PUPT-1169

January 1990

QUASI-QUANTUM GROUPS RELATED TO ORBIFOLD MODELS*

Robbert Dijkgraaf

Joseph Henry Laboratories,
Princeton University,
Princeton, NJ 08544

Vincent Pasquier

Institut de Recherche Fondamentale,
Service de Physique Théorique,
CEA-CEN Saclay,
91191 Gif-sur-Yvette Cedex, France

Philippe Roche

Centre de Physique Théorique,
Ecole Polytechnique,
91128 Palaiseau Cedex, France

ABSTRACT

We construct non-trivial quasi-quantum groups associated to any finite group G and any element of $H^3(G, U(1))$. We analyze the set of representations of these algebras and show that we recover the data of particular 3-dimensional topological field theories. We also give the R-matrix structure in non abelian RCFT orbifold models.

* Talk presented by V.Pasquier at the International Colloquium on Modern Quantum Field Theory, Tata Institute of Fundamental Research, 8-14 January 1990.

1. INTRODUCTION

This paper is part of an attempt to understand the connection between conformal field theory and quantum groups [AGS,MS,Pa]. It is now well known that the fusion rules of the WZW theories coincide with the tensor product decomposition of representations of the quantum Lie groups for q a root of unity. The aim of this work is to give such an interpretation to the fusion rules of orbifolds models considered in [DVVV] and generalized by [DW] using 3-dimensional topological field theory.

In this case, the quasi-Hopf algebras, recently introduced by Drinfeld turn out to be the adequate structure, the final result being a slight modification of the double construction applied to the algebra of functions over a finite group. An extensive version of this work will be published elsewhere [DPR].

2. QUANTUM DOUBLE CONSTRUCTION

2.1. Definition of the Hopf algebra structure

Let $\mathcal{A}$ be a Hopf algebra which is not necessarily quasitriangular. The quantum double construction of Drinfeld [Dr1] consists in building a quasitriangular Hopf algebra $\mathcal{D}(\mathcal{A})$ containing $\mathcal{A}$ as a Hopf subalgebra. Details of the construction are found in the original paper of Drinfeld [Dr1] and developed in the paper of Reshetikhin [Re]. This construction is an unvaluable tool for constructing the universal R matrix of ordinary quantum groups $\mathcal{U}_q(SL(n))$ as shown in [Ro] and [Bu]. In this section, we will apply this construction to $\mathcal{A} = \mathcal{F}(G)$ where G is a finite group and $\mathcal{F}(G)$ is the abelian algebra of complex functions defined on G.

$\mathcal{A}$ is endowed with its usual Hopf algebra structure and $\mathcal{A}^\star$ is equal to the Hopf algebra $\mathbf{C}[G]$ (the group ring of G). As a vector space $\mathcal{D}(\mathcal{A}) = \mathcal{A} \otimes \mathcal{A}^\star$. We shall use the notation $\mathcal{D}(G) = \mathcal{D}(\mathcal{A})$. A convenient basis of $\mathcal{D}(G)$ is $(\delta_g \otimes x)_{g,x\in G}$, where δ_g is the Dirac function at the point g. We will adopt the following notation:

$$\underset{x}{{}_g\llcorner\!\!\!\_\,} = \delta_g \otimes x \qquad \text{and} \qquad \underset{x}{{}_\varphi\llcorner\!\!\!\_\,} = \varphi \otimes x$$

for φ element of $\mathcal{F}(G)$. In this paper e is the unit element of the group G and $1 = \sum_{g\in G} \delta_g$ is the unit element of $\mathcal{F}(G)$.

From the definition of the quantum double, it is easy to obtain an explicit description of the structure of $\mathcal{D}(G)$. The algebra law is defined by:

$$\underset{x}{g\llcorner} \cdot \underset{y}{h\llcorner} = \delta_{g,xhx^{-1}} \quad \underset{xy}{g\llcorner} \tag{2.1}$$

and the unit element is $\underset{e}{1\llcorner}$. The comultiplication Δ is defined by:

$$\Delta(\underset{x}{g\llcorner}) = \sum_{\substack{h,k\in G \\ hk=g}} \underset{x}{h\llcorner} \otimes \underset{x}{k\llcorner}\,. \tag{2.2}$$

The counit ϵ and the antipode S are defined by:

$$\epsilon(\underset{x}{g\llcorner}) = \delta_{g,e} \ \underset{e}{1\llcorner} \qquad \text{and} \qquad S(\underset{x}{g\llcorner}) = \underset{x^{-1}}{x^{-1}g^{-1}x\,\llcorner}\,. \tag{2.3}$$

The R matrix of $\mathcal{D}(G)$ is

$$R = \sum_{g\in G} \underset{e}{g\llcorner} \otimes \underset{g}{1\llcorner}\,. \tag{2.4}$$

In the next sections, we will show that the study of the set of irreducible representations of this algebra will permit us to recover Moore and Seiberg's [MS] data (Verlinde algebra, F, B and S matrices and conformal weights) of a particular RCFT described in [DVVV].

2.2. Irreducible representations of $\mathcal{D}(G)$

Let $\{C_A\}_{A=0,\dots,p}$ be the set of conjugacy classes of G ($C_0 = \{e\}$). In each C_A we pick an element ${}^A g_1$ and we define N_A the centralizer of ${}^A g_1$ *i.e.* $N_A = \{h \in G, [h, {}^A g_1] = e\}$. We choose ${}^A x_1, \dots, {}^A x_q$ a set of representatives of the equivalence classes of G/N_A where, for convenience, we take ${}^A x_1 = e$. As a result $C_A = \{{}^A x_1^A g_1^A x_1^{-1} = {}^A g_1, \dots, {}^A x_q^A g_1^A x_q^{-1} = {}^A g_q\}$. We will often forget the label A.

Consider the subalgebra $\mathcal{B}_A$ of $\mathcal{D}(G)$ spanned as a vector space by the elements $(\underset{x}{g\llcorner})_{g,x}$ where g is in G and h is in N_A. Let π be an irreducible representation of N_A on a vector space V_π. We define a representation ρ_π of $\mathcal{B}_A$ on V_π in the following way:

$$\rho_\pi(\underset{x}{g\llcorner}) = \delta_{g,{}^A g_1} \ \pi(x). \tag{2.5}$$

This representation induces a representation of $\mathcal{D}(G)$, which we call ρ_π^A, acting on the vector space W_π^A. As a left module, we have $W_\pi^A = \mathcal{D}(G) \otimes_{\mathcal{B}_A} V_\pi$. Consider a basis $|e_i\rangle_{i=1,\dots,dimV_\pi}$ of V_π, then a basis of W_π^A is $(|{}^A x_j, e_i\rangle) = (\underset{{}^A x_j}{1\llcorner} \otimes |e_i\rangle)_{i,j}$.

The action of $\underset{x}{g\llcorner}$ on the basis is easily computable, and one finds the following formula:

$$\rho_{\pi}^{A}(\underset{x}{g\llcorner})|{}^{A}x_j, e_i\rangle = \delta_{g, x^A g_j x^{-1}} \ |{}^{A}x_k, \pi(h)(e_i)\rangle \tag{2.6}$$

where ${}^{A}x_k$ and h are defined by the relation $x^A x_j = {}^{A}x_k h$ and $h \in N_A$.

The set ρ_{π}^{A} is the complete list of irreducible representations of $\mathcal{D}(G)$. This can be shown with the use of the orthogonality relations:

$$\frac{1}{|G|}\sum_{g,x\in G} tr(\rho_{\pi}^{A}(\underset{x}{g\llcorner}))(tr(\rho_{\pi'}^{B}(\underset{x}{g\llcorner})))^{\star} = \delta_{A,B}\delta_{\pi,\pi'}. \tag{2.7}$$

Let consider the integers $N_{ABC}^{\alpha\beta\gamma}$ satisfying :

$$\rho_{\alpha}^{A}\otimes\rho_{\beta}^{B} = \bigoplus_{C,\gamma} N_{ABC}^{\alpha\beta\gamma}\rho_{\gamma}^{C} \tag{2.8}$$

These integers can be exactly computed using relation (2.7) and we found that they are exactly those defining the fusion rules of the RCFT considered in [DVVV] in the case $\sigma = 1$. Moreover if we define $R_{AB}^{\alpha\beta} = P(\rho_{\alpha}^{A}\otimes\rho_{\beta}^{B})(R)$ where P is the usual permutation operator and $K_{ABC}^{\alpha\beta\gamma}$ is the projection onto the isotypic component W_{C}^{γ} of $W_{A}^{\alpha}\otimes W_{B}^{\beta}$, we have shown that the following relation is satisfied:

$$K_{ABC}^{\alpha\beta\gamma}R_{BA}^{\beta\alpha}R_{AB}^{\alpha\beta} = \alpha({}^{A}g_1)\beta({}^{B}g_1)\gamma({}^{C}g_1^{-1})K_{ABC}^{\alpha\beta\gamma}. \tag{2.9}$$

Using the results of [MS] and [AGS] we find that the conformal weight $h_{A,\alpha}$ of the sector $[\phi_{\alpha}^{A}]$ satisfies:

$$e^{2i\pi h_{A,\alpha}} = \alpha({}^{A}g_1) \tag{2.10}$$

which is the relation found by purely CFT ideas in [DVVV].

In this section we have described what is the algebraic structure underlying special type of orbifold models with $\sigma = 1$. Using the point of view of topological field theory Witten and one of the authors have classified in [DW] this whole class of RCFT. The results was that these theories are classified by the group $H^3(G, U(1))$. One of our goal is to modify the definition of $\mathcal{D}(G)$ in order to recover Moore and Seiberg's data of these RCFT (or equivalently 3-dimensional topological field theories).

3. GROUP COHOMOLOGY AND QUASI HOPF ALGEBRAS

3.1. Construction of the quasi Hopf algebra $\mathcal{D}^\omega(G)$

A brief description of quasi Hopf Agebras is given in the appendix and a complete study of these algebras is given in the papers of Drinfeld [Dr2]. Let ω be an element of $H^3(G, U(1))$, we will always work with a representative of ω i.e a 3-cocycle. We are looking for an algebra $\mathcal{D}^\omega(G)$ and we will assume the following statements:

1) $\mathcal{D}^\omega(G) = \mathcal{D}(G)$ as a vector space and is a quasitriangular quasi Hopf algebra.

2) The algebra law and coalgebra law take the following form:

$$\underset{x}{g\llcorner} . \underset{y}{h\llcorner} = \delta_{g,xhx^{-1}} \ \underset{xy}{g\llcorner} \ \ \theta_g(x,y) \tag{3.1}$$

$$\Delta(\underset{x}{g\llcorner}) = \sum_{\substack{h,k\in G \\ hk=g}} \underset{x}{h\llcorner} \otimes \underset{x}{k\llcorner} \ \ \gamma_x(h,k) \tag{3.2}$$

where $\theta_g(x,y)$ and $\gamma_x(h,k)$ are phases such that if x,y or g is equal to e then $\theta_g(x,y)$ and $\gamma_g(x,y)$ are equal to one.

3) φ and R are given by:

$$\varphi = \sum_{g,h,k\in G} \omega(g,h,k)^{-1} \underset{e}{g\llcorner} \otimes \underset{e}{h\llcorner} \otimes \underset{e}{k\llcorner} \tag{3.3}$$

$$R = \sum_{g\in G} \underset{e}{g\llcorner} \otimes \underset{g}{1\llcorner} . \tag{3.4}$$

These assumptions determine completely $\mathcal{D}^\omega(G)$. We first remark that relation (A.5) of the appendix is equivalent to the 3-cocycle relation verified by ω.

Quasitriangularity implies that relations (A.12) and (A.13) are satisfied. When written in detail, these equations are equivalent to:

$$\theta_g(x,y) = \frac{\omega(g,x,y)\omega(x,y,(xy)^{-1}gxy)}{\omega(x,x^{-1}gx,y)} \tag{3.5}$$

$$\gamma_x(g,h) = \frac{\omega(g,h,x)\omega(x,x^{-1}gx,x^{-1}hx)}{\omega(g,x,x^{-1}hx)} \tag{3.6}$$

By a repeatedly use of the cocycle relation, the reader will verify the three following identities:

$$\theta_g(x,y)\theta_g(xy,z) = \theta_g(x,yz)\theta_{x^{-1}gx}(y,z) \tag{3.7}$$

$$\gamma_x(g,h)\gamma_x(gh,k)\alpha(x^{-1}gx, x^{-1}hx, x^{-1}kx) = \gamma_x(h,k)\gamma_x(g,hk)\alpha(g,h,k) \tag{3.8}$$

$$\theta_g(x,y)\theta_h(x,y)\gamma_x(g,h)\gamma_y(x^{-1}gx, x^{-1}hx) = \theta_{gh}(x,y)\gamma_{xy}(g,h). \tag{3.9}$$

These relations imply respectively that the algebra law is associative, the comultiplication is quasi coassociative and the comultiplication is an algebra morphism. This algebra also posess also a counit and an antipode:

$$\epsilon({}_{g}\underset{x}{\llcorner}) = \delta_{g,e}\ {}_{1}\underset{e}{\llcorner} \qquad \text{and} \qquad S({}_{g}\underset{x}{\llcorner}) = {}_{x^{-1}g^{-1}x}\underset{x^{-1}}{\llcorner}\ \ \theta_{g^{-1}}(x,x^{-1})^{-1}\gamma_x(g,g^{-1})^{-1}. \tag{3.10}$$

If $\delta\beta$ is any 3-coboundary then $\mathcal{D}^{\omega\delta\beta}(G)$ is obtained from $\mathcal{D}^{\omega}(G)$ by the twist element

$$F = \sum_{g,h\in G} \beta(g,h)^{-1}\, {}_{g}\underset{e}{\llcorner} \otimes {}_{h}\underset{e}{\llcorner}\,. \tag{3.11}$$

This remark implies that modulo natural transformations the set of representations of $\mathcal{D}^{\omega}(G)$ depends just on the class of ω in $H^3(G,U(1))$.

We note that when $[x,g]=[x,h]=e$ then $\gamma_x(g,h)$ and $\theta_x(g,h)$ both take the value

$$c_x(g,h) = \frac{\omega(x,g,h)\omega(g,h,x)}{\omega(g,x,h)} \tag{3.12}$$

which is the 2-cocycle appearing in [DW].

3.2. Irreducible representations of $\mathcal{D}^{\omega}(G)$

Irreducible representations of $\mathcal{D}^{\omega}(G)$ are easily constructed using the previous method. $\mathcal{B}_A$ is still a subalgebra of $\mathcal{D}^{\omega}(G)$. Let π be an irreducible projective representation of N_A with cocycle $c_{{}^{A}g_1}$. We define a representation ρ_π of $\mathcal{B}_A$ by the formula (2.5) and induce it to a representation ρ^A_π of $\mathcal{D}^{\omega}(G)$. Formula (2.6) is now changed into :

$$\rho^A_\pi({}_{g}\underset{x}{\llcorner})|{}^{A}x_j, e_i\rangle = \theta_g(x, {}^{A}x_j)\theta_g({}^{A}x_k, h)^{-1}\delta_{g,x^{A}g_jx^{-1}}\ |{}^{A}x_k, \pi(h)(e_i)\rangle. \tag{3.13}$$

Once again $N^{\alpha\beta\gamma}_{ABC}$ defined by equation (2.8) are shown to be the fusion rules computed in [DW].

4. CONCLUSION

In this paper we have shown that the study of representations of special kinds of quasi Hopf algebras could give new insights in the study of RCFT. We have found an interesting modification of the quantum double of $\mathcal{A}$ when $\mathcal{A}$ is equal to $\mathcal{F}(G)$. It is not difficult to construct such a modification in the case of an arbitrary Hopf algebra.

The conformal field theories we recovered are orbifolds of holomorphic CFT. What are the algebraic objects underlying orbifolds of more general CFT? An interesting step to the answer would be to analyze, from the (quasi)-quantum group approach the $c = 1$ orbifolds theories based on finite subgroups of $SU(2)$.

It also remains to understand the S and T matrices of modular transformations as Hopf algebra constructions [AGS].

Acknowledgements

Ph. R. thanks J. Lascoux and M. Rosso for interesting discussions. The work of R.D. is supported by NSF Grant PHY80-19754.

APPENDIX

In this appendix, we recall the most important properties of quasi Hopf algebras [Dr2]. Let $\mathcal{A}$ be an unital algebra over $\mathbf{C}$. A coproduct Δ is an algebra morphism $\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$. If $\mathcal{A}$ is endowed with a coproduct, then there is a natural notion of tensor product of representations of $\mathcal{A}$.Namely, if (V_1, π_1) and (V_2, π_2) are representations of $\mathcal{A}$, one defines the representation $\pi_1 \otimes \pi_2$ by acting on $V_1 \otimes_{\mathbf{C}} V_2$ with Δ.

In an ordinary Hopf algebra it is required that Δ is coassociative, which means that

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta. \qquad (A.1)$$

This relation implies strict associativity of tensor product of representations , i.e we have :

$$(\pi_1 \otimes \pi_2) \otimes \pi_3 = \pi_1 \otimes (\pi_2 \otimes \pi_3) \qquad (A.2)$$

where π_1, π_2 and π_3 are representations of $\mathcal{A}$.

The notion of quasi Hopf algebra arises naturally when one requires that $(\pi_1 \otimes \pi_2) \otimes \pi_3$ is equivalent to $\pi_1 \otimes (\pi_2 \otimes \pi_3)$. This is always satisfied if there exists an invertible element φ such that

$$(\Delta \otimes id)\Delta(a) = \varphi(id \otimes \Delta)\Delta(a)\varphi^{-1}, \forall a \in \mathcal{A}, \tag{A.3}$$

the intertwiner being $\varphi_{123} = (\pi_1 \otimes \pi_2 \otimes \pi_3)(\varphi)$. We shall say that the coproduct is quasi-coassociative.

Finally in order that the following pentagonal diagram be commutative :

$$\begin{array}{ccccc} ((V_1 \otimes V_2) \otimes V_3) \otimes V_4 & \longrightarrow & (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) & \longrightarrow & V_1 \otimes (V_2 \otimes (V_3 \otimes V_4)) \\ \downarrow & & & & \downarrow \\ (V_1 \otimes (V_2 \otimes V_3)) \otimes V_4 & & \longrightarrow & & V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) \end{array} \tag{A.4}$$

one requires the following relation:

$$(id \otimes id \otimes \Delta)(\varphi)(\Delta \otimes id \otimes id)(\varphi) = (1 \otimes \varphi)(id \otimes \Delta \otimes id)(\varphi)(\varphi \otimes 1). \tag{A.5}$$

In a quasi Hopf algebra there exists also analogues of the counit and antipode [Dr2,DPR].

A Hopf algebra is said to be quasitriangular if there exists an invertible element R of $\mathcal{A} \otimes \mathcal{A}$ such that :

$$\Delta'(a) = R\Delta(a)R^{-1} \ \forall a \in \mathcal{A} \tag{A.6}$$

$$(\Delta \otimes id)(R) = R_{13}R_{23} \tag{A.7}$$

$$(id \otimes \Delta)(R) = R_{13}R_{12} \tag{A.8}$$

Eq. (A.6) implies that:

$$\pi_1 \otimes \pi_2 \ \text{ is equivalent to } \pi_2 \otimes \pi_1$$

for any couple of representations of $\mathcal{A}$.

From eq (A.7) (resp.(A.8)) the following diagrams are commutative:

$$\begin{array}{ccccc} V_1 \otimes (V_2 \otimes V_3) & \longrightarrow & V_1 \otimes (V_3 \otimes V_2) & \longrightarrow & (V_1 \otimes V_3) \otimes V_2 \\ \downarrow & & & & \downarrow \\ (V_1 \otimes V_2) \otimes V_3 & \longrightarrow & V_3 \otimes (V_1 \otimes V_2) & \longrightarrow & (V_3 \otimes V_1) \otimes V_2 \end{array} \qquad (A.9)$$

$$\begin{array}{ccccc} (V_1 \otimes V_2) \otimes V_3 & \longrightarrow & (V_2 \otimes V_1) \otimes V_3 & \longrightarrow & V_2 \otimes (V_1 \otimes V_3) \\ \downarrow & & & & \downarrow \\ V_1 \otimes (V_2 \otimes V_3) & \longrightarrow & (V_2 \otimes V_3) \otimes V_1 & \longrightarrow & V_2 \otimes (V_3 \otimes V_1) \end{array} . \qquad (A.10)$$

Eq (A.6) and (A.7) (resp (A.8)) implies the Yang-Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \qquad (A.11)$$

A quasi Hopf algebras is called quasitriangular if there exists an invertible element R of $\mathcal{A} \otimes \mathcal{A}$ verifying Eq. (A.6) and two other relations replacing Eq. (A.7) and (A.8) which imply the commutativity of (A.9) and (A.10). These two relations take the following form: A

$$(\Delta \otimes id)(R) = \varphi_{312}R_{13}\varphi_{132}^{-1}R_{23}\varphi, \qquad (A.12)$$

$$(id \otimes \Delta)(R) = \varphi_{231}^{-1}R_{13}\varphi_{213}R_{12}\varphi^{-1}. \qquad (A.13)$$

(where we have used the notation : if $\varphi = \sum_i a_i^1 \otimes a_i^2 \otimes a_i^3$ then $\varphi_{s(1)s(2)s(3)} = \sum_i a_i^{s^{-1}(1)} \otimes a_i^{s^{-1}(2)} \otimes a_i^{s^{-1}(3)}$). From these relations follows the quasi Yang-Baxter equation:

$$R_{12}\varphi_{312}R_{13}\varphi_{132}^{-1}R_{23}\varphi = \varphi_{321}R_{23}\varphi_{231}^{-1}R_{13}\varphi_{213}R_{12}. \qquad (A.14)$$

If $(\mathcal{A}, \Delta, \varphi)$ is any quasi Hopf algebra, one can construct a new quasi Hopf algebra $(\mathcal{A}^F, \Delta^F, \varphi^F)$ associated to any F invertible element of $\mathcal{A} \otimes \mathcal{A}$, using the following definitions:

$\mathcal{A}^F$ as an algebra is isomorphic to $\mathcal{A}$,

$$\Delta^F(a) = F\Delta(a)F^{-1}, \qquad (A.15)$$

$$\varphi^F = (1 \otimes F)(id \otimes \Delta)(F)\varphi(\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1). \qquad (A.16)$$

One says that $\mathcal{A}^F$ is obtained from $\mathcal{A}$ by a twist F.

References

[AGS] L. Alvarez-Gaumé, C. Gomez and G. Sierra, *Duality and Quantum Groups*, CERN Preprint -TH .5369/89.

[Bu] N. Burroughs, *The universal R matrix of* $\mathcal{U}_q(sl(3))$ *and beyond* DAMTP-R.89-4 1989.

[$Dr1$] V.G. Drinfeld,*Quantum Groups*, in *Proceedings of the I.C.M. Berkeley*, (A.M.S, 1986).

[$Dr2$] V.G. Drinfeld, *Quasi Hopf algebras*, preprint in Russian.

V.G. Drinfeld, *Quasi Hopf algebras and Knizhnik-Zamolodchikov equations*, Preprint ITP-89-43E Kiev 1989.

[DPR] R. Dijkgraaf, V. Pasquier and Ph. Roche, Preprint to be published.

[$DVVV$] R. Dijkgraaf, C. Vafa, E. Verlinde and H. Verlinde, *Operator algebra of Orbifold models*, Commun. Math. Phys. **123** (1989) 485.

[DW] R. Dijkgraaf and E. Witten, *Topological gauge theories and group cohomology,* Commun. Math. Phys. to be published.

[Ji] M. Jimbo, *A q-difference analogue of* $\mathcal{U}(\mathcal{G})$ *and the Yang-Baxter equation,* Lett. Math. Phys. **10** (1985) 63-69.

[MS] G. Moore and N. Seiberg, *Classical and quantum conformal field theory*, Commun. Math. Phys. **123** (1989) 177.

[Pa] V. Pasquier, Thesis, unpublished.

[Re] N.Yu. Reshetikhin, *Quantized universal envelopping algebras, the Yang-Baxter Equation and Invariants of Links*, LOMI Preprints E-17-87.

[Ro] M. Rosso, *An analogue of P.B.W Theorem and the Universal R-matrix for* $\mathcal{U}_h sl(N+1)$, Commun. Math. Phys **124** (1989) 307.