa}^2 g^{\kappa\lambda}_{,\mu} + \kappa^2 \bar{g}^{\kappa\lambda}_{,\mu})k_{\kappa}k_{\lambda}\,, \tag{IX.27}$$

in which we have set:

$$\kappa^2 = g(k, k), \qquad \bar{\kappa}^2 = \bar{g}(k, k)\,. \tag{IX.28}$$

Although one can derive the corresponding geodesic equations that follow from setting:

$$\gamma^{\mu\nu} = \tfrac{1}{2}(\bar{\kappa}^2 g^{\mu\nu} + \kappa^2 \bar{g}^{\mu\nu})\,, \tag{IX.29}$$

nevertheless, since the process of inverting this matrix is not linear, a better way to understand the nature of the geodesic equations is by expressing the characteristic vector field in the form:

$$X_P = \bar{\kappa}^2 X_g + \kappa^2 X_{\bar{g}}\,, \tag{IX.30}$$

with:

$$X_g = \left(\tfrac{1}{2} g^{\mu\nu} k_{\nu}\right)\frac{\partial}{\partial x^{\mu}} - \left(\tfrac{1}{4} g^{\kappa\lambda}_{,\mu} k_{\kappa} k_{\lambda}\right)\frac{\partial}{\partial k_{\mu}}, \tag{IX.31}$$

and an analogous expression for $X_{\bar{g}}$.

One sees that, in a sense, there are two competing dynamical systems that represent the geodesic vector fields X_g and $X_{\bar{g}}$ on T^*M for the individual Lorentzian metrics g and $\bar{g}$ coupled in a linear combination by means of κ^2 and $\bar{\kappa}^2$ in a symmetric fashion. As a consequence, whenever the covector field k makes either coefficient vanish, the characteristic vector field is proportional to the geodesic vector field of the other metric.

3. Parallel translation. Since the bicharacteristic equations are apparently related to geodesic equations for the Levi-Civita connection that one obtains from the "metric" $\gamma^{\mu\nu}(x, k)$, and indeed agree with them when the dispersion polynomial $P[k]$ is quadratic, we naturally ought to examine how other geometric objects that just the velocity vectors to a geodesic congruence are translated along such curves.

a. Translation of k. First, let us return to the basic bicharacteristic equations (IX.14a, b). Once again, we express the first one as $v^\mu = \gamma^{\mu\nu} k_\nu$, with the usual notation. When one "inverts" the matrix $\gamma^{\mu\nu}$ this says $k_\mu = \gamma_{\mu\nu} v^n$, although when $P[k]$ is not quadratic the inverse only makes sense when one chooses a wave covector field k. When one substitutes this expression for one of the k's in (IX.14b), that equation takes the form:

$$0 = \frac{dk_\mu}{ds} + \frac{1}{r}\gamma_{\kappa p}\gamma^{\kappa\lambda}_{,\mu} v^\rho k_\lambda = v^\kappa \left[\frac{\partial k_\mu}{\partial x^\rho} + \frac{1}{r}\gamma_{\rho\kappa}\gamma^{\rho\lambda}_{,\mu} k_\lambda \right]. \tag{IX.32}$$

We have also generalized the degree of homogeneity of $P[k]$ in k to be r, although this will only equal 2 or 4 for our purposes.

By further manipulations, which include replacing $\gamma_{\rho\kappa}\gamma^{\rho\lambda}_{,\mu}$ with $-\gamma_{\rho\kappa,\mu}\gamma^{\rho\lambda}$ and observing that:

$$\gamma^{\rho\lambda}(\gamma_{\mu\rho,\kappa} - \gamma_{\kappa\mu,\rho}) v^\kappa k_\lambda = (\gamma_{\mu\rho,\kappa} - \gamma_{\kappa\mu,\rho}) v^\kappa v^\rho = 0, \tag{IX.33}$$

which follows from the conflict of symmetries in the indices κ and ρ, one finds that equation (IX.14b) actually says:

$$0 = v^\kappa \left[\frac{\partial k_\mu}{\partial x^\kappa} - \frac{2}{r}\Gamma^\lambda_{\kappa\mu} k_\lambda \right], \tag{IX.34}$$

with the same notation as above for the Christoffel symbols.

When $P[k]$ is quadratic ($r = 2$) the form that (IX.34) takes is precisely the equation of parallel translation when one uses the Levi-Civita connection for $\gamma^{\mu\nu}$, which then represents the Lorentzian structure. However, when $P[k]$ is a homogeneous quartic polynomial one sees that the connection that gives parallel translation associates infinitesimal Lorentz transformations that have half the magnitude of those in the quadratic case. One can also express this in the form:

$$\nabla_{\mathbf{v}} k_\mu = \begin{cases} 0 & (r = 2), \\ -\frac{1}{2}\Gamma^\lambda_{\kappa\mu} v^k k_\lambda & (r = 4). \end{cases} \tag{IX.35}$$

In either case, we conclude that the wave covector field k is parallel-translated along the null geodesics in manner that is even simpler than the way that the velocity vectors are subjected to. Of course, if one substitutes $k_\mu = \gamma_{\mu\nu} v^\nu$ in (IX.34), one sees that it is the differentiation of $\gamma_{\mu\nu}$ by k that can make the resulting equations of parallel translation for $\mathbf{v}$ more complicated than the ones for k.

*b. Translation of F and *F.* A immediate property of the bicharacteristic flow on M that is easy to verify is the fact that when the 2-forms F and $^*F$ – as well as f and $^*f$ – are associated with electromagnetic waves in the geometrical optics approximation in the geometrical optics approximation, they are constant along the flow of the geodesic vector field **v** that is associated with a given geodesic covector field k. This follows from the fact that since:

$$F = k \wedge E \qquad {}^*F = k \wedge B, \tag{IX.36}$$

in that approximation, one must have:

$$i_{\mathbf{v}}F = k(\mathbf{v})\, E - E(\mathbf{v})k = 0, \tag{IX.37a}$$
$$i_{\mathbf{v}}{}^*F = k(\mathbf{v})\, B - B(\mathbf{v})k = 0, \tag{IX.37b}$$

as **v** is transverse to the hyperplanes defined by E and B, for a suitable choice of E, B. We choose the E and B such that the triple $\{k, E, B\}$ is linearly independent and spans the annihilating hyperplane of **v** in each cotangent space; i.e., $E(\mathbf{v}) = B(\mathbf{v}) = 0$.

From this, and the sourceless field equations, we find that the Lie derivatives of F and $^*F$ along the flow of **v** are:

$$\mathrm{L}_{\mathbf{v}}F = i_{\mathbf{v}}dF + di_{\mathbf{v}}F = 0, \tag{IX.38a}$$
$$\mathrm{L}_{\mathbf{v}}{}^*F = i_{\mathbf{v}}d{}^*F + di_{\mathbf{v}}{}^*F = 0, \tag{IX.38b}$$

Hence, F and $^*F$ are, in a sense, convected by the flow of **v**.

Now, if we go back to the equations:

$$di_{\mathbf{v}}F = di_{\mathbf{v}}{}^*F = 0, \tag{IX.39}$$

and expand them in local components then we get, for the first one:

$$v^\mu\, dF_{\mu\nu} \wedge dx^\nu + F_{\mu\nu}\, dv^\mu \wedge dx^\nu = 0, \tag{IX.40}$$

The first term in this becomes:

$$v^\mu\, dF_{\mu\nu} \wedge dx^\nu = \tfrac{1}{2} v^\mu\, F_{\kappa\nu,\,\mu}\, dx^\kappa \wedge dx^\nu, \tag{IX.41}$$

in which we have used the fact that $dF = 0$ in order to obtain this.

In the second term of (IX.40) we note that if $v^\mu = \gamma^{\mu\lambda} k_\lambda$ then we can have two possible forms of the resulting expression depending upon whether $r = 2$ or 4.

i. Quadratic case ($r = 2$). In this case, $\gamma^{\mu\lambda}$ is independent of k and we obtain:

$$dv^\mu = k_\lambda\, \gamma^{\mu\lambda}{}_{,\kappa}\, dx^\kappa, \tag{IX.42}$$

which makes:

$$F_{\mu\nu}\, dv^\mu \wedge dx^\nu = \tfrac{1}{2}(F_{\mu\nu}\, k_\lambda\, \gamma^{\mu\lambda}{}_{,\kappa} - F_{\mu\kappa}\, k_\lambda\, \gamma^{\mu\lambda}{}_{,\nu})\, dx^\kappa \wedge dx^\nu. \tag{IX.43}$$

Substituting for $k_\lambda = \gamma_{\lambda\rho} v^\rho$ and rearranging puts the expression in parentheses into the form:

$$\gamma_{\lambda\rho}\,\gamma^{\mu\lambda}{}_{,\kappa}\,v^\rho F_{\mu\nu} + \gamma_{\lambda\rho}\,\gamma^{\mu\lambda}{}_{,\nu}\,v^\rho F_{\kappa\mu} = -\,\gamma^{\mu\lambda}\,\gamma_{\lambda\rho,\,\kappa} v^\rho F_{\mu\nu} - \gamma^{\mu\lambda}\,\gamma_{\lambda\rho,\,\nu}\,v^\rho F_{\kappa\mu}\ . \qquad \text{(IX.44)}$$

One can verify directly that if one substitutes $\Gamma^\mu_{\rho\kappa}$ for $\gamma^{\mu\lambda}\,\gamma_{\lambda\rho,\,\kappa}$ and similarly in the second term of the final expression in (IX.44) then the extra term that this introduces into the computations is symmetric in k and v, and therefore does not contribute to the components of the 2-form in (IX.43).

Ultimately, we find that when F has the form in question, **v** is the velocity vector field of a null geodesic congruence, and $P[k]$ is homogeneous of degree 2, one must have:

$$0 = v^\kappa\left[\frac{\partial F_{\mu\nu}}{\partial x^\kappa} - \Gamma^\lambda_{\kappa\mu}F_{\lambda\nu} - \Gamma^\lambda_{\kappa\nu}F_{\mu\lambda}\right] \equiv \nabla_{\mathbf{v}} F_{\mu\nu}\,. \qquad \text{(IX.45)}$$

That is, the electromagnetic field strength 2-form is parallel-translated by the Levi-Civita connection of the Lorentzian metric that is defined by $\gamma^{\mu\nu}$.

We could also use the property of the operator $\nabla_{\mathbf{v}}$ that it is a derivation with respect to the tensor product to see that:

$$\nabla_{\mathbf{v}} F_{\mu\nu} = \tfrac{1}{2}\nabla_{\mathbf{v}}(k_\mu E_\nu - k_\nu E_\mu) = \tfrac{1}{2}(k_\mu \nabla_{\mathbf{v}} E_\nu - k_\nu \nabla_{\mathbf{v}} E_\mu), \qquad \text{(IX.46)}$$

when one keeps in mind that k is parallel-translated along **v**.

In order for $\nabla_{\mathbf{v}} F_{\mu\nu}$ to vanish, we must have:

$$\nabla_{\mathbf{v}} E_\mu = \alpha k_\mu \qquad \text{(IX.47)}$$

for some real scalar α.

Hence, the 1-form E does not have to be parallel-translated, precisely, but its covariant derivative must be parallel to the wave covector field. This has the effect of allowing for a rotation of the polarization plane in the cotangent spaces, which is spanned the 1-forms k and E.

One finds that analogous results are arrived at for $*F = k \wedge B$; viz., $*F$ is parallel-translated along the null geodesics by means of the Levi-Civita connection, while B must have a covariant derivative that is parallel to k:

$$\nabla_{\mathbf{v}} *F_{\mu\nu} = 0, \qquad \nabla_{\mathbf{v}} B_\mu = \alpha k_\mu\,. \qquad \text{(IX.48)}$$

The fact that the proportionality constant is the same in either case follows from duality; in effect, E and B must remain perpendicular at all times.

The picture that emerges is that the 3-coframe $\{k, E, B\}$ moves along a null geodesic in such a manner that k remains parallel to itself while the 2-coframe $\{E, B\}$ can rotate in its plane. One sees that this generalizes the argument of Kline and Kay [**6**] in the first Appendix to Chapter V, in which they established, by means of vector calculus, that the 3-frame in question, or rather, its spatial projection, is parallel-translated along a null geodesic when the constitutive properties of the space are isotropic, but not necessarily

homogeneous, and the spatial metric that defines the connection is conformal to the Euclidian one, with a conformal factor that is the square of the index of refraction.

ii. Quartic case ($r = 4$). We return to (IX.42), only this time, we allow $\gamma^{\mu\lambda}$ to also be a differentiable function of k, so:

$$dv^{\mu} = \gamma^{\mu\lambda}{}_{,\kappa}\, k_{\lambda}\, dx^{\kappa} + \left[k_{\lambda} \frac{\partial \gamma^{\mu\lambda}}{\partial k_{\kappa}} + \gamma^{\mu\kappa} \right] dk_{\kappa}, \tag{IX.49}$$

before one pulls the dk_k down to M.

The term in brackets can be simplified by routine calculations that involve the definition of $\gamma^{\mu\lambda}$ and Euler's formula, and one gets:

$$dv^{\mu} = \gamma^{\mu\lambda}{}_{,\kappa}\, k_{\lambda}\, dx^{\kappa} + 3\gamma^{\mu\kappa} dk_{\kappa}, \tag{IX.50}$$

and after pulling the dk_k down to M by means of a section $k_k = k_k(x)$ – so $dk_{\kappa} = k_{\kappa,\lambda}\, dx^{\lambda}$ – one obtains the local 1-forms on M:

$$dv^{\mu} = \left[\gamma^{\mu\lambda}{}_{,\kappa} k_{\lambda} + 3\gamma^{\mu\lambda} k_{\lambda,\kappa} \right] dx^{\kappa}. \tag{IX.51}$$

Since this differs from the expression in (IX.42) only by the second term in brackets, we see that (IX.42) becomes:

$$\begin{aligned} &F_{\mu\nu}\, dv^{\mu} \wedge dx^{\nu} \\ &\qquad = (\text{R.H.S. of IX.42}) + \tfrac{3}{2}\, \gamma^{\mu\lambda}\, (k_{\lambda,\kappa} F_{\mu\nu} - k_{\lambda,\nu} F_{\mu\kappa})\, dv^{\kappa} \wedge dx^{\nu}. \end{aligned} \tag{IX.52}$$

However, from (IX.34) we can substitute $1/2\Gamma^{\rho}_{\lambda\kappa} k_{\rho}$ for $k_{\lambda,\kappa}$, which makes the components of the supplementary term in (IX.52) become:

$$3\gamma^{\kappa\lambda}\, (k_{\lambda,\mu}\, F_{\kappa\nu} - k_{\lambda,\nu} F_{\kappa\mu}) = -\tfrac{3}{2}\, \gamma^{\kappa\lambda}\, (\Gamma^{\rho}_{\lambda\mu} F_{\kappa\nu} - \Gamma^{\rho}_{\lambda\nu} F_{\kappa\mu})\, k_{\rho}\,. \tag{IX.53}$$

The ultimate effect of the extra term is then to replace (IX.45) with:

$$\nabla_{\mathbf{v}} F_{\mu\nu} = \tfrac{3}{2} (\gamma^{\kappa\lambda}\, \Gamma^{\rho}_{\lambda\mu} k_{\rho} F_{\kappa\nu} + \gamma^{\kappa\lambda}\, \Gamma^{\rho}_{\lambda\nu} k_{\rho} F_{\mu\kappa})\,. \tag{IX.54}$$

Hence, this is not parallel translation of F by means of the Levi-Civita connection of $\gamma^{\mu\nu}$, anymore.

In order to make more intuitive sense of the right-hand side of (IX.54), we introduce the notations:

$$\Gamma^{\kappa\rho}_{\mu} \equiv \gamma^{\kappa\lambda}\, \Gamma^{\rho}_{\lambda\mu}, \qquad \tilde{\omega}^{\kappa}_{\mu}(k) = \Gamma^{\kappa\rho}_{\mu} k_{\rho}, \tag{IX.55}$$

and re-write (IX.54) as:

$$\nabla_{\mathbf{v}} F_{\mu\nu} = \tfrac{3}{2}[\,\tilde{\omega}^{\kappa}_{\mu}(k)\,F_{\kappa\nu} + \tilde{\omega}^{\kappa}_{\nu}(k)\,F_{\mu\kappa}]\,. \qquad \text{(IX.54′)}$$

Replacing $F_{\mu\nu}$ with $k_\mu E_\nu - k_\nu E_\mu$ puts this into the form:

$$\begin{aligned}\nabla_{\mathbf{v}} F_{\mu\nu} &= \tfrac{3}{2}[\,\tilde{\omega}^{\kappa}_{\mu}(k)\,k_\kappa E_\nu - \tilde{\omega}^{\kappa}_{\nu}(k)\,k_\kappa E_\mu + k_\mu\,\tilde{\omega}^{\kappa}_{\nu}(k)\,E_\kappa - k_\nu\,\tilde{\omega}^{\kappa}_{\mu}(k)\,E_\kappa\,] \\ &= \tfrac{3}{2}[\,\tilde{\omega}(k)\,k \wedge E + k \wedge \tilde{\omega}(k)\,E]_{\mu\nu}\,, \qquad \text{(IX.56)}\end{aligned}$$

with some self-explanatory abbreviations.

If we apply $\nabla_{\mathbf{v}}$ to $k \wedge E$ and keep in mind (IX.35) then we also get:

$$\nabla_{\mathbf{v}}(k \wedge E) = \nabla_{\mathbf{v}} k \wedge E + k \wedge \nabla_{\mathbf{v}} E = -\tfrac{1}{2}\,\omega(\mathbf{v}) k \wedge E + k \wedge \nabla_{\mathbf{v}} E. \qquad \text{(IX.57)}$$

Hence, in order for this to be consistent with (IX.56), one must have:

$$\nabla_{\mathbf{v}} E_\mu = \alpha k_\mu + \tfrac{3}{2}\,\tilde{\omega}^{\kappa}_{\mu}(k) E_\kappa\,, \qquad [3\,\tilde{\omega}^{\kappa}_{\mu}(k) + \omega^{\kappa}_{\mu}(\mathbf{v})]\,k_\kappa = \beta E_\mu \qquad \text{(IX.58)}$$

for appropriate choices of real scalars α and β.

If we compare the first expression with the corresponding quadratic expression (IX.47) then we see that the net effect of the extension to a quartic dispersion polynomial is the addition of the second term on the right-hand side. The condition expressed by the second equation in (IX.58) gives a restriction on the connection itself, and if it is to be true for any conceivable E one sees that the proportionality constant β must vanish:

$$[3\,\tilde{\omega}^{\kappa}_{\mu}(k) + \omega^{\kappa}_{\mu}(\mathbf{v})]\,k_\kappa = 0. \qquad \text{(IX.59)}$$

This really just says that the effect of the infinitesimal transformation $3\,\tilde{\omega}^{\kappa}_{\mu}(k) + \omega^{\kappa}_{\mu}(\mathbf{v})$ on cotangent vectors should leave k fixed in any event.

The results for $^*F$ and B are obtained by analogous computations that essentially replace F with $^*F$ and E with B:

$$\nabla_{\mathbf{v}} {}^*F_{\mu\nu} = \tfrac{3}{2}[\,\tilde{\omega}(k)\,k \wedge B + k \wedge \tilde{\omega}(k)\,B]_{\mu\nu}, \qquad \text{(IX.60)}$$

$$\nabla_{\mathbf{v}} B_\mu = \alpha k_\mu + \tfrac{3}{2}\,\omega^{\kappa}_{\mu}(k) B_\kappa\,, \qquad [3\,\tilde{\omega}^{\kappa}_{\mu}(k) + \omega^{\kappa}_{\mu}(\mathbf{v})]\,k_\kappa = 0. \qquad \text{(IX.61)}$$

4. Huygens's principle. Huygens's principle is sufficiently fundamental and far-reaching in scope that it is presented in many forms in the literature (cf., [**3-11**]). The one that we shall first focus on is a modernization of the form that Huygens described graphically in his original treatise: Suppose we are given an initial hypersurface x_0: $N \to M$, $u \mapsto x_0(u)$, where N is an n–1-dimensional parameter manifold – e.g., hyperplane, sphere, ellipsoid, torus – and M is n-dimensional and path-connected. The time-t evolute of that initial hypersurface when it is in a state of wave motion takes the form of the

envelope of the (n–1)-parameter family of elementary hypersurfaces that emanate from each point of $x_0(u)$ and have the same value of t when the hypersurfaces are level surfaces of the function t.

a. The projectivized tangent bundle. The first obstacle to reconciling the traditional literature is the fact that most of it is set in a spatial manifold Σ of dimension two or three, not a spacetime manifold M of dimension four. Hence, as usual, one must proceed carefully in order to correctly account for the transition from spacetime to space. Indeed, we again find that the physically fundamental process that seems to assert itself is the projection of homogeneous coordinates for a projective space onto inhomogeneous coordinates, not merely the Cartesian projection that omits the temporal component.

In particular, it is the projection of any cotangent space T_x^*M onto its projectivized form PT_x^*M that first suggests this fact, since we are looking at the projection of the components $(\omega, -k_i)$ of any covector at $x \in M$ to the coordinates $(n_i = -k_i/\omega)$. One must then correctly account for the fact that although one regularly forms 1-forms such as:

$$n = n_i\, dx^i \tag{IX.62}$$

nevertheless, this is somewhat naïve since a local coordinate chart for PT^*U over an open subset $U \subset M$ would look like (t, x^i, n_i), in which the n_i are inhomogeneous coordinates for the real projective space $\mathbb{R}\mathrm{P}^3$ that models the fiber of PT^*U at each point of U, not components in the vector space $\mathbb{R}^3$.

It so happens that when M takes the form of the product manifold $\mathbb{R}\times\Sigma$ there is an accidental local equivalence of $PT^*M = PT^*(\mathbb{R}\times\Sigma)$ with $J^1(\Sigma; \mathbb{R})$, which is the manifold of 1-jets of differentiable functions on Σ; hence, such a construction has a distinctly non-relativistic sort of character. However, the use of functions of the form $t(x, y, z)$ in geometrical optics is so commonplace in the classical literature that we must unavoidably comment on this situation.

Locally, the manifold $J^1(\Sigma; \mathbb{R})$ looks like (x^i, f, f_i) and it projects onto Σ, $\mathbb{R}$, and $\Sigma\times\mathbb{R}$ in the predictable ways:

$$\begin{aligned} J^1(\Sigma; \mathbb{R}) &\to \Sigma, & j^1_x f &\mapsto x, \\ J^1(\Sigma; \mathbb{R}) &\to \mathbb{R}, & j^1_x f &\mapsto f, \\ J^1(\Sigma; \mathbb{R}) &\to \Sigma\times\mathbb{R}, & j^1_x f &\mapsto (x, f), \end{aligned}$$

It is not a fiber bundle under these projections, but a fibered manifold whose fibers locally look like the $\mathbb{R}\times\mathbb{R}^3$, $\mathbb{R}^3\times\mathbb{R}^3$, and $\mathbb{R}^3$, respectively.

The manifold $J^1(\Sigma; \mathbb{R})$ has a canonical 1-form $\theta = df - f_i\, dx^i$ that makes it a contact manifold. The significance of θ is based on the fact that when one looks at sections s: Σ

$\to J^1(\Sigma; \mathbb{R})$, $x \mapsto (x^i, f(x), f_i(x))$, such a section is integrable iff it is the 1-jet prolongation $j^1 f$ of the differentiable function f. Locally, this means that:

$$f_i = \frac{\partial f}{\partial x^i}, \tag{IX.63}$$

which is equivalent to requiring that:

$$0 = s^* \theta = df - f_i(x)\, dx^i. \tag{IX.64}$$

The contact structure on $J^1(\Sigma; \mathbb{R})$ then assigns each point of that manifold with the hyperplane $\theta = 0$ in its tangent space.

The manifold PT^*M also has a contact structure whose canonical 1-form $[\theta]$ we shall write in the form:

$$[\theta] = dt - n_i\, dx^i, \tag{IX.65}$$

when the local coordinate charts take the form (t, x^i, n_i).

We note that under the projection $T^*M \to PT^*M$, $k \mapsto [k]$, the canonical 1-form $k_\mu\, dx^\mu = \omega\, dt - k_i\, dx^i$ on T^*M projects to $\omega(dt - n_i\, dx^i)$ on PT^*M, which is proportional to the canonical 1-form, as long as one regards t as a function on Σ, not a coordinate of M. Hence, the vanishing of one of the canonical 1-forms is equivalent to the vanishing of the other.

Since $n_i = k_i / \omega$, we see that an integrable section $[k]$ of $PT^*M \to M$, i.e., an integrable line field on M, must satisfy the condition:

$$n_i = -\frac{\partial \phi / \partial x^i}{\partial \phi / \partial t} = \frac{\partial t}{\partial x^i} \tag{IX.66}$$

for some differentiable function ϕ on M and a differentiable function t on Σ. Note that changing the function ϕ to some other function ϕ' does not actually change the components n_i, since a change of ϕ is equivalent to a re-parameterization of $\mathbb{R}$.

The condition (IX.66) then leads to:

$$dt = n_i\, dx^i \tag{IX.67}$$

quite naturally.

We also find that if F is a differentiable function on T^*M then as long as it is homogeneous of some degree r in the fiber – i.e., $F[\lambda k] = \lambda^r F[k]$ – it will project to a function $[F]$ on PT^*M. Furthermore, this means that the characteristic vector field X_F on T^*M might be associated with a characteristic line field $[X_F]$ on PT^*M.

One any contact manifold (see Arnol'd [**11**]), the line in question is defined by the intersection of the contact hyperplane and the hyperplane that is annihilated by the 1-form $d[F]$. Hence, it will be a solution – up to a scalar multiple – of the equation:

$$i_{[X]} d\theta = d[F], \tag{IX.68}$$

in which we have abbreviated slightly.

Locally, this becomes the system:

$$[X]_t = 0 = \frac{\partial [F]}{\partial t}, \qquad [X]^i = \frac{\partial [F]}{\partial x^i}, \quad [X]_i = -\frac{\partial [F]}{\partial n_i}. \tag{IX.69}$$

Hence, we see immediately that not every homogeneous function on T^*M is projectable to something on PT^*M that will give a characteristic line field, but only the ones that are "time-invariant," in the sense that their first partial derivative with respect to t vanishes. Of course, when F refers to the dispersion law for electromagnetic wave propagation, this is a common assumption, since it amounts to assuming that the optical properties – or more generally, constitutive properties – of the medium are constant in time.

However, if F is homogeneous and time-invariant then the characteristic line field $[X_F]$ on PT^*M gives rise to a congruence of geodesics – up to parameterization – by integrating the system of ordinary differential equations that make $[X_F]$ a velocity vector field:

$$\frac{dx^i}{ds} = \frac{\partial [F]}{\partial n_i}, \qquad \frac{dn_i}{ds} = -\frac{\partial [F]}{\partial n_i}, \tag{IX.70}$$

in which the curve parameter is s, here.

Of course, when $M = \mathbb{R}\times\Sigma$, we see that essentially the same system of equations will come about by looking at the symplectic structure on $T^*\Sigma$ and the characteristic vector field that is defined by the analogous differentiable function $[F]$ on $T^*\Sigma$.

Furthermore, if we locally equate $PT^*(\mathbb{R}\times U)$ with $J^1(U; \mathbb{R})$ for some $U \subset \Sigma$, we see that we are also dealing with the system of bicharacteristic equations that follow from the first-order partial differential equation defined by:

$$[F][n] = 0, \qquad n = dt, \tag{IX.71}$$

which is the spatial version of the eikonal equation that one obtains from F, as we discussed in Section 1; however, we are now referring to the spatial phase function as t.

It is interesting that classical mechanics starts with objects in $J^1(\mathbb{R}; \Sigma)$ – viz., 1-jets of differentiable curves in Σ – while classical geometrical optics should model wave motion in terms of the dual objects in $J^1(\Sigma; \mathbb{R})$, especially since quantum physics seems to favor the representation of matter by waves rather than points. One can even define a form of *conjugacy* between curves in Σ and functions on Σ that is defined by the possibility that under composition of the curve γ: $\mathbb{R} \to \Sigma$ with the function f: $\Sigma \to \mathbb{R}$, one might obtain the identity map on $\mathbb{R}$:

$$f(\gamma(t)) = t \qquad \text{(for all } t \in \mathbb{R}\text{)}. \tag{IX.72}$$

Of course, given γ the choice of f is not unique, nor conversely, since one can see that a given curve in Σ can be embedded in many hypersurfaces, just as a given hypersurface admits many embedded curves.

Since the manifolds $J^1(\mathbb{R}; \Sigma)$ and $J^1(\Sigma; \mathbb{R})$ are topologically the same as $\mathbb{R} \times T(M)$ and $T^*M \times \mathbb{R}$, respectively, one sees that this sort of conjugacy is another form of dimension-codimension duality.

b. Elapsed-time functional. Now that we have made these prefatory remarks on the contact manifolds that we are concerned with, we see that some of the constructions of geometrical optics that lead to the principles of Huygens and Fermat follow quite naturally with no further restricting conditions.

For instance, given the spatial 1-form $n = n_i\, dx^i$ on Σ, one sees that its integral along any curve segment γ: $[0, 1] \to \Sigma$:

$$\Delta t[\gamma] = \int_\gamma n \tag{IX.73}$$

defines what we can call the *elapsed-time functional*, since in the case where n is exact and takes the form dt that is precisely what the integral will represent.

So far, our elapsed-time functional is a singular 1-cochain with coefficients in $\mathbb{R}$. If one is to resolve it to a two-point function on Σ that will represent a *characteristic function* for our geodesic flow on PT^*M, we have two basic possibilities:

1. If $\Delta t[\gamma]$ takes on the same values for any two homologous curves γ – i.e., ones that all have the same endpoints $\partial\gamma = y - x$ – then one can define:

$$\Delta t[x; y] = \Delta t[\gamma] \tag{IX.74}$$

unambiguously.

2. If there is some well-defined set of curves from x to y that all give the same value of $\Delta t[\gamma]$ then (IX.74) makes sense as long as γ is an element of that set.

In the first event, we must have the 1-form n is exact in order that the integral over curve reverts to the integral over its boundary, à la Stokes. The 1-cochain Δt then becomes a 1-coboundary.

In the second event, we see that we can take advantage of the fact that when one is given n: $\Sigma \to J^1(\Sigma; \mathbb{R})$, $x \mapsto (x, t(x), n_i(x))$, one can take advantage of the fact that then we have been given a spatial dispersion function $[F]$ on $J^1(\Sigma; \mathbb{R})$ there is a unique path from $n(x)$ to $n(y)$ in $J^1(\Sigma; \mathbb{R})$ – at least when they are sufficiently close – that is defined by the geodesic flow of $[F]$. The projection of this path onto Σ then gives the curve γ that we can integrate n over in order to make sense of (IX.74).

What makes $\Delta t[x; y]$ a characteristic function on $\Sigma \times \Sigma$ is the property that

$$n(x) = -\frac{\partial}{\partial x}\Delta t[x; y], \qquad n(y) = \frac{\partial}{\partial y}\Delta t[x; y]. \tag{IX.75}$$

Later, when we have discussed contact transformations, we shall see that this property also makes it a "generating functional" for the contact transformation that takes $(x, n(x))$ to $(y, n(y))$.

c. Elementary hypersurfaces. Since our elapsed-time functional is really just a generalization of the arc-length functional that one obtains in the case where $P[k]$ degenerates to the square of a quadratic polynomial, we can – at least for a sufficiently small elapsed time t – define a *geodesic sphere* about a given $x \in \Sigma$ of radius t to be the set $S_x(t)$ of all $y \in$ M such that $\Delta t[x, y]$ exists and is equal to t.

In the elementary geometrical case of Euclidian spatial geometry, the vanishing of the curvature of the Euclidian metric guarantees that geodesic spheres will exist about each point and have arbitrarily large radius. As long one considers only a constant speed of propagation for electromagnetic waves the elapsed time along any curve segment will be proportional to its arc length. One can then define a differentiable one-parameter family of geodesic spheres about each point for every $t > 0$.

However, when the curvature of the spatial manifold Σ is non-vanishing there will generally be a finite "radius of injectivity" for the exponential map that the chosen connection defines; that is, there will be pairs of points that are connected by geodesics of unequal length, such as non-antipodal points on a sphere.

Since the role of the dispersion polynomial $P[k]$ has been somewhat obscured by now, it is important to see that as long as one restricts oneself to only those 1-forms n on Σ that satisfy the requirement:

$$[P][n] = 0 \tag{IX.76}$$

one will find that one is dealing with 1-forms $k = \omega(dt - n)$ on $M = \mathbb{R}\times\Sigma$, where ω is non-zero, but arbitrary, that satisfy the requirement:

$$P[k] = 0. \tag{IX.77}$$

This means that elementary waves propagate in the characteristic hypersurfaces.

Hence, in the Minkowskian case, for instance, one sees that the elementary hypersurfaces in Σ are expanding spheres of radius t about each point x that lie on the light cone at the corresponding point (t, x) in Minkowski space.

d. The propagation of phase. The traditional formulation of Huygens's principle allows one to construct the time evolution of a momentary wave surface within its characteristic hypersurface from one moment to another by means of these elementary wave surfaces.

More precisely, let $\mathcal{F}_0 \subset \Sigma$ be a momentary wave surface at time $t = 0$; i.e., the image of an embedded two-dimensional submanifold. If one wishes to obtain the momentary wave front $\mathcal{F}_t$ at some later value of t then, as long as a geodesic sphere of radius t exists

about each $x_0 \in \mathcal{F}_0$, if we represent such a sphere by an embedding $\imath(x_0)$: $S^2 \to \Sigma$ then one can define a smooth two-parameter family of geodesic spheres of radius t by means of $\mathcal{F}_0 \times S^2 \to \Sigma$, $(x_0, y) \mapsto \imath(x_0)(y)$.

Of course, the image of $\mathcal{F}_0 \times S^2$ in Σ under this map will generally be a three-dimensional region. In order to obtain the evolute of the initial momentary wave surface $\mathcal{F}_0$ for this particular time interval, we need to consider only the boundary of that region – viz., the envelope of the family. A point y of this envelope $\mathcal{F}_t$ is characterized by the fact that if one moves along a curve through y whose tangent vector field is tangent to the envelope then the value of $\Delta t[x_0, y] = t$ does not change in an infinitesimal neighborhood of y. Hence, $y \in \mathcal{F}_t$ iff there is an $x_0 \in \mathcal{F}_0$ such that:

$$\Delta t[x_0, y] = t\,, \tag{IX.78a}$$

$$\frac{\partial}{\partial x_0}\Delta t[x_0, y] = n(x_0) = 0. \tag{IX.78b}$$

Between these equations, we obtain three local component equations for the three spatial coordinates of y as a function of x_0 . For instance, in the conventional case where Σ is Euclidian $\mathbb{R}^3$, when one uses:

$$\Delta t[x_0, y] = [\,(y^1 - x_0^1)^2 + (y^2 - x_0^2)^2 + (y^3 - x_0^3)^2\,]^{1/2}/c, \tag{IX.79}$$

with the points of $\mathcal{F}_0$ being parameterized by $x_0^i = x_0^i(u, v)$ the equations take the form:

$$(y^1 - x_0^1)^2 + (y^2 - x_0^2)^2 + (y^3 - x_0^3)^2 = (ct)^2, \tag{IX.80a}$$

$$\delta_{ij}(y^i - x_0^i)\frac{\partial x_0^j}{\partial u} = 0, \tag{IX.80b}$$

$$\delta_{ij}(y^i - x_0^i)\frac{\partial x_0^j}{\partial v} = 0. \tag{IX.80c}$$

The last two of these equations basically say that the line segment in $\mathbb{R}^3$ that goes from x_0 to y will always be orthogonal to the surface $\mathcal{F}_0$ and the first one says that it will have a length equal to ct. Since this defines a unique line segment (up to orientation) through each point of $\mathcal{F}_0$ there will be two well-defined endpoints $\pm y$ and thus two well-defined evolutes of $\mathcal{F}_0$, one of them a "forward" evolute $\mathcal{F}_{+y}$ and the other one $\mathcal{F}_{-y}$ a "backward" evolute. It was precisely this ambiguity in the solution that worried Huygens most. As it turned out, when one looks at the propagation of *amplitude*, the ambiguity is resolved. We illustrate the construction of the phase evolute by means of the envelope of geodesic phase spheres in Fig. 10.

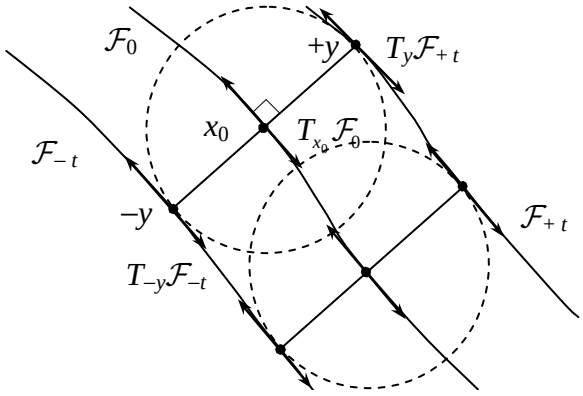

Figure 10. The construction of the evolved momentary wave surfaces by means of Huygens's principle.

e. Contact transformations. If we examine the construction of the evolved momentary wave surfaces then we see that we are not only mapping each point $x_0 \in \mathcal{F}_0$ to a pair of points on $\mathcal{F}_{+t}$ and $\mathcal{F}_{-t}$, but we also defining a map from the tangent space $T_{x_0}\mathcal{F}_0$ to the tangent spaces $T_{-y}\mathcal{F}_{-t}$ and $T_{+y}\mathcal{F}_{+t}$.

Now, since the tangent planes to any momentary wave surface $\mathcal{F}_t$ are hyperplanes in the tangent spaces to Σ at each point of $\mathcal{F}_t$, and a hyperplane in any tangent space $T_x\Sigma$ is associated with a unique 1-form n_x (up to non-zero scalar multiplication), we see that a momentary wave surface $\mathcal{F}_t$, along with its tangent spaces, defines a section $[n]$: $\mathcal{F}_t \to PT^*\Sigma$, $x \mapsto [n_x]$ of the projectivized cotangent bundle $PT^*\Sigma$ of Σ, that is, the bundle of lines through the origin in each fiber of $T^*\Sigma$. We can then extend this to a section n: $\mathbb{R}\times\mathcal{F}_t \to PT^*M$ by extending n to $dt - n$ at each (t, x).

What we now find is that since we have a canonical congruence of geodesics on PT^*M whenever $[F]$ has been chosen, it is actually unnecessary to go the route of first defining a family of geodesic spheres parameterized by the points of $\mathcal{F}_0$ and then going to the envelope of this family since, as long as a unique curve goes through each $n(x_0)$ in PT^*M over the points of $\mathcal{F}_0$ one can map the point x_0 *directly* to a unique point $y = x_0(t)$ for each t in the domain of definition for the local flow of X_F about $n(x_0)$.

One then sees that a basic property of the time evolution of momentary wave surfaces is that it must consist of a one-parameter family of *contact transformations* [1] whose parameter is t, in this case More precisely, a contact transformation is a diffeomorphism of PT^*M that preserves the hyperplanes in $T(PT^*M)$ that are annihilated by θ; note that this is not the same thing as preserving θ, as much as preserving θ up to multiplication by

[1] In addition to the references to Lie and Vessiot that were given in the Introduction to this book, one might also confer Eisenhart [**10**] or Arnol'd [**11**] on the subject of contact transformations..

a non-zero scalar function [2]. Hence, it will take a tangent plane at one point to a tangent plane at another point.

An immediate advantage of the use of contact transformations is that for a given t, one only produces *one* evolved momentary wave surface, namely, the one at $+t$, and not its somewhat unphysical twin at $-t$.

The elapsed-time function Δt: $\Sigma \times \Sigma \to \mathbb{R}$, $(x, y) \mapsto \Delta t[x, y]$ has a fundamental property in the eyes of contact transformations, namely that its partial derivatives at x and y define the 1-forms n_x and n_y when one is integrating n:

$$n_x = -\frac{\partial(\Delta t)}{\partial x} = -\frac{\partial(\Delta t)}{\partial x^i} dx^i, \qquad n_y = \frac{\partial(\Delta t)}{\partial y} = \frac{\partial(\Delta t)}{\partial y^i} dy^i; \tag{IX.81}$$

they also define the tangent planes to the momentary wave surfaces at each point. One then calls the function Δt a *generating function* for the contact transformation, since a given contact element (x, n_x) will get mapped to the corresponding (y, n_y) that one obtains from (IX.79).

f. The propagation of amplitude – general case. So far, we have only discussed Huygens's principle as it relates to the propagation of momentary wave surfaces; i.e., the propagation of the phase function for the wave. However, in elementary optics when one first encounters explanations for the phenomena of interference and diffraction one hears Huygens's principle applied to the propagation of wave amplitudes, or equivalently, intensities.

In that context, one finds that what one is really dealing with is the fact that elementary solutions of wave equations involve both phase functions and amplitude functions. We recall that the first step in the geometrical optics approximation was to consider only electromagnetic waves of the form $F(t, x^i) = e^{i\phi(t,x^i)} f(x^i)$, in which $\phi \in C^\infty(M)$ is a phase function and $f \in \Lambda^2_s M$, so it takes the form $1/2\, f_{\mu\nu}(x^i)\, dx^\mu \wedge dx^\nu$. This then makes $*F(t, x^i) = e^{i\phi(t,x^i)} * f(x^i)$.

Note that in order to make rigorous sense of the decomposition of F and $*F$, we have to explain how complex scalars act on real 2-forms. As it turns out, and we shall discuss this in greater depth in a later chapter, if the * isomorphism behaves like a Hodge * for a Lorentzian metric, then one has $*^2 = -I$, which means it defines an *almost-complex structure* on the real vector bundle $\Lambda^2 M$. One can then define the action of complex scalars by treating multiplication by * as the same thing as multiplication by i and:

$$(\alpha + i\beta)F = \alpha F + \beta *F. \tag{IX.82}$$

Hence:

$$e^{i\phi} F = \cos\phi\, F + \sin\phi\, *F. \tag{IX.83}$$

[2] According to Arnol'd [**11**], people who confuse the two conditions are "bad people," and pure mathematics could always use another moral principle.

One sees that the time evolution of the spatial amplitude f – or $*f$, for that matter – is confined to the orbit that it makes in the vector space $\Lambda^2 M$ under this action. Recall that this was precisely the issue that we discussed in the section of Chapter VIII on polarization, so we are again looking at duality rotations.

By exterior differentiation, the sourceless Maxwell equations $dF = d*F = 0$ give:

$$df + ik \wedge f = 0, \qquad d*f + ik \wedge *f = 0. \tag{IX.81}$$

The second step in the geometrical optics approximation is when one introduces the approximation that makes df and $d*f$ effectively zero. This would be like assuming that the spatial amplitudes f and $*f$ are constant, which would make the right-hand sides of (IX.83a, b, c) all vanish. Hence, one immediate way of going beyond the geometrical optics approximation, before one gets to the asymptotic series methods of diffraction theory, is to consider spatial 2-forms f and $*f$ that satisfy the differential equations (IX.81) instead of the algebraic equations that follow from omitting their differential terms.

One finds that the previous results that F and $*F$ are Lie-transported along the flow of $\mathbf{v}$ imply corresponding conditions on f and $*f$:

$$0 = \mathrm{L}_{\mathbf{v}} F = \mathrm{L}_{\mathbf{v}}(e^{i\phi} f) = e^{i\phi}(ik \wedge f + \mathrm{L}_{\mathbf{v}} f), \tag{IX.82a}$$
$$0 = \mathrm{L}_{\mathbf{v}} *F = \mathrm{L}_{\mathbf{v}}(e^{i\phi} *f) = e^{i\phi}(ik \wedge *f + \mathrm{L}_{\mathbf{v}} *f), \tag{IX.82b}$$

which gives:

$$\mathrm{L}_{\mathbf{v}} f = - ik \wedge f, \qquad \mathrm{L}_{\mathbf{v}} *f = - ik \wedge *f. \tag{IX.83}$$

Hence, the spatial 2-forms are no longer convected by the flow of $\mathbf{v}$.

d. The propagation of amplitude – linear case. Usually the propagation of amplitude in geometrical optics is addressed in the case of the linear wave equation. Since elementary wave solutions and fundamental solutions – in the Green function sense of the term – are closely-related concepts, one also finds that the basic construction associated with the propagation of amplitude functions from an initial wave surface $\mathcal{F}_0$ to some later surface $\mathcal{F}_t$ is essentially a graphical way of representing a linear operator [3] $K_{(0,t)} : \mathcal{L}^1(\mathcal{F}_0) \to \mathcal{L}^1(\mathcal{F}_t)$, $\psi \mapsto K_{(0,t)}\,\psi$ as an integral operator with a kernel $K(x, y)$:

$$(K_{(0,t)}\,\psi)(x) = \int_{\mathcal{F}_0} K(x, y)\psi(y)\,\mathcal{V}_y \,. \tag{IX.84}$$

Hence, this sort of construction is only useful for linear wave equations.

Since the kernel $K_{(0,t)} : \mathcal{F}_0 \times \mathcal{F}_t \to \mathbb{R}$ of the integral operator is a two-point function on a subset of $\Sigma \times \Sigma$, one naturally wonders how it relates to the elapsed-time $\Delta t(x, y)$.

[3] Here, we are using the notation $\mathcal{L}^1(N)$ to mean Lebesgue-integrable real functions on the manifold N. The Lebesgue measure on any manifold starts off on $\mathbb{R}^n$ in the coordinate charts and eventually defines the volume element $\mathcal{V}$ more globally.

One finds that, in practice, the kernel function is usually a function of the elapsed-time function, which usually takes the form of the geodesic distance function r: $\Sigma\times\Sigma \rightarrow \mathbb{R}$, $(x, y) \mapsto r(x, y)$ between pairs of points in Σ, as long as $r = ct$.

For example, consider the fundamental solutions $K(x, y)$ of the stationary linear wave equation, which takes the form of *Helmholtz's equation:*

$$(\Delta + k^2)\psi = 0, \tag{IX.85}$$

which one generally uses in the construction of interference patterns.

They take the form of expanding spherical wave solutions:

$$K(y - x) = \frac{e^{ik(y-x)}}{4\pi r(y - x)}, \tag{IX.86}$$

which implies that one must be dealing with a manifold Σ that has an affine structure, such as $\mathbb{R}^3$ if one is to make sense of the expression $y - x$. The distance function r is then the conventional Euclidian one.

It is important to note that $K(x - y)$ is not defined on the diagonal of $\Sigma\times\Sigma$ (i.e., all points of the form (x, x)). For this reason, fundamental solutions are often referred to as *fundamental singularities* (of the pole type).

The propagation of amplitude functions from a closed (= compact, without boundary) initial surface $\mathcal{F}_0$ to $\mathcal{F}_t$ is then given by *Helmholtz's theorem*: *Let:*

$$(K_{(0,t)}\,\psi)(y) = \int_{\mathcal{F}_0}\left[\frac{\partial K(y-x)}{\partial n}\psi(y) - K(y-x)\frac{\partial \psi(y)}{\partial n}\right]\mathcal{V}_y\,. \tag{IX.87}$$

with the kernel $K(y - x)$ defined as in (IX.86). One then has that:

$$(K_{(0,t)}\,\psi)(y) = \begin{cases}\psi(x) & x \text{ outside } \mathcal{F}_0,\\ 0 & x \text{ inside } \mathcal{F}_0.\end{cases} \tag{IX.88}$$

The fact that this integral vanishes for points inside the surface $\mathcal{F}_0$ accounts for the fact that the secondary solution $\mathcal{F}_{-t}$ to the propagation of the momentary wave surfaces is essentially physically irrelevant. That is, since the propagated wave surface $\mathcal{F}_{-t}$ lies inside the initial surface $\mathcal{F}_0$ the propagated amplitude of ψ will vanish on that surface.

This is the resolution of the dilemma that Huygens was worried about, since the existence of a secondary, inward-propagating wave front would not be physically significant if the amplitude of the wave on it was identically zero.

In order to restore the time evolution, one needs only to recall the separation of variables that produced the stationary solution:

$$\Psi(t, x) = T(t)\psi(x) = e^{-i\omega t}\psi(x). \qquad \text{(IX.89)}$$

However, there is an important subtlety associated with this reduction: By assuming that the time variation is governed by a single frequency ω, one is considering only *monochromatic* solutions of the full linear wave equation. Of course, as long as one is only concerned with the linear case, one can express the more general solutions in terms of a Fourier integral over all frequencies. However, the reduction does have the effect of converting a hyperbolic Cauchy problem into an elliptic boundary value problem. Hence, the values of ψ and its normal derivative on $\mathcal{F}_0$ are no longer independent of each and cannot be specified independently, as they would be in the hyperbolic Cauchy problem.

The reduction to a monochromatic stationary wave also implies that the solution space has been reduced from infinite-dimensional to finite-dimensional by essentially projecting onto the subspace that is singled out by a choice of ω. One can see that if each ω is associated with a distinct finite-dimensional vector space of solutions to the Helmholtz equation then clearly the generalized Cartesian product of this one-parameter family of vector spaces gives an infinite-dimensional vector space.

5. Diffraction. The foregoing discussion has been subordinate to the geometrical optics approximation, in which one regards the wave number (or frequency) of the wave in question as sufficiently large in comparison to the gradients of the electromagnetic field amplitudes that one can ignore those gradients and consider only a set of algebraic equations for the field amplitudes that amount to the passage from the field operator $\Box_\kappa$ to its symbol. As a consequence, one finds that the usual machinery of spacetime geometry – e.g., its Lorentzian metric, its geodesics, etc – are natural constructions that follow from the dispersion law, which gives to the characteristic hypersurfaces in the cotangent spaces. We have also seen that even within the geometrical optics approximation there is room for expansion in the geometry of spacetime when one considers birefringence and nonlinearity.

It is only natural to wonder how the geometry of spacetime would be further affected by eliminating, or at least weakening, the approximation that gave us this geometrical machinery. In particular, the system of linear, first-order, partial differential equations for f that we obtained in (VIII.35) would not reduce to algebraic equations if the differential contributions df and d^*f were no longer negligible compared to ω.

So far, the most definitive progress towards bridging the gap between wave optics and geometrical optics has been in the theory of diffraction. Since the original intent of wave mechanics was to make the transition from wave mechanics to classical mechanics analogous to the transition from wave optics to geometrical optics, there is also a close parallel between the methods of diffraction optics and the loop expansions of quantum field theory.

The starting point for most optical diffraction theory [4] is essentially the Cauchy problem for the Helmholtz equation for some C^2 function u on the spatial manifold Σ. As we saw, such a function gives the shape of stationary or time-harmonic waves. The

[4] The standard references on the subject of diffraction theory include Kline and Kay [**6**], Born and Wolf [**7**], Luneburg [**8**], and Baker and Copson [**9**].

Cauchy data u_0, $\partial u/\partial n$ are defined on some compact, possibly bounded, surface S in space, such as a rectangular slit or circular aperture in an opaque screen. Although it is usually emphasized that the waves propagating up to the slit or aperture are plane waves, actually, if the Cauchy problem is well-posed then that condition is irrelevant to the time evolution of the wave fronts past the opening. One must also keep in mind that since one is dealing with the elliptic equation that follows from the hyperbolic wave equation that the spatial Cauchy data are no longer independent of each other, but must satisfy a compatibility condition.

Since the Helmholtz operator $\Delta + k^2$ is linear and self-adjoint, one then solves the Cauchy problem for the Helmholtz equation by way of:

$$u(y) = -\frac{1}{4\pi}\int_S \left[u_0(x)\frac{\partial}{\partial n}\left(\frac{e^{ik(y-x)}}{\| y-x \|}\right) - \frac{\partial u_0(x)}{\partial n}\left(\frac{e^{ik(y-x)}}{\| y-x \|}\right)\right] \mathcal{V}_x . \qquad \text{(IX.90)}$$

However, like most integrals, this integral cannot generally be evaluated directly. Hence, one either goes the route of numerical integration or series approximations. The series approximation that is customarily employed is particularly difficult to fully comprehend, namely, the *asymptotic series* approximation (see, e.g., [**6-8**, **12-15**]). The justification for the use of such a series follows from the assumption that the Cauchy data was rapidly oscillating on the initial surface S, which amounts to the large ω approximation. In physics, this usually implies that the methods of diffraction theory are not useful until at least the microwave part of the electromagnetic wavelength spectrum, since the spatial dimensions of S can be quite appreciable for radio wave lengths, which might be in meters, or even kilometers.

Unlike the usual power series expansions of elementary calculus, an asymptotic series expansion is not presumed to converge in general; that is, it starts out as a *formal* power series. Furthermore, the series is also assumed to depend upon some parameter, such as k, in such a way that the series becomes an increasingly accurate approximation in the asymptotic limit as k grows arbitrarily large. One also frequently encounters the possibility that going to higher-order terms in the series might improve the approximation up some order, but then reduce the accuracy as the order goes beyond that point.

For the function $u(x)$, the expansion takes the form:

$$u(x) = e^{ik\psi(x)}\sum_{n=0}^{\infty}\frac{U_n(x)}{(ik)^n} . \qquad \text{(IX.91)}$$

Note that we have implicitly factored our phase function $\theta(x)$ into the product $k\psi(x)$, which assumes that we actually can define some characteristic wave number k, such as the Euclidian norm of the spatial wave number 1-form.

In order to get the functions $U_n(x)$ one can then substitute (IX.91) into (IX.90) and obtain a sequence of recurrence relations. However, in practice, one performs other transformations on the basic integral (IX.90) in order to obtain more manageable results. A further approximation that is often imposed is to assume that the dimensions of S are large compared to the wavelength of the waves passing though it, although diffraction

usually does not become noticeable until the dimensions of S are comparable to several wavelengths.

Generally, one expects that the series will start with the geometrical optics field $u^*(x) = e^{ik\psi(x)}U_0(x)$. The subsequent contributions to the series then represent successive orders of correction to the geometrical optics approximation. One can then regard the field $u - u^*$, which represents the terms of the asymptotic series beyond zeroth order, as the *diffracted field.*

The methods of asymptotic approximations get applied to quantum wave mechanics by using $1/\hbar$ in place of k as the large parameter. Presumably, one retrieves classical mechanics in the asymptotic limit as $\hbar$ goes to zero, which is like saying that the wave associated with a moving mass, such as an electron, ceases to diffract when passing though an opening. In fact, it has long since been established that low-energy electrons diffract in a wavelike manner when passing through openings whose dimensions are comparable to atomic spacings in crystal lattices, or about 1 Å, which was seen as a definitive proof of the de Broglie hypothesis of matter waves.

When one goes to first order in the asymptotic expansion, one obtains the WKB [5] approximation of quantum mechanics. This approximation has the advantage of being precise for the hydrogen spectrum, which also follows from the largely heuristic Bohr-Sommerfeld rules. In the eyes of quantum field theory, which uses asymptotic expansions for the evaluation of momentum-space Green functions for particle scattering operators, the powers of $\hbar$ represent the number of loops in the Feynman diagram for the scattering process, so the WKB approximation is regarded as a "one-loop" approximation to the Green function, while the classical scattering process is described by the zero-loop or "tree-level" diagrams.

From a geometrical standpoint, the most important question is that of what happens to the geodesics of spacetime – or even just space – as one goes to successive orders of diffraction. That is, can one still think in terms of transverse trajectories to the momentary wave surfaces, and, if so, how do the diffracted trajectories differ from the classical ones that represent the geometrical optics approximation? Here, one must clearly distinguish between the effect of diffraction on the propagation of phase, which is what the geodesics relate to, and its effect on the propagation of amplitude, which is what produces the diffraction patterns by way of interference patterns. We illustrate the situation that we have in mind in Fig. 11.

[5] The acronym WKB stands for Wentzel-Kramers-Brillouin. Sometimes one finds it referred to as the WKBJ approximation, where the J stands for Jeffreys.

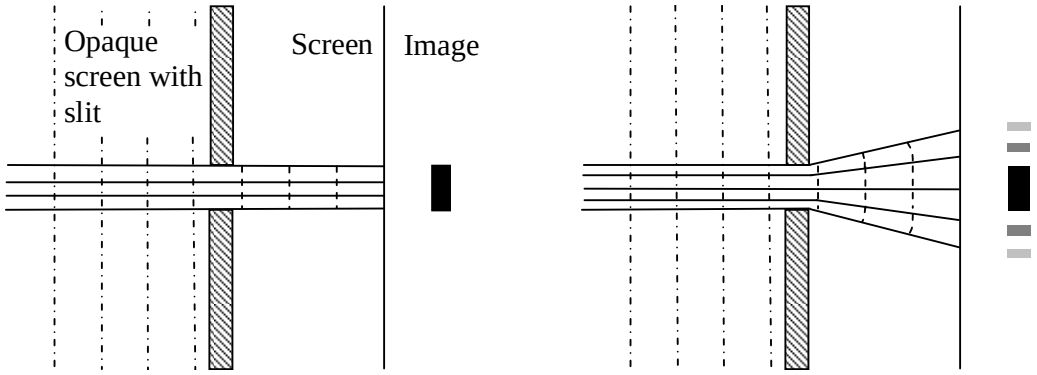

Figure 11. Geodesics in the geometrical optics approximation and in the diffracted case.

We shall conclude our discussion of the topic of diffraction with only these foregoing cursory observations, and the idea that since the spirit of pre-metric electromagnetism is to exhibit the manner by which spacetime geometry emerges from the way that electromagnetic waves propagate in that manifold, it appears that the most promising direction of further exploration in the eyes of understanding quantum physics is the examination of how diffraction affects the structure of geodesics.

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Chapter X

The calculus of variations and electromagnetism

One of the most established, accepted, and far-reaching foundations for theoretical physics – whether one is concerned with mechanics or field theories – takes the form of Hamilton's least action principle. In essence, it says that what characterizes the state of a system that represents its "natural" state is the fact that that the state in question is an extremum of some performance index on the state space that one calls the "action functional." It is a principle that resonates with even the most intuitive and elementary concepts in natural philosophy, such as the Panglossian belief that we live in the best of all possible worlds. Of course, to the scientist that is not a sufficient justification, since Ptolemy's conception of the place of the Earth in the cosmos had a high degree of intuitive appeal in its day, but one still prefers to believe that the most fundamental laws of nature should have that sort of basis in intuitive concepts that one sees manifested in all common phenomena.

To be sure, there is a considerable degree of ambiguity in the concept of "action" that probably accounts for the generality of its applications, just as Newton's second law of motion needs to be made more specific as far as the nature of the forces are concerned if one is to use it in practice. One finds that although the most common definition of action in physics gives it the units of energy-time, nevertheless, one also finds it used to describe things with the units of length, as in the geodesic problem, or time, as in Fermat's principle, which is the basis for Hamiltonian optics. Indeed, in more general optimization problems, the performance index can take almost any imaginable form.

One must also be advised that one of the recurring themes of Twentieth Century quantum physics, besides the ubiquitous role of vacuum polarization and the existence of non-zero ground states, is the idea that the classical least-action principle may represent essentially a first-order approximation to something more involved in nature. That is, in addition to considering the extremal states, which are analogous to the equilibrium or ground states of thermodynamics, one must consider the non-extremal states in the neighborhood of the extremals, as well. This is then analogous to the consideration of fluctuations about equilibrium states in non-equilibrium thermodynamics, which are then referred to in the quantum context as "quantum fluctuations about vacuum ground states." One can even consider global topological issues, such as phase transitions under large perturbations.

As an analogy, it is conceptually useful to view the calculus of variations as being something like "the calculus of infinity variables." Indeed, if the action functional represents a "differentiable" function on an infinite-dimensional "differentiable manifold" of system states then its first variation functional would represent its differential map and the extremal states would be the critical points of the function. Although there is a considerable body of mathematical literature that explore this possibility, generally under the mantle of "global analysis," one finds that even some pure mathematicians agree that this approach is not generally the most useful when it comes to the application of the calculus of variations to more tangible problems than the ones that global analysis poses. Hence, although we shall occasionally employ the

analogy as a heuristic device, we shall present the actual *calculus* in terms of local expressions.

Since the most natural mathematical formalism for the definitions of the calculus of variations seems to be the methods of jet manifolds [1] that we have already introduced to a limited extent, we shall add to the previous definitions with ones that are more specific to the present discussion. Although it might seem more conceptually logical to start with variational mechanics and then generalize to variational field theory, we shall present the topics in the opposite order, since our primary object under scrutiny is the pre-metric formulation of electromagnetic field theory, and we introduce mechanics only in the context of coupling the energy of an electromagnetic field to an "external" current.

One will also observe that the introduction of a metric into the field theory is generally unnecessary from the variational perspective since the only possible role it would play is in the association of field strengths with excitations. Since we have repeatedly emphasized that this sort of duality is best defined by the electromagnetic constitutive law of the medium, it is reassuring that we shall find that the introduction of a spacetime metric does not seem to be unavoidable until we discuss the notion of kinetic energy for point particles. Indeed, even in the case of extended matter, one must introduce a mechanical constitutive law in order to effect the association, and not necessarily a metric.

In the first section of this chapter, we shall discuss the energetics of electromagnetic fields. After that, we formulate the calculus of variations, first for sections of vector bundles in general and then for differential forms and multivector fields, more specifically. We then specialize the general expressions to the cases of electrostatics, magnetostatics, and electromagnetic fields, with particular attention to the case of electromagnetic waves. In that section, we also discuss how one makes a variational formulation of the problem of a charge distribution moving in an external electromagnetic field. Finally, we discuss the variational formulation of geometrical optics using Fermat's principle.

1. Electromagnetic energy. As a prelude to the discussion of the action functionals for electromagnetic fields, we first discuss the way that one associates potential energy with distributions of field sources, as well as the fields themselves, in the cases of electostatics, magnetostatics, and electromagnetic fields, more generally.

a. Electrostatic energy. One immediately recognizes that for static fields the only kind of energy that would be relevant is potential energy, or – more precisely – work. However, one has a choice of two directions to follow in this regard: the work done configuring a distribution of electric charges, electric dipoles, or magnetic dipoles, and the work done establishing the field that they generate as sources.

Of course, one expects that the total work done in one case should equal the total work done in the other, but the difference is that one generally desires that the total work

[1] For a good general treatment of the geometry of jet manifolds, one might confer Saunders [**1**]. Its application to physical field theories has been discussed to a considerable extent by Sardanashvily [**2**] and his colleagues.

done establishing the field be distributed over space as an energy density. Hence, there are more subtle issues to address in that case.

Since we are talking about work, we must immediately restrict our scope by requiring that the path functional in question be path-independent; i.e., the force in question must be conservative. This suggests that we are considering mostly "elementary" systems of charges or dipoles, since it is commonplace for dissipative forces to exist in the "complex" systems found in macroscopic matter, such as the heating of magnetic armatures exposed to time-varying magnetic fields. Indeed, a more comprehensive discussion of the non-conservative case brings one unavoidably to the issues of thermodynamics, which is beyond the scope of our present analysis.

The most elementary case in which one can define the potential energy of an electrostatic system consists of one point charge Q in an external field $\mathbf{E}$, which we assume to be conservative. The force of interaction that Q experiences at a point $x \in \Sigma$ is then $Q\mathbf{E}(x)$, with the usual caveat about the validity of the test-charge/external-field approximation; i.e., one also assumes that the field $\mathbf{E}$ is independent of Q.

The differential change in potential energy when Q goes from x to another point $x + dx$ – along any path $l(s)$ – is then:

$$dU(x) = \text{“}\mathbf{F}(x)\cdot d\mathbf{l}\text{”} = QE_i(x)\, dx^i = QE_i(s)\, v^i(s)\, ds\,. \qquad \text{(X.1)}$$

in which $v^i(s) = dx^i/ds|_s$ is the velocity of the parameterization for the chosen curve.

The reason for the quotation marks is to emphasize the fact that the introduction of the dot product is unnecessary if one regards the force as a 1-form to begin with. Instead of a scalar product of vectors one is then dealing with either the components F_i of a 1-form and the elements of a local coframe field dx^i or the bilinear pairing of a covector field $F_i(s)\, dx^i$ along a curve with a vector field $v^i(s)\, \partial/\partial x_i$ along it.

By integration along any curve γ from x to y, the change in potential energy of the charge-field system is then:

$$\Delta U[x, y] = Q\int_\gamma E(x(s)) = Q\int_0^1 E_i(s)v^i(s)\, ds = Q(U(y) - U(x)). \qquad \text{(X.2)}$$

If one brings Q in "from infinity" to y and expects that $U(y)$ is a finite number then this change in potential energy is finite iff $U(x)$ approaches a finite value as x goes to infinity. (Customarily, one lets this asymptotic value be zero, although any value of a potential function is ambiguous up to an arbitrary additive constant.) It is worth pointing out that if the external field is "constant" in space – assuming that the concept of constancy means something for the space in question – then the potential energy of the charge/field system is infinite no matter where you put Q, since it is proportional to the distance from any point "to infinity."

The second most elementary electrostatic system consists of two point charges Q_1 and Q_2 separated by a distance d. The essential difference between this case and the previous one is due to the fact that one cannot always regard the system as Q_1 in the external field due to Q_2, or vice versa, unless the test-charge/external-field approximation applies; i.e., the presence of Q_1 does not change the field of Q_2, or conversely. Hence, this assumption is more justifiable in the case of linear electrostatics than in the nonlinear case.

From the anti-symmetry of the electrostatic force $\mathbf{F}_{12}(d)$ between the two charges, combined with the change in sign when the line element $d\mathbf{l} = \mathbf{v}(s)\, ds$ for one direction changes to the opposite direction, one has the symmetry of the differential element of work done while changing their separation distance:

$$dF_{12} = \text{“}\mathbf{F}_{12} \cdot d\mathbf{l}\text{”} = F_i(l)\, dl^i = F_i(s)\, v^i(s)\, ds\,. \tag{X.3}$$

If one brings them together at a distance d by starting with one of them “at infinity” then the total work done is:

$$\Delta U_{12}[\infty, d] = \lim_{d_0 \to \infty} [d_0, d] = \int_{\infty}^{d} F_i(l) dl^i = \int_{\infty}^{d} F_i(s) v^i(s)\, ds\,. \tag{X.4}$$

If one uses the Coulomb expression for the force between them then the integration is immediate and one finds that the total potential energy associated with the configuration of Q_1 and Q_2 at a separation distance d is:

$$\Delta U_{12}[\infty, d] = \frac{1}{4\pi\varepsilon_0} \frac{Q_1 Q_2}{d^2}\,. \tag{X.5}$$

One sees that, in principle, the potential energy of the system grows without bound as the separation distance grows vanishingly small.

Implicit in all of this is the assumption that ε_0 is actually a constant and not a function of possibly Q_1, Q_2, and d, or simply the strength of the combined $\mathbf{E}$ field at each point. Hence, the aforementioned calculation has a distinctly linear field-theoretic character to it. In the nonlinear realm the total potential energy would involve a more complicated integration.

In order to extend the elementary case of two point charges at a distance d to N point charges Q_i, $i = 1, \ldots, N$ at points $x_i \in \Sigma$, we essentially use an inductive argument that assumes that if Q_k is any chosen one of the set of charges and $E(x_k)$ represents the field of the remaining $N - 1$ charges at the point x_k then the total potential energy of the system when one brings all of the charges in from infinity is simply:

$$U_{\text{tot}} = Q_k \int_{\infty}^{x_k} E(x(s)) = Q_k\, \phi_k(x_k). \tag{X.6}$$

Of course, this is based on the assumption that any of the possible choices for Q_k will produce the same value of the total energy by this process, which represents a consistency requirement.

When linear superposition is in effect, such as in the case of the Coulomb field, the total potential energy then takes the form of one-half the sum of all pairwise contributions over all pairs of charges:

$$U_{\text{tot}} = \tfrac{1}{2} \sum_{i=1}^{N} Q_i \phi_i(x_i)\,. \tag{X.7}$$

For each i, $\phi_i(x_i)$ represents the electric potential at x_i due to the remaining $N - 1$ charges, excluding Q_i ; the factor of ½ accounts for the fact the sum counts each pair of charges twice.

In order to extend this to a continuous distribution of charges that is described by a charge density $\rho(x)$, one basically replaces the finite charge Q_k with the infinitesimal charge $\rho(x)\mathcal{V}$, the electric potential at x due to the rest of the charge distribution by $\phi(x)$, and the summation by an integration over the support of ρ:

$$U_{\text{tot}} = \tfrac{1}{2}\int_{\text{supp}\,\rho} \rho\phi\,\mathcal{V}\,. \tag{X.8}$$

If one wishes to deduce the manner in which the total potential energy gets distributed through space as an energy density, one now imagines that the region described by supp ρ is dielectric in character. Hence, the electric field $E = -\,d\phi$ produces a response in the form of the electric excitation **D**, which couples to the charge density ρ by way of the fundamental equation:

$$\delta\mathbf{D} = \rho, \tag{X.9}$$

which we also express in the form:

$$d\#\mathbf{D} = \rho\mathcal{V}. \tag{X.10}$$

Hence, the integrand in (X.8) becomes:

$$\rho\phi\mathcal{V} = \phi\, d\#\mathbf{D} = d(\phi\#\mathrm{V}) - d\phi\wedge\#\mathbf{D} = d(\phi\#\mathrm{V}) + E\wedge\#\mathbf{D}\,. \tag{X.11}$$

This makes:

$$U_{\text{tot}} = \tfrac{1}{2}\int_{\partial(\text{supp}\,\rho)} \phi\,\#\mathbf{D} + \tfrac{1}{2}\int_{\text{supp}\,\rho} E\wedge\#\,\mathbf{D} = \tfrac{1}{2}\int_{\text{supp}\,\rho} E\wedge\#\,\mathbf{D}\,, \tag{X.12}$$

if we assume that the support of ρ has vanishing boundary.

From the form of this resulting expression, we see that we can define the energy density in supp ρ to be the 3-form:

$$w_E = \tfrac{1}{2}E\wedge\#\mathbf{D} = \tfrac{1}{2}E(\mathbf{D})\,\mathcal{V}. \tag{X.13}$$

In terms of local components, we have that:

$$E(\mathbf{D}) = E_i D^i = \varepsilon^{ij}E_i\,E_j = \tilde{\varepsilon}_{ij}\,D^i\,D^j\,. \tag{X.14}$$

We can also express this as a Euclidian scalar product:

$$E(\mathbf{D}) = \varepsilon(E, E) = \tilde{\varepsilon}\,(\mathbf{D}, \mathbf{D}). \tag{X.15}$$

Of course, the symbol $\tilde{\varepsilon}$ represents the scalar product on the tangent spaces that is inverse to the one that ε defines on the cotangent spaces.

Although we have defined w_E only on the support of ρ, where, by definition, ρ is non-vanishing, nevertheless, it is conventional to simply generalize the resulting expression for energy density (X.13) to the case of the classical vacuum, in which ρ vanishes, so one has:

$$w_E = \tfrac{1}{2}\,\varepsilon_0 E^2\,\mathcal{V}. \tag{X.16}$$

Apparently, the notion of energy being distributed throughout space according to the square of the field strength, and not simply a total value that was associated with the charge configuration was not always regarded as indisputably meaningful. According to Stratton [**3**], in a 1929 text on electromagnetic theory by Mayer and Weaver the authors argued that distributing the energy of a charge configuration throughout space made as much sense as distributing the beauty of a painting across the canvas! However, Stratton counters that certainly one distributes the energy of deformation in an elastic medium throughout the medium according to the distribution of strain, but he also admits that the continuum-mechanical analogy for electromagnetism was losing favor as a consequence of relativity.

Perhaps the best way to reconcile the distribution of energy with the notion of work done is to recall that each point is associated with an electric dipole moment by way of ε and it takes work to create an electric dipole. Of course, the formation of an electric dipole *in vacuo* only seems relevant for electric fields whose field strength approaches the critical value. However, the fact that ε_0 is not identically zero suggests that even the electric dipoles that eventually polarize into electron-positron pairs have a non-zero vacuum ground state, just as the space of electromagnetic fields does in the form of the zero-point field(s), and for sub-critical field strengths the change in the electric dipole moment is negligible.

b. Magnetostatic energy. The process of deriving the energy density of a static magnetic field is similar to the foregoing, although there is a fundamental difference: Since a current must be distributed over a chain that is at least 1-dimensional, instead of zero-dimensional, one cannot start with the work done moving a point in from infinity. Indeed, since a steady-state current can exist only in a closed circuit – i.e., a 1-cycle – one must first deal with the work done bringing a 1-cycle c_1 in from infinity when it carries a current vector field **I** when it moves in an external static magnetic field **B** distributed throughout space Σ.

First, look at the work done by the Lorentz force $f = \#(\mathbf{I} \wedge \mathbf{B})$ on the current in c_1 when one displaces it by a finite amount. We represent this displacement by the 2-chain $[-l, 0] \times c_1$, by which we really mean the embedding of the formal sum of squares formed from the products $[-l, 0] \times [0, 1]_a$, where a ranges through the intervals that define c_1. We represent the situation in Fig. 12:

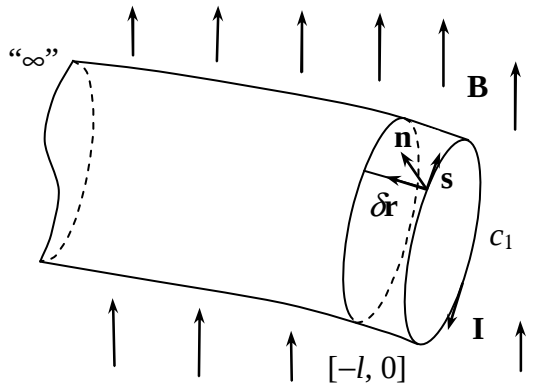

Figure 12. Bringing a current loop in from infinity in an external magnetic field.

First one must recall that actually f is a linear force density on c_1, not a force. Hence, the scalar function $\delta W = f(\delta\mathbf{r})$ on $[-l, 0] \times c_1$, which represents the linear work density due to the displacement $\delta\mathbf{r}$, must be integrated over c_1, as well as $[-l, 0]$. Hence, we must multiply it by the area element $\#\mathbf{n} = i_{\mathbf{n}}\mathcal{V}$. Now:

$$f(\delta\mathbf{r}) = \#(\mathbf{I} \wedge \mathbf{B})(\delta\mathbf{r}) = \mathcal{V}(\mathbf{B} \wedge \mathbf{I} \wedge \delta\mathbf{r}). \tag{X.17}$$

We define the linear frame field $\{\mathbf{s}, d\mathbf{r}, \mathbf{n}\}$ on $[-l, 0] \times c_1$, which we normalize to make:

$$\mathcal{V} = ds \wedge dr \wedge dn, \tag{X.18}$$

where $\{ds, dr, dn\}$ is the reciprocal coframe field.

In this frame:

$$\mathbf{B} = B^s\,\mathbf{s} + B^r\,\delta\mathbf{r} + B^n\,\mathbf{n}, \qquad \mathbf{I} = I\,\mathbf{s}, \tag{X.19}$$

so:

$$f(\delta\mathbf{r}) = IB^n \tag{X.20}$$

and:

$$f(\delta\mathbf{r})\,\#\mathbf{n} = I\,\#\mathbf{B}. \tag{X.21}$$

Hence, one ultimately finds that the total work done on c_1 by the displacement is:

$$W[-l, 0] = \int_{[-l,0]\times c_1} f(\delta\mathbf{r})\,\#\mathbf{n} = I\int_{[-l,0]\times c_1} \#\mathbf{B} = I\,\Phi_B[-l, 0], \tag{X.22}$$

in which $\Phi_B[-l, 0]$ represents the total magnetic flux through the 2-chain $[-l, 0] \times c_1$.

As long as the limit exists, we can then define the total work done bringing c_1 in from infinity as:

$$W_{\text{tot}}[c_1] = I\,\Phi_B[c_1] = I\,\lim_{l\to\infty}\Phi_B[-l, 0]\,. \tag{X.23}$$

Hence, the current couples to the total flux of the external field through the tube to infinity in the same way that charge couples to electric potential. Since the field **B** is

assumed to be conservative, the total flux will be the same for all homotopic tubes to infinity when c_1 is given as an initial loop.

For a configuration of N currents I_i, $i = 1, \ldots, N$, and only under the assumption of linear superposition, one then has:

$$W_{\text{tot}} = \tfrac{1}{2}\sum_{i=1}^{N} I_i \Phi_i[c_i], \tag{X.24}$$

In this expression, $\Phi_i[c_i]$ refers to the total magnetic flux linking the tube to infinity that has the current loop c_i as its initial loop and I_i as its current, as result of the magnetic field that is generated by the remaining $N - 1$ current loops.

We can also express the total work done in terms of the magnetic potential 1-form A ($B = \#\mathbf{B} = dA$) since:

$$\Phi_B[-l, 0] = \int_{[-l,0]\times c_1} dA = \int_{\{0\}\times c_1} A - \int_{\{-l\}\times c_1} A, \tag{X.25}$$

and:

$$\Phi_B[c_1] = \int_{\{0\}\times c_1} A = \mathcal{M}[c_1], \tag{X.26}$$

assuming that A goes to zero at infinity. Hence, the total flux linking the tube to infinity is the magnetomotive force around the loop c_1 .

In order to extend (X.24) to the continuum limit, we first point out that along any curve whose velocity vector field is $\mathbf{v}$, one has:

$$I\, A = I\, A(\mathbf{v})\, ds = A(\mathbf{I})\, ds. \tag{X.27}$$

However, if $\mathbf{I}$ is a current density that is distributed through space then we can also form a 3-form:

$$A(\mathbf{I})\, \mathcal{V} = A \wedge \#\mathbf{I} = A \wedge \#\delta\mathbf{H} = A \wedge d\#\mathbf{H}, \tag{X.28}$$

and since:

$$d(A \wedge \#\mathbf{H}) = dA \wedge \#\mathbf{H} - A \wedge d\#\mathbf{H} = B \wedge \#\mathbf{H} - A \wedge d\#\mathbf{H}, \tag{X.29}$$

when one forms the integral that corresponds to (X.24):

$$W_{\text{tot}} = \tfrac{1}{2}\int_{\text{supp}\,\mathbf{I}} A(\mathbf{I})\, \mathcal{V} = \tfrac{1}{2}\int_{\text{supp}\,\mathbf{I}} A \wedge \#\mathbf{I}, \tag{X.30}$$

an integration by parts puts this into the form:

$$W_{\text{tot}} = \tfrac{1}{2}\int_{\text{supp}\,\mathbf{I}} B \wedge \#\mathbf{H}. \tag{X.31}$$

Hence, the magnetostatic energy density takes the form:

$$w_B = \tfrac{1}{2} B \wedge \#\mathbf{H} = \tfrac{1}{2} B(\mathbf{H})\, \mathcal{V}, \tag{X.32}$$

and in component form one has:

$$B(\mathbf{H}) = B_i H^i = \tilde{\mu}^{ij} B_i B_j = \mu_{ij} H^i H^j, \tag{X.33}$$

which we can also express as:

$$B(\mathbf{H}) = \tilde{\mu}(B, B) = \mu(\mathbf{H}, \mathbf{H}). \tag{X.34}$$

One again, we see that the potential energy density takes the form of a quadratic form in either the field strengths or excitations whose components are defined by the magnetic constitutive law in effect.

c. Electromagnetic energy. Since the energy densities for both the electric field strength 1-form E and the magnetic field strength 2-form B are defined by quadratic expressions involving their respective constitutive laws, we naturally wish to examine the character of the quadratic expression $\kappa(F, F)$ when κ is the electromagnetic constitutive law for the four-dimensional spacetime manifold M, and F is the electromagnetic field strength 2-form.

We first assume that $T(M) = L(M) \oplus \Sigma(M)$ is a time-space splitting of the tangent bundle and that $\Lambda^2 = \Lambda^2_{\text{Re}} \oplus \Lambda^2_{\text{Im}}$ is the corresponding time-space splitting of the bundle of 2-forms. We can then express F as $dt \wedge E + \#\mathbf{B}$, where $dt \wedge E \in \Lambda^2_{\text{Re}}$ and $\#\mathbf{B} \in \Lambda^2_{\text{Im}}$. This allows us to write:

$$\kappa(F, F) = \kappa(dt \wedge E, dt \wedge E) + \kappa(dt \wedge E, \#\mathbf{B}) + \kappa(\#\mathbf{B}, dt \wedge E) + \kappa(\#\mathbf{B}, \#\mathbf{B}) . \tag{X.35}$$

If we revert the six-dimensional basis $\{b^i, \#\mathbf{b}_i, i = 1, 2, 3\}$ for Λ^2 so that κ can be expressed as a 6×6 block matrix:

$$\kappa^{IJ} = \left[\begin{array}{c|c} -\varepsilon^{ij} & \gamma^j_i \\ \hline \hat{\gamma}^i_j & \tilde{\mu}_{ij} \end{array}\right] \tag{X.36}$$

then:

$$\kappa(F, F) = -\varepsilon(E, E) + (\gamma + \hat{\gamma})(E, \mathbf{B}) + \tilde{\mu}(\mathbf{B}, \mathbf{B}) , \tag{X.37}$$

in which $\varepsilon(E, E)$ and $\tilde{\mu}(\mathbf{B}, \mathbf{B})$ represent the same quadratic expressions that we discussed in the static case while the middle term has the local expression:

$$(\gamma + \hat{\gamma})(E, \mathbf{B}) = (\gamma^i_j + \hat{\gamma}^j_i) E_i B^i . \tag{X.38}$$

One can similarly decompose the expression $F \wedge \#\mathfrak{h}$, with $\mathfrak{h} = \partial_t \wedge \mathbf{D} + \#^{-1} H$, into:

$$F \wedge \#\mathfrak{h} = dt \wedge E \wedge \#(\partial_t \wedge \mathbf{D}) + dt \wedge E \wedge H + \#\mathbf{B} \wedge \#(\partial_t \wedge \mathbf{D}) + \#\mathbf{B} \wedge H, \tag{X.38}$$

which then becomes:

$$F \wedge \#\mathfrak{h} = -[E(\mathbf{D}) - (\gamma + \hat{\gamma}^T)(E, \mathbf{B}) - H(\mathbf{B})]\, \mathcal{V}, \tag{X.39}$$

which equals $\kappa(F, F)\, \mathcal{V}$.

When the medium in question is not in a state of motion relative to the measurer/observer, and does not exhibit optical activity, the cross-term in the sum vanishes, and the remaining expression is:

$$F \wedge \#\mathfrak{h} = -\,[E(\mathbf{D}) - H(\mathbf{B})]\,\mathcal{V}. \tag{X.40}$$

For an isotropic medium, this takes the form:

$$F \wedge \#\mathfrak{h} = -\,[\varepsilon E^2 - (1/\mu)B^2]\,\mathcal{V}. \tag{X.41}$$

We then recognize the expressions in (X.39) as minus two times the usual electromagnetic Lagrangian density 4-form:

$$\mathcal{L}_{\text{em}} = -\tfrac{1}{2} F \wedge \#\mathfrak{h} = -\tfrac{1}{4} F_{\mu\nu} H^{\mu\nu}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \tag{X.42}$$

Interestingly, one can also account for the usual electromagnetic Hamiltonian using the quadratic form that is defined by κ. However, one must form the "complex conjugate" of F, namely:

$$\bar{F} = dt \wedge E - \#\mathbf{B}. \tag{X.42}$$

Note that, this operation of conjugation is meaningful only for a given choice of time-space splitting. Indeed, this is consistent with the idea that energy itself is not a relativistic invariant except in the context of rest energy.

We now find that:

$$\kappa(F, \bar{F}) = -\,[\varepsilon(E, E) + \tilde{\mu}(\mathbf{B}, \mathbf{B})\,] + (\gamma - \hat{\gamma}^T)(E, \mathbf{B}), \tag{X.43}$$

and when the last term vanishes, we see that we are again dealing with minus two times the usual electromagnetic field Hamiltonian.

2. The calculus of variations in terms of vector bundles. Although we have been assuming a certain familiarity with variational field theory all along, since we are mostly concerned with fields that take the specific form of multivector fields and exterior differential forms, it is necessary to show how one can define the basic notions and carry out the basic computations of the variational calculus using such fields, in particular.

a. Extremal fields. When one is given a particular class of fields, the starting point for the calculus of variations is the definition of an action functional on the space of fields in question. When the fields are all sections $\phi\colon M \to E$ of a vector bundle over M it is somewhat convenient that the space of fields $\Gamma(E)$ is a linear space, so the action functional becomes a function on that vector space. However, one is nonetheless dealing with an infinite-dimensional vector space that is not necessarily separable, so putting a topology and a differentiable manifold structure on the space becomes problematic

enough that it is usually only heuristically useful to imagine that the action functional is a differentiable function on an infinite-dimensional manifold.

A *first-order action functional* $S[.]$ on $\Gamma(E)$ is defined by a first-order *Lagrangian density* $\mathcal{L}$: $J^1(M, E) \to \mathbb{R}$, which is at least C^2, and such a functional takes the form:

$$S[\phi] = \int_M \mathcal{L}(j^1\phi)\mathcal{V}. \tag{X.44}$$

Let us briefly recall the terminology that we introduced in Chapter VI concerning the geometry of jet manifolds in the form that it takes in the present context:

The notation $J^1(M, E)$ refers to the manifold of 1-jets of sections of the vector bundle $E \to M$, and it has local coordinate systems that take the form $(x^\mu, \phi^A, \phi^A{}_\mu)$. In the present case, the projection $\pi_{1,0} : J^1(M, E) \to E$, $j^1_x\phi \mapsto \phi$ defines a vector bundle over E in its own right. The vector spaces that comprise its fibers have the $\phi^A{}_\mu$ for coordinates, so they are nN-dimensional ($n = \dim M$, $N = \text{rank } E$).

A section s of the projection $J^1(M, E) \to M$ takes the local form $(x^\mu, s^A(x), s^A{}_\mu(x))$. It is integrable iff s is the 1-jet prolongation $j^1\phi$ of some section ϕ of $E \to M$, in which case, it takes the local form $(x^\mu, s^A(x), s^A{}_{,\mu}(x))$. Hence, s is integrable iff one locally has:

$$s^A{}_\mu(x) = s^A{}_{,\mu}(x). \tag{X.45}$$

A first-order field Lagrangian can then be expressed in the local form $\mathcal{L} = \mathcal{L}(x^\mu, \phi^A, \phi^A{}_{,\mu})$ that is so established in the physics literature.

One now sees that in order for the integral in (X.44) to converge one must restrict either the class of sections that one uses or the topology of the manifold M. Indeed, requiring the sections to vanish smoothly "at infinity" is equivalent to saying that if "infinity" is a point that one uses to compactify M then such a section can be smoothly extended to the one-point compactification of M by setting its value at infinity equal to zero.

The most-discussed one-point compactification in elementary mathematics is probably the compactification of $\mathbb{R}^2$ into a 2-sphere by stereographic projection. However, we hasten to point out that the way that one compactifies a topological space is usually determined by the type of *geometry* that one is doing, in addition to topology. One-point compactification mostly relates to conformal Euclidian geometry, in which spheres play the same fundamental role as points do in affine geometry. In projective geometry, one compactifies the affine n-plane by the addition of a "hyperplane at infinity," and in conformal Lorentzian geometry, one adds a "light-cone at infinity."

We shall tacitly assume that suitable conditions are imposed on M or the field ϕ such that the action functional always exists.

The *first variation* δS of the action functional amounts to a linear functional on the (infinite-dimensional) vector space of vector fields on $J^1(M, E)$ that can be represented in the form of an integrated Lie derivative:

$$\delta S[X] = \int_M L_X(\mathcal{L}\mathcal{V}) = \int_M [(i_X d\mathcal{L})\mathcal{V} + \mathcal{L}\, d i_X \mathcal{V}]. \tag{X.47}$$

A (substantial) *variation* of a section ϕ is a vertical vector field $\delta\phi$ on the manifold E. That is, one varies values the of the field without varying the points of M that they are associated with. If a local trivialization $U \times V$ of E over $U \subset M$ has the local coordinates (x^μ, ϕ^A) then a general vector field on E will look like:

$$X = X^\mu \frac{\partial}{\partial x^\mu} + X^A \frac{\partial}{\partial \phi^A}, \tag{X.48}$$

in which the component functions X^μ and X^A are all smooth functions on E; i.e., $X^\mu = X^\mu(x^\mu, \phi^A)$, $X^A = X^A(x^\mu, \phi^A)$.

Hence, a vertical vector field, such as $\delta\phi$, will look like:

$$\delta\phi = \delta\phi^A \frac{\partial}{\partial \phi^A}. \tag{X.49}$$

One prolongs $\delta\phi$ to a vector field $\delta^1\phi$ on $J^1(M, E)$ by differentiation. Locally, it looks like:

$$\delta^1\phi = \delta\phi^A \frac{\partial}{\partial \phi^A} + \frac{\partial(\delta\phi^A)}{\partial x^\mu} \frac{\partial}{\partial \phi^A{}_\mu}. \tag{X.50}$$

The fundamental problem of the calculus of variations is to find the *extremal sections* [2] of the first variation functional under such first prolongations of substantial variations of the fields. This basically means that one is looking for all $\phi \in \Gamma(E)$ such that $\delta S[\delta^1\phi]$ vanishes for all $\delta^1\phi$ that represent first prolongations of variations of ϕ. This means:

$$0 = \delta S[\delta^1\phi] = \int_M (i_{\delta^1\phi} d\mathcal{L})\mathcal{V} \tag{X.51}$$

for all vertical $\delta\phi$ since the second term in the integrand in (X.47) vanishes for all prolongations of vertical vector fields.

In order to perform the traditional "integration by parts," we revert to the local formulation. One has:

$$d\mathcal{L} = \frac{\partial \mathcal{L}}{\partial x^\mu} dx^\mu + \frac{\partial \mathcal{L}}{\partial \phi^A} d\phi^A + \frac{\partial \mathcal{L}}{\partial \phi^A{}_\mu} d\phi^A{}_\mu, \tag{X.52}$$

so:

$$i_{\delta^1\phi} d\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi^A} \delta\phi^A + \frac{\partial \mathcal{L}}{\partial \phi^A{}_\mu} \delta\phi^A{}_{,\mu} = \frac{\delta \mathcal{L}}{\delta \phi^A} \delta\phi^A + \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi^A} \delta\phi^A\right), \tag{X.53}$$

[2] We avoid the term "extremal fields" because this has a completely distinct, but established, meaning in Hamilton-Jacobi theory, namely, geodesic congruences.

in which we have introduced the *variational derivative* of $\mathcal{L}$ with respect to ϕ:

$$\frac{\delta \mathcal{L}}{\delta \phi^A} = \frac{\partial \mathcal{L}}{\partial \phi^A} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial \phi^A{}_\mu}. \tag{X.54}$$

One can then say that:

$$\delta S[\delta^1 \phi] = \int_M \left(\frac{\delta \mathcal{L}}{\delta \phi^A} \delta \phi^A \right) \mathcal{V} + \int_{\partial M} \#\left(\mathbf{\Pi}_A \delta \phi^A \right), \tag{X.55}$$

in which we have introduced the *generalized momentum density* vector fields:

$$\mathbf{\Pi}_A = \frac{\partial \mathcal{L}}{\partial \phi^A{}_\mu}\frac{\partial}{\partial x^\mu}. \tag{X.56}$$

When one restricts oneself to either variations of ϕ that vanish on the boundary of M or variations that satisfy the transversality condition $\mathbf{\Pi}^A \delta \phi_A = 0$, the necessary and sufficient condition for an extremum is that:

$$\frac{\delta \mathcal{L}}{\delta \phi^A} = 0, \tag{X.57}$$

which represents the *Euler-Lagrange* equations.

These are generally N nonlinear partial differential equations for the components ϕ^A of the extremal fields. Hence, in order specify such fields uniquely one will generally have to pose boundary-value problems, in the elliptic case, or Cauchy problems, in the hyperbolic case.

If one introduces the *generalized force densities* f_A , along with the aforementioned generalized momentum density vector fields Π_A , on E by way of:

$$f_A = \frac{\partial \mathcal{L}}{\partial \phi^A}, \tag{X.58}$$

then the Euler-Lagrange equations can be put into the form:

$$\delta \Pi_A = f_A. \tag{X.59}$$

b. Case of differential forms. Since a k-vector field on a manifold M is a section of the vector bundle $\Lambda_k M \to M$ and a differential k-form on M is a section of the vector bundle $\Lambda^k M \to M$, one sees that all one needs to do in order to focus on the variational formulation of electromagnetism is to specialize the general results that were derived above.

One thing that needs to be addressed immediately is the fact that when considers the bundle $J^1(\Lambda^k)$ of 1-jets of exterior differential k-forms on M one sees that it is the differential of any section ϕ: $M \to \Lambda^k M$ that seems to factor most fundamentally in the

calculus of variations, whereas in the calculus of exterior differential forms, it is the completely antisymmetrized differential – viz., the exterior derivative – that plays the fundamental role. This suggests that one might wish to define a notion of "exterior 1-jet" in the predictable way – i.e., equivalence classes of k-forms that are defined in some neighborhood of a point $x \in M$ and have the same values at k as k-forms, along with the same exterior derivatives – but one eventually comes to see that this is basically unnecessary.

The real issue is differentiation with respect to fiber coordinates, and one finds that since components of k-forms are completely antisymmetrized sums a differentiation with respect to field components invariably singles out just one member of the sum. For instance, if $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$ then:

$$\frac{\partial F_{\mu\nu}}{\partial A_{\mu,\nu}} = \delta^{\mu}_{\mu}\delta^{\nu}_{\nu} - \delta^{\nu}_{\mu}\delta^{\mu}_{\nu} = 1, \tag{X.60}$$

since $\mu \neq \nu$, by anti-symmetry.

As a consequence, one has, in particular:

$$\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} = \frac{\partial \mathcal{L}}{\partial A_{\mu\nu}}. \tag{X.61}$$

Hence, whether one differentiates with respect to $F_{\mu\nu}$ or $A_{\mu\nu}$ is irrelevant.

We shall primarily be concerned with 1-forms A as the fundamental fields, so the relevant jet bundle is $J^1(\Lambda^1) \to M$. Local coordinate charts on the total space $J^1(\Lambda^1)$ take the form $(x^{\mu}, A_{\mu}, A_{\mu\nu})$. A section s: $M \to J^1(\Lambda^1)$ of this bundle then looks like $(x^{\mu}, A_{\mu}(x^{\mu}), A_{\mu\nu}(x^{\mu}))$ and it is integrable iff $s = j^1A$, which implies that:

$$A_{\mu\nu} = A_{\mu,\nu}. \tag{X.62}$$

A Lagrangian density on $J^1(\Lambda^1)$ locally looks like $\mathcal{L}(x^{\mu}, A_{\mu}, A_{\mu\nu})$ and for the prolongation of a 1-form A it takes the form $\mathcal{L}(x^{\mu}, A_{\mu}(x), A_{\mu,\nu}(x))$. The Euler-Lagrange equations then are expressed as:

$$0 = \frac{\delta \mathcal{L}}{\delta A_{\mu}} = \frac{\partial \mathcal{L}}{\partial A_{\mu}} - \frac{\partial}{\partial x^{\nu}}\frac{\partial \mathcal{L}}{\partial A_{\mu\nu}}. \tag{X.63}$$

By introducing the notations:

$$J^{\mu} = \frac{\partial \mathcal{L}}{\partial A_{\mu}}, \qquad \mathfrak{h}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial A_{\mu\nu}} \tag{X.64}$$

the fundamental equations for the field A_{μ} become:

$$\partial_\nu \mathfrak{h}^{\mu\nu} = J^\mu. \tag{X.65}$$

From (X.61), we see that it amounts to the same thing to define $\mathcal{L}$ as a function of $F_{\mu\nu}$, instead of $A_{\mu\nu}$. Hence, if we combine the integrability condition for $F_{\mu\nu}$ with the Euler-Lagrange equations in the latter form, and the second of equations (X.64), we get the system:

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}, \qquad \partial_\nu \mathfrak{h}^{\mu\nu} = J^\mu, \qquad \mathfrak{h}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}. \tag{X.66}$$

We can just as well express these equations in index-free form as equations in the 2-form F, the 1-form A, the bivector field $\mathfrak{h}$, and the vector field $\mathbf{J}$:

$$F = dA, \qquad \partial\mathfrak{h} = \mathbf{J}, \qquad \mathfrak{h} = \frac{\partial \mathcal{L}}{\partial F}. \tag{X.67}$$

As long as one can regard the last of these equations as essentially the constitutive law that relates F to $\mathfrak{h}$, these equations are formally identical with the pre-metric Maxwell equations. In the following sections, we shall pursue the extent to which this is a valid assumption.

3. Variational formulation of electromagnetic problems. Since the main issues in the variational formulation of the field equations for electromagnetism are the definition of a Lagrangian density and its relationship to the constitutive laws of the medium, we shall examine the specific form these matters take in the cases of static electric, static magnetic, and dynamic electromagnetic fields. We shall then discuss the variational formulation of the mechanical problem of determining the motion of charged mass distributions in external electromagnetic fields. (For some other treatments of the same basic topics, one might confer [**4**-**6**].)

a. Electrostatics. Whether one is dealing with linear or nonlinear electrostatics, one generally begins by defining a Lagrangian that is quadratic in the field strength $E \in \Lambda^1\Sigma$:

$$\mathcal{L}(x^i, E_i) = \tfrac{1}{2}\overline{\varepsilon}(E, E) = \tfrac{1}{2}\overline{\varepsilon}^{ij} E_i E_j \ . \tag{X.68}$$

For an integrable section, one has:

$$E = d\phi, \qquad (E_i = \phi_{,i}) \tag{X.69}$$

for some smooth function ϕ on Σ that is unique only up to an additive constant.

Note that even though we originally developed the formalism of calculus of variations by defining a Lagrangian density as a differentiable function $\mathcal{L}$ on $J^1(E)$, we see that as

long as we deal with gauge-invariant field Lagrangians – i.e., ones that do not depend upon ϕ explicitly – we can just as well define $\mathcal{L}$ on $\Lambda^1\Sigma$ itself.

When one derives the vector field **D** that is canonically conjugate to *E*, one obtains the electrostatic constitutive law in the form:

$$\#\mathbf{D} = \frac{\partial \mathcal{L}}{\partial E} = \#(\varepsilon^{ij} E_j \partial_i), \tag{X.70}$$

in which:

$$\varepsilon^{ij}(x^i, E_j) = \bar{\varepsilon}^{ij} + \tfrac{1}{2}\frac{\partial \bar{\varepsilon}^{kj}}{\partial E_i} E_k \,. \tag{X.71}$$

Hence, one sees that when one is concerned with nonlinear electrostatics there is a fundamental difference between the quadratic tensor field $\bar{\varepsilon}$ and the actual electrostatic constitutive tensor field ε; of course, they will coincide in the case of linear electrostatics.

The field equations that result from this choice of Lagrangian density are then:

$$E = d\phi, \qquad \delta\mathbf{D} = 0, \qquad \mathbf{D} = \varepsilon(E), \tag{X.72}$$

which can be combined into a single equation for ϕ:

$$\Delta_\varepsilon \phi = (\delta \cdot \varepsilon \cdot d)\phi = 0. \tag{X.73}$$

It is important to remember that we are really varying ϕ, not *E*, even though ϕ no longer figures explicitly in the Lagrangian density. Had we merely varied *E* directly, the only resulting field equation would be simply **D** = 0, which is a trivial outcome.

In order to include the source of the field, one must add another term to the Lagrangian density that couples the charge density ρ to the electric potential ϕ by way of

$$\mathcal{L}(x^i, \phi, \phi_i) = \tfrac{1}{2} E \wedge \#\mathbf{D} + \rho\phi\, \mathcal{V} = [\tfrac{1}{2}\bar{\varepsilon}(E,E) + \rho\phi]\, \mathcal{V} \,. \tag{X.74}$$

This makes the field equations take the form:

$$E = d\phi, \qquad \delta\mathbf{D} = \mathbf{J}, \qquad \mathbf{D} = \varepsilon(E), \tag{X.75}$$

or:

$$\Delta_\varepsilon \phi = (\delta \cdot \varepsilon \cdot d)\phi = \mathbf{J}, \tag{X.76}$$

which is of the generalized Poisson type.

b. Magnetostatics. Although the variational formulation of magnetostatics is analogous to that of electrostatics, nonetheless, there are significant differences that must be addressed. For one thing, the fundamental field *A* is a 1-form, not a 0-form, so the field strength $B = dA$ that it defines is a 2-form, not a 1-form. However, because the vector space A^2V of 2-forms over a three-dimensional vector space *V* is isomorphic to the vector space *V*, by means of Poincaré duality for a choice of volume element, it is more

conventional in classical magnetism to represent the 2-form $B = \#\mathbf{B}$ by the components B^i of the corresponding vector field **B**, and similarly, the bivector field $\mathbf{H} = \#^{-1}H$ gets represented by the components H_i of the 1-form H.

This also means that the magnetic constitutive law $\tilde{\mu} = \mu^{-1}$: $\Lambda^2\Sigma \to \Lambda_2\Sigma$ for the medium intrinsically couples 2-forms to bivector fields, although it is usually expressed by the matrix of a linear isomorphism $\tilde{\mu}: \Lambda_1\Sigma \to \Lambda^1\Sigma$ between their dual spaces. In effect, one is using:

$$\tilde{\mu}(B, B) = \tilde{\mu}(\#\mathbf{B}, \#\mathbf{B}) = (\#\tilde{\mu}\#)(\mathbf{B}, \mathbf{B}), \tag{X.77}$$

so the four-index components $\tilde{\mu}^{ijkl}$ of $\tilde{\mu}$ go to the two-index components of $\#\tilde{\mu}\#$:

$$\tilde{\mu}_{ij} = \varepsilon_{ikl}\varepsilon_{jmn}\tilde{\mu}^{klmn}. \tag{X.78}$$

From this, one defines the sourceless magnetostatic field Lagrangian density for the potential 1-form A by a quadratic form on B that then reverts to a quadratic form on **B**:

$$\mathcal{L}(x^i, B_{ij}) = \tfrac{1}{2}\bar{\mu}(\mathbf{B}, \mathbf{B}) = \tfrac{1}{2}\bar{\mu}_{ij}B^iB^j = \tfrac{1}{2}\bar{\mu}(B, B) = \tfrac{1}{4}\bar{\mu}^{ijkl}B_{ij}B_{kl}, \tag{X.79}$$

in which:

$$B^i = \tfrac{1}{2}\varepsilon^{ijk}B_{jk} = \tfrac{1}{2}\varepsilon^{ijk}(A_{j,k} - A_{k,j}) = \varepsilon^{ijk}A_{j,k} = (\nabla \times \mathbf{A})^i. \tag{X.80}$$

As before, we must emphasize that the second tensor field $\bar{\mu}$ that appears in this expression does not have to represent the magnetostatic constitutive law of the medium directly. Similarly, one sees that in the absence of sources, the Lagrangian density is independent of A – i.e., gauge-invariant – and can thus be defined as a differentiable function on $\Lambda^2\Sigma$, instead of $J^1(\Lambda^2\Sigma)$.

In order to derive the field equations for magnetostatics from this Lagrangian density, one needs only to take (X.78) into account in order to convert the partial derivative of $\mathcal{L}$ with respect to A_{ij} into a partial derivative with respect to B^i:

$$H^{ij} = \frac{\partial\mathcal{L}}{\partial A_{ij}} = \frac{\partial B_{mn}}{\partial A_{ij}}\frac{\partial B^k}{\partial B_{mn}}\frac{\partial\mathcal{L}}{\partial B^k} = \varepsilon^{ijk}\frac{\partial\mathcal{L}}{\partial B^k} = \varepsilon^{ijk}\tilde{\mu}_{kl}B^l = (\#^{-1}H)^{ij}, \tag{X.81}$$

in which:

$$H = \tilde{\mu}_{ij}B^j dx^i \tag{X.82}$$

represents the magnetic excitation 1-form that is Poincaré dual to the bivector field $\mathbf{H} = \tfrac{1}{2}H^{ij}\,\partial_i \wedge \partial_j$ and:

$$\tilde{\mu}_{ij} = \bar{\mu}_{ij} + \frac{\partial\bar{\mu}_{ij}}{\partial B^k}B^k, \tag{X.83}$$

which, as one sees, agrees with $\bar{\mu}_{ij}$ only in the case of a linear constitutive law.

The sourceless field equations for A that one derives from the given Lagrangian density take the form:

$$B = dA, \qquad \delta\mathbf{H} = 0, \qquad \mathbf{H} = \tilde{\mu}(B), \tag{X.84}$$

if one uses the 2-form B and the bivector field $\mathbf{H}$, or:

$$\mathbf{B} = \delta\#^{-1}A, \qquad dH = 0, \qquad H = \tilde{\mu}(\mathbf{B}), \tag{X.85}$$

if one uses the vector field $\mathbf{B}$ and the 1-form H.

One can also combine these equations into a single second-order system of partial differential equations for A:

$$\Delta_\mu A = (\delta \cdot \tilde{\mu} \cdot d)A = 0, \tag{X.86}$$

which is of a generalized Laplace type.

If one includes the coupling of the energy in the field $\mathbf{H}$ to the energy in the source current density $\mathbf{I}$ then one must add another term to the Lagrangian density to account for it:

$$\mathcal{L}(x^\mu, A_\mu, B_{\mu\nu}) = \tfrac{1}{2}\overline{\mu}(B,B) + A(\mathbf{I}) = \tfrac{1}{2}\overline{\mu}_{ij}B^iB^j + A_iI^i; \tag{X.87}$$

one can also express the field-source coupling term in the form:

$$A \wedge \#\mathbf{I} = A(\mathbf{I})\,\mathcal{V}. \tag{X.88}$$

The field equations now include a contribution from:

$$\frac{\partial\mathcal{L}}{\partial A} = \mathbf{I}, \tag{X.89}$$

namely:

$$B = dA, \qquad \delta\mathbf{H} = \mathbf{I}, \qquad \mathbf{H} = \tilde{\mu}(B), \tag{X.90}$$

or:

$$\mathbf{B} = \delta\mathbf{A}, \qquad dH = \#\mathbf{I}, \qquad H = \tilde{\mu}(\mathbf{B}). \tag{X.91}$$

The first set consolidates into a second-order equation for A:

$$\Delta_\mu A = \mathbf{I} \tag{X.92}$$

that has a generalized Poisson type.

c. Electromagnetism. When one puts electricity and magnetism together into a four-dimensional object, namely a 2-form F, the predictable form for a sourceless quadratic Lagrangian density is:

$$\mathcal{L}(x^\mu, F_{\mu\nu}) = \tfrac{1}{2}\overline{\kappa}(F,F) = \tfrac{1}{4}\overline{\kappa}^{\kappa\lambda\mu\nu}F_{\kappa\lambda}F_{\mu\nu}. \tag{X.93}$$

The excitation bivector field $\mathfrak{H}$ that is canonically conjugate to the 2-form F is then:

$$\mathfrak{h} = \kappa(F), \qquad (\mathfrak{h}^{\kappa\lambda} = \tfrac{1}{2}\,\kappa^{\kappa\lambda\mu\nu}F_{\mu\nu}) \tag{X.94}$$

in which:

$$\kappa^{\kappa\lambda\mu\nu} = \bar{\kappa}^{\kappa\lambda\mu\nu} + \frac{\partial \bar{\kappa}^{\kappa\lambda\mu\nu}}{\partial F_{\alpha\beta}} F_{\alpha\beta}\,. \tag{X.95}$$

As usual, this means that κ coincides with $\bar{\kappa}$ only in the case of a linear constitutive law.

The resulting field equations are then:

$$B = dA, \qquad \delta\mathfrak{h} = 0, \qquad \mathfrak{h} = \kappa(F), \tag{X.96}$$

which consolidate into a second-order equation for A:

$$\Box_\kappa A = (\delta \cdot \kappa \cdot d)A = 0 \tag{X.97}$$

that represents a generalized wave equation.

The coupling of a source current $\mathbf{J} = \rho\partial_t + \mathbf{I}$ to the field A proceeds in a manner that is analogous to the previous discussions in the electrostatic and magnetostatic cases. That is, the extra term in the Lagrangian that couples the energy of the source current to the energy of the field takes the form:

$$A \wedge \#\mathbf{J} = A(\mathbf{J})\,\mathcal{V} = \rho\phi\,\mathcal{V} + A_s \wedge \#\mathbf{I} = (\rho\phi + A_s(\mathbf{I}))\mathcal{V}, \tag{X.98}$$

which is then simply the sum of the electrostatic and magnetostatic terms.

The generalized force that appears in the field equations as a forcing term is then the charge flux 3-form:

$$\frac{\partial(A \wedge \#\mathbf{J})}{\partial A} = \#\mathbf{J}\,, \tag{X.99}$$

which makes the field equations take the form:

$$F = dA, \qquad \delta\mathfrak{h} = \mathbf{J}, \qquad \mathfrak{h} = \kappa(F), \tag{X.100}$$

or, in second-order form:

$$\Box_\kappa A = \mathbf{J}, \tag{X.101}$$

which is then a forced wave equation.

4. Motion of a charge in an electromagnetic field. The problem of determining the motion of a massive charge distribution ρ in the presence of an external electromagnetic field F brings one into the domain of mechanics rather than field theory. However, the formulation of the calculus of variations in terms of jet manifolds is sufficiently general in its application that one can adapt the methodology that was

presented above in the context of field theory to the context of mechanics. Indeed, one can treat both pointlike matter and continuously-extended matter within that formalism.

The basis for the external field approximation in this case is the assumption that the presence of ρ, which necessarily represents the source for a field of its own, does not affect the field F, except by linear superposition. Some situations in which this would break down might take the form of either nonlinear superposition, such as the combined field having a super-critical field strength in the eyes of the polarization of the medium or the cases in which the source **J** of the field F changes state, in some sense (e.g., position, shape, total charge) as a result of the presence of ρ.

First, we shall formulate the mechanical model for pointlike charged matter and then we shall discuss the issues that are associated extending the formalism to extended charged matter.

When a charge distribution is pointlike its support is along a smooth curve x: $[0, 1] \to M$. For the moment, rather than representing the distribution as a Dirac delta distribution centered on the points of x, we shall think of it as simply a real number q that is associated with γ; this also reduces the scope of the discussion to time-invariant charges. Similarly, we think of the rest mass of the point-particle as a positive real number m_0, rather than another delta distribution, and restrict the scope to the time-invariant case.

a. Variational formulation of point mechanics. Just as 1-jets of sections of vector bundles over M were the fundamental objects in the variational formulation of field theory, the 1-jets of curves in M are the fundamental objects in the variational formulation of point mechanics. However, one should note that there is certain simplification associated with point matter in the fact that the definition of the 1-jet $j^1_\tau x$ of a curve $x(\tau)$ in M at a point $x \in M$ reads the same way that the definition of the tangent vector to the curve at that point did, namely, the equivalence class of all differentiable curves though x that have the same first derivative – i.e., velocity – at that point.

We denote the manifold of all 1-jets of differentiable curves in M by $J^1(\mathbb{R}, M)$. Its source projection is $J^1(\mathbb{R}, M) \to \mathbb{R}$, $j^1_\tau x \mapsto \tau$, its target projection is $J^1(\mathbb{R}, M) \to M$, $j^1_\tau x \mapsto x$, and the projection $J^1(\mathbb{R}, M) \to \mathbb{R}\times M$, $j^1_\tau x \mapsto (\tau, x)$ plays a role that is analogous to the projection $J^1E \to E$ in the case of vector bundle. Indeed, one can regard a curve as a section of the (trivial) projection $\mathbb{R}\times M \to \mathbb{R}$, $(\tau, x) \mapsto \tau$ and then regard mechanics in general as a special case of a field theory.

Since the manifold $\mathbb{R}$ is contractible, the manifold $J^1(\mathbb{R}, M)$ is diffeomorphic to $\mathbb{R}\times T(M)$. Hence, a local coordinate chart on $J^1(\mathbb{R}, M)$ takes the form (τ, x^μ, v^μ), so a local section s: $\mathbb{R} \to J^1(\mathbb{R}, M)$ of the projection $J^1(\mathbb{R}, M) \to \mathbb{R}$, takes the local form $(\tau, x^\mu(\tau), v^\mu(\tau))$. One can also think of such a section as a differentiable curve in $T(M)$ that projects to the curve $x(\tau)$.

A section s of the latter projection is *integrable* iff it is the *1-jet prolongation* j^1x of a differentiable curve in M. In that case, it locally looks like $(\tau, x^\mu(\tau), v^\mu(\tau))$, in which the integrability condition is:

$$v^\mu(\tau) = \left.\frac{dx^\mu}{d\tau}\right|_\tau . \tag{X.102}$$

A (first-order) *Lagrangian* is a differentiable function $\mathcal{L}$ on $J^1(\mathbb{R}, M)$, which then takes the local form $\mathcal{L}(\tau, x^\mu, v^\mu)$. It defines an action functional on differentiable curves in M in the predictable way:

$$S[\gamma] = \int_\gamma \mathcal{L}(j^1\gamma)\, d\tau = \int_\gamma \mathcal{L}(\tau, x^\mu(\tau), \dot{x}^\mu(\tau))\, d\tau . \tag{X.103}$$

A *variation* δx of a curve x: $[0, 1] \to M$ is a vector field δx: $[0, 1] \to T(M)$, $\tau \mapsto \delta x(x(\tau))$ along that curve. One can think of it as the restriction to x of a vector field that is defined by differentiating a differentiable homotopy – i.e., a finite variation – of x in M. Its *prolongation* to a vector field $\delta^1 x$: $[0, 1] \to J^1(\mathbb{R}, M)$ has the local form:

$$\delta^1 x(\tau) = \delta x^\mu(\tau)\frac{\partial}{\partial x^\mu} + \left.\frac{d(\delta x^\mu)}{d\tau}\right|_\tau \frac{\partial}{\partial v^\mu} . \tag{X.104}$$

Note, in particular, that it does not have a component in the direction $\partial/\partial\tau$, which would amount to a variation of the curve parameterization.

The *first variation* of $S[.]$ is a linear functional $\delta S[X]$ on vector fields X on $J^1(\mathbb{R}, M)$ that is defined by:

$$\delta S[X] = \int_\gamma \mathrm{L}_X(\mathcal{L}\, d\tau) = \int_\gamma [(i_X d\mathcal{L})\, d\tau + \mathcal{L}\, di_X d\tau] , \tag{X.105}$$

analogously to (X.47).

Now:

$$d\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\tau}d\tau + \frac{\partial\mathcal{L}}{\partial x^\mu}dx^\mu + \frac{\partial\mathcal{L}}{\partial v^\mu}dv^\mu , \tag{X.106}$$

so when $X = \delta^1 x$, one has:

$$i_{\delta^1 x} d\mathcal{L} = \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial v^\mu}\frac{d(\delta x^\mu)}{d\tau} . \tag{X.107}$$

By the usual product rule ("integration by parts") trick, this takes the form:

$$i_{\delta^1 x} d\mathcal{L} = \frac{\delta\mathcal{L}}{\delta x^\mu}\delta x^\mu + \frac{d}{d\tau}\left(\pi_\mu \delta x^\mu\right) , \tag{X.108}$$

in which:

$$\frac{\delta \mathcal{L}}{\delta x^\mu} = \frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial x^\mu} \tag{X.109}$$

are the components of the variational derivative of $\mathcal{L}$ with respect to x^μ and:

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial v^\mu} \tag{X.110}$$

are the components of the *generalized momentum* 1-form that is canonically conjugate to the velocity vector field.

Hence, by an application of Stokes's theorem, the first variation of $S[.]$ in the direction δx becomes:

$$\delta S[\delta x] = \int_\gamma \left(\frac{\delta \mathcal{L}}{\delta x^\mu} \delta x^\mu \right) d\tau + \left[\pi_\mu \delta x^\mu \right]_{\tau=0}^{\tau=1}. \tag{X.111}$$

There are two basic variational problems in which the bracketed term vanishes:

1. *Fixed endpoint problems:* One considers only those differentiable curves between two fixed endpoints A and B, which then implies that $\delta x(A)$ and $\delta x(B)$ vanish.

2. *Variable endpoint-problems.* In this case, in order for the bracketed term to vanish one must restrict oneself to variations δx that satisfy the *transversality condition* that $\pi_\mu \, \delta x^\mu = 0$ at the endpoints of the curves considered.

In either case, the necessary and sufficient condition that a given curve γ be an *extremum* of the action functional – i.e., that $\delta S[\delta x]$ vanish for all allowable variations δx of γ– is that it satisfy the *Euler-Lagrange equations:*

$$\frac{\delta \mathcal{L}}{\delta x^\mu} = 0. \tag{X.112}$$

Although this condition is necessary for the extremum to be a local minimum of the action function, it is not sufficient. In order to find sufficient conditions, one must consider of the second variation of $\mathcal{L}$, which we shall not go into here.

Since the manifold $\mathbb{R}$ is one-dimensional, the Euler-Lagrange equations for point mechanics represent a system of ordinary differential equations for the curve γ. If one introduces the *generalized force* 1-form f on $J^1(\mathbb{R}, \mathrm{M})$:

$$f = \frac{\partial \mathcal{L}}{\partial x^\mu} dx^\mu \tag{X.113}$$

and sets $\pi = \pi_\mu \, dx^\mu$ then the Euler-Lagrange equations take the generalized Newtonian form:

$$f = \frac{d\pi}{d\tau}, \tag{X.114}$$

which can also be regarded as the "strong" form of the conservation law for linear momentum. (The "weak" form is Newton's first law, which says that when f vanishes π is constant along the extremal.)

b. Legendre transform. Since we have already defined equations of motion in terms of the dispersion polynomial $P[k]$ on the cotangent bundle, namely, the bicharacteristic equations, we need to relate those equations to the equations of motion for points that we just obtained for an arbitrary Lagrangian $\mathcal{L}$ on the tangent bundle. This is what the Legendre transformation accomplishes, although, as we shall see, when one does not use a quadratic dispersion polynomial, the form of the resulting Lagrangian is not quite as convenient.

Although we could have introduced the present topic above in the more general context of variational field theory, the resulting formalism seems more intuitively appealing in the context of point mechanics, in which the base manifold of the bundle in question is simply $\mathbb{R}$, so the bundle becomes the trivial one $\mathbb{R} \times M \to \mathbb{R}$, and the jet manifolds $J^1(\mathbb{R}, M)$ and $J^1(M, \mathbb{R})$ reduce to $\mathbb{R} \times T(M)$ and $T^*M \times \mathbb{R}$.

The first step in defining this transformation is to return to the association of tangent covectors with tangent vectors that one obtains from starting with a Hamiltonian function H: $T^*M \to \mathbb{R}$, which we assume to be continuously differentiable, and which was described in Chapter VIII in the section bicharacteristics. The differential dH defines a characteristic (i.e., Hamiltonian) vector field X_H by the process that was described in Chapter VIII. Any covector $k_x \in T^*M$ is then associated with a tangent vector $X_H(k_x)$ on T^*M, which then projects to a tangent vector $\mathbf{v}_x$ to M. This association of each $k_x \in T^*M$ with a corresponding $\mathbf{v}_x \in T(M)$ then defines a map $\imath_H : T^*M \to T(M)$, which we assume to be a diffeomorphism of each fiber of T^*M over $x \in M$ to the corresponding fiber T_xM. By the inverse function theorem, this implies that the differential map $d\imath_H|_{(x, k)}$ must be invertible at every point of T^*M.

Locally, the fiber map takes the form:

$$v^\mu(k_\nu) = \left.\frac{\partial H}{\partial k_\mu}\right|_{(x,k)} \tag{X.115}$$

at each x, so the local diffeomorphism constraint says that:

$$\left.\frac{\partial v^\mu}{\partial k_\nu} = \frac{\partial^2 H}{\partial k_\mu \partial k_\nu}\right|_{(x,k)} \tag{X.116}$$

must be an invertible matrix at each x.

By assumption, one can invert the relationship (X.115) to obtain $k = k(\mathbf{v})$, and one defines a Lagrangian on $T(M)$ by means of the classical Legendre transformation:

$$\mathcal{L}(x, \mathbf{v}) = k(\mathbf{v})(\mathbf{v}) - H_x(k(\mathbf{v})); \tag{X.117}$$

in local coordinate form this is:

$$\mathcal{L}(x^\mu, v^\mu) = k_\mu(\mathbf{v})\, v^\mu - H(k(\mathbf{v})). \tag{X.118}$$

The variational derivative of $\mathcal{L}$ with respect to x takes the local form:

$$\frac{\delta \mathcal{L}}{\delta x^\mu} = f_\mu - \frac{d\pi_\mu}{d\tau}, \tag{X.119}$$

in which:

$$f_\mu = \frac{\partial \mathcal{L}}{\partial x^\mu} = -\frac{\partial H}{\partial x^\mu}, \qquad \pi_\mu = k_\mu + \frac{\partial k_\nu}{\partial x^\mu}\left(v^\nu - \frac{\partial H}{\partial k_\nu}\right). \tag{X.120}$$

Hence, when one assumes the validity of the canonical equations for H, one finds that:

$$\pi_\mu = k_\mu, \qquad \frac{\delta \mathcal{L}}{\delta x^\mu} = 0, \tag{X.121}$$

and conversely; i.e., the Hamiltonian equations for H are equivalent to the Euler-Lagrange equations for $\mathcal{L}$ with the constraint that $\pi_\mu = k_\mu$.

Now, let us examine the form that the Legendre transformation takes when our Hamiltonian is defined by a homogeneous quartic polynomial:

$$P[k] = \tfrac{1}{4} P^{\kappa\lambda\mu\nu} k_\kappa k_\lambda k_\mu k_\nu. \tag{X.122}$$

The map $\imath_P : T^*M \to T(M)$, $k \mapsto \mathbf{v}(k)$, then has the local form:

$$v^\nu(k) = \frac{\partial P}{\partial k_\nu} = P^{\kappa\lambda\mu\nu} k_\kappa k_\lambda k_\mu. \tag{X.123}.$$

As we pointed out in the previous chapter, whereas this system of algebraic equations would be merely linear in the case of a quadratic $P[k]$, we see that we are presently concerned with a system of *cubic* equations, which implies that the very business of inverting them, which is unavoidable if one is to pull P over from T^*M to $T(M)$, is likely to be considerably more computationally involved than a simple matrix inversion. In particular, the inverse map i_P^{-1}, which we assume exists, must be homogeneous of degree 1/3, so it is not a polynomial map.

Locally, the invertibility of i_P means that the matrix:

$$\frac{\partial v^\nu}{\partial x^\mu} = 3P^{\kappa\lambda\mu\nu} k_\kappa k_\lambda \equiv 3\, \gamma^{\mu\nu}(x, k) \tag{X.124}$$

must be invertible for each (x, k). One sees that this matrix also represents the local components $\partial^2 P / \partial k_\mu \partial k_\nu$.

Once the invertibility assumption has been made, it is a simple matter to define a function Q: $T(M) \to \mathbb{R}$ by means of the inverse map $\imath_P^{-1}$: $T^*M \to T(M)$, $\mathbf{v} \mapsto k(\mathbf{v})$ by way of:

$$Q[\mathbf{v}] = P[k(\mathbf{v})]; \tag{X.125}$$

however, the specific nature of the resulting function is more involved than in the case where $\imath_P$ is linear on the fibers.

In particular, since the map $\imath_P$ is homogeneous of degree three, its inverse is homogeneous of degree 1/3. As the degree of P is four, this means that the degree of homogeneity of Q is 4/3:

$$Q[\lambda\mathbf{v}] = P[k(\lambda\mathbf{v})] = P[\lambda^{1/3}k(\mathbf{v})] = \lambda^{4/3}\, P[k(\mathbf{v})] = \lambda^{4/3}\, Q[\mathbf{v}]. \tag{X.126}$$

c. Variational formulation of the moving point charge problem. It is in attempting to define a Lagrangian for the motion of a charged mass point in an external electromagnetic field F that we find that reconsidering the role of a spacetime metric is also a crucial issue in mechanics, as well as electromagnetism. The key question is whether one can define the kinetic energy of motion in the absence of a metric. This is equivalent to the problem of how to associate a momentum 1-form $p(\tau)$ with the velocity vector field $\mathbf{v}(\tau)$ along a curve, since the kinetic energy will be proportional to $p(\mathbf{v})$.

Ultimately, the issue is again one of duality, in which the velocity and momentum have to be related to each other by a *mechanical constitutive law* in the same way that F and $\mathfrak{h}$ are related by an electromagnetic one and stress is related to strain by a different sort of mechanical constitutive law.

As long one deals with canonical momentum π the fact that its components are functions $\pi_\mu(\tau, x^\mu, v^\mu)$ suggests that the mechanical constitutive law is built into the form of π, or equivalently, the Lagrangian $\mathcal{L}$ that defines it. This is analogous to the way that one could define the constitutive law that coupled F to $\mathfrak{h}$ directly or indirectly by way of the Lagrangian density for F.

However, for the sake of computations in particular cases one still needs to be more specific about the form of either $\mathcal{L}$ or π and if one does not introduce a way of associating tangent vectors with covectors at some point then one gets into a vicious logical cycle by trying to avoid it. Hence, we shall assume that our manifold M is associated with an electromagnetic constitutive law, with a corresponding dispersion polynomial $P[k]$, and use the invertible map i_P: $T^*M \to T(M)$, $k \mapsto \mathbf{v}(k)$ that it defines to facilitate the definition of a Legendre transformation.

Under this transformation, the Hamiltonian $H(x, p)$ on T^*M will go to the Lagrangian:

$$\mathcal{L}(x, v) = p(\mathbf{v})(\mathbf{v}) - H(x, p(\mathbf{v})). \tag{X.127}$$

Previously, we associated frequency-wave number 1-forms with velocity vector fields. However, in order to discuss point mechanics, we must associate velocity fields along curves with energy-momentum 1-forms along those curves. Since we are dealing with the motion of a *point* particle the main problem conceptually is how to associate a frequency-wave number 1-form k with the particle when the electromagnetic field, being external, is essentially passive to the problem. We then see that, in a sense, the point particle approximation is too much of an approximation to suggest a notion of an associated wave covector field, in much the same way that point particles make the definition of angular momentum more involved than when one is dealing with extended matter.

Since the point particle concept is essentially classical – i.e., pre-quantum – we shall resolve the issue by assuming that the basic classical data of a rest energy E_0 and an energy-momentum covector field p are given and have their support on the curve of the particle. The wave covector k is then related to the energy-momentum covector field p by the de Broglie relation $p = \hbar k$, while the rest energy is related to a rest frequency ω_0 by $E_0 = \hbar\omega_0$. Furthermore, the energy-momentum covector field p satisfies the inhomogeneous characteristic equation:

$$P[p] = \hbar^4 P[k] = E_0^4 = \hbar^4 \omega_0^4; \tag{X.128}$$

hence, k satisfies:

$$P[k] = \omega_0^4. \tag{X.129}$$

In effect, the fact that the rest energy is non-vanishing has changed the nature of the dispersion law from the homogeneous one that relates to photons into the inhomogeneous one that relates to massive matter.

From the fact that the form of H is algebraically simpler than that of $\mathcal{L}$ when one is not dealing with quadratic polynomials – viz., $P[k]$ is a polynomial, while $Q[\mathbf{v}]$ is not, – one finds that it is mathematically more straightforward to deal with the Hamiltonian form of the geodesic equations, rather than their Lagrangian form.

We start with the fact that a pre-metric way of expressing kinetic energy is:

$$T = \tfrac{1}{d}\, p(\mathbf{v}), \tag{X.130}$$

in which d represents the degree of homogeneity of the function $p(x, \mathbf{v})$ in $\mathbf{v}$.

This expression can be interpreted in either the Lagrangian formalism or the Hamiltonian one.

In the Lagrangian formalism one must solve for the 1-form p as a function of the vector field $\mathbf{v}$:

$$T(x, v) = \tfrac{1}{d}\, p(\mathbf{v})(\mathbf{v}) = \tfrac{1}{d}\, p_\mu(x, v) v^\mu. \tag{X.131}$$

However, since it is $\mathbf{v} = \mathbf{v}(p)$ that is given by the elementary expression – namely, a system of homogeneous cubic equations – one sees that the inverse system will not generally admit such an elementary form.

Conversely, in the Hamiltonian formalism one must solve $\mathbf{v}$ for p:

$$T(x, p) = \tfrac{1}{d}\, p(\mathbf{v}(p)) = \tfrac{1}{d}\, p_\mu\, v^\mu(x, p). \tag{X.132}$$

Hence, since the equations that take p to $\mathbf{v}$ have the more elementary form in this case, one would regard the Hamiltonian formalism as being computationally preferable.

Now, let us contrast that with the situation regarding potential energy, which has the general form:

$$U = A(\mathbf{J}) = qA(\mathbf{v}). \tag{X.133}$$

One immediately sees that this expression has a distinctly Lagrangian character, since it depends most directly upon the velocity vector field. The local functional dependencies are then:

$$U(x, v) = qA_\mu(x)\, v^\mu(x). \tag{X.134}$$

In the Hamiltonian formalism, however, one must solve $\mathbf{v}$ for p:

$$U(x, p) = qA_\mu(x)\, v^\mu(x, p). \tag{X.135}$$

Although the map that takes $\mathbf{v}$ to p is not as conveniently represented as the inverse map, we shall present the derivation of the equations of motion for a charged mass point in an external electromagnetic field in both of the aforementioned formalisms in order to see how the change from a quadratic dispersion law to a quartic one affects the equations of motion.

In Lagrangian formalism, the Lagrangian has the form $\mathcal{L} = T - U$ and the equations of motion take the form:

$$\frac{\delta T}{\delta x^\mu} = \frac{\delta U}{\delta x^\mu}. \tag{X.136}$$

By substituting for T as in (X.131) and U as in (X.134), this becomes:

$$\frac{1}{d}\left[\frac{dp_\mu}{d\tau} - \frac{\partial p_\nu}{\partial x^\mu} v^\nu + \frac{\partial^2 p_\nu}{\partial x^\kappa \partial v^\mu} v^\kappa v^\nu + \frac{\partial^2 p_\nu}{\partial v^\kappa \partial v^\mu}\frac{dv^\kappa}{d\tau} v^\nu + \frac{\partial p_\nu}{\partial v^\mu}\frac{dv^\nu}{d\tau}\right] = F_{\mu\nu} J^\nu. \tag{X.137}$$

In the quadratic case, for which $d = 2$ and $p_\mu = m_0\, g_{\mu\nu} v^\nu$, one has:

$$\frac{\partial p_\nu}{\partial x^\mu} = m_0\, g_{\kappa\nu,\mu}\, v^\kappa, \qquad \frac{\partial p_\nu}{\partial v^\mu} = m_0\, g_{\mu\nu}, \qquad \frac{\partial^2 p_\nu}{\partial x^\kappa \partial v^\mu} = m_0\, g_{\kappa\nu,\mu}, \qquad \frac{\partial^2 p_\nu}{\partial v^\kappa \partial v^\mu} = 0, \tag{X.138}$$

and one finds that the equations of motion can be put into the form:

$$\nabla_{\mathbf{v}}\, p_\mu = F_{\mu\nu} J^\nu. \tag{X.139}$$

As we pointed out, when one chooses a quartic polynomial for $P[p]$, so $p(x, v)$ is homogeneous of degree 1/3 in v, there is no simplification of the left-hand side of (X.137).

Under a Legendre transform, a Lagrangian of the form $T(x, \mathbf{v}) - U(x, \mathbf{v})$ does not generally go to a Hamiltonian of the form $T(x, p) + U(x, p)$, so in order to get some idea of how to incorporate the coupling of the charge to the external field, we first look at the result of applying a Legendre transform to the quadratic Lagrangian of that form that pertains to a Lorentzian manifold, namely:

$$\mathcal{L}(x, v) = \tfrac{1}{2} m_0 \, g_{\kappa\lambda}(x) \, v^\kappa v^\lambda - qA_\nu(x) \, v^\nu. \tag{X.140}$$

We first find that the canonical momentum takes the form:

$$p_\mu = \frac{\partial \mathcal{L}}{\partial v^\mu} = m_0 \, g_{\mu\nu} \, v^\lambda - qA_\mu. \tag{X.141}$$

Hence, the Legendre transformation produces a Hamiltonian of the form:

$$H(x, p) = \tfrac{1}{2m_0} g^{\mu\nu}(x)(p_\mu + qA_\mu)(p_\nu + qA_\nu). \tag{X.142}$$

This exhibits the "minimal electromagnetic coupling" aspect of energy-momentum, namely, that in the Hamiltonian formalism the coupling of the free charge to the external field is by way of replacing the free particle energy-momentum 1-form p with $\tilde{p}_\mu(x, p) = p + qA$. One can also think of this as the Fourier transform of the replacement of the partial derivative ∂_μ with the covariant derivative $\partial_\mu - iqA_\mu$, in which A plays the role of the $U(1)$ connection form.

One then verifies that the canonical equations for this H can be reduced to the form:

$$m_0 \, g_{\mu\nu} \nabla_{\mathbf{v}} v^\nu = F_{\mu\nu} \, v^\nu. \tag{X.143}$$

In the quartic case, a minimal coupling of A to the energy-momentum of the particle by the replacement of p_μ with $\tilde{p}_\mu(x, p)$ gives the Hamiltonian:

$$H(x, p) = \tfrac{1}{4} P^{\kappa\lambda\rho\sigma}(x) \tilde{p}_\kappa \tilde{p}_\lambda \tilde{p}_\rho \tilde{p}_\sigma. \tag{X.144}$$

The canonical equations initially take the form:

$$\frac{dx^\mu}{d\tau} = v^\nu = P^{\kappa\lambda\rho\mu} \tilde{p}_\kappa \tilde{p}_\lambda \tilde{p}_\rho, \tag{X.145a}$$

$$\frac{dp_\mu}{d\tau} = -\frac{1}{4} \frac{\partial P^{\kappa\lambda\rho\sigma}}{\partial x^\mu} \tilde{p}_\kappa \tilde{p}_\lambda \tilde{p}_\rho \tilde{p}_\sigma - qA_{\kappa,\mu} P^{\kappa\lambda\rho\sigma} \tilde{p}_\lambda \tilde{p}_\rho \tilde{p}_\sigma, \tag{X.145b}$$

although with the substitution $p = \tilde{p} - qA$ the second equation becomes:

$$\frac{d\tilde{p}_\mu}{d\tau}+\left(\frac{1}{4}P^{\kappa\lambda\rho\sigma}{}_{,\mu}\tilde{p}_\sigma - qF_{\mu\nu}P^{\kappa\lambda\rho\nu}\right)\tilde{p}_\kappa\tilde{p}_\lambda\tilde{p}_\rho = 0, \qquad \text{(X.146)}$$

and the substitution of (X.145a) in (X.145b) gives:

$$\frac{d\tilde{p}_\mu}{d\tau}+\frac{1}{4}P^{\kappa\lambda\rho\sigma}{}_{,\mu}\tilde{p}_\kappa\tilde{p}_\lambda\tilde{p}_\rho\tilde{p}_\sigma = qF_{\mu\nu}v^\nu . \qquad \text{(X.147)}$$

Although presenting the equations of motion in this form allows us to see how the introduction of an external electromagnetic field affects the free-particle equations of motion, nonetheless, if one wishes to solve for $\tilde{p}$ (if only numerically), which then allows one to derive p and **v** directly, then one must use the form (X.146).

d. Radiation damping. One finds that attempting to include the effects of the radiation reaction on the motion of a point charge in an electromagnetic field is met with the same obstacles as in attempting to account for viscous drag forces in the medium (indeed, some refer to the reaction as *radiation damping*). That is, one is no longer considering a conservative mechanical system, so the very applicability of either Lagrangian or Hamiltonian methods is no longer justified. Basically, it comes down to a question of completeness in the model for the state space of the particle-field system, since the field itself potentially contributes an infinite number of degrees of freedom, which include the radiative modes, just as accounting for the energy that gets dissipated by friction in a mechanical system would imply adding an enormous number of dimensions to the state space to account for the internal motion of the molecules that are absorbing the energy that gets lost.

When one considers an extended charge distribution instead of a pointlike one, one also adds a potential infinitude of degrees of freedom to the state space in the form of the internal degrees of freedom that pertain to the distribution, such as the various deformations, and – more to the immediate point – the wavelike modes. Indeed, the very fact that elementary charges, such as electrons, are associated with a characteristic frequency $\omega_0 = m_e c^2/\hbar$ or wave number, in the form of the Compton wave number $k_0 = \omega_0/c$ (which assumes the Lorentzian dispersion law), suggests that there is some fundamental internal motion going on, probably of the standing-wave variety. The fact that massless waves can only exist as traveling waves with a characteristic speed is undoubtedly closely relevant to that fact.

e. The motion of extended charge distributions. Although it is certainly possible to treat continuum mechanics in general, and continuous distributions of charges, more specifically (see, e.g., [**7**, **8**]), within the framework of the calculus of variations and the geometry of jet manifolds, nevertheless, since our primary focus in this book is on examining the various topics in electromagnetism that can be formulated in a pre-metric manner, we shall not go further in that direction at the moment. Suffice it to say that, rather dealing with anti-symmetric second-rank tensor fields, such as F and $\mathfrak{h}$, one is dealing with symmetric ones, such stress σ and strain e, respectively. However, the

equations of motion can still be put into a form that is analogous to the pre-metric Maxwell equations, by postulating the integrability of the strain, the conservation of energy-momentum for stress, and a mechanical constitutive law that relates stress to strain:

$$e^i{}_j = \tfrac{1}{2}(u^i{}_{,j} + u^j{}_{,i}), \qquad \sigma^j_{i\,,j} = f_i\,, \qquad \sigma^j_i = C(e^i{}_j). \tag{X.138}$$

In these equations, the displacement vector field $\mathbf{u} = u^i\partial_i$, which describes an infinitesimal deformation of a region in a medium, plays a role that is analogous to the potential 1-form in electromagnetism, except that one is symmetrizing the derivative, instead of anti-symmetrizing it. Indeed, although the symmetry of the strain tensor field is based in the symmetry of the metric tensor field in the medium (both deformed and undeformed), the assumption of symmetry of the stress tensor field can be weakened, since it is based in the absence of internal torques or moments existing in the medium. Media in which such internal torques exist are called *Cosserat media* (see, [**9**, **10**]), and constitute an intriguing possibility for modeling the structure of spacetime by analogy, since one is basically looking at objects that are embedded in the bundle of orthonormal frames over a (pseudo-) Riemannian manifold, instead of the manifold itself.

5. Fermat's principle. No discussion of the application of the calculus of variations to electromagnetism is complete without a discussion of Fermat's principle as a basis for geometrical optics [**11-15**]. This is especially true in the present context of pre-metric electromagnetism, since the action functional that Fermat defined for the light rays in an optical medium is crucially related to its electromagnetic constitutive properties, and thus represents a simplification of the problem in its full generality.

Since the space Σ of geometrical optics is two-or-three-dimensional, in order to define the variational formulation of light rays in the manner of Fermat, one must consider the 1-jets of curves in Σ; we denote this fibered manifold by $J^1(\mathbb{R}, \Sigma)$. Fermat's principle is based on the specialization of Hamilton's principle that takes the form of saying that a light ray from point A in Σ to point B is a curve from A to B that minimizes the *elapsed time* dt along the curve. (Note that since light rays cannot be parameterized by proper time, this must be understood to represent an affine parameterization.)

A key to making the transition from spacetime to space, along with the expected Legendre transformation from a Hamiltonian function to a Lagrangian function is to note the local equivalence of $J^1(\mathbb{R}, \Sigma)$ with $PT(M)$ and $J^1(\Sigma, \mathbb{R})$ with $PT^*(M)$. This is easily seen by defining local coordinate systems for each pair of manifolds in the form of (t, x^i, V^i) for the former pair and (x^i, t, n_i) for the latter. Of course, in the latter case the time coordinate t must now be regarded as a differentiable function on an open subset of Σ, not an open subset of M.

We have previously encountered the elapsed-time functional in our discussion of Huygens's principle, where we found that it comes about naturally when one treats the components k_μ of the wave covector $k = \omega dt - k_i\, dx^i$ at any point $x \in M$ of spacetime as the homogeneous coordinates of a point $[k]$ in the projectivized cotangent space $PT^*_x M$,

such that the inhomogeneous coordinates are $n_i = k_i / \omega$. When $M = \mathbb{R}\times\Sigma$, these inhomogeneous coordinates then define a spatial 1-form on Σ by way of:

$$n = n_i \, dx^i, \tag{X.139}$$

and if k were exact – i.e., $k = d\phi = \partial\phi / \partial t \, dt + \partial\phi / \partial x^i \, dx^i$ –then this would mean that:

$$n = -\frac{\partial\phi / \partial x^i}{\partial\phi / \partial t} dx^i = \frac{\partial t}{\partial x^i} dx^i = dt. \tag{X.140}$$

Hence, the integral of n along any curve segment γ will represent the elapsed time that it takes to go from one endpoint to the other.

However, in order to obtain Fermat's principle one must use this spatial path functional to define an *action* functional on curve segments. This then reverts to the problem of defining a Lagrangian function on the jets of curve segments, which would then take the form of a differentiable function $\mathcal{L}(t, x^i, V^i)$, while so far we only have a section of $T^*\Sigma \to \Sigma$, namely, n. What is missing is the input from the dispersion law $P[k]$ on T^*M, which, as we saw, becomes an inhomogeneous polynomial $[P][n]$ on PT^*M that takes the form:

$$[P][n] = P_4[n] + P_2[n] + P_0 \tag{X.141}$$

for electromagnetic waves when one factors the ω out of $P[k]$; of course, the polynomial $[P][n]$ is no longer homogeneous in its independent variables n_i .

Since the Hamiltonian $H(x, k) = 1/4P(x)[k]$ on T^*M is supposed to vanish for the physically meaningful wave covectors, as well as $[P][n]$, any multiplicative factors are superfluous, and we define the Hamiltonian on PT^*M to be:

$$H(x^i, t, n_i) = \tfrac{1}{4}[P](x)[n]. \tag{X.142}$$

In order to make the transition from a Hamiltonian on the contact manifold PT^*M to a Lagrangian on $J^1(\mathbb{R}, \Sigma)$, at least locally, we first perform a Legendre transformation on H, as it is defined on T^*M, to a Lagrangian $\mathcal{L}$ on $T(M)$:

$$\mathcal{L}(x^\mu, v^\mu) = k_\mu(\mathbf{v})v^\mu - H(x^\mu, k_\mu(\mathbf{v})). \tag{X.143}$$

Since H is homogeneous in k of degree four, $k_\mu v^\mu = 4H$ and one can say:

$$\mathcal{L}(x^\mu, v^\mu) = \tfrac{3}{4} k_\mu(\mathbf{v})v^\mu. \tag{X.144}$$

The projection of this onto $PT(M)$ is $\tfrac{3}{4}\omega v^0[1 - n_i(V)V^i]$, and since it must vanish for physically meaningful velocity vectors, the leading scalar factor can be omitted. Similarly, differentiation to obtain the equations of motion makes the unity term irrelevant and the fact that the variational derivative is set to zero also makes the sign of

the second term irrelevant. Ultimately, there is no loss in generality in defining the Lagrangian on $PT(M)$ to be:

$$\mathcal{L}(t, x^i, V^i) = \tfrac{3}{4} n_i(x, V)V^i. \tag{X.145}$$

When one forms the action functional for differentiable curves in Σ, under the assumption that $n_i(x, V)$ is independent of t, one first finds that if the curve parameter is s then $V^i = dx^i / ds$ and:

$$\mathcal{L}(t, x^i, V^i)\, ds = n_i\, dx^i, \tag{X.146}$$

which equals the elapsed-time differential in the integrable case, where $t = t(x^i)$ and $n_i = \partial t / \partial x^i$.

One notes that is it equivalent to first project the Hamiltonian $H(x^\mu, k_\mu)$ on T^*M to a Hamiltonian $H(x^i, t, n_i) = 1/4 n_i V^i(n)$ on PT^*M and then perform a Legendre transformation:

$$\mathcal{L}(t, x^i, V^i) = n_i(V^i)V^i - H(x^i, t, n_i) = \tfrac{3}{4} n_i(V^i)V^i. \tag{X.147}$$

This can be summarized in a commutative diagram:

$$\begin{array}{ccc} T^*M & \longrightarrow & T(M) \\ \downarrow & & \downarrow \\ PT^*M & \longrightarrow & PT(M) \end{array}$$

in which the horizontal arrows are the Legendre transforms and the vertical ones are the projectivizations.

Although we have succeeded in deriving the elapsed-time path functional from the dispersion law in a more general manner than one usually obtains for the quadratic polynomial that gives a Lorentzian metric on spacetime, nonetheless, one immediately finds that it practical calculations, the advantage of this expansion of scope is diminished by the disadvantage that the functions $k_\mu(\mathbf{v})$ and $n_i(\mathbf{V})$ are no longer elementary linear isomorphisms, as they would be for the quadratic case, but a set of four homogeneous functions of degree 1/3 in the former case, and a set of three functions that are not even homogeneous in the latter. Hence, these functions are easier to work with in theory than they are in practice.

The inevitable conclusion seems to be that the formulation of geometrical optics is much more natural and straightforward when one deals with cotangent objects instead of tangent ones. In other words, the wave optics of Huygens is more straightforward than the ray optics of Fermat when one is concerned with dispersion laws that are not quadratic in nature.

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CHAPTER XI

Symmetry and electromagnetism

One of the most fundamental – some would say, *the* most fundamental – ways of formulating the first principles of physics is in terms of conservation laws. One can say that conservation laws can be phrased in a weak form and a strong form, where the weak form is simply an approximate version of the strong form.

The weak form of a conservation law pertains to a system that is closed in the sense of complete – i.e., having no well-defined "external" environment, or, at least no exchange of the physical quantity in question with the external environment. One must understand, though, that the "external" environment might very well be "internal" in nature, such as the unmodeled system modes due to the molecular structure of a material medium. The weak form of the conservation of the physical quantity in question states that its total value over the system is constant in time.

The strong form of a conservation law assumes that there is a well-defined distinction between the internal and external states of the system and an exchange of the physical quantity in question between the two. The strong form of the conservation law then states that the time derivative of the total value of the quantity over the internal states equals the total rate of transfer across the boundary between the internal and external systems. The weak form then represents the approximate form of the strong that one obtains by assuming that the system exterior is negligible, or, at least, the transfer of the physical quantity between the internal and external systems is negative.

In engineering mechanics, the strong form of a conservation law is called a *balance law*. One sees that a conservation law then represents a complete accounting of the *total amount* of something while a balance law represents a complete accounting of its *time rate of change.*

As an example of these two forms of a conservation law, consider Newton's first law of motion, which amounts to the weak form of the conservation of linear momentum: when the total of the external forces that act on a physical system vanishes, its total linear momentum remains constant in time. The second law of motion, by comparison, says that when the total external forces are non-vanishing the time derivative of the total linear momentum equals that resultant of the external forces, which then represents its strong form, namely the balance of linear momentum.

One of the most profound advances to the first principles of theoretical physics was made in the early Twentieth Century by Emmy Noether, who showed that the calculus of variations provides a direct way of associating transformations of the states of physical systems that preserve the action functional for the system with conserved currents – i.e., vector fields with vanishing divergence. The transformations are generally regarded as symmetries of the basic dynamical laws that govern the states of the system. For instance, symmetry under spatial translations is correlated with the conservation of linear momentum, or the vanishing of external forces. Conversely, the breaking of a symmetry of the action functional is then associated with the non-conservation of the quantity it is associated with, and implies a non-vanishing divergence of the vector field; for instance, there would be non-vanishing forces, in the case of linear momentum.

In the context of the sourceless Maxwell equations for electromagnetism, in their metric form, the problem of determining the Noether symmetries, when one formulates the field equations in Lagrangian form, was addressed by Lorentz and Poincaré, who found that the symmetries included the Poincaré group. This result was advanced in a crucial way by Bateman [**1**] and Cunningham [**2**], who found that the complete symmetry group of the action functional was the conformal Lorentz group. The extra symmetries that expansion of scope entailed were the homotheties, which are the non-zero scalar multiplications of the vectors in Minkowski space, and the inversions through the unit proper-time hyperboloid.

However, it was also becoming apparent from the work of Lie, Cartan, Vessiot, Riquier, Janet, and others[1], that not only were there systems of differential equations that could not be given a variational formulation – e.g., any non-conservative system – but there were also symmetries of systems of differential equations that were distinct from the symmetries of an action functional, should it exist. These symmetries came to play an increasingly fundamental role in modern mathematics and physics, especially in the context of nonlinear wave equations.

One of the more definitive applications of this methodology to the case of Maxwellian electromagnetism, in its metric form, was made by Harrison and Estabrook and [**4**]. As far as the author of this book is aware, to date, the only attempt to extend the Harrison-Estabrook results to pre-metric electromagnetism was his own effort [**5**]. Hence, this chapter will amount to an extended treatment of that earlier work, in one sense, while referring some of the details to the previous paper.

In the first section of this chapter, we discuss the basic terminology of Lie groups, Lie algebras, and the action of Lie groups on manifolds, for the sake of conceptual completeness. Then, we show how these notions can be applied to the calculus of variations to deduce Noether's theorem, and give several examples of physical interest. After that, we discuss the broader problem of the symmetries of systems of differential equations, which we finally specialize to the equations of pre-metric electromagnetism.

1. Transformation groups [6-9]. Since the most important role that groups play in physics seems to be in the context of groups of transformations that act on the various configuration manifolds of physics as motions, as well as acting on the fibers of bundles over them as internal symmetries – i.e., changes of gauge – we shall briefly discuss the subject of transformation groups for the sake of completeness in the presentation.

a. Lie groups. Many – but not all – of the groups of interest to physics consist of point sets that are associated with topologies and differential structures that are more interesting than merely the discrete topology and zero-dimensional manifolds. Counterexamples to this claim include the various finite groups that figure in the discussion of crystalline structures, as well as symmetries such as parity reversal, time reversal, and charge conjugation.

When the set underlying a group G is given a topology that is compatible with the group structure, in the sense that the group multiplication and inversion operations are

[1] For a historical survey of contributions to the theory of symmetries of systems of differential equations, one might read the introductory chapter of Pommaret [**3**].

both continuous maps, one calls G a *topological group*. When a topological group G is also given a differential structure – i.e., an atlas of charts with diffeomorphisms for coordinate transformations – that is compatible with the group structure, in the sense that the group operations are now required to differentiable, as well as continuous, that topological group is then called a *Lie group*.

Interestingly, the structures that the Norwegian mathematician Marius Sophus Lie was originally defining were not, in fact, Lie groups, in the sense that we just defined, but *Lie pseudogroups*, which have the more general properties that the multiplication of elements is not always defined, just as the composition of maps is defined only when the image of the first map is a subset of the domain of the second one, and furthermore there is usually more than identity. Actually, this sort of consideration becomes unavoidable when one is dealing with symmetries of systems of differential equations, which we shall touch upon later in this chapter, but for now we shall deal with the more elementary case of Lie groups.

Examples of Lie groups that are of interest to physics usually take the form of subgroups of the general linear group $GL(n; \mathbb{R})$, which consists of all invertible $n \times n$ real matrices or $GL(n; \mathbb{C})$, which consists of all invertible $n \times n$ complex matrices. In either case, the group multiplication is defined by matrix multiplication. When the scalar field is not ambiguous, we shall use the common abbreviation $GL(n)$.

As a topological space, $GL(n; \mathbb{R})$ is neither compact nor connected. Rather, it has two connected components, which correspond to the fact that a non-zero real determinant must be either positive or negative. However, this is no longer true for a non-zero complex determinant, so $GL(n; \mathbb{C})$ is non-compact, but connected.

The way that one defines subgroups of interest to physics is by imposing a condition on the matrices that often takes the form of a set of algebraic equations. Such groups are then referred to as *linear algebraic groups*. Topologically, they will all be closed subsets of $GL(n)$.

When one requires that a matrix A must satisfy the n^{th} degree polynomial equation $\det A = 1$, the subgroup that this defines is the *special linear group*, which is denoted $SL(n; \mathbb{R})$ or $SL(n, \mathbb{C})$, depending upon the field of scalars one is using. Such matrices represent linear transformations of $\mathbb{R}^n$ or $\mathbb{C}^n$ that preserve a choice of volume element. More briefly, one employs the notation $SL(n)$, when there is no risk of confusion. One sees that by demanding that the determinant of A be unity, one selects the identity component of $GL(n)$, so $SL(n)$ is a connected, but non-compact, topological space.

When the algebraic condition is the set of quadratic equations $A^T A = A A^T = I$, where T refers to matrix transposition, one defines either $O(n; \mathbb{R})$ or $O(n, \mathbb{C})$, which are the real and complex *orthogonal groups*, respectively. Such transformations preserve a choice of Euclidian scalar product on $\mathbb{R}^n$ or $\mathbb{C}^n$, resp. Topologically, the manifolds $O(n)$ have two components, since $(\det A)^2 = 1$ for orthogonal A, which makes the two components correspond to the two possibilities $\det A = \pm 1$. They are, moreover, compact in the real case and locally compact in the complex case.

When the scalar product is the Minkowski scalar product on $\mathbb{R}^4$, the orthogonal group that it defines is the *Lorentz group* $O(3, 1)$. As a topological space, $O(3, 1)$ has four components, one of which contains the identity matrix I. The other three components can be obtained by using the non-trivial elements of the finite Abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{I, P, T, PT\}$, where $P(x^0, x^i) = (x^0, -x^i)$ represents spatial parity inversion, $T(x^0, x^i) = (-x^0, x^i)$ represents the inversion of time orientation, and $PT(x^0, x^i) = (-x^0, -x^i) = -(x^0, x^i)$ then represents total parity inversion. The Lorentz group is not compact, but only locally compact. Indeed, as we shall discuss in more detail in a later chapter, the identity component of $O(3, 1)$ is isomorphic to $SL(2; \mathbb{C})$, as well as $SO(3; \mathbb{C})$.

By combining the set of equations that define $O(n)$ with $\det A = 0$, one defines the subgroup $SO(n)$, which is the *special orthogonal group*. Such transformations preserve both the scalar product and the volume element. In the case of Minkowski space, the resulting group is the *special Lorentz group* $SO(3, 1)$. Since this Lie group still has two connected components as a topological space, one can restrict oneself to the component $SO_0(3, 1)$ that contains the identity matrix by further requiring that the transformations preserve the time orientation of a vector in $\mathbb{R}^4$; i.e., the sign of its time component. This subgroup is often called the *proper orthochronous Lorentz group*.

In the case of complex scalars, one can also consider complex conjugation of matrix elements, in addition to transposition, and one then defines the *Hermitian conjugate* of a matrix $A \in GL(n; \mathbb{C})$ by $A^\dagger = (A^T)^* = (A^*)^T$, where * refers to the complex conjugation operator. The subgroup of $GL(n; \mathbb{C})$ that consists of invertible complex $n \times n$ matrices that satisfy the set of quadratic equations $A^\dagger A = AA^\dagger = I$ is then called the *unitary group* and is denoted by $U(n)$. These transformations of $\mathbb{C}^n$ preserve a Hermitian inner product, which differs from the Euclidian one by complex conjugation:

$$(z^i, w^j) = \delta_{ij} z^i \overline{w}^j . \tag{XI.1}$$

One of the properties of this inner product is the fact that (z^i, z^j) is always real, since each term $z^i \overline{z}^i = \| z \|^2$ in the sum is real.

The unitary groups are all compact connected topological groups.

The determinant of any unitary matrix has unit modulus since $1 = \det(A^\dagger A) = (\det A)^* (\det A) = \| \det A \|^2$. When one combines unitarity with the requirement that an element of $U(n)$ must also preserve the volume element on $\mathbb{C}^n$ – i.e., have unit determinant – one defines the subgroup $SU(n)$, which one calls the *special unitary group*. Interestingly, although one starts out by using complex matrices, the resulting differential manifold underlying $SU(n)$ is not a complex manifold, but a real one. In particular, $SU(2)$ is diffeomorphic to a real three-dimensional sphere.

An example of a Lie group that is of interest to physics, but not initially defined as a linear algebraic group, is the conformal Lorentz group $CO(3, 1)$, which consists of all

diffeomorphisms of Minkowski space that preserve the scalar product only up to a positive conformal factor Ω^2:

$$g(A\mathbf{v}, A\mathbf{w}) = \Omega^2\, g(\mathbf{v}.\ \mathbf{w}) = g(\Omega\mathbf{v}, \Omega\mathbf{w}). \tag{XI.2}$$

Although this group contains a linear subgroup – sometimes called the *Weyl group* – that consists of Lorentz transformations multiplied by dilatations, it also includes nonlinear transformations, such as the four-dimensional translation group $\mathbb{R}^4$, which act as affine transformations (but not linear ones), and transformations that represent inversion through the unit hyperboloid $g(\mathbf{v}, \mathbf{v}) = 1$. They take any vector $\mathbf{v}$ that does not lie on the light cone and map it to $g(\mathbf{v}, \mathbf{v})^{-1}\mathbf{v}$. Hence, it will lie on the same line through the origin, although its length will be contracted or expanded to the reciprocal of its length, except for vectors on the unit hyperboloid, which remain fixed.

One can represent the conformal Lorentz group as a linear group by taking advantage of the fact that the set of all light cones at all points of $\mathbb{R}^4$ (i.e., their defining coefficients) can be made into a six-dimensional real vector space that can be given a pseudo-Euclidian structure of signature type (–, –, +, +, +, +). One then finds that $CO(3, 1)$ can be represented by the linear algebraic group $O(2, 4)$.

Because Lie groups are manifolds, among other things, they have tangent spaces at each point, and the tangent bundle $T(G)$ to any Lie group G has the important property that it is always *parallelizable;* that is, there is always a global frame field on any Lie group. This fact follows the fact that if one chooses a frame $\{\mathbf{e}_i,\ i = 1, \ldots, n\}$ in any tangent space – say, the tangent space T_eG to the identity element $e \in G$ – then one can left-translate that frame to every other point $g \in G$ in such a manner that the resulting frame field on G is a differentiable section of the bundle $GL(G) \to G$ of linear frames in $T(G)$. Such a global frame field then allows one to express $T(G)$ as the trivial bundle $G \times \mathbb{R}^n \to G$ and $GL(G) \to G$ as the trivial bundle $G \times GL(n) \to G$.

b. Lie algebras. The tangent space T_eG at the identity plays a special role in the study of Lie groups since one can left-translate (or right translate, for that matter) not only tangent frames, but tangent vectors themselves. The result of choosing a tangent vector to a point of G – say, $\mathbf{v} \in T_eG$ – and left-translating it to every other point $g \in G$ is a *left-invariant vector field* on G:

$$\mathbf{v}(g) = dL_g|_e(\mathbf{v}). \tag{XI.3}$$

Hence, the components of $\mathbf{v}$ relative to a left-invariant frame field will be constant functions of $g \in G$. This means that the vector space of left-invariant vector fields on G is linearly isomorphic to $\mathbb{R}^n$, in general, and T_eG, in particular.

Similarly, one defines *right-invariant vector fields* on G by right-translation.

Since the vector space of vector fields on any differentiable manifold always has a Lie algebra structure that is defined by the Lie bracket of vector fields, one naturally wishes to examine how the constraint of left-invariance relates to that structure. In fact, it is preserved; i.e., the Lie bracket of left-invariant vector fields is left-invariant. One calls

this resulting Lie subalgebra $\mathfrak{G}$ of $\mathfrak{X}(G)$ the Lie algebra of G; it has the same dimension as a vector space as the dimension of G as a manifold. Hence, under the association of left-invariant vector fields with tangent vectors in T_eG one can define a Lie algebra on T_eG, as well.

We use the notations $\mathfrak{gl}(n; \mathbb{R})$, $\mathfrak{gl}(n; \mathbb{C})$, $\mathfrak{sl}(n; \mathbb{R})$, $\mathfrak{sl}(n; \mathbb{C})$, $\mathfrak{so}(n; \mathbb{R})$, $\mathfrak{so}(n; \mathbb{C})$, $\mathfrak{so}(3, 1)$, $\mathfrak{su}(n)$, $\mathfrak{co}(3, 1)$ to denote the Lie algebras that are associated with the Lie groups that were defined above, in an obvious way.

One can, of course, define differentiable curves in any Lie group G, and there is a particular class of curves through the identity element e that plays an essential role in understanding the relationship of G to its Lie algebra $\mathfrak{G}$. A differentiable curve g: $\mathbb{R} \to G$, $s \mapsto g(s)$, through e – say, $g(0) = e$ – is called a *one-parameter subgroup* of G iff $g(s_1 + s_2) = g(s_1)g(s_2)$ for all $s_1, s_2 \in \mathbb{R}$. That is, the map g is a homomorphism of the group $(\mathbb{R}, +)$ with its image in G. Such a subgroup of G will be one-dimensional, and therefore Abelian. As a consequence, it can only be diffeomorphic to S^1, when it is compact, or $\mathbb{R}$, when it is not. Merely starting with a compact Lie group G is not sufficient to restrict this choice, since there are subgroups of the 2-torus $S^1 \times S^1$ that are diffeomorphic to $\mathbb{R}$, namely, ones that give "irrational flows."

When one differentiates the curve $g(s)$ at e one obtains a tangent vector in T_eG:

$$\mathfrak{g} = \left.\frac{dg}{ds}\right|_{s=0}. \tag{XI.4}$$

When $g(s)$ satisfies a set of algebraic conditions that define the subgroup of $GL(n)$ that it lies in this differentiation along curves through the identity also allows us to obtain a corresponding set of algebraic conditions on $\mathfrak{g}$ that defines the Lie subalgebra of $\mathfrak{gl}(n)$ that it lies in. For instance, if $g(s) \in SL(n)$ for all s then $\det g(s) = 1$, so, by differentiation:

$$\left.\frac{d(\det g(s))}{ds}\right|_{s=0} = \mathrm{Tr}\, \mathfrak{g} = 0. \tag{XI.5}$$

That is, the elements of the Lie algebra $\mathfrak{sl}(n)$ consist of traceless $n \times n$ matrices (regardless of the choice of scalar field).

One obtains the algebraic condition for elements of $\mathfrak{so}(n)$ by differentiating the condition $A^TA = I$, which gives:

$$\omega^T + \omega = 0, \tag{XI.6}$$

in which we represent the tangent vector to the identity by ω, this time. This means that the matrices in $\mathfrak{so}(n)$ are all anti-symmetric. This is one reason why it is unnecessary to

distinguish between $\mathfrak{o}(n)$ and $\mathfrak{so}(n)$, since all anti-symmetric matrices have trace zero. One can also recall that $SO(n)$ is the connected component of the identity in $O(n)$.

Similarly, the condition $A^\dagger A = I$ on unitary matrices differentiates to the condition:

$$\omega^\dagger + \omega = 0, \tag{XI.7}$$

on elements of $\mathfrak{su}(n)$; that is, they must be *skew-hermitian.*

A skew-hermitian matrix does not need to have a vanishing trace, since in general its trace will be imaginary. Hence, one must clearly distinguish the Lie algebra $\mathfrak{u}(n)$ from the Lie algebra $\mathfrak{su}(n)$.

One can also take each tangent vector $s\mathfrak{g}$ along the line through the origin of $\mathfrak{G}$ that is generated by $\mathfrak{g}$ and associate it with a group element $g(s)$. The map exp: $\mathfrak{G} \to G$, $\mathfrak{g} \mapsto \exp(\mathfrak{g}) = g(1)$, is called the *exponential map* for G. It does not have to be either one-to-one or onto. In the former case, when a line through the origin of $\mathfrak{G}$ maps to a circle, the map exp will "wrap around" an infinite number of times, while, in the latter case, one finds that the image of exp can only be contained in the identity component of G. Hence, it cannot be surjective on a non-connected Lie group, such as $O(3, 1)$. It is, however, surjective on the identity component, which follows from its path-connectedness.

Because of the existence of this exponential map, one can think of the elements of any Lie algebra $\mathfrak{G} = T_eG$ as being the *infinitesimal generators* of one-parameter subgroups of G. Specifically, $\mathfrak{g} \in \mathfrak{G}$ generates the one-parameter subgroup $\exp(s\mathfrak{g}) \in G$.

When $\mathfrak{g}$ is represented by a matrix, one can use the power series expansion for the function e^x to define the matrix $\exp(\mathfrak{g})$:

$$\exp(\mathfrak{g}) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathfrak{g}^n , \tag{XI.8}$$

in which the product is matrix multiplication. Just as it does for real or complex numbers, this series converges for all $\mathfrak{g}$;

c. Group actions [**10**, **11**]. A Lie group G is said to be a *(differentiable) transformation group* for a differentiable manifold M iff there is a differentiable map $G \times M \to M$, $(g, x) \mapsto gx$, such that:

i) $ex = x$ for all x.

ii) $g(g'x) = (gg')x$, for all $g, g' \in G, x \in M$.

One also says that *G acts on M differentiably.*

As a consequence of the definition, for each $g \in G$, the map L_g: $M \to M$, $x \mapsto gx$ is differentiable and invertible with a differentiable inverse; hence, it is a diffeomorphism. One calls this map L_g *left-translation by g*. From conditions *i*) and *ii*) above, the map L: $G \to \mathrm{Diff}(M)$, $g \mapsto L_g$ is a group homomorphism. It does not have to be faithful, since,

for instance, if the action of G on M fixes every point of M, G will map to the identity diffeomorphism. The question of whether it is a Lie group homomorphism reverts to the question of whether Diff(M) is a Lie group. Since Diff(M) is infinite-dimensional, this is not always true, and one must be more careful about the vector space on which infinite-dimensional manifolds are modeled. One can say that when M is compact Diff(M) is a Banach Lie group; i.e., the manifold charts map to a Banach space and the coordinate transitions are Banach space diffeomorphisms. We shall not elaborate on this, however [2].

If one fixes $x \in M$ then there is a differentiable map $G \to M$, $g \mapsto gx$ whose image $G(x)$ is called the *orbit* of x under the action of G. Again, this map does not have to be one-to-one – i.e., an embedding since more than one element of G might take a give $x \in M$ to the same element. Due to the group structure on G, the question of how many elements of G take a given point x to the same point $y = gx$ reverts to the question of how many elements of G take x to itself. The set $G_x = \{g \in G | \, gx = x\}$ of all elements of G that fix a given $x \in M$ is called the *isotropy subgroup* [3] of x under the given group action. One finds that the coset space G/G_x with the quotient topology can be given a manifold structure that makes it diffeomorphic to the orbit $G(x)$ of x under the action of G. An important example of an isotropy subgroup is $G_x = G$, in which case x is a *fixed point* of the group action.

When the map $G/G_x \to M$ is a diffeomorphism, one calls the manifold M a *homogeneous space*. Since this means that any pair of points (x, y) in $M \times M$ are associated with at least one $g \in G$ such that $y = gx$, one also says that such an action is (multiply) *transitive*.

If $G_x = e$ – i.e., g is unique – then one sometimes hears the action referred to as *simply transitive*. For such a group action, the manifold M must be diffeomorphic to the manifold underlying the Lie group G.

In the multiply transitive case, one sees, by composition, that if $g \in G$ takes x to $y = gx$ then the set of all $g' \in G$ that take x to y is in one-to-one correspondence with either G_x or G_y. In particular, G_x and G_y must be isomorphic as Lie groups.

An elementary example of a homogeneous space is any affine space A^n, which has a simply transitive action of the translation group $\mathbb{R}^n$ on it. The (real or complex) n-sphere is diffeomorphic to the homogeneous space $SO(n+1)/SO(n)$, and the n-torus $T^n = S^1 \times \ldots \times S^1$ is the homogeneous space $\mathbb{R}^n / \mathbb{Z}^n$; in particular, $S^1 = \mathbb{R}/\mathbb{Z}$.

Isotropy subgroups at different points do not have to be isomorphic, although this is true for points on the same orbit. Conversely, if all isotropy subgroups are isomorphic then it does not have to be the case that there only one orbit; one refers to the set of all $x \in M$ that have the same isotropy subgroup, up to isomorphism, as the *stratum* of G_x. If one considers the example of $SO(n)$ acting on $\mathbb{R}^n - \{0\}$ then one sees that all of the isotropy subgroups are isomorphic to $SO(n-1)$, but the orbits are $n-1$-spheres of differing radii. In this example, the stratum of a given isotropy subgroup defines a manifold.

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[2] One might confer such references as [**12**] on this matter.

[3] It is also called the *stability subgroup* and the *little subgroup*.

An important example of a group action that is perhaps the best understood case in the theory of transformation groups, while also being quite pervasive in its application to physics, is that of a *linear representation*. In that case, the manifold is a vector space V and the action $G\times V \to V$ is required to be linear, in that the left-translation operator L_g: $V \to V$ is an invertible linear map for every $g\in G$. As a consequence, the image of the group homomorphism L: $G \to \mathrm{Diff}(V)$ is going to be a subgroup of $GL(V)$. When the kernel of this homomorphism is the identity element in G one calls the representation *faithful*, since the map L then becomes an isomorphism of G with its image in $GL(V)$.

By differentiation at the identity, a representation D: $G \to GL(V)$ gives a representation $\mathfrak{D}$: $\mathfrak{G} \to \mathfrak{gl}(V)$ of the Lie algebra of G in the Lie algebra of $GL(V)$. Hence, if $a, b \in G$ then:

$$[\mathfrak{D}(a), \mathfrak{D}(b)] = \mathfrak{D}[a, b]. \tag{XI.9}$$

When G acts on two differentiable manifolds M and N and one has a differentiable map f: $M \to N$, one might wish to compare the two group actions. The actions are said to be *equivariant* iff f takes every orbit of G in M to a subset of an orbit of G in N. One also says that f *commutes* with the group actions, since equivariance is equivalent to the condition that $gf(x) = f(gx)$ for every $g \in G$ and $x\in M$. Of particular interest is the case in which $N = V$ is a vector space and the action of G on N is linear, so one is considering actions of G on a manifold M that are equivariant to a linear action on some vector space. Actually, there is an equivariant embedding theorem that says that this is *always* possible if one goes to a large enough dimension of V.

Suppose that g: $(-\varepsilon, +\varepsilon) \to G$, $s \mapsto g(s)$ is a smooth curve through the identity, and its tangent vector at the identity is $\mathfrak{g} \in \mathfrak{G}$. If G acts on M then there is a smooth curve $g(s)x$ through each point $x \in M$ such that $g(0)x = x$.

By differentiation, there is then a tangent vector to x:

$$\tilde{\mathfrak{g}}(x) = \left.\frac{d(g(s)x)}{ds}\right|_{s=0}, \tag{XI.10}$$

and the association of x with $\tilde{\mathfrak{g}}(x)$ defines a vector field on M that one calls the *fundamental vector field* that is associated with $\mathfrak{g}$. In fact, the association of $\mathfrak{g}$ with $\tilde{\mathfrak{g}}$ defines a Lie algebra homomorphism $\mathfrak{G} \to \mathfrak{X}(M)$. That is, in any case, one has:

$$\widetilde{[\mathfrak{a},\mathfrak{b}]} = [\tilde{\mathfrak{a}},\tilde{\mathfrak{b}}]. \tag{XI.11}$$

Hence, the group action defines a representation of the Lie algebra $\mathfrak{G}$ in the Lie algebra $\mathfrak{X}(M)$. If this representation is faithful then each generator of $\mathfrak{G}$ – i.e., each member e_i, $i = 1, \ldots, \dim(\mathfrak{G})$ of a basis for $\mathfrak{G}$ – is associated with a fundamental vector field $\tilde{e}_i$ on E and the elements of the set $\{\tilde{e}_i\}$ are linearly independent.

d. Groups acting on vector bundles. Suppose that a Lie group G acts smoothly on the total space to a vector bundle E from the left and commutes with the projection p: $E \to M$. To say that the action commutes with the projection is to say that for each $x \in M$ it takes every element of a given fiber E_x to elements of the same fiber E_y. Hence, there is a well-defined action of G on M by projection: $G \times M \to M$, $(g, x) \mapsto p(g\phi(x))$, where ϕ is any element of E_xM; since the action of G on E commutes with projection, the choice of ϕ is immaterial. One can then express the condition of commutativity in either of the forms $gp = pg$ or $g\phi(x) = \phi(gx)$.

There are essentially two types of actions of G on E: *vertical actions* and *motions*, depending upon whether g takes $\phi \in E_xM$ to another element $g\phi$ of the same fiber or to an element of another fiber E_yM, respectively. The vertical actions then project to the identity on M, while the motions project to non-trivial diffeomorphisms of M.

Since the fibers have a linear structure, one can also further classify vertical actions as linear or nonlinear. In the linear case, one must always have:

$$g(\alpha\phi + \beta\psi) = \alpha\, g(\phi) + \beta g(\psi) \tag{XI.12}$$

for all scalars α, β and elements ϕ, ψ of any fiber of E. This then implies that there is an action of G on the vector space $\Gamma(E)$:

$$g(\phi(x)) = (g\phi)(x). \tag{XI.13}$$

One sees that a linear action of G on a fiber E_x of a vector bundle will be the same thing as a representation of G in $GL(E_x)$. Indeed, the vector bundle E is usually defined in physical applications by first starting with a representation of G in a vector space V and a G-principal bundle $P \to M$ that defines the G-gauge structure on M and obtaining E as the *associated vector bundle* to that G-principal bundle and representation of V. This means that one first forms the manifold $P\times V$ and lets G act on it by taking (p, x) to (pg^{-1}, gx) and defining E to be the orbit space of this action; i.e., each point of E is an orbit of the action of G on $P\times V$. Although this sounds somewhat abstruse, actually, if one lets G be $GL(n)$, P be the bundle $GL(M)$ of linear frames on M, and V be $\mathbb{R}^n$ then the orbits of the action of $GL(n)$ on $GL(M)\times\mathbb{R}^n$ consist of pairs $(\mathbf{e}_i, v^i)$ that all describe the same tangent vector $v^i\, \mathbf{e}_i$; i.e., the associated vector bundle to $GL(M)$ and $\mathbb{R}^n$ is simply $T(M)$.

A typical fundamental vector field for an action of G on E will have the local form:

$$\tilde{\mathfrak{g}} = X_{\mathfrak{g}}^{\ \mu} \frac{\partial}{\partial x^\mu} + X_{\mathfrak{g}}^{\ A} \frac{\partial}{\partial \phi^A}\,. \tag{XI.14}$$

2. Symmetries of the action functional. Now let us apply some of the aforementioned concepts regarding group actions on manifolds and vector bundles to the problems of the calculus of variations.

Let $E \to M$ be a vector bundle upon which the Lie group G acts on the left, so J^1E is the manifold of all 1-jets of sections of E. Although it is possible to define the

prolongation of the action of G on E to an action of G on J^1E, for our purposes it will be sufficient to deal with the infinitesimal case, since we are only dealing with variations of fields [4].

If $\mathfrak{g} \in \mathfrak{G}$ then there is a fundamental vector field $\tilde{\mathfrak{g}}$ on E defined by the group action. One can prolong $\tilde{\mathfrak{g}}$ to a vector field $j^1\tilde{\mathfrak{g}}$ on $J^1(M, E)$ by differentiation. If $\tilde{\mathfrak{g}}$ has the local form that is given by (XI.14) then $j^1\tilde{\mathfrak{g}}$ has the local form:

$$j^1\tilde{\mathfrak{g}} = X_{\mathfrak{g}}^{\mu}\frac{\partial}{\partial x^{\mu}} + X_{\mathfrak{g}}^{A}\frac{\partial}{\partial \phi^{A}} + \frac{\partial X_{\mathfrak{g}}^{A}}{\partial x^{\mu}}\frac{\partial}{\partial \phi^{A}_{\mu}}. \qquad \text{(XI.15)}$$

Furthermore, let there be given an action functional $S[\phi]$ on E that is defined by a Lagrangian density $\mathcal{L}$ on J^1E. Its first variation functional then takes the form:

$$\delta S[X] = \int_{\operatorname{supp}\phi}[i_X d(\mathcal{L}\mathcal{V}) + d i_{X_M}(\mathcal{L}\mathcal{V})] = \int_{\operatorname{supp}\phi}[d\mathcal{L}(X) + \mathcal{L}\operatorname{div}(X_M)]\mathcal{V}, \qquad \text{(XI.16)}$$

in which X_M refers to the part of $X \in \mathfrak{X}(J^1E)$ that projects onto $T(M)$ in a non-trivial way, and we now use the notation div for $\#^{-1}\mathrm{d}\#$, so as not to be confused with the symbol for variation.

Locally it then looks like:

$$X_M = X^{\mu}\frac{\partial}{\partial x^{\mu}}. \qquad \text{(XI.17)}$$

Previously, the variations $\delta\phi$ that we considered for the sake of deriving the Euler-Lagrange equations were necessarily vertical, so this part of X would be zero in such a case.

Now, let us apply the first-variation functional $\delta S[.]$ to the vector field $j^1\tilde{\mathfrak{g}}$ as in (XI.15).

When one evaluates the integrand in (XI.16) on this vector field, one obtains, with some tedious manipulation:

$$d\mathcal{L}(X) + \mathcal{L}\operatorname{div} X_M = \frac{\delta\mathcal{L}}{\delta\phi^{A}}X_{\mathfrak{g}}^{A} + \frac{\partial}{\partial x^{\mu}}\left(\mathcal{L}X_{\mathfrak{g}}^{\mu} + \frac{\partial\mathcal{L}}{\partial\phi^{A}_{\mu}}X_{\mathfrak{g}}^{A}\right). \qquad \text{(XI.18)}$$

If the section ϕ on which $\delta S[\, j^1\tilde{\mathfrak{g}}\,]$ is being evaluated is extremal, moreover, then the first term vanishes and when the first variation functional is evaluated on $j^1\tilde{\mathfrak{g}}$ for an extremal field ϕ one ultimately obtains:

$$\delta S[\, j^1\tilde{\mathfrak{g}}\,] = \int_{\operatorname{supp}\phi}\#\operatorname{div}[\mathbf{J}(\mathfrak{g})] = \int_{\partial\operatorname{supp}\phi}\#\mathbf{J}(\mathfrak{g}), \qquad \text{(XI.19)}$$

[4] More generally, one can confer Saunders [**13**].

in which we have defined the vector field on supp ϕ:

$$\mathbf{J}(\mathfrak{g}) = \left(\mathcal{L} X_{\mathfrak{g}}^{\ \mu} + \frac{\partial \mathcal{L}}{\partial \phi^{A}_{\ \ \mu}} X_{\mathfrak{g}}^{\ A} \right) \frac{\partial}{\partial x^{\mu}} . \tag{XI.20}$$

An element $\mathfrak{g} \in \mathfrak{G}$ is said to be *an infinitesimal symmetry* of the action functional $S[.]$ iff $\delta S[\, j^1 \tilde{\mathfrak{g}}\,]$ vanishes for every extremal section ϕ. Note that this does not imply that the integrand in (XI.19) must vanish identically, since one can also add an exact n-form $d\Lambda$ such that Λ vanishes on $\partial(\text{supp}\ \phi)$ to it without changing value of the integral.

We finally arrive that one of the most pervasive theorems of the variational calculus from the standpoint of modern physics. (Confer any of the variational treatments of electromagnetism, such as [**14-17**].)

Noether's theorem:

If $\mathfrak{g} \in \mathfrak{G}$ is an infinitesimal symmetry of an action functional for an extremal field ϕ then the vector field on M:

$$\mathbf{J}(\mathfrak{g}) = \left(\mathcal{L} X_{\mathfrak{g}}^{\ \mu} + \frac{\partial \mathcal{L}}{\partial \phi^{A}_{\ \ \mu}} X_{\mathfrak{g}}^{\ A} \right) \frac{\partial}{\partial x^{\mu}} \tag{XI.21}$$

has vanishing divergence.

Such a vector field on M is called the *Noether current* that is associated with $\mathfrak{g}$. When every element of $\mathfrak{G}$ is associated with a Noether current, one has a Lie algebra homomorphism $\mathfrak{G} \to \mathfrak{X}(M)$, $\mathfrak{g} \mapsto \mathbf{J}(\mathfrak{g})$. That is, $[\mathbf{J}(\mathfrak{g}_1), \mathbf{J}(\mathfrak{g}_2)] = \mathbf{J}([\mathfrak{g}_1, \mathfrak{g}_2])$.

Now, suppose that G acts on M, as well as on E. Hence, one has a homomorphism L: $G \to \text{Diff}(M)$, $g \mapsto L_g$ of the Lie group G and a homomorphism σ: $\mathfrak{G} \to \mathfrak{X}(M)$, $\mathfrak{g} \mapsto \sigma(\mathfrak{g}) = \tilde{\mathfrak{g}}$ of the Lie algebra $\mathfrak{G}$. If we assume, moreover, that the representation is linear then we can represent the fundamental vector field $\tilde{\mathfrak{g}}$ in local form as:

$$\tilde{\mathfrak{g}} = (\sigma_a^{\mu} \mathfrak{g}^a) \partial_{\mu} , \tag{XI.22}$$

in which the components of the $n \times \dim(\mathfrak{G})$ matrices σ_a^{μ} are smooth function on $U \subset M$.

When the vector field $X_{\mathfrak{g}}$ on E is the push-forward $d\phi(\tilde{\mathfrak{g}})$ of a fundamental vector field $\tilde{\mathfrak{g}}$ on M by a section ϕ: $M \to E$, it takes the local form:

$$X_{\mathfrak{g}} = (\sigma_a^{\mu} \mathfrak{g}^a) \frac{\partial}{\partial x^{\mu}} + \left(\frac{\partial \phi^A}{\partial x^{\mu}} \sigma_a^{\mu} \mathfrak{g}^a \right) \frac{\partial}{\partial \phi^A} ; \tag{XI.23}$$

i.e.:

$$X_{\mathfrak{g}}^{\ \mu} = \sigma_a^{\mu}\mathfrak{g}^a\,, \qquad\qquad X_{\mathfrak{g}}^{\ A} = \frac{\partial \phi^A}{\partial x^{\mu}} X_{\mathfrak{g}}^{\ \mu} = \phi^A{}_{,\mu}\sigma_a^{\mu}\mathfrak{g}^a\,. \tag{XI.24}$$

More generally, the components $X_{\mathfrak{g}}^{\ A}$ also include a purely vertical contribution $\overline{X}_{\mathfrak{g}}^{\ A}$ when G acts linearly on the fibers of E as a structure – or *gauge* – group. There is then a representation D: $G \to GL(V)$, $g \mapsto D(g)$ and a corresponding representation $\mathfrak{D}$: $\mathfrak{G} \to \mathfrak{gl}(V)$, $\mathfrak{g} \mapsto \mathfrak{D}(\mathfrak{g})$. The purely vertical contribution then takes the form:

$$\overline{X}_{\mathfrak{g}}^{\ A} = \mathfrak{D}_a^A\mathfrak{g}^a = X_{\mathfrak{g}}^{\ A} + \phi^A_{,\mu}X_{\mathfrak{g}}^{\ \mu}\,, \tag{XI.25}$$

When we substitute this, along with (XI.24), into (XI.21), we see that the Noether current that is associated with $\mathfrak{g} \in \mathfrak{G}$ has the local form:

$$J^{\mu}(\mathfrak{g}) = \left[T_{\nu}^{\mu}\sigma_a^{\nu} + S_a^{\mu}\right]\mathfrak{g}^a\,, \tag{XI.26}$$

in which:

$$T_{\nu}^{\mu} = \mathcal{L}\delta_{\nu}^{\mu} - \frac{\partial \mathcal{L}}{\partial \phi^A{}_{\mu}}\frac{\partial \phi^A}{\partial x^{\nu}} = \mathcal{L}\delta_{\nu}^{\mu} - \Pi_A^{\mu}\phi^A{}_{,\nu}\,. \tag{XI.27}$$

define the components of the *canonical energy-momentum tensor* on M, and the remaining term in the bracket:

$$S_a^{\mu} = -\,\Pi_A^{\mu}\mathfrak{D}_a^A \tag{XI.28}$$

is referred to as the *spin* tensor.

Hence, we can say that the matrix components of the Noether homomorphism J: $\mathfrak{G} \to \mathfrak{X}(U)$, $\mathfrak{g} \mapsto \mathbf{J}(\mathfrak{g})$, relative to the bases for both vector spaces that have been using, are:

$$J_a^{\mu} = T_{\nu}^{\mu}\sigma_a^{\nu} + S_a^{\mu}\,. \tag{XI.29}$$

3. Examples of Noether symmetries. We shall examine some examples of the Noether representation for various groups of motions that might act on a manifold M and gauge groups that act on the fibers of a vector bundle E.

a. Translations. The first example of the Noether representation that we examine is the case of the translation group acting on M. Of course, when M is not an affine space, one must keep in mind that the very concept of translation has a largely local and approximate character on such a manifold, and that it is more natural to start with local *convections*, which are local diffeomorphisms about its points. This is equivalent to regarding extended matter distributions as physically fundamental, while regarding pointlike matter as an asymptotic approximation that implies a high degree of rigidity to the distribution.

That notwithstanding, let $\mathfrak{G} = \mathbb{R}^n$. The representation $\sigma\colon \mathbb{R}^n \to \mathfrak{X}(U)$ simply takes ε^μ to the local vector field $\varepsilon^\mu \partial_\mu$ on $U \subset M$: hence, $\sigma^\mu_\nu = \delta^\mu_\nu$. One generally does not have the translation group for a manifold acting on the fibers of a vector bundle, although it is reasonable in the case of an affine bundle. Hence, one lets $\mathfrak{D}^A_\mu = 0$.

These substitutions reduce the matrix J^μ_a of the Noether homomorphism to simply T^μ_ν. At each point $x \in U$, the vector field $J^\mu(\varepsilon^\nu) = T^\mu_\nu \varepsilon^\nu$ is then proportional to the stress that acts on a unit area tangent plane to x whose normal points in the direction specified by ε^ν. In the case of a four-dimensional spacetime manifold M, one can also interpret:

$$T^0_0 = \mathcal{L} - \Pi^0_A \phi^A{}_{,0} = -\mathcal{H} \tag{XI.30}$$

as an energy density and:

$$T^i_0 = -\Pi^i_A \phi^A{}_{,0} = -p^i \tag{XI.31}$$

as a spatial momentum flux.

This equivalence of components comes about due to the somewhat coincidental equality of the units for energy density, momentum flux, and stress (i.e., pressure), despite the fact that they describe a 3-form and two 2-forms with values in a vector space, respectively.

The fact that $\mathbf{J}(\varepsilon)$ has vanishing divergence for any ε and the fact that ε represents a constant gives the consequence that:

$$T^\mu_{\nu,\mu} = 0, \tag{XI.32}$$

which often serves as one of the fundamental sets of partial differential equations for continuum mechanics. Since the divergence on the left-hand side represents a generalized force density, one sees that the invariance of the action functional under translations implies the vanishing of all forces in spacetime. By contraposition, one can then say that presence of a force field f_ν in M "breaks" the translation invariance of the action; hence, $T^\mu_{\nu,\mu} = f_\nu$, in that case.

It is traditional at this point to note that since the component matrix T^μ_ν is not generally symmetric one can obtain a symmetric stress-energy-momentum tensor field by the Belinfante-Rosenfeld process. However, as we have defined things, it becomes clear that in a pre-metric formalism, it is meaningless to discuss the symmetry of a second-rank tensor that is of mixed type except as a component matrix when one chooses a frame, but the resulting definition of symmetry is not frame-invariant. One could only define symmetry in a frame-invariant manner by raising or lowering one index, and without a metric to define the isomorphism, the process of raising or lowering indices becomes ill-defined.

It is important to realize that the matrix T^μ_ν does not, by any means, have to be invertible. Indeed, its only non-zero component might very well be a rest energy density for T^0_0. Hence, it does not define a linear isomorphism of tangent vectors at each point, but something more like an infinitesimal deformation of tangent vectors.

Of course, one cannot avoid mentioning that the expression for T_0^0 in (XI.30) bears a less-than-coincidental resemblance to the Legendre transformation by which one associates a Hamiltonian density $\mathcal{H}$ with a Lagrangian density $\mathcal{L}$. Once again, the only way that can single out a specific component of T_ν^μ to represent an energy density is by choosing a time-space splitting of $T(M)$, which is a recurring theme in any relativistic treatment of energy, in general.

b. Rotations. Now suppose we consider only the three-dimensional spatial manifold Σ, with $G = SO(3)$, and we use the basis for $\mathfrak{so}(3)$ that is defined by $[e_i]_k^j = \varepsilon_{ijk}$, $i, j, k = 1, 2, 3$. Hence, an element $\omega \in \mathfrak{so}(3)$ can be represented in the form of the matrix $\omega^j \varepsilon_{ijk}$.

For any coordinate chart (U, x^i), one can represent the basis elements of $\mathfrak{so}(3)$ by the three fundamental vector fields:

$$L_k = \varepsilon_{ijk}\, x^i \partial_j \,, \tag{XI.33}$$

which look like:

$$L_x = y\partial_z - z\partial_y \,, \qquad L_y = z\partial_x - x\partial_z \,, \qquad L_z = x\partial_y - y\partial_x \,, \tag{XI.34}$$

explicitly.

Hence, the fundamental vector field that corresponds to ω is:

$$\begin{aligned} \tilde{\omega} &= \omega^k L_k = \varepsilon_{ijk}\, \omega^k x^i \partial_j \\ &= (\omega_y z - \omega_z y)\, \partial_x + (\omega_z x - \omega_x z)\, \partial_y + (\omega_x y - \omega_y x)\, \partial_z \,, \end{aligned} \tag{XI.35}$$

which has the same form as $\boldsymbol{\omega} \times \mathbf{r}$ from classical rotational mechanics. Hence, the fundamental vector field associated with ω represents the tangential velocity to the circle through each point of U that represents one of the orbits of the one-parameter family of rotations $g(t) = \exp(\omega t)$ that is generated by ω.

From (XI.33), we see that the matrix of the representation σ: $\mathfrak{so}(3) \to \mathfrak{X}(U)$ that we are using is:

$$\sigma_j^i = \varepsilon_{ijk}\, x^k \,. \tag{XI.36}$$

As for the action of $SO(3)$ on the fibers of E, we assume only that it is non-trivial, so the matrix $\mathfrak{D}_j^A$ is non-vanishing. By substituting (XI.36) into (XI.29), we see that the matrix J_j^i of the Noether homomorphism J: $\mathfrak{so}(3) \to \mathfrak{X}(U)$ takes the form:

$$J_j^i = T_k^i \sigma_j^k + S_j^i = T_k^i (\varepsilon_{kjl} x^l) + S_j^i = L_j^i + S_j^i \,, \tag{XI.37}$$

in which:

$$L_j^i = -\tfrac{1}{2} \varepsilon_{jkl} (T_k^i x^l - T_l^i x^k) \tag{XI.38}$$

represents the *orbital angular momentum* part of the homomorphism and:

$$S_j^i = -\,\Pi_A^i \mathfrak{D}_j^A \tag{XI.40}$$

represents the *intrinsic angular momentum* – or *spin* – part of the homomorphism.

It is important to see that the spin does not generally represent a contribution from "internal" orbital angular momenta, such as the rotation of the Earth as it orbits around the Sun, but a contribution that is more related to the nature of the action of the group in question on the fibers of E. In general, when the action is linear, so one is dealing with a representation of the Lie group in the typical fiber of E, the eigenvalues of the first-order differential operators $S_j = S^i_j \partial_i$ will be closely related to the "weight" of the representation. In particular, when ϕ is a scalar field, the weight is zero and S^i_j must vanish.

The extension from $SO(3)$ acting on Σ locally to the special Lorentz group $SO(3, 1)$ acting on M locally is straightforward. In addition to extending the coordinate indices to $\mu = 0, \ldots, 3$, etc., one must add three more generators (i.e., basis elements) K_i, $i = 1, 2, 3$ to the Lie algebra of $\mathfrak{so}(3)$ that represent the infinitesimal boosts in the spatial coordinate directions. Hence, if we index the generators of $\mathfrak{so}(3, 1)$ by $a = 1, \ldots, 6$ then we have to express the matrix of the Noether homomorphism J: $\mathfrak{so}(3, 1) \rightarrow \mathfrak{X}(U)$ in the form J^μ_a that we defined in (XI.29).

c. Homotheties. The group of *homotheties* – or *dilatations*, as they are called in continuum mechanics – is a one-dimensional subgroup of $GL(n)$ for any n that is isomorphic to the multiplicative group $(\mathbb{R}^+, \times)$ of positive real numbers, so all of its elements take the form λI, $\lambda > 0$. Hence, any non-trivial representation of it acting locally on $U \subset M$ would also have to be one-dimensional.

In effect, any vector field $\mathbf{v}$ might serve as a choice for that representation, which means that one can use the intuition gained from the geometrical representation of systems of ordinary differential equations and their singularities narrow down the choice. For one thing, as long as $\mathbf{v}$ has no zeroes, the integral curves of $\lambda\mathbf{v}$ will describe the same points in M; the only possible difference will be in the parameterization. Hence, the interesting case is when $\mathbf{v}$ has zeroes, especially isolated zeroes, such as sources and sinks.

For the representation of homotheties, the single infinitesimal generator of a dilatation centered on $x_0 \in U$, which is a zero of "source" type, is the *radius vector field* that (U, x^μ) defines:

$$\mathbf{r}(x) = x^\mu \partial_\mu . \tag{XI.41}$$

The one-parameter family of dilatations that $\mathbf{r}$ defines on U is then:

$$\exp(\lambda \mathbf{r}(x)) = e^\lambda x^\mu . \tag{XI.42}$$

Since $\mathfrak{G}$ is one-dimensional in this case, the representation matrix σ takes the form $\sigma^\mu = x^\mu$.

The action of $(\mathbb{R}^+, \times)$ on the fibers of any real vector bundle is by scalar multiplication. The fundamental vector field that it generates is a vertical vector field $\mathbf{R}$

on E – (zero section) that is sometimes called the *Liouville vector field* and assigns the radius vector $\mathbf{R}(v)$ – within the fiber – to each $v \in E$; i.e., the displacement in the fiber that takes the origin to v. For a local trivialization $U \times V$ of $E(U)$ it then takes the form:

$$\mathbf{R}(x, v) = v^A \frac{\partial}{\partial v^A}. \tag{XI.43}$$

Hence, the matrix of $\mathfrak{D}$ takes the form $\mathfrak{D}^A = v^A$, and the spin tensor becomes:

$$S^\mu = -\frac{\partial L}{\partial v^A_\mu} v^A. \tag{XI.44}$$

The Noether homomorphism J: $\mathbb{R} \to \mathfrak{X}(U)$, $\lambda \mapsto J(\lambda)$ that is associated with infinitesimal spacetime dilatations then has the components:

$$J^\mu(\lambda) = T^\mu_\nu x^\nu + S^\mu. \tag{XI.45}$$

The vanishing of its divergence implies that:

$$T^\mu_\mu = 0. \tag{XI.46}$$

Hence, when the action functional for a field theory is invariant under dilatation the trace of the energy-momentum tensor must vanish. Since this trace amounts to the rest energy density, the conservation law is basically conservation of mass-energy.

d. Gauge transformations. Now, let us consider a purely vertical action of a group G on the fibers of E, which makes G essentially a *gauge group*. More precisely, one usually has a right action of G on the "associated principal bundle" to E, namely, the bundle $GL(E)$ of all linear frames in the fibers of E. For instance, one often considers the bundles $GL(M)$ of all linear frames in $T(M)$ and $GL^*(M)$, which consists of linear frames in T^*M.

The right action then comes about, in most cases, by representing G as a subgroup of $GL(V)$, where V is the model vector space for the typical fiber of E. That is, one represents $g \in G$ by a matrix g^A_B in $GL(V)$ and its action on a frame e_A in any fiber of E gives the linear frame $e_A \tilde{g}^A_B$, where the tilde again represents the inverse of the matrx.

The action of G on the components of any vector $v = v^A e_A$ in E is then a left action that takes v^A to $g^B_A v^A$. This "contragredient" action of G on frames and components is necessary to insure that the vector v remains an invariant object in the process.

Hence, we shall think of the representation $\mathfrak{D}$: $\mathfrak{G} \to \mathfrak{gl}(V)$ as being more relevant to the action of G on $GL(E)$ than on E itself. However, due to the linear structure on each fiber one can still speak of the direct action of $\mathbb{R}^+$ by scalar multiplication and each fiber

acts on itself by translations; also, there are often various finite groups that can act naturally, such as $\mathbb{Z}_2$ acting as multiplication by ± 1.

When E is a complex vector bundle – so its fibers are complex vector spaces – one can also think of a natural action of $U(1)$ on the fibers as part of complex scalar multiplication, since the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers is isomorphic to $\mathbb{R}^+ \times U(1)$ by polar representation. The subgroup $U(1)$ then consists of the points of the unit circle in $\mathbb{C}$, which are usually represented in the form $e^{i\theta}$.

If the action of G on E is the vertical action of a gauge group on the fibers then the matrix of the Noether homomorphism J: $\mathfrak{G} \to \mathfrak{X}(U)$ reduces to:

$$J_a^\mu = S_a^\mu . \tag{XI.47}$$

In the case of a one-dimensional $\mathfrak{G}$, the $\mathfrak{D}^A$ represent the components of some section of $E \to M$ and the single conserved current associated with any $\lambda \in \mathfrak{G}$ is the vector field whose local components are:

$$\lambda J^\mu = -\lambda \Pi_A^\mu \mathfrak{D}^A . \tag{XI.48}$$

One generally thinks of the conserved quantities that are described by the gauge symmetries of an action functional as "charges." When we treat the case of electromagnetic actions, in particular, we shall see that this interpretation is indeed appropriate, although in the case of gauge groups of dimension greater than one, one must be careful to note that the charge in question does not have to be electric in character. For instance, in quantum chromodynamics, which is regarded as an $SU(3)$ gauge field theory, the charges associated with the eight generators of the Lie algebra of $\mathfrak{su}(3)$ are regarded as "colors."

4. Symmetries of electromagnetic action functionals. We now apply the general methods that we just described to the case of electrostatics, magnetostatics, and electromagnetism, more generally. In each case, we start with the field Lagrangian and compute the resulting matrices in the Noether homomorphism. We particularly examine the eigenvectors and eigenvalues of the energy-momentum-stress tensor.

a. Electrostatics. The Lagrangian density for electrostatics takes the form:

$$\mathcal{L}(x, E) = \tfrac{1}{2} D^i E_i = \tfrac{1}{2} \varepsilon^{ij} E_i E_j . \tag{XI.49}$$

The matrix of the Noether homomorphism for this particular Lagrangian density is:

$$J_a^i = T_j^i \sigma_a^j + S_a^i , \tag{XI.50}$$

with:

$$T^i_j = \mathcal{L}\delta^i_j - \frac{\partial \mathcal{L}}{\partial \phi_{,i}}\phi_{,j} = \tfrac{1}{2}(D^k E_k)\delta^i_j - D^i E_j\,, \qquad S^i_a = -\,D^i\,\mathfrak{D}_a\,. \tag{XI.51}$$

One can express this construction in basis-free form as:

$$T = \mathcal{L}I - \mathbf{D}\otimes E, \qquad S = -\,D\otimes\mathfrak{D}. \tag{XI.51}$$

in which I represents the identity transformation that acts on tangent vectors.

If one thinks of T as an infinitesimal transformation of tangent vectors then its effect, when applied to a tangent vector **n** can be expressed as:

$$T(\mathbf{n}) = \mathcal{L}\mathbf{n} - E(\mathbf{n})\mathbf{D}, \tag{XI.53}$$

Since ε defines a scalar product on tangent vectors, we can express **n** as:

$$\mathbf{n} = \frac{\varepsilon(\mathbf{n},\mathbf{D})}{\varepsilon(\mathbf{D},\mathbf{D})}\mathbf{D} + \mathbf{n}_\perp = \frac{2}{\mathcal{L}}E(\mathbf{n})\mathbf{D} + \mathbf{n}_\perp, \tag{XI.54}$$

where $\mathbf{n}_\perp$ represents the part of **n** that is orthogonal to **D** under ε. One recalls that the planes that are orthogonal to the vector field **D** will be tangent to the equipotential surfaces of ϕ since they will be annihilated by the 1-form E.

This makes (XI.53) take the form:

$$T(\mathbf{n}) = \mathcal{L}\mathbf{n}_\perp - \tfrac{1}{2}E(\mathbf{n})\mathbf{D}. \tag{XI.55}$$

Note that **D**, as well as any vector field $\mathbf{n}_\perp$ that is tangent to the equipotential surfaces, is an eigenvector of T. In the former case, the corresponding eigenvalue is $-\mathcal{L}$, while in the latter case it is $+\mathcal{L}$. Indeed, from the form (XI.52) of T the eigenvectors of T are the same as the eigenvectors of $\mathbf{D}\otimes E$, while the eigenvalues of T are $\mathcal{L} - \lambda$, with $\lambda = 0$ or $E(\mathbf{D}) = 2\mathcal{L}$.

Generally, the matrix T^i_j is not symmetric, and its polarization into a symmetric and an anti-symmetric part is:

$$\overset{+}{T}{}^i_j = \tfrac{1}{2}(T^i_j + T_i{}^j) = \tfrac{1}{2}[(D^k E_k)\delta^i_j - D^i E_j - D^j E_i], \tag{XI.56a}$$

$$\overset{-}{T}{}^i_j = \tfrac{1}{2}(T^i_j - T_i{}^j) = -\tfrac{1}{2}(D^i E_j - D^j E_i). \tag{XI.56b}$$

A necessary and sufficient condition for T^i_j to be symmetric is that there be a non-zero real number α such that $D^i = \alpha E_i$, which is equivalent to saying that the dielectric is isotropic. (Sufficiency is obvious. To see necessity, contract both sides of $D^i E_j = D^j E_i$ with D^j, and note that both $D^j D^j = \delta_{ij} D^i D^j$ and $E_j D^j = E(\mathbf{D})$ are non-vanishing when **D** is.

Although the introduction of an auxiliary Euclidian metric has no physical meaning, it does serve to prove the theorem.)

The trace of T^i_j is:

$$T^i_i = \tfrac{1}{2} D^i E_i = \mathcal{L}, \tag{XI.57}$$

which is consistent with the sum of the eigenvalues, including multiplicity. Hence, the one-parameter family of transformations that T^i_i generates does not consist of volume-preserving transformations.

b. Magnetostatics. The Lagrangian for magnetostatics takes the form:

$$\mathcal{L}(x, B) = \tfrac{1}{4} H^{ij} B_{ij} = \tfrac{1}{4} \tilde{\mu}^{ijkl} B_{ij} B_{kl}\,, \tag{XI.58}$$

when expressed in terms of the 2-form B and the bivector field $\mathbf{H}$, or:

$$\mathcal{L}(x, B) = \tfrac{1}{2} H^i B_i = \tfrac{1}{2} \tilde{\mu}^{ij} B_i B_j \tag{XI.59}$$

in terms of the vector field $\mathbf{B}$ and the 1-form H.

The Noether homomorphism for this particular Lagrangian density takes the form:

$$J^i_a = T^i_j \sigma^j_a + S^i_a \tag{XI.60}$$

with:

$$T^i_j = \mathcal{L}\delta^i_j - \frac{\partial \mathcal{L}}{\partial A_{ki}} A_{k,j} = \tfrac{1}{4}(H^{kl} B_{kl})\delta^i_j - H^{ki} B_{kj}\,, \tag{XI.61a}$$

$$S^i_a = H^{ij}\, \mathfrak{D}_{ja}\,. \tag{XI.61b}$$

in terms of B and $\mathbf{H}$ or:

$$T^i_j = \mathcal{L}\delta^i_j - \varepsilon^{jkm}\varepsilon_{ikn} \frac{\partial \mathcal{L}}{\partial B^m} B^n = -\tfrac{1}{2}(B^k H_k)\delta^i_j + B^i H_j\,, \tag{XI.62a}$$

$$S^i_a = H_j \mathfrak{D}^{ij}_a \;(\mathfrak{D}^{ij}_a \equiv \varepsilon^{ijk}\mathfrak{D}_{ka}) \tag{XI.62b}$$

in terms of $\mathbf{B}$ and H.

The basis-free representation of the tensor field T is now:

$$T = -\mathcal{L}\, I + \mathbf{B} \otimes H, \tag{XI.63}$$

which has the same form as minus the result (XI.52) that we obtained above for the electrostatic case. Hence, the eigenspaces are the same as before, except that the signs of the eigenvalues are switched. Of course, the tangent planes that are orthogonal to $\mathbf{B}$ under the scalar product that is defined by $\tilde{\mu}$ are no longer tangent to equipotential surfaces, since the potentials for B are 1-forms, not 0-forms.

As before in the electrostatic case, the matrix T is not usually symmetric, and its polarization into a symmetric and an anti-symmetric part is:

$$\overset{+}{T}{}^{i}_{j} = \tfrac{1}{2}[-(B^k H_k)\delta^i_j + B^i H_j + B^j H_i], \qquad \text{(XI.64a)}$$

$$\overset{-}{T}{}^{i}_{j} = \tfrac{1}{2}(B^i H_j - B^j H_i), \qquad \text{(XI.64b)}$$

and its trace is:

$$T^i_i = -\tfrac{1}{2} B^i H_i = -\mathcal{L}, \qquad \text{(XI.65)}$$

so the one-parameter family of transformations that it generates does not consist of volume-preserving transformations in this case either. As before in the electrostatic case, symmetry is equivalent to the isotropy of the magnetic material.

c. Electromagnetism. The electromagnetic field Lagrangian has the form:

$$\mathcal{L}(x, F) = \tfrac{1}{4} H^{\mu\nu} F_{\mu\nu} = \tfrac{1}{4} \kappa^{\kappa\lambda\mu\nu} F_{\kappa\lambda} F_{\mu\nu}. \qquad \text{(XI.66)}$$

The matrix of the Noether homomorphism is then:

$$J^\mu_a = T^\mu_\nu \sigma^\nu_a + S^\mu_a \qquad \text{(XI.67)}$$

with:

$$T^\mu_\nu = \mathcal{L}\delta^\mu_\nu - \frac{\partial \mathcal{L}}{\partial A_{\kappa\mu}} A_{\kappa,\nu} = \tfrac{1}{4}(\mathfrak{h}^{\kappa\lambda} F_{\kappa\lambda})\delta^\mu_\nu - \mathfrak{h}^{\kappa\mu} F_{\kappa\nu}, \qquad \text{(XI.68a)}$$

$$S^\mu_a = -H^{\mu\nu} \mathfrak{D}_{\nu a}. \qquad \text{(XI.68b)}$$

One can use the local frame field ∂_μ and its reciprocal coframe field dx^μ to define four vector fields $\mathfrak{h}_\kappa$ ($\kappa = 0, \ldots, 3$) and four 1-forms F^κ by way of:

$$\mathfrak{h}_\kappa = \mathfrak{h}^{\kappa\mu} \partial_\mu, \qquad F^\kappa = F_{\kappa\nu}\, dx^\nu, \qquad \text{(XI.69)}$$

Of course, these quadruples of vectors and covectors do not, by any means, have to be linearly independent, and some of them might very well vanish.

We can now express the stress-energy-momentum tensor T as:

$$T = \mathcal{L} I - \mathfrak{h}_\kappa \otimes F^\kappa. \qquad \text{(XI.70)}$$

As far as eigenvectors and eigenvalues are concerned, one immediately sees that the eigenvectors of T are identical with those of $\mathfrak{h}_\kappa \otimes F^\kappa$, although the eigenvalues λ of T will differ from those λ' of $\mathfrak{h}_\kappa \otimes F^\kappa$ by the relation:

$$\lambda = \mathcal{L} - \lambda'. \qquad \text{(XI.71)}$$

Hence, one can redirect one's attention to the eigenvectors of $\mathfrak{h}_\kappa \otimes F^\kappa$.

Here, it helps to keep in mind that on a four-dimensional vector space V a non-zero 2-form F – hence, a bivector $\mathfrak{h}$ – can have a rank that equals either two or four, but nothing else. That is, in the rank-two case they can be expressed in the form:

$$F = \alpha \wedge \beta, \qquad \mathfrak{h} = \mathbf{a} \wedge \mathbf{b}, \tag{XI.72}$$

but not uniquely, and in the rank-four case, one has:

$$F = \alpha \wedge \beta + \gamma \wedge \delta, \qquad \mathfrak{h} = \mathbf{a} \wedge \mathbf{b} + \mathbf{c} \wedge \mathbf{d}. \tag{XI.73}$$

In both cases, all of the 1-forms and vectors in question are linearly independent. Hence, a rank-two 2-form defines a 2-frame $\{\mathbf{a}, \mathbf{b}\}$ in V and a 2-coframe $\{\alpha, \beta\}$ in V^*, neither of which are unique. Similarly, a rank-four 2-form defines a 4-frame $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ – i.e., a basis – for V and a 4-coframe $\{\alpha, \beta, \gamma, \delta\}$ in V^*. We shall refer to their elements generically as $\mathbf{e}_a$ and θ^a, respectively, with the understanding that the range of indices is defined by the nature of the problem at hand.

In the rank-2 case, one has:

$$\mathfrak{h}_a = i_{\theta^a}\mathfrak{h} = \theta^a(\mathbf{e}_1)\, \mathbf{e}_2 - \theta^a(\mathbf{e}_2)\, \mathbf{e}_1, \tag{XI.74a}$$

$$F^a = i_{\partial_a} F = \theta^a(\mathbf{e}_1)\, \theta^2 - \theta^a(\mathbf{e}_2)\, \theta^1. \tag{XI.74b}$$

One finds that it is particularly convenient if the 2-frame and the 2-coframe are *projectively* reciprocal, in the sense that:

$$\theta^a(\mathbf{e}_b) = \begin{cases} \neq 0 & a = b, \\ = 0 & a \neq b. \end{cases} \tag{XI.75}$$

This makes:

$$\mathfrak{h}_1 = \gamma_2\, \mathbf{e}_2, \qquad \mathfrak{h}_2 = -\gamma_1\, \mathbf{e}_1, \quad F^1 = \gamma_2\, \theta^2, \qquad F^2 = -\gamma_1\, \theta^1, \tag{XI.76}$$

in which we have defined:

$$\gamma_1 = \theta^2(\mathbf{e}_2), \qquad \gamma_2 = \theta^1(\mathbf{e}_1). \tag{XI.77}$$

One now has:

$$F^1(\mathfrak{h}_1) = (\gamma_2)^2 \gamma_1, \qquad F^2(\mathfrak{h}_2) = (\gamma_1)^2 \gamma_2, \qquad F^1(\mathfrak{h}_2) = F^2(\mathfrak{h}_1) = 0. \tag{XI.78}$$

In this case, one finds that:

$$T = \mathfrak{h}_1 \otimes F^1 + \mathfrak{h}_2 \otimes F^2, \tag{XI.79}$$

and its eigenvectors are $\mathfrak{h}_1$ and $\mathfrak{h}_2$, with associated eigenvalues $F^1(\mathfrak{h}_1)$ and $F^2(\mathfrak{h}_2)$, resp.

One has a similar situation in the rank-four case, although generally one must deal with it in terms of special situations since it admits many possibilities.

Let us look at the two rank-2 cases that we already examined above, namely, electrostatics and magnetostatics. In the former case, one has:

$$F = dt \wedge E, \qquad \mathfrak{h} = \partial_t \wedge \mathbf{D}, \tag{XI.80}$$

which makes:

$$\mathfrak{h}_0 = \mathbf{D}, \qquad \mathfrak{h}_1 = -\partial_t, \qquad F^0 = E, \qquad F^1 = -E(\mathbf{D})\, dt, \tag{XI.81}$$

since one sees that the 2-frame $\{\partial_t, \mathbf{D}\}$ is projectively reciprocal to the 2-coframe $\{dt, E\}$.

This makes:

$$\begin{aligned} T &= \tfrac{1}{2} E(\mathbf{D}) I - \mathbf{D} \otimes E - E(\mathbf{D})\, \partial_t \otimes dt \\ &= -\tfrac{1}{2} E(\mathbf{D})\, \partial_t \otimes dt + \tfrac{1}{2} E(\mathbf{D})\, \partial_i \otimes dx^i - \mathbf{D} \otimes E. \end{aligned} \tag{XI.82}$$

We express this in space-time block matrix form:

$$T = \left[\begin{array}{c|c} -\mathcal{L}_E & 0 \\ \hline 0 & T^i_j(E) \end{array}\right], \tag{XI.83}$$

in which $T^i_j(E)$ the 3×3 electrostatic stress matrix (XI.49) that we derived above.

One notes that the trace of T is now zero, whereas the trace of the spatial part was found to be $\mathcal{L}_E$.

The magnetostatic case is similar, but still requires special treatment, since the role of E and $\mathbf{B}$ are actually played by $\mathbf{B}$ and H, respectively, not vice versa. One starts with:

$$F = -\#\mathbf{B} = -B^i \#\partial_i, \qquad \mathfrak{h} = -\#^{-1}H = -H_i \#^{-1} dx^i, \tag{XI.84}$$

from which it follows that:

$$F^i = \varepsilon_{ijk} B^j dx^k, \qquad \mathfrak{h}_i = \varepsilon^{ijk} H^j \partial_k, \tag{XI.85}$$

which makes:

$$\mathfrak{h}_i \otimes F^i = H(\mathbf{B})\, \partial_i \otimes dx^i - \mathbf{B} \otimes H. \tag{XI.86}$$

and:

$$\begin{aligned} T &= \tfrac{1}{2} H(\mathbf{B})\, \partial_\mu \otimes dx^\mu - H(\mathbf{B})\, \partial_i \otimes dx^i + \mathbf{B} \otimes H \\ &= \tfrac{1}{2} H(\mathbf{B})\, \partial_t \otimes dt - \tfrac{1}{2} H(\mathbf{B})\, \partial_i \otimes dx^i + \mathbf{B} \otimes H. \end{aligned} \tag{XI.87}$$

We express this in space-time block matrix form as:

$$T = \left[\begin{array}{c|c} \mathcal{L}_B & 0 \\ \hline 0 & T^i_j(B) \end{array}\right], \tag{XI.88}$$

in which $T^i_j(B)$ represents the spatial magnetostatic stress tensor (XI.61a) or (XI.62a) that we derived previously. Once again, the trace comes out to zero.

In the generic case of a rank-four 2-form, one has:

$$F = dt \wedge E - \#\mathbf{B}, \qquad \mathfrak{h} = \partial_t \wedge \mathbf{D} - \#^{-1}H. \tag{XI.89}$$

This makes:

$$F^0 = i_{\partial_t} F = E, \qquad F^i = i_{\partial_i} F = -E_i\, dt - i_{\partial_i} \#\mathbf{B} = -E_i\, dt + \#(\partial_i \wedge \mathbf{B}), \tag{XI.90a}$$

$$\mathfrak{h}_0 = i_{dt}\mathfrak{h} = \mathbf{D}, \qquad \mathfrak{h}_i = -D^i\, \partial_t - i_{dx^i} \#^{-1} H = -D^i\, \partial_t + \#^{-1}(dx^i \wedge H). \tag{XI.90b}$$

We can then compute the components of T:

$$T^0_0 = \mathcal{L}_{\text{em}} - E(\mathbf{D}), \qquad T^i_0 = -\#^{-1}(E \wedge H)^j, \tag{XI.91a}$$

$$T^0_j = \#(\mathbf{D} \wedge \mathbf{B})_j, \qquad T^i_j = \mathcal{L}_{\text{em}}\delta^i_j - D^iE_j + B^iH_j. \tag{XI.91b}$$

When the constitutive law has vanishing couplings between E and $\mathbf{B}$, so $\mathcal{L}_{\text{em}} = \frac{1}{2}[E(\mathbf{D}) - H(\mathbf{B})]$, these components take the form:

$$T^0_0 = -\tfrac{1}{2}[E(\mathbf{D}) + H(\mathbf{B})], \qquad T^i_0 = -\#^{-1}(E \wedge H)^j, \tag{XI.92a}$$

$$T^0_j = \#(\mathbf{D} \wedge \mathbf{B})_j, \qquad T^i_j = T^i_j(E) + T^i_j(\mathbf{B}). \tag{XI.92b}$$

One sees that the time-space and space-time components of the stress-energy-momentum tensor field take the form of the elementary Poynting vector $\mathbf{E} \times \mathbf{B}$. However, two points must be made concerning this: First, it is clear that when one does not identify vectors and covectors, as in elementary electromagnetism, the vector field $\#^{-1}(E \wedge H)$ is quite distinct from the covector field $\#(\mathbf{D} \wedge \mathbf{B})$, even when one identifies components using a Euclidian metric. Second, it is only in the case of propagating waves that it is physically meaningful to identify these terms with energy flux or momentum flux. Of course, in order to convert the energy flux to a momentum flux, one must divide by a propagation speed. One sees that in the general case, for which this speed is not a universal constant, it would probably make the most sense to divide by the speed of propagation in the direction of wave motion.

This brings us to the fact that one of the most important rank-2 cases, namely, the fields of electromagnetic waves, is probably better treated within the general framework of the rank-4 case that we just described. The relevant 2-form and bivector field take the form:

$$F = k \wedge A, \qquad \mathfrak{h} = \mathbf{k} \wedge \mathbf{A} = \kappa(k \wedge A), \tag{XI.93}$$

in which A is defined up to a gauge transformation of the form $A \mapsto A + \lambda k$ for some λ; similarly $\mathbf{A}$ can be replaced by $\mathbf{A} + \lambda' \mathbf{k}$ for some λ'.

In order for F to be wavelike it is necessary, but not sufficient, that one have:

$$F \wedge F = F \wedge \#\mathfrak{h} = 0. \qquad \text{(XI.94)}$$

If we put everything in time-space form, with:

$$k = \omega\, dt - k_s, \qquad \mathbf{k} = \omega \partial_t + \mathbf{k}_s\,, \qquad \text{(XI.95)}$$

then we have:

$$F = \omega\, dt \wedge A - k_s \wedge A, \qquad \mathfrak{h} = \omega \partial_t \wedge \mathbf{A} + \mathbf{k}_s \wedge \mathbf{A}\,. \qquad \text{(XI.96)}$$

This allows us to identify the E, $\mathbf{D}$, H, and $\mathbf{B}$ that are necessary in order to identify terms in (XI.92a, b):

$$E = \omega A, \qquad \#\mathbf{B} = k_s \wedge A, \quad \mathbf{D} = \omega \mathbf{A}, \qquad H = \#(\mathbf{k}_s \wedge \mathbf{A}). \qquad \text{(XI.97)}$$

One verifies directly – with an appropriate choice of gauge – that the triples $\{k_s, E, H\}$ and $\{\mathbf{k}_s, \mathbf{D}, \mathbf{B}\}$ are projectively reciprocal. That is, $k_s(\mathbf{k}_s)$, $E(\mathbf{B})$, and $H(\mathbf{B})$ are non-zero, while all of the cross terms are already zero or can be made zero by a choice of gauge for A and $\mathbf{A}$. In particular, one chooses λ and λ' such that:

$$k_s(\mathbf{A}) = A(\mathbf{k}_s) = 0. \qquad \text{(XI.98)}$$

By substitution of (XI.98) into (XI.92a, b), we obtain more specific forms for the time-space and space-time components:

$$T_0^i = \omega A(\mathbf{A})\, k^i, \qquad T_j^0 = -\,\omega A(\mathbf{A})\, k_j\,. \qquad \text{(XI.99)}$$

Hence, we see the direction of flow of the energy in the wave is defined by the wave vector or covector.

In the case of a constitutive law with no cross couplings, the energy density takes the form:

$$T_0^0 = \tfrac{1}{2}[\,\omega^2 + k_s(\mathbf{k}_s)]\, A(\mathbf{A}). \qquad \text{(XI.100)}$$

Here, one can point out that k must satisfy the dispersion relation $P[k] = 0$ that is associated with the constitutive law. Hence, in the simplest – viz., quadratic – case, where $\omega^2 = k_s(\mathbf{k}_s)$, one can express the energy density as $\omega^2 A(\mathbf{A})$. This provides another opportunity for specifying A and $\mathbf{A}$; recall that so far we have only specified the spatial parts.

5. Symmetries of systems of differential equations [18-20]. In addition to considering the symmetries of the action functional for a system of differential equations, one can also consider the symmetries of that system itself. Indeed, the latter class of symmetries is potentially richer than the former one since not every system of differential equations can be represented as the Euler-Lagrange equations for some action functional. In physics, this is especially true when one is considering non-conservative forces in mechanical systems, but even when one is still considering systems that follow from a

variational principle, one can often find symmetries beyond the ones that follow from Noether's theorem.

Loosely speaking, a transformation is said to be a *symmetry* of a system of differential equations if it takes every solution of the system to another solution of the system. Of course, in order to make this concept precise in the language of transformation groups that was discussed above one must first be more precise about how one represents the system of differential equations and its space of solutions. Of the various ways of representing such things that we discussed in Chapter VI the ones that are most naturally adapted to the problem of finding symmetries of systems of differential equations seem to be the methods of jet manifolds and exterior differential systems, so we first present the general notions in both forms and then apply them to the specific problem of the system of differential equations that is associated with pre-metric electromagnetism.

It is worth pointing out that the problem that first motivated Lie to develop his theory of differentiable transformation groups was the problem of how to adapt the methods of Galois theory, which involved associating finite groups of transformations with the solutions of systems of algebraic equations, to the problem of investigating the solutions of systems of differential equations by means of continuously infinite groups of transformations of those solutions. It is for that reason that one must recognize that the problem and methodology that he was first introducing was actually more involved than the simpler problems of Lie groups acting globally as transformations on manifolds, namely the problems of Lie pseudogroups and Lie groupoids acting locally. The obstruction to the extension of the local methods to the global ones generally involves the limits of the existence and uniqueness of the solutions to the differential equations.

a. Symmetries in terms of jets. If we recall the definitions associated with the manifold $J^1(M, N)$ of 1-jets of differentiable maps from a manifold M to a manifold N that were discussed in Chapter VI then we see that the formalism is naturally adapted to the problem of describing symmetries of systems of differential equations since both the system of equations and its solutions can be represented as submanifolds in one sense or the other.

If M is m-dimensional and N has dimension n then a system of m first-order partial differential equations for n unknown functions can be described implicitly by a submanifold in $J^1(M, N)$ of codimension one – i.e., a hypersurface – by means of some differentiable function F: $J^1(M, N) \to \mathbb{R}$. The equation then takes the local form of a level hypersurface of F:

$$F(x^i, y^a, y^a_i) = \text{const.} \tag{XI.101}$$

One generally expects that one can solve F for the derivatives y^a_i, at least in principle:

$$y^a_i = f(x^i, y^a), \tag{XI.102}$$

so this implies that one restricts F by means of the implicit function theorem to satisfy:

$$\frac{\partial F}{\partial y_i^a} \neq 0, \tag{XI.103}$$

by which we mean that not all of the derivatives vanish identically.

One might also define a differentiable function F: $J^1(M, N) \to \mathbb{R}^k$ that would define a system of k implicit differential equations.

A *solution* to the differential equation defined by such an F is then a continuously-differentiable function ϕ: $M \to N$ such that its 1-jet prolongation $j^1\phi$, which locally looks like $(x^i, y^a(x), y_a^i(x))$, satisfies the equation (XI.101):

$$F(j^1\phi) = F(x^i, y^a(x), y_a^i(x)) = \text{const.} \tag{XI.104}$$

Hence, one can regard a solution ϕ as a submanifold of M that also defines a submanifold $j^1\phi$: $M \to J^1(M, N)$ of $J^1(M, N)$.

One calls a section s: $M \to J^1(M, N)$ integrable iff $s = j^1\phi$ for some C^1 map ϕ: $M \to N$. A necessary and sufficient condition for this is that when one pulls back the contact forms:

$$\theta^a = dy^a - y_i^a dx^i \tag{XI.105}$$

by means of s the result is a set of n vanishing 1-forms:

$$0 = s^*\theta^a(x) = (y_{,i}^a - y_i^a)dx^i . \tag{XI.106}$$

That is, s is integrable iff its components satisfy the compatibility condition:

$$\frac{\partial y^a}{\partial x^i} = y_i^a . \tag{XI.107}$$

The vanishing of $\theta^a(X)$ for all $a = 1, \ldots, n$ when X is a tangent vector to $s \in J^1(M, N)$ defines n hypersurfaces in the tangent space T_sJ^1, whose intersection then defines a linear subspace C_sJ^1 of T_sJ^1 of dimension $m + n + mn - n = m(n + 1)$; i.e., of codimension n. The set of all such linear subspaces C_sJ^1 defines a vector sub-bundle $C(J^1)$ of $T(J^1)$ that we shall call the *contact bundle* to $J^1(M, N)$.

Since any vector sub-bundle of a tangent bundle can be regarded as a differential system on the manifold in question, one must naturally inquire about the integrability of the sub-bundle. From Frobenius's theorem, this is equivalent to asking whether there are 1-forms η_b^a on $J^1(M, N)$ such that:

$$-\Omega^a = d\theta^a = \eta_b^a \wedge \theta^b . \tag{XI.108}$$

By substituting (XI.105), we first get:

$$\Omega^a = dy_i^a \wedge dx^i , \tag{XI.109}$$

and (XI.108) takes the form:

$$0 = (dy_i^a - y_i^b \eta_b^a) \wedge dx^i + \eta_b^a \wedge dy^b. \qquad \text{(XI.110)}$$

Since this is not generally the case, we must conclude that in general the contact sub-bundle is not integrable.

We then call a diffeomorphism Φ: $J^1(M, N) \to J^1(M, N)$ a contact transformation iff its differential map at each point $d\Phi|_s : T_s J^1 \to T_{\Phi(s)} J^1$ takes each $C_s J^1$ to $C_{\Phi(s)} J^1$ isomorphically. Note that this is not equivalent to saying that it preserves the contact 1-forms, except up to a linear isomorphism A_b^a of $\mathbb{R}^n$ that may vary from point to point:

$$\Phi^* \theta^a|_{\Phi(s)} = A_b^a(s)\theta^b\Big|_s, \qquad \text{(some } A_b^a : J^1(M, N) \to GL(n)\text{)}. \qquad \text{(XI.111)}$$

It now becomes clear that a diffeomorphism of $J^1(M, N)$ that takes solutions of a differential equation that is defined by a level hypersurface of a C^1 function F on $J^1(M, N)$ must be, at the very least, a contact transformation, since it must take integrable sections to other integrable sections. A *symmetry* of the differential equation that is defined by F is then defined to be a contact transformation of $J^1(M, N)$ that also takes points of the level hypersurface to other points of the level hypersurface. Hence, it must also preserve F:

$$\Phi^* F = F, \qquad \text{(XI.112)}$$

which has the local form:

$$F(x^i, y^a, y_i^a) = F(\Phi^i, \Phi^a, \Phi_i^a), \qquad \text{(XI.113)}$$

in which each of the coordinate functions on the right-hand side is a function of (x^i, y^a, y_i^a), in general.

A one-parameter family of contact transformations Φ_σ, $\sigma \in (-\varepsilon, +\varepsilon)$ of F will be called *differentiable* iff for each $s \in J^1(M, N)$ the curve $\Phi_\sigma(s)$ in $J^1(M, N)$ is a differentiable function of σ. By differentiation, one can then define a vector field on $J^1(M, N)$:

$$X(s) = \frac{d\Phi_\sigma(s)}{d\sigma}\bigg|_{\sigma=0}. \qquad \text{(XI.114)}$$

However, since each $\Phi_\sigma(s)$ must preserve each θ^a only up to $A_b^a(s)$, as in (XI.108), we see that by Lie derivation the infinitesimal condition takes the form:

$$\mathrm{L}_X \theta^a = a_b^a(s)\theta^b, \qquad \text{(XI.115)}$$

in which $a_b^a : J^1(M, N) \to \mathfrak{gl}(n)$, this time.

When a vector field X on $J^1(M, N)$ satisfies this condition we shall it an *infinitesimal contact transformation.*

By an application of Cartan's formula for L_X, we see that (XI.115) also takes the form:

$$di_X\theta^a + i_X d\theta^a = a^a_b(s)\theta^b\,, \tag{XI.116}$$

and by substitution of (XI.105), this gives the following set of equations for the components of X:

$$\frac{\partial X^a}{\partial x^i} - X^a_i = y^a_j \frac{\partial X^j}{\partial x^i} - y^b_j a^a_b\,, \tag{XI.117a}$$

$$\frac{\partial X^a}{\partial y^b} - y^a_i \frac{\partial X^i}{\partial y^b} = a^a_b\,, \tag{XI.117b}$$

$$\frac{\partial X^a}{\partial y^j_b} = y^a_i \frac{\partial X^i}{\partial y^b_j}\,. \tag{XI.117c}$$

By substituting (XI.117b) into (XI.117a), the arbitrary matrix a^a_b ceases to play any role and the equations for the components of X become:

$$\frac{\partial X^a}{\partial y^b_i} = y^a_j \frac{\partial X^j}{\partial y^b_i}\,, \tag{XI.118a}$$

$$X^a_i = D_i X^a - y^a_j D_i X^j\,, \tag{XI.118b}$$

in which the operator D_i takes the form of a total derivative:

$$D_i = \frac{\partial}{\partial x^i} + y^a_i \frac{\partial}{\partial y^a}\,. \tag{XI.119}$$

Hence, from (XI.118b) we see that only the components X^i and X^a remain undetermined, except as solutions to the system of partial differential equations (XI.118a).

If X is the prolongation $j^1\xi$ of a vector field ξ on $M \times N$ then (XI.118b) gets replaced by:

$$X^a_i = X^a{}_{,i} \tag{XI.120a}$$

$$y^b_i X^a_{,b} = y^a_j D_i X^j\,. \tag{XI.120b}$$

It is illuminating to see what happens to vector fields that are annihilated by the θ^a; i.e., vector fields that are tangent to the integrable sections of $J^1(M, N) \to M$. If $i_X\theta^a = 0$ then $i_X\Omega^a = -\,a^a_b\theta^b$ (although the minus sign is unnecessary, since the matrix a^a_b is ambiguous), and upon expanding this into component equations, one gets:

$$-X^a_i dx^i + X^i dy^a_i = -a^a_b y^b_i dx^i + a^a_b dy^b\,, \tag{XI.121}$$

which implies that $a^a_b = 0$, which then makes X itself identically zero.

One concludes that any non-zero contact transformation must be transversal to any integrable section.

When we also have a differential equation defined by a C^1 function F on $J^1(M, N)$, we call the infinitesimal contact transformation X an *infinitesimal symmetry* of the differential equation that is defined by F if the local one-parameter family of contact transformations Φ_σ of $J^1(M, N)$ that it generates by integration all lie within the level hypersurface of F that defines the equation. Since each Φ_σ preserves F, by differentiating along the curves $\Phi_\sigma(s)$, we see that the Lie derivative of F with respect to X must vanish:

$$0 = \mathrm{L}_X F = XF = X^i \frac{\partial F}{\partial x^i} + X^a \frac{\partial F}{\partial y^a} + X_i^{\ a} \frac{\partial F}{\partial y_i^a} . \qquad \text{(XI.122)}$$

b. Symmetries in terms of exterior differential systems [**18-21**]. Now, let us consider a system of partial differential equations that is defined by a finite set $\{\theta^\kappa, \kappa = 1, \ldots, k\}$ of exterior differential forms on a manifold N of varying degrees as an exterior differential system:

$$\theta^\kappa = 0 \qquad \text{(all } \kappa\text{)}. \qquad \text{(XI.123)}$$

A solution to this system is a differentiable map y: $M \to N$ such that:

$$y^* \theta^\kappa = 0 \qquad \text{(all } \kappa\text{)}. \qquad \text{(XI.124)}$$

If the forms θ^κ take the local form:

$$\theta^\kappa = \frac{1}{r!} \alpha^\kappa_{a_1 \cdots a_r}(y) dy^{a_1} \wedge \cdots \wedge dy^{a_r} \qquad \text{(XI.125)}$$

then the pull-backs to M (with local coordinates x^i) take the form:

$$y^* \theta^\kappa = \frac{1}{r!} \alpha^\kappa_{a_1 \cdots a_r}(y(x)) y^{a_1}_{i_1} \cdots y^{a_r}_{i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r} . \qquad \text{(XI.126)}$$

A *(finite) symmetry* of the exterior differential system is a diffeomorphism f: $N \to N$ that takes solutions to other solutions. Note that this does not imply that it takes the θ^κ to themselves – i.e., $f^* \theta^\kappa = \theta^\kappa$ – only that $y^*(f^* \theta^\kappa) = 0$ iff $y^* \theta^\kappa = 0$. This means that there is an invertible matrix $A^\kappa_\nu \in GL(k)$ such that:

$$f^* \theta^\kappa = A^\kappa_\nu \theta^\nu . \qquad \text{(XI.127)}$$

An *infinitesimal symmetry* of the exterior differential system is then a vector field $X \in \mathfrak{X}(N)$ such that for some matrix $a^\kappa_\nu \in \mathfrak{gl}(k)$ one has:

$$\mathrm{L}_X \theta^\kappa = d i_X \theta^\kappa + i_X d\theta^\kappa = a^\kappa_\nu \theta^\nu . \qquad \text{(XI.128)}$$

Hence, the vector field X will be a solution to a system of linear first-order partial differential equations of a type that one calls *Lie equations*, since their solutions represent infinitesimal symmetries. In fact, since Lie himself was not using the global, basis-free formalism of modern Lie groups, he was more concerned with algebraic structures that were defined by solving such local systems of partial differential equations. To this day, the study of Lie pseudogroups and Lie equations still represents a class of problems that are considerably more complicated to address than those of finite-dimensional Lie groups and Lie algebras, which are certainly non-trivial, in their own right [5].

When the solutions to an exterior differential system are to take the form of sections s: $M \to E$ of a vector bundle $E \to M$ over a manifold M, the exterior differential system will be defined by a set of differential forms on E. If a local trivialization has coordinates of the form (x^i, y^a) then differential forms on E will be exterior products of factors of both the 1-forms dx^i and the 1-forms dy^a, while vector fields on E will have the local form:

$$X = X^i \frac{\partial}{\partial x^i} + X^a \frac{\partial}{\partial y^a}. \tag{XI.129}$$

c. Symmetries of the equations of pre-metric electromagnetism [**5**]. Since the equations of pre-metric electromagnetism are concerned with differential forms on the space or spacetime manifold, it seems clear that representing them as an exterior differential system on a vector bundle would be the most straightforward path to take. However, one must note that the form that we have been addressing them relates to differential operators on sections of vector bundles, not an exterior differential system. Hence, we must convert them to such a system. For the sake of specificity, we shall consider the case of spacetime.

Since the field that we are solving for is a section F: $M \to \Lambda^2 M$, we see that what we need to do first is to define the pre-metric Maxwell equations in terms of a set of differential forms on $\Lambda^2 M$ that pull down to the previous equations on M by way of a section. The way that we do this is to start with the local expression for the 2-form F as $\frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$ and regard the components $F_{\mu\nu}$, not as functions on M, but as coordinate functions on $\Lambda^2 M$. We then have the local expression for a 2-form F on the total space $\Lambda^2 M$ of the bundle that becomes a 2-form on M when we pull it down by way of the section F; i.e., when one replaces the components $F_{\mu\nu}$ with functions $F_{\mu\nu}(x)$ on M.

Hence, when we form the 3-form on $\Lambda^2 M$:

$$\Theta^1 = 2dF = dF_{\mu\nu} \wedge dx^\mu \wedge dx^\nu \tag{XI.130}$$

we see that the first of the Maxwell equations can be represented by the exterior differential system on $\Lambda^2 M$:

$$\Theta^1 = 0. \tag{XI.131}$$

[5] For some idea of the possible role of Lie pseudogroups and Lie groupoids in physics, see Pommaret [**3**].

We absorb the algebraic constitutive law $\mathfrak{h} = \kappa(F)$ into the differential equation for $\mathfrak{h}$ in the absence of sources, which we express in the form $d{*}F = 0$, in which ${*}F = (\kappa\cdot\#)F$; i.e., ${*} = \kappa\cdot\#$. We have not yet normalized the * diffeomorphism, so we only assume that ${*}^2 = -\lambda^2 I$.

If ${*}F$ then takes the form $\frac{1}{2}{*}F_{\mu\nu}\, dx^\mu \wedge dx^\nu$ then we see that the differential equation for ${*}F$ can be expressed by the vanishing of the 3-form on $\Lambda^2 M$:

$$\Theta^2 = 2d{*}F = d{*}F_{\mu\nu}\ dx^\lambda \wedge dx^\mu \wedge dx^\nu, \tag{XI.132}$$

that is:

$$\Theta^2 = 0. \tag{XI.133}$$

Since the coordinate functions on $\Lambda^2 M$ involve $F_{\mu\nu}$, not ${*}F_{\mu\nu}$, we see that next we must expand the differentiation:

$$d{*}F_{\mu\nu} = d(\kappa\cdot\#)F = \tfrac{1}{4} d(\kappa_{\kappa\lambda\alpha\beta}\, \varepsilon^{\alpha\beta\mu\nu})F_{\mu\nu}\,. \tag{XI.134}$$

Here, we see that the representation of the pre-metric Maxwell equations as an exterior differential system on $\Lambda^2 M$ has one natural advantage in the eyes of constitutive laws: nonlinear constitutive laws are simply the ones for which the component functions $\kappa_{\kappa\lambda\alpha\beta} = \kappa_{\kappa\lambda\alpha\beta}(x, F)$ are fully general, while linear laws have components of the form $\kappa_{\kappa\lambda\alpha\beta} = \kappa_{\kappa\lambda\alpha\beta}(x)$. This suggests that nonlinear electrodynamics is more conveniently addressed in the present form.

As a further simplification, we combine the components of κ and # into the components:

$$\kappa_{\kappa\lambda}{}^{\mu\nu} = \tfrac{1}{2}\, \kappa_{\kappa\lambda\alpha\beta}\, \varepsilon^{\alpha\beta\mu\nu}, \tag{XI.135}$$

of *.

Since:

$$d\kappa_{\kappa\lambda}{}^{\mu\nu} = \kappa_{\kappa\lambda}{}^{\mu\nu}{}_{,\rho}\ dx^\rho + \tfrac{1}{2}\, \kappa_{\kappa\lambda}{}^{\mu\nu,\rho\sigma}\, dF_{\rho\sigma}\,, \tag{XI.136}$$

we get:

$$\begin{aligned} d{*}F_{\mu\nu} &= \tfrac{1}{2}(d\kappa_{\mu\nu}{}^{\kappa\lambda}\, F_{\kappa\lambda} + \kappa_{\mu\nu}{}^{\kappa\lambda}\, dF_{\kappa\lambda}) \\ &= \tfrac{1}{2}\, \kappa_{\mu\nu}{}^{\kappa\lambda}{}_{,\rho}\ F_{\kappa\lambda}\, dx^\rho + \tfrac{1}{2}(\kappa_{\mu\nu}{}^{\kappa\lambda} + \tfrac{1}{2}\, \kappa_{\mu\nu}{}^{\rho\sigma,\,\kappa\lambda}\, F_{\rho\sigma})\, dF_{\kappa\lambda}\,. \end{aligned} \tag{XI.137}$$

As usual, we introduce the notation:

$$\tilde{\kappa}_{\mu\nu}{}^{\kappa\lambda} = \kappa_{\mu\nu}{}^{\kappa\lambda} + \tfrac{1}{2}\, \kappa_{\mu\nu}{}^{\rho\sigma,\,\kappa\lambda}\, F_{\rho\sigma}, \tag{XI.138}$$

and put (XI.137) into the form:

$$d{*}F_{\mu\nu} = \tfrac{1}{2}(\kappa_{\mu\nu}{}^{\kappa\lambda}{}_{,\rho}\ F_{\kappa\lambda}\, dx^\rho + \tilde{\kappa}_{\mu\nu}{}^{\kappa\lambda}\, dF_{\kappa\lambda}\,). \tag{XI.139}$$

This allows us to set:

$$\Theta^2 = \kappa_{\mu\nu}{}^{\kappa\lambda}{}_{,\rho}\ F_{\kappa\lambda}\ dx^\rho \wedge dx^\mu \wedge dx^\nu + \tilde{\kappa}_{\mu\nu}{}^{\kappa\lambda} dF_{\kappa\lambda} \wedge dx^\mu \wedge dx^\nu \ . \qquad \text{(XI.140)}$$

We next address the infinitesimal symmetries of the exterior differential system:

$$\Theta^a = 0, \qquad a = 1, 2, \qquad \text{(XI.141)}$$

which will then be solutions X of the system of Lie equations:

$$\mathrm{L}_X\, \Theta^a = di_X\, \Theta^a = a^a_b \Theta^b \, . \qquad \text{(XI.142)}$$

Since $\Theta^1 = dF$ and $\Theta^2 = d{*}F$ are both exact forms, this takes the form:

$$di_X dF = a^1_1 dF + a^1_2 d*F \, , \qquad di_X d{*}F = a^1_1 dF + a^1_2 d*F \, . \qquad \text{(XI.143)}$$

The possible solutions of these equations include the possibility that the underdetermined functions a^a_b are constants, in which case, the exterior derivative operator commutes with them in the right-hand sides of (XI.143), and we are left with the equations:

$$di_X dF = d(a^1_1 F + a^1_2 * F) \, , \qquad di_X d{*}F = d(a^2_1 F + a^2_2 * F) \, , \qquad \text{(XI.144)}$$

which can be solved by:

$$i_X dF = a^1_1 F + a^1_2 * F + \varepsilon_{\mu\nu} \, , \qquad i_X d{*}F = a^2_1 F + a^2_2 * F + {*}\varepsilon_{\mu\nu} \, , \qquad \text{(XI.145)}$$

in which $\varepsilon_{\mu\nu}$ and ${*}\varepsilon_{\mu\nu}$ are the components of closed 2-forms ε and ${*}\varepsilon$ on $\Lambda^2 M$.

If we expand the left-hand sides in latter equations in components we get the following algebraic equations for the components X^μ and $X_{\mu\nu}$:

$$X_{\mu\nu}\, dx^\mu \wedge dx^\nu - 2X^\mu\, dF_{\mu\nu} \wedge dx^\nu = (a^1_1 + a^1_2 *) F_{\mu\nu} dx^\mu \wedge dx^\nu \, , \qquad \text{(XI.146a)}$$

$$(3\kappa_{\mu\nu}{}^{\kappa\lambda}{}_{,\rho} F_{\kappa\rho} X^\lambda + \tilde{\kappa}_{\mu\nu}{}^{\kappa\lambda} X_{\kappa\lambda}) dx^\mu \wedge dx^\nu - 2\tilde{\kappa}_{\mu\nu}{}^{\kappa\lambda} X^\mu dF_{\kappa\lambda} \wedge dx^\nu$$
$$= (a^2_1 + a^2_2 *) F_{\mu\nu} dx^\mu \wedge dx^\nu \, . \qquad \text{(XI.146b)}$$

From the first one we deduce:

$$X^\mu = 0, \qquad X_{\mu\nu} = (a^1_1 + a^1_2 *) F_{\mu\nu} + \varepsilon_{\mu\nu}, \qquad \text{(XI.147)}$$

and when we substitute these results in the second equation of (XI.146b), we get:

$$\tilde{\kappa}_{\mu\nu}{}^{\kappa\lambda} X_{\kappa\lambda} = (a^1_1 + a^1_2 *) \tilde{\kappa}_{\mu\nu}{}^{\kappa\lambda} F_{\mu\nu} + \tilde{\kappa}_{\mu\nu}{}^{\kappa\lambda} \varepsilon_{\mu\nu} = (a^2_1 + a^2_2 *) F_{\mu\nu} + {*}\varepsilon_{\mu\nu} . \qquad \text{(XI.148)}$$

Hence, the only remaining issue to be resolved in order to conclude that (XI.147) represents a class of infinitesimal symmetries is whether one can find suitable constants a^a_b, and components $*\varepsilon_{\mu\nu}$ of a closed 2-form that make:

$$(a^1_1 + a^1_2 *)\tilde{\kappa} = a^2_1 + a^2_2 *, \qquad *\varepsilon = \tilde{\kappa}\varepsilon, \tag{XI.148}$$

in which we have reverted to the component-free expressions for the operators in question. Of course, the second of these equations gives us $*\varepsilon_{\mu\nu}$ directly.

In the linear case, we find that $\tilde{\kappa} = \kappa = *$, and the first equation says simply:

$$-\lambda^2 a^1_2 + a^1_1 * = a^2_1 + a^2_2 *, \tag{XI.149}$$

which admits a two-parameter family of solutions:

$$a^2_1 = -\lambda^2 a^1_2, \qquad a^1_1 = a^2_2. \tag{XI.150}$$

From (XI.147), the symmetries that we have been describing represent vertical transformations of $\Lambda^2 M$. More precisely, when the constitutive law described by κ defines a complex structure on each fiber of the real vector bundle $\Lambda^2 M$ (which is then an *almost-complex* structure on $\Lambda^2 M$), as long as the field equations are complex-linear the solution space is a complex vector space in its own right, and the symmetries of the solution space will include complex affine transformations. This follows directly from the fact that if F is a solution to $dF = d*F = 0$ then so is $\alpha F + \beta *F + \varepsilon = (\alpha + i\beta)F + \varepsilon$ for any constant real scalars α and β and any closed 2-form ε. Since any non-zero complex number can be expressed in polar form the multiplication of F by the complex scalar $\alpha + i\beta$ involves a rotation in the plane of F and $*F$ that one calls a *duality rotation.*

Of course, in the nonlinear case ($\tilde{\kappa} \neq \kappa = *$) one does not expect the space of solutions to be a vector space of any sort, in general. Hence, the question of the extent to which the pre-metric Maxwell equations admit the aforementioned class of symmetries must be dealt with in terms of specific cases of nonlinear constitutive laws.

Returning to the general case (XI.142), let us first examine the form of the general element on the right-hand side:

$$\begin{aligned} &a^a_b \Theta^b \\ &\quad = (a^a_1 \delta^{[\kappa}_\mu \delta^{\lambda]}_\nu + a^a_2 \tilde{\kappa}_{\mu\nu}{}^{\kappa\lambda}) dF_{\kappa\lambda} \wedge dx^\mu \wedge dx^\nu + a^a_2 \kappa_{\mu\nu}{}^{\kappa\lambda}{}_{,\lambda} F_{\kappa\lambda} dx^\lambda \wedge dx^\mu \wedge dx^\nu . \end{aligned} \tag{XI.151}$$

Our first clue as to how we can reduce the results of expanding the left-hand side of (XI.142) is to note that only 3-forms involving at most one factor of $dF_{\kappa\lambda}$ can appear with a non-zero coefficient. Furthermore, we see that the coefficients of $dF_{\kappa\lambda} \wedge dx^\mu \wedge dx^\nu$ take the general form of complex scalar multiplication, as long as the nonlinear-modified constitutive law $\tilde{\kappa}$ defines an almost complex structure, although, so far, we have only defined that operation on 2-forms over M, not 3-forms over $\Lambda^2 M$.

Taking $a = 1$, we find:

$$\mathrm{L}_X\Theta^1 = X_{\mu\nu,\lambda}\, dx^\lambda \wedge dx^\mu \wedge dx^\nu + \left(\tfrac{1}{2} X_{\mu\nu}{}^{,\kappa\lambda} + X^{[\kappa}_{\ ,\mu}\delta^{\lambda]}_\nu\right) dF_{\kappa\lambda} \wedge dx^\mu \wedge dx^\nu$$
$$- X^{\mu,\kappa\lambda}\, dF_{\kappa\lambda} \wedge dF_{\mu\nu} \wedge dx^\nu . \qquad \text{(XI.152)}$$

Hence, the first general conclusion we can infer about infinitesimal symmetries of the pre-metric electromagnetic field equations is the fact that:

$$\frac{\partial X^\mu}{\partial F_{\kappa\lambda}} = 0, \qquad \text{(XI.153)}$$

which says that the X^μ must be functions of only position, but not field strength. Therefore, they are the lifts of infinitesimal generators of spacetime diffeomorphisms; i.e., motions.

The other equations that follow from (XI.142), (XI.151), and (XI.152) are:

$$\frac{\partial X_{\mu\nu}}{\partial x^\lambda} = a_2^1 \kappa_{\mu\nu}{}^{\kappa\rho}{}_{,\lambda} F_{\kappa\sigma}\,, \qquad \frac{\partial X_{\mu\nu}}{\partial F_{\kappa\lambda}} + X^{[\kappa}_{\ ,\mu}\delta^{\lambda]}_\nu = a_1^1 \delta^{[\kappa}_\mu \delta^{\lambda]}_\nu + a_2^1 \tilde{\kappa}_{\mu\nu}{}^{\kappa\lambda}\,. \qquad \text{(XI.154)}$$

When one chooses the arbitrary function a_2^1 to be a function of only F, the first set of equations can be solved by:

$$X_{\mu\nu} = a_2^1 \kappa_{\mu\nu}{}^{\kappa\lambda} F_{\kappa\lambda} + \varepsilon_{\mu\nu}\,, \qquad \text{(XI.155)}$$

which is already included on the infinitesimal generators of complex affine transformations of the fibers of $\Lambda^2 M$, so it tells us nothing new. However, if one considers a homogeneous medium then the right-hand side of the first equations in (XI.154) vanishes, and this says that in such a case $X_{\mu\nu}$ must be functions $\varepsilon_{\mu\nu}(F)$ of only F; i.e., it is the infinitesimal generator of purely vertical translations. Between this and the previous observation about the character of X_μ , we see that the infinitesimal symmetries of the field equations for a homogeneous medium are the sums of infinitesimal generators of pure motions and pure vertical transformations; i.e.:

$$X(x, F) = X^\mu(x) \frac{\partial}{\partial x^\mu} + X_{\mu\nu}(F) \frac{\partial}{\partial F_{\mu\nu}}\,. \qquad \text{(XI.156)}$$

As for the second set of equations in (XI.154), it clear that it requires deeper analysis. It includes the special cases of pure motions and pure vertical transformations. The pure vertical transformations again give us complex scalar multiplication and translation in the fibers, but the pure motions give the equation:

$$X^{[\kappa}_{\ ,\mu}\delta^{\lambda]}_\nu = a_1^1 \delta^{[\kappa}_\mu \delta^{\lambda]}_\nu + a_2^1 \tilde{\kappa}_{\mu\nu}{}^{\kappa\lambda}\,. \qquad \text{(XI.157)}$$

The solutions to it include the possibility that $a_2^1 = 0$ and a_1^1 is a constant, which then gives the class of solutions:

$$X_\mu(x) = \alpha x^\mu + \varepsilon_\mu, \tag{XI.158}$$

in which the ε_μ are constants. These vector fields represent infinitesimal spacetime dilatations and translations.

When a_2^1 is non-vanishing, the motions thus defined are harder to interpret, unless $\tilde{\kappa}$ is expressible as an anti-symmetrized tensor product, such as in the Lorentzian case, where κ takes the form $g \wedge g$, so when one raises one index on the components of each g, the resulting components are simply $\delta_\mu^{[\kappa}\delta_\nu^{\lambda]}$, which gives the previous transformation.

It is in addressing (XI.142) for $a = 2$ that we find the most complexity arising, since we are differentiating the constitutive law in the process. The equations for the components of X that we obtain consist of three basic types: equations for the coefficients of $dF_{\alpha\beta} \wedge dF_{\kappa\lambda} \wedge dx^\mu$, equations for the coefficients of $dx^\lambda \wedge dx^\mu \wedge dx^\nu$, and equations for the coefficients of $dF_{\kappa\lambda} \wedge dx^\mu \wedge dx^\nu$.

The first type of coefficients must vanish, which gives the equations:

$$0 = X^\mu \tilde{\kappa}_{\mu\nu}{}^{\kappa\lambda,\alpha\beta} = \frac{\partial}{\partial F_{\alpha\beta}}\left(X^\mu \tilde{\kappa}_{\mu\nu}{}^{\kappa\lambda} \right), \tag{XI.159}$$

which says that $X^\mu \tilde{\kappa}_{\mu\nu}{}^{\kappa\lambda}$ must be a function of only x. Although this is always true when κ describes a linear medium, it must imply that $\tilde{\kappa}$ is a function of only x in the nonlinear case. Hence, if we differentiate (XI.138) with respect to $F_{\gamma\delta}$, while suppressing the irrelevant indices, we get the following condition on κ:

$$\frac{\partial \kappa^{\delta\gamma}}{\partial F_{\alpha\beta}} F_{\alpha\beta} = -3\,\kappa^{\gamma\delta}, \tag{XI.160}$$

which says that $\kappa^{\cdot,\gamma\delta}$ is homogeneous of degree – 3 in F.

If κ does not have this property then one must have $X^\mu = 0$, which means that the general nonlinear medium will have no symmetries due to spacetime motions.

The second type of coefficients gives equations of the form:

$$3(X^\sigma \kappa_{\mu\nu}{}^{\kappa\rho}{}_{,\sigma} F_{\kappa\rho})_{,\lambda} + X_{\kappa\rho}\, \kappa_{\mu\nu}{}^{\kappa\rho}{}_{,\lambda} = a_1^2 (\kappa_{\mu\nu}{}^{\kappa\rho} F_{\kappa\rho})_{,\lambda}. \tag{XI.161}$$

Of course, for homogeneous media these equations become trivial.

In the inhomogeneous case, when we consider $X_{\kappa\rho}$ that are purely vertical – i.e., functions of only F – these equations reduce to the form:

$$3X^\sigma \kappa_{\mu\nu}{}^{\kappa\rho}{}_{,\sigma} F_{\kappa\rho} + X_{\kappa\rho}\, \kappa_{\mu\nu}{}^{\kappa\rho} = a_1^2\, \kappa_{\mu\nu}{}^{\kappa\rho} F_{\kappa\rho} + \varepsilon_{\mu\nu}(F). \tag{XI.162}$$

This includes the possibility:

$$X^\sigma \kappa_{\mu\nu}{}^{\kappa\rho}{}_{,\sigma} = \tfrac{1}{3} a_1^2\, \kappa_{\mu\nu}{}^{\kappa\rho}, \qquad X_{\kappa\rho} = \kappa^{\mu\nu}{}_{\kappa\rho}\, \varepsilon_{\mu\nu}, \tag{XI.163}$$

in which we have made use of the invertibility of κ:

Since the $\varepsilon_{\mu\nu}$ are arbitrary functions of F, the second set of equations says that $X_{\kappa\rho}$ depend on x only by way of κ:

The first set of equations is somewhat more intriguing, since it says that the components of κ must be eigenfunctions of the first-order differential operator defined by X. This related to the more general condition:

$$L_X \kappa = \lambda \kappa; \tag{XI.164}$$

which we shall return to shortly.

The third type of coefficients gives the equations:

$$(3\kappa_{\mu\nu}{}^{\kappa\rho}{}_{,\lambda}{}^{\cdot\alpha\beta} F_{\alpha\beta} + 3\kappa_{\mu\nu}{}^{\alpha\beta}{}_{,\lambda} + \tilde{\kappa}_{\lambda\nu}{}^{\alpha\beta}{}_{,\mu})X^{\lambda} + 2X^{\lambda}{}_{,\mu}\,\kappa_{\lambda\nu}{}^{\alpha\beta} + (X_{\kappa\lambda}\tilde{\kappa}_{\mu\nu}{}^{\kappa\lambda})^{\cdot\alpha\beta}$$
$$= a_1^2 \delta_{\mu}^{[\kappa}\delta_{\nu}^{\lambda]} + a_2^2 \kappa_{\mu\nu}{}^{\alpha\beta}. \tag{XI.165}$$

Once again, in a homogeneous medium, we see that the motions include dilatations and translations, as expected. The inhomogeneous case is clearly more involved, and requires further analysis.

The purely vertical transformations that satisfy these equations amount to complex scalar multiplications and complex translations of the fibers of $\Lambda^2 M$.

One can get more of an intuition for some of the symmetries of the field equations if one applies the Lie derivative operator L_X to the expressions dF and $d{*}F$, which then give $dL_X F$ and $dL_X {*}F$, respectively. The 3-forms dF and $d{*}F$ are eigenforms of the first-order differential operator L_X iff:

$$dL_X F = \lambda' dF, \qquad dL_X {*}F = \lambda'' d{*}F \tag{XI.166}$$

for some real constants λ', λ''.

These conditions become:

$$L_X F = \lambda' F + (\text{closed 2-form}),\ L_X {*}F = \lambda'' {*}F + (\text{another closed 2-form}). \tag{XI.167}$$

Hence, F and ${*}F$ are eigenforms of L_X, modulo a closed 2-form.

One sees that these conditions are equivalent iff:

$$L_X {*} = \lambda {*}. \tag{XI.168}$$

for some real function λ.

Since ${*} = \# \cdot \kappa$, this becomes:

$$L_X \# \cdot \kappa + \# \cdot L_X \kappa = \lambda \# \cdot \kappa, \tag{XI.169}$$

which then demands that:

$$L_X \# = \alpha \#, \qquad L_X \kappa = \beta \kappa; \tag{XI.170}$$

for suitable real functions α, β.

The first equation is equivalent to:

$$L_X \mathcal{V} = (\delta X)\mathcal{V} = \alpha \mathcal{V}, \tag{XI.171}$$

for some a, which is always satisfied by setting $\alpha = \delta X$. Hence, it is not necessary for one to restrict oneself to volume-preserving diffeomorphisms.

The second equation in (XI.171) is a generalization of the conformal Killing equation for the Lorentzian metric g:

$$L_X g = \Omega^2 g, \tag{XI.172}$$

as one can see by expressing κ in the form $g \wedge g$.

Since that equation gives infinitesimal generators of diffeomorphisms of Minkowski space that preserve the light cone, it appears that the corresponding group for the more general case of pre-metric electromagnetism is defined by diffeomorphisms of $\mathbb{R}^4$ that preserve the characteristic hypersurface that is defined by the dispersion law. This would include a subgroup that preserves the characteristic polynomial itself, which would then be analogous to the Lorentz group in the quadratic case. The deeper nature of these two groups then merits further study, since its role in physics is possibly as fundamental as that of the Lorentz group.

One finds that in practice the study of symmetries of the pre-metric field equations involves first choosing a specific constitutive law κ; or at least a specific class of them, such as homogeneous linear, inhomogeneous linear, homogeneous nonlinear, etc. For some progress in this direction, one might confer Delphenich [**5**].

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CHAPTER XII

PROJECTIVE GEOMETRY AND ELECTROMAGNETISM

One of the lasting contributions that general relativity made to the foundations of physics was the idea that such an everyday natural phenomenon as gravitation could nonetheless ultimately be a manifestation of something as abstract as the geometric structure of the spacetime manifold itself. Up until Einstein succeeded in showing the link between the distribution of matter in that manifold and its curvature, the study of non-Euclidian geometries had been of interest primarily to pure mathematicians, some of whom, such as William Kingdon Clifford and Henri Poincaré, had already speculated on the possibility of such a link.

A predictable consequence of the revelation that spacetime geometry was at the root of gravitation was that theoretical physics devoted more attention to the "geometrization" of all of its other fundamental principles. Mechanics drifted further into the geometry of contact and symplectic manifolds, gauge field theory focused more on the description of gauge structures for field theories in terms of connections on principal bundles, and the conjecture emerged that perhaps one could generalize spacetime geometry in some way that would encompass the other fundamental forces (or fundamental *interactions*, to the quantum field theorists).

In the original form of the field unification problem, Einstein conjectured that there was such a broadening of the scope of spacetime geometry that would subsume both gravitation and electromagnetism. He made many attempts along these lines, along with Mayer, Cartan, Oskar Klein, Kaluza, Jordan, Schrödinger, Veblen, Vranceanu, and others [1]. However, the most common results of these theories were either elegant mathematical theories that nevertheless admitted unphysical solutions or consistent unifications of gravitation and electromagnetism that said nothing new about either; one might call such a theory a "concatenation" of the two field theories.

The scope of the unification problem eventually expanded to include all of the fundamental interactions, presumably in a gauge theory for a suitably large gauge group, but the new obstacle emerged that the mathematical formalism of quantum field theories was mostly oriented towards describing the interaction of matter in the scattering approximation, not the specific details of the time evolution of the system during the actual time interval in which the interaction takes place. Hence, since general relativity does not spend much time addressing the scattering of gravitating bodies, any more than most quantum field theories address the Cauchy problem for field solutions, simply trying to reconcile the mathematical methodologies become a complex problem, in its own right [2].

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[1] A good historical discussion of some of these attempts can be found in the book by Vizgin [**1**], and a more mathematically detailed discussion of two them – viz., the Kaluza-Klein and Einstein-Schrödinger theories – can be found in Part II of Lichnerowicz [**2**].

[2] An illuminating insight into the difference between the general relativistic approach to fundamental interactions and the quantum field theoretic approach can be gleaned from reading the Feynman lectures on

We have seen how the theory of electromagnetism can be formulated in the absence of a background spacetime metric, since the Lorentzian structure eventually "emerges" as a consequence of the dispersion law that follows from the constitutive law, by way of the field equations. This suggests that perhaps the unification problem that Einstein posed, although reasonable by analogy to the unification of electricity and magnetism into electromagnetism, might still be the wrong problem to pose. Indeed, we already have a definitive link between the two theories by way of the fact that the Lorentzian structure that accounts for gravitation consists of *light* cones, not *gravity* cones; i.e., cones that pertain to the propagation of electromagnetic waves, not merely the propagation of gravitational waves.

The question then arises of how one might still geometrize pre-metric electromagnetism in the absence of a spacetime metric. In order to address this question, one might consider the geometrical hierarchy that was proposed by Felix Klein, who once decreed that "projective geometry is all geometry." From that level of generality, one could then reduce one's scope to affine geometry or metric geometry by considering the hyperplane at infinity or introducing a metric on the projective space, respectively. One could also introduce the metric on an affine space, which is more akin to the approach of Riemannian geometry and general relativity. The resulting hierarchy can then be schematically described as in Fig. 13.

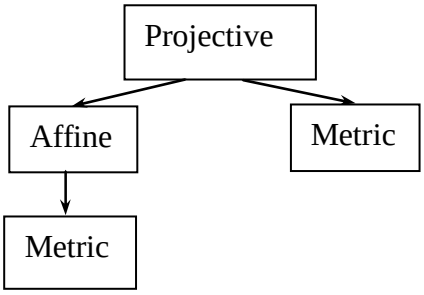

Figure 13. Klein's hierarchy of geometries.

What we shall attempt to do in this chapter is establish that projective geometry plays a role at the fundamental level of physics in both mechanics and field theories. We begin by summarizing some of the relevant notions of projective analytic geometry [**4-7**], and then show how some of these notions play a role in mechanics. We then go on to discuss the use of the algebra of exterior forms and multivector fields to represent projective geometric situations by way of the Plücker-Klein embedding, and then apply that to the exterior forms and multivector fields that are of interest in electromagnetism.

It is our hope that by establishing the role of projective geometry in physics at the fundamental level, this might bring about a shift of emphasis in physics from the Riemannian geometry defined by a metric on the tangent bundle to the Kleinian geometry of subspaces in the tangent spaces, as they are represented by the exterior algebras over the tangent and cotangent bundles. The fundamental physical concept that supplants the

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gravitation [**3**], in which he attempts to extend the methodology of quantum field theory to imply the Einstein equations by regarding the graviton as an elementary particle like any other, while avoiding the necessity of differential geometry, which he apparently regarded as "fancy-schmancy mathematics."

distribution of massive matter is then the electromagnetic constitutive law, which might itself be determined by the distribution of charged matter, by way of polarization. It is entirely reasonable, considering the relative magnitudes of the coupling constants, that gravity is then simply the shadow cast by electromagnetism.

1. Elementary projective geometry. Something else that Klein once said, which has received considerably more attention by the mathematics community at large, was the suggestion that he made in his Erlangen Programme that geometries could be best characterized by the invariants of group of transformations that were distinctive to the class of problems at hand. For instance, the fundamental concept in affine geometry is parallelism, so the relevant group is the group of transformations of affine space that preserve the parallelism of lines, which is then the group of translations. In metric geometry, the key concept is that of the distance between points of the space, which implies that the relevant group is the group of isometries of the space; in conformal geometry, distance is replaced by angle, and the isometry group is then enlarged to the conformal isometry group. Note that in order to make sense of an angle one requires the presence of *two* lines, which then suggests that one is implicitly considering the points of a projective space; i.e., and angle between two lines through a point in an affine space is equivalent to a distance between two points in the corresponding projective space.

As Hilbert and Cohn-Vossen envisioned the situation [**8**], the key concept in projective geometry was that of *incidence*, which refers to the way that subspaces of projective space intersect. Hence, the relevant group that pertains to projective geometry should be the group of transformations of a projective space that preserve the incidence of subspaces. We shall then begin by clarifying the concepts of projective subspaces and incidence.

a. Projective subspaces and incidence. For the sake of (physically useful) generality, we first assume our field $\mathbb{K}$ of scalars is either $\mathbb{R}$ or $\mathbb{C}$, since many of the definitions and results of projective geometry work the same in either case. In most cases, the differences do not appear until one starts looking at solving algebraic equations or making use of the conjugation operator.

Although a projective space, such as $\mathbb{K}\mathrm{P}^n$, is defined by a process of projectivization, i.e., projecting from a vector space, such as $\mathbb{K}^{n+1}$, to the set of all lines through its origin, for the purposes of projective geometry it is better to regard the elements of that set as points of a projective space, not lines in a vector space. Similarly, under the projection $\mathbb{K}^{n+1} - \{0\} \rightarrow \mathbb{K}\mathrm{P}^n$, a 2-plane through the origin projects to a line in a projective space, and, more generally, a k+1-plane through the origin of $\mathbb{K}^{n+1}$ projects to a k-plane in $\mathbb{K}\mathrm{P}^n$. We shall then call a subset of $\mathbb{K}\mathrm{P}^n$ that is the image of a a k+1-plane through the origin of $\mathbb{K}^{n+1}$ under this projection a *k-dimensional (projective) subspace* of $\mathbb{K}\mathrm{P}^n$.

As subsets, projective subspaces can be partially ordered by inclusion; i.e., one can speak of a subspace as being a subspace of some other subspace or not. However, this is not precisely the relation of incidence, which, unlike a partial ordering, is symmetric. In particular, two subspaces are *incident* iff one of them is a subspace of the other one [3]. For instance, a point and a line are incident iff the point in question is one of the points of the line, while two lines are incident iff they coincide. One then sees that it is possible for two subspaces to be neither incident nor disjoint, such as when a line intersects a plane without being contained in it.

One of the binary operations that pertain to the partial ordering of subsets that carries over to the partial ordering of subspaces is that of intersection. However, one refers to the intersection of two subspaces S_1 and S_2 as their *meet* and denotes it by $S_1 \wedge S_2$. Under the defining projection of $\mathbb{K}^{n+1} - \{0\}$ onto $\mathbb{K}\mathrm{P}^n$, the meet of two projective subspaces will be the image of the intersection of the linear subspaces that projected onto them. One also sees that $-1 \leq \dim(S_1 \wedge S_2) \leq \min\{\dim(S_1), \dim(S_2)\}$; by convention, $\dim\{\varnothing\} = -1$.

For example, the intersection of a line and a plane in $\mathbb{K}\mathrm{P}^3$ can be either a point or a line, in which case, the line is incident on the plane. The intersection of two planes can be a line or a plane since they are projections of 3-planes in $\mathbb{K}^4$, which can intersect in a 2-plane or a 3-plane. The only pairs of projective subspaces of $\mathbb{K}\mathrm{P}^3$ that can intersect vacuously are a point and some subspaces of any dimension and some pairs of lines.

One sees why the concept of parallelism is not as useful in the context of projective geometry by considering the intersection of two lines in $\mathbb{K}\mathrm{P}^2$. Since they both represent the projections of 2-planes through the origin of $\mathbb{K}^3$, and all such 2-planes must intersect in a linear subspace of dimension one or two the projection of their intersection will be either a point or a line in $\mathbb{K}\mathrm{P}^2$, but a non-vacuous subset, in either event. In other words, any two lines in $\mathbb{K}\mathrm{P}^2$ must intersect, so no lines can be parallel to each other. However, as we shall discuss below, if one regards a projective space as an affine space plus a hyperplane "at infinity" then one is saying that all of the parallel lines in the affine space will intersect at points in the hyperplane at infinity. Hence, one sees how projective spaces can be regarded as the completions of affine spaces, in one sense. This makes the claim of Klein about the level of generality that projective geometry represents seem more reasonable.

By contrast, one does not usually consider the union of two subspaces, as much as their *join*, which is denoted by $S_1 \vee S_2$. This is then the smallest subspace (with respect to inclusion) that contains both subspaces. Under the defining projection, the join of two projective subspaces will be the projection of the join of the linear subspaces that projected onto them, in the sense of the smallest linear subspace that contained both of them. This join will then be generated by all finite linear combinations of vectors from both spaces. Hence, $\max\{\dim(S_1), \dim(S_2)\} \leq \dim(S_1 \vee S_2) \leq \dim(S_1) + \dim(S_2)$. In fact:

[3] More elaborate axioms for incidence than the ones give here can be found in Van der Waerden [**9**].

$$\dim(S_1 \vee S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \wedge S_2). \qquad \text{(XII.1)}$$

For example, the join of two distinct points is a line and the join of a line and a point not on that line is a plane. When two linear subspaces V_1 and V_2 of $\mathbb{K}^{n+1}$ are transversal, in the sense that $V_1 \wedge V_2 = 0$ and $V_1 \vee V_2 = \mathbb{K}^{n+1}$ the corresponding condition on projective subspaces S_1 and S_2 of $\mathbb{K}\mathrm{P}^n$ is that $S_1 \wedge S_2 = \varnothing$ and $S_1 \vee S_2 = \mathbb{K}\mathrm{P}^n$. That is, they are disjoint and their join is the entire space.

Although one cannot speak of linear combinations of elements in $\mathbb{K}\mathrm{P}^n$, and consequently, bases for subspaces, one can speak of "projective frames." A set $\{p_0, \ldots, p_k\}$ of k+1 elements $p_i \in \mathbb{K}\mathrm{P}^n$ is said to be a *projective frame* [4] for the k-dimensional subspace S_k iff S_k is the join $p_0 \vee \ldots \vee p_k$ of all its elements (one can define that notion by recursion since the join operation is associative) and no proper subset of that set will span S_k. Hence, two distinct points frame a line, three non-collinear points frame a plane, and so on. The pre-images of the points in a projective frame then define a linear frame for the pre-image of S_k .

When given the meet and the join operations, the partially ordered set of projective subspaces of $\mathbb{K}\mathrm{P}^n$ becomes a *lattice* (see, e.g., Birkhoff [**10**]). It also has a greatest element and a least element in the form of $\mathbb{K}\mathrm{P}^n$ and $\varnothing$, respectively. Hence, every subspace is a subspace of $\mathbb{K}\mathrm{P}^n$ and $\varnothing$ is a subspace of every subspace.

This lattice is, moreover, a *complemented* lattice in the sense that for every subspace S there is a subspace S' such that $S \vee S' = \mathbb{K}\mathrm{P}^n$. However, this complement is by no means unique, which would be the case for an *orthocomplemented* lattice. An example of such a lattice is given by the linear subspaces of an orthogonal space, since one can uniquely specify an orthogonal complement to any subspace in that event.

In one sense, we can regard projective geometry as being primarily concerned with transformations of $\mathbb{K}\mathrm{P}^n$ that preserve its lattice of projective subspaces, in the sense that meets go to meets and joins go to joins. Consequently, such a transformation will also preserve incidence.

b. Duality. One can just as well define the projectivization of $\mathbb{K}^{n*} = (\mathbb{K}^n)^*$ by looking at all of the lines through its origin. The resulting projective space will then be denoted by $\mathbb{K}\mathrm{P}^{n*}$, although it will be projectively equivalent to $\mathbb{K}\mathrm{P}^n$, in a sense that we shall define shortly. We shall refer to $\mathbb{K}\mathrm{P}^{n*}$ as the *dual* of the projective space $\mathbb{K}\mathrm{P}^n$.

[4] The classical term for a projective frame was a "reference simplex," or "reference tetrahedron," in the three-dimensional case. We shall not use that terminology since the concept of a frame is closer to the group-theoretic methods of geometry.

The lattice of projective subspaces of $\mathbb{K}\mathrm{P}^{n*}$ will be in one-to-one correspondence with that of $\mathbb{K}\mathrm{P}^n$, except that a k-dimensional projective subspace of $\mathbb{K}\mathrm{P}^{n*}$ will define an $n–k$-dimensional projective subspace of $\mathbb{K}\mathrm{P}^n$, by annihilation. That is, all of the elements of the linear subspace V_{k+1} of $\mathbb{K}^{n+1}$ that is the pre-image of some k-dimensional projective subspace S_k of $\mathbb{K}\mathrm{P}^n$ will be annihilated by all of the elements of some unique subspace V^*_{n-k} of $(\mathbb{K}^{n+1})^*$ that is the pre-image of a unique $n–k$-dimensional subspace S^*_{n-k} of $\mathbb{K}\mathrm{P}^{n*}$. As we shall see, the fact that any k-dimensional subspace in $\mathbb{K}\mathrm{P}^n$ is associated with a unique $n–k$-dimensional subspace in $\mathbb{K}\mathrm{P}^{n*}$ is at the root of the Poincaré duality between k-vectors and $n–k$-forms that we have been using all along.

The one-to-one correspondence between subspaces of $\mathbb{K}\mathrm{P}^n$ and subspaces of $\mathbb{K}\mathrm{P}^{n*}$ does not, however, preserve the operations of meet and join. Rather, it inverts them, just as subset complementation inverts the operations of intersection and union. That is, the dual of the meet of two subspaces in $\mathbb{K}\mathrm{P}^n$ is the join of the duals of the subspaces, and conversely.

One must be cautioned at this point not to assume that the one-to-one correspondence between *subspaces* in $\mathbb{K}\mathrm{P}^n$ and subspaces of its dual implies that there is an actual one-to-one correspondence between the *points* of these spaces. Indeed, under the present duality, a point in one space is associated with a *hyperplane* in the other. Such an association of points would be called a "correlation," which we shall discuss below.

c. Hyperplane at infinity. As we pointed out in chapter II, the projective spaces $\mathbb{K}\mathrm{P}^n$ cannot be covered by a single coordinate chart, since they are all compact, while $\mathbb{K}^n$ is not. However, it is in examining the homogeneous coordinate charts for $\mathbb{K}\mathrm{P}^n$ that we gain a deeper insight into the relationship between affine and projective spaces.

Let $(x^0, x^1, \ldots, x^n)$ be one such chart that projects onto the corresponding inhomogeneous chart $(X^1, \ldots, X^n)$ by the prescription $X^i = x^i/x^0$. For each $x^0 \neq 0$, this map is a diffeomorphism of the affine subspace of $\mathbb{K}^{n+1}$ that is parameterized by setting x^0 equal to a constant and allowing the remaining coordinates to vary arbitrarily in the vector space $\mathbb{K}^n$.

However, as x^0 goes to 0 the inhomogeneous coordinates all become indefinitely large. Hence, none of the points of $\mathbb{K}^{n+1}$ that take the form $(0, x^1, \ldots, x^n)$ project to inhomogeneous coordinates. One refers to the points of this hyperplane as the *hyperplane at infinity.* Nevertheless, they still describe points of $\mathbb{K}\mathrm{P}^n$, even though they

do not define inhomogeneous coordinates. Indeed, one can regard $\mathbb{K}\mathrm{P}^n$ as consisting of the affine space $\mathbb{K}^n$ completed by the addition of the hyperplane at infinity, which also renders the resulting space compact.

For instance, if one looks at $\mathbb{K}\mathrm{P}^1$, which consists of lines through the origin of the plane, in terms of the homogeneous coordinates (x^0, x^1) and represents the inhomogeneous coordinates by $(1, X)$ with $X = x^1/x^0$ then it becomes clear that the point of $\mathbb{K}\mathrm{P}^1$ that gets left out by the omission of $x^0 = 0$ is the vertical line in the plane. The inhomogeneous coordinate X is, in fact, the tangent of the angle between the line through the point (x^0, x^1) and the x^0 axis. Hence, to completely describe all of the lines through the origin, one must add the "point at infinity" that is described by the vertical line. We illustrate this situation in Fig. 14. One should also notice that since all of the lines through the origin in this case are described by either homogeneous coordinates with $x^0 > 0$ or $x^0 < 0$ individually, the projection of $\mathbb{K}^2 - (0, x^1)$ onto $\mathbb{K}\mathrm{P}^1$ is two-to-one and thus represents a double covering map. This fact is at the root of the spin representations of the orthogonal groups.

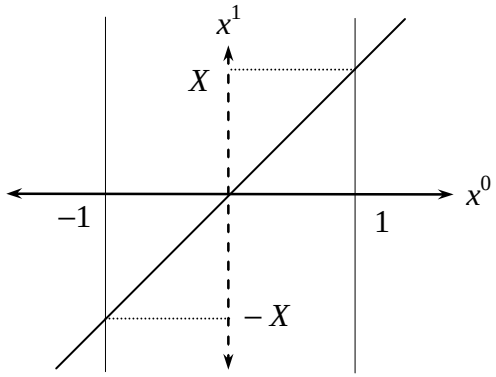

Figure 14. The projective line as an affine line plus a point at infinity.

d. Projective transformations. We shall call a map f: $\mathbb{K}\mathrm{P}^n \to \mathbb{K}\mathrm{P}^m$ of an n-dimensional projective space to an m-dimensional one a *projective transformation* iff it maps the lattice of projective subspaces of $\mathbb{K}\mathrm{P}^n$ into the lattice of projective subspaces of $\mathbb{K}\mathrm{P}^m$ consistently. That is, if $S \subset S'$ in $\mathbb{K}\mathrm{P}^n$ then $f(S) \subset f(S')$, and for any subspaces S_1, S_2 one has $f(S_1 \wedge S_2) \subset f(S_1) \wedge f(S_2)$, $f(S_1 \vee S_2) \subset f(S_1) \vee f(S_2)$.

Under the projectivization $\mathbb{K}^{n+1} - \{0\} \to \mathbb{K}\mathrm{P}^n$, linear subspaces of $\mathbb{K}^{n+1}$ go to projective subspaces of $\mathbb{K}\mathrm{P}^n$, and one finds that a projective transformation of $\mathbb{K}\mathrm{P}^n$ to $\mathbb{K}\mathrm{P}^m$ is covered by a map of $\mathbb{K}^{n+1}$ to $\mathbb{K}^{m+1}$ that takes linear subspaces of $\mathbb{K}^{n+1}$ to linear subspaces of $\mathbb{K}^{m+1}$ in a manner that preserves the lattice of linear subspaces.

This way of characterizing projective transformations subsumes the classical concept of a *collineation*, which takes lines in $\mathbb{K}^{n+1}$ to other lines in $\mathbb{K}^{n+1}$, hence, points of $\mathbb{K}\mathrm{P}^n$ to other points. However, since a projective transformation must, among other things, preserve all joins, and any k-dimensional projective subspace can be expressed as the join of $k+1$ points, one sees that it is sufficient to specify the effect on points.

Of course, in order to preserve the dimension of subspaces, one must be dealing with invertible projective transformations. We call such invertible projective transformations *projective equivalences.*

When one considers how a projective transformation must take projective frames to projective frames, hence, it must be covered by a map that takes linear frames to linear frames, one sees that projective transformations must be covered by linear transformations. Of course, they are not unique, but are only defined up to scalar multiplication. The usual way of representing these linear transformations is:

$$\rho y^a = A^a_i x^i, \quad \rho \neq 0. \tag{XII.2}$$

One then sees that a projective transformation of $\mathbb{K}\mathrm{P}^n$ to $\mathbb{K}\mathrm{P}^m$ corresponds to a line through the origin in the vector space $L(n+1, m+1)$ of linear maps from $\mathbb{K}^{n+1}$ to $\mathbb{K}^{m+1}$. That is, it defines a point in the projective space that is defined by that vector space.

In the invertible case, one must have $n = m$ to begin with, and the image of a projective k-frame under the map must be a projective k-frame for all $0 < k \leq n$. As for the linear transformation that covers the projective transformation, the determinant of the matrix A^i_j must be non-vanishing (for any linear frame on $\mathbb{K}^{n+1}$). Hence, one is looking at the intersection of a line through the origin in the $\mathfrak{gl}(n+1; \mathbb{K})$ with the elements of the group $GL(n+1; \mathbb{K})$. One finds that this set $PGL(n; \mathbb{K})$ of line intersections is a group under the multiplication of matrices since invertible matrices that are defined up to a multiplicative scalar constant will multiply to produce another invertible matrix that is also defined up to a multiplicative scalar constant. This group is referred to as the *projective linear group* in dimension n.

One sees that each of these equivalence classes contains an element of $SL(n+1; \mathbb{K})$ that is unique, up to sign, since:

$$\det(-A) = (-1)^{n+1} \det A. \tag{XII.3}$$

Hence, when $n+1$ is even – i.e., when n is odd – a change of sign will make no change in the determinant. One can then say that:

$$PGL(n; \mathbb{K}) \cong \begin{cases} SL(n+1; \mathbb{K}), & n \text{ even}, \\ SL(n+1; \mathbb{K})/\mathbb{Z}_2, & n \text{ odd}. \end{cases}$$

These invertible projective transformations are called, variously, *homographies, collineations,* and *projectivities* in the literature. They are ultimately generated by finite products of elementary transformations called *perspectivities*, which show us how projective geometry relates to the older study of perspective in visual perception. A typical example of how a perspectivity, relative to a point *P*, takes a projective 3-frame *ABC* for a projective plane to another projective 3-frame *A′ B′ C′* is illustrated in Fig. 15. One can think of the three-dimensional space in which the connecting lines *PAA′, PBB′, PCC′* exist as the space of homogeneous coordinates.

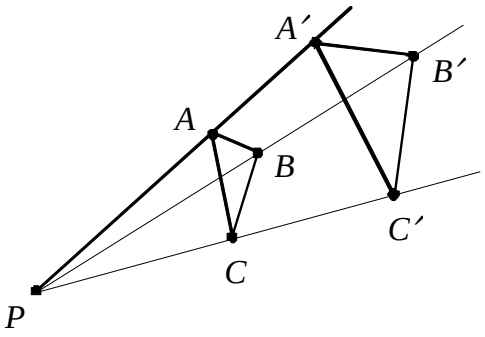

Figure 15. An example of a perspectivity in the projective plane.

It is important to see how the linear transformations of homogeneous coordinates relate to transformations of inhomogeneous coordinates because one finds that generally they will not be linear, but rational transformations. In particular, let $(x^0, x^1, \ldots, x^n)$ be homogeneous coordinates on $\mathbb{K}\mathrm{P}^n$ with corresponding inhomogeneous coordinates $(X^1, \ldots, X^n)$, and similarly one has $(y^0, y^1, \ldots, y^m)$ and $(Y^1, \ldots, Y^m)$ for $\mathbb{K}\mathrm{P}^m$. For convenience, we assume that x^0 and y^0 are non-zero, so $X^i = x^i/x^0$ and $Y^a = y^a/y^0$.

If a linear map A: $\mathbb{K}^{n+1} \to \mathbb{K}^{m+1}$ is represented by a matrix A^α_μ then the transformation of X^i into Y^a that is induced by A is:

$$Y^a = y^i/y^0 = \frac{A^a_0 x^0 + A^a_i x^i}{A^0_0 x^0 + A^0_i x^i} = \frac{A^a_0 + A^a_i X^i}{A^0_0 + A^0_i X^i}. \qquad \text{(XII.4)}$$

In the invertible case, for which $m = n$, these transformations, which are sometimes called *fractional bilinear* transformations, then fall into four elementary categories:

1) *Homotheties:* In this case, the only non-zero submatrices of A^μ_ν are A^0_0 and A^i_j, which equals δ^i_j, so X^i goes to $A^0_0 X^i$.

2) *Translations:* For these, the non-zero matrices are $A^0_0 = 1$, $A^i_j = \delta^i_j$, and at least one component of A^i_0. The transform of X^i is then $X^i + A^i_0$.

3) *Inversions:* Now, $A^0_0 = A^i_0 = 0$, $A^i_j = \delta^i_j$, and at least one A^0_j is non-vanishing. These transformations take X^i to $X^i / A^0_j X^j$, as long as X^i does not live in the hyperplane $A^0_j X^j = 0$. One sees that points of the affine hyperplane $A^0_j X^j = 1$ will be fixed by these transformations. Hence, the inversion in question is an inversion *through* the affine hyperplane $A^0_j X^j = 1$.

4) *Linear transformations:* $A^0_0 = 1$ and A^i_j are the only non-zero submatrices.

In general, one can express A^{μ}_{ν} in block matrix form as:

$$A^{\mu}_{\nu} = \left[\begin{array}{c|c} A^0_0 & A^0_j \\ \hline A^i_0 & A^i_j \end{array}\right]. \qquad \text{(XII.5)}$$

The direct sum decomposition of $\mathbb{K}^{n+1}$ that this corresponds to is $\mathbb{K} \oplus \mathbb{K}^n$, in which the first summand consists of all (x^0, 0, …, 0) while the second summand represents all (0, x^1, …, x^n), which is the hyperplane at infinity. One then sees that the affine subgroup $A(n; \mathrm{K})$ of $SL(n+1; \mathbb{K})$ consists of those elements that take the hyperplane at infinity to itself, which will then have $A^0_j = 0$; the stray factor of A^0_0 can be eliminated by the constraint that the determinant of A^{μ}_{ν} is unity. The affine group in n dimensions can also be represented by the transformations that take the affine hyperplane $x^0 = 1$ to itself, which implies that A^0_0 must be set to unity.

e. Correlations. As pointed out above, the duality that associates each k-plane in $\mathbb{K}\mathrm{P}^n$ with a unique $n–k$-plane in $\mathbb{K}\mathrm{P}^{n*}$ is not sufficiently fine-grained to also associate a point of $\mathbb{K}\mathrm{P}^n$ with a point of $\mathbb{K}\mathrm{P}^{n*}$. Such an invertible map $[C]$: $\mathbb{K}\mathrm{P}^n \to \mathbb{K}\mathrm{P}^{n*}$, is called a *correlation*. Under projectivization, it will be covered by an invertible linear map C: $\mathbb{K}^{n+1} \to (\mathbb{K}^{n+1})^*$, which is unique up to a non-zero scalar multiple.

Even though the points of $\mathbb{K}\mathrm{P}^{n*}$ no longer represent linear functionals, as the elements of $(\mathbb{K}^{n+1})^*$ do, nonetheless one can still speak of the evaluation of an element $[\alpha] \in \mathbb{K}\mathrm{P}^{n*}$ on an element $[\mathbf{v}] \in \mathbb{K}\mathrm{P}^n$. The trick is to understand that the "field of projective scalars" associated with $\mathbb{K}$ consists of 0 and "$\neq 0$." That is, the only issue in the eyes of projective geometry is whether the number $\alpha(\mathbf{v})$ is zero or not. Since any choices of representive non-zero vectors and covectors $\mathbf{v}$ and α for the points $[\mathbf{v}]$ and $[\alpha]$ will differ by non-zero scalar multiples, so will all possible values of the number $\alpha(\mathbf{v}) \in \mathbb{K}$. Hence, $\alpha(\mathbf{v})$ will either be zero for all representatives or non-zero for all representatives and the evaluation $[a][\mathbf{v}]$ will either be zero or non-zero unambiguously.

The geometric interpretation for the vanishing or non-vanishing of $[\alpha][\mathbf{v}]$ is then based in incidence: $[\alpha][\mathbf{v}] = 0$ iff the point $[\mathbf{v}]$ is incident on the hyperplane $[\alpha]$.

One can then define the evaluation $[C][\mathbf{v}][\mathbf{w}]$ of $[C][\mathbf{v}] \in \mathbb{K}\mathrm{P}^{n*}$ on the point $[\mathbf{w}] \in \mathbb{K}\mathrm{P}^n$, which allows one to define the "projectively bilinear" functional on $\mathbb{K}\mathrm{P}^n$:

$$[C]([\mathbf{v}], [\mathbf{w}]) = [C][\mathbf{v}][\mathbf{w}], \qquad \text{(XII.6)}$$

whose value will either be zero or not. Geometrically, one sees that $[C]([\mathbf{v}], [\mathbf{w}])$ will vanish iff the line $[\mathbf{w}]$ is incident on the hyperplane $[C][\mathbf{v}]$.

This functional on $\mathbb{K}\mathrm{P}^n$ will be covered by a bilinear functional on $\mathbb{K}^{n+1}$:

$$C(\mathbf{v}, \mathbf{w}) = C(\mathbf{v})(\mathbf{w}). \tag{XII.7}$$

Since the linear map C: $\mathbb{K}^{n+1} \rightarrow (\mathbb{K}^{n+1})^*$ is defined only up to a nonzero scalar multiple, so is the bilinear functional that it defines. Hence, we are now dealing with a point in the projective space that is obtained from the vector space $(\mathbb{K}^{n+1})^* \otimes (\mathbb{K}^{n+1})^*$.

One can then consider the symmetry of the bilinear form $[C]$. When the functional $[C]$ is symmetric one calls the map $[C]$ itself a *polarity*. Hence, $[\mathbf{v}]$ is incident on $[C][\mathbf{w}]$ iff $[\mathbf{w}]$ is incident on $[C][\mathbf{v}]$. Furthermore, if a point $[\mathbf{v}] \in \mathbb{K}\mathrm{P}^n$ is regarded as the *pole* of a polarity then the point $[C][\mathbf{v}] \in \mathbb{K}\mathrm{P}^{n^*}$ is then regarded as its *polar*, in classical terminology.

The bilinear map C that projects to $[C]$ can be either symmetric or anti-symmetric. Either are called *involutions* since the expression $[C]([\mathbf{v}], [\mathbf{w}])$ will not have a sign, and will then be sensitive to the symmetry of C only by means of the absolute value; i.e., $[C]([\mathbf{v}], [\mathbf{w}])$ is symmetric iff $|C(\mathbf{v}, \mathbf{w})| = |C(\mathbf{w}, \mathbf{v})|$ for any representative non-zero vectors $\mathbf{v}, \mathbf{w} \in \mathbb{K}^{n+1}$. When C is anti-symmetric, the map $[C]$ is called a *null correlation*.

Of particular interest is the case where $[C][\mathbf{v}][\mathbf{v}]$ vanishes, since this means that $[\mathbf{v}]$ is incident on the hyperplane $[C][\mathbf{v}]$; such a point will be called *isotropic*, since it is the projection of a non-zero vector $\mathbf{v} \in \mathbb{K}^{n+1}$ such that $C(\mathbf{v}, \mathbf{v}) = 0$. Clearly, when C is anti-symmetric this will always be the case. The set of all isotropic vectors in $\mathbb{K}^{n+1}$ is therefore a quadric hypersurface. If all $\mathbf{v}$ are isotropic then C must be anti-symmetric since:

$$0 = C(\mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w}) = -[C(\mathbf{w}, \mathbf{v}) + C(\mathbf{v}, \mathbf{w})] \tag{X.8}$$

when all vectors are isotropic.

f. Homogeneous functions. The usage of the words "homogeneous" and "inhomogeneous" in the context of coordinates for projective space is actually quite consistent with their usage in the context of functions. This is due to the fact that any homogeneous function on $\mathbb{K}^{n+1}$ is associated with a unique inhomogeneous function on $\mathbb{K}\mathrm{P}^n$ and conversely, since if $f(\mathbf{x})$ is homogeneous of degree r on $\mathbb{K}^{n+1}$ then, by definition:

$$f(\lambda\mathbf{x}) = \lambda^r f(\mathbf{x}) \tag{X.9}$$

for all scalars $\lambda \in \mathbb{K}$. This implies that $f(\mathbf{x}) = 0$ iff $f(\lambda\mathbf{x}) = 0$ for all λ. Hence, f has the same value as a projective scalar (viz., 0 or $\neq 0$) for all representatives $\mathbf{x}$ of the point $[\mathbf{x}] \in \mathbb{K}\mathrm{P}^n$. This means that the level hypersurface $f(\mathbf{x}) = 0$ in $\mathbb{K}^{n+1}$ projects to a level hypersurface $[f][\mathbf{x}] = 0$ in $\mathbb{K}\mathrm{P}^n$.

The case in which $r = 0$ defines a class of homogeneous functions that one calls *ray functions*, since they are constant on the lines generated by the non-zero $\mathbf{x}$ in $\mathbb{K}^{n+1}$. They therefore define functions on $\mathbb{K}\mathrm{P}^n$ by the association $[f][\mathbf{x}] = f(\mathbf{x})$ for any $\mathbf{x}$ that generates $[\mathbf{x}]$.

The way that one defines the inhomogeneous function $[f]$ on $\mathbb{K}\mathrm{P}^n$, when it is described by inhomogeneous coordinates $X^i = x^i/x^0$ is:

$$[f](X^1, \ldots, X^n) = f(x^0, x^i) = (x^0)^r f(1, X^i), \tag{X.10}$$

when one has chosen a non-zero value of x^0; of course, if f is a ray function then this choice is immaterial.

Conversely, when one is given $[f]$ one can define the homogeneous function f by means of:

$$f(x^0, x^i) = [f](x^i/x^0), \tag{X.11}$$

as long as x^0 is non-vanishing.

One finds that a polynomial function of degree d on $\mathbb{K}^{n+1}$ is homogeneous iff it is the sum of monomials of degree d. In particular, quadratic forms are homogeneous of degree two.

Of particular interest in classical geometry are the quadratic functions. If $\mathbb{K}$ represents the space of inhomogeneous coordinates X for $\mathbb{K}\mathrm{P}^1$ then an inhomogeneous quadratic polynomial:

$$[P][X] = aX^2 + bX + c \tag{X.12}$$

corresponds to the homogeneous polynomial on $\mathbb{K}^2$:

$$P[x, y] = ax^2 + bxy + cy^2 \tag{X.13}$$

when one omits the factor $1/y^2$; as long as the only issue is the vanishing of the polynomials, this is permissible. If the level hypersurface $[P] = 0$ consists of two points X_1 and X_2 in $\mathbb{K}\mathrm{P}^1$ – viz., the roots of the polynomial – then the level hypersurface $P = 0$ consists of two lines in $\mathbb{K}^2$ that are generated by any points (x_1, y_1) and (x_2, y_2) such that $X_i = y_i/ x_i$, $i = 1, 2$.

Going to the next dimension, an inhomogeneous polynomial on $\mathbb{K}\mathrm{P}^2$:

$$[P][X, Y] = aX^2 + bXY + cY^2 + dX + eY + f \tag{X.14}$$

becomes, when one sets $X = x/z$ and $Y = y/z$:

$$P[x, y, z] = ax^2 + bxy + cy^2 + dxz + eyz + fz^2; \tag{X.15}$$

again, we have eliminated the irrelevant multiplicative factor $1/z^2$.

We now see that the quadratic polynomials in two variables, which describe conic sections, also correspond to quadratic forms on $\mathbb{K}^3$. Indeed, that is the space in which the cone is defined, and a section of the cone is defined by a choice of intersecting affine plane in $\mathbb{K}^3$. The fact that one is dealing with a cone is due to the fact that the homogeneity of the function $[P]$ implies that the level hypersurface $[P] = 0$ contains the lines through each of its points and the origin. When $\mathbb{K} = \mathbb{R}$, one sees that the quadratic form $[P]$ must be hyperbolic in order for $[P] = 0$ to consist of anything but the origin.

We shall discuss the case $\mathbb{K} = \mathbb{R}$, $n = 3$ in the section since it is directly relevant to special relativity.

2. Projective geometry and mechanics. In order to justify the importance of a purely mathematical concept or technique in physics, one must show its point of application. Furthermore, that point of application can be either specialized or fundamental in character, where a specialized application of a mathematical technique is generally of no interest outside of some particular problem. For instance, a trick that makes an integral more manageable would fall into this category.

The purpose of this chapter is to show that projective geometry can be applied to the mathematical modeling of physical phenomena in many contexts, including at the most fundamental level of mechanics and field theories. In effect, in this section we will be going beyond the scope of pre-metric electromagnetism into "pre-metric mechanics" to show that the issues in physics that touch upon the methods and concepts of projective geometry are found at the very root level of physics itself, namely, the process of measurement that serves as the empirical, phenomenological basis for the theories that emerge eventually.

a. The geometry of measurement. The simplest level at which relativistic physics and quantum physics overlap seems to be at the level of physical measurements and observations. Indeed, one can think of an observation as a special kind of measurement, namely, a *passive* measurement. By this, we mean that the measurer/observer is not interacting with the source of the information that is being measured, only the information that is being received. The obvious example of such measurements would be the ones that are made by astronomers concerning the photons that are emitted by distant stars.

By contrast, an *active* measurement involves the measurer emitting information that is intended to interact with the system in question so that the measurement will involve the way that the information that comes back from the system has been altered by the state of

the system. This is the spirit of elementary particle physics, which depends upon collisions of test particles with target particles to deduce information about the structure of the state space – i.e., the field space – in which the particle/fields live. It also accounts for most of the measurements of geophysics that are used to construct models for the Earth's interior, although the mere act of recording an unprovoked earthquake would constitute a passive measurement.

Of course, in the eyes of quantum mechanics, or rather, the statistical interpretation of wave mechanics (see, e.g., Dirac [**11**]), the concept of a measurement includes both types. Hence, although it is absurd to say that observing a distant star through a telescope has changed the state of the star since the event being observed is outside of your causal future, nonetheless, it is correct to say that the state of the star changed as a result of the emission of the photons that are being observed, however slightly. Indeed, the necessity of passing from classical to quantum physics seems to be most unavoidable when one can no longer justify the existence of "test" particles, in the sense of particles whose interaction with the system in question will not change its state appreciably. For instance, a collision with a microwave photon from a traffic policemen's radar gun will not change the state (e.g., position and momentum) of an oncoming motor vehicle appreciably, but it might very well change the state of an atomic electron.

In order to see how this all relates to projective geometry, one must start with the observation of Max Born that all measurements are carried out in the rest space of the measuring device. One must then take a closer look at the very nature of a rest space, along with the process of measurement itself, which is where projective geometry becomes applicable, because, in a sense, it is the geometry of perception. The recurring theme in what follows is that geometrically a rest space is not an affine space, but a projective space, so the affine spaces of conventional physics represent either the hyperplane at infinity in the projective space or the vector space that it projects onto it in its definition; i.e., the space of homogeneous coordinates.

First, let us examine the geometry of vision, which amounts to the statement that the effect of a converging lens on light rays is analogous to the effect of projecting from homogeneous to inhomogeneous coordinates. We refer to Fig. 16 in our explanation.

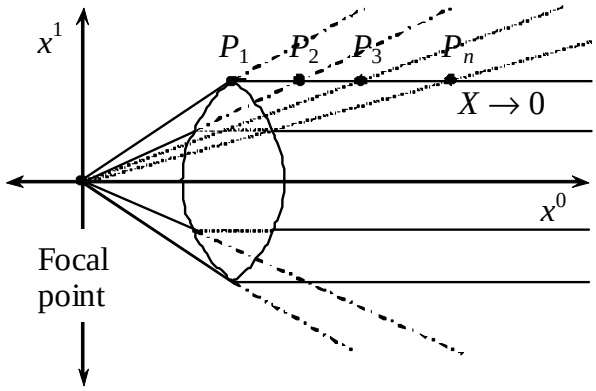

Figure 16. The effect of a converging lens on parallel lines.

The key geometrical attribute of a converging lens is the fact that it bends parallel lines into lines that all intersect in a single point, which is called the *focal point*. One can see that, for all practical purposes, the parallel lines "live" in the affine space $\mathbb{R}^2$, while the lines through the focal point "live" in the projective line $\mathbb{R}\mathrm{P}^1$. Furthermore, the projection of points in $\mathbb{R}^2$ to points in $\mathbb{R}\mathrm{P}^1$ that is implied by the process of refraction is precisely the projection of homogeneous coordinates for $\mathbb{R}\mathrm{P}^1$ onto inhomogeneous coordinates. For instance, if one considers the sequence of points P_1, P_2,

… along the top incoming light ray in the diagram then one sees that their homogeneous coordinates (x^0, x^1) will all have the same value of x^1 and increasing values of x^0, while the corresponding sequence of inhomogeneous coordinates $X^1, ^2 \ldots$ with $X = x^1/x^0$ will be converging to zero. Indeed, this same fact would be true for any other light ray that is parallel to the one considered. In effect, they all converge to a "point at infinity" – the *vanishing point*, as they say in the language of perspective – in the same way that the refracted rays intersect at the focal point. Indeed, in terms of homogeneous coordinates the x^1 axis becomes the point at infinity relative to the projection of $\mathbb{R}^2 - \{0\}$ onto $\mathbb{R}\mathrm{P}^1$ and the focal point is the ideal point $\{0\}$ that is not projected. Hence, the way that the scene to the right of the lens appears to the observer at the focal point is like an inversion through the vertical line through the center of the lens that maps the focal point to the vanishing point.

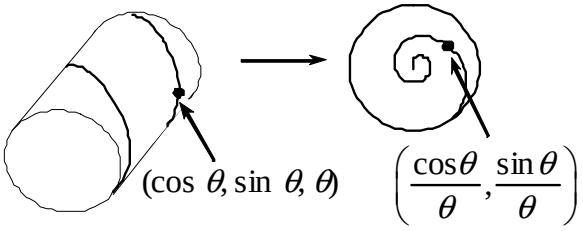

Figure 17. The projection of a helix from homogeneous to inhomogeneous coordinates.

As an example of the analogous situation in a three-dimensional affine space projecting onto a two-dimensional projective space, consider the way that a cylindrical helix about the *z*-axis in $\mathbb{R}^3$, which we rename the θ axis, appears when viewed along that axis, namely, as a spiral. We illustrate this in Fig. 17.

b. Special relativity. Certainly no one would deny the applicability of projective geometric concepts to the problem of measurement/observation in the purely optical sense that we just described. However, the somewhat surprising fact is that one can also apply precisely the same considerations to the problem of projecting events in a four-dimensional spacetime manifold M into the three-dimensional rest space Σ of a measurer/observer.

The key to understanding this is in understanding the difference between the proper time parameter τ of a worldline $x(\tau)$ in M and the time coordinate $t = x^0/c$ associated with a coordinate chart $(x^0, \ldots, x^3)$ on an open subset U in M. One must understand that proper time is valid only in the rest space of a measurer/observer. Hence, the parameter τ refers to the rest space of the point whose worldline is described by $x(\tau)$. One must then think of the time coordinate t as simply being the proper time parameter for another measurer/observer whose rest space involves the coordinates of the chart as measurements.

Since:

$$\frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \tag{XII.16}$$

one sees that the parameter τ and the coordinate t will differ iff the measurer/observer that is described by the coordinate chart is not comoving with the object that is described by the worldline $x(\tau)$, which would make the relative speed v vanish.

As a result of this, if:

$$\mathbf{v}(\tau) = v^{\mu}(\tau)\frac{\partial}{\partial x^{\mu}} \qquad [g(\mathbf{v}, \mathbf{v}) = c^2] \tag{XII.17}$$

is the velocity vector field along $x(\tau)$ then if one wishes to describe how it will appear in the rest space of the coordinates, one cannot simply project $(v^0, \ldots, v^3)$ onto its spatial components (v^1, v^2, v^3), which would be appropriate if the tangent spaces to M looked like $\mathbb{R}\oplus\Sigma$, but one must also change from the proper-time parameter τ to the time coordinate t as a parameter. (In (XII.17), we have assumed a Lorentzian metric g so the constraint on **v** that we make corresponds to the constraint that $x(\tau)$ is parameterized by proper time.)

By the chain rule for differentiation, this means that the spatial components of **v**, as seen by the measurer/observer that defines the chart will be:

$$V^i(t) = \frac{dx^i}{dt} = \frac{d\tau}{dt}\frac{dx^i}{d\tau} = \frac{v^i}{v^0}. \tag{XII.18}$$

Hence, the projection of $(v^0, \ldots, v^3)$ onto the rest space described by the chart gives the inhomogeneous coordinates (V^1, V^2, V^3), instead of the homogeneous ones (v^1, v^2, v^3).

We then conclude that, at least as far as velocity is concerned, rest spaces are projective spaces, not affine ones. The affine space that usually gets employed to describe the projection of velocity four-vectors onto three-vectors amounts to the plane at infinity in $\mathbb{R}P^3$, which corresponds to the hyperplane $v^0 = 0$ in the space of homogeneous coordinates.

One can give this situation a coordinate-free description by defining the *projectivized tangent bundle* $PT(M)$ to M to be set of all lines through the origins of the tangent spaces in $T(M)$. Hence, there will be a map $T(M) - Z(M) \to PT(M)$, $\mathbf{v} \mapsto [\mathbf{v}]$ that takes each non-zero tangent vector to the line through the origin in its tangent space that it generates; we are, of course, using the notation $Z(M)$ to refer to the zero section of $T(M)$.

Since we have established the result above only for velocity vectors, it is illuminating to examine the form that it takes for other four-dimensional measurements. In particular, one must examine the effect of projecting the coordinates $(x^0, \ldots, x^3)$ themselves onto the corresponding inhomogeneous coordinates $X^i = x^i/x^0$, under the assumption that $x^0 = ct$ is non-null. One sees that the only thing that has changed is to rescale the distance measurements x^i into dimensionless units, such as light-seconds. This amounts to using light rays as one's universal yardsticks for the measurement of distance. Hence, geometry again emerges from the propagation of electromagnetic waves.

We naturally need to examine the effect of a coordinate transformation $\overline{x}^{\mu} = \overline{x}^{\mu}(x^{\nu})$ on the inhomogeneous coordinates V^i of the velocity four-vector **v**. As one knows, from the chain rule for differentiation the homogeneous coordinates transform as:

$$\overline{v}^{\mu} = \frac{d\overline{x}^{\mu}}{d\tau} = \frac{\partial \overline{x}^{\mu}}{\partial x^{\nu}}\frac{dx^{\nu}}{d\tau} = A^{\mu}_{\nu}v^{\nu}, \tag{XII.19}$$

in which we have abbreviated the matrix $\partial\overline{x}^{\mu}/\partial x^{\nu}$ of the differential of the coordinate transformation by A^{μ}_{ν}.

We then see that the resulting effect on the inhomogeneous coordinates is to take $V^i = v^i/v^0$ to:

$$\overline{V}^i = \frac{A^i_0 + A^i_j V^j}{A^0_0 + A^0_j V^j}. \tag{XII.19}$$

We recover the conventional Galilean transformations of velocity 3-vectors by setting A^i_0 equal to the components V^i_r of the relative velocity of the measurer/observers described by the two charts, and getting $A^i_j = \delta^i_j$, $A^0_0 = 1$, and $A^0_j = 0$.

Of course, there is a relative speed limit in spacetime that originates in the fact that if two measurer/observers were moving with a relative speed greater than that of light there would be no way for them to communicate information anymore. In conventional relativity, this means that one must erect light cones in the tangent spaces of M and restrict oneself to relative velocities vectors that lie inside them. Of course, as we have seen, when the dispersion law for electromagnetic waves is quartic, not quadratic, the light cones become more involved as algebraic manifolds.

In the quadratic case, one introduces the light cone:

$$c^2(v^0)^2 - g_{ij}\, v^i\, v^j = 0 \tag{XII.20}$$

in the space of homogeneous coordinates for velocity four-vectors. One can also write this in the form:

$$v^0 = \frac{1}{c}(g_{ij} v^i v^j)^{1/2}, \tag{XII.21}$$

which shows that in the Galilean limit as c grows infinite the coordinate v^0 goes to 0. Hence, in the Galilean limit the light cone for a finite c converges to the hyperplane at infinity for $\mathbb{R}\mathrm{P}^3$. Perhaps for this reason, the light cone is referred to as the *absolute quadric* in projective geometry. In a sense, it represents a deformation of the hyperplane at infinity into a quadric hypersurface "at infinity."

Now, let us absorb the factor of c into the units of the homogeneous coordinate x^0 so that the components $g_{\mu\nu}$ are dimensionless. Under projection, the homogeneous quadratic polynomial $P[x^{\mu}] = g_{\mu\nu}\, x^{\mu}\, x^{\nu}$ in $\mathbb{R}^4$ goes to the inhomogeneous polynomial in $\mathbb{R}\mathrm{P}^3$:

$$[P][X^i] = g_{00} + 2g_{0i}\, X^i + g_{ij}\, X^i\, X^j. \tag{XII.22}$$

One finds that there is always a frame (indeed, an infinitude of frames) in $\mathbb{R}^4$ that makes g_{0i} disappear, since this is the case for any orthonormal frame relative to g, and therefore any frame that is obtained from it by an invertible spatial transformation, which does not have to be orthogonal. Furthermore, we rescale the components of the $[P][X]$ by

factoring out the g_{00} and absorbing it, too, into the coordinates X^i. We now have $[P][X]$ in the form:

$$[P][X^i] = 1 + G_{ij}\, X^i\, X^j \qquad (G_{ij} = g_{ij}/\, g_{00}). \qquad \text{(XII.22)}$$

Since our transformation from $P[x]$ to $[P][X]$ has involved only non-zero scalar multiplications, we see that the hypersurface $P[x] = 0$ in $\mathbb{R}^4$ corresponds to the hypersurface $[P][X] = 0$ in $\mathbb{R}\mathrm{P}^3$. In particular, the light cone:

$$P[x] = g_{00}(x^0)^2 - g_{ij}\, x^i\, x^j = 0 \qquad \text{(XII.23)}$$

in $\mathbb{R}^4$ corresponds to the unit sphere in $\mathbb{R}\mathrm{P}^3$:

$$G_{ij}\, X^i\, X^j = 1. \qquad \text{(XII.22)}$$

Hence, we can just as well regard the transition from introducing a Euclidian metric on $\mathbb{R}^3$ to introducing a Minkowski metric on $\mathbb{R}^4$ as being like replacing the affine space $\mathbb{R}^3$ with the projective space $\mathbb{R}\mathrm{P}^3$ and introducing the Euclidian metric on $\mathbb{R}\mathrm{P}^3$ directly.

It is also essential to understand what happens to Lorentz transformations of $\mathbb{R}^4$ when they are projected to fractional bilinear transformations of $\mathbb{R}\mathrm{P}^3$. In fact, one can just as well extend to the Weyl group $\mathbb{R}^*\times O(3, 1)$, by including the non-zero scalar factor ρ; as we mentioned before, this is the linear subgroup of the conformal Lorentz group. We think of the elements of this group as being invertible linear transformations of $\mathbb{R}^4$ that preserve the light cone, in the sense that if $\overline{x}^\mu = A^\mu_\nu x^\nu$ are the transforms of the x^μ then $P[\overline{x}^\mu] = 0$ iff $P[x] = 0$. Hence, the corresponding transformations of $\mathbb{R}\mathrm{P}^3$ are invertible fractional bilinear transformations that preserve the unit sphere defined by G.

It is important to note that this does not mean that they preserve the metric G itself. It means that $G(AX, AX) = 1$ iff $G(X, X) = 1$. Indeed, only the $O(3)$ subgroup ($A^0_0 = 0$, $A^0_j = A^i_0 = 0$, $A^i_j \in O(3)$) will preserve the metric G for all $X \in \mathbb{R}\mathrm{P}^3$.

As for the Lorentz boosts, it is illuminating to see what they become when represented as collineations of $\mathbb{R}\mathrm{P}^3$. For instance, a boost along the x^1 axis takes the following form in homogeneous coordinates ($x^0 = ct$):

$$\overline{x}^0 = \gamma(x^0 - v/c\, x^1), \qquad \overline{x}^1 = \gamma(-\, v/c\, x^0 + x^1), \quad \overline{x}^i = x^i \ \ (i = 1, 2), \qquad \text{(XII.23)}$$

in which we have introduced the Fitzgerald-Lorentz contraction factor:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} . \qquad \text{(XII.24)}$$

The matrix of this as a linear transformation of $\mathbb{R}^4$ is then:

$$A^{\mu}_{\nu} = \left[\begin{array}{c|ccc} \gamma/c & -\gamma v/c & 0 & 0 \\ \hline -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] . \qquad \text{(XII.25)}$$

In terms of projective transformations, such a boost is then going to involve all four types of transformations, namely, homotheties, inversions, spatial translations, and linear transformations.

The corresponding fractional bilinear transformation of $\mathbb{R}\mathrm{P}^3$ is:

$$\bar{X}^1 = \frac{\bar{x}^1}{\bar{x}^0} = \frac{-v + cX^1}{1 - (v/c)X^1} , \qquad \bar{X}^i = \frac{\bar{x}^i}{\bar{x}^0} = \frac{\gamma^{-1} X^i}{1 - (v/c)X^1} \qquad (i = 2, 3). \qquad \text{(XII.26)}$$

Note, in particular, that the coordinates X^2 and X^3 are altered, even though the coordinates x^2 and x^3 were not. Moreover, although the factor γ has dropped out of $\bar{X}^1$, it appears in the other two coordinates.

These transformations admit an immediate physical interpretation when the $X^i = V^i/c$ represent the components of a spatial velocity vector in dimensionless units. They then take the form:

$$\bar{V}^1 = \frac{v + V}{1 + vV/c^2} , \qquad \bar{V}^i = \frac{(c\gamma)^{-1} V^i}{1 + vV/c^2} . \qquad \text{(XII.27)}$$

The first expression is, of course, nothing but the usual relativistic rule for the addition of spatial velocities, which we now see to be projective-geometric in its origin [5].

c. The SL(2; $\mathbb{C}$) *representation of the Lorentz group.* One of the fundamental refinements that quantum wave mechanics, both relativistic and non-relativistic, introduced into special relativity was the fact that the empirically-verified existence of electron spin suggested that there was something more fundamental about the representation of the Lorentz group as the group *SL*(2; $\mathbb{C}$), since the two-to-one homomorphism of that group with the identity component of the Lorentz group seemed

[5] Many of these associations between special relativity and projective geometry were made in the early years of relativity when projective geometry was more mainstream in mathematics than it seems to be nowadays. One might confer the treatment it is given in Kommerell [**5**], for instance. More recently, it has been discussed by Gschwind [**12**] and, of course, the author of the present work [**13**].

to account for the existence of up and down spin states in electron wavefunctions. Since we now see that the groups $SL(n+1; \mathbb{R})$ describe the groups of collineations of the projective spaces $\mathbb{R}P^n$, we should now examine the fact that $SL(2; \mathbb{C})$ has a similar significance in terms of the complex projective space $\mathbb{C}P^1$.

The complex projective space $\mathbb{C}P^1$ is, of course, defined by the set of all *complex* lines through the origin of $\mathbb{C}^2$. That is, if the complex vector $\mathbf{v} \in \mathbb{C}^2$ is represented by the pair (z_1, z_2) then the corresponding point $[\mathbf{v}] \in \mathbb{C}P^1$ that it defines is the set of all complex scalar multiples $\lambda(z_1, z_2) = (\lambda z_1, \lambda z_2)$, $\lambda \in \mathbb{C}$.

Now, a complex line can also be regarded as a real plane, just as $\mathbb{C}^2$ can be regarded as the real vector space $\mathbb{R}^4 = \mathbb{R}^2 \oplus i\mathbb{R}^2$. (Here, we use the symbol i more as an identifier than anything else). However, not every real plane in $\mathbb{R}^4$ represents a complex line, since the complex line spanned by any $(z_1, z_2) = (x_1, x_2) + i(y_1, y_2)$ is the orbit of the action of all scalar multiples by scalars of the form $\lambda = \alpha + i\beta$:

$$\lambda(z_1, z_2) = (\alpha x_1 - \beta y_1, \alpha x_2 - \beta y_2) + i(\alpha y_1 + \beta x_1, \alpha y_2 + \beta x_2). \qquad \text{(XII.28)}$$

For instance, the real plane spanned by all points of the form $(x, y) + i(0, 0)$ is not a complex line in $\mathbb{R}^4$. Hence, there is a difference between the real manifold $\mathbb{C}P^1$ and the Grassmanian manifold V_2^4 of all planes through the origin in $\mathbb{R}^4$.

In fact, one easily sees that the real dimension of $\mathbb{C}P^1$ is two, since the complex dimension is one. One simply regards an element (z_1, z_2) of $\mathbb{C}^2 - \{0\}$ as the homogeneous coordinates of a complex line through the origin while the single inhomogeneous coordinate is then either:

$$Z = z_2 / z_1 \qquad \text{(XII.29)}$$

when z_1 is non-zero or the reciprocal when it is zero.

In real form, (XII.29) looks like:

$$X + iY = \frac{x_1 x_2 + y_1 y_2}{|z_1|^2} + i\frac{x_1 y_2 - y_1 x_2}{|z_1|^2} = \left\|\frac{z_2}{z_1}\right\|(\cos\psi + i\sin\psi), \qquad \text{(XII.30)}$$

in which ψ is the angle between the real lines through (x_1, x_2) and (y_1, y_2).

Since the only point that is being omitted by the set of all inhomogeneous coordinates corresponds to $z_1 = 0$, and the completion of the complex line by a point is the Riemann

sphere, one sees how the two-dimensional real manifold that describes $\mathbb{C}P^1$ is actually the 2-sphere.

Physically, one can think of this 2-sphere as the celestial sphere; i.e., a sphere in "space" at infinity. Under the projection of $\mathbb{C}^2 - \{0\}$ onto $\mathbb{C}P^1$, one also sees that each point of the celestial sphere gets associated with a complex line – i.e., a real plane – as a fiber in $\mathbb{C}^2 - \{0\}$. In fact, this complex line bundle is sometimes referred to as the *canonical line bundle* on $\mathbb{C}P^1$.

As we observed above, the 2-sphere also shows up in special relativity as the projection of the light cone in Minkowski space onto $\mathbb{R}P^3$, although in that case, the resulting radius was finite (viz., unity). However, since the points of the hyperplane at infinity ($x^0 = 0$) are in one-to-one (perspective) correspondence with the points of the hyperplane at $x^0 = 1$, one can see how the celestial sphere is essentially the same as the light sphere.

d. Time-space splittings. The concept of a time-space splitting $[\mathbf{t}](M) \oplus \Sigma(M)$ of the tangent bundle $T(M)$ to a manifold M or an analogous splitting $[\tau](M) \oplus \Sigma^*(M)$ of T^*M is deeply rooted in projective-geometric notions. For the sake of simplicity, we confine ourselves to a (finite-dimensional) vector space V and its dual V^*, in general, rather than the tangent spaces to a particular manifold; however, the transition from vector spaces to vector bundles is entirely straightforward.

If one is given a direct sum decomposition of V into $[\mathbf{t}] \oplus \Sigma$, where $[\mathbf{t}]$ is the line generated by a non-zero vector $\mathbf{t} \in V$ and Σ is a complementary hyperplane, then since any hyperplane, such as Σ, in V defines a unique element $[\tau] = \#^{-1}\Sigma \in PV^*$, we see that in order to complete $[\tau]$ to a direct sum decomposition $[\tau] \oplus \Sigma^*$ of V^*, we need only to specify the complementary hyperplane Σ^*. However, $\Sigma^* = \#[\mathbf{v}]$ is dual to a unique element $[\mathbf{v}]$ in PV, which we then choose to be $[\mathbf{t}]$. Hence, one can either uniquely characterize the splitting $[\mathbf{t}] \oplus \Sigma$ by means of the pair $([\mathbf{t}], [\tau]) \in PV \times PV^*$ or the pair $(\Sigma, \Sigma^*) \in PV^* \times PV$.

The transversality requirements on the aforementioned two splittings of V and V^* can be written in terms of the bilinear pairings of vectors and covectors as:

$$\tau(\mathbf{t}) \neq 0, \qquad \tau(\mathbf{v}) = \nu(\mathbf{t}) = 0, \qquad \mathbf{v} \in \Sigma, \nu \in \Sigma^*. \tag{XII.31}$$

Since the conditions only pertain to zero, it is immaterial which representative vector one uses from the subspaces involved. However, in practice, one might choose τ and $\mathbf{t}$ to make $\tau(\mathbf{t}) = 1$.

The physical process by which a splitting comes about generally begins with the selection of $\mathbf{t}$ to be a velocity vector field on the spacetime manifold M, whose congruence of integral curves represents the motion of a particular measurer/observer. Hence, for practical reasons, one does not generally expect the support of $\mathbf{t}$ to be all of M,

but only some proper subset, such as the diffeomorphic image of a *world tube* of the form $\mathbb{R}\times O$, where O is a three-dimensional "object" such as a 3-chain or compact 3-manifold.

In the absence of a Lorentzian structure, one cannot choose the complementary spatial sub-bundle $\Sigma(M)$ of $T(M)$ uniquely, but in the Lorentzian case, one can choose it to be the orthocomplement of $[\mathbf{t}]$. Since $\Sigma(M)$ is dual to a line field $[\tau]$ in T^*M – i.e., a section of $PT^*M \to M$ – we can also choose the line field $[\tau]$ arbitrarily, except for the transversality conditions in (XII.31), when one defines Σ to be $\#^{-1}[\tau]$ and Σ^* to be $\#[\mathbf{t}]$; hence, only the first condition is not automatic. We shall then refer to the pair $([\mathbf{t}], [\tau])$ as a (pre-metric) choice of measurer/observer.

A choice of measurer/observer $([\mathbf{t}], [\tau])$ defines a splitting $\Lambda_k V = \Lambda_k^{\mathrm{Re}} \oplus \Lambda_k^{\mathrm{Im}}$ ($\Lambda^k V = \Lambda^k_{\mathrm{Re}} \oplus \Lambda^k_{\mathrm{Im}}$, resp.) into time-space k-vectors (k-forms, resp.) and purely spatial ones. In the former case, one can factor out $\mathbf{t}$ (τ, resp.) as an exterior factor, while in the latter case, one has $\Lambda_k^{\mathrm{Im}} = \Lambda_k\Sigma$ ($\Lambda^k_{\mathrm{Im}} = \Lambda^k\Sigma$, resp.) (see Delphenich [**14**] for more discussion of this subject). The reason that there are no time-time-space, etc. summands in $\Lambda_k V$ or its dual is because subspaces $[\mathbf{t}]$ or $[\tau]$ are one-dimensional, so only one exterior factor of $\mathbf{t}$ or τ can appear without driving the resulting expression to zero identically.

By now, we are getting some inkling that whenever our perception of space is based in local – i.e., tangential – objects, we should really be thinking of the space we perceive as a projective space, not an affine one. Indeed, to enlarge the scope of Max Born's observation, one should think of all measurements as local measurements that are carried out in the rest space of the measuring device, and the appropriate representation for the rest space of the measurer/observer at a point of spacetime is the projectivization of the tangent space at that point. The actual splitting of the manifold itself is a deeper matter that involves subtle question of integrability.

e. Wave motion. If we regard the wave covector field $k = \omega\, dt - k_i\, dx^i$ as the fundamental kinematical object in wave mechanics that is analogous to the velocity vector field $\mathbf{v}$ in continuum mechanics then we find that wave motion is dual to point motion in the sense of the word "duality" that projective geometry recognizes. That is, a non-zero tangent vector $\mathbf{v}$ defines a point $[\mathbf{v}]$ in the projectivized tangent bundle $PT(M)$, while a non-zero cotangent vector k defines a point $[k]$ in the projectivized cotangent bundle PT^*M. Hence, whereas a non-zero tangent vector generates a tangent line, a non-zero cotangent vector generates a tangent hyperplane.

We have already observed in Chapter VIII that the effect of projecting the four-dimensional components k_μ of the wave covector k as if they were homogeneous coordinates projecting onto inhomogeneous coordinates is to produce components that have the dimensions of 1/ (phase) velocity:

$$K_i = \frac{k_i}{\omega} = \frac{1}{v_p^i}, \tag{XII.32}$$

although the dimensionless numbers cK_i are not necessarily the principal indices of refraction n_i that one obtains from Fresnel analysis.

Note that if one defines the energy-momentum 1-form p as $\hbar k$ then this has no effect on the corresponding inhomogeneous coordinates.

If we look at the vector field $\mathbf{v} = (\partial P[k(x)] / \partial k_\mu)\, \partial/\partial x^\mu$ then we see that, although we have previously identified the components as belonging to a four-velocity vector field, nonetheless its components actually have the dimensions of time (period) and distance (wavelength). However, the inhomogeneous coordinates that it defines are the components of group velocity, up to sign:

$$V^i = \frac{\partial P / \partial k_i}{\partial P / \partial \omega} = -v_g^i . \tag{XII.33}$$

Hence, we see that whereas the components K_i describe a point in PT^*M the components V^i describe a point in $PT(M)$.

Since the characteristic polynomial $P[k]$ is homogeneous of degree four in k, under the projection of the k_μ onto the K_i , it will define a inhomogeneous polynomial $[P][K]$ of degree four on $\mathbb{R}\mathrm{P}^3$. The characteristic hypersurface that $P[k]$ defines by its vanishing will then produce a corresponding hypersurface in $\mathbb{R}\mathrm{P}^3$. Of course, the geometry of this situation will be more complicated to describe than the projection of a cone onto a sphere.

Furthermore, by homogeneity, we have:

$$P[k] = \tfrac{1}{4}\frac{\partial P}{\partial k}(k) = \tfrac{1}{4}k(\mathbf{v}) \tag{XII.34}$$

so we see that the effect of the dispersion law $P[k] = 0$ is to make $k(\mathbf{v})$ vanish, as well. That is, the vectors $\mathbf{v}(x)$ are incident on the tangent hyperplanes that are annihilated by k. However, we shall still regard $\mathbf{v}$ as the dual vector field to k, even though we do not have the transversality that would make $k(\mathbf{v})$ non-vanishing, as we assumed for $\tau(\mathbf{t})$, above. This "isotropy" condition is based in the physical reality that there is no rest space for electromagnetic wave motion.

We recall the important property of a homogeneous function $P[k]$ – of any degree r – that since Euler's formula takes the form:

$$k_\mu v^\mu = rP[k], \tag{XII.35}$$

the hypersurface $P[k] = 0$ in $\mathbb{R}^{4*}$ corresponds to the hypersurface:

$$k_\mu v^\mu = 0 \tag{XII.36}$$

in $\mathbb{R}^{4*} \times \mathbb{R}^4$ and the hypersurface:

$$K_i V^i = 1 \tag{XII.37}$$

in $\mathbb{R}\mathrm{P}^{3*} \times \mathbb{R}\mathrm{P}^3$ (Here, the components km are of the form $(\omega, -k_i)$).

We previously encountered this situation in the context of the Fresnel normal surface and ray surface in the context of geometrical optics.

3. The Plücker-Klein embedding. Although projective geometry begins by dealing with lines through the origin in a vector space – say, $\mathbb{K}^{n+1}$ – nonetheless, as we saw above, it is also concerned with 2-planes, 3-planes, and so on up to hyperplanes in $\mathbb{K}^{n+1}$. In the case of dimension four, one is then concerned with points, lines, and planes in $\mathbb{K}P^3$. Furthermore, by duality, they will also be represented by planes, lines, and points in $\mathbb{K}P^{3*}$, respectively.

a. Plücker-Klein embedding. The connection between projective geometry and electromagnetism comes about once one discovers that one can represent any k-plane through the origin in $\mathbb{K}^n$ by either a decomposable k-vector in $\Lambda_k(\mathbb{K}^n)$ or a decomposable n–k-form in $\Lambda^{n-k}(\mathbb{K}^n)$; both are unique up to a non-zero scalar factor. Hence, the representation of a k-plane by a point in the projective spaces $P\Lambda_k(\mathbb{K}^n)$ or $P\Lambda^{n-k}(\mathbb{K}^n)$ is unique. Furthermore, the operations of exterior and interior multiplication are closely related to the meet and join operations, although not isomorphically.

Suppose V_k is a k–plane through the origin in $\mathbb{K}^n$. Let $\mathbf{e}_i$, $i = 1, \ldots, k$ be a k-frame that spans it. Since the members of any frame are linearly independent, the k-vector $\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_k$ is non-zero. Now, observe the effect of changing to another frame $\mathbf{f}_i$ for V_k that is related to the first one by the formula $\mathbf{f}_i = A_i^j \mathbf{e}_j$. When one forms the new k-vector $\mathbf{f}_1 \wedge \ldots \wedge \mathbf{f}_k$, one finds:

$$\mathbf{f}_1 \wedge \ldots \wedge \mathbf{f}_k = (\det A)\, \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_k . \qquad \text{(XII.38)}$$

That is, the two k-vectors differ only by a non-zero scalar multiple.

We define the *Grassmanian manifold* $V_{k,n}(\mathbb{K})$ to be set of all k-planes through the origin in $\mathbb{K}^n$. (Previously, we encountered this concept in the more specific context of 2-planes in $\mathbb{R}^4$.) When $\mathbb{K} = \mathbb{C}$ ($\mathbb{R}$, resp.) it can be given the topology of a complex (real, resp.) differentiable manifold of dimension equal to kn. Indeed, the most direct way of representing the coordinates of chart is by choosing a basis $\mathbf{e}_a$, $a = 1, \ldots, n$ for $\mathbb{K}^n$, such as the canonical basis, and noting that since the members of any k-frame, such as $\mathbf{f}_i$, can be represented as linear combinations of the basis elements $\mathbf{e}_a$ as $\mathbf{f}_i = A_i^a \mathbf{e}_a$, the most direct way of associating the k-plane spanned by the $\mathbf{f}_i$ as a set of coordinates is to associate it with the matrix A_i^a.

The map $V_{k,n+1}(\mathbb{K}) \to \mathrm{P}\Lambda_k(\mathbb{K}^n)$, $\mathrm{span}\{\mathbf{f}_i\} \mapsto [\mathbf{f}_1 \wedge \ldots \wedge \mathbf{f}_k]$ is not only one-to-one, it is, in fact an embedding that one calls the *Plücker-Klein embedding* [6]. The aforementioned image of this embedding consists of all decomposable k-vectors. If one defines a basis $\mathbf{e}_i$ for $\mathbb{K}^n$ and expresses the decomposable k-vector $\mathbf{f}_1 \wedge \ldots \wedge \mathbf{f}_k$ in terms of the basis $\mathbf{e}_i$:

$$\mathbf{f}_1 \wedge \ldots \wedge \mathbf{f}_k = \frac{1}{k!} V^{i_1 \cdots i_k} \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k} \tag{XII.39}$$

then the components of the $V^{i_1 \cdots i_k}$ can be regarded as the homogeneous coordinates of the corresponding point in $\mathrm{P}\Lambda_k(\mathbb{K}^n)$; they are usually called the *Plücker coordinates* of the k-plane.

Dually, one can also represent k-planes through the origin of $\mathbb{K}^{n*}$, which collectively define a manifold that we denote by $V^*_{n-k,n}(\mathbb{K})$, by k-forms, up to a non-zero scalar multiple. Let V^*_k be such a k-plane and let θ^i, $i = 1, \ldots, k$ be a k-frame that spans it. The k-form $\theta^1 \wedge \ldots \theta^k$ is then unique up to a non-zero scalar factor, as before, and we also have an embedding of $V^*_{n-k,n}(\mathbb{K})$ in $\mathrm{P}\Lambda^k(\mathbb{K}^n)$ that takes $\mathrm{span}\{\theta^i\}$ to $[\theta^1 \wedge \ldots \theta^k]$.

By duality, since one has both the diffeomorphism of $V_{k,n}(\mathbb{K})$ with $V^*_{n-k,n}(\mathbb{K})$ and the projective equivalence of $\mathrm{P}\Lambda_k(\mathbb{K}^n)$ with $\mathrm{P}\Lambda^{n-k}(\mathbb{K}^n)$, one can either represent a k-plane through the origin of $\mathbb{K}^n$ as a projective equivalence class of either k-vectors or n–k-forms over the vector space $\mathbb{K}^n$.

b. Line geometry. In the case of 2-planes in $\mathbb{K}^n$, one sees that they will be represented by either decomposable 2-vectors on $\mathbb{K}^n$ or decomposable n–2-forms. Since a 2-plane in $\mathbb{K}^n$ projects to a line in $\mathbb{K}\mathrm{P}^{n-1}$, the study of 2-planes by their representation as 2-vectors is sometimes referred to as *line geometry* [**18**].

One can further characterize the decomposable 2-vectors by the fact that their *rank* equals two. The rank of a k-vector $\mathbf{A}$ on $\mathbb{K}^n$ can be defined, equivalently, as the minimum number of linearly independent vectors in that it takes $\mathbb{K}^n$ to express $\mathbf{A}$ as a linear combination of k-fold exterior products of vectors or as the minimum integer $r \geq 0$ such that $\mathbf{A} \wedge \ldots \wedge \mathbf{A} = 0$.

In the case of 2-vectors, the rank must be an even integer, and the set of all 2-vectors $\mathbf{A}$ of rank 2 is then characterized by the quadratic equation:

[6] In addition to the mathematical references on projective geometry one might also confer [**16**, **17**] for applications to physics.

$$\mathbf{A} \wedge \mathbf{A} = 0. \qquad \text{(XII.40)}$$

Hence, the set of all decomposable 2-vectors in the vector space $\Lambda_2(\mathbb{K}^n)$ is a quadric hypersurface that one calls the *Klein quadric.*

The exterior product is actually closely related to the question of incidence of projective subspaces, although, unfortunately, it does not give an isomorphic representation of the lattice of projective subspaces in the sense of representing the meet and join operations precisely. What one can say is that if a k-plane V_k in $\mathbb{K}^n$ is represented by a decomposable k-vector $\mathbf{A}$ and an m-plane V_m is represented by a decomposable m-vector $\mathbf{B}$ then $\mathbf{A} \wedge \mathbf{B} = 0$ iff the meet (i.e., intersection) of V_k and V_m has a dimension that is greater than zero. For instance, when 2-planes are represented by 2-vectors the vanishing of their exterior product is equivalent to the statement that the 2-planes intersect in a line. Hence, when regarded as projective subspaces in $\mathbb{K}\mathrm{P}^{n-1}$, one is describing two lines that intersect at a point.

Since the (exterior) polynomial that (XII.40) defines is homogeneous of degree two, it also defines a quadric hypersurface in the associated projective space $\mathrm{P}\Lambda_2(\mathbb{K}^n)$.

4. Projective geometry and electromagnetism. As we have seen from the outset, the mathematics of electromagnetism, in its most general formulation, involves bivector fields and 2-forms. In the previous section, we pointed out that the exterior algebra over any vector space has a close relationship with the lattice of linear subspaces of that vectors space, as well as the lattice of projective subspaces of its associated projective space. Therefore, in this section we shall apply the one result to the other to establish the extent to which projective geometry relates to pre-metric electromagnetism as metric geometry relates to gravitation.

When $n = 4$ and the field of scalars is $\mathbb{R}$, the basic vector space is $\mathbb{R}^4$, and its associated projective space is $\mathbb{R}\mathrm{P}^3$. The only linear subspaces of $\mathbb{R}^4$ that one needs to consider are lines, planes, and hyperplanes. These, in turn, correspond to points, lines, and planes in $\mathbb{R}\mathrm{P}^3$. We have already seen that representation of lines and hyperplanes is natural to $\Lambda_1(\mathbb{R}^4)$ and $\Lambda^1(\mathbb{R}^4)$, respectively, so we see that the only remaining k-vectors and k-forms that produce projective subspaces are bivectors and 2-forms, which represent 2-planes in $\mathbb{R}^4$ or lines in $\mathbb{R}\mathrm{P}^3$, in the decomposable case.

Hence, in order to be sure that the projective geometry – in fact, *line* geometry – of spacetime is rooted in the physics of electromagnetism, one must first gain an intuition for the physical nature of electromagnetic fields that can be described by a decomposable 2-form F and a decomposable bivector $\mathfrak{h}$. What one finds is that they generally represent the "elementary" fields, while the electromagnetic fields that are described by 2-forms and bivector fields of rank four are more elaborate linear superpositions of elementary

fields. For instance, any static electric field is described by a 2-form $F = dt \wedge E$ and a bivector field $\mathfrak{h} = \partial_t \wedge \mathbf{D}$, any static magnetic field is represented by a 2-form $F = \#_s\mathbf{B}$ and a bivector field $\mathfrak{h} = \#_s^{-1} H$, and an electromagnetic wave field has $F = k \wedge u$, $\mathfrak{h} = \mathbf{k} \wedge \mathbf{u}$.

The best way to distinguish between static electric, static magnetic, and wavelike 2-forms is given by the constitutive law κ: $\Lambda^2 M \to \Lambda_2 M$, $F \mapsto \mathfrak{h} = \kappa(F)$, at least in the linear case. In that case, one can define a bilinear pairing on $\Lambda^2 M$ by means of:

$$(F, G) \equiv \kappa(F)(G) = \tfrac{1}{4}\kappa^{\kappa\lambda\mu\nu} F_{\kappa\lambda} G_{\mu\nu}. \qquad \text{(XII.41)}$$

In general, this pairing is non-degenerate, but not symmetric, so in order to obtain a symmetric, non-degenerate, bilinear pairing – i.e., a scalar product – one must either restrict κ to its symmetric part or assume that the constitutive law has the property of symmetry to begin with. We shall call a κ with the property:

$$(F, G) = (G, F) \qquad \text{(XII.42)}$$

self-adjoint.

If one wishes to consider only the quadratic form (F, F) then it is unnecessary to make such a restriction, since only the symmetric part of κ will be involved; i.e., the skewon part plays no role.

Furthermore, as we pointed out in the discussion of constitutive laws, there is another non-degenerate bilinear pairing on $\Lambda^2 M$ that is defined by any volume element $\varepsilon \in \Lambda_4 M$, namely:

$$\langle F, G\rangle \equiv \#(F)(G) = \tfrac{1}{4}\varepsilon^{\kappa\lambda\mu\nu} F_{\kappa\lambda} G_{\mu\nu}. \qquad \text{(XII.43)}$$

As we have seen, the following are equivalent:

1. $\langle F, F\rangle = 0$,
2. F is decomposable,
3. F has rank two,
4. F lies on the Klein quadric.

This pairing is always symmetric, and therefore defines a scalar product on $\Lambda^2 M$. Hence, in order to properly separate the effects of κ from those of #, one must use only the principal part of κ in the definition (XII.41). We shall assume that this is the case from now on.

One sees that scalar products can just as well be defined on $\Lambda_2 M$ using the inverse isomorphisms to # and κ:

$$\langle \mathbf{A}, \mathbf{B}\rangle = \#^{-1}(\mathbf{A})(\mathbf{B}) = \tfrac{1}{4}\varepsilon_{\kappa\lambda\mu\nu} A^{\kappa\lambda} B^{\mu\nu}. \qquad \text{(XII.44a)}$$

$$(\mathbf{A}, \mathbf{B}) = \kappa^{-1}(\mathbf{A})(\mathbf{B}) = \tfrac{1}{4}\kappa_{\kappa\lambda\mu\nu} A^{\kappa\lambda} B^{\mu\nu}. \qquad \text{(XII.44b)}$$

One can also express the scalar products $\langle F, G\rangle$ and (F, G) as the Poincaré duals of the 4-forms $F \wedge G$ and $F \wedge \#\kappa(G)$:

$$\langle F, G\rangle\varepsilon = F \wedge G, \qquad (F, G)\varepsilon = F \wedge \#\kappa(G). \qquad \text{(XII.45)}$$

One can now distinguish three distinct type of 2-forms F, according to the quadratic form (F,F):

1. Electric: $(F,F) > 0$,
2. Isotropic: $(F,F) = 0$,
3. Magnetic: $(F,F) < 0$.

The reason that we did not call the isotropic 2-forms "wavelike" is because it is conceivable that an isotropic 2-form might represent a superposition of static electric and magnetic fields.

Now suppose we have a measurer/observer $(\mathbf{t}, \tau)$ that defines a time-space splitting of $T(M)$ and T^*M, with corresponding splittings $\Lambda_2 = \Lambda_2^{\mathrm{Re}} \oplus \Lambda_2^{\mathrm{Im}}$, $\Lambda_2 = \Lambda^2_{\mathrm{Re}} \oplus \Lambda^2_{\mathrm{Im}}$. We see that when $F = \tau \wedge E + \#_s\mathbf{B}$, $\mathfrak{h} = \mathbf{t} \wedge \mathbf{D} + \#_s^{-1} H$ the quadratic forms take the form [7]:

$$\langle F, F\rangle = 2\mathbf{V}(\tau \wedge E, \#_s\mathbf{B}) = -2E_iB^i, \tag{XII.47a}$$

$$\langle \mathfrak{h}, \mathfrak{h}\rangle = 2V(\mathbf{t} \wedge \mathbf{D}, \#_s^{-1} H) = -2H_iD^i, \tag{XII.47b}$$

$$\begin{aligned}(F, F) &= \kappa(\tau \wedge E, dt \wedge E) + 2\kappa(\tau \wedge E, \#_s\mathbf{B}) + \kappa(\#_s\mathbf{B}, \#_s\mathbf{B})\\ &= \varepsilon(E, E) + 2\gamma(E, \mathbf{B}) - \mu^{-1}(\mathbf{B}, \mathbf{B}),\\ &= \varepsilon^{ij}E_iE_j + 2\gamma^i_jE_iB^j - \tilde{\mu}_{ij}B^iB^j,\end{aligned} \tag{XII.48c}$$

$$\begin{aligned}(\mathfrak{h}, \mathfrak{h}) &= \kappa^{-1}(\mathbf{t} \wedge \mathbf{D}, \partial_t \wedge \mathbf{D}) + 2\kappa^{-1}(\mathbf{t} \wedge \mathbf{D}, \#_s^{-1} H) - \kappa^{-1}(\#_s^{-1} H, \#_s^{-1} H)\\ &= \varepsilon^{-1}(\mathbf{D}, \mathbf{D}) + 2\gamma^{-1}(\mathbf{D}, H) - \mu(H, H),\\ &= \tilde{\varepsilon}_{ij}D^iD^j + 2\gamma^i_jH_iD^j - \mu^{ij}H_iH_j.\end{aligned} \tag{XII.48d}$$

In the isotropic case, the last two take the form:

$$(F, F) = \varepsilon_0 E^2 - 1/\mu_0 B^2, \qquad (\mathfrak{h}, \mathfrak{h}) = 1/\varepsilon_0 D^2 - \mu_0 H^2, \tag{XII.49}$$

which is the form that gets the most attention from theoreticians.

One sees that the quadratic forms (F, F) and $\langle F, F\rangle$ are proportional to the Lorentz-invariant field invariants $\mathcal{F}$ and $\mathcal{G}$, resp., that were introduced by Mie [**19**] and currently play such a crucial role in phenomenological Lagrangians, such as the Born-Infeld and Heisenberg-Euler ones.

In the rank-two case, we see that the 2-form F and the bivector field $\mathfrak{h}$ define 2-planes, which we denote by $[F]$ and $[\mathfrak{h}]$, in the tangent spaces of the spacetime manifold. The main issue in the eyes of projective geometry is how they intersect. Naively, they can intersect transversally ($[F] \wedge [\mathfrak{h}] = 0$), in a line ($[F] \wedge [\mathfrak{h}] = [\mathbf{k}]$), or they can be

[7] We now represent our volume elements by $V \in \Lambda^4$ and $\mathbf{V} \in \Lambda_4$ to avoid confusion with the electric part of κ;

concurrent ($[F] = [\mathfrak{h}]$). However, the last possibility is not realized, which follows from the fact that $\mathfrak{h} = \kappa(F)$. Hence, the ultimate question is whether or not:

$$F \wedge \#\mathfrak{h} = \kappa(F, F)\, V \tag{XII.50}$$

vanishes; i.e., whether the scalar product (F, F) vanishes. Its vanishing is, in turn, equivalent to the possibility that the intersection is a line.

We then see that the homogeneous quadratic equation on $\Lambda^2 M$:

$$(F, F) = 0 \tag{XII.51}$$

defines not only a second quartic hypersurface in each fiber, in addition to the Klein quadric, but also a hypersurface in each projectivized tangent space $PT_x(M)$, namely, the set of all points $[\mathbf{k}]$ that are defined by the intersections of the lines $[F]$ and $[\mathfrak{h}]$ in each projective tangent space.

Now, let us try to get a better physical intuition for the nature of the 2-planes $[F]$ and $[\mathfrak{h}]$ by examining the form that they take in various elementary cases.

In the electrostatic case, they are of the form $[\tau \wedge E]$ and $[\mathbf{t} \wedge \mathbf{D}]$ and they intersect transversally. Hence, if $\mathbf{D} = D_x\, \partial_x$, while $E = E_x\, dx$ then the plane of $[\tau \wedge E]$ is the yz-plane, while the plane of $[\mathbf{t} \wedge \mathbf{D}]$ is the tx-plane. When one intersects them with the spatial subspaces $\Sigma(M)$ of $T(M)$, for the relevant choice of measurer/observer, the plane of $[\mathbf{t} \wedge \mathbf{D}]$ becomes the line generated by $\mathbf{D}$, when it is non-zero, and the plane $[\tau \wedge E]$ is tangent to the equipotential surfaces for $E = d\phi$.

In the magnetostatic case, the situation is reversed. If $F = \#_s\mathbf{B}$, $\mathfrak{h} = \#_s^{-1} H$, where $\mathbf{B} = B_x\, \partial_x$ and $H = H_x\, dx$ then $\#_s\mathbf{B} = B_x\, dy \wedge dz$, $\#_s^{-1} H = H_x\, \partial_y \wedge \partial_z$, and the plane of $[\#_s\mathbf{B}]$ is the tx-plane while the plane of $[\#_s^{-1} H]$ is the yz-plane. The spatial intersections are then the line generated by $\mathbf{B}$, when it is non-zero, and the plane that is annihilated by the 1-form H, when it is non-zero.

If $\mathbf{B} = \delta_s\mathbf{A} = \#_s^{-1} d_s \#_s \mathbf{A}$, where $\mathbf{A} \in \Lambda^2(\Sigma)$ is the magnetic potential bivector field then $\mathbf{A}$ spans a plane in each tangent space when it is non-zero, but the resulting rank-two sub-bundle of $\Sigma(M)$ does not have to be integrable, as a differential system, since the issue at stake, from Frobenius, is whether the 3-form:

$$A \wedge d_s A = A \wedge B \tag{XII.52}$$

vanishes, in which we have set $A = \#_s\mathbf{A}$, $B = \#_s\mathbf{B}$, which are then a 1-form and a 2-form, respectively. We see that this 3-form is of Chern-Simons type, if one regards A as a $\mathfrak{u}(1)$ connection form.

Of course, A is defined only up to the addition of a closed 1-form α, so $A \wedge B$ is defined only up to the addition of a 3-form $\alpha \wedge B$. Hence, the question of whether the rank-two sub-bundle defined by $\mathbf{A}$ is integrable into magnetostatic equipotential surfaces is equivalent to the question of whether a suitable choice of gauge A will make $A \wedge B$

vanish. All uniform magnetic fields have this property, as one verifies in the example $A = -B_0 y\, dx + B_0 x\, dy$, $B = B_0\, dx \wedge dy$, where B_0 is a non-zero constant. An example of a non-integrable B is given by starting with a potential 1-form A of the form $A_x(y, z)\, dx + A_y(x, z)\, dy$, which then makes $A \wedge dA = (A_x A_{y,z} - A_y A_{x,z})\, dx \wedge dy \wedge dz$, which does not have to vanish.

When $F = k \wedge u$ and $\mathfrak{h} = \mathbf{k} \wedge \mathbf{u}$ are isotropic, one has:

$$0 = F \wedge \#\mathfrak{h} = k(\mathbf{k})u(\mathbf{u}) - k(\mathbf{u})u(\mathbf{k}) = -k(\mathbf{u})u(\mathbf{k}); \qquad \text{(XII.53)}$$

the vanishing of the first term follows from the dispersion law for k. Hence, either $k(\mathbf{u})$ vanishes or $u(\mathbf{k})$ vanishes.

As a consequence of (XII.53), the planes $[k \wedge u]$ and $[\mathbf{k} \wedge \mathbf{u}]$ intersect in a line $[\mathbf{l}]$, where one can express the vector $\mathbf{l}$ as $\alpha\mathbf{k} + \beta\mathbf{u}$ for appropriate scalars α, β. Since $\mathbf{l}$ is incident on the plane $[k \wedge u]$, one must have:

$$0 = i_{\mathbf{l}}(k \wedge u) = \beta k(\mathbf{u})\, u - [\alpha u(\mathbf{k}) + \beta u(\mathbf{u})]k \qquad \text{(XII.54)}$$

after one includes the dispersion law for $k(\mathbf{k})$. This gives the conditions:

$$0 = \beta k(\mathbf{u}) = \alpha u(\mathbf{k}) + \beta u(\mathbf{u}). \qquad \text{(XII.55)}$$

Now, since the vector field $\mathbf{u}$ is defined only up to the addition of a scalar multiple of $\mathbf{k}$, one can choose it to be any vector in $[\mathbf{k} \wedge \mathbf{u}]$ except $\mathbf{k}$. Similarly, the 1-form u can be replaced with any 1-form in the plane spanned by k and u, except k itself. However, these choices of gauge are not independent, since F and $\mathfrak{h}$ are connected by the constitutive law κ. We then choose $\mathbf{u}$ and u to make:

$$0 = k(\mathbf{u}) = u(\mathbf{k}), \qquad u(\mathbf{u}) \neq 1, \qquad \text{(XII.56)}$$

which then forces β to vanish. Hence, with this choice of "gauge" for $\mathbf{u}$ and u, one must have $[\mathbf{l}] = [\mathbf{k}]$. That is, the line of intersection of the planes $[\mathbf{k} \wedge \mathbf{u}]$ and $[k \wedge u]$ is the direction of motion for the electromagnetic wave fronts, when the field in question is wavelike.

Since the planes $[k \wedge u]$ and $[\mathbf{k} \wedge \mathbf{u}]$ are not identical, but only intersect in the line $[\mathbf{k}]$, their join is three-dimensional. In order to find a third linearly-independent vector field $\mathbf{v}$ and covector field v, we consider the bivector field $\#^{-1}F$, which is also decomposable, and defines a plane that intersects $[\mathfrak{h}]$ in a line since:

$$\#^{-1}F \wedge \mathfrak{h} = F(\mathfrak{h})\mathbf{V} = 0. \qquad \text{(XII.57)}$$

Since the line of intersection is also the line $[\mathbf{k}]$, one can express $\#^{-1}F$ and $\#\mathfrak{h}$ in the form:

$$\#^{-1}F = \mathbf{k} \wedge \mathbf{v}, \quad \#\mathfrak{h} = k \wedge v \qquad \text{(XII.58)}$$

for a suitable vector field **v** and covector field *v*.

The vector field **v** must be non-collinear with both **k** and **u**, and must span the three-dimensional space [**k** ∧ **u**] ∨ [**k** ∧ **v**]. Hence, **k** ∧ **u**, **k** ∧ **v**, and **u** ∧ **v** must all be non-vanishing, as well as **k** ∧ **u** ∧ **v**; one derives analogous statements for *k*, *u*, and *v*.

In order to see how the 3-frame {**k**, **u**, **v**} relates to the 3-coframe {*k*, *u*, *v*}, we note that the vanishing of the expressions $F \wedge F$, $\mathfrak{h} \wedge \mathfrak{h}$, $F \wedge \mathfrak{h}$, and $\#^{-1}F \wedge \mathfrak{h}$ implies that:

$$0 = k(\mathbf{v})\, u(\mathbf{k}) = k(\mathbf{u})\, v(\mathbf{k}) = k(\mathbf{u})\, u(\mathbf{k}) = k(\mathbf{v})\, v(\mathbf{k}), \tag{XII.59}$$

which have the solution:

$$0 = k(\mathbf{v}) = u(\mathbf{k}) = k(\mathbf{u}) = v(\mathbf{k}). \tag{XII.60}$$

Hence, although the 3-frame {**k**, **u**, **v**} and the 3-coframe{*k*, *u*, *v*} are not projectively reciprocal, since *k*(**k**) vanishes, due to the dispersion law, they are dual, in the three-dimensional sense that the plane [**k** ∧ **u**] is the intersection of the hyperplane [*v*] with the three-dimensional space that is spanned by {**k**, **u**, **v**}, with analogous statements for [**k** ∧ **v**] and [**u** ∧ **v**]. We illustrate this situation in Fig. 18.

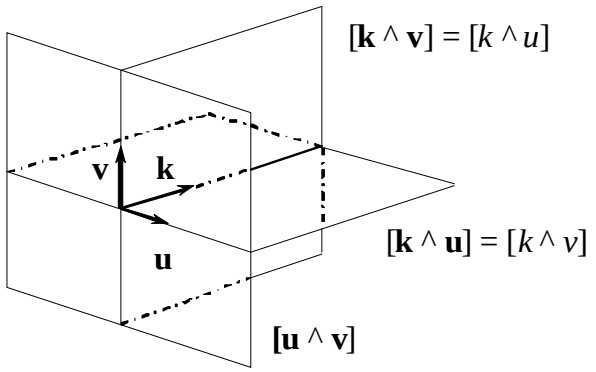

Figure. 18. The planes defined by an isotropic *F*.

Note that so far we have characterized [**u** ∧ **v**] as only the intersection of the hyperplane [*k*] with the three-dimensional space that is spanned by {**k**, **u**, **v**}, but not in terms of the Poincaré dual of a 2-form. This is where we must notice a geometric subtlety: whereas the planes [**k** ∧ **u**] and [**k** ∧ **v**] are uniquely defined by *F* and $\mathfrak{h}$, nevertheless, the choice of **u** and **v**, as well as the plane that they spanned, was an arbitrary gauge choice, except that neither could be collinear with **k**. Hence, the Poincaré dual to the plane [**u** ∧ **v**] also becomes open to a choice of gauge, in the form of a choice of measurer/observer, as defined by the pair (**t**, τ), where we choose $\tau(\mathbf{t}) = 1$. The pair (**t**, τ) allows us to decompose *k* into $\omega\tau - k_s$ and **k** into $\omega' \mathbf{t} + \mathbf{k}_s$, ($k(\mathbf{k}) = 0$) which then makes:

$$F = \omega\tau \wedge u - k_s \wedge u = \omega\tau \wedge u + \#(\omega' \mathbf{t} \wedge \mathbf{v}), \tag{XII.61a}$$

$$\mathfrak{h} = \omega' \mathbf{t} \wedge \mathbf{u} + \mathbf{k}_s \wedge \mathbf{u} = \omega' \mathbf{t} \wedge \mathbf{u} + \#^{-1}(\omega\tau \wedge v). \tag{XII.61b}$$

The "electric" parts of the *F* and $\mathfrak{h}$ are then proportional to *u* and **u**, respectively:

$$E = i_{\mathbf{t}} F = \omega u, \qquad \mathbf{D} = i_{\tau} \mathfrak{h} = \omega' \mathbf{u}, \tag{XII.62}$$

and we can consistently define **v** and *v* by their "magnetic" parts:

$$\mathbf{B} = i_{\tau}\#^{-1}F = -\omega'\mathbf{v}, \qquad H = i_{\mathbf{t}}\#\mathfrak{h} = \omega\nu. \tag{XII.63}$$

That is, **u** is in the direction of **D**, **v** is in the direction of **B**, *u* is in the direction of *E*, and *v* is in the direction of *H*, relative to this measurer/observer.

Since **u**, **v**, *u*, and *v* follow naturally when one chooses (**t**, τ), we see that the choice of a gauge for an isotropic electromagnetic field *F* is closely related to a choice of measurer/observer. We also see that the bivector **u** ^ **v** is proportional to the Poynting bivector field **D** ^ **B**, while the 2-form *u* ^ *v* is proportional to the Poynting 2-form *E* ^ *H*.

By contrast, the 2-planes [*k* ^ *u*] and [**k** ^ **u**] are defined independently of this choice of gauge or measurer/observer. One calls either the *polarization plane* for the electromagnetic wave in question.

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CHAPTER XIII

COMPLEX RELATIVITY AND LINE GEOMETRY

Since the primary focus of this work has been to investigate the mathematical and physical considerations that strictly precede the introduction of a spacetime metric, the subject of the present chapter may seem out of place. However, since a recurring theme all along has been that is it is necessary to shift one's geometric intuition from the tangent bundle to the bundle of 2-forms on spacetime, it is important to see how the geometry that follows from the definition of a Lorentzian structure can be represented in terms of things that are more closely associated with the bundle of 2-forms.

The key to making the transition from $T(M)$ to $\Lambda^2(M)$ is in the isomorphism of the identity component of the Lorentz group with the Lie group $SO(3; \mathbb{C})$. It has the effect of saying that there is a one-to-one correspondence between oriented, time-oriented Lorentzian frames on Minkowski space and complex orthogonal frames in $\mathbb{C}^3$ that have unit volume. As long as one gives the real vector bundle $\Lambda^2(M)$ an almost-complex structure, which turns its fibers into three-dimensional complex vector spaces, any choice of complex frame in the fiber will define a complex-linear isomorphism of the fiber with $\mathbb{C}^3$. If one introduces a complex orthogonal structure on the fiber, which follows naturally from the introduction of a complex structure, then any complex orthogonal frame in a fiber defines an isometry of the fiber with $\mathbb{C}^3$, when it is given the complex Euclidian inner product. If one further introduces a complex volume element on the bundle $\Lambda^2(M)$ – which is not to be confused with a real volume element on $T(M)$ – then one can represent the Lie group $SO(3; \mathbb{C})$ by its action on complex orthogonal frames in $\Lambda^2(M)$ that have unit volume.

One can associate a principal fiber bundle with any vector bundle, and its elements are the linear frames in the fibers of that vector bundle. When the vector bundle is the tangent bundle $T(M)$ to a manifold, the associated bundle is the bundle $GL(M)$ of linear frames. One can reduce this bundle to various sub-bundles whose structure groups G are subgroups of $GL(n)$, and are called "G-structures," in general. For instance, unit-volume frames correspond to $G = SL(n)$, orthonormal frames, to $G = O(p, q)$, if the signature type of the metric is (p, q), and a global frame field would correspond to a complete reduction to $G = \{e\}$.

If one starts with the associated principal bundle to $\Lambda^2(M)$, which we denote by $GL(\Lambda^2)$, and whose structure group is $GL(6; \mathbb{R})$, then one can carry out similar reductions of this bundle by examining the frames that correspond to the various subgroups of $GL(6; \mathbb{R})$. These subgroups include $GL(3; \mathbb{C})$ and $SO(3; \mathbb{C})$, which correspond to the complex linear frames in the fibers of $\Lambda^2(M)$ and the complex orthogonal ones with unit complex

volume, respectively. It is in this latter reduction that we find the representation of Lorentzian relativity as something that also lives in the structure of the bundle $\Lambda^2(M)$.

In section 1 we first discuss the elementary concepts that we shall be applying to the context of vector bundles that first appear in complex linear algebra. Then, in section 2 we show how this relates to the representations of the Lorentz group by complex orthogonal transformations with unit determinant.

Since the literature of general relativity only occasionally presents the differential geometry that it uses in the language of connection 1-forms on the bundle of linear frames over spacetime, in section 3 we provide a section of this chapter that serves that purpose. The transition to expressing the same concepts in terms of the bundle of complex linear frames in $\Lambda^2(M)$ then becomes more natural.

Finally, in section 4 we bring the various elementary pieces together into the representation of general relativity in terms of the geometry of $\Lambda^2(M)$. Along the way, we point out that when this bundle has been given an almost-complex structure it is actually unnecessary to further complexify it, as is commonly done in complex relativity, and upon closer inspection, one sees that one is using only half of the resulting six-complex-dimensional vector space.

Once one has recast the geometry of gravitation as a sub-geometry of the geometry of electromagnetism, it is only natural to ponder the question of whether there is as much ultimate physical significance to better understanding the geometry of electromagnetism as their seems to be in the geometry of gravitation. Since pre-metric electromagnetism seems to be most relevant to the realm of strong electromagnetic fields, this also suggests possible inroads into the realm of quantum electrodynamics, which has always been obscured from geometry by its phenomenological formalism that deals with interactions and scattering amplitudes more that it deals with electromagnetic fields or spacetime structures directly.

1. Complex structures on real vector spaces [1, 2]. Although sometimes it is simpler to merely deal with complex scalars from the outset and consider complex vector spaces as being modeled on $\mathbb{C}^n$, nevertheless, it is sometimes more illuminating to regard the field $\mathbb{C}$ of complex numbers as an algebra over the real vector space $\mathbb{R}^2$ and then treat complex vector spaces as real vector spaces that have been given a "complex structure," when suitably defined. In order to motivate the concept of a complex structure on a real vector space, we first show how one exhibits $\mathbb{C}$ as an algebra over $\mathbb{R}^2$.

Recall that a $\mathbb{K}$-*algebra* over a vector space V (for some specified field of scalars $\mathbb{K}$) is a bilinear map $V \times V \to V$, $(a, b) \mapsto ab$; that is, it is a binary operation that respects the linear structure on each factor. Hence, since V already has an Abelian group structure defined by vector addition, and bilinearity amounts to the statement that the algebra multiplication (left and right) distributes over addition:

$$a(b + c) = ab + ac, \qquad (a + b)c = ac + bc, \tag{XIII.1}$$

we see that any algebra can also be regarded as a ring in the eyes of abstract algebra.

As a ring, an algebra does not need to be multiplicatively commutative, or even associative. Lie algebras are the primary example of a non-associative algebra, as far as physics is concerned, but the Cayley algebra over $\mathbb{R}^8$ has this property, as well. Similarly, an algebra does not need to have a unity – i.e., a multiplicative identity – and elements of an algebra do not need to possess multiplicative inverses. In the event that they do, they are called *units*, and if all non-zero elements of an algebra are units then one calls it a *division algebra*. In the extreme case where the multiplicative structure defines a group structure on the non-zero elements of the algebra, one calls it a *field*.

When the field is $\mathbb{R}$ and the vector space is $\mathbb{R}^2$, we can define a bilinear multiplication by imitating the effect of complex multiplication. Since:

$$(x + iy)(u + iv) = (xu - yv) + i(xv + yu) \qquad \text{(XIII.2)}$$

we define the product of two elements $(x, y), (u, v) \in \mathbb{R}^2$ to be:

$$(x, y)(u, v) = (xu - yv, xv + yu). \qquad \text{(XIII.3)}$$

As an $\mathbb{R}$-algebra, the complex numbers represent a commutative field. The multiplicative identity is $1 = (1, 0)$, and if $z = a + ib \neq 0$ then its inverse is $z^{-1} = 1/z = \overline{z} / \| z \|^2$, in which we have introduced the complex conjugate $\overline{z}$ of z and its modulus $\| z \|$ in the usual fashion. In its real representation, this says:

$$(a, b)^{-1} = \frac{1}{a^2 + b^2}(a, -b)\,. \qquad \text{(XIII.4)}$$

Hence, if we regard $\mathbb{C}$ as a *real* vector space (by restricting the scalars to real numbers) then the map $\mathbb{C} \to \mathbb{R}^2$, $x + iy \mapsto (x, y)$ is not only a linear isomorphism of real vector spaces, which we describe by saying that the map is an $\mathbb{R}$-linear isomorphism, but, from (XIII.2), it also an isomorphism of $\mathbb{R}$-algebras.

Since the algebra product is bilinear when one fixes one of its factors – say the left factor a – the remaining map L_a: $V \to V$, $b \mapsto ab$ is $\mathbb{K}$-linear. One calls this map *left multiplication* by a; one can define *right multiplication* by a analogously. In the event that the algebra product is commutative, it is unnecessary to distinguish between left and right multiplication. From (XIII.3), if $a = x + iy$ then the matrix of left (or right) multiplication by a, relative to the canonical basis on $\mathbb{R}^2$, is:

$$[L_a] = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} = xI + yJ, \tag{XIII.5}$$

in which I is the 2×2 identity matrix and:

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{XIII.6}$$

represents multiplication by the imaginary i.

One sees that:

$$\det L_a = x^2 + y^2 = \| a \|^2. \tag{XIII.7}$$

Hence, as long as $a \neq 0$, L_a will be invertible. We pause to note how the Euclidian quadratic form on $\mathbb{R}^2$ has been replaced by a determinant on a matrix algebra.

Since the product $L_a L_b$ of the matrices associated with $a, b \in \mathbb{C}$ is L_{ab}, we have a homomorphism L_a: $\mathbb{C}^* \to GL(2; \mathbb{R})$ of the group $(\mathbb{C}^*, \times)$ of non-zero complex numbers under multiplication in the group of invertible real 2×2 matrices under matrix multiplication. As the only element in $\mathbb{C}^*$ that goes to the identity element I is 1, the homomorphism is injective and we can call L_a a faithful representation of the group $(\mathbb{C}^*, \times)$ in $GL(2; \mathbb{R})$. Since not all matrices in $GL(2; \mathbb{R})$ can be represented in the form (XIII.5), the representation is not an isomorphism, except onto its image, which is a two-dimensional Abelian Lie subgroup of the four-dimensional non-Abelian Lie group $GL(2; \mathbb{R})$.

Now, one can generalize (XIII.5) to define the action of the *real* algebra $\mathbb{C}$ on $V \times V$ for any $\mathbb{K}$-vector space [1] V by defining L_a: $V \times V \to V \times V$ to have a matrix of the form:

$$[L_a] = \left[\begin{array}{c|c} xI & -yI \\ \hline yI & xI \end{array}\right] = xI + yJ, \tag{XIII.8}$$

in which I now represents the $2n \times 2n$ identity matrix and:

$$J = \left[\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right] \tag{XIII.9}$$

[1] We are tacitly assuming that $\mathbb{R}$ is a sub-field of $\mathbb{K}$ and the injection is by way of $a \mapsto a1$, where $1 \in \mathbb{K}$ is the unity element; as long as the only other field of interest to us is $\mathbb{K} = \mathbb{C}$, this is no problem.

has replaced the multiplication by i. In particular, J: $V\times V \to V\times V$ is a linear isomorphism such that:

$$J^2 = -I. \tag{XIII.10}$$

We shall call the vector space $V\times V$, together with a J that has this property a *complexification* of V. One can define such a structure for any $V\times V$ by making $J(v, w) = (-w, v)$.

By contrast, when J: $V \to V$ is an $\mathbb{R}$-linear isomorphism such that (XIII.10) is valid one says that J defines a *complex structure* on V. However, whereas any real vector space can be complexified, not every real vector space admits a complex structure. In particular, in order such a J will exist iff dim V is even. Clearly, $V\times V$ will always admit a complex structure

When a real vector space V has a complex structure J one can define complex scalar multiplication on V by:

$$(a + ib)v = av + bJv. \tag{XIII.11}$$

If V has dimension $2n$ as a real vector space then a basis $\{\mathbf{e}_i, i = 1, \ldots, 2n\}$ for V is called a *complex basis* iff it is compatible with J:

$$\mathbf{e}_{i+n} = J\mathbf{e}_i, \qquad i = 1, \ldots, n. \tag{XIII.12}$$

Hence, if $\mathbf{v} = v^i\, \mathbf{e}_i + v^{i+n}\, \mathbf{e}_{i+n}$ with real components v^i, v^{i+n} we can also say that:

$$\mathbf{v} = (v^i + iv^{i+n})\mathbf{e}_i \tag{XIII.13}$$

in which the components $v^i + iv^{i+n}$ are now complex numbers.

Hence, whereas the set $\{\mathbf{e}_i, i = 1, \ldots, 2n\}$defines an $\mathbb{R}$-linear isomorphism $V \to \mathbb{R}^{2n}$, as long as this set satisfies (XIII.12), the set $\{\mathbf{e}_i, i = 1, \ldots, n\}$ defines a $\mathbb{C}$-linear isomorphism $V \to \mathbb{C}^n$. This justifies our terminology in calling the set a complex basis on V.

Another way of defining the *complexification* of a real vector space is by defining the complex scalar multiplication on elements of $V^{\mathbb{C}} = V\otimes\mathbb{C}$, which one regards as the *real* vector space of $\mathbb{R}$-linear maps of V into the *real* vector space $\mathbb{C} = \mathbb{R}^2$; strictly speaking, we should probably use $V^*\otimes\mathbb{C}$ for this purpose, but it seems to be convention that one always sees the present notation.

One then defines complex scalar multiplication on the elements of $V^{\mathbb{C}}$ by:

$$\begin{aligned}(a + ib)\, v\otimes z &= v\otimes(a + ib)z = v\otimes az + v\otimes ibz \\ &= \alpha v\otimes z + \beta v\otimes iz.\end{aligned} \tag{XIII.14}$$

We can then see that the complex structure on $V\otimes\mathbb{C}$ also comes from:

$$J(v\otimes z) = v\otimes iz\ . \tag{XIII.15}$$

If V has real dimension n then $V^{\mathbb{C}}$ has complex dimension n and real dimension $2n$.

One can define an $\mathbb{R}$-linear isomorphism of $V\otimes\mathbb{C}$ with $V\times V$ that takes $v\otimes(a + ib)$ to (av, bv). Since, from (XIII.16), $J(v\otimes(a + ib)) = v\otimes(-b + ia)$ one can then define $J(v, w) = (-w, v)$, as before, and see that the two ways of complexifying a real vector space are equivalent.

Some elementary examples of complexification are given by $\mathbb{R}^{\mathbb{C}} = \mathbb{C}$ and $(\mathbb{R}^n)^{\mathbb{C}} = \mathbb{C}^n$. A somewhat more confusing example is given by $\mathbb{C}^{\mathbb{C}} = \mathbb{C}^2$, which only makes sense if one regards $\mathbb{C}$ as a *real* vector space of real dimension 2 and $\mathbb{C}^2$ as a complex vector space of complex dimension two. The element $(x + iy)\otimes(u + iv)$ in $\mathbb{C}^{\mathbb{C}}$ goes to $(x + iy, u + iv)$ in $\mathbb{C}^2$ or (x, y, u, v) in $\mathbb{R}^4$. Note that the complex vector space of $\mathbb{C}$-linear maps from $\mathbb{C}$ to itself has complex dimension one, which means real dimension two; hence, the possible confusion.

Elements of $\mathbb{C}^{\mathbb{C}}$ can also be represented by 2×2 real matrices, and, in fact, the representation is an $\mathbb{R}$-linear isomorphism that takes the basis elements $1\otimes1$, $i\otimes1$, $1\otimes i$, $i\otimes i$ to the basis elements:

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},$$

resp.

2. Complex orthogonal representations of the Lorentz group [3]. When it comes to alternative ways of representing the Lorentz group, most of the attention to date has been focused on the two-to-one homomorphism from $SL(2;\mathbb{C})$ to $SO_0(3, 1)$, which, you will recall, denotes the identity component of the Lorentz; viz., the proper orthochronous Lorentz group. Most of the justification for the popularity of that representation comes from its use in the representation of the relativistic quantum-mechanical wave equation, namely, the Dirac equation.

In the previous chapter, we discussed the projective-geometric significance of the group $SL(2;\mathbb{C})$ as the group of projective transformations of $\mathbb{C}P^1$, which is diffeomorphic to the two-sphere as a real manifold. In this section, we shall discuss another isomorphic

representation of $SO_0(3, 1)$ that is quite naturally suggested by the representation of 2-planes in $\mathbb{R}^4$ by decomposable bivectors and 2-forms.

a. Isomorphism of $SO_0(3, 1)$ *and* $SO(3; \mathbb{C})$. In addition to the isomorphism $SO_0(3, 1)$ and $SL(2; \mathbb{C})$ there is also an isomorphism of Lie groups between $SO_0(3, 1)$, $SL(2; \mathbb{C})$, and $SO(3; \mathbb{C})$. The latter group is the subgroup of complex invertible matrices in $GL(3; \mathbb{C})$ that not only preserve a volume element but also the complex Euclidian scalar product:

$$<\alpha, \beta> = \delta_{ij}\, \alpha^i\, \beta^j, \qquad (\alpha, \beta \in \mathbb{C}^3,\ \alpha^i, \beta^j \in \mathbb{C}). \tag{XIII.16}$$

Hence, the only difference between the geometry of $\mathbb{C}^3$ with this metric and $\mathbb{R}^3$ with the same metric is in the field to which the scalars and vector components belong. In particular, a matrix $A \in SO(3; C)$ satisfies:

$$\det A = 1, \qquad AA^{\mathrm{T}} = A^{\mathrm{T}}A = I. \tag{XIII.17}$$

One sees that the Lie algebra $\mathfrak{so}(3; \mathbb{C})$ of $SO(3; \mathbb{C})$ then consists of complex 3×3 matrices ω that satisfy:

$$\mathrm{Tr}\ \omega = 0, \qquad \omega^{\mathrm{T}} + \omega = 0; \tag{XIII.18}$$

although the scalars are complex, the first requirement follows automatically from the anti-symmetry that is implied by the second one. A matrix $\omega \in \mathfrak{so}(3; \mathbb{C})$ is then the infinitesimal generator of a one-parameter subgroup of complex rotations in $\mathbb{C}^3$; namely, $e^{t\omega}$.

It follows immediately that $\mathfrak{so}(3; \mathbb{C}) = \mathfrak{so}(3; \mathbb{R})\otimes\mathbb{C}$; i.e., $\mathfrak{so}(3; \mathbb{C})$ is the complexification of $\mathfrak{so}(3; \mathbb{R})$. Hence, if I_i, $i = 1, 2, 3$ is the basis for $\mathfrak{so}(3; \mathbb{R})$ that consists of the elementary anti-symmetric matrices $[I_i]_{jk} = \varepsilon_{ijk}$:

$$I_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad I_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{XIII.19}$$

then these matrices also define a basis for $\mathfrak{so}(3; \mathbb{C})$. Note that these matrices also give the adjoint representation of the Lie algebras $\mathfrak{so}(3; \mathbb{R})$ and $\mathfrak{so}(3; \mathbb{C})$.

Since we defined $\mathfrak{so}(3; \mathbb{C})$ as a matrix Lie algebra – i.e., a sub-algebra of $\mathfrak{gl}(3; \mathbb{C})$ – we observe that the isomorphism of $\mathfrak{so}(3; \mathbb{R})$ with $\mathbb{R}^3$, when it is given the vector cross product, also complexifies to an isomorphism of $\mathfrak{so}(3; \mathbb{C})$ with $\mathbb{C}^3$, when it is given the same cross product. This is really somewhat amusing since one of the selling points of using the exterior product in place of the cross product is the fact that the exterior product generalizes to vector spaces of any dimension, while the cross product is only defined in the vector spaces of dimension three. However, we now see that the transition from non-relativistic physics to relativistic physics does not have to define the transition from a real three-dimensional Euclidian space to a real four-dimensional Minkowski space, but simply to a complex three-dimensional Euclidian space. In that sense, relativistic physics is the complexification of non-relativistic physics.

We then see that if $\mathbf{z} = z^i\, \mathbf{e}_i \in (\mathbb{C}^3, \times)$ then its representation by a complex 3×3 matrix in $\mathfrak{so}(3; \mathbb{C})$ is:

$$\mathrm{ad}(\mathbf{z}) = z^i\, \mathrm{ad}(\mathbf{e}_i) = z^i\, I_i \tag{XIII.20}$$

since:

$$\mathrm{ad}(\mathbf{z})(\mathbf{v}) = [\mathbf{z}, \mathbf{v}] = (\varepsilon_{ijk}\, z^j\, v^k)\, \mathbf{e}_i = \mathbf{z} \times \mathbf{v}. \tag{XIII.21}$$

This makes the components of the 3×3 complex matrix ad(**z**) equal to:

$$[\mathrm{ad}(\mathbf{z})]_{ij} = \varepsilon_{ijk}\, z^k. \tag{XIII.22}$$

The isomorphism of $SO_0(3, 1)$ with $SO(3; \mathbb{C})$ is easiest to see by first defining the isomorphism of the corresponding Lie algebras, which implies that $\mathfrak{so}(3; \mathbb{C})$ must be regarded as a real Lie algebra, not a complex one, and exponentiating them. The isomorphism of Lie algebras is then simplest to describe in terms of real bases for both.

We use the basis $\{J_i, K_i, i = 1, 2, 3\}$ for $\mathfrak{so}(3, 1)$, where the J_i are of the form:

$$J_i = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & I_i \end{array}\right], \quad i = 1, 2, 3, \tag{XIII.23}$$

and the K_i are the elementary infinitesimal boosts:

$$K_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{XIII.24}$$

By direct computation, the structure constants of this Lie algebra are obtained from:

$$[J_i, J_j] = \varepsilon_{ijk} J_k\,, \qquad [J_i, K_j] = \varepsilon_{ijk} K_k\,, \qquad [K_i, K_j] = -\,\varepsilon_{ijk}\, J_k\,. \qquad \text{(XIII.25)}$$

Notice that the last of these commutation relations implies that the three-dimensional vector subspace of $\mathfrak{so}(3, 1)$ that is spanned by the infinitesimal boosts is not a Lie subalgebra. It indirectly leads to Thomas precession, since it says, in effect, that the composition of boosts along different axes will produce a rotation in addition to the combined boost. However, the first set does imply that the Lie algebra $\mathfrak{so}(3; \mathbb{R})$ is included in $\mathfrak{so}(3, 1)$ by the replacement of the I_i with the J_i.

By complexification, the I_i also define a complex basis for the complex Lie algebra $\mathfrak{so}(3; \mathbb{C})$. The commutation rules for this basis are then identical to those of so(3; $\mathbb{R}$):

$$[I_i, I_j] = \varepsilon_{ijk} I_k\,, \qquad \text{(XIII.26)}$$

The simplest way to expand this into a real basis is by way of $\{I_i, iI_i, i = 1, 2, 3\}$. From the $\mathbb{C}$-bilinearity of the Lie bracket, the commutation rules for this real basis are then:

$$[I_i, I_j] = \varepsilon_{ijk} I_k\,, \qquad [I_i, iI_j] = i\varepsilon_{ijk} I_k\,, \qquad [iI_i, iI_k] = -\,\varepsilon_{ijk} I_k\,, \qquad \text{(XIII.26)}$$

which are formally the same as (XIII.25). Hence, the $\mathbb{R}$-isomorphism of $\mathfrak{so}(3, 1)$ with $\mathfrak{so}(3; \mathbb{C})$ is then effected by the association of the J_i with the I_i and the K_i with the iI_i.

This amounts to regarding a boost as essentially a rotation through an imaginary angle. Of course, this is really the opposite of what one should think, as one sees in the elementary case of the isomorphism of $\mathbb{R} \oplus \mathfrak{so}(2; \mathbb{R})$ with $\mathbb{C}$ (= $\mathbb{R}^2$), when one represents the $\mathbb{R}$ summand by matrices of the form αK with:

$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \qquad \text{(XIII.27)}$$

The isomorphism simply takes $\alpha K + \beta J$ to $\alpha + i\beta$. Under exponentiation, since the Lie algebra is Abelian, one has:

$$\exp(\alpha K + \beta J) = \exp(\alpha K)\exp(\beta J) = \begin{bmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \qquad \text{(XIII.28)}$$

and:

$$e^{\alpha + i\beta} = e^{\alpha}\, e^{i\beta} = e^{\alpha}(\cos\beta + i \sin\beta). \qquad \text{(XIII.29)}$$

Hence, it is more precise to say that rotations are the imaginary entities.

The isomorphism of the Lie group $SO_0(3, 1)$ with $SO(3; \mathbb{C})$ is then obtained by exponentiation of the isomorphism of Lie algebras. The actual association of a 4×4 real matrix with a complex 3×3 matrix is more involved at the group level than it was at the algebra level, though. Although one first exponentiates the basis matrices to obtain generators of the group, the remaining elements of the group are obtained by taking matrix products, not matrix sums.

b. The representation of $SO(3, 1)$ *in* $\Lambda_2(\mathbb{R}^4)$ *or* $\Lambda^2(\mathbb{R}^4)$. Both $\Lambda_2(\mathbb{R}^4)$ and $\Lambda^2(\mathbb{R}^4)$ are real vector spaces that are isomorphic to $\mathbb{R}^6$ by a choice of frame or coframe, resp. Hence, the set of all real linear frames for either vector space is described by the elements of the Lie group $GL(6; \mathbb{R})$.

However, this set of frames ignores the fact that both vector spaces have an additional structure that is defined by the fact that both of them are exterior products of four-dimensional real vector spaces: $\Lambda_2(\mathbb{R}^4) = \mathbb{R}^4 \wedge \mathbb{R}^4$, $\Lambda^2(\mathbb{R}^4) = \mathbb{R}^{4*} \wedge \mathbb{R}^{4*}$. Hence, we can identify a subgroup of $GL(6; \mathbb{R})$ that amounts to the image of $GL(4; \mathbb{R})$ under the anti-symmetrized tensor product representation.

We describe this representation by starting with a 4-frame $\{\mathbf{e}_\mu\,,\ \mu = 0, \ldots, 3\}$ on $\mathbb{R}^4$ and associating it with the 6-frame $\{\mathbf{E}_I,\ I = 1, \ldots, 6\}$ on $\Lambda_2(\mathbb{R}^4)$ that we defined previously:

$$\mathbf{E}_i = \mathbf{e}_0 \wedge \mathbf{e}_i\,, \qquad \mathbf{E}_{i+3} = \tfrac{1}{2}\varepsilon_{ijk}\,\mathbf{e}_j \wedge \mathbf{e}_k\,. \tag{XIII.30}$$

When $\mathbf{e}_\mu$ is subjected to a linear frame change to $\overline{\mathbf{e}}_\mu = A^\nu_\mu \mathbf{e}_\nu$, the resulting transformation of $\mathbf{E}_I$ is obtained from expanding:

$$\overline{\mathbf{E}}_i = \overline{\mathbf{e}}_0 \wedge \mathbf{e}_i = A^\mu_0 A^\nu_i \mathbf{e}_\mu \wedge \mathbf{e}_\nu\,, \qquad \overline{\mathbf{E}}_{i+3} = \tfrac{1}{2}\varepsilon_{ijk}\,\overline{\mathbf{e}}_j \wedge \overline{\mathbf{e}}_k = \tfrac{1}{2}\varepsilon_{ijk} A^\mu_j A^\nu_k \mathbf{e}_\mu \wedge \mathbf{e}_\nu \tag{XIII.31}$$

and identifying submatrices.

If D: $GL(4; \mathbb{R}) \to GL(6; \mathbb{R})$, $A \mapsto D(A)$ is the resulting representation and the matrices of $GL(6; \mathbb{R})$ are represented in the block matrix form:

$$D(A) = \left[\begin{array}{c|c} D_{tt}(A)^i_j & D_{ts}(A)^i_j \\ \hline D_{st}(A)^i_j & D_{ss}(A)^i_j \end{array}\right] \tag{XIII.32}$$

then the submatrices take the form:

$$D_{tt}(A)^i_j = A^0_0 A^i_j - A^i_0 A^0_j\,, \qquad D_{ts}(A)^i_j = \varepsilon_{jkl} A^0_k A^i_l\,, \tag{XIII.33a}$$

$$D_{st}(A)^i_j = \varepsilon_{ikl} A^k_0 A^l_j, \qquad D_{ss}(A)^i_j = \tfrac{1}{2}\varepsilon_{ikl}\varepsilon_{jmn} A^k_m A^l_n. \qquad \text{(XIII.33b)}$$

It is instructive to examine the form that these matrices take for various subgroups of $GL(4;\mathbb{R})$.

For the homotheties, one will have $A^0_0 = \lambda \neq 0, A^0_j = A^i_0 = 0, A^i_j = \delta^i_j$, and the corresponding matrix for $D(A)$ will take the form:

$$\left[\begin{array}{c|c} \lambda\delta^i_j & 0 \\ \hline 0 & \delta^i_j \end{array}\right].$$

For the $GL(3;\mathbb{R})$ subgroup that is represented by $A^0_0 = 1, A^0_j = A^i_0 = 0, A^i_j \in GL(3;\mathbb{R})$, one has a matrix of the form:

$$\left[\begin{array}{c|c} A^i_j & 0 \\ \hline 0 & A^i_j \end{array}\right].$$

Naturally, any subgroup of $GL(3;\mathbb{R})$, such as $SO(3;\mathbb{R})$, will be represented in this form, as well.

For the translations of $\mathbb{R}^3$, with $A^0_0 = 1, A^0_j = 0, A^i_0 = a^i, A^i_j = \delta^i_j$ the matrix looks like:

$$\left[\begin{array}{c|c} \delta^i_j & 0 \\ \hline -\varepsilon_{ijk}a^k & \delta^i_j \end{array}\right].$$

For the inversions of $\mathbb{R}^3$ that are described by $A^0_0 = 1, A^0_j = b^i, A^i_0 = 0, A^i_j = \delta^i_j$, it looks like:

$$\left[\begin{array}{c|c} \delta^i_j & -\varepsilon_{ijk}b^k \\ \hline 0 & \delta^i_j \end{array}\right].$$

Of particular interest in relativistic electromagnetism is the representation of a boost along the x axis. If the 4×4 matrix takes the form:

$$A = \begin{bmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \text{(XIII.34)}$$

in which $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Fitzgerald-Lorentz contraction factor, then its image under the exterior product representation is:

$$D(A) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cosh\zeta & 0 & 0 & 0 & \sinh\zeta \\ 0 & 0 & \cosh\zeta & 0 & -\sinh\zeta & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\sinh\zeta & 0 & \cosh\zeta & 0 \\ 0 & \sinh\zeta & 0 & 0 & 0 & \cosh\zeta \end{array}\right]. \qquad \text{(XIII.35)}$$

In order to transform the components b^I of a bivector $\mathbf{b} = b^I \mathbf{E}_I$, one must use the inverse matrix to $D(A)$, which is:

$$D(A^{-1}) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cosh\zeta & 0 & 0 & 0 & -\sinh\zeta \\ 0 & 0 & \cosh\zeta & 0 & \sinh\zeta & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \sinh\zeta & 0 & \cosh\zeta & 0 \\ 0 & -\sinh\zeta & 0 & 0 & 0 & \cosh\zeta \end{array}\right]. \qquad \text{(XIII.36)}$$

When one applies this matrix to $\mathfrak{h} = [D^i, H^i]^{\mathrm{T}}$ the new components are:

$$\bar{D}^1 = D^1, \qquad \bar{D}^i = \gamma[\mathbf{D} + 1/c(\mathbf{v} \times \mathbf{H})]^i, \qquad i = 2, 3, \qquad \text{(XIII.37a)}$$
$$\bar{H}^1 = H^1, \qquad \bar{H}^i = \gamma[-\mathbf{D} + 1/c(\mathbf{v} \times \mathbf{H})]^i, \qquad i = 2, 3, \qquad \text{(XIII.37b)}$$

which agrees with the usual formulae, as one might find in Jackson [**4**].

The representation of $GL(4; \mathbb{R})$ in $\Lambda^2(\mathbb{R}^4)$ is contragredient to the one in $\Lambda_2(\mathbb{R}^4)$ that we just described. That is, if $\mathbf{e}_\mu$ transforms by way of A^μ_ν then if θ^μ is the reciprocal coframe to $\mathbf{e}_\mu$, in order to preserve the relation $\theta^\mu(\mathbf{e}_\nu) = \delta^\mu_\nu$ the coframe θ^μ must transform by means of the inverse matrix $\tilde{A}^\mu_\nu$. Thus, $\theta^\mu \wedge \theta^\nu$ must transform as:

$$\bar{\theta}^\mu \wedge \bar{\theta}^\nu = \tilde{A}^\mu_\kappa \tilde{A}^\nu_\lambda \, \theta^\kappa \wedge \theta^\lambda . \qquad \text{(XIII.38)}$$

Since we are defining our 6-coframe for $\Lambda^2(\mathbb{R}^4)$ by means of:

$$E^i = \theta^0 \wedge \theta^i, \qquad E^{i+3} = \tfrac{1}{2} \varepsilon_{ijk} \, \theta^j \wedge \theta^k, \qquad \text{(XIII.39)}$$

this serves to define the representation, which is then D': $GL(4;\ \mathbb{R}) \rightarrow GL(6,\ \mathbb{R})$, $A \mapsto D(A^{-1})$. Hence, the matrix of $D'(A) = D(A^{-1})$ is similar to (XIII.32), except that the components of A are replaced by the components of A^{-1}. The transformation of the components of a 2-form $b = b_I E^I$ is then by way of $D'(A^{-1}) = D(A)$, whose matrix is then given by (XIII.32) directly.

The transformation of $F = [E_i, B_i]$ is then entirely analogous to (XIII.37a, b), except that one right-multiplies by $D(A)$, instead of left-multiplying by $D(A^{-1})$. However, the end result is a similar set of equations to (XIII.37a, b), with the symbols for the components of $\mathfrak{h}$ replaced with those of F.

c. The representation of SO(3; $\mathbb{C}$) *in* $\Lambda_2(\mathbb{R}^4)$ *or* $\Lambda^2(\mathbb{R}^4)$. Another structure that one can impose on the six-dimensional real vector space $\Lambda_2(\mathbb{R}^4)$ is a complex structure, since its dimension is even. This time, we denote the complex structure by an $\mathbb{R}$-linear isomorphism $*$: $\Lambda_2(\mathbb{R}^4) \rightarrow \Lambda_2(\mathbb{R}^4)$, $\mathbf{b} \mapsto *\mathbf{b}$, such that:

$$*^2 = -I. \tag{XIII.40}$$

We can then define complex scalar multiplication by the usual means:

$$(\alpha + i\beta)\mathbf{b} = \alpha\mathbf{b} + \beta *\mathbf{b}. \tag{XIII.41}$$

An $\mathbb{R}$-linear frame$\{\mathbf{E}_I\ ,\ I = 1\ ,\ \ldots.,\ 6\}$ is *complex* iff:

$$\mathbf{E}_{i+3} = *\mathbf{E}_i = i\mathbf{E}_i\ , \qquad i = 1,\ 2,\ 3. \tag{XIII.42}$$

This terminology is then consistent with (XIII.41) and one can regard $\{\mathbf{E}_i\ ,\ i = 1\ ,\ 2,\ 3\}$ as a complex 3-frame on the complex vector space $\Lambda_2(\mathbb{R}^4)$, so any $\mathbf{b} \in \Lambda_2(\mathbb{R}^4)$ can be expressed in the form:

$$\mathbf{b} = b^i\ \mathbf{E}_i = \alpha^i\ \mathbf{E}_i + \beta^i\ *\mathbf{E}_i\ , \qquad i = 1,\ 2,\ 3, \tag{XIII.43}$$

in which the components $b^i = \alpha^i + i\beta^i$ are generally complex numbers.

Not all $\mathbb{R}$-linear frames on $\Lambda_2(\mathbb{R}^4)$ will have the property (XIII.42) that makes them complex. This is simple enough to show if one considers the transformation from one complex 3-frame $\{\mathbf{E}_i\ ,\ i = 1\ ,\ 2,\ 3\}$ on $\Lambda_2(\mathbb{R}^4)$ to another one $\{\bar{\mathbf{E}}_i\ ,\ i = 1\ ,\ 2,\ 3\}$. Since the former set defines a complex frame, one must have a 3×3 complex matrix A^i_j of components that makes $\bar{\mathbf{E}}_i = A^j_i \mathbf{E}_j$. Since $\bar{\mathbf{E}}_i$ is another complex 3-frame, the matrix A^i_j will be invertible. Hence, the matrix A^i_j is an element of the Lie group $GL(3;\ \mathbb{C})$. One can

then see that the set of real 6-frames on $\Lambda_2(\mathbb{R}^4)$ is in one-to-one correspondence with the elements of $GL(6; \mathbb{R})$, while the set of complex 3-frames is one-to-one correspondence with the elements of $GL(3; \mathbb{C})$. As the real dimension of $GL(6; \mathbb{R})$ is 36 and that of $GL(3; \mathbb{C})$ is 18 (complex dimension = 9), it is clear that there are "more" real frames than complex ones.

In order for an $\mathbb{R}$-linear map A: $V \to V$ on a real vector space V that has been given a complex structure *: $V \to V$ to behave like a $\mathbb{C}$-linear map, it must commute with * in the same way that $A(i\mathbf{v}) = iA\mathbf{v}$ when $\mathbf{v}$ belongs to a complex vector space; i.e.:

$$A* = *A. \tag{XIII.44}$$

In fact, this is not only necessary, but sufficient for a real 6×6 matrix $A \in GL(6; \mathbb{R})$ to be associated with a complex 3×3 matrix $C \in GL(3; \mathbb{C})$. The association of the one with the other is quite straightforward: If $C^i_j = \alpha^i_j + i\beta^i_j$ is the matrix of C then the corresponding A has a matrix that is given in block form by:

$$A^I_J = \left[\begin{array}{c|c} \alpha^i_j & -\beta^i_j \\ \hline \beta^i_j & \alpha^i_j \end{array}\right] = \left[\begin{array}{c|c} \alpha^i_j & 0 \\ \hline 0 & \alpha^i_j \end{array}\right] + * \left[\begin{array}{c|c} \beta^i_j & 0 \\ \hline 0 & \beta^i_j \end{array}\right]. \tag{XIII.45}$$

with respect to a complex basis.

If one goes back to (XIII.5) then one sees that the way that one constructs a real 6×6 matrix that represents a complex 3×3 matrix is closely analogous to the way that one makes a real 2×2 matrix out of a complex number. In particular, the identity map and * take the block matrix form:

$$I = \left[\begin{array}{c|c} \delta^i_j & 0 \\ \hline 0 & \delta^i_j \end{array}\right], \qquad * = \left[\begin{array}{c|c} 0 & -\delta^i_j \\ \hline \delta^i_j & 0 \end{array}\right]. \tag{XIII.46}$$

If $\{\mathbf{E}_I, I = 1, \ldots, 6\}$ is a complex frame for $\Lambda_2(\mathbb{R}^4)$ then one can also decompose $\Lambda_2(\mathbb{R}^4)$ into a direct sum $\Lambda_2^{\mathrm{Re}} \oplus \Lambda_2^{\mathrm{Im}}$ of 3-dimensional subspaces that are spanned by the frames $\{\mathbf{E}_i, i = 1, 2, 3\}$ and $\{*\mathbf{E}_i, i = 1, 2, 3\}$, respectively, since they then have the property:

$$*\Lambda_2^{\mathrm{Re}} = \Lambda_2^{\mathrm{Im}}, \qquad *\Lambda_2^{\mathrm{Im}} = \Lambda_2^{\mathrm{Re}}.$$

More precisely, if $\mathbf{b}_{\mathrm{R}} \in \Lambda_2^{\mathrm{Re}}$ and $\mathbf{b}_{\mathrm{I}} \in \Lambda_2^{\mathrm{Im}}$ then $*\mathbf{b}_{\mathrm{R}} \in \Lambda_2^{\mathrm{Im}}$ and $*\mathbf{b}_{\mathrm{I}} \in \Lambda_2^{\mathrm{Re}}$.

This situation seems to justify our terminology of "Re" and "Im" since these subspaces correspond to real and imaginary subspaces under the complex structure.

However, it is important to understand that when one is given a complex structure * on a vector space V the decomposition of V into real and imaginary subspaces is not generally canonical, as it is when $V = \mathbb{R}^n$, so one must choose a real+imaginary splitting in the same way that one might choose a frame.

If $[\mathbf{t}] \oplus \Sigma$ is a time+space splitting of $\mathbb{R}^4$ then one has an induced splitting [2] of $\Lambda_2(\mathbb{R}^4)$ into $[\mathbf{t}] \wedge \Sigma \oplus \Lambda_2(\Sigma)$, in which the subspace $[\mathbf{t}] \wedge \Sigma$ is spanned by all elements of the form $\mathbf{t} \wedge \mathbf{v}$, where $\mathbf{v} \in \Sigma$ and the subspace $\Lambda_2(\Sigma)$ is spanned by all elements of the form $\mathbf{v} \wedge \mathbf{w}$ with $\mathbf{v}, \mathbf{w} \in \Sigma$. Whether the splitting is an actual real+imaginary decomposition will depend upon the choice of time+space splitting, since not all of them will induce a splitting with the property $*([\mathbf{t}] \wedge \Sigma) = \Lambda_2(\Sigma)$, $*(\Lambda_2(\Sigma)) = [\mathbf{t}] \wedge \Sigma$.

Previously, we decomposed an electromagnetic bivector field into an electric part and a magnetic part. Now, we can now see that this latter decomposition is closely related to imposing a complex structure on $\Lambda_2(\mathbb{R}^4)$. It is amusing to regard electric bivectors as real and magnetic ones as imaginary, since one knows that elementary magnetic fields are, in a sense, "fictitious" fields that appear as a result of the choice of rest space in much the same way that Coriolis and centripetal forces appear.

Conversely, it is not true that any real+imaginary splitting of $\Lambda_2(\mathbb{R}^4)$ will imply a corresponding time+space splitting of $\mathbb{R}^4$. The main issue is an algebraic one: in order for a three-dimensional subspace of $\Lambda_2(\mathbb{R}^4)$ to take the form $[\mathbf{t}] \wedge \Sigma$ all of the elements must have a common exterior factor that is a scalar multiple of some $\mathbf{t} \in \mathbb{R}^4$, which is not always the case. Moreover, it is not the case that if $\Lambda_2(\mathbb{R}^4)$ is decomposed into a pair of three-dimensional summands then at least one of then must be of the form $[\mathbf{t}] \wedge \Sigma$ for some $[\mathbf{t}] \in \mathbb{R}^4$.

As we have seen before, if $\mathbb{R}^4$ has been given a volume element $V \in \Lambda^4$ then one can define a scalar product on $\Lambda_2(\mathbb{R}^4)$ by way of $\langle \mathbf{A}, \mathbf{B} \rangle = V(\mathbf{A} \wedge \mathbf{B})$. Furthermore, if one also has a complex structure * then one can define a second scalar product by means of $(\mathbf{A}, \mathbf{B}) = \langle \mathbf{A}, *\mathbf{B} \rangle = V(\mathbf{A} \wedge *\mathbf{B})$, as long as * is self-adjoint; i.e., $\mathbf{A} \wedge *\mathbf{B} = *\mathbf{A} \wedge \mathbf{B}$ for all $\mathbf{A}, \mathbf{B} \in \Lambda_2(\mathbb{R}^4)$.

One can define a complex orthogonal structure on $\Lambda_2(\mathbb{R}^4)$ by means of:

$$\langle \mathbf{A}, \mathbf{B} \rangle_{\mathbb{C}} = (\mathbf{A}, \mathbf{B}) + i\langle \mathbf{A}, \mathbf{B} \rangle. \qquad \text{(XIII.47)}$$

[2] The details of this statement are discussed in greater rigor in Delphenich [**1**].

Actually, one could also define such a structure by means of $\langle \mathbf{A}, \mathbf{B}\rangle + i(\mathbf{A}, \mathbf{B})$, but it is clear that this is equal to the complex conjugate of $-i\langle \mathbf{A}, \mathbf{B}\rangle_{\mathbb{C}}$, which is not fundamentally different. This amounts to the observation that complex orthogonal quadratic forms all have the Euclidian signature type.

One can then further reduce the set of complex 3-frames on $\Lambda_2(\mathbb{R}^4)$ to the set of complex orthogonal 3-frames, which then satisfy:

$$\langle \mathbf{E}_i, \mathbf{E}_j\rangle_{\mathbb{C}} = \delta_{ij} \,. \tag{XIII.48}$$

A 3×3 complex matrix A^i_j that relates a complex orthogonal 3-frame $\bar{\mathbf{E}}_i$ to another one $\mathbf{E}_i$ by way of $\bar{\mathbf{E}}_i = A^j_i \mathbf{E}_j$ will then be an element of $O(3; \mathbb{C})$. Hence, $\langle A\mathbf{a}, A\mathbf{b}\rangle_{\mathbb{C}} = \langle \mathbf{a}, \mathbf{b}\rangle_{\mathbb{C}}$ for any elements $\mathbf{a}, \mathbf{b} \in \Lambda_2(\mathbb{R}^4)$ and, as a result, $A^{-1} = A^{\mathrm{T}}$.

In order reduce this set of complex orthogonal frames further to a subset of complex orthogonal 3-frames with unit-volume one must define a volume element on the complex vector space $\Lambda_2(\mathbb{R}^4)$. Here, we must be careful not to confuse this with a volume element on $\mathbb{R}^4$, since a volume element on the three-dimensional complex vector space $\Lambda_2(\mathbb{R}^4)$ will be a non-zero complex 3-form on $\Lambda_2(\mathbb{R}^4)$ itself, not a non-zero 4-form on $\mathbb{R}^4$. Hence, if $\mathbf{E}_i$ is a complex 3-frame on $\Lambda_2(\mathbb{R}^4)$ and E^i is its reciprocal 3-frame on $\Lambda_2(\mathbb{R}^4)$ then one can define such a volume element by means of:

$$\mathcal{V} = E^1 \perp E^2 \perp E^3 = \frac{1}{3!}\, \varepsilon_{ijk}\, E^i \perp E^j \perp E^k, \tag{XIII.49}$$

in which the symbol $\perp$ is used to denote the exterior product in the exterior algebra over the complex vector space $\Lambda_2(\mathbb{R}^4)$ in order to minimize the confusion with the exterior algebra over $\mathbb{R}^4$. Since the complex dimension of the vector space $\Lambda_2(\mathbb{R}^4)$ is three, its exterior algebra will consist of complex scalars, vectors, bivectors, and trivectors. The volume of the parallelepiped spanned by a complex linear 3-frame $\mathbf{F}_i = A^j_i \mathbf{E}_j$ is then:

$$\mathcal{V}(\mathbf{F}_1 , \mathbf{F}_2 , \mathbf{F}_3) = \det A\, \mathcal{V}(\mathbf{E}_1 , \mathbf{E}_2 , \mathbf{E}_3) = \det A. \tag{XIII.50}$$

Hence, the transformation preserves the volume of the 3-frame iff $\det A = 1$.

This allows us to associate complex unit-volume 3-frames with elements of $SL(3; \mathbb{C})$ and complex unit-volume orthogonal 3-frames with elements of $SO(3; \mathbb{C})$. Hence, we have finally arrived at a representation of the proper orthochronous Lorentz group $SO_0(3;$

1) as an isomorphic subgroup of the group that is responsible for the transformations of frames in the vector space of bivectors on $\mathbb{R}^4$.

Since we made no mention of defining a Lorentzian structure on $\mathbb{R}^4$ that would induce the complex orthogonal structure on $\Lambda_2(\mathbb{R}^4)$ the question arises whether the complex orthogonal structure on $\Lambda_2(\mathbb{R}^4)$ will induce a Lorentzian structure on $\mathbb{R}^4$.

We have already seen that a complex orthogonal structure on $\Lambda_2(\mathbb{R}^4)$ will allow us to construct a light cone out of the lines of intersection of all isotropic decomposable bivectors and their duals under *, which then determines a conformal class of Lorentzian metrics. We now see that since $SO(3; \mathbb{C})$ is isomorphic to $SO_0(3, 1)$ every unit-volume complex orthogonal 3-frame on $\Lambda_2(\mathbb{R}^4)$ is associated with a unique time-oriented unit-volume Lorentzian frame on $\mathbb{R}^4$, as long as one has agreed on the way that 4-frames in $\mathbb{R}^4$ relate to complex 3-frames in $\Lambda_2(\mathbb{R}^4)$; the one that we have been using all along would suffice, as long as the 6-frame defined by $\mathbf{e}_\mu \wedge \mathbf{e}_\nu$ is complex under *. Hence, the isomorphism of $SO(3; \mathbb{C})$ and $SO_0(3, 1)$, together with one 4-frame $\mathbf{e}_\mu$ on $\mathbb{R}^4$ whose corresponding 6-frame $\mathbf{E}_I$ on $\Lambda_2(\mathbb{R}^4)$ is complex orthogonal will define the set of all time-oriented unit-volume Lorentzian frames on $\mathbb{R}^4$, which is equivalent to a choice of Lorentzian structure.

Of course, analogous constructions to the foregoing ones can be made for the vector space $\Lambda^2(\mathbb{R}^4)$ that is dual to $\Lambda_2(\mathbb{R}^4)$. One must use the transpose of * to define a complex structure on $\Lambda^2(\mathbb{R}^4)$ in order to make $*A(\mathbf{A}) = A(*\mathbf{A})$ for every $A \in \Lambda^2(\mathbb{R}^4)$ and $\mathbf{A} \in \Lambda_2(\mathbb{R}^4)$.

d. Hermitian structures on $\Lambda_2(\mathbb{R}^4)$ *and* $\Lambda^2(\mathbb{R}^4)$. Since we are starting with such a large group of frame transformations on $\Lambda_2(\mathbb{R}^4)$ – namely, $GL(6; \mathbb{R})$ – we have potentially more subgroups to consider than when we start with $GL(4; \mathbb{R})$, which pertains to frames in $\mathbb{R}^4$ itself. This is especially true when one puts a complex structure on either of the vector spaces $\Lambda_2(\mathbb{R}^4)$ or $\mathbb{R}^4$, which allows one to consider complex frames and the groups $GL(3; \mathbb{C})$ and $GL(2; \mathbb{C})$, respectively. Of course, the physical significance of such a complex structure is more established in the case of $\Lambda_2(\mathbb{R}^4)$ than it seems to be in the case of $\mathbb{R}^4$.

One of the intriguing consequences of introducing a complex structure * on $\Lambda_2(\mathbb{R}^4)$ is that in addition to the complex orthogonal structure that such an isomorphism defines one can also define a Hermitian structure, as long as one also has a choice of real+imaginary decomposition of $\Lambda_2(\mathbb{R}^4)$, as well. Furthermore, the physical significance of the construction is actually quite fundamental to electromagnetism.

The reason that one needs to choose a real+imaginary decomposition $\Lambda_2^{\mathrm{Re}} \oplus \Lambda_2^{\mathrm{Im}}$ of $\Lambda_2(\mathbb{R}^4)$ is because otherwise there is no unambiguous way to define the operator of complex conjugation. However, when one is given such a decomposition the definition is immediate. If $\mathbf{a} = \mathbf{a}_R + *\mathbf{a}_I$ with $\mathbf{a}_R, \mathbf{a}_I \in \Lambda_2^{\mathrm{Re}}$ then its complex conjugate relative to this splitting is:

$$\overline{\mathbf{a}} = \mathbf{a}_R - *\mathbf{a}_I , \tag{XIII.51}$$

We now examine what happens to the complex Euclidian structure when we include the effect of conjugation. We define:

$$(\mathbf{a}, \mathbf{b})_{\mathbb{C}} = <\mathbf{a}, \overline{\mathbf{b}}>_{\mathbb{C}} = (\mathbf{a}, \overline{\mathbf{b}}) + i <\mathbf{a}, \overline{\mathbf{b}}> . \tag{XIII.52}$$

Suppose $\mathbf{b}_i$, i = 1, 2, 3 is a complex orthonormal frame for $\Lambda_2(\mathbb{R}^4)$, so we assume:

$$<\mathbf{b}_i, \mathbf{b}_j>_{\mathbb{C}} = \delta_{ij}, \tag{XIII.53}$$

and define the bilinear form associated with the complex Euclidian structure to be:

$$\delta = \delta_{ij}\, b^i \otimes b^j, \tag{XIII.54}$$

where b^i, i = 1, 2, 3 is the reciprocal coframe to $\mathbf{b}_i$. As long as we understand the notation $\overline{b}^i$ to mean the composition of the $\mathbb{C}$-linear functional b^i with the complex conjugation operation on $\mathbb{C}$ then we should have:

$$(\mathbf{b}_i, \mathbf{b}_j)_{\mathbb{C}} = <\mathbf{b}_i, \overline{\mathbf{b}}_j>_{\mathbb{C}} = <\mathbf{b}_i, \mathbf{b}_j>_{\mathbb{C}} = \delta_{ij} , \tag{XIII.55}$$

since $\mathbf{b}_i$ is "real."

This allows us to set the bilinear form h that is associated with the new inner product equal to:

$$h = \delta_{ij} b^i \otimes \overline{b}^j , \tag{XIII.56}$$

in this frame and:

$$h = h_{ij} \beta^i \otimes \overline{\beta}^j , \tag{XIII.57}$$

for a general frame.

Hence, when $\mathbf{A} = A^i \mathbf{b}_i$ and $\mathbf{B} = B^j \mathbf{b}_j$ the value of the inner product is:

$$(\mathbf{A}, \mathbf{B})_{\mathbb{C}} = h(\mathbf{A}, \mathbf{B}) = \delta_{ij} A^i \overline{B}^j . \qquad \text{(XIII.58)}$$

This shows that what we have defined is indeed a Hermitian structure on the complex vector space $\Lambda_2(\mathbb{R}^4)$, and the $\mathbb{C}$-linear isomorphism of $\Lambda_2(\mathbb{R}^4)$ with $\mathbb{C}^3$ that is defined by a Hermitian frame becomes a unitary isomorphism when one gives $\mathbb{C}^3$ the usual Hermitian structure $\delta_{ij}\theta^i \otimes \overline{\theta}^j$ with θ^i being the canonical coframe.

Due to the assumption of the self-adjointness of *, when one computes $(\mathbf{a}, \mathbf{a})_{\mathbb{C}}$, one obtains:

$$(\mathbf{a}, \mathbf{a})_{\mathbb{C}} = [(\mathbf{a}_R, \mathbf{a}_R) + (\mathbf{a}_I, \mathbf{a}_I)] + i[\langle \mathbf{a}_R, \mathbf{a}_R \rangle + \langle \mathbf{a}_I, \mathbf{a}_I \rangle] = (\mathbf{a}_R, \mathbf{a}_R) + (\mathbf{a}_I, \mathbf{a}_I). \qquad \text{(XIII.59)}$$

The vanishing of the imaginary part follows from the fact that $\mathbf{a}_R$ and $\mathbf{a}_I$ belong to a three-dimensional (real) vector space, while $\langle \mathbf{a}_R, \mathbf{a}_R \rangle$ and $\langle \mathbf{a}_I, \mathbf{a}_I \rangle$ are both defined by 4-vectors, which must vanish identically, since $\mathbf{a}_R$, $\mathbf{a}_I$ belong to three-dimensional vector spaces.

If we apply this to the case of the electromagnetic excitation bivector $\mathfrak{h} = \mathbf{t} \wedge \mathbf{D} + *(\mathbf{t} \wedge \mathbf{H})$ then we see that:

$$\begin{aligned}(\mathfrak{h}, \mathfrak{h})_{\mathbb{C}} &= (\mathbf{D}, \mathbf{D}) + 2(\mathbf{D}, \mathbf{H}) + (\mathbf{H}, \mathbf{H}) \\ &= \varepsilon^{-1}(\mathbf{D}, \mathbf{D}) + 2\gamma^{-1}(\mathbf{D}, \mathbf{H}) + \mu(\mathbf{H}, \mathbf{H}). \qquad \text{(XIII.60)}\end{aligned}$$

In the case where the electromagnetic couplings (= γ) vanish this becomes:

$$(\mathfrak{h}, \mathfrak{h})_{\mathbb{C}} = \varepsilon^{-1}(\mathbf{D}, \mathbf{D}) + \mu\,(\mathbf{H}, \mathbf{H}) = E(\mathbf{D}) + H(\mathbf{B}), \qquad \text{(XIII.61)}$$

and in the isotropic case:

$$(\mathfrak{h}, \mathfrak{h})_{\mathbb{C}} = 1/\varepsilon D^2 + \mu H^2 = \varepsilon E^2 + 1/\mu B^2. \qquad \text{(XIII.62)}$$

Hence, we see that the Hermitian structure that we defined is intimately related to the energy density of the electromagnetic field. The fact that it can only be defined after one has made a choice of real+imaginary decomposition is entirely consistent with the fact that one cannot speak of energy in relativistic mechanics until one has made a choice of time+space splitting of spacetime.

As is well-known, quadratic expressions such as (XIII.62) are associated with the Hamiltonian of the simple harmonic oscillator in mechanics, and electromagnetic wave motion is envisioned to involve a continuous distribution of simple harmonic oscillators throughout space. Hence, the generalization defined by (XIII.60) shows that one can generalize the oscillators accordingly to three-dimensional anisotropic harmonic oscillators.

What is truly intriguing about the appearance of a Hermitian structure on the (complex) three-dimensional vector spaces of interest to electromagnetism is that this naturally allows one to reduce the group $GL(3;\mathbb{C})$ of complex linear frame transformations to the group $U(3)$ of unitary frame transformations, and further to $SU(3)$, if one requires that they preserve the volume element, as well. Ordinarily, the group $SU(3)$ does not begin to figure in physics until one is dealing with strong interactions, but now we see that it is clearly relevant to the energetics of electromagnetism, as well. Since the strong interaction was introduced into physics in order to account for the stability of most nuclei in the face of the mutual electrostatic repulsion of their constituent protons, this makes one wonder if there is some natural transition from the theory of electromagnetism into the theory of the strong interaction within the pre-metric formalism. After all, we have already seen that it affords a natural transition from electromagnetism into gravitation by way of the electromagnetic dispersion law for the medium.

For more musings along these lines, the reader is invited to peruse the author's paper [**5**]

3. General relativity in terms of Lorentzian frames. In order to facilitate the transition from conventional Lorentzian general relativity, which deals with Lorentzian frames in the tangent bundle, to the form that things take in the context of complex orthogonal frames in the bundle of 2-forms, we briefly summarize the formulation of general relativity using the formalism of connection 1-forms on the bundle of oriented, time-oriented, Lorentzian frames on spacetime. The approach to differential geometry that involves defining connection 1-forms on principal bundles is quite commonplace in gauge field theories, but apparently still not completely accepted in general relativity and gravitation theory [3].

a. Reductions of the bundle of linear frames. In order to define an oriented, time-oriented, Lorentzian frame in a tangent space T_xM to a point x in real four-dimensional spacetime manifold M, one will clearly need three things: a volume element $V \in \Lambda^4$, a time+space splitting of $T(M)$ into $[\mathbf{t}]\oplus\Sigma$, and a Lorentzian metric g.

The presence of a volume element allows one to reduce from the bundle $GL(M) \to M$ of linear frames in $T(M)$ to the bundle $GL^+(M) \to M$ of oriented frames to the bundle $SL(M) \to M$ of unit-volume frames. Of course, one must assume that M – or rather, $T(M)$ – is orientable to begin with, which we will.

One can also regard a volume element as a real-valued function V: $GL(M) \to \mathbb{R}$, $\mathbf{e}_\mu \mapsto V(\mathbf{e}_\mu) = V(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, where we are abusing the notation by using the same symbol for the function and the non-zero 4-form. It is, moreover, required to be equivariant under the right action of $GL(4;\mathbb{R})$ on $GL(M)$ and its action on $\mathbb{R}$ by way of the determinant. That is, when the frame $\mathbf{e}_\mu$ is changed to $\mathbf{e}_\nu A^\nu_\mu$ the numerical value of $V(\mathbf{e}_0,$

[3] Some of the references on general relativity that use this formalism to a greater or lesser degree are [**6-12**].

$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) is changed to $\det(A)V(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. One can then reconstruct the 4-form at $x \in M$ by means of:

$$V_x = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^2 = \frac{1}{4!}\,\varepsilon_{\kappa\lambda\mu\nu}\,\theta^\kappa \wedge \theta^\lambda \wedge \theta^\mu \wedge \theta^\nu. \qquad \text{(XIII.63)}$$

The actual reduction of $GL(M)$ to $GL^+(M)$ then follows from considering the subset of $GL(M)$ that consists of linear frames for which $V(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is positive. Since each fiber of $GL(M)$ consists of two components – namely, $V(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) < 0$ and $V(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) > 0$ – one can see that the homogeneous space $\mathcal{O}_x = GL_xM\,/GL_x^+$ consists of two points, up to homotopy equivalence. We can then regard the bundle $\mathcal{O}(M) \to M$ as the *orientation bundle* over M, so an orientation on M is a global section of this bundle. Hence, the manifold $\mathcal{O}(M)$ is diffeomorphic to the orientable covering manifold of M.

The reduction from $GL^+(M)$ to $SL(M)$ follows from looking at the level hypersurface of unity for the function V. That is, one chooses a set of oriented frames at each point that one wishes to call *unit-volume* frames. Since the homogeneous space $GL_x^+/\,SL_x$ is homotopically equivalent to $\mathbb{R}$ (by way of the determinant), which is contractible, this reduction is never obstructed by homotopy.

The time+space splitting allows one to further reduce from $SL(M)$ to the bundle $SL_0(M) \to M$ of time-oriented unit-volume frames. Again, one must assume that the line bundle $[\mathbf{t}](M) \to M$ is orientable, which amounts to the existence of a *non-zero* vector field $\mathbf{t} \in \mathfrak{X}(M)$ that generates the line $[\mathbf{t}](x)$ at each $x \in M$. If M is compact then it is necessary that its Euler-Poincaré characteristic vanish in order for this to be possible. However, requiring that a given line field admit a non-zero section is a stronger condition than requiring that the tangent bundle admit a non-zero section, so it is not a sufficient condition in this case.

The equivariant maps on $SL(M)$ and its dual $SL^*(M)$ – which is then the bundle of unit-volume coframes on M – that define the time+space splitting of $T(M)$ are then the maps $\mathbf{t}$: $SL(M) \to \mathbb{R}^4$, $\mathbf{e}_\mu \mapsto \mathbf{t}(\mathbf{e}_\mu) = t^\mu$ and τ: $SL^*(M) \to \mathbb{R}^{4*}$, $\theta^\mu \mapsto \tau(\theta^\mu) = \tau_\mu$, which are equivariant under the right action of $SL(4; \mathbb{R})$ on the bundles and its left action on $\mathbb{R}^4$ and its dual in a manner that is consistent with regarding $\mathbf{t}$ as a vector field and τ as a covector field. As usual, we assume that $\tau(\mathbf{t}) \neq 0$.

The Lorentzian metric g then allows one to reduce $SL_0(M)$ to the bundle $SO_0(3, 1)(M)$ M of oriented, time-oriented, Lorentzian frames in $T(M)$. When M is compact the introduction of a Lorentzian metric is also obstructed by the non-vanishing of the Euler-Poincaré characteristic, but for a non-compact M, such a metric always exists [4].

One can regard the metric g as also defining a map g: $GL(M) \to \mathrm{Lor}(4)$, $\mathbf{e}_\mu \mapsto g(\mathbf{e}_\mu) = g_{\mu\nu}(x)$, which takes any linear frame $\mathbf{e}_\mu$ in T_xM to the matrix $g_{\mu\nu}(x)$ of components of g with respect to this frame. The manifold $\mathrm{Lor}(4) = GL(4; \mathbb{R})/O(3, 1)$ is the homogeneous

[4] See the paper of Markus [**13**] on the subject of the obstructions to the existence of Lorentzian metrics. Some general remarks were also made in Steenrod [**14**].

space of all invertible 4×4 real matrices that can serve as the component matrices of Lorentzian metrics. That is, in addition to their invertibility they must also be symmetric and congruent to the matrix $\eta_{\mu\nu}$ = diag[+1, −1, −1, −1]. Furthermore, one requires that the map g, as just defined, must be equivariant under the right action of $GL(4;\mathbb{R})$ on $GL(M)$ and the left action on Lor(4). This is really just the usual tensorial transformation law of the metric components, since it says that when the frame $\mathbf{e}_\mu$ is changed to $\mathbf{e}_\nu A^\nu_\mu$ the component matrix $g_{\mu\nu}$ goes to $A^\kappa_\mu A^\lambda_\nu g_{\kappa\lambda}$.

If θ^μ is reciprocal coframe to $\mathbf{e}_\mu$ then one can reconstruct the metric $g(x)$ at the point x M by means of:

$$g(x) = g_{\mu\nu}(x)\, \theta^\mu \otimes \theta^\nu . \tag{XIII.64}$$

Since g is equivariant under frame changes, this construction will be unambiguous at each point of M even when no global frame field on M exists.

The actual reduction from $GL(M)$ to $O(3, 1)(M)$ then comes about by defining $O(3, 1)(M)$ to be the level hypersurface of $\eta_{\mu\nu}$ under the map g. Since the map g is not unique, neither is this reduction. Another way of seeing this is to observe that any linear frame $\mathbf{e}_\mu$ in T_xM can be regarded as orthonormal for some Lorentzian metric; i.e.:

$$g(\mathbf{e}_\mu, \mathbf{e}_\nu) = \eta_{\mu\nu} . \tag{XIII.65}$$

In fact, one simply *defines* that metric by means of:

$$g = \eta_{\mu\nu}\, \theta^\mu \otimes \theta^\nu . \tag{XIII.66}$$

The frame $\mathbf{e}_\mu$ defines an orbit in the manifold GL_xM under the action of $O(3, 1)$, but not all linear frames in GL_xM will belong to a common orbit. For instance, frames that differ by a non-trivial dilatation will not be on the same orbit. Hence, one can see that a choice of Lorentzian metric on M is also equivalent to a global section of the fiber bundle $\mathrm{Lor}(M) \to M$ whose fibers are diffeomorphic to the orbit spaces we just described. The fact that a global section does not have to exist follows from the fact this orbit space is not contractible; in fact, it is homotopically equivalent to $\mathbb{R}\mathrm{P}^3$. More precisely, Lor_xM is homotopically equivalent to PT_xM. Hence, the existence of a Lorentzian metric is homotopically equivalent to the existence of a global line field.

b. Canonical 1-forms on frame bundles. The bundle $\pi\colon GL(M) \to M$ has a *canonical 1-form* θ^μ with values in $\mathbb{R}^4$ that is defined on it. If $x \in M$ and $\mathbf{e} \in GL_xM$ then for any tangent vector $\mathbf{v} \in T_{\mathbf{e}}GL_x(M)$ the numbers $\theta^\mu(\mathbf{v})$ are the components v^μ of the tangent vector $\pi_*\mathbf{v}$ in T_xM:

$$\theta^\mu(\mathbf{v}) = \theta^\mu(\pi_*\mathbf{v}) = v^\mu \qquad (\text{so } \pi_*\mathbf{v} = v^\mu \mathbf{e}_\mu). \tag{XIII.67}$$

Again, we are abusing notation on the grounds that the 1-forms θ^μ on GL_xM are associated with the reciprocal coframe field in T_xM.

The canonical 1-form θ^μ on $GL(M)$ has a predictable variance under the action of $GL(4;\mathbb{R})$ on $GL(M)$ to the right, if one looks at the parenthetical comment in (XIII.66). That is, when the linear frame $\mathbf{e}_\mu$ goes to $\mathbf{e}_\nu A^\nu_\mu$, since the components v^μ of the tangent vector $\pi_*\mathbf{v}$ must go to $\tilde{A}^\mu_\nu v^\nu$ (tilde = matrix inverse) and $v^\mu = \theta^\mu(\mathbf{v})$, one must have that θ^μ goes to $\tilde{A}^\mu_\nu \theta^\nu$ in order to be consistent.

Although this probably sounds devilishly esoteric and confusing to many relativity theorists, who, to this day, seem to prefer the local component formulation of differential geometry to the exclusion of its modern formulation, it is really quite elementary when one goes to local components, since when one has a local frame field $\mathbf{e}$: $U \to GL(U)$, $x \mapsto \mathbf{e}_\mu(x)$ the 1-forms θ^μ on $GL(U)$ pull down to the reciprocal coframe field $\theta^\mu(x)$ to $\mathbf{e}_\mu(x)$.

The canonical 1-form θ^μ on $GL(M)$ also defines a canonical 1-form on all of its reductions by restriction. The only thing that changes is the subgroup of $GL(4;\mathbb{R})$ that one uses to make the frame changes.

Since so many important geometric constructions can be associated with reductions of the bundle $GL(M) \to M$ that are defined by reductions of the structure group to a subgroup $G \subset GL(4;\mathbb{R})$, differential geometry long since evolved a general notion for this process. When G is a Lie subgroup of $GL(n)$ a reduction of the bundle $GL(M) \to M$ of linear frames on an n-dimensional differentiable manifold M to a bundle $G(M) \to M$ is called a *G-structure*. In addition to being fundamental in the purely geometric context (cf., e.g., [**15-17**]), this notion is also quite fundamental in physics, as well (cf., [**18**]).

In general, it relates to a particular form of the general process of the spontaneous breaking of gauge symmetries in gauge field theories, which has been emerging over the years as a natural process that is more general than simply its application to elementary particle physics would suggest. Indeed, its application in condensed matter physics is just as important, since many of the phenomena of that branch of physics are more intuitively tractable than those of the sub-visible microcosmos.

c. Connections on frame bundles. There are many ways of introducing a connection on a fiber bundle, depending upon which ultimate class of problems one intends upon addressing, and they almost all relate to each other at varying levels of generality. When the bundle in question is a frame bundle – i.e., a G-structure $G(M) \to M$ – over a manifold M, some of them are more natural than others. Since we mostly wish to define a connection 1-form ω on $G(M)$ that will allow us to define the parallel translation of frames along curves in M, at least locally, we shall start with that as the definition.

If G is a Lie subgroup of $GL(4;\mathbb{R})$ and $\mathfrak{g}$ is its Lie algebra then a $\mathfrak{g}$-*connection* on a G-structure $G(M) \to M$ is a 1-form ω: $T(G(M)) \to \mathfrak{g}$, $\mathbf{v} \mapsto \omega^\mu_\nu(\mathbf{v})$ such that its restriction to the vertical [5] sub-bundle $V(G(M))$ is a linear isomorphism of each $V_\mathbf{e}$ with the vector

[5] Recall that a tangent vector to any fiber bundle $\pi: B \to M$ is *vertical* iff it projects to zero under the differential map $d\pi$. Such a tangent vector will also be tangent to some fiber of B. In the case of $GL(M)$ vertical tangent vector fields generate one-parameter families of linear frame transformations with the fibers of $GL(M)$.

space on which $\mathfrak{g}$ is defined, and when the frame $\mathbf{e}_\mu$ is changed to $\mathbf{e}_\nu A^\nu_\mu$ the components of the matrix ω^μ_ν go to $\tilde{A}^\mu_\kappa \omega^\kappa_\lambda A^\lambda_\nu$.

This last requirement eventually leads to the usual transformation law for a connection matrix when one chooses a local G-frame field $\mathbf{e}$: $U \to G(U)$ and pulls the 1-form ω on $G(U)$ down to a $\mathfrak{g}$-valued 1-form on U that we also denote by ω^μ_ν. In such a case, a change of local frame field involves a smooth transition function A: $U \to G$, $x \mapsto A^\mu_\nu(x)$. If one considers how the differential map $D\mathbf{e}$ to the frame field $\mathbf{e}$ relates to the differential map $D\overline{\mathbf{e}}$ to the frame field $\overline{\mathbf{e}} = \mathbf{e}A$ then one finds that:

$$D\overline{\mathbf{e}} = D(\mathbf{e}A) = (D\mathbf{e})A + \mathbf{e}\otimes dA = [D\mathbf{e} + \mathbf{e}\otimes(dAA^{-1})]A. \qquad \text{(XIII.68)}$$

That is, the transformation of the differential is not *covariant* – i.e., effected by means of A alone. Hence, one must *replace* the differential operator D with the *covariant differential* operator:

$$\nabla\mathbf{e}_\mu = D\mathbf{e}_\mu + \mathbf{e}_\nu \otimes \omega^\nu_\mu. \qquad \text{(XIII.69)}$$

One now has:

$$\nabla\overline{\mathbf{e}}_\mu = D\overline{\mathbf{e}}_\mu + \overline{\mathbf{e}}_\nu \otimes \overline{\omega}^\nu_\mu = [D\mathbf{e}_\nu + \mathbf{e}_\kappa \otimes (dA^\kappa_\lambda \tilde{A}^\lambda_\nu + A^\kappa_\lambda \overline{\omega}^\lambda_\rho \tilde{A}^\rho_\nu)]A^\nu_\mu. \qquad \text{(XIII.70)}$$

Hence, as long as:

$$\omega^\nu_\mu = A^\nu_\kappa \overline{\omega}^\kappa_\lambda \tilde{A}^\lambda_\mu + dA^\nu_\kappa \tilde{A}^\kappa_\mu \qquad \text{(XIII.71)}$$

one will have

$$\nabla\overline{\mathbf{e}}_\mu = \nabla\mathbf{e}_\nu\, A^\nu_\mu; \qquad \text{(XIII.72)}$$

i.e., the covariant differential will indeed be covariant.

Thus, the local version of the Ad^{-1}-equivariance of the 1-form ω on $G(M)$ will take the form:

$$\omega = A\overline{\omega}A^{-1} + dAA^{-1}. \qquad \text{(XIII.73)}$$

The matrix of local 1-forms ω^μ_ν on U can be related to the coframe θ^μ that is reciprocal to the chosen frame field $\mathbf{e}_\mu$ by means of:

$$\omega^\mu_\nu = \Gamma^\mu_{\kappa\nu}\theta^\kappa \qquad \text{(XIII.74)}$$

for a unique set of smooth functions $\Gamma^\mu_{\kappa\nu}$ on U that are analogous to the Riemann-Christoffel symbols of the Levi-Civita connection, which we shall discuss shortly.

The introduction of the covariant derivative allows us to introduce the notation of parallel translation along a curve $\gamma(t)$ in U. The local frame field $\mathbf{e}_\mu$ on U is *parallel along* the curve $\gamma(t)$ iff the covariant derivative of $\mathbf{e}_\mu$ in the direction of $\mathbf{v} = d\gamma/dt$ vanishes; i.e.:

$$0 = \nabla_{\mathbf{v}}\mathbf{e}_\mu = i_{\mathbf{v}}(\nabla\mathbf{e}_\mu) = v^\nu \frac{\partial \mathbf{e}_\mu}{\partial x^\nu} + \Gamma^\nu_{\kappa\mu} v^\kappa \mathbf{e}_\nu \,. \tag{XIII.75}$$

If **w** is a vector field on U then one can say that if $\mathbf{e}_\mu$ is parallel along $\gamma(\tau)$ then **w** is parallel along $\gamma(\tau)$ iff its components w^μ relative to $\mathbf{e}_\mu$ are constant. More generally, we define the covariant derivative of **w** along **v** to be:

$$\nabla_{\mathbf{v}}\mathbf{w} = v^\mu \nabla_{\mathbf{e}_\mu}(w^\nu \mathbf{e}_\nu) = v^\mu(\mathbf{e}_\mu w^\nu + w^\nu \nabla_{\mathbf{e}_\mu}\mathbf{e}_\nu) = v^\mu(\mathbf{e}_\mu w^\nu + \Gamma^\nu_{\mu\kappa} w^\kappa)\mathbf{e}_\nu \tag{XIII.76}$$

for an arbitrary G-frame field $\mathbf{e}_\mu$ on U. In these computations, we have set:

$$\nabla_{\mathbf{e}_\mu}\mathbf{e}_\nu = \Gamma^\kappa_{\mu\nu}\mathbf{e}_\kappa. \tag{XIII.77}$$

This means that the components of $\nabla_{\mathbf{v}}\mathbf{w}$ with respect to this local frame field are:

$$(\nabla_{\mathbf{v}}\mathbf{w})^\mu = v^\nu \mathbf{e}_\nu w^\mu + \Gamma^\mu_{\kappa\nu} v^\kappa w^\nu \,. \tag{XIII.78}$$

The vector field **w** is then parallel along the curve $\gamma(\tau)$ iff $\nabla_{\mathbf{v}}\mathbf{w}$ vanishes.

If the local frame field in question is the natural frame field ∂_μ for a local coordinate system (U, x^μ), with reciprocal coframe field dx^μ then (XIII.78) will take the form:

$$(\nabla_{\mathbf{v}}\mathbf{w})^\mu = v^\nu \frac{\partial w^\mu}{\partial x^\nu} + \Gamma^\mu_{\kappa\nu} v^\kappa w^\nu = \frac{dw^\mu}{d\tau} + \Gamma^\mu_{\kappa\nu} v^\kappa w^\nu \,. \tag{XIII.79}$$

Of particular interest are *geodesics*, which are curves along which the velocity vector $\mathbf{v}(t)$ is itself parallel-translated. One will then have the vanishing of the *proper acceleration* of the curve:

$$\mathbf{a}(\tau) = \nabla_{\mathbf{v}}\mathbf{v}, \tag{XIII.80}$$

which has the component form:

$$0 = v^\nu \mathbf{e}_\nu v^\mu + \Gamma^\mu_{\kappa\nu} v^\kappa v^\nu \tag{XIII.81}$$

with respect to an arbitrary local frame field and:

$$0 = \frac{dv^\mu}{d\tau} + \Gamma^\mu_{\kappa\nu} v^\kappa v^\nu \,. \tag{XIII.82}$$

with respect to a natural one.

The map that takes any G-frame $\mathbf{e}_\mu$ to its reciprocal G-coframe θ^μ defines a canonical isomorphism of the bundle $G(M) \to M$ of G-frames on M with the bundle $G^*(M) \to M$ of G-coframes on M. It too has a canonical 1-form with values in $\mathbb{R}^4$ that we also denote by θ^μ, because for a local coframe field $\theta\colon U \to G^*(M)$, $x \mapsto \theta^\mu_x$ the canonical 1-form pulls down to the coframe field θ^μ itself.

This isomorphism also allows one to map the connection ω on $G(M)$ over to a corresponding $\mathfrak{g}$-connection on $G^*(M)$. The covariant derivative of a local frame field θ^μ on $U \subset M$ along a curve $\gamma(\tau)$ in U whose velocity vector field is $\mathbf{v}(\tau)$ is defined to be:

$$\nabla_{\mathbf{v}}\theta^\mu = i_{\mathbf{v}} D\theta^\mu - \omega^\mu_\nu(\mathbf{v})\theta^\nu ; \tag{XIII.83}$$

Hence, this coframe field is parallel along the curve in question iff this covariant derivative vanishes.

The covariant derivative of a covector field $\alpha = \alpha_\mu \theta^\mu$ is then defined to have components with respect to θ^μ that are equal to:

$$(\nabla_{\mathbf{v}}\alpha)_\mu = \mathbf{v}\alpha_\mu - \omega^\nu_\mu(\mathbf{v})\alpha_\nu \tag{XIII.84}$$

with respect to an arbitrary coframe field and:

$$(\nabla_{\mathbf{v}}\alpha)_\mu = \frac{d\alpha_\mu}{d\tau} - \Gamma^\nu_{\kappa\mu} v^\kappa \alpha_\mu \tag{XIII.85}$$

with respect to a natural one.

One can define a stronger condition on vector fields, frame fields, and the like by eliminating the requirement that the field in question be parallel only along a specified curve and generalizing it to the requirement that the field itself be parallel; i.e., parallel along all curves. If one wishes that a local frame field $\mathbf{e}_\mu$ on $U \subset M$ be *parallel* then it is necessary and sufficient that its covariant differential:

$$\nabla \mathbf{e}_\mu = D\mathbf{e}_\mu + \omega^\nu_\mu \otimes \mathbf{e}_\nu \tag{XIII.86}$$

must vanish at all points of U. That is:

$$D\mathbf{e}_\mu = - \omega^\nu_\mu \otimes \mathbf{e}_\nu . \tag{XIII.87}$$

This condition amounts to a system of partial differential equations for the members of the local frame field while the condition that it be parallel along a specified curve gave a system of ordinary differential equations for the restriction of those members to the curve in question. Hence, the question of the integrability of the system of differential equations is more involved in the present case. In fact, one finds that a connection cannot admit parallel local frame fields unless its curvature vanishes; we shall discuss curvature in the next subsection, but, for now, we simply characterize it as an obstruction to the integrability of the equation (XIII.87).

One can also say that a vector field $\mathbf{v} = v^\mu \mathbf{e}_\mu$ on U is parallel on U if its covariant differential, which we define by:

$$\nabla \mathbf{v} = dv^\mu \otimes \mathbf{e}_\mu + v^\mu \nabla \mathbf{e}_\mu = (dv^\mu + \omega^\mu_\nu v^\nu) \otimes \mathbf{e}_\mu + v^\mu D\mathbf{e}_\mu , \tag{XIII.88}$$

vanishes at all points of *U*.

In the case of a natural frame, for which $D\mathbf{e}_\mu = 0$, we find that the covariant differential of **v** takes the form:

$$\nabla \mathbf{v} = \left(\frac{\partial v^\mu}{\partial x^\nu} + \Gamma^\mu_{\kappa\nu} v^\kappa \right) dx^\nu \otimes \partial_\mu . \tag{XIII.89}$$

To clarify what we mean by the expression $D\mathbf{e}_\mu$, let ∂_μ be a natural frame field on *U*, so we can represent $\mathbf{e}_\mu$ as $e^\nu_\mu(x)\partial_\nu$ for some unique invertible matrix of smooth functions $e^\nu_\mu(x)$. We then define:

$$D\mathbf{e}_\mu = de^\nu_\mu \otimes \partial_\nu = e^\nu_{\mu,\kappa} dx^\kappa \otimes \partial_\nu . \tag{XIII.90}$$

Hence, the vanishing of $D\mathbf{e}_\mu$ means that $\mathbf{e}_\mu$ differs from a natural frame field only by a constant transition function on *U*. In other words, the local frame field $\mathbf{e}_\mu$ could be called *integrable*, or, in physics terminology, *holonomic*; the non-integrable local frame fields are then *anholonomic.*

One can similarly define the covariant differential of a local coframe field θ^μ:

$$\nabla \theta^\mu = D\theta^\mu - \omega^\mu_\nu \otimes \theta^\nu , \tag{XIII.91}$$

and a covector field $\alpha = \alpha_\mu \theta^\mu$:

$$\nabla \alpha = d\alpha_\mu \otimes \theta^\mu + \alpha_\mu \nabla \theta^\mu = (d\alpha_\mu - \omega^\nu_\mu \alpha_\nu) \otimes \theta^\mu + \alpha_\mu D\theta^\mu . \tag{XIII.92}$$

In a natural frame ($D\theta^\mu = 0$), we then have:

$$\nabla \alpha = \left(\frac{\partial \alpha_\mu}{\partial x^\nu} - \Gamma^\kappa_{\mu\nu} \alpha_\kappa \right) dx^\mu \otimes dx^\nu . \tag{XIII.93}$$

As above, if dx^μ is a natural coframe field on *U* and $\theta^\mu = \theta^\mu_\nu(x)\, dx^\nu$ then we are defining:

$$D\theta^\mu = d\theta^\mu_\nu \otimes dx^\nu = \theta^\mu_{\nu,\kappa}\, dx^\kappa \otimes dx^\nu . \tag{XIII.94}$$

There is then an analogous notion of integrability and anholonomity that is associated with local coframe fields.

The corresponding notion of parallelism in the above cases is also the vanishing of the various covariant differentials.

The relationship between the covariant differential and the covariant derivative along a curve – or, more generally, a vector field **v** on *U* – is simply that of the relationship of a differential to a directional derivative; i.e.:

$$\nabla_{\mathbf{v}} = i_{\mathbf{v}}\nabla. \tag{XIII.95}$$

Since every possible tensor field on U can be expressed as a finite linear combination of tensor products of frame fields and coframe fields on U, the covariant differential operator can be extended to the full tensor algebra over $T(U)$ and T^*M by requiring that it behave like a derivation. That is, if $T = a \otimes b$ then:

$$\nabla T = \nabla a \otimes b + a \otimes \nabla b. \tag{XIII.96}$$

d. Torsion and curvature. In addition to the covariant form of the differential operator on tensor fields there is also a covariant form of the exterior derivative operator on differential forms, at least when the forms take their values in vector spaces.

For this situation, it becomes more convenient to represent a tensor field of rank $r+s$ on a manifold M as a smooth map:

$$T\colon G(M) \to \mathbb{R}^{n^*} \otimes \ldots \otimes \mathbb{R}^{n^*} \otimes \mathbb{R}^n \otimes \ldots \otimes \mathbb{R}^n,\ \mathbf{e} \mapsto T(\mathbf{e}) = T^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s}(x)$$

that is equivariant under the right action of G on frames and the tensor product representation of G in $GL(\mathbb{R}^{n^*} \otimes \ldots \otimes \mathbb{R}^{n^*} \otimes \mathbb{R}^n \otimes \ldots \otimes \mathbb{R}^n)$. That is, T associates a G-frame $\mathbf{e}_\mu$ at $x \in M$ with the components $T^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s}(x)$ of a tensor field on M with respect to that frame in a manner that obeys the usual rules of transformation for the components of a tensor field:

The tensor field on M is then defined globally, even in the absence of a global frame field, by the construction rule:

$$T(x) = T^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s}(x)\ \mathbf{e}_{\mu_1} \otimes \ldots \mathbf{e}_{\mu_r} \otimes \theta^{\nu_1} \otimes \ldots \otimes \theta^{\nu_s}. \tag{XIII.97}$$

As an example, suppose $G = GL(n; \mathbb{R})$. One can then represent a metric tensor field g on M as a smooth function $g\colon GL(M) \to \mathbb{R}^n \otimes \mathbb{R}^n$, $\mathbf{e} \mapsto g(\mathbf{e}) = g_{\mu\nu}(x)$ and reconstruct the tensor field g by way of:

$$g(x) = g_{\mu\nu}(x)\ \theta^\mu \otimes \theta^\nu. \tag{XIII.98}$$

Although a smooth map is essentially a 0-form, this definition of a tensor field generalizes immediately to the case of k-forms on M with values in vector spaces, and, in particular, the vector spaces that are expressible as the tensor product of other vector spaces. For instance, we can regard the canonical 1-form θ^μ on $G(M)$ as a 1-form on $G(M)$ with values in $\mathbb{R}^n$ and a connection form ω is a 1-form on $G(M)$ with values in $\mathfrak{g}$.

The exterior derivative operator that acts on elements of $\Lambda^*(G(M))$ can be extended to an *exterior covariant derivative* operator on $\Lambda^*(G(M)) \otimes V$, which is how we generically represent equivariant differential forms on $G(M)$ with values in a vector space V on which G acts linearly. Since the definition will depend upon the choice of action, rather than

define the general case, we simply define the way that it works for the cases of immediate interest to us.

In the case of the canonical 1-form θ^μ, as we observed above, $V = \mathbb{R}^n$, and when G is represented as a subgroup of $GL(n; \mathbb{R})$ the action of G on $\mathbb{R}^n$ is simply the defining representation; viz., the multiplication of a matrix and a vector or covector. We then define the exterior covariant derivative of the $\mathbb{R}^n$-valued 1-form θ^μ on $G(M)$ to be the $\mathbb{R}^n$-valued *torsion* 2-form on $G(M)$:

$$\Theta^\mu = \nabla^\wedge \theta^\mu = d\theta^\mu + \omega^\mu_\nu \wedge \theta^\nu . \tag{XIII.99}$$

We clarify that the second term on the right-hand side of this definition takes any pair of vector field **v**, **w** on $G(M)$ to the vector in $\mathbb{R}^n$:

$$\omega^\mu_\nu \wedge \theta^\nu (\mathbf{v}, \mathbf{w}) = \tfrac{1}{2}[\omega^\mu_\nu(\mathbf{v})\theta^\nu(\mathbf{w}) - \omega^\mu_\nu(\mathbf{w})\theta^\nu(\mathbf{v})] = \tfrac{1}{2}(\Gamma^\mu_{\kappa\nu} - \Gamma^\mu_{\nu\kappa})v^\kappa w^\nu . \tag{XIII.100}$$

When one defines a local G-frame field **e**: $U \to G(M)$, one can pull all of the differential forms in (XIII.99) down to forms on U that we denote by the same symbols. In particular, θ^μ represents the reciprocal local coframe field to $\mathbf{e}_\mu$ on U.

The first term on the right-hand side of (XIII.99) is the *anholonomity* of the local coframe field θ^μ. Since all k-forms on U can be represented in terms of θ^μ we have:

$$d\theta^\mu = -\tfrac{1}{2}\, c^\mu_{\kappa\lambda}\ \theta^\kappa \wedge \theta^\lambda, \tag{XIII.101}$$

for a set of functions $c^\mu_{\kappa\lambda}$ on U that we call the *structure functions* of θ^μ; the reason for the minus sign is rooted in the fact that one also has:

$$[\mathbf{e}_\kappa, \mathbf{e}_\lambda] = c^\mu_{\kappa\lambda}\, \mathbf{e}_\mu . \tag{XIII.102}$$

One sees that the anholonomity amounts to an obstruction to the integrability of the local system of differential equations $\theta^\mu = dx^\mu$ on U, since its vanishing is a necessary and sufficient condition for such functions x^μ to exist.

If $\theta^\mu = h^\mu_\nu dx^\nu$ for some natural coframe field then we can represent the $c^\mu_{\kappa\lambda}$ by:

$$c^\mu_{\kappa\lambda} = h^\mu_{\kappa,\lambda} - h^\mu_{\lambda,\kappa} . \tag{XIII.103}$$

If we represent ω^μ_ν as $\Gamma^\mu_{\kappa\nu}\theta^\kappa$ then the torsion 2-form takes the form:

$$\Theta^\mu = \tfrac{1}{2} S^\mu_{\kappa\lambda}\, \theta^\kappa \wedge \theta^\lambda, \tag{XIII.104}$$

with:

$$S^\mu_{\kappa\lambda} = -c^\mu_{\kappa\lambda} + \Gamma^\mu_{\kappa\lambda} - \Gamma^\mu_{\lambda\kappa} . \tag{XIII.105}$$

In a holonomic local frame field, the $c^{\mu}_{\kappa\lambda}$ vanish and the torsion 2-form then represents the "anti-symmetric part" of the connection. In such a frame field, the vanishing of torsion is equivalent to the symmetry of the connection components $\Gamma^{\mu}_{\kappa\nu}$ in their lower indices.

Although (XIII.104) and (XIII.105) were defined on $G(M)$, they take the same form when pulled down to $U \subset M$ by means of a local G-frame field.

Another way of looking at torsion is that when it vanishes the equation:

$$d\theta^{\kappa} = -\omega^{\mu}_{\nu} \wedge \theta^{\nu} \tag{XIII.106}$$

says that the exterior differential system $\theta^{\kappa} = 0$ on $G(M)$ is completely integrable. As the 1-form θ^{κ} vanishes precisely when evaluated on vertical tangent vectors one sees that connections with vanishing torsion will always exist since the integral submanifolds to this exterior differential system will be the fibers of G, which always exist. Hence, non-vanishing torsion is only weakly an obstruction to the integrability of this exterior differential system.

Since the action of G on the Lie algebra $\mathfrak{g}$ is the adjoint action, we define the exterior covariant derivative of the $\mathfrak{g}$-valued connection 1-form ω to be the $\mathfrak{g}$-valued *curvature* 2-form:

$$\Omega^{\mu}_{\nu} = \nabla\hat{}\,\omega^{\mu}_{\nu} = d\omega^{\mu}_{\nu} + \omega^{\mu}_{\kappa} \wedge \omega^{\kappa}_{\nu}. \tag{XIII.107}$$

In the right-hand side of this expression, we clarify that the term $\omega^{\mu}_{\kappa} \wedge \omega^{\kappa}_{\nu}$, when evaluated on two vector fields **v**, **w** on $G(M)$, will have the value:

$$\omega^{\mu}_{\kappa} \wedge \omega^{\kappa}_{\nu}(\mathbf{v}, \mathbf{w}) = \tfrac{1}{2}[\omega^{\mu}_{\kappa}(\mathbf{v})\omega^{\kappa}_{\nu}(\mathbf{w}) - \omega^{\mu}_{\kappa}(\mathbf{w})\omega^{\kappa}_{\nu}(\mathbf{v})] = -\tfrac{1}{2}[\omega(\mathbf{v}), \omega(\mathbf{w})]. \tag{XIII.108}$$

When ω^{μ}_{ν} is expressed as $\Gamma^{\mu}_{\kappa\nu}\theta^{\kappa}$, the curvature 2-form can be expressed as:

$$\Omega^{\mu}_{\nu} = \tfrac{1}{2} R^{\mu}_{\kappa\lambda\nu}\, \theta^{\kappa} \wedge \theta^{\lambda}, \tag{XIII.109}$$

in which:

$$R^{\mu}_{\kappa\lambda\nu} = \Gamma^{\mu}_{\lambda\nu,\kappa} - \Gamma^{\mu}_{\kappa\nu,\lambda} + \Gamma^{\mu}_{\kappa\sigma}\Gamma^{\sigma}_{\lambda\nu} - \Gamma^{\mu}_{\lambda\sigma}\Gamma^{\sigma}_{\kappa\nu} \tag{XIII.110}$$

gives the components of the curvature tensor field in its conventional local form.

One can regard the curvature 2-form as an obstruction to the complete integrability of the exterior differential system $\omega^{\mu}_{\nu} = 0$ since its vanishing would make:

$$d\omega^{\mu}_{\nu} = -\omega^{\mu}_{\kappa} \wedge \omega^{\kappa}_{\nu}. \tag{XIII.111}$$

The 1-forms ω^{μ}_{ν} vanish precisely when they are evaluated on tangent vectors to $G(M)$ that are, by definition, *horizontal*. if one denotes the vector space of all horizontal tangent vectors at $\mathbf{e} \in G(M)$ by $H_{\mathbf{e}}(G(M))$ and the resulting horizontal sub-bundle of $T(G(M))$ by $H(G(M))$ then one will have a Whitney sum decomposition:

$$T(G(M)) = H(G(M)) \oplus V(G(M)).$$

However, unlike the vertical sub-bundle $V(G(M))$, which is always integrable, the horizontal sub-bundle $H(G(M))$ does not always have to be integrable. Since its fibers have dimension n, and, in fact, the 1-form θ^μ defines a linear isomorphism of each horizontal tangent space with $\mathbb{R}^n$, the integral submanifolds of the sub-bundle $H(G(M))$ would have to represent diffeomorphic copies of M in $G(M)$ that are transverse to the fibers; i.e., global sections of the fibration $G(M) \to M$. This possibility is obstructed by topology, though, so the existence of a $\mathfrak{g}$-connection with vanishing curvature is necessary for the triviality of $G(M)$; viz., its parallelizability.

Often (see, e.g., [**7, 9, 10, 12, 15, 16**]), equations (XIII.99) and (XIII.106) are collectively referred to as the *Cartan structure equations* for the connection ω. As we have presented them, they are simply the definitions of the torsion and curvature of the connection, but if one starts with a more general definition of a $\mathfrak{g}$-connection, such as a complementary horizontal sub-bundle to the vertical sub-bundle of $T(G(M))$, then one can derive the equations as a consequence.

If one is given an $\mathbb{R}^n$-valued 2-form Θ^μ on $G(M)$ and a $\mathfrak{g}$-valued 2-form Ω^μ_ν that have the same transformation properties as the torsion and curvature 2-forms then the integrability conditions for the systems of partial differential equations for a connection 1-form ω^μ_ν that the structure equations represent are given by computing the square of the exterior covariant derivative operator, which we express generically as:

$$\nabla^\wedge\nabla^\wedge\alpha = d(d\alpha + \omega^\wedge \alpha) + \omega^\wedge (d\alpha + \omega^\wedge \alpha) = \Omega \wedge \alpha. \qquad \text{(XIII.112)}$$

When this formula is applied to the canonical 1-form and the connection 1-form, one derives the *Bianchi identities:*

$$\nabla^\wedge\Theta^\mu = \Omega^\mu_\nu \wedge \theta^\nu, \qquad \nabla^\wedge\Omega^\mu_\nu = 0. \qquad \text{(XIII.113)}$$

One can associate the curvature 2-form Ω^μ_ν on $G(M)$ with a 1-form $\mathcal{R}_\mu$ that takes its values in $\mathbb{R}^{n*}$, which we shall the *Ricci curvature* tensor, even though that tensor usually gets defined in the context of metric connections. It takes the G-frame $\mathbf{e}_\mu \in G(M)$ to the 1-form:

$$\mathcal{R}_\mu = i_{\mathbf{e}_\nu}\Omega^\nu_\mu = R_{\mu\nu}\theta^\nu, \qquad \text{(XIII.114)}$$

in which:

$$R_{\mu\nu} = R^\kappa{}_{\mu\kappa\nu}. \qquad \text{(XIII.115)}$$

In the general case of a $\mathfrak{g}$-connection, these components do not have to be symmetric, though.

An aspect of the definition of a $\mathfrak{g}$-connection on $G(M)$ that is important to the structure of the spacetime manifold is the fact that even though the manifold M does not

have to be parallelizable, nonetheless, the manifold $GL(M)$ is always parallelizable. This is because the 1-form θ^μ defines a linear isomorphism of each horizontal subspace on $GL(M)$ with $\mathbb{R}^n$, while the connection 1-form ω^μ_ν defines a linear isomorphism of each vertical subspace with $\mathfrak{gl}(n; \mathbb{R})$. Collectively, the set $\{\theta^\mu, \omega^\mu_\nu\}$ defines a linear isomorphism of each tangent space to $GL(M)$ with the vector space $\mathbb{R}^n \oplus \mathfrak{gl}(n; \mathbb{R})$. If one also chooses a basis for $\mathfrak{gl}(n; \mathbb{R})$ then one can think of the set $\{\theta^\mu, \omega^\mu_\nu\}$ as defining a linear isomorphism of each $T_{\mathbf{e}}G(M)$ with $\mathbb{R}^{n(n+1)}$; i.e., a global coframe field on $GL(M)$. By restriction, the same is true for any reduction of $GL(M)$, as well, with a suitable adjustment to the dimension.

The set $\{\theta^\mu, \omega^\mu_\nu\}$ can be thought of as a 1-form $\tilde{\theta}^a$, $a = 1, \ldots, n + \dim \mathfrak{g}$ on $G(M)$ with values in the vector space $\mathbb{R}^{n+\dim \mathfrak{g}}$. Hence, one can represent $\tilde{\theta}^a$ in block matrix form as:

$$\tilde{\theta}^a = \left[\begin{array}{c} \theta^\mu \\ \hline \omega^\mu_\nu \end{array}\right], \tag{XIII.116}$$

although it would be more precise to denote the elements of the matrix ω^μ_ν as a singly-indexed column vector by choosing a basis E_α, $\alpha = 1, \ldots, \dim \mathfrak{g}$, so that ω gets expressed in the form $\omega^\alpha E_\alpha$.

We define the connection on $GL(G(M)) = G(M) \times \mathbb{R}^{n+\dim \mathfrak{g}}$ that makes the global frame field $\tilde{\theta}^a$ parallel; one calls such a connection either a *teleparallelism connection*, in general, or a *Cartan connection*, in the present case.

It is no longer necessary to specify both the torsion and curvature of such a connection 1-form, since both the torsion and the curvature 2-forms of ω^μ_ν get absorbed into the torsion 2-form $\tilde{\Theta}^a$:

$$\tilde{\Theta}^a = d\tilde{\theta}^a = \left[\begin{array}{c} \Theta^\mu \\ \hline \Omega^\mu_\nu \end{array}\right], \tag{XIII.117}$$

while the curvature vanishes identically, due to parallelizability.

In physics, teleparallelism usually refers to making the assumption that the spacetime manifold M is parallelizable and working with the geometry of such manifolds in the hopes of unifying the theories of gravitation and electromagnetism, since the manifold $GL(4; \mathbb{R})$ has the same dimension – viz., sixteen – as the sum of the dimensions of the manifold of Lorentzian structures Lor(4), namely, ten, and the vector space $\Lambda^2(\mathbb{R}^4)$, which has dimension six. Interestingly, when Einstein, Mayer, and others [**19**] were doing this work in the late 1920's, they were using purely local expressions for frame fields and made no mention of the global topological obstructions to parallelizability.

Indeed, the first definitive work on topology and teleparallelism, which was published by Stiefel [**20**], did not appear until 1935. Hence, one can only wonder whether a more topologically thorough analysis of the situation might extend the physical theory accordingly, especially since the obstruction to parallelizability involves non-vanishing curvature, which is bound to contribute to the physics when it could not in the parallelizable case.

e. Metric connections. Now that we have defined the covariant differential of tensor fields in general, we can apply it to the various fundamental tensor fields that relate to the reduction of the bundle of linear frames to its various G-structures.

The first reduction that we encountered was from $GL(M)$ to $GL^+(M)$, but that did not involve a fundamental tensor field, only a choice of orientation. It was in the next reduction to $SL(M)$ that we had to introduce a tensor field in order to define this reduction, in the form of a volume element V on $T(M)$. Hence, a connection ω on $GL(M)$ that is compatible with this reduction must preserve the volume of a linear frame under parallel translation along a curve. In order for this to happen it is necessary and sufficient that ω must be Ad^{-1}-equivariant under the action of $SL(n;\mathbb{R})$ on $SL(M)$, which is equivalent to saying that ω must take its values in the Lie algebra $\mathfrak{sl}(n;\mathbb{R})$.

Now, we can draw upon a general result from the geometry of G-structures [**15-17**] that when a reduction from $GL(M)$ to a G-structure $G(M)$ is effected by means of a fundamental tensor field τ: $GL(M) \to V$ the necessary and sufficient condition for a $\mathfrak{gl}(n)$-connection 1-form ω on $GL(M)$ to be reducible to a $\mathfrak{g}$-connection on $G(M)$ is that τ be covariantly constant with respect to the connection ω; i.e.:

$$0 = \nabla^{\wedge}\tau = d\tau + \omega^{\wedge}\tau . \qquad \text{(XIII.118)}$$

In the case of V, we regard it, not as a 4-form on $GL(M)$, but as a 0-form with values in $\Lambda^4(\mathbb{R}^4)$. This makes dV vanish by dimensionality, and the remaining condition on ω is derived from:

$$\begin{aligned} 0 = \omega^{\wedge}\tau &= \frac{1}{4!}\varepsilon_{\kappa\lambda\mu\nu}(\omega^{\kappa}_{\alpha}\theta^{\alpha}\wedge\theta^{\lambda}\wedge\theta^{\mu}\wedge\theta^{\nu} + \theta^{\mu}\wedge\omega^{\lambda}_{\alpha}\theta^{\alpha}\wedge\theta^{\mu}\wedge\theta^{\nu} \\ &\qquad + \theta^{\kappa}\wedge\theta^{\lambda}\wedge\omega^{\mu}_{\alpha}\theta^{\alpha}\wedge\theta^{\nu} + \theta^{\kappa}\wedge\theta^{\lambda}\wedge\theta^{\mu}\wedge\omega^{\nu}_{\alpha}\theta^{\alpha}) \\ &= 4\,\mathrm{Tr}(\omega)\,V. \end{aligned} \qquad \text{(XIII.119)}$$

The last step follows from the fact that:

$$\frac{1}{4!}\varepsilon_{\kappa\lambda\mu\nu}\omega^{\kappa}_{\alpha}\theta^{\alpha}\wedge\theta^{\lambda}\wedge\theta^{\mu}\wedge\theta^{\nu} = \omega^{\alpha}_{\alpha}\frac{1}{4!}\varepsilon_{\kappa\lambda\mu\nu}\theta^{\kappa}\wedge\theta^{\lambda}\wedge\theta^{\mu}\wedge\theta^{\nu}, \text{ etc.} \qquad \text{(XIII.120)}$$

Hence, since V is, by assumption, non-vanishing, one must have:

$$0 = \mathrm{Tr}\ \omega = \omega^{\mu}_{\mu}. \qquad \text{(XIII.121)}$$

This is, of course, the condition for the matrix ω to have membership in $\mathfrak{sl}(n; \mathbb{R})$.

In the case of spacetime, the next reduction is from $SL(M)$ to $SO(3, 1)(M)$, which is associated with the fundamental tensor field g: $GL(M) \rightarrow SL(4)/SO(3,1)$; i.e., the Lorentzian metric on $T(M)$. From (XIII.118), the necessary and sufficient condition that an $\mathfrak{sl}(4; \mathbb{R})$-connection be reducible to an $\mathfrak{so}(3;1)$-connection is then:

$$0 = \nabla g_{\mu\nu} = dg_{\mu\nu} + \omega^{\kappa}_{\mu} g_{\kappa\nu} + \omega^{\kappa}_{\nu} g_{\mu\kappa}. \qquad \text{(XIII.122)}$$

One often finds the 1-form $Q_{\mu\nu} = \nabla g_{\mu\nu}$ on $GL(M)$ with values in $\mathrm{Lor}(4) = SL(4)/SO(3,1)$ referred to as the *non-metricity* tensor of ω (cf., e.g., [**11**]).

When g is restricted to the bundle $SO(3,1)(M)$ of Lorentzian frames, $g_{\mu\nu} = \eta_{\mu\nu}$, and the compatibility condition for ω to be a metric connection is:

$$0 = \omega^{\kappa}_{\mu}\eta_{\kappa\nu} + \omega^{\kappa}_{\nu}\eta_{\mu\kappa} = \omega_{\mu\nu} + \omega_{\nu\mu}\ . \qquad \text{(XIII.123)}$$

Again, this is simply the condition that ω take its values in $\mathfrak{so}(3, 1)$.

If we wish to further reduce to $SO_0(3, 1)(M)$ then we need a global timelike vector field $\mathbf{t}$. However, this is not a fundamental tensor field for the reduction from $SO(3,1)(M)$ to $SO_0(3,1)(M)$ since the homogeneous space $SO(3,1)/SO_0(3, 1)$ consists of two points. We note that when one has such a vector field $\mathbf{t}$, it does represent the fundamental tensor field for the reduction from $SO(3,1)(M)$ to $SO(3)(M)$ since the homogeneous space $SO(3,1)/SO(3)$ is diffeomorphic to $\mathbb{R}^3$. Hence, we can regard $\mathbf{t}$ as an $SO(3,1)$-equivariant map t: $SO(3,1)(M) \rightarrow \mathbb{R}^3$, $\mathbf{e}_{\mu} \mapsto t(\mathbf{e}_{\mu}) = t^i$, where the t^i represent the spatial components of $\mathbf{t}$ relative to $\mathbf{e}_{\mu}$. Hence, the reduction from $SO(3,1)(M)$ to $SO(3)(M)$ is defined by $t^i = 0$, which means all oriented Lorentzian frames with $\mathbf{e}_0 = \mathbf{t}$. This means that one is reducing to the rest space of a measurer/observer that is defined by $\mathbf{t}$ and its g-orthogonal complementary hyperspace, which has the 1-form $\tau = i_{\mathbf{t}} g = t_{\mu}\theta^{\mu}$.

The compatibility condition for this latter reduction of ω is that $\mathbf{t}$ be a parallel vector field for ω:

$$0 = \nabla \mathbf{t} = \nabla t^{\mu} \otimes \mathbf{e}_{\mu}. \qquad \text{(XIII.124)}$$

As a consequence, its flow must be geodesic.

Even though all $\mathfrak{gl}(n)$-connection 1-forms take their values in a vector space – namely, $\mathfrak{gl}(n)$ – nonetheless, the set $\Gamma(GL(M))$ of all $\mathfrak{gl}(n)$-connections on $GL(M)$ is not a vector space, but only an affine space. This is because for any two $\mathfrak{gl}(n)$-connection 1-forms ω and ω' the sum of the connections does not have to be a connection. However, their *difference 1-form*:

$$\tau = \omega' - \omega, \qquad \text{(XIII.125)}$$

is tensorial. Hence, the vector space on which the affine space $\Gamma(GL(M))$ is modeled is the space of equivariant 1-forms on $GL(M)$ that take their values in $\mathfrak{gl}(n)$, which is, as one sees, an infinite-dimensional vector space.

One of the big differences between affine spaces and vector spaces is that in an affine space there is usually no distinguished point that could serve as a canonical choice of origin. In the case of connections, one finds that the set of connections with vanishing torsion can sometimes define such a distinguished point in $\Gamma(GL(M))$. Although the set of torsionless connections on $GL(M)$ and $SL(M)$ is of dimension higher than zero, one finds that the space of $\mathfrak{so}(3, 1)$-connections does have a distinguished element, which one calls the *Levi-Civita* connection. By definition, it will be the unique connection ω that has both vanishing torsion and metricity:

$$0 = \Theta^{\mu}, \qquad 0 = Q_{\mu\nu}\,. \tag{XIII.126}$$

The local component equations for these conditions are then:

$$\Gamma^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\nu\mu}\,, \qquad g_{\mu\nu,\,\kappa} = -\,\Gamma_{\mu\nu\kappa} - \Gamma_{\kappa\nu\mu}\,. \tag{XIII.127}$$

in a holonomic local frame field.

The components of the Levi-Civita connection – viz., the *Riemann-Christoffel* symbols – are then:

$$\Gamma^{\kappa}_{\mu\nu} = \tfrac{1}{2} g^{\kappa\alpha}(\, g_{\mu\alpha,\,\nu} + g_{\alpha\nu,\,\mu} - g_{\mu\nu,\,\alpha}\,). \tag{XIII.128}$$

The Riemann curvature tensor $\Omega^{\mu}_{\nu} = \nabla\omega^{\mu}_{\nu}$ that is obtained from the Levi-Civita connection differs from the general curvature 2-form for a linear connection mostly by the fact that it takes its values in $\mathfrak{so}(3, 1)$.

However, the Ricci curvature 1-form $\mathcal{R}_{\mu} = R_{\mu\nu}\theta^{\nu}$ that we defined above is now symmetric:

$$R_{\mu\nu} = R_{\nu\mu}\,, \tag{XIII.129}$$

and one can also associate Ω^{μ}_{ν} with a *scalar curvature:*

$$R(\mathbf{e}_{\mu}) = g^{\mu\nu} i_{\mathbf{e}_{\mu}} \mathcal{R}_{\nu} = g^{\mu\nu} R_{\mu\nu}\,. \tag{XIII.130}$$

From equivariance, this function on $G(M)$ is associated with a corresponding function $R(x)$ on M.

When Einstein was first looking for a way of coupling the spacetime curvature to the symmetric energy-momentum tensor $\tau_{\mu} = T_{\mu\nu}dx^{\nu}$, he had to take into account that, whereas the conservation of energy-momentum would dictate that the covariant divergence of $\tau^{\mu} = g^{\mu\nu}\,\tau_{\nu}$ should vanish:

$$0 = \nabla_{\mu}\,\tau^{\mu} = T^{\mu}{}_{\nu;\,\mu}\,dx^{\nu} \qquad (T^{\mu}{}_{\nu} = g^{\mu\kappa}T_{\kappa\nu}) \tag{XIII.131}$$

nonetheless, from the Bianchi identities, one should have:

$$\nabla_\mu \mathcal{R}^\mu = R^\mu{}_{\nu;\mu}\ dx^\nu = \tfrac{1}{2} R_{,\nu}\, dx^\nu \qquad (R^\mu{}_\nu = g^{\mu\kappa} R_{\kappa\nu}) \qquad \text{(XIII.132)}$$

Hence, the divergenceless part of the Ricci curvature tensor is called the *Einstein tensor*, which we can represent as an equivariant 1-form $\mathcal{E}^\mu$ on $GL(M)$ with values in $\mathbb{R}^4$:

$$\mathcal{E}^\mu(\mathbf{e}) = (R^\mu{}_\nu - \tfrac{1}{2} R\, \delta^\mu_\nu)\ \theta^\nu . \qquad \text{(XIII.133)}$$

The Einstein field equations for gravitation then express the idea that $\mathcal{E}^\mu$ should be proportional to τ^μ:

$$\mathcal{E}^\mu = 8\pi\kappa\tau^\mu, \qquad \text{(XIII.134)}$$

or:

$$R^\mu{}_\nu - \tfrac{1}{2} R\, \delta^\mu_\nu = 8\pi\kappa T^\mu{}_\nu, \qquad \text{(XIII.135)}$$

which also implies:

$$R = -\,8\pi\kappa T^\mu{}_\mu\ . \qquad \text{(XIII.136)}$$

Hence, the scalar curvature is directly proportional to minus the trace of the energy-momentum tensor. This is particularly relevant to the case of an energy distribution that is purely due to electromagnetic radiation, since the Faraday energy-momentum tensor then has that property. In one of the early stages of the Big Bang, the matter in the universe was assumed to be purely radiation-dominated.

This implies that (XIII.135) can also be written in the form:

$$R^\mu{}_\nu = 8\pi\kappa(T^\mu{}_\nu - \tfrac{1}{2} T^\mu{}_\mu\, \delta^\mu_\nu). \qquad \text{(XIII.137)}$$

Hence, the sourceless field equations for g are simply the vanishing of the Ricci curvature tensor:

$$R^\mu{}_\nu = 0. \qquad \text{(XIII.138)}$$

That this is consistent with the equations:

$$R^\mu{}_\nu = -\tfrac{1}{2} R\, \delta^\mu_\nu \qquad \text{(XIII.139)}$$

then follows from the vanishing of scalar curvature with the vanishing of $T^\mu{}_\nu$, as one sees from (XIII.136).

In differential geometry, the condition for a manifold with a metric g to be an *Einstein space* is usually weakened slightly to the condition that $R_{\mu\nu}$ be proportional to $g_{\mu\nu}$.

One can give the definition of the Einstein tensor $\mathcal{E}_\mu$ and the scalar curvature R a somewhat more concise and elegant formulation by means of the *curvature 3-form* $\mathcal{E}_\mu$ on $GL(M)$, which takes a frame $\mathbf{e}_\mu$ to:

$$\mathcal{E}_\mu(\mathbf{e}) = -\,\theta^\nu \wedge {}^*\Omega_{\mu\nu} = -\,\varepsilon_{\mu\nu\kappa\lambda}\, \theta^\nu \wedge \Omega^{\kappa\lambda}\ , \qquad \text{(XIII.140)}$$

in which we have defined:

$$\Omega_{\mu\nu} = g_{\mu\nu}\,\Omega^{\kappa}_{\nu}\,, \qquad \Omega^{\kappa\lambda} = g^{\kappa\alpha}\Omega^{\lambda}_{\alpha} = \tfrac{1}{4}R^{\kappa\lambda}{}_{\alpha\beta}\,\theta^{\alpha}\wedge\theta^{\beta}. \qquad \text{(XIII.141)}$$

Substituting (XIII.141) into (XIII.140) gives:

$$\begin{aligned}\mathcal{E}_{\mu}(\mathbf{e}) &= -\tfrac{1}{4}\,\varepsilon_{\mu\nu\kappa\lambda}\,R^{\kappa\lambda}{}_{\alpha\beta}\,\theta^{\nu}\wedge\theta^{\alpha}\wedge\theta^{\beta} = -\tfrac{1}{4}\,\varepsilon_{\mu\nu\kappa\lambda}\,\varepsilon^{\nu\alpha\beta\rho}\,R^{\kappa\lambda}{}_{\alpha\beta}\,\#\mathbf{e}_{\rho}\\ &= -\tfrac{1}{4}\,\delta^{\alpha\beta\rho}_{\kappa\lambda\mu}\,R^{\kappa\lambda}{}_{\alpha\beta}\,\#\mathbf{e}_{\rho} = (R^{\nu}_{\mu} - \tfrac{1}{2}R\delta^{\nu}_{\mu})\,\#\mathbf{e}_{\nu} = \#(E^{\nu}_{\mu}\mathbf{e}_{\nu})\end{aligned} \qquad \text{(XIII.142)}$$

In these computations, we have made use of the fact that the identity map on 3-forms has the components:

$$\delta^{\alpha\beta\rho}_{\kappa\lambda\mu} = \delta^{\alpha}_{\kappa}\delta^{\beta}_{\lambda}\delta^{\rho}_{\mu} + \delta^{\beta}_{\kappa}\delta^{\rho}_{\lambda}\delta^{\alpha}_{\mu} + \delta^{\rho}_{\kappa}\delta^{\alpha}_{\lambda}\delta^{\beta}_{\mu}\,. \qquad \text{(XIII.143)}$$

Hence, the curvature 3-form $\mathcal{E}_{\mu}(\mathbf{e})$, which takes its values in $\mathbb{R}^4$, is Poincaré dual to a set of four vector fields:

$$\mathbf{E}_{\mu} = E^{\nu}_{\mu}\mathbf{e}_{\nu} \qquad \text{(XIII.144)}$$

that describe the Einstein tensor for this choice of frame. They are not generally linearly independent, though, so they do not usually define a 4-frame.

If we form the 4-form on $GL(M)$:

$$\theta^{\mu}\wedge\mathcal{E}_{\mu}(\mathbf{e}) = -\,\theta^{\mu}\wedge\theta^{\nu}\wedge{*}\Omega_{\mu\nu} \qquad \text{(XIII.145)}$$

then we find that since θ^{μ} is reciprocal to $\mathbf{e}_{\mu}$ (XIII.142) gives:

$$\theta^{\mu}\wedge\theta^{\nu}\wedge{*}\Omega_{\mu\nu} = -\,(R^{\mu}_{\mu} - \tfrac{1}{2}R\delta^{\mu}_{\mu})\,V = R\,V, \qquad \text{(XIII.146)}$$

which is precisely the *Einstein-Hilbert* Lagrangian for the gravitational field that is described by the Lorentzian metric g.

Thus, we see that much of the geometric information that pertains to the physics of gravity is concisely summarized in the forms Ω^{μ}_{ν}, $\theta^{\mu}\wedge{*}\Omega_{\mu\nu}$, and $\theta^{\mu}\wedge\theta^{\nu}\wedge{*}\Omega_{\mu\nu}$.

4. General relativity in terms of complex orthogonal frames [1, 21, 22]. Now that we have discussed the isomorphism of $SO_0(3, 1)$ with $SO(3; \mathbb{C})$ and the corresponding association of oriented, time-oriented, Lorenztian frames in $T(M)$ with oriented, complex, orthogonal frames in $\Lambda^2(M)$, and presented general relativity in terms of connection 1-forms on the bundle $SO_0(3,1)(M) \to M$, it will be relatively straightforward to recast it in terms of the bundle $SO(3; \mathbb{C})(\Lambda) \to M$.

a. The isomorphism of $SO_0(3,1)(M)$ with $SO(3;\mathbb{C})(\Lambda)$. Since the Lie groups $SO_0(3,1)$ and $SO(3;\mathbb{C})$ are diffeomorphic as manifolds, the principal bundles $SO_0(3,1)(M)$ and $SO(3;\mathbb{C})(\Lambda)$ have diffeomorphic fibers; in fact, the bundles themselves are also isomorphic. The diffeomorphism of the fibers at a given $x \in M$ is not canonical, but simply takes an oriented, time-oriented, Lorentzian 4-frame $\mathbf{e}_\mu$ to an oriented, complex, orthogonal 3-frame Z_i. When both objects are local frame fields on $U \subset M$, one has:

$$Z_i = \tfrac{1}{2} Z_i^{\mu\nu}(x)\mathbf{e}_\mu \wedge \mathbf{e}_\nu . \tag{XIII.147}$$

For instance, the choice that we have been making all along has:

$$Z_i^{0j} = \delta_i^j , \qquad Z_i^{jk} = \varepsilon^{ijk}. \tag{XIII.148}$$

There is a corresponding isomorphism of the bundles of coframes, $SO_0(3,1)^*(M)$ and $SO(3;\mathbb{C})^*(\Lambda)$, that takes a coframe θ^μ to a coframe Z^i with:

$$Z^i = \tfrac{1}{2} Z^i_{\mu\nu}(x)\theta^\mu \wedge \theta^\nu \tag{XIII.149}$$

locally.

Since the bundle $SO_0(3,1)(M)$ is isomorphic to the bundle $SO_0(3,1)^*(M)$ by the association of any frame with its reciprocal coframe, and similarly for the bundles $SO(3;\mathbb{C})(\Lambda)$ and $SO(3;\mathbb{C})^*(\Lambda)$, one has the isomorphism of all four bundles.

b. Canonical 1-form on $SO(3;\mathbb{C})(\Lambda)$. Just as the frame bundle $GL(M)$ has a canonical real-valued 1-form θ^μ defined on it, the complex frame bundle $GL_{\mathbb{C}}(\Lambda^2)$ has a canonical complex-valued 1-form Z^i defined on it. One finds that the origin of either canonical form is based in the idea that one can regard a frame E_i, $i = 1, \ldots, n$ in a vector space V over a field of scalars $\mathbb{K}$ as a linear isomorphism $E_i : \mathbb{K}^n \to V$, $(v^1, \ldots, v^n) \mapsto v^i E_i$, while a coframe is a linear isomorphism $E^i : V \to \mathbb{K}^n$, $\mathbf{v} \mapsto v^i = E^i(\mathbf{v})$. Hence, a frame and a coframe are reciprocal iff these isomorphisms are inverse to each other.

If $E \to M$ is a vector bundle over M whose fiber is isomorphic to $\mathbb{K}^n$ then any frame E_i in E_x is associated with a set of set of n 1-forms E^i in $\Lambda^1_x(M)$. The canonical 1-form E^i on the bundle $\pi: GL(E) \to M$ of E-frames is obtained by pulling back the E^i along π:

$$E^i(\mathbf{v}) = \pi^* E^i(\mathbf{v}) = E^i(\pi_*\mathbf{v}). \tag{XIII.150}$$

Again, the reason that we are abusing notation by denoting both the canonical 1-form and the coframe with the same symbol is because when one chooses a local E-frame field over $U \subset M$ the canonical 1-form pulls down to the local reciprocal coframe field.

In the case where the elements of E are complex 3-frames in $\Lambda_2 M$, the canonical complex-valued 1-form Z^i on $GL_{\mathbb{C}}(\Lambda_2)$ pulls down to a set of three local real *2-forms* Z^i on $U \subset M$ by way of a local section Z_i: $U \to GL_{\mathbb{C}}(\Lambda_2)$. The 2-forms Z^i can then be expressed with respect to a local coframe field on U by expressions of the form (XIII.149). One way of seeing how the 1-form on $GL_{\mathbb{C}}(\Lambda_2)$ turns into a 2-form on U is to note that the vectors of the fibers of $\Lambda_2 M$ are actually *bivectors* to begin with. Hence, linear functionals on such vectors are 2-forms.

The canonical 1-form Z^i on $GL_{\mathbb{C}}(\Lambda_2)$ restricts to a canonical 1-form on any G-reduction of $GL_{\mathbb{C}}(\Lambda_2)$ where G is a Lie subgroup of $GL(3; \mathbb{C})$. Similarly, it induces a corresponding canonical 1-form on the coframe bundle $GL^*(3; \mathbb{C})$ by the isomorphism of those two bundles.

c. Connections on $GL_{\mathbb{C}}(M)$ and its reductions. The reason for introducing a connection on the bundle $GL_{\mathbb{C}}(M) \to M$ is analogous to the reason for introducing one on $GL(M) \to M$. That is, if $\mathbf{F}(x(\tau)) = F^i Z_i$ is a complex 3-vector along a differentiable curve $x(\tau)$ in $U \subset M$ one wishes to give some precise sense to the condition that this complex 3-vector be "parallel" along the curve. By differentiating with respect to τ, we get, with $\mathbf{v}(\tau) = dx/d\tau$:

$$\frac{d\mathbf{F}}{d\tau} = \frac{dF^i}{d\tau} Z_i + F^i (i_{\mathbf{v}} dZ_i) = \left(\frac{dF^i}{d\tau} + F^j \varpi^i_j(\mathbf{v}) \right) Z_i , \qquad \text{(XIII.151)}$$

in which:

$$\varpi^i_j = dZ_j \otimes_{\mathbb{C}} Z^i \qquad \text{(XIII.152)}$$

represents the generalized complex "angular velocity" of the 3-frame Z^i along the curve in question; here, we use the notation $\otimes_{\mathbb{C}}$ to indicate that we are concerned with the tensor product over the complex vector space that Z^i lives in, not the real vector space of 2-forms that it is associated with. One can see that it represents a 1-form on U with values in the Lie algebra $\mathfrak{gl}(3; \mathbb{C})$. Although one could propose to say that $\mathbf{F}$ is constant along $x(\tau)$ if this derivative vanishes, the condition would not be true for all choices of local frame field Z^i.

In order to make the vanishing of the derivative covariant under all changes of local frame fields, including ones for which the transition function is non-constant, one must replace the ordinary derivative with a covariant derivative that is defined by making a

choice of $\mathfrak{gl}(3;\mathbb{C})$-connection 1-form σ^i_j on the bundle $GL_{\mathbb{C}}(M) \to M$. Hence, this will be a 1-form on $GL_{\mathbb{C}}(M)$ with values in $\mathfrak{gl}(3;\mathbb{C})$ that is Ad^{-1}-equivariant under the right action of $GL(3;\mathbb{C})$ on complex 3-frames:

$$\sigma^i_j\big|_{Z_k A} = A(\sigma|_{Z_i})A^{-1} \tag{XIII.153}$$

and defines a linear isomorphism of each vertical tangent space on $GL_{\mathbb{C}}(M)$ with $\mathfrak{gl}(3;\mathbb{C})$ in such a way that if $a \in \mathfrak{gl}(3;\mathbb{C})$ and $\tilde{a}$ is the fundamental vector field on $GL_{\mathbb{C}}(M)$ that is associated with a then one has:

$$\sigma(\tilde{a}) = a \tag{XIII.154}$$

under this linear isomorphism.

If Z_i: $U \to GL_{\mathbb{C}}U$ is a local complex 3-frame field over U and Z^i is its reciprocal complex coframe field then σ can be expressed as a 1-form on U with values in $\mathfrak{gl}(3;\mathbb{C})$ by way of:

$$\sigma^i_j = \sigma^i_{\mu j}\theta^\mu, \tag{XIII.155}$$

in which the $\sigma^i_{\mu j}$ are smooth functions on U. Note that in general the choice of Z_i does not imply an unambiguous choice of θ^μ, although this will be true in the reduced case of oriented, complex orthonormal frames and oriented, time-oriented, Lorentzian frames.

The 1-forms σ^i_j transform to another coframe field $\bar{Z}^i = A^i_j Z^j$ by way of:

$$\bar{\sigma}^i_j = \tilde{A}^i_k \sigma^k_m A^m_j + \tilde{A}^i_k dA^k_j. \tag{XIII.156}$$

One can reduce the bundle $GL_{\mathbb{C}}(\Lambda_2)$ to the bundle $SL_{\mathbb{C}}(\Lambda_2)$ by defining a complex volume element $\mathcal{V}$ on $\Lambda_2 M$, which can also be defined as the 3-form on $GL_{\mathbb{C}}(\Lambda_2)$:

$$\mathcal{V} = \frac{1}{3!}\varepsilon_{ijk} Z^i \perp Z^j \perp Z^k, \tag{XIII.157}$$

in which Z^i is the canonical 2-form; again, the notation $\perp$ is used to distinguish the exterior algebra over the complex vector space in which one finds Z^i, not the real vector space of 2-forms that the Z^i are associated with.

A $\mathfrak{gl}(3;\mathbb{C})$-connection σ on $GL_{\mathbb{C}}(\Lambda_2)$ is reducible to an $\mathfrak{sl}(3;\mathbb{C})$-connection iff the exterior covariant derivative of $\mathcal{V}$ using σ vanishes:

$$\nabla\mathcal{V} = d\mathcal{V} + \sigma\mathcal{V} = \mathrm{Tr}(\sigma)\mathcal{V}. \qquad \text{(XIII.158)}$$

Hence, the connection is reducible iff the trace of the matrix σ^i_j vanishes identically.

One can define a complex orthogonal structure γ on $\Lambda_2 M$ by means of an equivariant 0-form γ_{ij} on $GL_{\mathbb{C}}(\Lambda_2)$ with values in the homogeneous space $GL(3;\mathbb{C})/O(3;\mathbb{C})$:

$$\gamma(Z_i) = \gamma_{ij}\, Z^i \otimes_{\mathbb{C}} Z^j. \qquad \text{(XIII.159)}$$

The reduction is then achieved by restricting to all complex orthonormal 3-forms, which will then have $\gamma_{ij} = \delta_{ij}$.

The $\mathfrak{gl}(3;\mathbb{C})$-connection σ is reducible to an $\mathfrak{o}(3;\mathbb{C})$-connection iff the tensor field γ is covariantly constant:

$$0 = \nabla\gamma_{ij} = d\gamma_{ij} - \sigma^k_i\gamma_{kj} - \sigma^k_j\gamma_{ki}\,. \qquad \text{(XIII.160)}$$

For a complex orthonormal frame, this says that:

$$0 = \sigma_{ij} + \sigma_{ji}\,; \qquad \text{(XIII.161)}$$

i.e., σ_{ij} is anti-symmetric.

Of course, in order for σ to be an $\mathfrak{so}(3;\mathbb{C})$-connection, the matrix σ must satisfy both conditions that it have vanishing trace and satisfy (XIII.160).

c. Curvature and torsion. Now let G be a Lie subgroup of $GL(3;\mathbb{C})$ and let $\mathfrak{g}$ be its Lie algebra. Suppose furthermore that σ is a $\mathfrak{g}$-connection on $GL_{\mathbb{C}}(\Lambda^2)$.

Since Z^i can be interpreted as either a 1-form on $GL_{\mathbb{C}}(\Lambda^2)$ with values in $\mathbb{C}^3$ or a local 2-form on some $U \subset M$, we need to clarify what it means to define the torsion of a $\mathfrak{g}$-connection σ on $GL_{\mathbb{C}}(\Lambda^2)$ to be the "exterior covariant derivative" of the canonical 1-form:

$$\Psi^i = \nabla^{\wedge} Z^i = dZ^i + \sigma^i_j \wedge Z^j\,. \qquad \text{(XIII.162)}$$

In particular, one needs to explain what its two terms represent.

It seems simplest to use the local interpretation, since it is inevitable that physics will require the necessity of going to local expressions. Hence, we shall regard Z^i as a 2-form on U with values in $\mathbb{C}^3$ that is given by (XIII.149), so Ψ^i will be a 3-form with values in $\mathbb{C}^3$, along with both of the terms in its definition. This makes:

$$dZ^i = \tfrac{1}{2}[dZ^i_{\mu\nu} - Z^i_{\kappa\nu}c^{\kappa}_{\lambda\mu}\theta^{\lambda}] \wedge \theta^{\mu} \wedge \theta^{\nu}\,, \qquad \text{(XIII.163)}$$

in which we have substituted the appropriate expression for $d\theta^\mu$, and:

$$\sigma^i_j \wedge Z^j = \tfrac{1}{2}\sigma^i_{[\kappa j} Z^j_{\mu\nu]}\, \theta^\kappa \wedge \theta^\mu \wedge \theta^\nu\,, \qquad \text{(XIII.164)}$$

in which the [.] notation implies that we have completely anti-symmetrized the components $\kappa\mu\nu$:

$$\sigma^i_{[\kappa j} Z^j_{\mu\nu]} = \tfrac{1}{3}(\sigma^i_{\kappa j} Z^j_{\mu\nu} + \sigma^i_{\mu j} Z^j_{\nu\kappa} + \sigma^i_{\nu j} Z^j_{\mu\kappa})\,. \qquad \text{(XIII.165)}$$

Combining expressions, we see that the local form of the torsion 3-form is:

$$\Psi^i = \tfrac{1}{2}[\mathbf{e}_\kappa Z^i_{\mu\nu} + \sigma^i_{[\kappa j} Z^j_{\mu\nu]} - Z^i_{\lambda\nu} c^\lambda_{\kappa\mu}]\theta^\kappa \wedge \theta^\mu \wedge \theta^\nu\,; \qquad \text{(XIII.166)}$$

that is, its components with respect to this local coframe field are:

$$\Psi^i{}_{\kappa\mu\nu} = \mathbf{e}_\kappa Z^i_{\mu\nu} + \sigma^i_{[\kappa j} Z^j_{\mu\nu]} - Z^i_{\lambda\nu} c^\lambda_{\kappa\mu}\,. \qquad \text{(XIII.167)}$$

Hence, the vanishing of torsion becomes an algebraic condition on σ.

Two simplifying special cases present themselves: local coframe fields in which the $Z^i_{\mu\nu}$ are constant, which we call *canonical* coframe fields, and anholonomic coframe fields, for which the structure functions $c^\kappa_{\mu\nu}$ vanish. In a holonomic canonical coframe field the vanishing of torsion takes the form of the algebraic identity:

$$0 = \sigma^i_{[\kappa j} Z^j_{\mu\nu]}\,. \qquad \text{(XIII.168)}$$

In the case of curvature, since only σ is involved, if we express it locally as a 1-form on U with values in $\mathfrak{g}$ then the definition of its exterior covariant derivative is more conventional. It is a 2-form on U with values in $\mathfrak{g}$:

$$\Sigma^i_j = \nabla\hat{}\sigma^i_j = d\sigma^i_j + \sigma^i_k \wedge \sigma^k_j\,, \qquad \text{(XIII.169)}$$

which has the components:

$$\Sigma^i_{j\mu\nu} = \mathbf{e}_\mu \sigma^i_{\nu j} - \mathbf{e}_\nu \sigma^i_{\mu j} + \sigma^i_{\mu k}\sigma^k_{\nu j} - \sigma^i_{\nu k}\sigma^k_{\mu j}\,. \qquad \text{(XIII.170)}$$

The Bianchi identities work the same way as before:

$$\nabla\hat{}\Psi^i = \nabla\hat{}\nabla\hat{}Z^i = \Sigma^i_j \wedge Z^j, \qquad \text{(XIII.171a)}$$

$$\nabla\hat{}\Sigma^i_j = \nabla\hat{}\nabla\hat{}\sigma^i_j = 0. \qquad \text{(XIII.171b)}$$

Of course, the expression $\nabla\hat{}\Psi^i$ is locally a 4-form, this time.

d. Penrose-Debever decomposition of the Riemann curvature tensor. The curvature 2-form Σ^i_j, at least when it is considered locally, can be regarded as a linear map Σ_x :

$(\Lambda_2)_x M \to \mathfrak{so}(3;\mathbb{C})$, $\mathbf{a} \wedge \mathbf{b} \mapsto [\Sigma_x]^i_j(\mathbf{a},\mathbf{b})$, from one real vector space of dimension six to another; hence, $\Sigma_x \in \Lambda^2_x M \otimes_{\mathbb{R}} \mathfrak{so}(3;\mathbb{C})$. Furthermore, both real vector spaces are assumed to be given complex structures, which are defined by * in the case of $\Lambda^2_x M$ and i in the case of $\mathfrak{so}(3;\mathbb{C})$.

i. The space of curvature tensors. The basis for the Penrose-Debever decomposition of the Riemann curvature tensor is that since $\Lambda^2_x M$ and $\mathfrak{so}(3;\mathbb{C})$ have the same dimension as vector spaces – whether real or complex – one can identify their elements by choosing a basis for each. Of course, this is somewhat naïve, as far as the algebraic structures on the two vector spaces are concerned, and comes down to the fact that one can associate any element $\omega^\mu_\nu \in \mathfrak{so}(p, q)$ in a matrix representation of an orthogonal Lie algebra with an anti-symmetric matrix $\omega_{\mu\nu} = g_{\mu\kappa}\omega^\kappa_\nu$, regardless of the signature type (p, q) of the metric $g_{\mu\nu}$.

In particular, since the Riemann curvature tensor Ω^μ_ν takes its values in $\mathfrak{so}(3, 1)$ in the case of general relativity, one can associate its value $\Omega^\mu_\nu(\mathbf{v}_x, \mathbf{w}_x)$ when applied to tangent vectors $\mathbf{v}_x$, $\mathbf{w}_x$ at a point $x \in M$ with a 2-form:

$$\Omega(\mathbf{v}_x, \mathbf{w}_x) = \tfrac{1}{2}\Omega_{\mu\nu}(\mathbf{v}_x, \mathbf{w}_x)\, dx^\mu \wedge dx^\nu, \qquad \text{(XIII.172)}$$

by setting:

$$\Omega_{\mu\nu}(\mathbf{v}_x, \mathbf{w}_x) = g_{\mu\nu}(x)\,\Omega^\mu_\nu(\mathbf{v}_x, \mathbf{w}_x). \qquad \text{(XIII.173)}$$

Since Ω^μ_ν is a 2-form to begin with, this means that we can regard Ω as defining linear maps from each vector space $\Lambda_{2,x}M$ to the corresponding vector space $\Lambda^2_x M$; i.e., Ω is a section of $\Lambda^2 M \otimes \Lambda^2 M \to M$ that one can locally represent as:

$$\Omega = \tfrac{1}{4} R_{\kappa\lambda\mu\nu}(dx^\kappa \wedge dx^\lambda) \otimes (dx^\mu \wedge dx^\nu). \qquad \text{(XIII.174)}$$

In fact, from the symmetry of $R_{\kappa\lambda\mu\nu}$ in the first and last index pairs, one can write this as:

$$\Omega = \tfrac{1}{4} R_{\kappa\lambda\mu\nu}(dx^\kappa \wedge dx^\lambda)(dx^\mu \wedge dx^\nu), \qquad \text{(XIII.175)}$$

with the symmetrized tensor product implied by the absence of a multiplication symbol.

If one goes the route of the other isomorphic copy of both $\Lambda^2_x M$ and $\mathfrak{so}(3;\mathbb{C})$ – namely, $\mathbb{C}^3$ – by way of the isomorphisms Z^i, then one can represent the Riemann curvature tensor (or really its value at any point $x \in M$) by means of an element Σ of $\mathbb{C}^3 \odot \mathbb{C}^3$:

$$\Sigma = \Sigma_{ij}\, Z^i\, Z^j, \qquad \Sigma_{ij} = \gamma_{ik}\Sigma^k_j\,. \tag{XIII.176}$$

Here, we can pause to point out that even though the Euclidian metric on $\mathbb{C}^3$ that is defined by γ_{ij} can be obtained by first defining a Lorentzian structure $g_{\mu\nu}$ on $T(M)$, actually, that definition is not *necessary*. It is *sufficient* to define a linear electromagnetic constitutive tensor $\kappa \in \Lambda^2 M \otimes \Lambda^2 M$ and then obtain the Euclidian scalar product by means of the association:

$$\gamma_{ij} = \tfrac{1}{4}\,\kappa_{\kappa\lambda\mu\nu}\, Z_i^{\kappa\lambda} Z_j^{\mu\nu}\,. \tag{XIII.177}$$

Note that under symmetrization the skewon part of κ will not contribute to γ_{ij}.

Therefore, just as the observation that the Lorentzian metric contributed to the Maxwell equations only by way of the isomorphism * represented the starting point for pre-metric electromagnetism, so does this latter association represent the starting point for "pre-metric gravitation." Of course, what this would really mean is not that there is no metric on anything involved, but only that the more fundamental metric is a complex Euclidian metric on the bundle of 2-forms over spacetime, not a Lorentzian metric on the bundle of tangent vectors. Moreover, the fundamental character of the metric γ_{ij} is then purely electromagnetic in origin.

ii. The double dual operator. When one is given two real vector spaces V and W that have complex structures * and $\overline{*}$ defined on them, a natural problem to investigate is the way that a given real-linear map, such as Σ_x, affects the two complex structures.

Indeed, the basic question is whether an $\mathbb{R}$-linear map A: $V \to W$ between two real vector spaces V and W that have been given complex structures * and $\overline{*}$ also defines a $\mathbb{C}$-linear map. The necessary and sufficient condition for this is that A must commute with both the isomorphisms * and $\overline{*}$:

$$A* = \overline{*}A. \tag{XIII.178}$$

We can then define an $\mathbb{R}$-linear map $\overline{(\cdot)}$: $\mathrm{Hom}_{\mathbb{R}}(V, W) \to \mathrm{Hom}_{\mathbb{R}}(V, W)$, $A \mapsto \overline{A}$, where:

$$\overline{A} = \overline{*}A*, \tag{XIII.179}$$

which we refer to as the *double dual* operator.

Since $*^2 = -I$ and $\overline{*}^2 = -I$, one sees that this map $\overline{(\cdot)}$ is then an involutory isomorphism on $\mathrm{Hom}_{\mathbb{R}}(V, W)$:

$$\overline{\overline{A}} = A. \tag{XIII.180}$$

This also says that the eigenvalues of the map $\overline{(\cdot)}$ are ± 1. Indeed, the real vector space $\mathrm{Hom}_{\mathbb{R}}(V, W)$ splits into a direct sum $\mathrm{Hom}_+ \oplus \mathrm{Hom}_-$ of two subspaces of equal dimension that correspond to the positive and negative eigenvalues, respectively. In fact, the map that takes A to $A_+ + A_-$ is just polarization using the involution $\overline{(\cdot)}$:

$$A_+ = \tfrac{1}{2}(A + \overline{A}), \qquad A_- = \tfrac{1}{2}(A - \overline{A}). \tag{XIII.181}$$

It is traditional to call the elements of the subspaces Hom_+ and Hom_- *self-dual* and *anti-self-dual*, resp., but we can see from (XIII.178) that the elements of Hom_- represent $\mathbb{C}$-linear maps, while the elements of Hom_+ are $\mathbb{C}$-antilinear. One must note that the signs behave in the opposite manner to one's intuition regarding $\mathbb{C}$-linearity.

The way this double dual map is defined in most of the literature of complex relativity is to consider the space of curvature tensors as sections of the vector bundle $\Lambda^2 M \otimes \mathfrak{so}(3; \mathbb{C}) \to M$ and give the fibers of the factors $\Lambda^2 M$ and $\mathfrak{so}(3; \mathbb{C})$ the complex structures defined by * and i, respectively.

One then defines:

$$\overline{Z}^i = i{*}Z^i, \qquad (\text{so } {*}Z^i = -i\overline{Z}^i) \tag{XIII.182}$$

which can be interpreted as meaning that * takes the 2-form Z^i to its dual 2-form and i takes its value $Z^i(\mathbf{b})$ in $\mathfrak{so}(3; \mathbb{C})$, when evaluated on a bivector $\mathbf{b}$ to the complex 3-vector $iZ^i(\mathbf{b})$. When $\overline{Z}^i = \pm Z^i$ one has ${*}Z^i = \mp iZi$, which makes the self-dual (anti-self-dual, resp.) 2-forms take the form of eigenvectors of * with eigenvalue $-i$ ($+i$, resp.).

iii. The Einstein equations. In order to account for the symmetry of $R_{\kappa\lambda\mu\nu}$ in its first last index pairs, we see that it becomes more convenient to represent Riemann curvature tensors as elements of either $\Lambda^2 M \odot \Lambda^2 M$ or $\mathbb{C}^3 \odot \mathbb{C}^3$, which we then refer to as the *real* and *complex* representations, respectively.

In the complex representation, the decomposition into self-dual and anti-self-dual parts allows one to express Σ in the form:

$$\Sigma = C'_{ij} Z^i Z^j + iE_{ij} Z^i \overline{Z}^j. \tag{XII.183}$$

This means that the complex matrix C'_{ij} is symmetric and E_{ij} is Hermitian.

Furthermore, since we already have a symmetric second rank tensor on $\mathbb{C}^3$ that is defined by γ, which we express in the form:

$$\gamma = \gamma_{ij} Z^i Z^j, \tag{XII.184}$$

we can project Σ onto that one-dimensional subspace of $\mathbb{C}^3 \odot \mathbb{C}^3$ by taking its trace:

$$\mathrm{Tr}(\Sigma) = \Sigma^i_i = \gamma^{jk}\Sigma_{ki} = \mathrm{Tr}(C'). \tag{XII.185}$$

The traceless part of C'_{ij} is then denoted by:

$$C_{ij} = C'_{ij} - \tfrac{1}{3}\mathrm{Tr}(\Sigma)\,\gamma_{ij}\,. \tag{XII.186}$$

Therefore, we have the complex representation of the *Penrose-Debever decomposition* of Σ into:

$$\Sigma = C + E + \tfrac{1}{2}(\mathrm{Tr}\,\Sigma)\gamma. \tag{XII.187}$$

This can be given a corresponding real form as a decomposition of $R_{\kappa\lambda\mu\nu}$ by means of the isomorphisms defined by Z^i. It can be shown (cf., e.g., [**21, 22**]) that:

$$\mathrm{Tr}\,\Sigma = \tfrac{1}{4}R\,, \tag{XII.188}$$

where R is the scalar curvature of Ω, and E_{ij} corresponds to:

$$E_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{4}Rg_{\mu\nu}, \tag{XII.189}$$

which is the traceless part of the Ricci curvature tensor.

Hence, although this differs from the Einstein tensor by a term equal to $-\tfrac{1}{4}Rg_{\mu\nu}$, nevertheless, since the vacuum Einstein equations implied that R had to vanish, we see that the vacuum Einstein equations are equivalent to the equations:

$$E_{ij} = 0. \tag{XII.190}$$

The remaining part of Σ that is defined by C corresponds to the *Weyl curvature tensor* of the connection ω, which is also called the *conformal curvature tensor*. Its vanishing implies that the metric g is conformal to the Minkowskian one:

$$g_{\mu\nu} = \alpha^2\eta_{\mu\nu}\,, \tag{XII.191}$$

with α^2 as the conformal factor.

One should observe that so far we have obtained the complex form of only the vacuum Einstein equations. Hence, we need to extend the analysis give to the complex form of the Einstein equations with sources. This step then depends upon first finding the complex form of the energy-momentum tensor.

As pointed out by Krasnov [**23**], in the Plebanski [**24**] formulation of complex relativity one can define the complex form of the energy-momentum tensor T^μ_ν by first splitting it into its trace $T = T^\mu_\mu$ and its traceless part $\tilde{T}^\mu_\nu = T^\mu_\nu - \tfrac{1}{4}T\delta^\mu_\nu$ and then defining:

$$\tilde{T}^i_j = \tfrac{1}{4}\tilde{T}^\kappa_\mu Z^i_{\nu\kappa}\bar{Z}^{j\mu\nu}, \qquad \bar{Z}^{i\mu\nu} = \gamma^{ij}\bar{Z}^{\mu\nu}_j . \tag{XIII.192}$$

Note that since the T^μ_ν form of the energy-momentum tensor is pre-metric in its definition, and involves a mechanical constitutive law as the mechanism for associating tangent objects with cotangent ones, the metric is not introduced into the definitions until one uses γ^{ij}, which, as we pointed out above, can be defined by the electromagnetic constitutive law alone.

The Einstein equations, with the source term included then take the form:

$$E^{ij} = -2\pi G\tilde{T}^{ij}, \qquad \mathrm{Tr}(\Sigma) = -\tfrac{1}{4}\Lambda - 2\pi GT, \tag{XIII.193}$$

if one includes the cosmological constant Λ.

In order to make the comparison more immediate, one should use following real form of the Einstein equations:

$$R^\mu_\nu - \tfrac{1}{4}R\delta^\mu_\nu = 8\pi G\tilde{T}^\mu_\nu, \qquad R = -\Lambda - 8\pi GT, \tag{XIII.194}$$

in place of the usual formulation in terms of the divergenceless part of the Ricci tensor.

5. Discussion. In its conventional formulation (e.g., [**21, 22**]), complex relativity generally involves a slight redundancy in its basic definitions. In particular, although one assumes an almost-complex structure on Λ^2M, in the form of *, one also complexifies this bundle to $\Lambda^2M\otimes\mathbb{C}$ in order to treat the mathematical expression $*F = \pm iF$ as an eigenvalue equation that allows one to define a decomposition of $\Lambda^2M\otimes\mathbb{C}$ into a direct sum of self-dual and anti-self-dual 2-forms, which then correspond to the positive and negative imaginary eigenvalues, respectively.

However, since * defines an almost-complex structure on Λ^2M it would seem unnecessary to complexify it again. One need only regard the expression $*F = \pm iF$ as giving two possible *definitions* of how the imaginary unit i acts on the fibers of Λ^2M, and thus complex scalars, more generally.

The concept of self-duality reasserts itself in the context of the Debever-Penrose decomposition of the curvature 2-form. However, at that point one can observe that an $\mathbb{R}$-linear map from Λ_2M to either $\mathfrak{so}(3;\mathbb{C})$ or $\mathbb{C}^3$ can commute with the complex structures on both or not. Hence, we can define a (double) duality operator on $\Lambda^2M\otimes\mathfrak{so}(3;\mathbb{C})$ – or $\Lambda^2M\otimes\Lambda^2M$, for that matter – that allows one to define the aforementioned decomposition, just the same. One then finds that the anti-self-dual $\mathbb{R}$-linear maps are precisely the $\mathbb{C}$-linear maps.

Since the theme of this book all along has been that the electromagnetic structure of the spacetime manifold, as encoded in the constitutive map κ, implies the Lorentzian geometry as a consequence of the dispersion law that follows from κ by way of the field

equations, we first observe that even the primary emphasis in the relativistic theory of gravitation can be shifted from the Lorentzian metric $g_{\mu\nu}$ on tangent vectors to the complex Euclidian metric γ_{ij} in 2-forms. Since this metric can be defined in terms of only κ, one sees that in a sense one can also define a sort of "pre-metric gravitation" as well as pre-metric electromagnetism. Of course, as we have observed a number of times, there is a difference between the algebraic sort of metric structure that comes from κ directly and the differential sort that comes from defining field equations that involve κ and passing to the symbol of the differential operator $\Box_\kappa$ that defines them.

A deep question then arises whether perhaps the field κ, which is like a generalization of the metric tensor, might be itself subject to field equations. One sees that in order to make physical sense of this coupling one would have to go into deeper detail about the nature of the medium in question and how electric polarizability and magnetization can come about in it. If one is dealing with macroscopic media, such as optical ones, this resolution of the macroscopic picture to a microscopic one is generally more straightforward – e.g., crystal lattices, electron orbitals – than when one is addressing the electromagnetic vacuum state of quantum electrodynamics. In that case, one must mostly rely upon effective models, such as the Heisenberg-Euler model.

Of course, if one is attempting to account for the gravitational fields of astronomical bodies then one must realize that the energy, momentum, and stress that couple to the spacetime metric in the form of κ must be essentially affecting the state of electric and magnetic polarization of the vacuum of space, at least in the immediate neighborhood of elementary massive matter, in order to produce a spacetime metric that is not merely obtained from a linear, isotropic, homogeneous electromagnetic constitutive law that is based on the constants ε_0 and μ_0. Of course, this is precisely what one gets from the Heisenberg-Euler dispersion law, which attributes the perturbation of the Minkowski metric in a region of spacetime to the Faraday stress-energy-momentum tensor of an electromagnetic field that permeates it. Although the electric and magnetic field strengths are generally close to the critical values only in the small neighborhoods of elementary charge distributions, nevertheless, one can still think of those distributions as the ultimate sources of gravitational fields, as well. Perhaps the gravitational field of a star or planet is then best viewed as the macroscopic effect of a large number of microscopic perturbations to the metric at the elementary level due to vacuum polarization.

The main objective of this chapter was therefore to embed the usual Lorentzian geometry of general relativity in the framework of the geometry that pertains to the bundle of 2-forms instead of the tangent bundle. At this point, one can only speculate on what form the field equations of κ might take, although undoubtedly the answer to that problem will probably follow from a better understanding of the role that connections on the bundle of frames in $\Lambda^2 M$ and its reductions and their curvatures take in physics.

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