# Chapter 2 Smoke Rings

*Do not think that mathematics is hard and crabbed,*
*and repulsive to common sense.*
*It is merely the etherealization of common sense.*
William Thomson

Knot theory is one of the hottest topics in contemporary mathematics, right there at the interface between a range of exciting research fronts in pure mathematics, with seminal contributions made by leading string theorist Edward Witten. We shall see more of how it fits the higher-dimensional outlook in later chapters. Here, I want to take the time to consider the peculiar constellation of circumstances which brought about knot theory's early development. Knowledge of the history of a discipline is vital for philosophical reflection. Thoughtful practitioners of a discipline are usually well read in its history. It reminds us that ideas we hold dear had once to start to take a grip on us, and that theories which seemed highly plausible at one time appear now as old curiosities, wrong-headed yet uncannily presaging modern viewpoints. Although the spirit of a time might seem quite foreign to us today, one can only wonder at the way so many contemporary ideas are melded together from earlier fragments.

One evening in the middle of January 1867, two eminent scientists could be found in animated discussion within one of the lecture halls of the University of Edinburgh. The elder and more famous of the two men was paying his colleague a customary visit. But whereas it had been their long-standing project to prepare a text-book of natural philosophy, or physics as we now call it, that regularly induced him to travel the fifty miles from his home in Glasgow, the purpose of their meeting on this winter's night was quite different.

His host was busily preparing a packing crate which had been placed on its side on a table. A piece of heavy cloth had been tacked onto the box so as to cover the open face, and out of the opposite panel a disc around eight inches in diameter had been removed. Reaching in through the hole he tossed a piece of zinc into a beaker containing nitric acid. Alongside the beaker there lay a cloth soaked in ammonia. Rapidly the box began to fill with smoke from the ensuing reaction. Satisfying himself that enough smoke had been produced, the experimenter sharply struck the cloth-covered side with the palm of his hand and out through the opening emerged a quivering smoke ring. So began the demonstration that over the next twenty years would intrigue audiences the length and breadth of Britain.

During that evening the two of them conducted all manner of experiments on these tubular apparitions. Their attempts to cut one open with a knife were doomed to failure—the ring would simply slide past any blade put in its path. A solid object, such as a suspended globe, placed to one side of a smoke ring's trajectory would appear to attract it, causing it to deviate from a straight line. Still more unexpected was the effect a pair of such rings had on one another. Two meeting obliquely would bounce off each other, shuddering violently before regaining their form. And when two were propelled in the same direction, the one lagging behind would quicken its pace and contract, while the leader slowed and dilated, allowing the second to pass through the aperture of the first and take the lead. Now that their positions had been reversed, their behaviour switched and so the game of chase continued.

The inspiration for this experiment had come from a study of *Wirbelbewegungen*, or vortex motion, carried out by a German scientist, Hermann von Helmholtz, and contained in a paper on hydrodynamics, the study of the motion of fluids. This paper had recently been translated by the host, Peter Guthrie Tait, Professor of Natural Philosophy in Edinburgh, in whose lecture hall the demonstration was taking place. The visitor was Sir William Thomson, later to become Lord Kelvin, one of Britain's most illustrious scientists. Thomson was still basking in the glory of having successfully completed the project of laying several thousands of miles of the first telegraphic cable across the Atlantic sea floor. He had been rewarded with a knighthood by Queen Victoria for an achievement which had allowed her to exchange pleasantries instantaneously with McKinsey, the American President, at a rate of thirteen words per minute.

As typified a leading scientist of his day, Thomson's range of intellectual interests was immense. Alongside his writings in physics on electricity, magnetism, light and matter, he had published tracts on the origin of the solar system, the habitation of the Earth and the transmutation of species. But he was a practical man and proud of his design activities, which would later include a fathoming device for sounding the depths off a ship while maintaining maximum speed at sea. So, while Thomson was happy to speculate on the fundamental scientific issues of the day, he liked to think in terms of concrete analogies from the everyday world. Vortex motion so appealed to him that within days of Tait's demonstration he was writing to Helmholtz that the latter's *Wirbelbewegungen* had 'displaced everything else' in his mind. He now had smoke rings on the brain.

Thomson had long desired to find a fundamental physical rôle for vortices, having observed twenty years earlier, as he stirred his *chocolat au lait* in a Paris café, that they possess a rigidity and elasticity beyond what might be expected from the properties of the liquid itself. The whirlpools he had created in his cup had appeared to bounce off its sides as a ball rebounds off a wall. This conviction of his about the stability of spinning matter had once cost Helmholtz a hat when, in his presence, Thomson had struck with a hammer a heavy metal disc that was rotating very rapidly upon a point. Thomson had predicted that the disc would wobble slightly before regaining a horizontal position. Instead it had flown off directly towards the headpiece, shredding it in the process.

When it came to imagining physical processes, rotating systems were never far from Thomson's mind, but it was the vivid impression made by Tait's smoke rings which brought about in him the sudden thought that he had at last found the secret of the structure of the atom. Here was a wonderful new way of picturing the fundamental constituents of matter, which would allow for a far deeper understanding than did the standard atomic theory of the day. He felt a strong aversion to the most popular current theory, which presented matter as formed of infinitely hard, elastic, indivisible, indestructible, minuscule, spherical atoms. Thomson believed that scientists were projecting onto these hard spheres whatever properties they felt would allow agreement with experimental results. For example, to account for the capacity of a gas to apply pressure to a chamber wall, these atomic spheres were imagined to be endowed with elastic properties. This might be reasonable for rubber balls, but it seemed implausible to him that entities understood to be without internal structure should possess any elasticity.

Even more tellingly, how could such featureless corpuscles vary sufficiently to account for the range of properties possessed by the chemical elements: colourless, gaseous, life-giving oxygen; light, metallic, combustible sodium; carbon as graphite or as diamond; mercury as a shiny, liquid poison; or any other of the elements to have been isolated up to that time? The number of elements that had been discovered steadily increased through the nineteenth century. By the late 1860s, sixty-three had been found and their properties tabulated. A further one, fluorine, was known to exist, but could not be isolated due to its reactivity. What could be the source of such variation?

Of course, smoke rings disperse over the course of a minute or so, while atoms were known to have endured for millions of years and perhaps all eternity. But this presented no problem to Thomson's idea. After all, the smoke rings were indicating regions of the room in which the air was rapidly rotating, and air is a medium in which frictional effects would quickly diffuse these pockets of rotational flow. What Helmholtz had shown was that in a *perfectly frictionless* medium, vortices would persist indefinitely. So the permanence of atoms could be explained, Thomson reasoned, were atoms taken to be tubular vortex motions of a fluid 'destitute of viscosity'. Now when it came to accounting for an atom's elasticity, a far less *ad hoc* explanation could be given, since vortex-atoms had a more complex internal structure than had sphere-atoms. For example, one would expect to be able to calculate how the dimensions of the ring and its speed of rotation would relate to the elasticity of an atomic collision, rather than merely assign this quantity a value derived from macroscopic measurements of pressure.

Unlike spheres, there were many ways in which these tubes might vary. For instance, Helmholtz had shown them ‘invariable as to strength’, in the sense that at any point on the ring and at any moment the cross-sectional area multiplied by the speed of rotation there was a fixed quantity. In other words, if a section of the ring became narrower, it rotated more rapidly. Vortices of different strengths might be expected to behave differently.

More intriguingly, seeing that a ring is a string-like object, wave motions could be expected to ripple around it at particular frequencies and this, Thomson argued, would account for a phenomenon which had long defied explanation. It had been found that any chemical element when suitably excited gives off light with a characteristic pattern of frequencies, referred to as its *spectrum*, and that each such spectrum is unique. With this major discovery, mankind could at last say something about the make-up of the heavenly bodies on the basis of hard scientific evidence, rather than rely on the imaginings of poets or the speculations of philosophers. Indeed, by comparing the light arriving from distant stars with spectral data obtained in the laboratory, scientists were to discover that the principal constituents of stellar bodies are hydrogen and a gas which came to be called helium, from *helios* the Greek word for the sun. The orange-yellow colour of one of the lines in helium’s spectrum was first picked up during a solar eclipse in 1868, the year following Tait’s smoke ring demonstration.

Now, an element’s spectrum of emissions could only reasonably be attributed to some structural feature of the atom of that element which constrained it to vibrate in a limited number of ways. But how could miniature balls be thought to have these multiple internal vibrations? Only some of the frequencies of the emitted light fall in the visible range, our eyes allowing us to detect only a narrow band of radiation, which we experience as running from red to violet. Sodium is seen to emit a yellow light, but to much surprise it had been found that this is caused by emissions of two very close frequencies. Might this occur, Thomson wondered, due to the sodium atom being formed of two nearly identical linked tubes, vibrating at slightly different rates? Could the equivalent of interlocking smoke rings have been formed?

These and other features of the vortex-atom hypothesis meant that it remained a serious contender for physicists over the next twenty years. But besides the physical advantages to his model, Thomson was also taken with the idea that it might lead to evidence for the existence of God. This was no small matter for a religious scientist of the second half of the nineteenth century, threatened by the rise of an atheistic materialism which claimed to base itself on nothing other than scientific principles. Coincidently, it was the sphere model Thomson was so set against that had been devised precisely to exclude divine presence in the world. Four centuries before Christ, the Greek philosopher Democritus had devised his atomic theory so as to remove the need to invoke the gods to explain untoward events in the universe. In the belief that all that existed were atomic spheres and the empty space between them, mankind might lead a life untroubled by the fear of natural calamities induced by some vengeful Immortal, slighted by a perceived shortfall in the sacrifices offered up to him. For Thomson, on the other hand, scientific theories did not entail a rejection of a divine being. As he put it, “if you think strongly enough you will be forced by science to the belief in God.”

The vortex-atom theory was well suited to allow this inference to be made. Vortices in a frictionless fluid could neither be generated nor destroyed once the fluid was set in motion, except by an act of creative power. Thomson had seen how he himself could act in such a way by causing a smoke ring to disintegrate by encircling its tube with his forefinger and thumb. This in itself was an act of free will on his part, something possessed by all living beings, and every action of which was to be seen as “a miracle to physical and chemical and mathematical science”. But who else except God could create and masterfully direct the perfect fluid? It was He, then, who must be the original cause and directing power of all matter present in the universe.

Theological considerations similarly drove Tait. Indeed, they drove him even more forcefully than they did Thomson. While Thomson was the perfect choice to co-author a treatise on natural philosophy, he was not the man to share the writing of a cosmological work. Many scientists of the Victorian age strove to reconcile their faith with an acceptance of the findings of evolutionary theory, chemistry and physics, but few went as far as Tait and his fellow physicist, Balfour Stewart, in their efforts to wed contemporary scientific knowledge to a theology grounded in the Testaments. We shall look at this work, *The Unseen Universe*, after discussing Tait’s part in the vortex-atom project, for it was Tait who should be named the ‘father of knot theory’.

While it was Thomson who devoted himself to demonstrating the stability of vortices by examining their equations of motion, Tait dedicated himself to another of their features. If vortex tubes could adopt different knotted configurations, then they might be expected to behave differently. Imagine, for instance, a tube tied in the form of what came to be known as the trefoil knot.

Figure 1

It seems evidently clear that a tube shaped like this cannot unknot itself without one portion of the ring passing through another portion. This possibility had, however, been precluded by Helmholtz' results: regions of rotational flow remain impenetrable to one another. This had been illustrated by that part of Tait's demonstration in which two smoke rings bounce off each other, rather than passing through each other. Without this possibility, the way in which a tube was knotted could be taken as a permanent property of that tube. Might then the different ways of knotting a tube correspond to the different varieties of atom? Furthermore, blending the idea of knotted tubes with Thomson's idea of linked tubes, atoms in the form of knotted links could also be envisaged.

Figure 2

But where to begin the task of finding the five dozen or so varieties of knotted and linked tubes to match up to the then known chemical elements? Playing about with a piece of string, or better still a piece of electrical flex, you will soon come to realise the apparent limitlessness of the possible forms. How was Tait to find a way of telling one from another?

First he had to find a way of representing them on paper. Knotted tubes, or simply *knots* as mathematicians call them, are most easily represented by drawing them on a piece of paper as curves whose ends meet up. Where from your point of view one piece of the knot passes behind another, a small break is shown in the corresponding section of the curve. Such a diagram is known as a projection of the knot. By rearranging the knot, twisting it or moving parts of it relative to each other, an infinite number of projections of the same knot are possible. What was now required was a way of saying whether given two projections, they arose from the same knot, so that one could know, for example, that the following diagrams correspond to the same knot.

Figure 3

Tait's writings on knots are contained in three papers published in the Transactions of the Royal Society of Edinburgh. In his first paper, published in 1876, Tait outlined two approaches to knot classification. His first step was to define a knot's *knottiness* as the least number of places where portions of the knot cross each other, the minimum being taken over all possible projections. The picture of the trefoil knot (Fig. 1) shows three crossings and it seems clear that, however it is rearranged, no fewer than three could occur. Indeed, your intuition serves you well here; you cannot represent a trefoil knot with fewer than three crossings, though it is a surprisingly difficult result to prove to a mathematician's satisfaction. The drawing on the right of Fig. 3 has five crossings, but it corresponds to a knot which may be represented as having four crossings, as shown on the left of the figure. This in fact is the minimum possible number and so the knot, the figure-eight knot as it is known, has knottiness four.

Tait's major efforts were devoted to *alternating* knots. These are knots which have a projection in which crossings alternate between underpasses and overpasses. Fig. 1 and the left drawing of Fig. 3 show that the trefoil and figure-eight knots are alternating.

Fig. 4(a) shows a non-alternating knot projection with six crossings, while the one shown in Fig. 4(b) is alternating. They correspond in fact to the same knot of knottiness 6.

Figure 4

This knot reveals an interesting feature: it is 'composed' of two simpler knots. Two trefoil knots, one left-handed and the other right-handed, have been cut open and their loose ends attached to one another (Fig 4(c)). Tait realised that any two knots could be composed in this way to produce a third, just as any two whole numbers may be multiplied together to produce a third. Now, once a mathematician has located an apparent analogy, she will always push it hard to see how far it can be made to run. In that way, the structural properties of a situation already worked out in one domain can be sought in a second domain without the need to reinvent the wheel. So, a feature of the positive whole numbers, or natural numbers, is that each of them can be written uniquely as a product of prime numbers, those divisible by no other number than themselves or one. For example, 504 may be expressed as 2 x 2 x 2 x 3 x 3 x 7, and there is no other way of building up to 504 from prime numbers aside from rearranging the order of the multiplication. In a very similar way knots may be expressed uniquely as the composition of a certain collection of *prime* knots. These prime knots can thus be seen as the building blocks out of which all other knots are formed.

It might have been the case that all such prime knots were alternating, but this is not so. The non-alternating prime knot of least knottiness has eight crossings. So Tait realised that by restricting himself to alternating knots his list would not be complete. Alternating knots are simpler to deal with. For a given projection which does not display the nature of each of the crossings there are just two ways of assigning them. Starting from a point on the projection and proceeding in a chosen direction, the first crossing encountered can either be an overcrossing or an undercrossing.

Once this is chosen the nature of the remaining crossings is now fixed by the requirement of alternation. Fig. 5(a) shows the diagram underlying the trefoil knot. The two alternating knots associated with it are the left- and right-hand trefoils (Figs. 5(b) and (c)). Without the requirement of alternation the corresponding knot can easily collapse, as Fig. 5(d) shows.

Figure 5

Tait's idea here was to label each of the intersections by a letter, so that a projection is represented by the sequence of letters, or *scheme*, corresponding to a passage around the knot in a specific direction from a specific starting point. Each letter will occur twice in a scheme since each crossing is composed of two arcs.

ABCDEAFCDEBF

Figure 6

The two copies of a letter should not occur next to each other, however, for otherwise you will find that you are dealing with a simple twist in the knot which may be removed.

ACBACBDD ACBACB

Figure 7

Some doodling will also show you that in knot-worthy schemes the two copies of a letter will have an even number of letters between them. Otherwise you meet with the following problem:

ABCDABCD...

Figure 8

The string has had to cross in and out of the loop at B, C and D, but this has left one end of it inside the loop, unable to cross out again to join up to the other end.

Taking these findings into consideration, Tait proceeded to label *every other* crossing A, B, C... He then listed the ways of feeding copies of these letters into the spaces while avoiding adjacent occurrences of the same letter. For example, schemes representing three crossings arise from feeding a copy each of A, B and C into the spaces of A _ B _ C _ . There is only one way of doing so: A $\underline{\text{C}}$ B $\underline{\text{A}}$ C $\underline{\text{B}}$ . This scheme corresponds to the trefoil knot.

For four crossings, we find two possibilities: A $\underline{\text{C}}$ B $\underline{\text{D}}$ C $\underline{\text{A}}$ D $\underline{\text{B}}$ or A $\underline{\text{D}}$ B $\underline{\text{A}}$ C $\underline{\text{B}}$ D $\underline{\text{C}}$. But these can only represent the same knot since shifting the last three letters of the first scheme to the beginning produces the second scheme. They represent the figure-eight knot.

ACBDCADB ADBACBDC

Figure 9

Once the different ways of doing this have been listed—and for 7 letters the number of these has already reached 579—several questions have to be asked about each scheme:

(1) Does it represent a knot at all?

Take a sequence running between a pair of letters and containing no other pair, for example, ...ADBA... This represents a loop Fig. 10(a).

Figure 10

Now, between any two occurrences of another letter, say, C, there had better either be both D and B or neither of them, since to return to C requires an even number of crossings of the loop. Thus a scheme beginning, say, ADBACEDC... as in Fig. 10(b) should be discarded.

(2) Was the scheme minimal?

There would be what Tait termed a *nugatory* crossings, if the scheme had the form

...X(combination of a set of pairs)X...

For example, AFX$\underline{\text{BDCBDC}}$XEAFE corresponds to the left side of Fig. 11. A simple twist would remove the crossing X, and so reduce the length of the scheme.

AFXBDCBDCXEAFE AFBDCBDCEAFE

Figure 11

Schemes of this kind correspond to composite knots, in this case the sum of two trefoils. All composite knots have the property that their schemes contain a subsequence comprised of a combination of pairs.

(3) If already minimal, might it be equivalent to a knot already discovered?
We have seen that cyclical permutations of the letters of a scheme correspond to the same knot, but there are more subtle equivalences.

ADBECADCEB ADBACEDBEC ADBECBDAEC

Figure 12

Answering these questions for each scheme in the initial list, there resulted a much reduced list of distinct knots. Out of the 80 possible schemes for six-crossings, 20 represent distinct configurations, 8 of which correspond to alternating knots. These pair up to form four distinct knots, one of which is the composite of two trefoils, leaving three prime alternating knots of knottiness six.

In this first paper, Tait managed to find all distinct knots with seven or less crossings. His work here turned out to be accurate, but he was aware of the problems which would have to be faced for higher orders of knottiness. The dangers were twofold: first, a knot might have been overlooked and, second, two diagrams listed separately might in fact correspond to the same knot. But while care had to be taken to avoid these errors, faster means of generating knot diagrams were required. Already for seven-crossings nearly six hundred schemes had to be sifted through to locate the seven prime knots of that knottiness. For eight-crossing knots this figure had risen to more than four thousand, while for nine-crossing knots over forty thousand schemes would have to be processed. He needed a second way of producing knot diagrams to which his first method could act as an independent check.

Tait managed to do this by his observation that knot projections could be shaded like a chess board.

Figure 13

Taking the black regions only, one could count the number of curved sides surrounding them, along with the number of corners at which they come into contact with each other. This Tait thought of as a way of partitioning the total number of curved sides. Fig. 13 shows three knots shaded in this way along with their 'partitions'. Each number corresponds to the number of curves sides of a shaded region. The lines between two numbers denote the number of corners at which the corresponding regions meet. A useful observation to make here is that the sum of the numbers occurring in the partition is equal to twice the number of crossings in the diagram, e.g., for the trefoil, $2 + 2 + 2 = 6 = 2 \times 3$. This follows from the fact that each connecting line in the partition corresponds to one crossing and to one corner of each of the two regions meeting there. The number of corners in a region clearly equals the number of curves forming its boundary.

Thus, to find knot diagrams with a given number of crossings, say 5, split twice this number, in this case 10, into numbers between 2 and 5, and draw in linking lines.

Figure 14

Fig. 14 shows eight partitions of 10 and the corresponding diagrams of five crossings. Notice that some of the diagrams ((c), (d), (g) and (h)) represent not knots but knotted links, which as potential atomic forms would also have to be tabulated. Furthermore, the diagrams pair up. If in figure (a) you reverse the shading, considering the region outside the diagram now as one large black region (Fig. 15), you should be able to observe that it corresponds to partition (f). Similarly, (b) and (e), (c) and (d), and (g) and (h) are seen to correspond to the same knot or link.

Figure 15

This pairing of diagrams is related to an operation which may be carried out on knots, that Tait called 'flyping'. Tait notes when he introduces this term that there is no closer English equivalent to the Scots 'to flype' than the cumbersome expression 'to turn outside-in' as one does a glove. To flype a piece of string knotted as in Fig. 14(a) turn it inside-out by placing your thumbs within one of the five sided regions and using your fingers to bring the rest of the knot within that region. A little rejigging will now result in a knot configured as in (f). In fact by applying this process to a real knot, you can discover how the system of over- and under-passes is affected. What transpires is something rather interesting. Take the figure-eight knot shown in Fig. 16(a). If you flype it from region (i) you arrive at the knot shown in Fig. 16(b). If, however, you flype it from region (ii), the result is the knot shown in Fig. 16(c), identical to (a) except that the crossings have been reversed. By contrast, from whichever region you flype the trefoil, you can never arrive at the opposite handed trefoil with reversed crossings.

Figure 16

The figure-eight knot has been shown to be *amphicheiral*, that is, it can be transformed in three-dimensional space into its mirror image. The fact that the right-hand trefoil cannot be flyped into a left-hand mirror image is suggestive of the fact that it is not amphicheiral, although this result was not established conclusively until 1910.

Although Tait had successfully listed the prime knots up to 7 crossings, he was left with the impression that the task of knot hunting

> is a very much more difficult and intricate one than at first sight one is inclined to think, and I feel that I have not succeeded in catching the key-note.

Eight years were to elapse before he was to publish further results of his knot investigations and by this time he had enrolled the services of the Reverend Thomas Penyngton Kirkman.

Kirkman's mathematical career tells us something of the state of British mathematics in the nineteenth century. He had left school at the age of fourteen to join his father's business, but nine years later he could no longer stand this life, and so moved to Dublin to study for a degree. After graduating he decided to become a priest and at the age of thirty he moved to Southworth with Croft in Lancashire, where he was to spend the next fifty years of his life as its rector. Despite the arrival of seven children and the demands of his ministry, at the age of forty Kirkman developed an obsession for mathematical research, and to this activity he dedicated the best part of his spare time for the remainder of his long life.

His greatest love was of the area of mathematics known as combinatorics. Combinatorics deals with problems of arranging objects bearing certain features into given patterns. Some of these problems are depicted in such striking concrete settings that they come to take on their inventor's name. So it is that, in spite of his considerable contributions to mathematics, Kirkman is best remembered for the seemingly frivolous combinatorial puzzle he posed of how to arrange the positions of fifteen schoolgirls in such a way that they could walk out three abreast on seven successive days with no two girls walking abreast on more than one occasion. While a psychoanalyst might care to linger over the details of this scenario, from the perspective of the mathematician, a specific problem such as this, even when abstracted from the schoolground setting, is of little consequence in itself. It is the general theory devised to allow the solution of this and related classes of problem that has contributed to a lively branch of mathematics with many varied connections to computer science in recent decades.

Even though an obscure Lancashire clergyman, Kirkman's efforts did not go unnoticed. The leading British exponents of combinatorics and other related branches of pure mathematics of this era were J. J. Sylvester and Arthur Cayley. Neither had the straightforward academic career which their talents deserved. Sylvester was Jewish and so, while allowed to attend lectures at Cambridge, he was not permitted a degree. He had a troubled career shifting between academic institutions in America and England, interspersed with legal work. Cayley, on the other hand, was unhindered by creed, yet not wishing to take holy orders, as Cambridge fellows were required to do until 1871, he decided on graduating to pursue a legal vocation. Fortunately, this occupation was not too arduous to prevent him from writing a large number of papers, and after 15 years he was invited to take up a professorship at Cambridge. So at a time when French and German mathematicians had established niches for themselves in the élite universities of their country, their leading British counterparts were made to struggle.

The centre of British mathematics had long been Cambridge University, an institution which managed to produce a series of very capable, internationally famous, mathematical physicists, including Thomson and Tait, but few leading lights in pure mathematics. Thomson and Tait were very much products of the Cambridge system in which to succeed one was required to absorb little of the latest continental abstract mathematics. In 1845 Thomson had come second in the gruelling Mathematics Tripos examinations, while Tait had gone one better seven years later, passing his degree as Senior Wrangler.

As one might imagine, with few like-minded compatriots Cayley was open to the ideas of amateurs and it was he who orchestrated the admission of Kirkman as a Fellow of the Royal Society in London in 1857. This recognition spurred Kirkman into a decision to enter two competitions organised by the Paris Academy. Such mathematical competitions, usually endowed with monetary prizes, were common in the last century. The first competition called for a study of an aspect of group theory, a field we shall encounter in the following chapter. Nobody was awarded the prize on this occasion, yet the two French entrants were more highly commended than Kirkman. There may well have been an element of Chauvinism here, but Kirkman was also handicapped by not having been trained to write in an established mathematical language. Idiosyncratic terminology and style can make a mathematical paper exceedingly hard to read.

The second competition required a study to be made of polyhedra—figures, such as a cube or a pyramid, bounded by an arrangement of polygonal faces. On learning that this had been chosen as the topic for the competition Kirkman had grown excited, for this was one of his favourite topics and he had already carried out a significant quantity of relevant research. His plan was first to send his findings to the Royal Society for them to be publish in the Society's *Proceedings*, and only then to submit an entry to Paris. However, in view of its prolixity and unconventional terminology, the editing committee were not overly enthusiastic about the idea of publishing Kirkman's paper in its entirety, and they would only agree to print a small portion of it. Once again Kirkman had been rebuffed and from that day on his relations with the scientific establishment were seriously impaired. This resulted in his including diatribes against the Paris Academy and Royal Society in the introduction to his later articles.

Luckily for Tait, Kirkman's research on polyhedra was just what he needed to extend his knot tables to higher orders of knottiness. If you imagine drawing a knot diagram on the surface of a ball, the unbounded region which appeared exterior to the outline of the diagram is now a bounded portion of surface just like any other. Here in Fig. 17, outside the trefoil knot there is just another three-sided region.

Figure 17

A knot diagram can then be seen as a kind of polyhedron in which four faces meet at each corner. An example of an ordinary polyhedron, that is, one with straight edges, which satisfies this condition is the octahedron.

Figure 18

In the case of knot projections, however, the faces are bounded by curved edges, allowing the possibility of faces bounded by only two edges, something impossible for ordinary polygons. The trefoil knot, for instance, corresponds to the polyhedron resulting from gluing together the three 2-sided shapes and the two 3-sided shapes shown in Fig. 19.

Figure 19

Kirkman's constructions could be employed in a process resembling Tait's partition method, allowing the nearly eighty-year-old rector to produce a list of potential eight- and nine-crossing knots for Tait to distill into distinct knots. Tait's census of these knots appeared in 1884.

The following year Kirkman listed 364 potential ten-crossing knots, which Tait whittled down to 124 prime alternating knots. It was now with some surprise that they discovered that they had been joined in their task by an unknown American mathematician, C. N. Little, who had also listed the ten-crossings in an obscure journal, Transactions of the Connecticut Mathematical Society. Little's list suffered from a few omissions but it did show Tait that he had included one duplication in his own list, allowing him to publish an accurate list of the 123 prime alternating knots of knottiness ten.

With the arrival of an enthusiastic colleague, Tait could now take a back seat. By 1890 Little had gone on to produce a tabulation of 11-crossings on the basis of results forwarded by Kirkman, which were found eighty years later to have 11 omissions and one duplication. Over a period of six years between 1893 to 1899 Little also found 54 *non-alternating* knots of knottiness eight, nine and ten. As we shall see in a later chapter, in the 1960s a mathematician would be able to repeat this feat in the space of a single afternoon, demonstrating in a fraction of the time that Little's list was indeed complete. This increase in the speed of classification had been foreseen by Tait:

> Besides, it is probable that modern methods of analysis may enable us (by a single "happy thought" as it were) to avoid the larger part of the labour. It is in matters like this that we have the true "raison d'être" of mathematicians.

More than one "happy thought" would be required, however, as we shall see over the next chapters.

The publication by Tait, Kirkman and Little of their lists of knots marks the end of the first stage of knot theory. Tables had been produced which would be used as a source of examples for knot theorists over the next seven decades and even the most up-to-date ones used today reflect the originals. From here on a more subtle study of the way knots sit in the space they inhabit would be required.

But before pressing on with our tale, let us take the opportunity to consider a question that vexes many a schoolchild: What reasons are there for doing mathematics aside for passing exams to impress universities and employers? The important point to remember here, one lost on governments who seem to think of mathematics solely in terms of the importance of numeracy in the workplace to maximise GDP, is that there is no single answer to this question. (a) It is beneficial to develop an unfamiliar mode of thinking. The vast majority of people are not going to use more than a tiny fraction of the mathematics they learn at secondary school in their later lives. The discipline of this unique controlled form of creativity can only extend one's mental repertoire.
(b) It is exciting to think of yourself in contact with a long line of thinkers stretching back over centuries. No other form of knowledge has the survival rate of mathematics. One of the most advanced techniques taught to our brighter sixteen-year-olds was available to the Babylonians four millennia ago
(c) It encourages contemplation of the mystery of the applicability of mathematics.
Of course, not all children will be persuaded of these reasons, but it wouldn't do any harm to try to persuade some children of some of them. The trouble is that the uninspired, overly pragmatic syllabuses that confront them can offer them little sense of these matters.

In Thomson, Tait and Kirkman we may discern considerable differences in the nature of the motivations causing each of them to engage in mathematical activity. Thomson, for example, would never have dreamed of carrying out mathematical research for its own sake. Mathematics for him was simply a tool to be used in the service of physical science. Kirkman, on the other hand, saw fit to devote countless hours of his spare time to calculations that bore no immediate relation to the current needs of science. The fact that, as in the case we are considering, his ideas could help a colleague develop a mathematical theory which might assist the anti-materialist cause probably encouraged him, but this was not his primary motivation. Tait, meanwhile, should be seen as somewhere in the middle of these two extremes. After all, it was he who embarked upon research in the novel field of knot theory, while Thomson was to remain working with differential equations, the staple diet of mathematical physicists. Moreover, Tait and Thomson carried on a friendly dispute over thirty-eight years as to whether to introduce into the volumes of their textbook a new way of representing physical equations in a four-dimensional number system known as the quaternions. Thomson could not be made to see the point of a calculus of evident mathematical elegance, but whose adoption entailed little or no additional calculating power. Tait, on the other hand, was thoroughly converted to the quaternion cause. And yet, hearing about some investigations of Cayley, like Kirkman very much the pure mathematician, he could remark "Is it not a shame that such an outstanding man puts his abilities to such entirely useless questions?"

So, it is certain that Tait would not have gone ahead with his knot studies had not a physical goal, and beyond that a spiritual goal, been in sight. As he wrote in his first paper on knots:

> The development of this subject promises absolutely endless work—but work of a very interesting and useful kind—because it is intimately connected with the theory of knots, which (especially as applied in Sir W. Thomson's Theory of Vortex Atoms) is likely soon to become an important branch of mathematics.

So, however pure the mathematics, Tait remained in the realm of usefulness, still searching, as had a long line of British physicists descending from Sir Isaac Newton, for the language in which the Book of the Universe had been written. Science was to provide mankind with clues to the workings of God's creation, which for Tait was not one that He would have left to its own devices. Direction from the immaterial world was continual.

It was in fact a common anti-materialist cause that brought Kirkman and Tait together. Kirkman had attracted Tait's attention by a pamphlet in which he had expressed his disagreement with the Darwinian, Herbert Spencer. Now, Spencer, although author of the catchy phrase "the survival of the fittest", was in many ways a liability to the evolutionary camp. A key issue in the debates following the publication of *The Origin of Species* was how life could have begun on Earth. Even were one to accept the validity of the transmutation of species, from what had they transmuted? It was generally accepted that at some point in the past no living thing existed on Earth. What nature of event could possibly have caused life to begin? Spencer had contributed to this debate with what he intended to be the helpful pronouncement that there had occurred 'a change from an indefinite incoherent homogeneity to a definite coherent heterogeneity'. Tait had especially enjoyed Kirkman's parody of this woolly phrase: 'a change from a nohowish untalkaboutable all-likeness to a somehowish and in-general-talkaboutable not-all-likeness'.

It was easy to laugh at any contribution to the debate and Thomson himself did not escape ridicule. While he could not envisage life arising undirected from inorganic matter, he thought to push back the miraculous moment by speculating that 'countless seed-bearing meterioric stones moving about through space' might have bombarded our planet supplying the organisms from which all life on Earth had evolved. This idea, which is topical again today, was not in itself deemed absurd, but when it was conjoined to Thomson's assertion that the Creator continuously directs the development of life in the universe, it presented a golden opportunity for the Darwinians to attack. Thomas Huxley, known as "Darwin's Bulldog" due to his ferocious style of campaigning for the evolutionary cause, mocked Thomson for his "creation by cockshy", and in a letter to an ally he facetiously marvelled at the image of

> God Almighty sitting like an idle boy at the seaside and shying aerolites mostly missing, but sometimes hitting a planet!

Tait's spiritual goal was to establish beyond doubt the impossibility of materialism, that there was not only a universe filled with matter and radiation but an *unseen universe*, and this, as I mentioned earlier, formed the title of the book he wrote with Balfour Stewart. This tome may be seen in part as a response to a lecture given by John Tyndall, then President of the British Association for the Advancement of Science, in Belfast in 1874. Tyndall was a Victorian success story. From very humble roots in Ireland, he had risen to the highest echelons of London society, communing with the likes of William Gladstone. While not a confirmed atheist, Tyndall was certainly opposed to what he saw as the pernicious influence of Christian faith upon the spirit of inquiry. Religious appeals to supernatural causes in place of scientific exploration were for him the cause of intellectual death. So he was most disappointed when during the severe cattle plague of 1870 prayers for divine deliverence were uttered from every pulpit of the land. To his friend the liberal Dean of Westminster Abbey, Arthur Stanley, Tyndall wrote

> I hope and think - and pardon me for thus hoping and thinking - that you will not pray as others will for the staying of this plague, but will ask on the contrary for strength of heart and clearness of mind to meet it manfully and fight against it intelligently.

Stanley duly followed his advice.

For Tyndall, theology must submit to the control of science. Science had shown that within a closed system all energy is conserved, the universe is such a closed system, and so there is no place within it for divine intervention. It was in fact over the question of who had first formulated this principle of the conservation of energy that Tyndall had had the first of many intellectual clashes with Tait. The contents of the Belfast address were to provide another.

The central axiom on which the argument of The Unseen Universe rested, Stewart and Tait named the *Law of Continuity*. This law maintained that we may rely on natural phenomena whose patterns have been recognised not to deviate drastically from what might be expected. For example, the capability for predicting the motions of the planets had been refined over the centuries to a point at which by the late nineteenth century conjunctions and eclipses could be foreseen years in advance. What was forbidden in this case by the Law of Continuity was an abrupt irreversible change to random chaotic motion. Temporary deviations from the laws of physics were possible, but these were only indications of a greater order.

What we have here is something which goes beyond Réné Descartes' belief, expressed over three hundred years earlier, that God as a perfect being is not One who would deceive us. Here we have a God who actually offers us helpful hints.

> In fine, the visible universe was plainly intended to be something which we are capable of investigating, and the few apparent breaks are in reality so many partially concealed avenues leading up to the unseen.

Scientific investigation, they claimed, may show us these breaks. By way of a concrete illustration of how science could reveal these 'avenues leading to the unseen', Tait and Stewart now allowed the vortex-atom theory to enter the picture. Smoke rings are composed of molecules and are of short duration. Molecules, on the other hand, are of much longer duration. Might it not be, they wondered, that molecules are vortices of another finer substance in which rotational flows would take much longer to dissipate? While Thomson saw these rotations to be eternal, Stewart and Tait took them to be of long but finite duration. This would be due to the imperfections of the fluid supporting these vortices. But then this fluid in turn might be composed of vortices in an even finer fluid, and so on indefinitely.

The visible universe would then occur as the final layer of a stratified universe, which taken as a whole would require a directing power. "In fine, our conclusion is, that the visible universe has been developed by an intelligent resident in the Unseen." This resident is not the Absolute Deity, God the Father, who is not within the Universe but stands outside it as its creator. The authors had now to turn to the scriptures and other religious writings for hints as to its nature, with the understanding that such knowledge cannot be made available by scientific investigation. Theological speculations on the Trinity suited their needs admirably. The first person of the Trinity was the unapproachable Creator. The second was the Son, conditioned by the Father to develop the various universes and to become the type and pattern of each order. Ascent and descent between the universes was possible for Him, and it was in the figure of Jesus that he had entered and departed from the visible universe. The third person of the Trinity, the Holy Spirit, could also be given a rôle. It too inhabited the Unseen, organising this time not energy, but life itself, which was taken to be present at all levels of the universe. At a stroke the problem of the origin of life had been solved.

The vortex-atom theory was introduced by Stewart and Tait as a way of providing a concrete illustration for the discussion, and the authors do also suggest the possibility that the visible universe is a three-dimensional projection of a higher dimensional entity, most of whose dimensions remain unseen. It is clear, however, that much was resting for Tait on the former theory. Only this can explain why we find the trefoil knot appearing on the title page as the emblem of the whole book. Tait must have been behind the choice of this, the simplest knot in his tables. Knot theory was pointing the way to the Trinity, as it would a century later for the French psychoanalyst Jacques Lacan.

Reactions to *The Unseen Universe* and to its sequel, *Paradoxical Philosophy*, were mixed. Some clergymen deplored it as heretical, others were enchanted. Views amongst scientists were equally varied, some arguing that, even if the theological speculations were nonsense, the books had been good vehicles through which to convey leading scientific ideas to a new audience. One of Tait's friends, James Clerk Maxwell, the greatest British physicist of the era, even composed a gently humorous poem, *A Paradoxical Ode (After Shelley)*, which opened

My soul's an amphicheiral knot
Upon a liquid vortex wrought
By intellect in the unseen residing,
...

By the time he had classified prime alternating knots of less than ten crossings Tait had 73 models to put into correspondence with atoms of the chemical elements. These already sufficed as less than seventy elements had been isolated by the 1880s. In any case, the next two orders of knottiness provided hundreds of further models. Unfortunately, however, Thomson's part of the project was not faring so well. The knotted vortex-atom was an imaginative idea, which had been part of a theory of some plausibility. No less a figure than Maxwell had stated in the entry for 'atom' in the *Encyclopaedia Britannica* that the vortex atom "satisfies more of the conditions than any atom hitherto considered".

Slowly, however, the realisation dawned on Thomson that his vortices could not be made to rotate stably. He stretched his imagination to picture vortices with hollow cores and vortices composed of twisted bundles of miniature filaments, but by 1883 he still had to admit that "Hitherto I have not succeeded in rigorously demonstrating the stability of the Helmholtz ring in any case." Coupled to this, vortex-atoms offered no 'finger-posts', as he put it, towards an understanding of the phenomena displayed by material bodies—chemical affinity, electricity, magnetism, gravitation, and inertia. Although others were to cling to the vortex atom picture into this century, by the second half of the 1880s Thomson had admitted defeat.

This failure may well have contributed to Thomson's later pessimism as he went on to join in the fin-de-siècle gloom common amongst elderly scientists, with his pronouncement that all that there was left for physicists to do was to calculate additional decimal points to the physical constants. This was a singularly bad prediction given the discovery of the correct structure of the atom and the development of relativity theory and quantum mechanics over the next few decades. Before the turn of the eighteenth century, Lagrange, a leading French mathematician, had proclaimed that "the mine of mathematics has been exhausted", again a hopelessly wrong prediction. Similar forecasts were once more heard before the millennium a few years ago. They have proved to be universally unwise. The philosopher Karl Popper explained why in his critique of

Marxism and all other historicist theories claiming to know what the future held in store for mankind. Since we cannot know what scientists will find, we cannot predict what new technologies will be developed, and so we cannot know how societies will be structured.

As for Tait, I should not wish to give the impression that he abandoned all other scientific pursuits for knot theory and theology. As a product of the Cambridge Mathematical Tripos, he had the requisite tools to tackle many other research projects of a more mundane nature. One of these arose from his realisation that the standard theory of projectiles could not account for the great distances travelled by a golf-ball. However, his attempts to model the extra length gained by the imparting of backspin by a club were sorely tested by the prodigious driving ability of his son, winner of the Championship on several occasions. Tait's final years were blighted when this his only son, a lieutenant in the Black Watch, slipped early into the unseen having receiving a bullet from the rifle of a Boer.

Despite Thomson's disappointment, much had been gained from his idea. As so often happens, a very important branch of applied mathematics, in this case fluid dynamics, receives a boost from a theoretical model we now take to be obsolete. Indeed, the modern study of the generation of turbulent flows in fluids embraces not just vortices, but cascades of vortices within vortices. The mathematical theory of knots, on the other hand, as we shall see, was to lead a life of its own, touched only by other branches of pure mathematics for the best part of a century. We may pause to wonder where knot theory would be today had not a Holy Grail appeared to Tait. Would anyone have spent innumerable hours pencilling in and erasing knots without such divine motivation? In a secular world, it may seem that we have lost such sources of inspiration, but they are seldom far from the surface, emerging in recent times in such phrases as Stephen Hawking's repeated wish to 'know the mind of God'. And the pure mathematician? Even if, delighting solely in the play of abstract structures, she does not have the goal of understanding the language in which the universe is written so firmly in her sights, for her there are still blessed moments of ecstatic fulfilment when finally everything fits perfectly into place. It is to the pure mathematicians of the Continent that we now turn.