Math 396. Stokes' theorem with corners

1. Motivation

The version of Stokes' theorem that has been proved in the course has been for oriented manifolds with boundary. However, the theory of integration of top-degree differential forms has been defined for oriented manifolds with corners. In general, if M is a manifold with corners then ∂M is not a manifold with corners. For example, if $M = [0,1]^3$ is the unit solid cube in $\mathbf{R}^3$ then ∂M is problematic at the vertices. Nonetheless, in our early development of the theory of manifolds with corners we saw how to naturally stratify M into locally closed subsets M_r consisting of points whose index of singularity is r (with index r at a point meaning that the point has a neighborhood that is isomorphic to a neighborhood of the origin in $[0,\infty)^r \times \mathbf{R}^n$ for some $n \geq 0$). In particular, we saw that the locus $M_{\leq r}$ of points with index $\leq r$ is an open subset of M that is naturally a manifold with corners.

As a special case, $M_{\leq 1}$ is a manifold with boundary. For example, if M is a polygonal region in the plane then ∂M is the boundary polygon that is *not* a manifold with corners (since a 1-manifold with corners has to be a 1-manifold with boundary), but $M_{\leq 1}$ is the complement of the vertices and its boundary is the union of open boundary segments of the polygon with the vertices deleted. Hence, by working with $M_{\leq 1}$ we are led to formulate a natural version of Stokes' theorem in the setting of manifolds with corners:

Theorem 1.1. *Let (M,μ) be an oriented manifold with corners and with constant dimension $n \geq 1$. Choose a compactly supported $\omega \in \Omega_M^{n-1}(M)$, and give $\partial(M_{\leq 1})$ the induced orientation $\partial\mu$ as the boundary of the manifold-with-boundary $M_{\leq 1}$. The $(n-1)$-form ω is absolutely integrable on $\partial(M_{\leq 1})$ and $\int_{M,\mu} \mathrm{d}\omega = \int_{\partial(M_{\leq 1}),\partial\mu} \omega$.*

The hypothesis of compact support for ω on M is inherited by $\mathrm{d}\omega$, so $\mathrm{d}\omega$ is absolutely integrable over M. However, the pullback of ω to $\partial(M_{\leq 1})$ is generally *not* compactly supported, and so the absolutely integrability of this $(n-1)$-form over $\partial(M_{\leq 1})$ has content. (As an example, if $M = [0,1]^2$ is the unit square then $\partial(M_{\leq 1})$ is the non-compact union of the four open edges with endpoints deleted. A 1-form $f(x,y)\mathrm{d}x$ pulled back to $\partial(M_{\leq 1})$ generally does not have compact support.) The preceding theorem, which one may call "Stokes' theorem with corners", is used all the time in practice without hesitation. For example, one computes an integral over an oriented polygonal planar region as an integral over the oriented bounding polygon (with vertices deleted).

Since the locus $M_{\geq 2}$ of points with index ≥ 2 is a closed set of measure zero in ∂M, it is immaterial for the calculation of integrals (in the sense discussed in the handout on integration over manifolds) how we treat $M_{\geq 2}$ for computational purposes. It is reasonable to ask if there is a Stokes' theorem on more singular spaces, such as cones or general algebraic sets (zero loci of multivariable polynomials) in Euclidean space. One can partially axiomatize exactly what properties are required to push through the method of proof, but in practice one usually just adapts the argument to the case at hand via a suitable limit process. We will say nothing further here on formulation of Stokes' theorem beyond the context of manifolds with corners. In the next section, we explain how to prove Theorem 1.1 by copying the proof in the case of manifolds with boundary.

2. Proof of Theorem 1.1

The main problem in the proof of Theorem 1.1 is that we have to also verify the absolute integrability of ω over $\partial(M_{\leq 1})$ during the proof. Other than this issue, the method of proof will be rather much the same as in the case of manifolds with boundary.

Let $\{U_i\}$ be a locally finite open covering by connected coordinate charts and let $\{\phi_i\}$ be a subordinate partition of unity with compact supports. The boundaries $\partial((U_i)_{\leq 1})$ are a locally finite open covering for $\partial(M_{\leq 1})$ and $\{\phi_i|_{\partial((U_i)_{\leq 1})}\}$ is a subordinate partition of unity with generally *non-compact* supports (since $\partial(M_{\leq 1})$ is generally not closed in ∂M, except when M is a manifold with boundary). However, we know from the earlier handout on integration over manifolds that ω is absolutely integrable over $\partial(M_{\leq 1})$ if and only if each $\omega_i = (\phi_i\omega)|_{\partial((U_i)_{\leq 1})}$ is absolutely integrable over $\partial((U_i)_{\leq 1})$ ((or equivalently, over $\partial(M_{\leq 1})$) with $\sum_i \int_{\partial((U_i)_{\leq 1})} |\omega_i| < \infty$, in which case $\sum_i \int_{\partial((U_i)_{\leq 1}),\partial\mu} \omega_i$ is absolutely convergent and equal to $\int_{\partial(M_{\leq 1}),\partial\mu} \omega$.

Let us suppose that Theorem 1.1 is known for the U_i's, so since $\phi_i\omega$ is compactly supported in U_i (with support contained in the compact support of ϕ_i) it follows from our assumption of the theorem for U_i that $\phi_i\omega$ has absolutely integrable pullback over $\partial((U_i)_{\leq 1})$ with

$$\int_{\partial((U_i)_{\leq 1}),\partial\mu} \phi_i\omega = \int_{U_i,\mu} \mathrm{d}(\phi_i\omega) = \int_{M,\mu} \mathrm{d}(\phi_i\omega).$$

Since the support of ω is compact in M, it meets only finitely many U_i's. Hence, $\phi_i\omega$ vanishes for all but finitely many i. For each i we know that $\phi_i\omega$ over $\partial(M_{\leq 1})$ is supported inside of the open subset $\partial((U_i)_{\leq 1})$, so absolute integrability over this open subset is equivalent to absolute integrability over $\partial(M_{\leq 1})$. We conclude that $\phi_i\omega$ is absolutely integrable over $\partial(M_{\leq 1})$ for all i. Adding this up over the finitely many i such that $\phi_i\omega$ is not identically zero on M, it follows that the *finite* sum $\sum \phi_i\omega = \omega$ has absolutely integrable pullback over $\partial(M_{\leq 1})$ and that

$$\int_{\partial(M_{\leq 1}),\partial\mu} \omega = \int_{\partial(M_{\leq 1}),\partial\mu} \sum_i \phi_i\omega = \sum_i \int_{\partial(M_{\leq 1}),\partial\mu} \phi_i\omega = \sum_i \int_{\partial((U_i)_{\leq 1}),\partial\mu} \phi_i\omega,$$

where we suppress the "pullback" notation and we note that all but finitely many terms in each sum is equal to 0.

By the assumption that Theorem 1.1 is known for the U_i's, we may rewrite our formula as

$$\int_{\partial(M_{\leq 1}),\partial\mu} \omega = \sum_i \int_{U_i,\mu} \mathrm{d}(\phi_i\omega) = \sum_i \int_{U_i,\mu} (\mathrm{d}\phi_i \wedge \omega + \phi_i \mathrm{d}\omega) = \sum_i \int_{M,\mu} \mathrm{d}\phi_i \wedge \omega + \sum_i \int_{M,\mu} \phi_i \mathrm{d}\omega$$

since $\phi_i\mathrm{d}\omega$ and $\mathrm{d}\phi_i \wedge \omega$ vanish for all but finitely many i and are compactly supported in U_i. The final sum is $\int_{M,\mu} \mathrm{d}\omega$, so it remains to prove that the sum of the $\int_{M,\mu} \mathrm{d}\phi_i \wedge \omega$'s is equal to 0. This goes exactly as in the proof of the usual Stokes' theorem, namely using the equality of the finite sum $\sum_i (\mathrm{d}\phi_i \wedge \omega)$ with the wedge product $(\sum_i \mathrm{d}\phi_i) \wedge \omega$ against the *locally finite* sum $\sum_i \mathrm{d}\phi_i = \mathrm{d}(\sum_i \phi_i) = \mathrm{d}(1) = 0$. Hence, we get the desired result over M when it is assumed to hold over the U_i's.

We have now reduced the problem to the case when M is a connected open subset U in a standard sector $\Sigma = [0,\infty)^r \times \mathbf{R}^{n-r}$ in $\mathbf{R}^n$ for some $0 \leq r \leq n$, and since ω has *compact* support in U we can use "extension by zero" to consider it as a compactly supported form on Σ. By connectivity of U, $\pm\mu$ are the only two orientations on U. Hence, replacing μ with $-\mu$ if necessary allows us to assume that the orientation on U is the standard one arising from how it sits as an open in a sector in $\mathbf{R}^n$. Since $\partial(M_{\leq 1}) = \partial(\Sigma_{\leq 1}) \cap M$, it follows that we may replace U with Σ exactly like in our earlier proof of Stokes' theorem for manifolds with boundary.

The boundary $\partial(\Sigma_{\leq 1})$ is covered by the disjoint open loci

$$\partial\Sigma_i = (0,\infty)^{i-1} \times \{0\} \times (0,\infty)^{r-i} \times \mathbf{R}^{n-r}$$

for $1 \leq i \leq r$ in $\partial\Sigma$, so from the handout on integration over manifolds we have that ω is absolutely integrable over $\partial(\Sigma_{\leq 1})$ if and only if it is absolutely integrable over each $\partial\Sigma_i$ (with $1 \leq i \leq r$)

that that in such cases $\int_{\partial(\Sigma_{\le 1}),\partial\mu}\omega = \sum_i \int_{\partial\Sigma_i,\partial\mu}\omega$. Hence, our goal is to show that ω is absolutely integrable over $\partial\Sigma_i$ for $1 \le i \le r$ and that

$$\sum_{i=1}^{r}\int_{\partial\Sigma_i,\partial\mu}\omega = \int_{\Sigma,\mu}\mathrm{d}\omega$$

where μ is the standard orientation arising from $\mathbf{R}^n$. We may uniquely write

$$\omega = \sum_{j=1}^{n} f_j \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_j} \wedge \cdots \wedge \mathrm{d}x_n$$

with each f_j a smooth compactly supported function on Σ, so $\mathrm{d}\omega = \sum_{i=1}^{n}(-1)^{j-1}\partial_{x_j} f_j \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ and hence

$$\int_{\Sigma,\mu}\mathrm{d}\omega = \int_{[0,\infty)^r \times \mathbf{R}^{n-r}} \sum_{i=1}^{n} \partial_{x_j} f_j.$$

The only term in

$$\omega|_{\partial\Sigma_i} = \sum_{j=1}^{n} f_j|_{\partial\Sigma_i} \mathrm{d}(x_1|_{\partial\Sigma_i}) \wedge \cdots \wedge \mathrm{d}\widehat{(x_j|_{\partial\Sigma_i})} \wedge \ldots \mathrm{d}(x_n|_{\partial\Sigma_i})$$

that does not necessarily vanish is for $j = i$ (since $x_i|_{\partial\Sigma_i}$ is constant), and the sign of the coordinate system $\{x_1|_{\partial\Sigma_i}, \ldots, x_{i-1}|_{\partial\Sigma_i}, x_{i+1}|_{\partial\Sigma_i}, \ldots, x_n|_{\partial\Sigma_i}\}$ with respect to the boundary orientation on the open locus $\partial\Sigma_i$ in $\partial(\Sigma_{\le 1})$ is $(-1)^n$ times the sign $(-1)^{n-i}$ of the orientation of the coordinate system $\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_i\}$ with respect to the orientation μ on Σ arising from the standard orientation (i.e., trivialization of tangent bundle) on $\mathbf{R}^n$. This combines to give a sign of $(-1)^i$, so

$$\sum_{i=1}^{r}\int_{\partial\Sigma_i,\partial\mu}\omega = \sum_{i=1}^{r}\int_{(0,\infty)^{i-1}\times(0,\infty)^{r-i}\times\mathbf{R}^{n-r}} (-1)^i f_i(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n).$$

For $f \in C^\infty([0,\infty)^r \times \mathbf{R}^{n-r})$ with compact support we need that $\int_{[0,\infty)^r\times\mathbf{R}^{n-r}}(-1)^{i-1}\partial_{x_i} f = 0$ if $i > r$ and that

$$\int_{[0,\infty)^r\times\mathbf{R}^{n-r}}(-1)^{i-1}\partial_{x_i} f = (-1)^i \int_{(0,\infty)^{i-1}\times(0,\infty)^{r-i}\times\mathbf{R}^{n-r}} f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$

otherwise. For $i > r$, we use Fubini's theorem to compute the integral over $[0,\infty)^r \times \mathbf{R}^{n-r}$ (or really just over a big compact box containing the support of f) as an interated integral with innermost integration taken over x_i-lines with all other coordinates held fixed. Since $i > r$, these inner integrals are instances of the integral $\int_{-\infty}^{\infty} h'(t)\mathrm{d}t$ with $h \in C^\infty(\mathbf{R})$ a compactly supported smooth function. Such integrals vanish, by the Fundamental Theorem of Calculus.

For $1 \le i \le r$, our problem is to prove

$$\int_{[0,\infty)^r\times\mathbf{R}^{n-r}}\partial_{x_i} f = -\int_{(0,\infty)^{i-1}\times(0,\infty)^{r-i}\times\mathbf{R}^{n-r}} f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n).$$

We may replace $[0,\infty)^r$ with $(0,\infty)^r$ on the left side since $[0,\infty)^r \times \mathbf{R}^{n-r}$ has boundary that is a closed set of measure zero in $\mathbf{R}^n$. Using Fubini's theorem to shift the integration over the ith coordinate to the inside, it remains to note that (by the Fundamental Theorem of Calculus)

$$\int_{(0,\infty)}(\partial_{x_i} f)(c_1, \ldots, c_{i-1}, t, c_{i+1}, \ldots, c_n) = -f(c_1, \ldots, c_{i-1}, 0, c_{i+1}, \ldots, c_n)$$

for $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_r > 0$ and $c_{r+1}, \ldots, c_n \in \mathbf{R}$.